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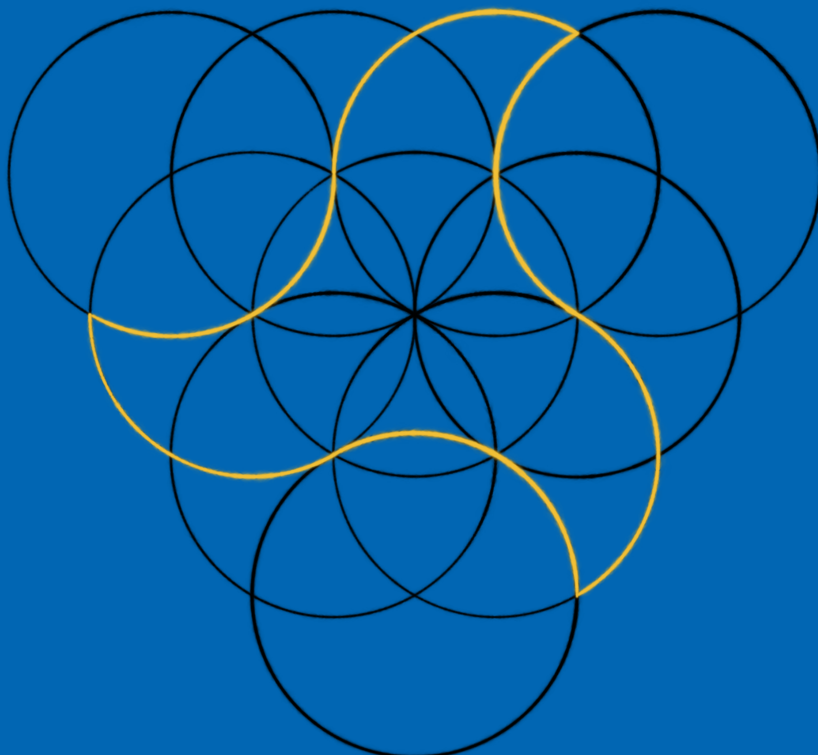
Jacek Jendrej
Recent progress on the problem of soliton resolution

Miguel Ortega
An introduction to the mathematics in the Alhambra of Granada

Stéphane Vinatier and Reg Alcorn
Mathematics as seen by an artist: Inspiring mathematical objects



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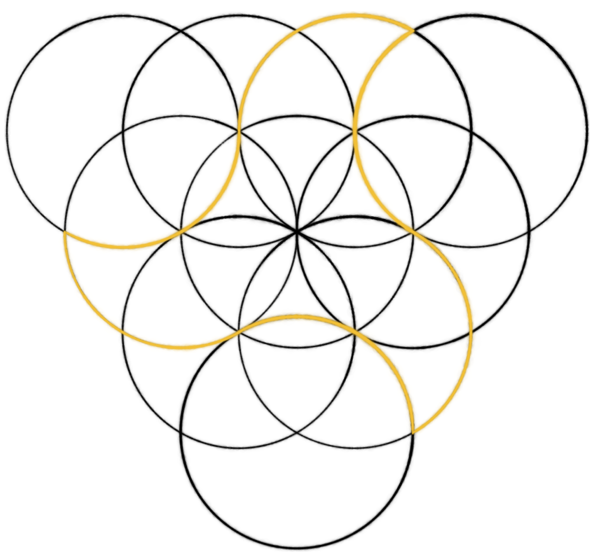

The cover illustration by A. B. Araújo shows the construction of a Nasrid *pajarita*, referencing the article on the mathematics in the Alhambra of Granada, authored by Miguel Ortega in the present issue.

Photo by Jim Høyer, University of Copenhagen.

Since my last message for the Magazine, the world has seen much change and the world of academia is not going unaffected. It may be argued that academic freedom is being challenged in new and unsuspected ways. This will of course have a ripple effect on the entire academic world, but mathematics will also be affected, and the EMS views these developments with concern. It is crucial for EMS to stand up for academic freedom and academic values and to unite with academics all over the world where such values are being questioned. The EMS also sees diversity, in particular, regarding gender, race, geography, and scientific subjects, as part of its core values. This view has guided some of our recent successful initiatives such as the formation of the young academy EMYA, the topical activity groups (TAGs), and the general support within the EMS for open science.

For now, Europe is only indirectly affected, but changing policies and financial priorities may soon have an effect on us too. We, of course, have our own problems in Europe. British universities are seeing serious financial problems. Some mathematics departments have been singled out for severe personnel cuts. EMS and many of our member societies have joined the London Mathematical Society in attempts to prevent this. I hope we can all make a difference. This is probably much more difficult in Ukraine where the situation is unfortunately not improving and the conditions for everyone, including academics, are hard to imagine.

Speaking of diversity let me turn for a moment to an issue that has been a focal point of discussion within the EMS. Our society has always held the ambition to encompass all aspects of mathematics, from the purest to the most applied. I am actually not particularly fond of these terms as I believe adding adjectives to "mathematics" to create classifications divides rather than unites. I understand that these terms may sometimes be useful in defending underrepresented groups, and this is an important reason not to disregard them entirely, though they may prove less productive in the long term, and in general we hope to see a celebration of mathematics that embraces all its beauty and diversity without the need for narrow categorization. Therefore, I am a strong supporter of EMS being diverse and inclusive, and I am proud that it is. There has been some criticism that the EMS could do even more for diversity and that we should keep broadening our perspective on mathematics. This is certainly our ambition and I hope we will continue to be successful in being a society for all mathematicians and that all mathematicians feel that we speak for them and have something to offer across the full spectrum of mathematics. We can of course only be successful in doing so if we all start thinking more of mathematics as a unified field and less of its fragmentation.

EMS is considering its social media presence in particular regarding the platforms X and Facebook. We have opened an account on Bluesky with the goal of possibly leaving X. The EMS is not off X yet, and I would like to take this opportunity to encourage all of you to follow us on Bluesky whether or not you are already following us on X.

In my last message I thanked all the new chairs of our standing committees. Since then Ann Dooms, professor at the Department of Mathematics & Data Science at the Vrije Universiteit Brussel, has accepted to chair the Education Committee. I welcome Ann as chair of the committee, and I am looking forward to working with her on the many important challenges facing mathematics education across Europe and the world in general.

Finally, I hope you have enjoyed receiving the email updates with news from the EMS. I am grateful to Enrico Schlitzer, our Community Engagement Manager, for putting these together.

Jan Philip Solovej
President of the EMS

Dear readers of the EMS Magazine, in this issue you can find many interesting contributions in the usual sections on discussions, societies, education, book reviews, and one more article by one of the 2024 EMS Prize winners, this time written by Jacek Jendrej.

In no way disparaging the other contributions, I would like to highlight the articles *"An introduction to the mathematics in the Alhambra of Granada"* by Miguel Ortega and *"Mathematics as seen by an artist: Inspiring mathematical objects"* by Stéphane Vinatier and Reg Alcorn. With these papers the Magazine joins the celebration of the International Day of Mathematics,[1] whose theme for 2025 is: Mathematics, Art and Creativity. In this regard, I also invite you to take a look at the illustration created by Coni Rojas-Molina and at the Magazine cover illustration by António B. Araújo.

Finally, with the current issue three new editors are starting their duties in the editorial board: Bruno Teheux, who will reinforce the Features and Discussion team, Michele Coti Zelati, who will be responsible for the Book Review section, and Vladimir Tsanov, who will take care of the Solved and Unsolved Problems section. A short biographical note of each of them can be found at the end of the issue. I thank them and I am grateful to them for accepting to dedicate part of their time to our Magazine.

Happy reading!

Donatella Donatelli
Editor-in-chief

[1] https://www.idm314.org

Recent progress on the problem of soliton resolution

Jacek Jendrej

Dispersive partial differential equations are evolution equations whose solutions decay in large time due to the fact that various frequencies propagate with distinct velocities. In some cases, there exist special solutions called solitons*, which do not change their shape as time passes. The* soliton resolution conjecture *predicts that solitons are the only obstruction to the decay of solutions. More precisely, every solution eventually decomposes into a superposition of solitons and a decaying term called* radiation*. We discuss the conjecture in the context of the wave maps equation, which is the analog of the wave equation for sphere-valued maps.*[1]

1 The phenomenon of dispersion

This section is devoted to standard introductory material. For a comprehensive introduction to the topic, the reader can consult for instance [27, 44].

1.1 The wave equation

Consider the *wave equation* in dimension $1+2$,

$$c^{-2}\partial_t^2\psi(t,x_1,x_2) = \partial_{x_1}^2\psi(t,x_1,x_2) + \partial_{x_2}^2\psi(t,x_1,x_2), \tag{W}$$

where $(t,x_1,x_2) \in \mathbb{R}^{1+2}$. The positive number $c > 0$ is the wave speed. Let us assume for simplicity that ψ is real-valued, but it could just as well be vector-valued. We will always write $\mathbb{R}^{1+2}$ instead of $\mathbb{R}^3$ in order to stress that one deals with one time dimension and two space dimensions. Equation (W) is equivalent to requiring that ψ is a *critical point* of the Lagrangian

$$\mathscr{L}(\psi) := \frac{1}{2}\int_{\mathbb{R}^{1+2}} \big(c^{-2}(\partial_t\psi)^2 - (\partial_{x_1}\psi)^2 - (\partial_{x_2}\psi)^2\big). \tag{1}$$

The precise meaning of this assertion is the following. Let $\zeta\colon \mathbb{R}^{1+2} \to \mathbb{R}$ be a smooth compactly supported function and $\psi_\varepsilon := \psi + \varepsilon\zeta$, where ε is a small real number. We then have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{L}(\psi_\varepsilon) &= \int_{\mathbb{R}^{1+2}} \big(c^{-2}\partial_t\psi\partial_t\zeta - \partial_{x_1}\psi\partial_{x_1}\zeta - \partial_{x_2}\psi\partial_{x_2}\zeta\big) \\ &= -\int_{\mathbb{R}^{1+2}} \zeta\big(c^{-2}\partial_t^2\psi - \partial_{x_1}^2\psi - \partial_{x_2}^2\psi\big),\end{aligned} \tag{2}$$

where the last step is integration by parts. The left-hand side can be interpreted as the directional derivative of $\mathscr{L}$ at ψ in the direction ζ. Hence, we see that all the directional derivatives vanish if and only if ψ satisfies (W).

Equation (W) appears in many physical contexts, the most familiar being the evolution in time of a small disturbance of the membrane of a drum. It should be understood that the membrane extends to infinity and occupies the whole horizontal plane, and $\psi(t,x_1,x_2)$ is the vertical displacement at time t of the element of the membrane whose horizontal coordinates are (x_1,x_2).

Recalling that the Lagrangian density is the difference of the kinetic and the potential energy densities, from the form of (1) we find that the total energy is given by

$$\begin{aligned}E(\psi) &:= E_{\text{kinetic}} + E_{\text{potential}} \\ &= \frac{1}{2}\int_{\mathbb{R}^2} c^{-2}(\partial_t\psi)^2 + \frac{1}{2}\int \big((\partial_{x_1}\psi)^2 + (\partial_{x_2}\psi)^2\big),\end{aligned} \tag{3}$$

and is a conservation law (a quantity independent of time).

Remark 1.1. By an appropriate choice of units, one can assume that $c = -1$, which we will always do in the sequel. We will also write

$$x := (x_1,x_2), \qquad \nabla := (\partial_{x_1},\partial_{x_2}), \tag{4}$$

$$\Delta := \partial_{x_1}^2 + \partial_{x_2}^2, \qquad \mathrm{d}x := \mathrm{d}x_1\mathrm{d}x_2. \tag{5}$$

Mechanical intuition suggests that in order to determine the evolution in time of the disturbance of the membrane we need to specify the *initial conditions* consisting of the initial positions $\psi(0,x)$ and the initial velocities $\partial_t\psi(0,x)$ of all the elements of the drum. One can indeed prove that for any such initial conditions, equation (W) has a unique solution for all time, which moreover depends continuously on the initial conditions (in an appropriate sense that we will not make precise here), which is referred to as *global well-posedness*.

[1] This note is based on the talk given by the author at the 9th European Congress of Mathematics.

Having once more recourse to the intuition from mechanics, we can expect that, if the membrane is initially perturbed only in a bounded region and flat elsewhere, then this disturbance will propagate in various directions, resulting in a decay of its amplitude, namely

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^2} |\psi(t,x)| = 0, \tag{6}$$

which is referred to as *radiative behavior*.

1.2 A few generalities on linear dispersive PDEs

Since it is hard to give a rigorous definition of a linear dispersive PDE which would cover all the interesting cases, we limit ourselves to the following heuristic definition.

Definition 1.2. A linear PDE is called *dispersive* if

(i) it is an evolution equation: it involves the time variable t and the space variable x,

(ii) various frequencies propagate with distinct velocities.

Examples of linear dispersive PDEs include:

- the wave equation (W) and its analogs in higher space dimensions,
- the Schrödinger equation,
- the Klein–Gordon equation,

but the list could be made longer. All these examples are *time-reversible* and have a *conserved energy*, and yet smooth localized initial conditions lead to *radiative behavior* as $t\to\infty$.

Remark 1.3. Radiative behavior crucially depends on the fact that the spatial domain is the whole Euclidean space (or in any case that it is unbounded).

Remark 1.4. In some sense, for a linear dispersive PDE, the trivial solution $\{\psi\equiv 0\}$ is the *global attractor* of the flow.

2 The nonlinear setting: dispersion, solitons, soliton resolution

For a comprehensive introduction to the topic of this section, the reader can consult the monographs [42] and [36].

2.1 Wave maps

Wave maps are nonlinear, geometric analogs of linear waves in the case of maps taking values in a Riemannian manifold, rather than in a Euclidean space. We consider here wave maps $\Psi\colon \mathbb{R}^{1+2}\to\mathbb{S}^2\subset\mathbb{R}^3$. In classical mechanics, a constrained mechanical system is obtained from the same Lagrangian as for the ambient system, see [1, Chapter 4]. Following this principle and recalling (1), we say that a map $\Psi\colon \mathbb{R}^{1+2}\to\mathbb{S}^2\subset\mathbb{R}^3$ is a *wave map* if it is a critical point of the Lagrangian

$$\mathscr{L}(\Psi) = \frac{1}{2}\int_{\mathbb{R}^{1+2}} (|\partial_t\Psi|^2 - |\partial_{x_1}\Psi|^2 - |\partial_{x_2}\Psi|^2), \tag{7}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$. Similarly, as in Section 1.1, we can consider $\Psi_\varepsilon = \Psi+\varepsilon Z$, where $Z\colon \mathbb{R}^{1+2}\to\mathbb{R}^3$ is smooth and compactly supported. In order not to violate at main order the condition that Ψ_ε takes values in $\mathbb{S}^2$, it is necessary and sufficient to require that

$$Z(t,x)\perp\Psi(t,x) \qquad \text{for all } (t,x). \tag{8}$$

As in Section 1.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{L}(\Psi_\varepsilon) = -\int_{\mathbb{R}^{1+2}} Z\cdot(\partial_t^2\Psi - \Delta\Psi). \tag{9}$$

This quantity vanishes for all Z satisfying (8) if and only if

$$\partial_t^2\Psi(t,x) - \Delta\Psi(t,x) = \mu(t,x)\Psi(t,x) \tag{10}$$

for some $\mu(t,x)\in\mathbb{R}$. Differentiating twice the identity $\Psi\cdot\Psi=1$, we obtain

$$\Psi\cdot(\partial_t^2\Psi-\Delta\Psi) = -|\partial_t\Psi|^2 + |\partial_{x_1}\Psi|^2 + |\partial_{x_2}\Psi|^2, \tag{11}$$

so we can write the wave map equation $\mathbb{R}^{1+2}\to S^2$ as

$$\partial_t^2\Psi-\Delta\Psi = -(|\partial_t\Psi|^2 - |\partial_{x_1}\Psi|^2 - |\partial_{x_2}\Psi|^2)\Psi. \tag{WM}$$

Similarly, as in the linear case, the *total energy*

$$E(\Psi) \coloneqq \int_{\mathbb{R}^2}\Big(\frac{1}{2}|\partial_t\Psi|^2 + \frac{1}{2}|\partial_{x_1}\Psi|^2 + \frac{1}{2}|\partial_{x_2}\Psi|^2\Big) \tag{12}$$

is a conservation law for wave maps.

Equation (WM) has trivial constant in space-time solutions $\Psi(t,x)=\omega_0\in\mathbb{S}^2$. If we linearize around such a solution, that is, if we write $\Psi=\omega_0+\varepsilon\Phi$ with $\varepsilon\ll 1$ and plug into (WM), at main order we obtain $\partial_t^2\Phi-\Delta\Phi=0$; thus, each component of Φ satisfies the wave equation (W), which indicates that small perturbations of a constant solution should exhibit a radiative behavior, so that the whole wave map should converge to a constant. It was proved in the works of Tataru [43] and Tao [41] that, in an appropriate sense, this is indeed the case.

For the study of the long-time behavior of solutions of (WM), *criticality* is an important (and helpful) property. Let $\lambda>0$ and consider

$$\Psi_\lambda(t,x) \coloneqq \Psi(t/\lambda, x/\lambda). \tag{13}$$

It is clear from (WM) (or from the Lagrangian) that Ψ is a wave map if and only if Ψ_λ is a wave map. Moreover,

$$E(\Psi_\lambda) = E(\Psi). \tag{14}$$

For this reason, equation (WM) is called *energy critical*, and its solutions *critical wave maps*.

Remark 2.1. In general, a problem is *subcritical* if it becomes a "small data problem" when rescaling (zooming) to a small region. It is called *supercritical* if such a zoom makes it large. It is called *critical* if the size of the data remains unchanged.

2.2 Harmonic maps

One might wonder if every, not necessarily small, solution of (WM) has radiative behavior. The answer is "no" for a simple reason: there exist non-trivial static (time-independent) solutions. Namely, inserting $\Psi(t,x) = \omega(x)$ into (WM), we obtain the *critical harmonic map equation*:

$$-\Delta\omega = |\nabla\omega|^2\omega, \qquad \omega\colon \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3. \tag{HM}$$

Its solutions are called *harmonic maps* $\mathbb{R}^2 \to \mathbb{S}^2$. It was proved by Eells and Wood [13], and Hélein [17] that harmonic maps of finite energy correspond to *rational functions* $\mathbb{S}^2 \to \mathbb{S}^2$ and their complex conjugates (we identify $\mathbb{R}^2$ with $\mathbb{S}^2$ using the stereographic projection).

Remark 2.2. It was proved by Krieger, Schlag and Tataru [26], Rodnianski and Sterbenz [32], and Raphaël and Rodnianski [31], that solutions of large energy can even cease to exist in finite time. Equation (WM) is thus locally well-posed, but not globally well-posed.

2.3 A few generalities on nonlinear dispersive PDEs

A nonlinear PDE is called dispersive if it is related to a linear dispersive PDE. Most frequently, "related" means "obtained through linearization around trivial solutions," like in the case of wave maps discussed above.

Nonlinear dispersive PDEs appear frequently in physics, for example in the study of water waves and nonlinear optics, see [44, Chapters 12, 13, 16, 17]. Typical examples are Hamiltonian systems, which, in particular are *time-reversible* and have a *conserved energy*.

One is often interested in the *dynamical behavior* of solutions of a given nonlinear PDE, by which we mean their asymptotic description as time becomes large (for solutions defined for all time; if they are not, one studies the limit as the time tends to the maximal time of existence of the solution). Among the most common questions of this type is the *problem of stability*, which can be formulated as follows.

Problem. *Do small solutions of a nonlinear dispersive PDE exhibit radiative behavior? In other words, does the flow restricted to small solutions have a trivial attractor, like in the linear case (see Remark 1.4)?*

The intuitive reasoning is that small solutions should behave in the same way as the solutions of the linearized problem, which have radiative behavior as we saw in Section 1.2.

2.4 Solitons and soliton resolution

The notion of a soliton is somewhat controversial, see [28, Section 1.5]. We adopt the following definition.

Definition 2.3. A soliton is a solution of an evolution PDE which does not change its shape in the course of time (it can, however, change its position).

Harmonic maps from Section 2.2 are examples of solitons for (WM). Solitons moving at constant velocity can be obtained using the Lorentz invariance of (WM).

Solitons do not exhibit radiative behavior. The problem of *soliton resolution* is to prove that they are the only obstruction to radiative behavior. However, it would be too naive to expect that every solution is either radiative or a soliton. Rather, one expects that every solution eventually decomposes into a superposition of solitons which interact sufficiently weakly (for example, they could travel with distinct velocities). Such superpositions are called *multisoliton configurations*.

Problem (Soliton resolution conjecture). *For a given nonlinear dispersive PDE, does every solution converge in large time to the sum of a multisoliton and a radiative term? In other words, does the flow have a simple global attractor related to multisoliton configurations?*

The soliton resolution is inspired by

- numerical simulations, see Fermi, Pasta and Ulam [14], Zabusky and Kruskal [45],
- the theory of *completely integrable systems*, see Segur and Ablowitz [34], Eckhaus and Schuur [12],
- analogous *elliptic* and *parabolic* problems (*bubbling*), see Sacks and Uhlenbeck [33], Struwe [39].

Remark 2.4. Even with such a vague formulation, the soliton resolution is not expected to hold for all nonlinear dispersive PDEs. For example, the sine-Gordon equation has so-called breather solutions, which do not fall into the regime of soliton resolution.

Remark 2.5. Strictly speaking, soliton resolution only provides an "upper bound" on the global attractor, in the sense that it does not say anything on the types of multisoliton configurations which can be realized by the evolution.

Our main goal is to provide an example of "natural" nonlinear dispersive PDEs for which we can prove soliton resolution. Even though we cannot provide a full description of the global attractor, we will see that it contains configurations consisting of more than one soliton.

3 Soliton resolution for equivariant energy-critical wave maps

3.1 Equivariant maps

The governing PDE will be obtained from (WM) by restricting the flow to a certain subclass of all the maps $\mathbb{R}^2 \to \mathbb{S}^2$ which is preserved by the flow. They are called *equivariant maps* and are defined in the following way. We fix $k \in \{1, 2, \dots\}$ and consider maps of the form

$$\begin{aligned} &\Psi(t, r\cos\theta, r\sin\theta) \\ &\quad = \big(\sin(\psi(t,r))\cos(k\theta), \sin(\psi(t,r))\sin(k\theta), \cos(\psi(t,r))\big), \end{aligned} \tag{15}$$

where $r > 0$ and $\psi(t,r) \in \mathbb{R}$. Plugging this expression into (WM), we find the *scalar equation*

$$\partial_t^2\psi(t,r) - \partial_r^2\psi(t,r) - \frac{1}{r}\partial_r\psi(t,r) + \frac{k^2}{2r^2}\sin(2\psi(t,r)) = 0, \tag{WM$_k$}$$

where $r > 0$ is the radial coordinate. Note that ψ and $\psi + 2\pi\ell$ represent the same map Ψ for any $\ell \in \mathbb{Z}$.

Remark 3.1. In the non-geometric context, it is common to consider spherically symmetric solutions. Equivariant solutions are analogous objects in the geometric setting of (WM). More generally, whenever symmetries of a given equation lead to invariance of a certain class of states, it is a well-known technique to study the restriction of the system to this subclass.

One can check that under the substitution (15), the Lagrangian (7) becomes

$$\mathscr{L} := \pi \iint \Big((\partial_t\psi)^2 - (\partial_r\psi)^2 - \frac{k^2\sin(\psi)^2}{r^2}\Big)\, r\mathrm{d}r\,\mathrm{d}t. \tag{16}$$

Its critical points are thus k-equivariant wave maps, a fact that can easily be checked directly. The kinetic energy and the potential energy are

$$E_{\text{kinetic}} := \pi\int_0^\infty (\partial_t\psi)^2\, r\mathrm{d}r, \tag{17}$$

$$E_{\text{potential}} := \pi\int_0^\infty \Big((\partial_r\psi)^2 + \frac{k^2\sin(\psi)^2}{r^2}\Big)\, r\mathrm{d}r. \tag{18}$$

Their sum is the total energy, and it is a conserved quantity.

We always consider strong solutions of finite energy (that is, strong limits of sequences of smooth solutions in the topology induced by the energy, locally uniformly in time). Their existence and uniqueness for any finite-energy initial conditions was obtained in [15,35]. It can be deduced from Strichartz estimates for the wave equation, see for example [5, Section 2] in the case $k \in \{1, 2\}$. If $k \geq 3$ is large, Strichartz estimates from [30] can be applied, see [19, Section 2]. Finite-energy solutions of (WM$_k$) are not guaranteed to exist for all time. We denote $(T_-, T_+) \subset \mathbb{R}$ the maximal time interval on which the solution exists.

3.2 Multibubble (multisoliton) configurations

Recalling the discussion from Section 2.2, the only k-equivariant harmonic maps correspond to rational functions $az^k, az^{-k}, a\bar{z}^k$ and $a\bar{z}^{-k}$, with $a > 0$. In order to represent these maps in the context of the scalar equation (WM$_k$), it is convenient to denote

$$Q_\lambda(r) := 2\arctan\Big(\frac{r^k}{\lambda^k}\Big), \qquad \lambda > 0. \tag{19}$$

Then the stationary solutions of (WM$_k$) are

$$\begin{aligned} &Q_\lambda + 2\pi\ell, && -Q_\lambda + 2\pi\ell, \\ &\pi + Q_\lambda + 2\pi\ell, && \pi - Q_\lambda + 2\pi\ell, \end{aligned} \tag{20}$$

for any $\lambda > 0$ and $\ell \in \mathbb{Z}$. In this context of equation (WM$_k$), solitons are also called *bubbles*. Note that, with our notational conventions, λ is the spatial scale of the bubble Q_λ. For a given number of bubbles M, an integer m, and scales $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_M$, we define a *multibubble configuration* by

$$Q(\lambda_1, \dots, \lambda_M) := m\pi + \sum_{j=1}^M \pm Q_{\lambda_j} \tag{21}$$

(in the notation, we skip the dependence on m, which is not going to be essential here). One should think of the scales as satisfying $\lambda_1 \ll \dots \ll \lambda_M$, so that each bubble is separated in scale from all the others. Figure 1 shows a multibubble configuration with $(\lambda_1, \lambda_2, \lambda_3) = (\frac{1}{10}, \frac{1}{2}, 5)$.

3.3 Soliton resolution

Our main result can be formulated as follows.

Theorem 3.2 (Jendrej–Lawrie [21]). *Let ψ be a solution of* (WM$_k$) *defined for all $t \in (0, \infty)$. As $t \to \infty$, $(\psi(t,\cdot), \partial_t\psi(t,\cdot))$ decomposes into a superposition of*

- *multibubble configuration,*
- *radiation, corresponding to a solution of the linear wave equation* (W)*,*
- *remainder whose energy converges to* 0*.*

Remark 3.3. We also prove a similar result in the case of a finite maximal time of existence of the solution.

Remark 3.4. The case $k = 1$ was settled by Duyckaerts, Kenig, Martel and Merle [8] using a different approach (so-called channels of energy, see below).

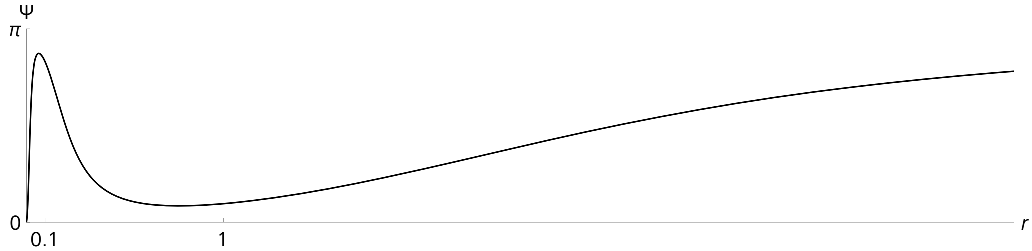

Figure 1. A multibubble configuration with three bubbles.

Remark 3.5. It was proved in [18] that for all $k \geq 2$, there exists a solution containing two bubbles. For these solutions, the radiation component vanishes. It was proved in [21] that for $k = 1$ multi-bubble solutions with vanishing radiation component do not exist. Existence of multibubble solutions with a non-vanishing radiation component is an open problem for all k.

The history of the progress on understanding the dynamical behavior of large solutions of (WM_k) is quite long. Fundamental results were obtained in [2, 3, 37, 38], see also [36, Chapter 8], the main conclusion being the *decay of energy at the self-similar scale*, which in particular excludes self-similar blow-up, but also, as proved by Struwe [40], leads to *bubbling*: if ψ is a solution of (WM_k) which blows up in finite time T_+, then there exist sequences $t_n \to T_+$ and $0 < \lambda_n \ll T_+ - t_n$ such that

$$(\psi(t_n, \lambda_n \cdot), \lambda_n \partial_t \psi(t_n, \lambda_n \cdot)) \to m\pi \pm Q, \tag{22}$$

the convergence being understood in the topology induced by the energy *locally* (on bounded sets).

The bubbling also implies that a solution whose energy is smaller than the energy of Q cannot blow up. It was proved by Côte, Kenig, Lawrie and Schlag [5] that such a solution actually has radiative behavior. Above this threshold energy, finite time blow-up can occur, as was proved in the works [26, 31, 32] already mentioned above. Important progress toward the soliton resolution conjecture was made in [6]. *Sequential* soliton resolution, that is convergence to a superposition of solitons for a sequence of times, was proved by Côte [4] for $k \in \{1, 2\}$, and Jia and Kenig [22] for $k \geq 3$. Similar results without imposing equivariant symmetry assumptions, but with a less precise description of the radiation, were obtained by Grinis [16].

In [19], continuous in time resolution was proved at the minimal possible energy level allowing for existence of a two-bubble. As a relatively simple consequence, continuous in time resolution was proved in [20] under the assumption that the solution contains at most two bubbles.

For the closely related energy-critical wave equation, scattering below the ground state energy threshold was proved by Kenig and Merle [24], establishing together with [23] the so-called *Kenig–Merle route map*. In the radially symmetric case, the soliton resolution conjecture was proved by Duyckaerts, Kenig and Merle in space dimension 3 in [9], in any odd space dimension in [10], and in dimension 4 in [8] (in collaboration with Martel). All these works used the *channels of energy* introduced in [9].

In the non-radial case, sequential soliton resolution was proved by Duyckaerts, Jia, Kenig and Merle [7].

3.4 Main ideas of the proof

Let ψ be a solution of (WM_k) defined for all $t \in (0, \infty)$. Thanks to the sequential soliton resolution results [4, 22], we know that there exists a sequence $t_n \to \infty$ such that $(\psi(t_n, \cdot), \partial_t \psi(t_n, \cdot))$ decomposes into a superposition of a multibubble configuration, radiation and a small remainder. It thus suffices to prove a *no-return lemma*: if a multibubble configuration is destroyed (we say that a *collision* takes place), it cannot recover its shape (note the analogy with non-existence of homoclinic/heteroclinic orbits). A similar idea is present in the works of Duyckaerts and Merle [11], Nakanishi and Schlag [29], Krieger, Nakanishi and Schlag [25] for a *single soliton* which is linearly unstable. In our case, interactions between solitons play a similar role as the linear instability in those works. This idea has already been used in [19] in the special case where there are only two bubbles and the radiative component vanishes.

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Jacek Jendrej is a professor at Sorbonne University. He studies dynamics of solutions of nonlinear wave equations, with a special emphasis on the role of solitons, multisolitons and dispersion.

jendrej@imj-prg.fr

An introduction to the mathematics in the Alhambra of Granada

Miguel Ortega

Every visitor to the Alhambra of Granada is captivated by the beauty of the geometric decoration in the Nasrid Palaces, which are spread across floors, ceilings, windows, doors and walls. In this text, we aim to provide a brief introduction to the mathematical secrets that the artisans and artists likely employed, under modern point of views. We will review some Nasrid decorative elements, rosettes and realizations of plane crystallographic groups.

1 Introduction

Muhammad I ibn al-Ahmar (1195–1273), also known as *Alhamar*, founded the Nasrid dynasty in 1238, establishing the Emirate of Granada in the south of present-day Spain. It became the only surviving Islamic kingdom in the Iberian Peninsula, while others like Jaén, Córdoba and Seville were defeated by the army led by Ferdinand III of Castile. In fact, Alhamar became a vassal of Ferdinand III, ensuring the survival of the Emirate of Granada for another 250 years. Naturally, everything related to these years in Granada is called *Nasrid*. King Alhamar first ordered the construction of a fortress on top of a hill. Along the next two centuries, the Nasrid dynasty expanded this original castle into a complex of towers, walls, gardens and palaces, known as *the Alhambra*. Nasrid artisans and artists brilliantly decorated plenty of places, achieving breathtaking results. In fact, some offices of the Museum of the Alhambra contain several thousand pieces, ranging from single tiles to complete decorative tilings.

Nasrid artisans and artists used basic tools, compared to our modern technology, and a lot of inventiveness to make extremely beautiful decorative creations out of plaster and tiles. From a mathematical point of view, they used a deep knowledge of Euclidean geometry, and intuitive ideas of the four rigid movements of the plane, namely, *translations*, *rotations*, *reflections* and *glides* (or glide reflections).

We will pay attention to Nasrid decorative elements, rosettes and examples of crystallographic groups. Also, we will provide some hints of how to construct them. However, the reader should be aware that we cannot go into much detail. Instead, one can consult various books that summarize endless hours of work by many authors. We just cite a few. The contributions of Rafael Pérez-Gómez are important, see for example [7] and [1]. Also, the three secondary school teachers Francisco Fernández, Joaquín Valderrama and Antonio Fernández wrote the book [3] to summarize a life-long task of showing the many mathematical secrets of the Alhambra to their students, in *just* 471 pages. In addition, Manuel Martínez Vela's book [5] can be seen as an introductory textbook of the drawings in the Alhambra, with 226 pages, in which the author explains step by step how to draw the most famous tilings found there. By the same author, the book [4] is a second and much bigger project with the same aim, but 608 pages. Most of the drawings in the present paper are based on Manuel Martínez Vela's books.

Figure 1. Walls in the Court of the Myrtles, full of pajaritas.

2 Nasrid decorative elements

One can reasonably assert that all geometric, decorative constructions in the Alhambra are based on lattices of squares, equilateral triangles, or a mixture of squares and rectangles. In the following subsections, we select just three of the typical Nasrid decorative elements, among many.

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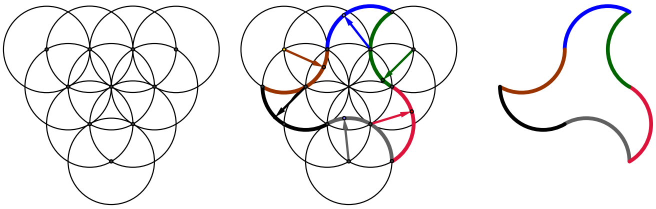

Figure 2. Steps to draw one pajarita.

2.1 The Nasrid pajarita (bow-tie)

This is the very *symbol* of the Alhambra. In fact, when entering some villages close to Granada, the typical advertisement '*Welcome to…*' is illustrated with three or more Nasrid *pajaritas*. For example, there are a few walls in the Court of the Myrtles full of pajaritas. See Figure 1. In order to draw them, we make a lattice of circles with the same radius. Finally, we choose the suitable arcs. See Figure 2.

2.2 The bone

This pattern can be found in the Chamber of Ambassadors, which is one of the main rooms of the Nasrid Palaces. One way to draw it is to make a lattice of squares, and then choose the right sides. The *heads* are just made by drawing the diagonals of the correct squares. See Figures 3 and 4.

Figure 3. Wall in the Chamber of Ambassadors.

2.3 The airplane

At the entrance to the Chamber of Ambassadors, on both walls, there are two nicely decorated *alcoves* called *tacas*. Inside, you can see a tiling of black and white figures called *airplanes*, see Figure 5.

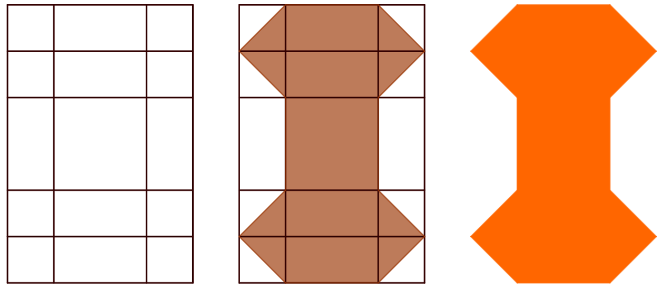

Figure 4. Steps to construct the Nasrid *bone*.

Figure 5. A tiling of black and white *airplanes*, Chamber of Ambassadors.

We start with a regular octagon. Then, we expand the sides until they meet, and draw some diagonals. Next, we remark the desired sides. We will return to this example later. See Figure 6.

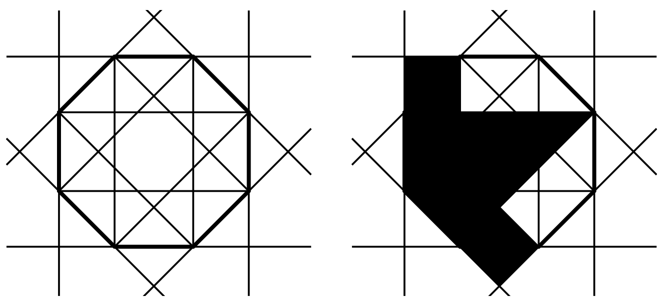

Figure 6. A regular octagon with expanded sides and some diagonals, and a black airplane.

3 Rosettes

The main idea is to draw a figure which rotates around a fixed point (the *center*) a finite number of times, returning to the original position. A reflection axis containing the center is also possible. In the Alhambra, there are rosettes almost everywhere, which were used to decorate walls, windows, false windows, ceilings, doors, etc.

Definition 3.1. Let $\mathrm{Iso}(\mathbb{R}^2)$ be the group of (affine) isometries of the Euclidean plane. A *rosette* is a plane figure F whose symmetry group

$$G = \{f \in \mathrm{Iso}(\mathbb{R}^2) : f(F) = F\}$$

is finite, with at least two elements. A *group of Leonardo* is a finite subgroup of $\mathrm{Iso}(\mathbb{R}^2)$ with at least two elements.

Note that the identity map is an isometry of any plane figure, regardless of its shape. This is why we assume that there exists at least another isometry.

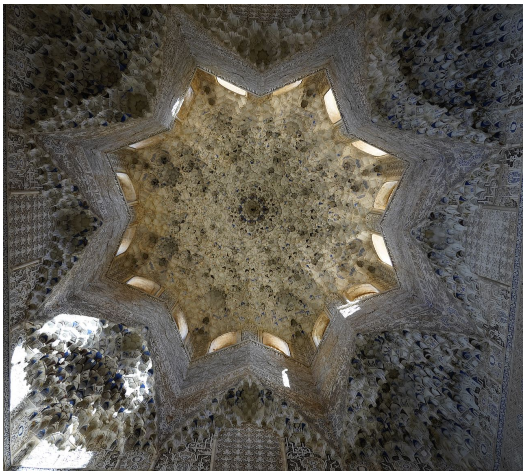

Figure 7. Ceiling of the Room of Abencerrajes.

Theorem 3.2. *The only possible groups of Leonardo are (isomorphic to) either:*

1. *A cyclic group generated by a rotation of center O and angle $2\pi/n$, where $n \in \mathbb{N}$, $n \geq 2$.*
2. *A dihedral group generated by a rotation of center O and angle $2\pi/n$, where $n \in \mathbb{N}$, $n \geq 2$, and a reflection whose axis contains the center O.*

We will call the natural number n the *order* of the rosette. In this way, there exist only two types of rosettes, namely cyclic or dihedral. The first one can only admit rotations, and the second one both rotations and symmetries. We show a few examples in Figures 7, 8, 9, 10 and 11.

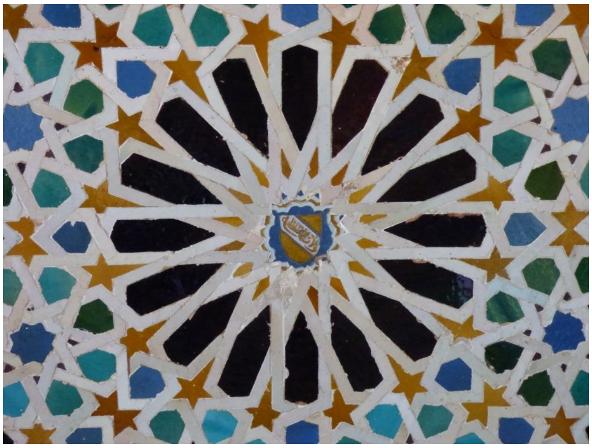

Figure 8. Rosette at Mexuar.

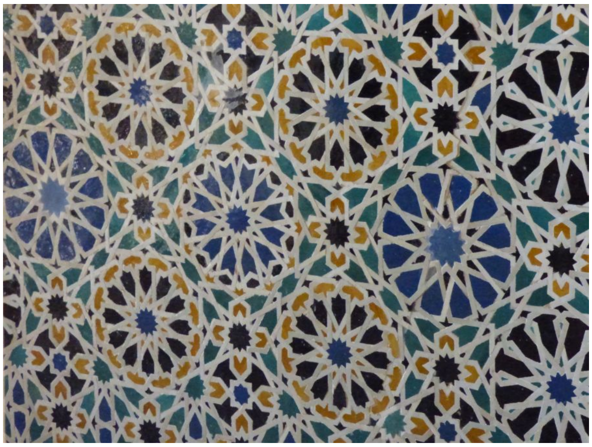

Figure 9. Chamber of Ambassadors.

3.1 The incredible case of the regular nonagon

It is well known that the regular nonagon is not constructible with ruler and compass. However, it is possible to find examples of rosettes of order nine in the Nasrid Palaces. We show four examples in Figures 13, 15, 14, 16.

Before the wide adoption of personal computers, architects and draftsmen used two right triangles known as *set squares*, usually of two types: The first one with two angles of $\pi/4$ and the second one with angles of $\pi/3$ and $\pi/6$. Arabic artisans also used right triangles to make their designs, according to the book written by Diego López de Arenas between 1613 and 1619, see [6] and [1]. In fact, there were two families of right triangles, namely, *cartabones* and *ataperfiles*. For each cartabon there was the corresponding ataperfil, whose smaller angle was half of the smaller angle of the cartabon. With them, it was *possible* to divide the straight angle into several equal angles, taking the possible values 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18 and 20, see [1].

Figure 10. The Partal (on the left) and Court of the Lions (on the right).

Figure 13. Rosette of order 9, wooden ceiling.

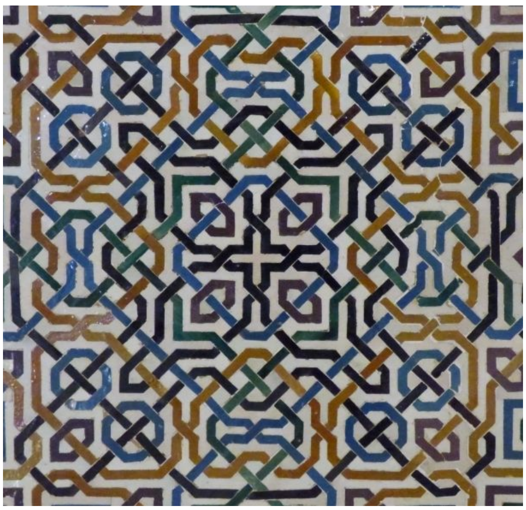

Figure 11. Room of the Two Sisters.

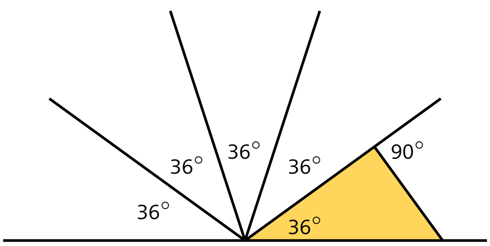

Figure 12. Splitting a straight angle into five equal angles with a cartabon.

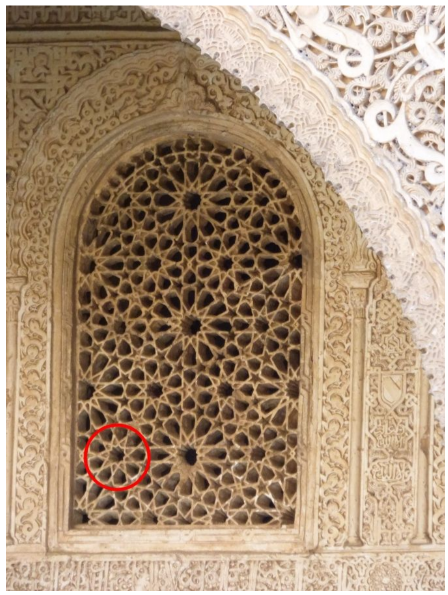

Figure 14. Rosette of order 9, false window, Court of the Myrtles.

We know that it is mathematically impossible to obtain some of such angles with ruler and compass, according to the theorem of classification of constructible regular polygons. However, we should recall that in the 20th century, it was possible to construct protractors, which divided a straight angle in 180 *equal* parts, by accepting a small enough error. Now, in the 21st century, the user of some software can set the accuracy to make drawings.

We note that Manuel Martínez Vela in [4, pp. 366–371] obtained a method to construct a rosette of order nine with an error of just 0.2 degrees, which is a very good achievement.

The amazing fact is that the Nasrid artisans were able to construct all these *cartabones* and *ataperfiles* with high precision, bearing in mind the available technology in the Middle Ages.

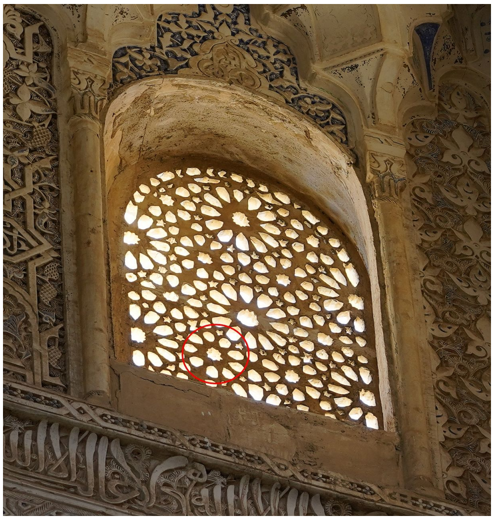

Figure 15. Rosette of order 9, Room of Abencerrajes.

Figure 16. Rosette of order 9, Point of Observation of Daraxa.

4 Crystallographic groups

Definition 4.1. A plane *crystallographic group* is a discrete subgroup of the affine isometries of $\mathbb{R}^2$ whose subgroup of translations is generated by two linearly independent minimal translations.

Theorem 4.2 (Crystallographic restriction). *Let $r_{\theta,c} : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation of angle $\theta \in]0, 2\pi[$ and center $c \in \mathbb{R}^2$. If $r_{\theta,c}$ is an element of a plane crystallographic group, then there exists $m \in \{2, 3, 4, 6\}$ such that $r_{\theta,c} \circ \overset{m}{\cdots} \circ r_{\theta,c} = \mathrm{Id}$ (identity map).*

As an element of a group, the order of $r_{\theta,c}$ can only be $m = 2, 3, 4$ or 6. The point c is called a *center of rotation or order m*. Note that the minimal angle of rotation can only be π $(m = 2)$, $2\pi/3$ $(m = 3)$, $\pi/2$ $(m = 4)$ or $\pi/3$ $(m = 6)$. For our purposes, we introduce now Conway's notation, [2], with a little change. In the book [2], the glide reflections are called *miracles*, but we avoid this term.

o: Only translations.

$*$: Reflections with respect to an axis.

$\times$: Glide reflection (axis and translation vector).

$*n$: When n different axes of reflection intersect at one point, called *rotation point*.

$*np$: When n axes of reflection intersect at one point, and also p axes of reflection intersect at another point.

m: A center of rotation of order m not included in any reflection axis, called *gyration point*.

Always, $n, m, p \in \{2, 3, 4, 6\}$, according to the Crystallographic Restriction Theorem. With this notation, we completely determine a group by its generators. To simplify notation, we discard the translations. A suitable description is called a *signature*.

Examples.

$**$ Two parallel axes of reflection.

$*\times$ An axis of reflection and a glide reflection whose axis is parallel to the first axis.

$2{*}22$ First, the blue 2 stands for a gyration point of order 2. Next, the red $*22$ indicates that there are two reflection axes at one point, but also another pair of axes intersecting at a different point.

632 A gyration point of order 6, another gyration point of order 3, and finally a gyration point of order 2. There are no reflection axes in this case.

2222 Four different gyration points of order 2.

Theorem 4.3 (Classification of plane crystallographic groups). *Up to isometry, there are 17 different plane crystallographic groups, described by their signature:* o, $**$, $*\times$, $\times\times$, 2222, $*2222$, $2{*}22$, $22\times$, $22*$, 333, $*333$, $3{*}3$, 442, $*442$, $4{*}2$, 632, $*632$.

Some authors complain that, sometimes, the painting, tiling, or the decorative motive made of plaster is not big enough to be properly considered an example or realization of a plane crystallographic group. However, in all the examples shown in this paper, the main idea is there, up to the size of the tiling or plaster artwork. That is to say, even though the given realization or construction is not a perfect crystallographic group, I would like to think that, indeed, just by chance or by good luck, these geniuses were able to find all possible combinations without knowing it.

In the following list of Figures from 17 until 33, I provide the Conway signature, a basic explanation, and the place. To be honest, the last example (Figure 33) in our list is a bit complicated. It is unknown where this example was exactly located. According to [1, p. 500], there is an original hexagonal piece of ceramic, with register number 1295.

Figure 17. (1) $\times\times$ – Glide reflections with parallel axes. Painting, Wine Gate.

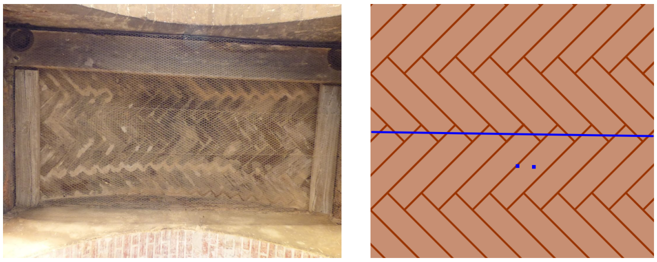

Figure 18. (2) $22\times$ – A glide and two gyration points of order 2. Ceiling, Wine Gate.

Figure 19. (3) $3*3$ – A gyration point of order 3, and three axes intersecting at a point. Paint, Wine Gate.

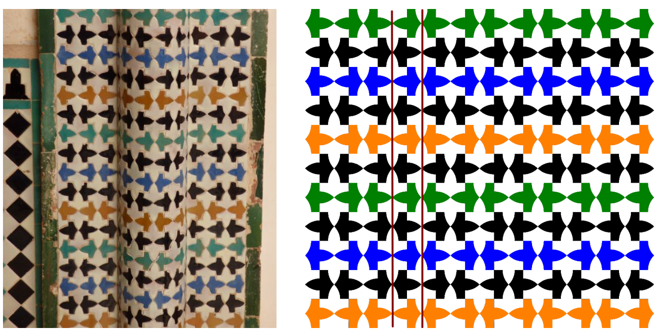

Figure 20. (4) $**$ – Parallel reflection axes. Column, Golden Patio.

Figure 21. (5) $22*$ – Two gyration points of order 2, and a reflection axis. Fountain, Golden Patio.

Figure 22. (6) o – Only translations. Court of the Myrtles.

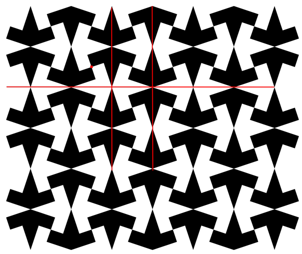

Figure 23. (7) Recall Figure 5. $2*22$ – One gyration point of order 2, and perpendicular axes of reflection. Chamber of the Ambassadors.

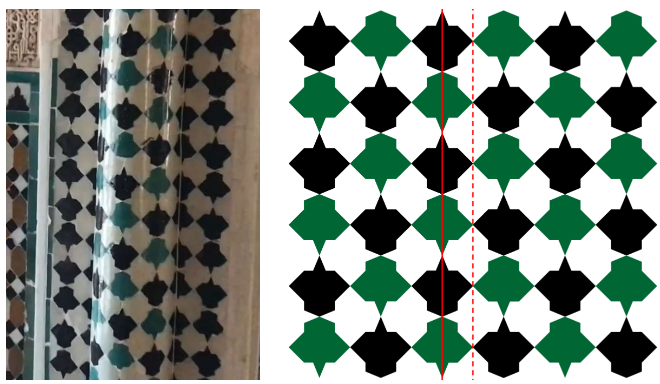

Figure 24. (8) $*\times$ – A reflection and a glide reflection with parallel axes. Chamber of Ambassadors.

Figure 25. (9) ∗442 – Four reflection axes intersecting at one point. Room of the Ship.

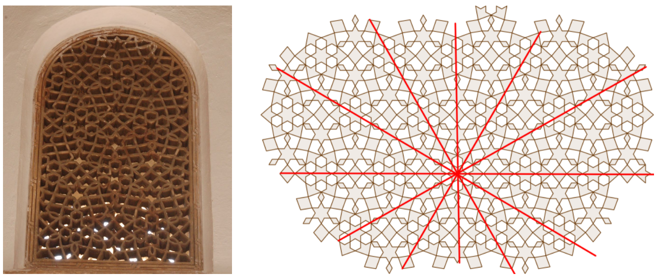

Figure 26. (10) ∗632 – Six reflection axes intersecting at one point. Court of the Myrtles.

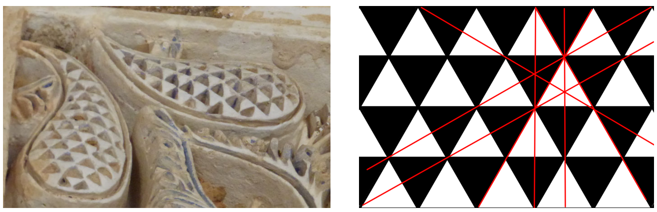

Figure 27. (11) ∗333 – Three points are the intersection of three reflection axes. Court of the Lions.

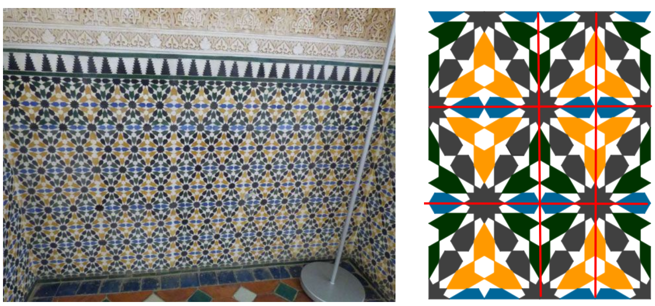

Figure 28. (12) ∗2222 – Four points are the intersection of two orthogonal reflection axes. Hall of the Kings.

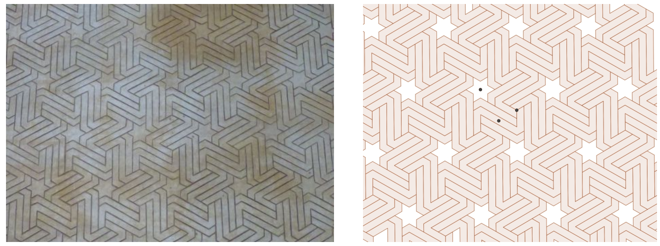

Figure 29. (13) 632 – Gyration points of orders 6, 3 and 2. Hall of the Kings.

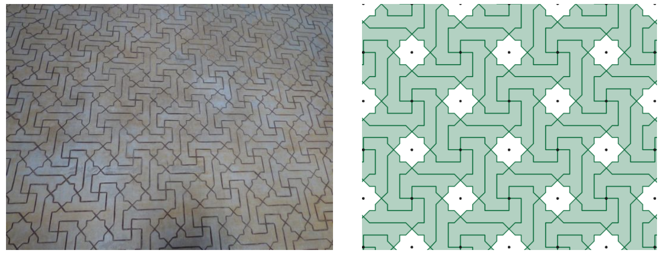

Figure 30. (1) 442 – Gyration points of orders 4 and 2. Hall of the Kings.

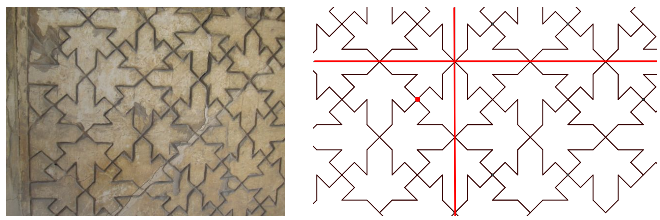

Figure 31. (15) 4 ∗ 2 – A gyration point of order 4, and two reflection axes intersecting at a point. The Infant's Tower.

Figure 32. (16) 2222 – Gyration points of order 2. Generalife, Tower of Ismail I.

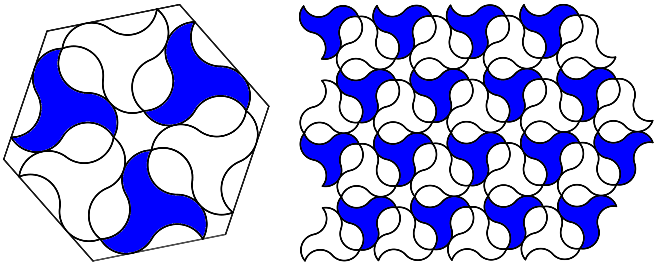

Figure 33. (17) 333 – Gyration points of order 3. The structure of the drawing and the hexagonal shape can only lead to this 333 example.

5 A few words of gratitude

When the author was a young assistant professor in Granada University, professors Rafael Pérez Gómez and Ceferino Ruiz Garrido introduced him to the marvelous world of the mathematics in the Alhambra. Bearing in mind the background of the 'World Mathematical Year 2000,' professor Pérez Gómez made up a first *mathematical visit*, and this author participated as one bumbling mathematical guide. Since then, the author has been teaching this subject to his students, as well as promoting it in talks for the general public.

Finally, the author wishes to thank the *Patronato de la Alhambra y Generalife*, and especially Silvia Pérez López, who works at the *Fondo del Museo*, for her kindness. Without her help, his student Elora Prados Raya would have never made the video about the crystallographic groups in the Alhambra.[1]

Remark. All photos were taken by Miguel Ortega or Raquel Máiquez Sáez. All drawings made by Miguel Ortega.

[1] https://www.youtube.com/watch?v=wl4h0Wot6cY

Funding. M. Ortega was partially supported the Spanish MICINN and ERDF project PID2020-116126GB-I00.

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Miguel Ortega graduated in mathematics at the University of Granada in 1995. He worked at the University of Huelva (Spain) for one academic year, until September 1998, when he obtained a research grant in Japan and moved to Okayama in October 1998. He defended his PhD in Granada in March 1999, and then stayed at the Shimane University (Japan) until September 1999. In October 1999, he began working as an assistant professor in Granada, where he is currently an associate professor.

miortega@ugr.es

Mathematics as seen by an artist: Inspiring mathematical objects

English translation of the article entitled "Les mathématiques vues par un artiste: des objets mathématiques qui inspirent" and published in *La Gazette de la Société Mathématique de France* **181** (July 2024)

Stéphane Vinatier and Reg Alcorn

This article presents a series of hundreds of paintings produced by the painter Reg Alcorn[1] since 2017, entitled *Transitions*, as well as the two mathematical objects he uses to determine the structure of the paintings in this series, one ancient and the other more recent, the Truchet tile and the Ulam spiral.

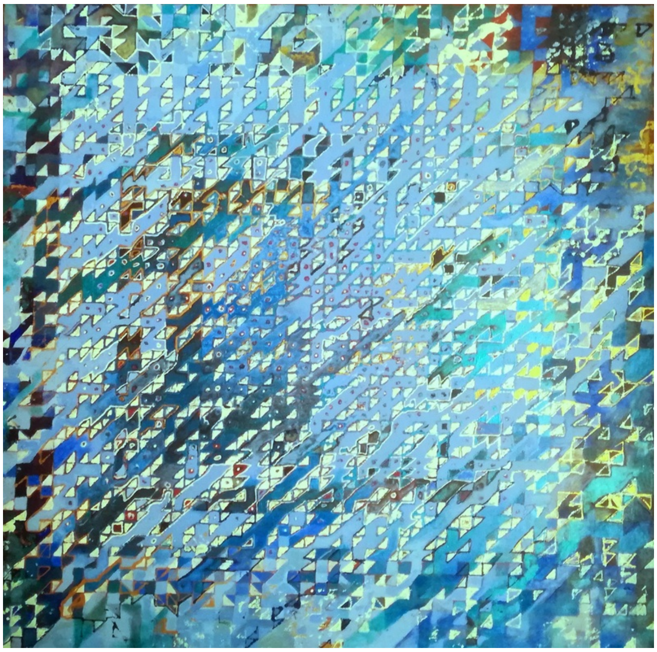


Figure 1. Let All Blues Rejoice, acrylic on canvas, 120 × 120 cm, Reg Alcorn (2019).

The title of our article echoes the one previously published by the same authors in the same journal [19], in which we presented an initiative by the IREM of Limoges[2] and the artist Reg Alcorn, in collaboration with the CCSTI[3] of Limousin *Récréasciences*, to disseminate mathematical culture through artistic media: several periods in the history of art were used to illuminate and highlight mathematical concepts (Antiquity and proportion, the Arab-Andalusian Middle Ages and paving, the Renaissance and perspective). Reg Alcorn's art was then serving the topics explored in this initiative by producing paintings that were used to explain the artistic and mathematical concepts involved. In the series *Transitions*, the artist reverses the perspective: he chooses objects from his study of mathematics and its history and uses them for his personal artistic explorations, resulting in unique works with purely artistic aims.[4]

This may well be an effect of the work previously carried out jointly by Reg Alcorn and the IREM of Limoges, which greatly enriched the mathematical (or scientific) and artistic knowledge of those involved. The artist has immersed himself in numerous mathematical works (aimed at the general public, students and even researchers), which has aroused in him a genuine fascination for some of the objects or concepts encountered. So much so that, after a maturing phase, two of them, from very different eras and contexts, found themselves interwoven in the series of paintings *Transitions*.

We will describe them in turn in the next two sections, taking the opportunity to develop certain mathematical or historical aspects beyond what would be strictly necessary to understand the artist's use of them, in the hope that these digressions will be of interest to our readers. Without pretending to be historians and keeping to the surface of mathematical theories, we thought it would be a good idea to present these objects through the circumstances of their discovery, surprising in both cases, to show how they were apprehended and how they relate to key mathematical issues at the beginning of the 18th century or today. Finally, for those who would rather focus on the links between art and mathematics, the definitions of the objects at the start of Sections 1 and 2 should suffice to appreciate Reg Alcorn's recipe for the paintings in his *Transitions* series, which we will give in the third and final section.

[1] https://www.reg-alcorn.fr

[2] Institute of Research on Mathematics Education.

[3] Center for Scientific, Technical and Industrial Culture.

[4] That is why Reg Alcorn is listed as author of this article, almost entirely written by the first author and based essentially on the artistic investigation of the second.

1 Truchet tiles

The first mathematical ingredient in the series *Transitions*, the Truchet tile, is itself at the crossroads of science and art, as we shall see. It takes us back to an important milestone in the development of combinatorics, when Father Sébastien Truchet (1657–1729), mathematician, typographer, "clockmaker, great canal specialist, inventor of countless machines (cannons, tree-transplanting machines, sundials, etc.) including the famous mechanical tables of Marly" [1], became interested in the different relative positions of two squares, both divided into two colors: given two identical squares, split in two along the diagonal, with one half white and the other black (see Figure 2), in how many ways can they be arranged in relation to each other?

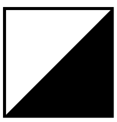


Figure 2. Truchet tile.

This question is systematically discussed by Truchet in a 1704 paper for the Académie royale des sciences [18].

Combinatorics and tessellations. Before presenting his answer, let us note that Truchet's aim is to study ornamental pavings that can be made from this tile (he proposes a selection of them in his dissertation). This is indeed the motivation he himself gives for his study, by way of an introduction to his *Mémoire* [18, p. 363]:

> *On the last trip I made to the Orléans Canal by order of His Royal Highness, I found in a castle named La Motte S. Lyé, 4 leagues from Orléans, several square fayence tiles divided into two colors by a diagonal line, which were intended to tile a chapel & several other rooms. In order to form pleasant patterns & shapes by arranging these tiles, I first examined how many ways two of these tiles could be joined together, always arranging them in a chessboard.*

This work, in which chance and curiosity seem to have played an important role, makes him a precursor in the history of the study of tessellations, which was little practiced by scientists before the 19th century and the development of crystallography (although he had Kepler[5] as a very illustrious predecessor in this field [10]). It is also based on a combinatorial study, a field to which his contribution is described in [16, p. 377] as follows:

> *Truchet's treatise is of considerable importance for it is in essence a graphical treatment of combinatorics, a subject that, under the influence of Pascal, Fermat and Leibniz, was at the forefront of mathematics at the time.*

The originality of his approach, which links combinatorics and tessellations, is also underlined by André and Girou [3, p. 11].

Truchet's treatise. He begins by presenting 64 combinations obtained by considering that the two tiles are different: the first has four possible orientations, as does the second, which can moreover be placed against any of the four sides of the first. This number of combinations is immediately reduced to 32 if we no longer differentiate the two tiles, and further reduced to ten if, in addition, we disregard the pairs of tiles that only differ by rotation.[6] These combinations and reductions are illustrated by two tables in [18], reproduced in Figure 3.

The paper [15], in addition to examining the patrimonialization in various ways (mathematical, playful, pedagogical) of the Truchet tiles, rightly notes that ten equals the number of combinations with possible repetition of two objects among four. This equality does not seem immediately meaningful to us: if we have to choose two of the four squares, with possible repetition, to form the patterns considered by Truchet, and if we can arrange them in a line, regardless of pairwise rotation, the order in which we arrange them has an importance (ab does not produce the same result as ba, regardless of pairwise rotation) and some of the resulting patterns are identical after rotation, such as ab and dc (a, b, c, d denote the tiles obtained by successive quarter-turn rotations of the Truchet tile in Figure 2, see also Figure 9). However, these two effects compensate each other: the involutions defined on $\{a,b,c,d\}^2$ by $(x,y) \mapsto (y,x)$ and $(x,y) \mapsto (r(y),r(x))$, where r is the "half-turn" that changes a to c and b to d, both have four fixed points and six pairs of related couples, i.e., they have the same number of orbits; yet those of the first involution correspond to combinations with possible repetition, and those of the second to patterns of two tiles regardless of pairwise rotation.

The "graphical treatment" of the combinatorial problem evoked by [16] also consists in the spatial organization of the possible configurations, which reflects the systematic study carried out by Truchet: in each column of his Table I (see Table I in Figure 3), the "first" tile is in the orientation drawn in the first row, while that of the "second" varies according to the row; finally, the four relative positions appear in the four corners of each box of the table. The arrangement of the configurations shows the completeness of the initial search, so all that remains is to identify those that are alike according to the set criteria (Tables II and III in Figure 3).

The author adds that he has begun work on combinations of three, four and five bisected tiles, but that he is not satisfied with

[5] Johannes Kepler, German astronomer (1571–1630).

[6] See [16, p. 384, note 5.] for a discussion of these reductions.

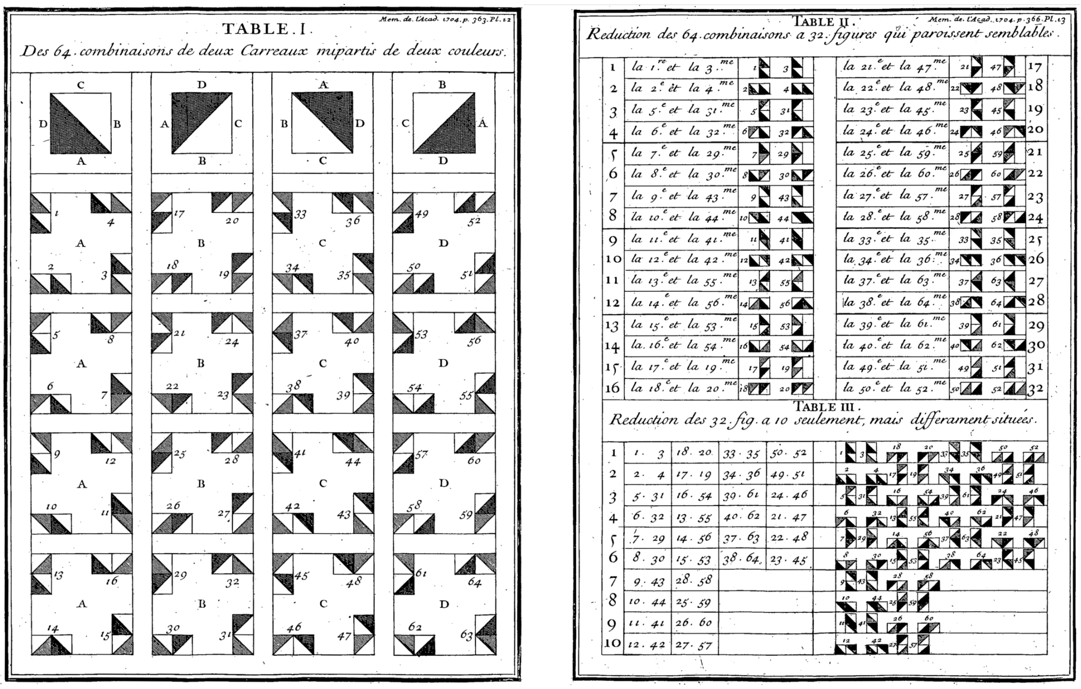


Figure 3. Tables of the combinations obtained by Truchet [18]. (Source: gallica.bnf.fr / Bibliothèque nationale de France)

it and intends to publish it at a later date (which will not be the case). Of course, complexity increases very rapidly with the number of tiles, as can be seen in Reg Alcorn's paintings, if only because there are several spatial arrangements of three, four or five tiles: in a line, in an 'L,' in a square, in a 'T,' in a cross…

Lastly, Truchet presents seven plates of tiling patterns obtained by connecting some of the 64 tile combinations he has listed: they include 24 tilings of size 12×12 tiles and six of size 24×18, chosen from among 100 completed tilings, themselves selected among patterns in "too large a quantity to report them all" [18, p. 364].

Another reverend of the same religious order as Father Truchet, Dominique Doüat, took up this work and published a *Method for making an infinite number of drawings* in 1722 [8] containing 72 tilings produced with Truchet's bisected tiles. We show one of his plates in Figure 4.

Sources. For all Truchet-related information, Jacques André's website[7] is an invaluable mine of documents and references, in addition to his own publications on the subject [1, 3]. In particular, he mentions the possibility of consulting Truchet's treatise [18] on the Bibliothèque nationale de France's Gallica documentary database (with plates of engravings ahead of the memoir), as well as in English translation in [16]. Doüat's work [8] (which André makes available, preceded by a rich introduction) and the article *Carreau (Architecture)* [7] from Diderot and d'Alembert's *Encyclopédie*, discussed below, are also available on Gallica in facsimile. For all these documents, links to Gallica are provided in the bibliography. André also points to the transcribed version of the *Encyclopédie* on the ARTFL project site,[8] see also the *Édition Numérique Collaborative et CRitique de l'Encyclopédie* which combines transcription and facsimile of the first edition.[9]

On Jacques André's website, one can also find a document [2] featuring illustrations of Truchet tilings from several sources: by Truchet himself, in his treatise [18] and in the *Description des arts et métiers*, an encyclopedia whose project was launched by Colbert, directed in particular by Réaumur, to which he contributed[10]; by Doüat [8]; from the *Encyclopédie* [7], whose *Table des 64 combinaisons* is reproduced (while pointing out an error); and from a later work by an architect (Lemaire, 1862), from which two similarly inspired plates are featured.

In the Encyclopédie. Diderot faithfully reproduces the contents of Truchet's treatise in the article 'Carreau' of the *Encyclopédie* [7]: he presents the 64 combinations obtained by differentiating the two tiles under consideration, as well as the reductions to 32 and ten combinations we have described. He goes on to add a final

[7] https://jacques-andre.fr/faqtypo/truchet/

[8] https://encyclopedie.uchicago.edu

[9] https://enccre.academie-sciences.fr/encyclopedie/

[10] The plates date from 1705, but this encyclopedia only began to be published in fascicules from 1761 onwards.

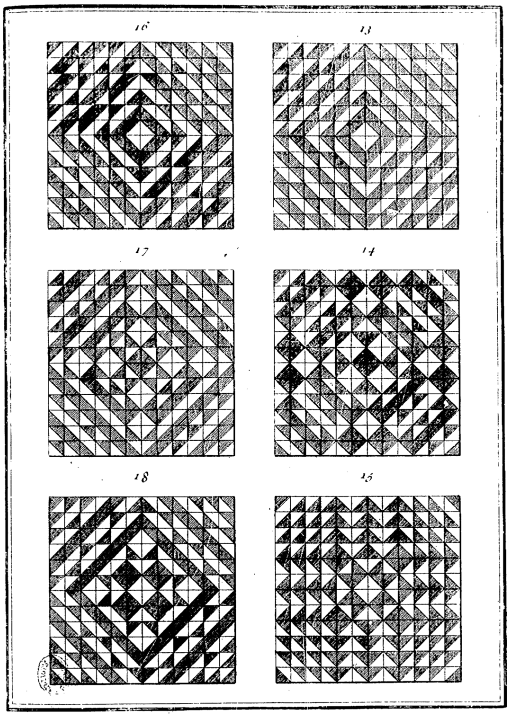


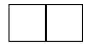
Figure 4. A plate of tilings by Father Doüat [8].
(Source: gallica.bnf.fr / Bibliothèque nationale de France)

(double) reduction, consisting of identifying identical combinations by color inversion or by flipping (the latter identification is described as follows: "if we suppose them traced on transparent paper, we will see some of them by looking through the paper, the same way we see the others on the paper itself" [7, p. 700]). This leaves only four classes of combinations. He concludes the reduction as follows:

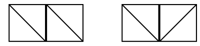
> *Perhaps if we had looked for a way of arranging the combinations of these tiles on paper, we would have found a law that would have dispensed with the previous enumeration: but this is what no one has yet attempted, nor the combination of several tiles, and even less the combination of tiles of several colors.*

It is noteworthy that Diderot concludes this study by highlighting a number of gaps in the knowledge of the time on this subject, which can be considered open questions to the scientific community; and it is just as noteworthy that the artist Reg Alcorn happily exploited a variant of this last gap, by declining at will the colors of the halves of the squares he laid out on his canvas, taking full advantage of his artistic freedom and finding there a field in which to express his talent. To describe the way he arranges them, we'll introduce a second, more recent and complex mathematical object a little further down, in Section 2.

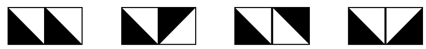
Another enumeration of combinations. Before that, let us try to answer Diderot's first question and show how we can arrive at the four terminal combinations more quickly than by Truchet's and Diderot's successive reductions. We start with two empty squares, and there is, disregarding rotation, only one way to assemble them side by side:



We now add a diagonal to each: there are, a priori, two possible choices of the diagonal in each of the two squares, so four possibilities; but they are two by two identical by flipping, which leaves us with the two configurations:



We now need to color one half out of two in each square. Again, in each of the two configurations, there are two choices for the half to be colored in each of the two squares, i.e., a priori four possibilities for each configuration, which would make eight possibilities in total. But each of these is accompanied by one where the colors are reversed, so if we identify the identical combinations by reversing the colors, this divides the number of possibilities by two, and we get the following four combinations:



The result points the way to another, even more straightforward demonstration: allowing rotation, the two tiles can be arranged in a line and, allowing flipping and color inversion, we get the following pattern first.

It remains to choose the second motif from the possible four.

2 The Ulam spiral

Now we come to our second mathematical ingredient.

Creativity from boredom. According to Martin Gardner,[11] who tells the story in an article for the popular science magazine *Scientific American* [11], the mathematician Stanislaw Ulam, bored while attending a conference in the autumn of 1963, began to draw a grid on a sheet of paper to represent a chessboard; changing his mind, he began to number the intersections starting from the center of the grid and spiraling around it in an anti-clockwise direction; then he began to circle the prime numbers and, to his great surprise, saw lines forming along the diagonals (Figure 5).

As Gardner points out, this fact is not very significant for a small spiral, insofar as all the integers belonging to a line of diagonal

[11] American writer (1914–2010), great popularizer of mathematics.

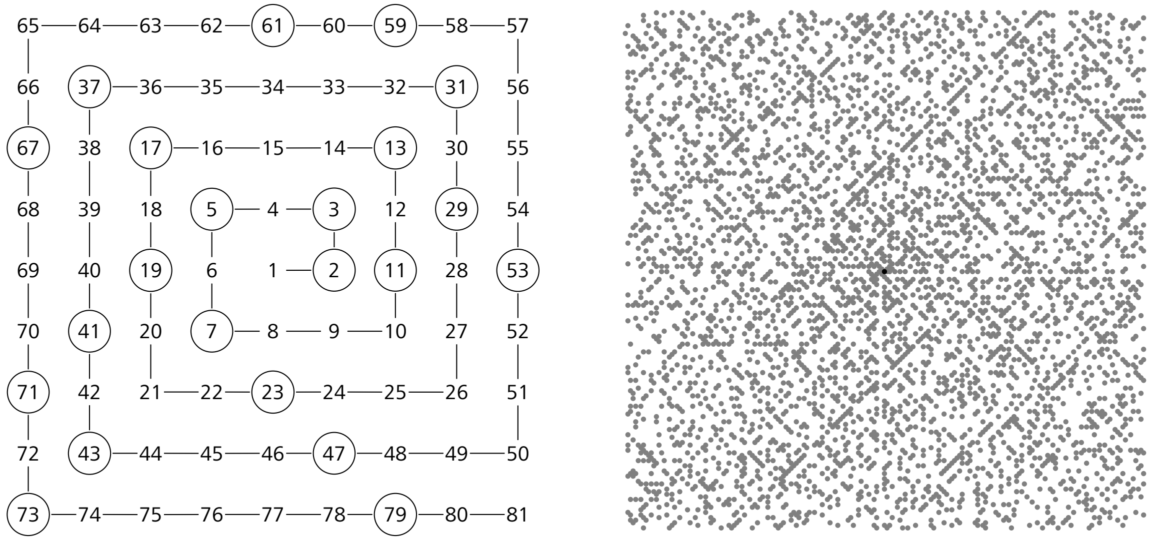


Figure 5. Ulam spirals 9×9 (prime numbers are circled) and 200×200 (prime numbers are represented by gray dots, 1 in the center by a black dot).

direction have the same parity, all the primes (except one[12]) are odd and the density of primes is high among the smallest integers. So in the 9×9 spiral shown in Figure 5, the 21 odd primes are distributed among just 41 possible positions, forming lines parallel to the diagonals…

Consolidation. However, Ulam was able to verify with two collaborators, Stein and Wells [17], that the observed phenomenon was not confined to small integers, but appeared to be a general property of prime numbers. To do this, they used the MANIAC II computer at their research institution, the *Los Alamos* laboratory of the University of California, as well as magnetic tapes containing tables of prime numbers, to obtain spirals covering much larger numbers of integers (photos of spirals obtained with 10000 and 65000 integers are reproduced in [11], they resemble the spiral on the right in Figure 5, at different scales).

As impressive as these images of unexpected prime number alignments may be, so much so that Ulam's spiral made the front page of the issue of *Scientific American* in which Gardner's article appeared, Ulam and his collaborators explain that they are due to the large numbers of prime values taken by certain quadratic functions of the integers, of the type $n \mapsto an^2 + bn + c$ with $a, b, c \in \mathbb{Z}$, such as Euler's famous formula

$$E(n) = n^2 + n + 41 \tag{1}$$

for which all the values for n between 0 and 39 are distinct primes (!) and which takes about 47.5% of prime values among its values up to 10000000 [17, p. 520]. Ulam and his colleagues have identified other quadratic forms with high rate τ of prime values up to 10000000, including

$$\begin{gathered} 4n^2 + 170n + 1847 \quad (\tau \approx 46.6\%), \\ 4n^2 + 4n + 59 \quad (\tau \approx 43.7\%). \end{gathered} \tag{2}$$

However, the reporter of the article pointed out that the integer values of the first are included in the sequence of numbers produced by Euler's formula (on even integers):

$$4n^2 + 170n + 1847 = E(2n + 42).$$

The rate of 46.6% of prime values among values up to 10^7 of $E(2n + 42)$ is roughly the rate of prime values among values up to 10^7 of E at even integers; by comparison with the overall rate, that at odd integers must be 48.4%. We do not know whether this slight difference between the rates for even and odd integers is asymptotically confirmed. We shall see below that our knowledge of the asymptotic behavior of the number of prime values of polynomials is essentially conjectural.

Alignments parallel to semi-diagonals. It is easy to position oneself in Ulam's spiral by observing (on Figure 5) that the piece of spiral that goes from 1 to $(2n + 1)^2$ for an integer n forms a square, each side of which contains $2n + 1$ integers, centered at 1, with the number $(2n + 1)^2 = 4n^2 + 4n + 1$ in the bottom right-hand corner. We immediately deduce the expressions of the numbers at the other corners of the square of size $2n + 1$. They form the four semi-diagonals starting from 1, and are plotted on Figure 6.

For example, the numbers of the southeast semi-diagonal are the squares of odd integers, of the form $4n^2 + 4n + 1$, so

[12] Or rather except two!

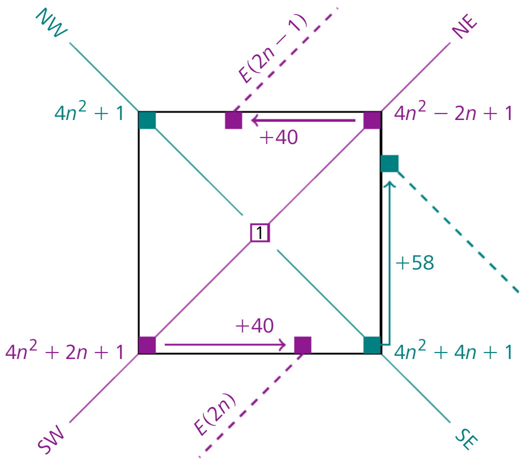


Figure 6. Alignments parallel to semi-diagonals.

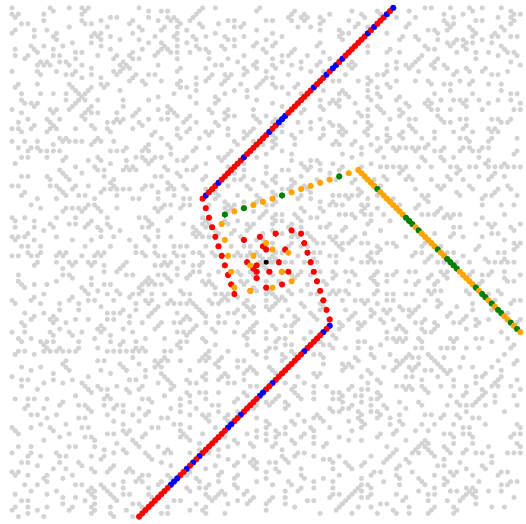


Figure 7. In the Ulam spiral 160 × 160, $E(n)$ for $n \leq 160$: in red (prime values) and blue (other values); $4n^2 + 4n + 59$ for $n \leq 80$: in orange (prime values) and green (other values).

that the values of the second quadratic form mentioned in (2), $(4n^2 + 4n + 1) + 58$, occupy, for n large enough, the translation of this semi-diagonal by 58 upwards (we turn counter-clockwise, following Ulam's example).

Similarly, the numbers of the northeast (resp. southwest) semi-diagonal are integers of the form $4n^2 - 2n + 1$ (resp. $4n^2 + 2n + 1$), so that the values of Euler's formula (1) at odd integers, $E(2n-1) = (4n^2 - 2n + 1) + 40$ (resp. at even ones, $E(2n) = (4n^2 + 2n + 1) + 40$), occupy, for n large enough, the translation of the northeast (resp. southwest) semi-diagonal by 40 to the left (resp. right).

We therefore see in Figure 7 alignments of prime numbers in directions parallel to the diagonals, produced by the quadratic forms (1) and (2), which according to [17] take many prime values at integers, up to 10^7. To find out whether this phenomenon persists beyond this value, when the size of the spiral is further increased, we need to look at the asymptotic behavior of these polynomials.

Asymptotic behavior. Hardy and Littlewood's "Conjecture F" (1923) predicts the asymptotic behavior of the number of values of the variable, natural number less than a positive real x, for which r polynomials with integer coefficients simultaneously take prime values, in the form of a constant depending on the respective polynomials multiplied by the function $x/(\ln x)^r$. See [5] for more details in the general case.

In [13], Jacobson and Williams study polynomials of degree 2 that generalize Euler's formula, of the form $f_A(x) = x^2 + x + A$ with $A \in \mathbb{Z}$. Denoting by $P_A(n)$ the number of prime values taken by f_A at natural integers at most n and by $\Delta = 1 - 4A$ the discriminant of f_A, they state Conjecture F as follows:

$$P_A(n) \sim 2C(\Delta) \int_0^n \frac{dx}{\ln f_A(x)}, \tag{3}$$

where the *Hardy–Littlewood constant* $C(\Delta)$ is defined by the Euler product:

$$C(\Delta) = \prod_{p \geq 3} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p-1}\right).$$

It is a product on odd primes p, with $\left(\frac{\Delta}{p}\right)$ denoting the Legendre symbol for Δ on p. One can check by calculation (via integration by parts) or by using the formulation of the conjecture given in [5] (note that the constant is defined here by an Euler product over *all* primes) that the equivalence (3) can also be written as

$$P_A(n) \sim C(\Delta) \frac{n}{\ln n}. \tag{4}$$

Under this conjecture, the constant $C(\Delta)$ has a decisive influence on the asymptotic behavior of the number of prime values taken by f_A.

The numerical results support Conjecture F. For $A = 41$, we have $C(-163) = 3.3197732$ and the Euler polynomial $x^2 + x + 41$ takes 87% of prime values up to 100 (estimated value using (4): 72%) and 47.5% up to $3162 = \lfloor \sqrt{10^7} \rfloor$ according to [17] (estimated value: 41.2%). We know from [13] that this rate is 22.08% up to 10^7 (estimated value: 20.6%) and that, for $A' = 33997146285531 18047$, we have

$$C(-135988585142124 72187) = 5.3670819$$

and 25.17% of prime values up to 10^7; even if this rate is lower than the estimated value (33.3%), it is higher than the corresponding rate for $A = 41$, which seems to corroborate the fact that a larger Hardy–Littlewood constant corresponds asymptotically to a larger number of prime values.

The value of $C(\Delta)$ associated with the integer A' considered above was the largest known before the calculations presented in [13]. The prepublication [4, §4.2] explains how to determine a high-precision approximate value of $C(\Delta)$ and also gives as an example the much smaller value for $A = 75$, $C(-299) = 0.3109767$. The maximum value for $C(\Delta)$ found by Jacobson and Williams is 5.65726388 (under the generalized Riemann hypothesis), corresponding to an integer with 71 decimal digits.

Arithmetic progressions. One may also see in [5] that, while the general Conjecture F is supported by numerical calculations and sieve methods, it is only proven in the case of one polynomial of degree 1: it then comes down to Dirichlet's *arithmetic progressions* theorem (in the quantitative version proved by de La Vallée Poussin), which states that for all coprime integers a, b the polynomial $ax + b$ takes infinitely many prime values (uniformly distributed in the invertible classes modulo a). More precisely, denoting by $P_{a,b}(n)$ the number of prime values of $ax + b$ at natural numbers at most n, we have the equivalence

$$P_{a,b}(n) \sim \frac{a}{\varphi(a)} \frac{n}{\ln n}$$

where φ is the Euler totient function; this confirms the equivalence (4) and specifies the Hardy–Littlewood constant in this particular case. This result in degree 1 also has a graphic interpretation: let us write the integers in a rectangular grid of given width, starting at the top left and filling the grid line by line. The numbers $an + b$ for $n \in \mathbb{N}$ then lie on a straight line (cut into pieces by the grid), so the corresponding prime values again produce alignments.

In all other cases, we cannot even prove that the number whose asymptotic behavior the conjecture predicts, tends towards infinity with x. For example, the integer values of the two polynomials x and $x + 2$ that are simultaneously prime are the twin primes, and we do not know whether they are infinite or not.

In degree 1, we know even more thanks to Green and Tao's Theorem [12]: not only do arithmetic progressions contain an infinite number of primes, but there are arithmetic progressions of primes of any length: for any natural number k, there are primes $p_1, p_2, \ldots, p_k$ which are consecutive terms of an arithmetic sequence, i.e., such that the differences $p_{i+1} - p_i$ are constant. The proof of the theorem is not constructive, and finding such sequences is difficult, as the common difference is likely to be very large.[13]

Small variable values. Several authors have studied the first values of other polynomials of degree 2, with smaller coefficients and for smaller values of the variable, and therefore more likely to explain the lines that show on the small-scale plots of Ulam's spiral. The polynomial of degree 2:

$$36x^2 + 18x - 1801 \tag{5}$$

holds the current record for the number of consecutive distinct prime values: 45 (between -33 and 11, established by Ruby in 1989); it therefore surpasses the Euler polynomial (only two other polynomials are known for which this is true, due to Fung and dating from 1988). In addition, it takes 49 distinct prime values in 50 consecutive values of the variable, with five possible starts between -41 and -33. See the introduction and Section 2 of [9] for details.

In that paper, Dress and Olivier present the results of their numerical search for polynomials with a large number of prime values, counting only the *distinct* prime values taken for 50, 100, 500 or 1000 consecutive values of the variable. In the latter case, they significantly improve on previous results with the polynomials

- $x^2 + x - 1354363$: 698 prime values (from $x = 1139$ to $x = 2138$);
- $x^2 + x - 752293$ and $4x^2 + 2x - 349513$: 685 prime values;
- $4x^2 + 2x - 501229$: 684 prime values.

Note that $36x^2 + 18x - 1801 = E(6x + 1) - 1844$ and that, for any $A \in \mathbb{Z}$:

$$x^2 + x + A = E(x) + A - 41$$

and

$$4x^2 + 2x + A = E(2x) + A - 41,$$

so that the polynomials we have just discussed, like Euler's formula, produce alignments in Ulam's spiral parallel to the northeast and/or southwest semi-diagonals, provided the value of the variable is sufficiently large. A number of other polynomials taking a large number of prime values appear in [9], which are neither of this form nor of the second identified by Ulam and his collaborators (2), including in degree 2. A precise description of the alignments that may appear in Ulam's spiral is found in [14], along with figures showing the distribution of values of some of these other polynomials (in the spiral-wound plane as for Ulam's spiral), such as the one in Figure 8.

3 The genesis of the *Transitions* series

At the origin of the *Transitions* series, there is … a desire for geometry! Like several other artists before him, Reg Alcorn was looking for a geometric process to structure the canvases of his new series. Among the artists who followed this path, some of them having links to mathematics, are Mondrian (1872–1944), Kandinsky (1866–1944), Vasarely (1906–1997), Morellet (1926–2016) and, of course, Picasso (1881–1973) and the Cubist movement in general, for the analysis of perception and the geometrization of space.

[13] The current record is for $k = 27$, set by Gahan in 2019, see https://oeis.org/A327760.

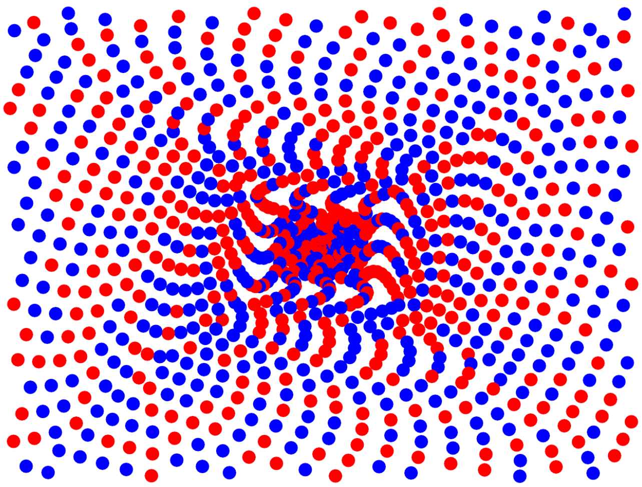
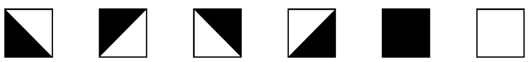


Figure 8. Fung polynomial values $103n^2 + 31n - 3391$ for $n \leq 1000$ (prime values in red).

3.1 Reg Alcorn's approach

For his series of paintings *Transitions*, the artist has developed a systematic procedure[14] determined by the two mathematical ingredients presented above: Truchet tiles and Ulam's spiral. In fact, to the four tiles obtained by turning the Truchet tile in Figure 2, he adds two: an all-white square and an all-black square. It is convenient to codify them as in Figure 9.

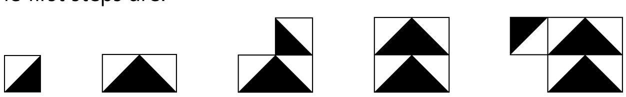


a *b* *c* *d* *k* *w*

Figure 9. The expanded Truchet tile inventory.

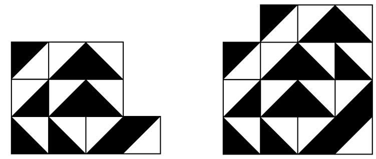
He then chooses a sequence of some of these tiles, of any length, for example *daadb*, and – instead of arranging it in a line, which would look like this:

– he winds it in a spiral from the center of the painting, repeating the sequence identically until the painting is filled. In our example, the first steps are:

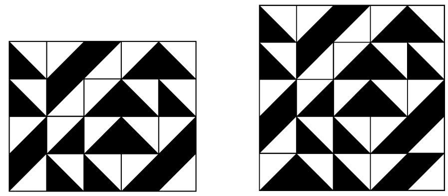


[14] Although he might occasionally deviate from it, over the course of the two hundred paintings in this series.

for the first occurrence of the sequence, then:



and



with two, three, four and five occurrences respectively (a complete square is obtained). Figure 10 shows the result with 20 occurrences of the sequence (i.e., 100 tiles forming a square of 10×10).

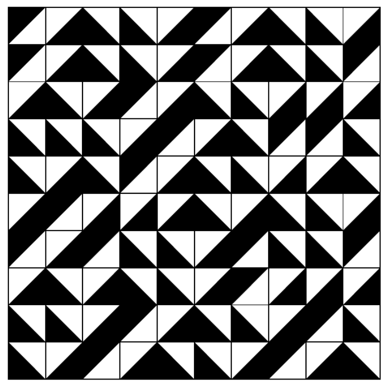


Figure 10. The pattern *daadb* wound in a spiral (repeated 20 times).

We realize immediately that the very simple process we have implemented quickly produces complex, unpredictable patterns, even though they are determined by the sequence chosen at the outset and, in particular, are highly dependent on its length. For example, it can be seen in Figure 11 that sequences with lengths in multiples of four produce much more regular patterns than the example of length five chosen above, notably because the numbers belonging to any of the pattern's semi-diagonals are always congruent modulo 4. This means that the same tile can be found all along (resp., at least every other time) a semi-diagonal of a pattern produced by a sequence of length four (resp., eight). It is worth noting that, even in this case, reversing two tiles will produce very different effects.

An atlas of sequences is in preparation, which promises to show the incredible richness of this process, complementing the numerous canvases created by the artist, more examples of which are given below.

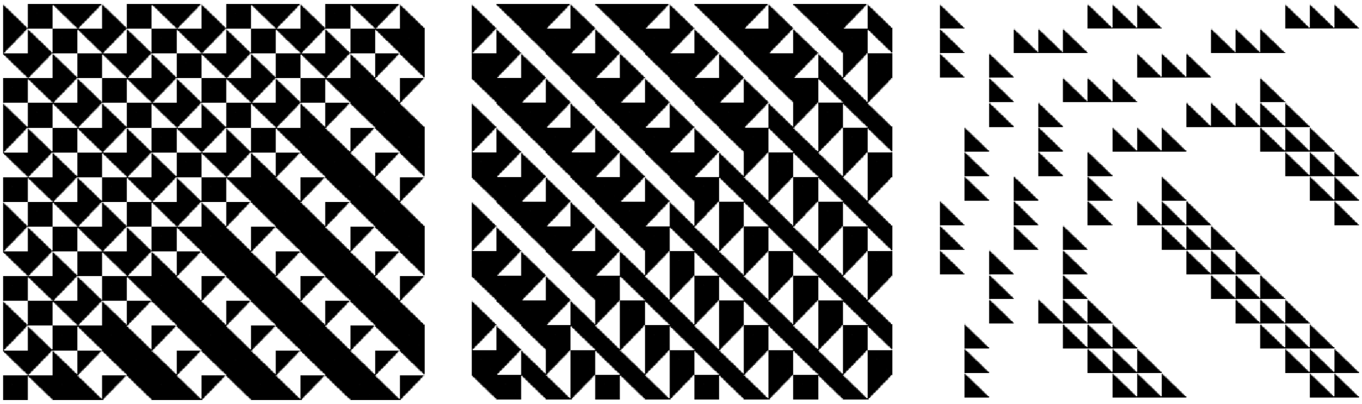


Figure 11. $abck$, $abkc$ and $aaawwwww$.

3.2 Playing with color

Once the sequence has been selected and the configuration of the painting determined, Reg Alcorn's art consists in choosing the colors that will replace the black and white of the areas drawn by the completed paving, accentuating particular motifs, sometimes revealing different ones depending on the distance from the viewer,[15] provoking amazing visual effects as in the paintings displayed in Figure 12, where the complex juxtapositions of colors puzzle and draw the eye in, and can even, in full size, make us doubt the reality of what we are actually looking at.

Combining colors. The basic rule is to choose two main colors, red and green for example, which complement each other well, and assign one to the white areas created by the paving, and the other to the black areas. However, in the world of painting, things are not all black and white, and different combinations of the two colors can replace either one or the other for a given area, obtained either by mixing them in varying proportions, or by superimposing one on the other. The choice of one variant or another may be guided, for example, by the shapes that appear and whether or not the artist wishes to emphasize them, or simply by the color harmony of the painting.

Then, during the slow, delicate work of filling in the areas, the artist gives free rein to his inspiration to introduce a third, warmer color here, a cooler one there, as the overall picture takes shape in his mind, and he senses whether or not to broaden his palette. In the end, there are no real rules when it comes to the colors: the constraint of the paving patterns is the framework within which the artist's freedom to choose, arrange and interact with them can fully expand. In his words:

[15] It is striking that, with a completely different take on Truchet's tilings, [16] also stresses the differences in perception depending on distance from the motif or the scale at which it is viewed.

Figure 12. The paintings *Golden Orbweaver*, oil on canvas, 130 × 130 cm (2018); *Garden Raga*, acrylic on canvas, 130 × 130 cm (2019); *Stripe Break*, acrylic on canvas, 120 × 120 cm (2020) – Reg Alcorn.

Coming across a picture of the Ulam spiral in a completely different context, I saw the potential. The creation of the paintings at first depends largely on these calculations, but in the elaboration, the eye and intuition also come into play.

Precision. Long an adept of fast painting, where brushstrokes amalgamate colors into rich, vibrant textures with slightly blurred outlines, Reg Alcorn adopted a different way of painting for the *Transitions* series: slow, precise work, very sharp contours, which he had previously experimented with in geometrically patterned canvases inspired in particular by Penrose tilings.

Although the colors are well delineated on the canvas, predicting the effect their juxtaposition will produce in the patterns created by the sequence of Truchet tiles is a challenge that only the artist's long experience and sense of color can meet. And there are surprising outcomes in certain cases, when the colors "vibrate" in contact with each other, whenever the eye is at the right distance from the canvas. All this is the fruit of long-term research, as he explains:

I spent a long time trying to develop a formal grammar capable of exploiting these combinations, in order to create compelling yet legible configurations for any harmony, and to highlight the colors' distinct identities.

Pigments. Note that the chemical composition of the pigments is very important for this work: for example, the red-green sector shows significant variations in visual impact depending on whether, for example, cadmium red dark is used in combination with phthalocyanine green, or iron oxide red with green earth.

There are actually a dozen top-of-the-range reds, according to paint manufacturers, made of either mineral, vegetable or synthetic pigments, with differences in hue, transparency or opacity. Their texture depends on the thickness of the paint, the additives and tools used, the texture of the canvas fabric and its preparation for the color. Let us take two colors as examples: cadmium red light, an opaque, vivid color with great visual impact, and alizarin crimson red, a transparent, very dark color with a violet-red tendency. Their juxtaposition enhances their identities, while their overlay modifies both colors.

The paintings in the "Transitions" series invite us to discover a chromatic perspective, giving the illusion that colors are at different distances. Composing these paintings, I embarked on an exciting odyssey, exploiting the alchemy of colors and multiple combinations without being committed to representativeness. This is the heart of "Transitions."

3.3 Variations

The rule we have described for determining the paintings' motifs, that is, the choice of a sequence of Truchet tiles wound into a spiral, has also been subject to exceptions, even transgressions, in the course of the very many canvases that have been created. These have often proved to be judicious and enriching in terms of new perspectives. Some paintings, for example, are constructed from several spirals starting from different foci, which can give the impression of a more realistic painting: with five foci, one thinks almost instinctively of the more or less symmetrical five-pointed shapes so familiar in the natural environment, like that of a silhouette with a head, two arms and two legs, somehow blurred by the pictorial process, unless the colors chosen by the painter make it appear.

In other paintings, Truchet tiles are replaced by "Pythagorean" rectangles cut diagonally into two triangles of sides 3, 4, 5, wound into a spiral in the same way as before. The first consequence is that the canvases are no longer necessarily square, which was almost always the case before, as a matter of construction; the second is that the lines delimiting the spiral are less blended into the overall motif, revealing more of the painting's structure. Finally, the elongated triangles give rise to smoother, more fluid shapes than Truchet's tiles. Of course, the artist also played with mixing "Greek" tiles with Truchet's, or even with squares cut into four triangles along the diagonals – there are countless combination possibilities… An example is shown in Figure 13.

3.4 From so simple a beginning

A stunning characteristic of Reg Alcorn's process is that, despite its great simplicity, it produces extraordinarily rich and complex patterns. Music is another artistic field where this principle is frequently found.

Figure 13. Snow step, oil on canvas, 120 × 120 cm, Reg Alcorn (2023).

Electronic musician *Grand Ciel* experimented with it, notably by mixing loops of very simple sounds of different lengths (on three, four and five beats, for example), on the occasion of a conference-performance organized by the IREM de Limoges for the Fête de la Science 2019, at the Musée national Adrien Dubouché in Limoges, as part of the *Year of Mathematics*. This show entitled *From so simple a beginning* sought to dramatize that principle, with a live performance by the painter and musician improvising together, following a two-part lecture by the first author and his colleague Olivier Prot, to introduce the theme, the Ulam spiral and Reg Alcorn's method.

During a second, longer performance of the show, we also proposed an example of a mathematical question in which complexity appears unexpectedly in a problem that is easy to state, in this case the Jacobsthal function that associates with any integer n the largest difference between two consecutive integers prime to n, and for which only about sixty terms are known when applied to primorials.[16] The firsts are

$$2, 4, 6, 10, 14, 22, 26, 34,$$

and the following is not 38 but 40!

The show's title quotes Charles Darwin [6], to reflect the fact that the mentioned characteristic, far from applying only to the arts or mathematics, is intrinsic to the evolution of life:

> *From so simple a beginning, endless forms most beautiful and most wonderful have been, and are being, evolved.*

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Stéphane Vinatier is a Professor in Mathematics at the University of Limoges, France. His research is conducted within the XLIM research institute, UMR CNRS 7252. Specialized in algebraic number theory, focusing on the Galois action on invariants in number field extensions under specific ramification conditions. He is also interested in other mathematical domains, such as combinatorics. He has contributed to the French network of research institutes in mathematics education, serving as its head for two years. He continues to engage in experimental education projects.

stephane.vinatier@unilim.fr

William Reginald Alcorn was born in Zambia in 1950 to Irish and Scottish parents. He developed a passion for painting at a very early age. After studying humanities in England, he traveled around Europe, eventually settling in France. With projects in music, languages, science and, more recently, mathematics, he divides his time between his painting studio and popularizing the arts, notably as part of the activities of the IREM of Limoges.

regalcorn.frescoevent@gmail.com

The International Day of Mathematics 2025

Constanza (Coni) Rojas-Molina

March 14 marks the International Day of Mathematics, an initiative started in 2020 and led by the International Mathematical Union. The 2025 theme is "Mathematics, Art, and Creativity."[1]

Coni is a mathematician at CY Cergy Paris University, France. She is a science communicator and illustrator. Her preferred formats are sketchnotes and comics.

She is also member of the EMS Outreach and Engagement Committee, and the recipient of the 2024 Prize for Science Communication at CY Cergy Paris University. You can see her work at crojasmolina.com.

hello@crojasmolina.com

[1] For more details, visit https://www.idm314.org.

The circular restricted three-body problem: a modern symplectic viewpoint

Agustin Moreno

We give an overview of recent advances on the circular restricted three-body problem, from the perspective of modern symplectic geometry, and describe a "symplectic toolkit" created in connection to spacecraft trajectory design. This is based on the author's recent book draft.

1 Introduction

The three-body problem is the dynamical system corresponding to three masses under gravitational interaction, as described by Newton's law. It is one of those ancient conundrums that has withstood the ages, maturing like a complex wine. Its study goes back at least to the times of Newton, and its history is tied up with some of the most brilliant scientific figures, like Kepler, Poincaré, Arnold, Moser, and so many others, who had the courage to try to delve into its well-hidden secrets. The aim of this short note is to describe some of the recent advances in a special case of this problem, made possible through the modern methods of symplectic geometry. In what follows, we will restrict our attention to the *circular* and *restricted* three-body problem (or CR3BP for short), corresponding to the case where one of the masses is negligible, and the other two move in circles around their common center of mass. We will focus on the *spatial* problem, where the negligible mass moves in three-space, as opposed to the *planar* problem, where it moves in the plane. Despite the simplifications and approximations, this is still an outstanding open challenge.

The CR3BP is not only interesting from a theoretical point of view, but also from a practical perspective, due to its deep connections to astronomy and space exploration. Namely, the CR3BP is one of the most basic models approximating the motion of a spacecraft under the influence of a planet–moon system. This is a modern interpretation: unlike the times of Newton, when space travel was but a wild opium dream, in the current day and age, when mission proposals to remote regions of our expanding Universe are common currency, the CR3BP is one of the preeminent models used for spacecraft trajectory design. In the context of astrodynamics, the CR3BP is then the theoretical starting point supporting the high-fidelity (or *ephemeris*) numerical studies which go into actual mission proposals. While finding trajectories that meet the requirements of an actual mission is a very complicated art, the families of periodic orbits found in the CR3BP, as well as the stable/unstable manifolds of some of them, can be used as building blocks for designing the desired trajectories, and to transfer between them.

We will first discuss some of the theoretical aspects, and then those aspects which are closer to applications. In particular, we will describe a "symplectic toolkit" created with the needs of trajectory design in mind, the result of a collaboration of the author with NASA engineers. More details can be found in the author's recent book draft [13] (see also Quanta Magazine's recent article [19]).

2 The CR3BP

We consider three bodies: Earth (E), the Moon (M) and a satellite (S), with masses m_{E}, m_{M}, and m_{S}. One has the following cases and assumptions:

- *(Restricted case)* $m_{\mathrm{S}} = 0$, i.e., the satellite is negligibly small when compared with the *primaries* E and M;
- *(Circular assumption)* Each primary moves in a circle, centered around the common center of mass of the two (as opposed to general ellipses);
- *(Planar case)* S moves in the ecliptic plane containing the primaries;
- *(Spatial case)* The planar assumption is dropped, and S is allowed to move in three-dimensional space.

We denote the *mass ratio* by $\mu = \frac{m_{\mathrm{M}}}{m_{\mathrm{E}}+m_{\mathrm{M}}} \in [0,1]$, and we normalize so that $m_{\mathrm{E}} + m_{\mathrm{M}} = 1$, and so $\mu = m_{\mathrm{M}}$ can be thought of as the mass of the Moon. In rotating coordinates, in which both primaries are at rest, the Hamiltonian describing the problem is actually autonomous. If the positions of the Earth and the Moon are $E = (\mu, 0, 0)$ and $M = (-1+\mu, 0, 0)$, the Hamiltonian is

$$H\colon \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R},$$

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q-M\|} - \frac{1-\mu}{\|q-E\|} + p_1 q_2 - p_2 q_1.$$

By conservation of energy, this means that H is a preserved quantity of the Hamiltonian motion. The planar problem is the subset

$\{p_3 = q_3 = 0\}$, which is clearly invariant under the Hamiltonian dynamics. The two parameters in the problem are the *Jacobi constant* c (the energy value), and μ.

As computed by Euler and Lagrange, there are precisely five critical points of H, called the *Lagrangian points* $L_i = L_i(\mu)$, $i = 1, \dots, 5$, ordered so that $H(L_1) < H(L_2) < H(L_3) < H(L_4) = H(L_5)$. The *low-energy* range corresponds to $c < H(L_1)$ (or slightly above).

For $c \in \mathbb{R}$, consider the energy hypersurface $\Sigma_c = H^{-1}(c)$. If

$$\pi\colon \mathbb{R}^3 \setminus \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{E, M\}, \quad \pi(q, p) = q,$$

is the projection onto the position coordinate, we define the *Hill region* of energy c as

$$\mathcal{K}_c = \pi(\Sigma_c) \in \mathbb{R}^3 \setminus \{E, M\}.$$

This is the region in space where the satellite of energy c is allowed to move. If $c < H(L_1)$, then $\mathcal{K}_c$ has three connected components: a bounded one around the Earth, another bounded one around the Moon, and an unbounded one. Denote the first two components by $\mathcal{K}_c^{\mathrm{E}}$ and $\mathcal{K}_c^{\mathrm{M}}$, and set $\Sigma_c^{\mathrm{E}} = \pi^{-1}(\mathcal{K}_c^{\mathrm{E}}) \cap \Sigma_c$, $\Sigma_c^{\mathrm{M}} = \pi^{-1}(\mathcal{K}_c^{\mathrm{M}}) \cap \Sigma_c$. As c crosses the value $H(L_1)$, $\mathcal{K}_c^{\mathrm{E}}$ and $\mathcal{K}_c^{\mathrm{M}}$ get glued together into a new connected component, $\mathcal{K}_c^{\mathrm{E,M}}$, topologically their connected sum. Then, the satellite in principle has enough energy to transfer between Earth and the Moon. See Figure 1.

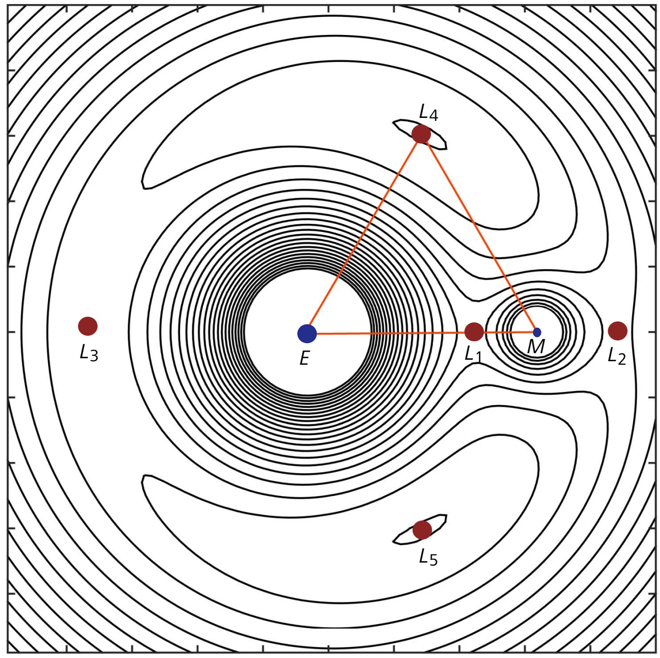

Figure 1. The Hill regions and the Lagrange points for the planar problem.

3 Theoretical aspects

3.1 The Poincaré–Birkhoff theorem, and the planar CR3BP

Poincaré [16, 17] reduced the problem of finding periodic orbits in the planar CR3BP to:

(1) finding a global surface of section for the dynamics;

(2) proving a fixed point theorem for the resulting first-return map.

This is the setting for the celebrated Poincaré–Birkhoff theorem.

3.2 Collision regularization

The 5-dimensional energy hypersurfaces in the spatial CR3BP are non-compact, due to collisions of the massless body S with one of the primaries. Two-body collisions can be regularized via Moser's recipe. The bounded components Σ_c^{E} and Σ_c^{M} (for $c < H(L_1)$), as well as $\Sigma_c^{\mathrm{E,M}}$ (for $c \in (H(L_1), H(L_1) + \varepsilon)$), are thus compactified to compact manifolds $\overline{\Sigma}_c^{\mathrm{E}} \cong \overline{\Sigma}_c^{\mathrm{M}} \cong S^3 \times S^2$, and $\overline{\Sigma}_c^{\mathrm{E,M}} \cong S^2 \times S^3 \# S^2 \times S^3$. In the planar problem, we obtain copies of $\mathbb{R}P^3$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$. We use the notation $\overline{\Sigma}_{\mathrm{P},c}^{\mathrm{E}}$, $\overline{\Sigma}_{\mathrm{P},c}^{\mathrm{M}}$ and $\overline{\Sigma}_{\mathrm{P},c}^{\mathrm{E,M}}$ for the corresponding planar regularized energy level sets.

3.3 The advent of contact geometry in the CR3BP

It was only recently that the modern techniques from contact and symplectic geometry (holomorphic curves, Floer theory, etc.) have been made to bear on the CR3BP. This is due to the following result.

Theorem 1 ([1] (planar problem), [5] (spatial problem)). *If $c < H(L_1)$, the regularized hypersurfaces $\overline{\Sigma}_c^{\mathrm{E}}, \overline{\Sigma}_c^{\mathrm{M}}, \overline{\Sigma}_{\mathrm{P},c}^{\mathrm{E}}, \overline{\Sigma}_{\mathrm{P},c}^{\mathrm{M}}$ carry contact structures. The same holds for $\overline{\Sigma}_c^{\mathrm{E,M}}$ and $\overline{\Sigma}_{\mathrm{P},c}^{E,M}$, if $c \in (H(L_1), H(L_1) + \varepsilon)$ for sufficiently small $\varepsilon > 0$.*

3.4 Open book decompositions

We have the following fundamental notion from smooth topology.

Definition 3.1 (Open book decomposition). Let M be a closed manifold. A (concrete) *open book decomposition* on M is a fibration $\pi\colon M \setminus B \to S^1$, where $B \subset M$ is a closed, codimension-2 submanifold with trivial normal bundle. We further assume that $\pi(b, r, \theta) = \theta$ along some collar neighborhood $B \times \mathbb{D}^2 \subset M$, where (r, θ) are polar coordinates on the disk factor.

The submanifold B is called the *binding*, and the closures of the fibers $P = P_\theta = \overline{\pi^{-1}}(\theta)$ are called the *pages*, which satisfy $\partial P_\theta = B$ for every θ. We usually denote a concrete open book by the pair (π, B), but also use the abstract notation $M = \mathbf{OB}(P, \varphi)$, where φ is a diffeomorphism $\varphi\colon P \to P$ with $\varphi = id$ near B (the *monodromy*). See Figure 2.

If M is oriented and endowed with an open book decomposition, then the natural orientation on the circle induces orientations on the pages, which in turn induce the boundary orientation on the binding.

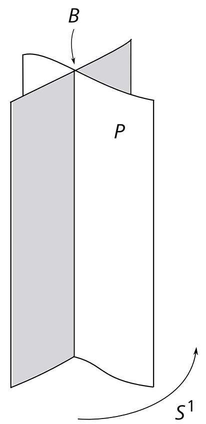

Figure 2. A neighborhood of the binding look precisely like the pages of an open book, whose front cover has been glued to its back cover via some gluing map (the monodromy).

Definition 3.2 (Giroux). Let (M,ξ) be an oriented contact manifold, and (π, B) an open book decomposition on M. Then ξ is *supported* by the open book if one can find a positive contact form α for ξ (called a *Giroux form*) such that:

(i) $\alpha_B := \alpha|_B$ is a positive contact form for B;

(ii) $d\alpha|_P$ is a positive symplectic form on the interior of every page P.

Here, a *positive* contact form is a contact form α on M^{2n-1} such that the orientation induced by the volume form $\alpha \wedge d\alpha^{n-1}$ coincides with the given orientation on M.

We say that the open book is *adapted to the dynamics* of α if (i) and (ii) hold. One has the following fundamental result.

Theorem 2 (Giroux [8]). *Every open book decomposition supports a unique isotopy class of contact structures. Any contact structure admits a supporting open book decomposition.*

Here, two contact structures are said to be isotopic if they can be joined by a smooth path ξ_t of contact structures. By *Gray's stability result*, isotopic contact structures are *contactomorphic*, i.e., there exists a diffeomorphism which carries one to the other.

Remark 3.3. In fact, Giroux's result is stronger, as there is in fact a correspondence between contact structures up to isotopy and open books up to a notion of positive stabilization. Giroux proved this in dimension 3, and the result was recently established in higher dimensions by Breen, Honda, and Huang [3].

We usually write $(M,\xi) = \mathbf{OB}(P,\varphi)$ to indicate that the contact structure ξ is supported by the open book (P,φ).

The Giroux correspondence reduces the topological study of contact manifolds to the topological study of open books. *However,* this result holds only when the (isotopy class of the) contact *structure* is fixed, and the contact form (and hence the dynamics) is auxiliary; Giroux's result is *not* dynamical, but rather topological/geometrical. But this will serve as motivation for what comes next.

3.5 Open books in the CR3BP

Let $\overline{\Sigma}_c$ stand for either $\overline{\Sigma}_c^{\mathrm{E}}$, $\overline{\Sigma}_c^{\mathrm{M}}$ or $\overline{\Sigma}_c^{\mathrm{E,M}}$ (for the spatial problem). The following result generalizes the approach of Poincaré in the planar problem (i.e., step (1)) to the *spatial* problem. Combining Theorem 1 with the Giroux correspondence, we know there exist supporting open book decompositions on $\overline{\Sigma}_c$ when c belongs to the low-energy range. However, as we emphasized already, this correspondence does not give adapted open books whenever the dynamics is fixed. The content of the following result is that the *given* dynamics of the spatial CR3BP in the low-energy range, and near the primaries, is given by a contact form which is a Giroux form for some concrete open book.

Theorem 3 (Moreno–van Koert [15]). *For any $\mu \in [0,1]$, if c lies in the low-energy range, $\overline{\Sigma}_c$ admits a supporting open book decomposition for energies $c < H(L_1)$ that is adapted to the dynamics. Furthermore, if $\mu < 1$, then there is $\varepsilon > 0$ such that the same holds for $c \in (H(L_1), H(L_1)+\varepsilon)$. The open books have the following abstract form:*

$$\overline{\Sigma}_c \cong \begin{cases} (S^*S^3, \xi_{\mathrm{std}}) = \mathbf{OB}(\mathbb{D}^*S^2, \tau^2), & \text{if } c < H(L_1), \\ (S^*S^3, \xi_{\mathrm{std}}) \# (S^*S^3, \xi_{\mathrm{std}}) & \\ \quad = \mathbf{OB}(\mathbb{D}^*S^2 \natural \mathbb{D}^*S^2, \tau_1^2 \circ \tau_2^2), & \text{if } c \in (H(L_1), H(L_1)+\varepsilon), \\ & \quad \mu < 1. \end{cases}$$

In all cases, the binding is the planar problem is

$$B = \overline{\Sigma}_{\mathrm{P},c} = \begin{cases} (S^*S^2, \xi_{\mathrm{std}}), & \text{if } c < H(L_1) \\ (S^*S^2, \xi_{\mathrm{std}}) \# (S^*S^2, \xi_{\mathrm{std}}), & \text{if } c \in (H(L_1), H(L_1)+\varepsilon), \\ & \quad \mu < 1. \end{cases}$$

Here, $\mathbb{D}^*S^2$ is the unit cotangent bundle of the 2-sphere, τ is the positive Dehn–Seidel twist, and $\mathbb{D}^*S^2 \natural \mathbb{D}^*S^2$ denotes the boundary connected sum of two copies of $\mathbb{D}^*S^2$. The monodromy of the second open book is the composition of the square of the positive Dehn–Seidel twists along both zero sections (they commute).

4 Practical aspects

A given Hamiltonian system usually depends on parameters (e.g., energy or mass parameters), which one may vary. Under such deformations, periodic orbits may undergo *bifurcation*, a mechanism by which new families of periodic trajectories arise. The way different

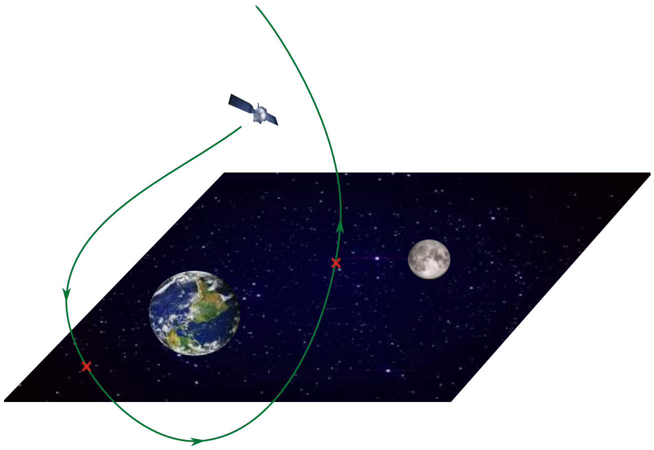

Figure 3. Theorem 3 admits a physical interpretation: away from collisions, the orbits of the negligible mass point intersect the plane containing the primaries transversely. The "pages" $\{q_3 = 0, p_3 > 0\}$, $\{q_3 = 0, p_3 < 0\}$ of the "physical" open book $(q, p) \mapsto \frac{q_3 + ip_3}{\|q_3 + ip_3\|} \in S^1$, are global hypersurfaces of section outside of the collision locus.

families can connect to each other is encoded in the topology of a *bifurcation graph*. The aim of this chapter is then to introduce a "symplectic toolkit," extracted from the modern methods of symplectic geometry, and designed to systematically map out how different orbit families merge together.

4.1 Symplectic data analysis

The "symplectic toolkit" consists of the following elements:

(1) *Floer numbers:* Integers that are invariant under bifurcation, and so can help predict the existence of orbits.

(2) *The B-signs* [7]: a $\pm$ sign associated to each elliptic or hyperbolic orbits, which helps predict bifurcations, and generalizes the classical Krein theory [9, 10].

(3) *Global topological methods:* the (geometric invariant theory) GIT-*sequence* [7], a sequence of spaces whose global topology encodes bifurcations, and refines Broucke's stability diagram [4] by adding the *B*-signs.

(4) *CZ-index* [6, 18]: a winding number associated to non-degenerate orbits, extracted from the topology of the symplectic group. It can be used to determine which families connect to which.

4.2 Numerical work

We now describe some numerical work where we put the symplectic toolkit into practical use. This is based on the article [14].

We consider the *Jupiter–Europa system* (JE), which corresponds to a CR3BP with $\mu = 2.5266448850435e^{-05}$, and the *Saturn–Enceladus system* (SE), $\mu = 1.9002485658670e^{-07}$. These two systems are of tremendous current interest for space agencies such as NASA, as they may have conditions suitable for real life problems. Starting from the Hill 3BP, we deform to JE and to SE. One symmetry is broken, and families behave more generically. In what follows, we use $\Gamma = -2c$.

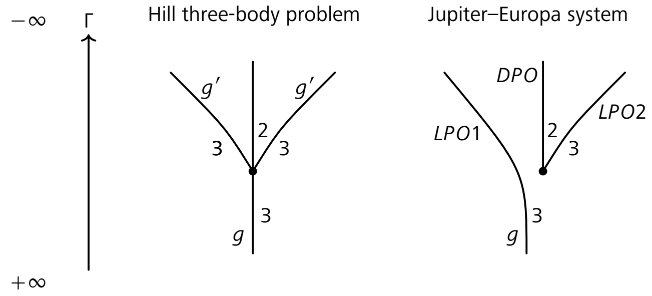

Figure 4. Bifurcations of planar direct/prograde orbits with corresponding CZ-index.

4.3 Planar direct/prograde orbits

The planar pitchfork bifurcation described by Hénon [12] in the Hill 3BP (concerning planar orbit families f, g, g') becomes a generic broken bifurcation in the planar JE CR3BP, see Figure 4.

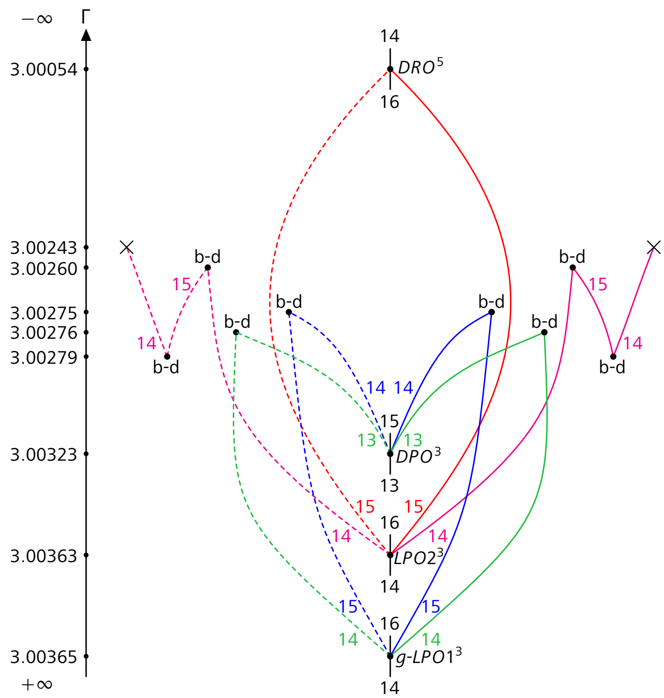

Figure 5. Bifurcation graph for JE, between g-$LPO1^3$, DPO^3, $LPO2^3$, and DRO^5.

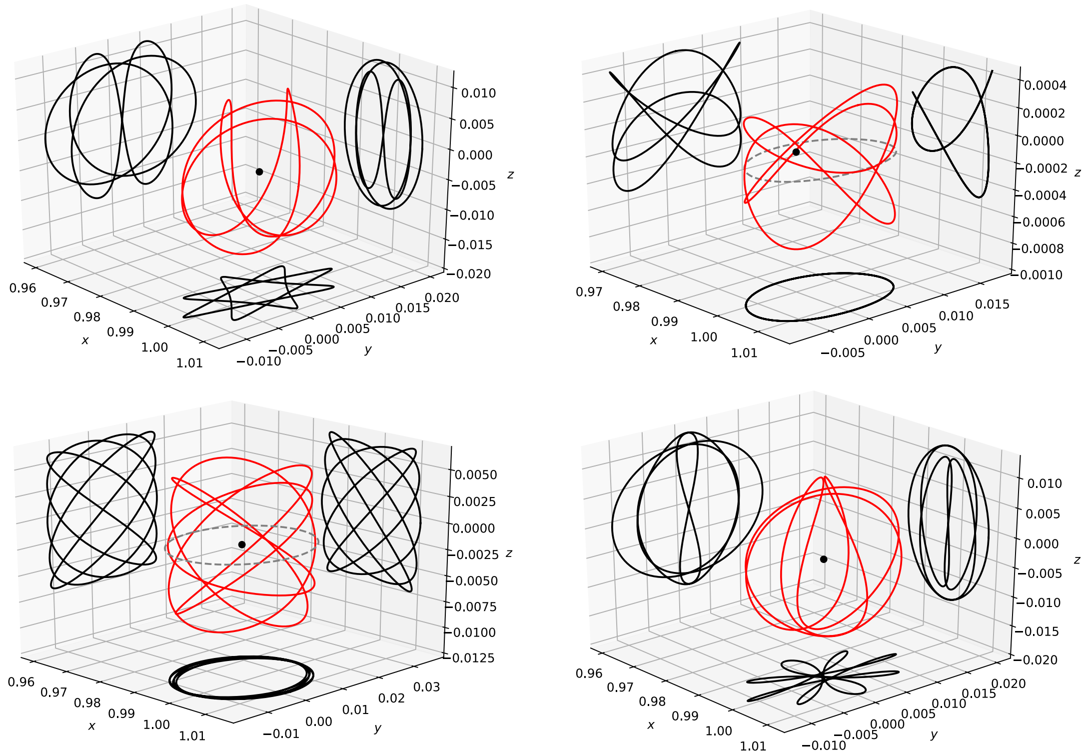

Figure 6. Prograde to retrograde spatial connection, red CZ-index 15 in Figure 5.

4.4 Spatial bifurcation graphs between planar prograde and retrograde orbits

A bifurcation graph relating third covers of the direct orbits g, g', and fifth covers of planar retrograde orbits f, connected via spatial families, was obtained by Aydin [2]. We deformed it to JE in Figure 5.

4.5 Halo orbits in SE

We consider halo orbits coming out of the Lagrange point L_2 in SE. This family appears also in NASA's technical report on the Enceladus Orbilander [11], and is meant to be used to visit the poles in future missions. The corresponding family for the Earth–Moon system is currently very popular, as it will be where NASA's Gateway Space Station will be parked. The most interesting part of the family in SE occurs just after the CZ-index jumps from 3 to 4, where orbits are stable and close to the water plumes.

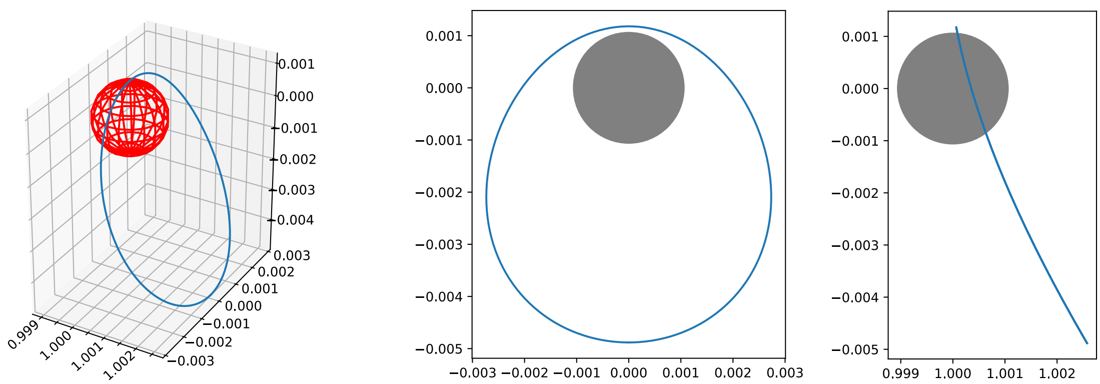

Figure 7. Halo-polar orbit ($\Gamma = 3.000034709155895$) with an altitude of 29 km. The CZ-index has just jumped to 4, and is doubly elliptic.

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Agustin Moreno is a junior professor in the Department of Mathematics at Heidelberg University in Germany, as well as a former member of the School of Mathematics at the Institute for Advanced Study, Princeton. He holds degrees from the University of Cambridge, and from the Humboldt University in Berlin, where he finished his PhD (started at the University College London, UK). He has been affiliated with the Berlin Mathematical School, and has been a research fellow at the Mittag-Leffler Institute in Sweden – an institution famously tied to the work of Poincaré on the three-body problem – where much of his work on the very same problem was carried out. Agustin Moreno's research lies at the crossroads of geometry, topology, dynamics and mathematical physics, with further interest in applications to space mission design, which he pursues in collaboration with navigational engineers at NASA and aerospace engineers at the University of Colorado, Boulder, USA.

agustin.moreno2191@gmail.com

Explainable artificial intelligence and mathematics: What lies behind? Let us focus on this new research field

Massimiliano Ferrara

The growing complexity inherent in modern artificial intelligence (AI) models has necessitated an increased focus on the demand for explainability, commonly referred to as explainable artificial intelligence *(XAI). The primary objective of XAI is to render the decision-making processes of AI systems not only transparent, but also understandable to human users, thereby fostering greater trust and comprehension among stakeholders. As AI systems become more sophisticated and are deployed in critical areas such as healthcare, finance, and autonomous vehicles, the demand for clarity surrounding their operations intensifies.*

This paper delves deeply into the vital relationship between XAI and mathematics, asserting that mathematical principles are foundational to enhancing the interpretability, transparency, and overall trustworthiness of AI models. We will investigate the key mathematical constructs that underlie various XAI techniques, providing insights into how they function and contribute to explainability.

To illustrate the practical significance of these principles, we will examine specific case studies where mathematical frameworks have successfully improved the elucidation of AI model predictions. Furthermore, this paper will outline potential future avenues for research that aim to further integrate mathematical methodologies within XAI frameworks. By doing so, we hope to contribute to the development of more robust and interpretable AI systems that can be trusted and effectively utilized by humans in a multitude of applications.

1 Introduction

In recent years, artificial intelligence (AI) has undergone revolutionary advancements, leading to its integration into numerous critical applications spanning industries such as healthcare, finance, transportation, and beyond. With its capabilities to analyze vast datasets, recognize patterns, and make informed decisions, AI systems have become invaluable in promoting efficiency and innovation. However, as AI technologies evolve, so too has the complexity of the models driving their decision-making processes.

Simultaneously, this sophistication raises significant concerns regarding transparency and interpretability issues encapsulated in the term *black box* [7]. Many modern AI algorithms, particularly those based on deep learning, operate in ways that are not easily understandable by humans, rendering their decision-making processes opaque. This lack of clarity poses serious risks, especially in high-stakes environments. For instance, in healthcare, algorithmic decisions can influence clinical diagnoses and treatment plans, where a misinterpretation or erroneous model output could have drastic consequences on patient outcomes. In finance, automated systems determining creditworthiness must comply with regulations requiring transparency. When applicants are denied loans, they must be provided with understandable explanations.

To address these concerns, the field of explainable artificial intelligence (XAI) has emerged as a critical research area. XAI aims to develop methodologies and frameworks that enable humans to comprehend, trust, and exploit AI systems effectively. More than simply improving model interpretability, XAI encompasses a proactive approach to ensuring accountability and ethical standards in AI deployment. Increasingly, stakeholders are demanding that AI not only be high-performing, but also accessible and explainable to users and regulators alike.

This paper delves into the vital interplay between mathematics and XAI, asserting that a robust understanding of mathematical foundations is essential to fostering clearer interpretations of AI model behavior. Mathematics provides the frameworks necessary for developing advanced interpretability techniques, ranging from Shapley values to feature importance scores. The synthesis of these mathematical tools with AI narrows the gap between model complexity and user understanding, ultimately driving innovation in more reliable AI systems.

Through this exploration, the paper aims to outline the critical connections between mathematics and XAI methodologies while providing concrete case studies that illustrate these principles in action. By addressing the landscape of XAI in conjunction with its mathematical backbone, we aim to promote a comprehensive understanding of how these two domains can and should intersect.

2 Motivation

To appreciate the significance of explainability in artificial intelligence, it is essential to understand the historical context of AI development and the challenges that have accompanied the rise of complex algorithms. The roots of AI can be traced back to the mid-20th century, with early endeavors focused on rule-based systems that simulated basic reasoning capabilities. These systems relied primarily on human-crafted instructions and logic, making them relatively interpretable. However, as computational capacity burgeoned alongside data availability, the emergence of machine learning algorithms marked a paradigm shift, enabling systems to learn from data rather than rely solely on predefined rules [6].

Machine learning models, particularly those utilizing neural networks and deep learning architectures, have since demonstrated unparalleled performance in tasks such as image recognition, natural language processing, and game playing. However, this success has come at the cost of interpretability. As these models grow in complexity, comprising multiple hidden layers, millions of parameters, and intricate interactions, their internal workings become increasingly opaque. Users cannot easily ascertain how inputs are transformed into outputs, resulting in a sense of discomfort and mistrust, particularly in critical applications.

This "black box" nature of AI has prompted a renewed focus on explainability over the last decade. Researchers and practitioners recognize that building public trust in AI systems requires elucidating how and why decisions are made. Moreover, explainable AI is not merely a technical challenge, but also a societal imperative. Ethical implications abound when algorithms govern fundamental aspects of human lives, such as health and financial stability.

In this context, key concepts within XAI have emerged, defining a spectrum of approaches and frameworks designed to enhance interpretability. These include model-agnostic methods, which offer insights applicable across various algorithms, and instance-based explanations, which delve into the specifics of individual predictions. Researchers have utilized methods from diverse fields, including statistics, game theory, and information theory, to craft explanations that resonate with end-users. The importance of explainability is underscored by industry efforts and regulatory requirements, highlighting the critical need for interpretable models to ensure ethical practices [4, 9].

Simultaneously, this new domain raises discussions about the mathematical foundations of XAI techniques. Understanding the underlying mathematics is crucial for developing robust explanations that carry both technical accuracy and meaningful human insights. By integrating mathematical reasoning into XAI, we can promote the development of systems that not only function well, but also provide clear and actionable explanations for their behavior. As we transition into exploring the mathematical foundations of AI and XAI methodologies in the following sections, it becomes evident that the innovative approaches we seek will rely heavily on our understanding of the mathematical concepts that underpin this technology. A deeper engagement with these connections will pave the way for enhanced trust, usability, and societal acceptance of artificial intelligence.

3 Mathematical foundations of XAI

Mathematics provides the bedrock upon which many XAI methods are built. From linear algebra and calculus to more complex fields like information theory and topology, mathematical concepts facilitate the extraction of meaningful information from AI models.

3.1 Linear algebra and matrix decompositions

Linear algebra is fundamental in model interpretation, particularly in techniques like principal component analysis (PCA) and singular value decomposition (SVD). These methods reduce data dimensionality while preserving variance, making it easier to visualize and interpret high-dimensional data.

Principal component analysis (PCA)

PCA transforms data by projecting it onto orthogonal vectors that maximize variance. The transformation of a dataset X using PCA involves computing its covariance matrix Σ, and then deriving its eigenvalues and eigenvectors. The principal components are the eigenvectors corresponding to the largest eigenvalues:

$$\Sigma = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)(x_i - \mu)^{\mathsf{T}},$$

$$\Sigma v = \lambda v.$$

Here, v represents the eigenvectors (principal components), and λ the eigenvalues.

Singular value decomposition (SVD)

SVD generalizes PCA and decomposes a matrix into singular vectors and singular values. For a given matrix A, SVD can be represented as

$$A = U \Sigma V^{\mathsf{T}},$$

where U and V are orthogonal matrices, and Σ is a diagonal matrix of singular values.

3.2 Calculus and optimization

Gradient-based optimization techniques, derived from calculus, are essential for training AI models. Understanding gradients and Hessian matrices helps in explaining how models learn from data, and in identifying critical features and decision boundaries.

Gradient descent
Gradient descent minimizes a function $f(\theta)$ by iteratively moving in the direction of the steepest descent, defined by the negative gradient. The update rule is given by

$$\theta_{t+1} = \theta_t - \eta \nabla f(\theta_t),$$

where η is the learning rate, and $\nabla f(\theta_t)$ is the gradient of the function at θ_t.

Hessian matrices and curvature
The Hessian matrix H of a function $f(\theta)$ at point θ is a square matrix of second-order partial derivatives, representing the local curvature:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial \theta_1^2} & \cdots & \frac{\partial^2 f}{\partial \theta_1 \partial \theta_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial \theta_n \partial \theta_1} & \cdots & \frac{\partial^2 f}{\partial \theta_n^2} \end{bmatrix}.$$

3.3 Information theory
Information theory quantifies uncertainty and information gain, aiding in the development of metrics such as entropy and mutual information. These metrics are vital for feature selection and model interpretability.

Entropy and Information Gain
Entropy $H(X)$ measures the uncertainty in a random variable X:

$$H(X) = -\sum_{i=1}^{n} P(x_i) \log P(x_i).$$

Information gain measures the reduction in entropy when a dataset is split based on an attribute:

$$\mathrm{IG}(Y \mid X) = H(Y) - H(Y \mid X).$$

3.4 Mutual information
Mutual information $I(X;Y)$ quantifies the amount of information obtained about one random variable through another:

$$I(X;Y) = \sum_{x \in X} \sum_{y \in Y} P(x,y) \log \frac{P(x,y)}{P(x)P(y)}.$$

4 The contribution of game theory: New perspectives

Game theory is a branch of mathematics that investigates the strategic interactions among rational agents and explores their wide-ranging applications across diverse fields, including artificial intelligence (AI). Within the domain of explainable artificial intelligence (XAI), game theory offers a foundational methodology for enhancing our understanding and improving the transparency of AI models.

A pivotal concept in game theory is the representation of strategic interactions as "games," where participants engage in rational decision-making to optimize their objectives. By applying these principles to AI explainability, we can regard the decision-making processes of AI models as a game involving the artificial system and human users attempting to comprehend its actions.

Game theory furnishes a conceptual framework for examining the strategies deployed by AI models to convey their decisions in a clear and comprehensible manner. For instance, utilizing concepts such as Nash equilibrium allows us to analyze how AI models and human users can collaborate effectively to facilitate meaningful explanations of the system's decisions.

Furthermore, game theory can assist in modeling situations where the explainability of AI may conflict with other objectives, such as computational efficiency or predictive accuracy. By evaluating multi-agent games and identifying strategic trade-offs, we can devise strategies that reconcile these competing considerations and create explainable AI frameworks that satisfy a variety of requirements.

In conclusion, integrating game theory into the XAI realm can offer novel insights and methodologies for addressing challenges related to the transparency and interpretability of artificial systems. By leveraging fundamental concepts from game theory to analyze and optimize the interactions between AI models and human users, we can foster the development of intelligent systems that are not only powerful and accurate, but also comprehensible and acceptable to society.

Shapley values and their role
Shapley values, which originate from cooperative game theory, guarantee a fair distribution of payoffs among participants [10]. In the context of XAI, Shapley values quantify the contribution of each feature to the overall prediction. The Shapley value for a feature i is defined as

$$\phi_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} [v(S \cup \{i\}) - v(S)],$$

where N represents the complete set of features, and $v(S)$ is the value function that denotes the prediction when the subset S of features is utilized.

Application in SHAP
Shapley additive explanations (SHAP) apply Shapley values to provide consistent and verifiable feature attributions. Delve into the mathematical formulation of SHAP using the Shapley value equation above and demonstrate with an example.

5 Case studies

To illustrate the synergy between mathematics and XAI, we consider several case studies where mathematical techniques have enhanced explainability.

LIME and SHAP
Local interpretable model-agnostic explanations (LIME) and Shapley additive explanations (SHAP) are popular XAI methods that rely on mathematical principles. LIME uses locally weighted linear regression to approximate a model's behavior around a specific prediction, while SHAP leverages cooperative game theory to distribute contributions of features fairly.

LIME
Detail the mathematical methodology behind LIME, including the optimization of local surrogates and interpretability of linear approximations. Provide a detailed example showcasing a step-by-step application of LIME to a specific prediction instance.

SHAP
Discuss SHAP's foundation in Shapley values from cooperative game theory. Highlight the mathematical derivation of Shapley values and their contribution to fair attribution of feature importance. Include a case study that rigorously applies SHAP to a real-world dataset, illustrating how feature contributions are computed and interpreted.

Decision trees and rule extraction
Decision trees, inherently interpretable models, use recursive partitioning based on feature values to generate easily understandable rules. Techniques like decision tree surrogate models create interpretable approximations of complex models.

Recursive partitioning
Explain the mathematical basis of recursive partitioning, including impurity measures like Gini impurity and entropy in the context of decision trees. Provide a case study that demonstrates the construction of a decision tree and the derivation of decision rules from the model [8]:

$$\mathrm{Gini}(S) = 1 - \sum_{i=1}^{n} (p_i)^2.$$

Rule extraction methods
Detail methods for extracting rules from black-box models, such as model distillation and surrogate decision trees, with mathematical explanations of each approach. Include examples of rule extraction processes, illustrating the transformation of complex model outputs into human-understandable rules.

Bayesian networks
Bayesian networks utilize probability theory to represent and reason about the dependencies among variables. These networks simplify the visualization and understanding of probabilistic relationships, aiding in the interpretability of predictions.

Probabilistic graphical models
Discuss the mathematical foundation of Bayesian networks, including concepts of conditional independence and factorization of joint distributions. Provide an example application of Bayesian networks in a specific domain, highlighting how probabilistic dependencies are modeled and interpreted:

$$P(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} P(X_i \mid \mathrm{Parents}(X_i)).$$

6 The role of mathematics in future XAI developments

As the field of artificial intelligence continues to evolve, the integration of advanced mathematical techniques into explainable artificial intelligence (XAI) is becoming increasingly critical. This intersection not only enhances model interpretability, but also opens new avenues for research and application, contributing to the overall trustworthiness of AI systems. The role of mathematics in future XAI developments can be categorized into several key areas: the exploration of advanced modeling techniques, the establishment of quantitative metrics for explainability, the application of optimization methods, and the potential contributions from emerging fields such as topological data analysis and information theory [3].

6.1 *Advanced modeling techniques*

Traditional machine learning algorithms have relied on wellestablished mathematical frameworks, such as linear regression and decision trees. However, with the rise of deep learning and other complex models, researchers are exploring innovative mathematical representations that enhance explainability. For instance, neural networks can be enhanced by integrating concepts from calculus, specifically through techniques such as those mentioned below.

Gradient-based explanation methods
The gradients of the loss function concerning input features are paramount for understanding model behavior. The backpropagation algorithm, expressed mathematically as

$$\delta^l = \nabla_a C \odot \sigma'(z^l),$$

is vital for calculating error derivatives across hidden layers, where δ^l represents the error term, C is the cost function, a is the activation output, σ is the activation function, and z is the weighted input. Through gradient calculations, we can gain insights into which features influence the most the model's predictions.

Interpretability via attention mechanisms

Attention mechanisms in neural architectures, particularly transformer models, allow the model to focus on specific parts of the input sequence. Mathematically, the attention score can be defined as

$$\mathrm{Attention}(Q, K, V) = \mathrm{softmax}\Big(\frac{QK^\top}{\sqrt{d_k}}\Big)V,$$

where Q (queries), K (keys), and V (values) are derived from the input representations. Understanding the attention weights can help determine the importance of various input components in the model's predictions, making it easier to devise explanations that correspond to the input features responsible for specific outputs.

6.2 Quantitative metrics for explainability

To evaluate and compare the effectiveness of different explanation methods, it is imperative to establish rigorous quantitative metrics. Mathematics plays a crucial role in developing these metrics, which can quantify various aspects such as the following.

Fidelity and consistency

The fidelity of an explanation refers to how accurately it reflects the behavior of the underlying model. One way to mathematically validate this is through measures based on approximating the original model f with an interpretable model g.

Simplicity and completeness

Explainability metrics often emphasize the trade-off between complexity and comprehensiveness. For instance, a metric S to evaluate the simplicity of an explanation might be defined as

$$S(g) = \frac{1}{|g|}\sum_{i=1}^{|g|} \mathrm{Length}(g_i),$$

where g_i are the components of the explanation. Here, a lower score indicates that an explanation is simpler, which is typically desirable.

6.3 Optimization methods

Mathematics is fundamental in optimizing models for both performance and explainability. Model interpretability often requires trade-offs that can be addressed through optimization techniques, such as those mentioned below.

Multi-objective optimization

This paradigm allows the simultaneous optimization of multiple conflicting objectives, for instance, maximizing model accuracy while minimizing complexity. An example objective function is provided by

$$\text{minimize} \quad \sum_{i=1}^{N} L(y_i, f(x_i)) + \lambda \cdot \mathrm{Complexity}(f),$$

where L is the loss function measuring the error, y_i is the true output, x_i is the input data, and λ is a trade-off parameter for model complexity.

Regularization techniques

Regularization techniques, such as L1 (lasso) and L2 (ridge) regularization, help prevent overfitting while enhancing interpretability by encouraging sparsity in the model weights. The L1 regularization term can be mathematically expressed as

$$R(\theta) = \lambda \sum_{j=1}^{p} |\theta_j|,$$

where θ_j are the model parameters and p is the number of features. Sparse solutions lead to simpler models that are easier to interpret.

6.4 Contributions from emerging fields

As AI progresses, emerging mathematical fields are beginning to influence how we understand and develop explainable models, as in the examples below.

Topological data analysis (TDA)

TDA focuses on the shape of data and has been proposed as an avenue to reveal insights into high-dimensional datasets that may inform model behavior. Techniques such as persistent homology can provide a geometric understanding of the data manifold, potentially revealing relationships that enhance interpretability.

Information theory

Applying concepts from information theory allows researchers to quantify the information gains achieved through XAI methods. Measures such as mutual information can be leveraged to determine how much information an explanation conveys about the prediction

$$I(X; Y) = H(X) + H(Y) - H(X, Y),$$

where H represents entropy. Understanding the mutual information between input features X and predictions Y can guide the development of more informative explanations.

7 Future directions

The integration of advanced mathematical techniques into XAI is an ongoing field of research [1, 2, 5]. Future work may involve the following.

7.1 Topological data analysis (TDA)

TDA applies concepts from algebraic topology to uncover the shape and structure of data. Persistent homology, a key tool in TDA, can reveal robust features that contribute to model explanations.

Persistent homology

Explain persistent homology's mathematical foundation and its utility in identifying significant data features that persist across multiple scales. Include examples of how TDA has been applied to complex datasets and the insights it has provided.

Causal inference

Mathematical techniques from causal inference can help distinguish causation from correlation in AI models, providing deeper insights into the underlying mechanisms driving predictions.

Causal models

Introduce causal models and the mathematical formulation of causal relationships (e.g., do-calculus). Discuss applications in interpreting model decisions, providing examples of causal inference techniques applied to real-world AI predictions.

Information geometry

Information geometry examines the differential-geometric structure of statistical models. This perspective can enhance our understanding of model parameter spaces and improve interpretability.

Geometric understanding of models

Explain the mathematical principles of information geometry, including divergence measures and their role in interpreting statistical models. Provide examples of how information geometry can be applied to examine and understand deep learning models.

8 Conclusions

The integration of advanced mathematical techniques into future XAI developments is not merely a theoretical exercise, but a practical necessity. The mathematical tools underpinning AI systems will continue to shape the evolution of methods designed to promote transparency and interpretability. As we navigate a landscape of increasingly intricate models, the role of mathematics will persist as a cornerstone in the quest to unravel the complexities of AI, ensuring these systems serve humanity while adhering to standards of trust and accountability.

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Massimiliano Ferrara is a full professor of mathematical economics, business analytics, decision support systems, and economic intelligence at the Mediterranean University of Reggio Calabria, and a research affiliate at ICRIOS – the Invernizzi Center for Research on Innovation, Organization, Strategy and Entrepreneurship, Bocconi University, Milan. He served as a member of the AMASES Scientific Committee (2017–2022), and is a council full member delegate at the European Mathematical Society (EMS) since 2020. He is an editor, co-editor, and associate editor of various international scientific journals, notably: *Nature – Scientific Reports* (member of the editorial committee), *Soft Computing, Dynamic Games and Applications,* and *Journal of Dynamics and Games* (associate editor). He has served as a referee for over 250 international scientific journals on economics, pure and applied mathematics, indexed by SCOPUS, WoS, and MathSciNet. His research interests include mathematical economics, game theory, explainable artificial intelligence (XAI), machine and deep learning modeling, business analytics, optimization theory, nonlinear analysis, dynamical systems, and epidemic modeling.

massimiliano.ferrara@unirc.it

My memories with László Fuchs

Luigi Salce

I will try in these notes to retrace the most important moments I spent with László Fuchs, or at least those which resurface most easily in my memory, and also some moments spent together with his wife Shula and our families.

I met László for the first time in 1972 in Rome, at the Istituto Nazionale di Alta Matematica, where Adalberto Orsatti had organised a conference on Abelian groups. Several mathematicians from the United States attended the conference, including László, who just a few years ago had moved from Budapest to the United States, where he was a professor at Tulane University in New Orleans.

In Rome, Adalberto introduced me to László and asked him if I could go to Tulane for a year to work with him. László's answer was positive. I was a little excited about the idea, having never been to the United States, and a little apprehensive, being still a novice in research, whereas László was already then considered the foremost expert in Abelian group theory, having published the monograph *Abelian Groups* in 1958, and his two volumes *Infinite Abelian Groups* having come out in the early 1970s. Moreover, the twenty-two-year age difference made me see László as a very successful famous mathematician, far above the level of a young researcher at the beginning of his career, as I was then.

In 1972, I was Adalberto Orsatti's algebra assistant in Ferrara. Three years later I took advantage of a 'ternata' in a competition for an assistant professor of analysis post to return to Padua, my home town where I had graduated and where I was to marry Paola in April 1975. I applied for funding from the CNR for a one-year scholarship to spend in the States doing research with Fuchs. The application was accepted, so at the end of October 1975 I left for New Orleans, where Paola, who was then teaching at a middle school, joined me before Christmas. When I arrived at the New Orleans airport, I found László who had come to pick me up and who, fearing I would not recognise him after three years, had a flashy orange Mathematical Reviews issue under his arm.

Of that year spent in New Orleans in the student Rosen House, where we had a small two-room flat as accommodation, and at Gibson Hall, a beautiful neo-Gothic building where Tulane University's mathematics department was located, Paola and I have the vivid memory of a happy time. The majestic oak trees of Audubon Park near the department filled us with admiration, and visits to the French Quarter and Central Grocery selling Italian delicacies brightened our weekends. László taught a course on Abelian groups to PhD students, whose notes I still keep, filling several notepads. In that course I learnt the main results on totally projective groups, which made me passionate about the theory of Abelian p-groups. I then became interested in and studied for many months the existing literature on this theory, hitherto almost completely unknown to me.

One evening we were invited to dinner at László's house on Laudun Street, where we got to know Shula, who was pregnant with Terry. Paola and I certainly did not think on that occasion that we would become close friends with Shula and László, as in fact happened a few years later.

Thanks to László's great willingness to help, we tackled together, with him leading and me following, some problems on Abelian groups, so that by the end of the year spent at Tulane I had three published papers with László as co-author. The last of these three papers concerned *cotorsion groups*, to which I continued to devote myself once I returned to Padua in 1976. So much so that in December 1977, at the second conference organised by Orsatti at the Istituto di Alta Matematica in Rome, I presented the work *Cotorsion theories for Abelian groups*, later published in 1979 in the conference proceedings, which was to be so successful from 2000 onwards after Göbel and Shelah brought it back into the limelight. Naturally, László attended that conference, which was an opportunity for us to meet Rüdiger Göbel, an Essen mathematician with a background in physics, who would become a close friend of both of us, as well as a prestigious algebraist.

Of the years following our return to Padua in 1976 I have many overlapping memories. Among these is the bridge that was created between Padua and New Orleans, since in those years first Claudia Metelli went to work with László at Tulane, in two different one-year terms, joined one year by Silvana Bazzoni, and then Elisabetta Monari Martinez, all professors at Padua. I remember a nice dinner at Claudia's house with László and Shula and Tulane algebraist John Dauns. A second group of memories concerns the countless visits that László made to Padua to work with his Paduan colleagues, often bringing with him his family, which had meanwhile grown due to the arrival first of Terry and then of David. For accommodation, I found flats for rent on various occasions. For those who know a little about Padua, they were first located in Via S. Rosa, then in Via XX Settembre, then in Via Crescini (the house that Shula appreciated more than any other) and many times in La Nave, the university's guest quarters in Portello, a historic Paduan neighbourhood near the mathematics department.

There is an amusing episode concerning the first visit László and Shula made to Padua in 1977, with Terry only a few months old. I had found for them a flat owned by family friends, as I have already mentioned, on Via S. Rosa in the centre of the town. However, I had not thought about Terry, so the first night Shula, not knowing where to put her to sleep, put her in a wardrobe drawer on top of a blanket. The next day I managed to get a camp bed. I also remember that during one of my stays in Padua I took László and Shula for a drive in the Euganean Hills; at one point a group of guinea fowls crossed the road; when Shula asked me what kind of birds they were, I answered that they were *special chickens*, which amused Shula very much, who then always remembered the *special chickens*.

An important moment in our scientific collaboration, which I also recalled recently in the lecture I gave in Budapest in June 2024 on the occasion of the *Big Five Centenary*, which I will discuss later, took place in July 1979. We were with many other mathematicians in Montreal for a summer school organised by the local department of mathematics, which lasted a month and in which László was one of the lecturers. During a picnic on a weekend, László proposed that I begin the systematic study of modules on valuation domains with him, which had been studied up to then by various algebraists, but not in a systematic way. Of this particular episode there remains as a memento a photograph (see Figure 1) that shows, around a picnic table, László with Mario Fiorentini, a recently deceased Roman surveyor who was well known in Italy not only for his mathematical merits, with Adalberto Orsatti and with me.

Figure 1. Picnic with László Fuchs, Mario Fiorentini, Adalberto Orsatti, and Luigi Salce (from right to left).

That was the beginning of a much more intense collaboration than the one we had had until then. We tackled a series of problems together in which László's great experience and my enthusiasm produced blossoming results and work, in which first Paolo Zanardo and then Silvana Bazzoni were involved. So much so that after only five years, we arranged the knowledge based on the previous work of great algebraists, among whom I cannot fail to mention Irving Kaplansky and Robert Warfield, and on the developments we had achieved in five years, in the first of the two books we wrote together, *Modules over Valuation Domains*, published in 1985 with Marcel Dekker, Inc.

I already had the fatigue and satisfaction of having published my own book behind me. In fact, in 1980, the monograph *Struttura dei p-Gruppi Abeliani* had come out in the Quaderni dell'Unione Matematica Italiana, and it was perhaps for this reason that László proposed that we write the book on modules on valuation domains together. The preparation of this first book was useful and valuable, allowing me to learn 'from the inside' the way László organised his books. We would start with the 'skeleton,' i.e., the list of chapters, each divided into its various sections, and for each chapter we would list the articles in the literature that related to the topics to be covered and that we would have to consult or study from scratch. For both this book and the next one, which was much more challenging, we agreed to write half chapters each. However, my English was far from perfect, to put it mildly, so László revised the chapters I had written and put them into good English. Moreover, and not unimportantly, the valuable 'Notes' at the end of each chapter were almost always his work, his algebraic knowledge being far superior to mine, while for the list of open problems at the end of the chapter our contributions were almost equal. These dynamics were repeated when we worked on our second book, *Modules over Non-Noetherian Domains*, published in 2001 by the American Mathematical Society.

I now focus my memories on this second book. I recall that we were in Colorado Springs at the conference organised by Rangaswamy in the summer of 1996, which many of us from Padua attended. One afternoon, after the lectures were over, I found myself with László, who proposed that we write a book together on modules on integral domains, looking almost exclusively at non-Noetherian domains, since existing books on Noetherian domains and their modules numbered in the dozens. I had just come back from work done together with Silvana Bazzoni on modules on particular non-Noetherian domains, which we had called *Warfield domains*, in honour of the late Robert B. Warfield, Jr., while László had long-standing experience on certain classes of non-Noetherian rings, having worked on them since the late 1940s. Given the positive experience of the previous book, I gladly accepted László's proposal, warning him, however, that until the end of that year I was still totally absorbed by my position as faculty dean in Padua. So with the new year 1997 we began to think about the new book, first defining its skeleton, consisting of 16 chapters divided into over a hundred sections, which eventually became 126, and then by drawing up the list of papers to be examined; here the professionalism of László, who regularly updated his *database* of dedicated literature, was decisive.

For three consecutive years, from 1997 to 1999, I spent a week in January in New Orleans as a guest of Shula and László on Laudun Street. We worked many hours a day on the new book, also studying subjects unfamiliar to us. Our laborious and challenging endeavour lasted four years, and the book came out in 2001 with a beautiful dedication written by László for our respective parents. In the preparation of the book, I had an advantage over László, because I could confront myself with Paolo Zanardo and Silvana Bazzoni, who had worked with me for years. Of the two research topics I have studied most in recent years, in fact, the one of finitely generated modules had seen me side by side with Paolo, while the one of non-standard uniserial modules had been investigated with Silvana. Of the latter strand of research, I wrote an article entitled *Fascinating modules over valuation domains*, based on the conference I held in Budapest in June 2024. In that article I told of the time when with László, during a conference I organised in Udine in 1984, we involved Saharon Shelah in studying the problem of the existence of non-standard uniserial modules, which he solved overnight, and how for many years this problem fascinated some of the most brilliant scholars of rings and modules.

Changing genre of memories, some trips made together with Shula and László come to mind. Two trips I remember with particular pleasure: the first to Port Arthur, Tasmania, during the conference in Hobart that followed the Perth conference in Australia organised by Phil Schultz in 1987; the second to Ireland, at the end of the conference organised in Dublin in 1998 by Brendan Goldsmith.

Port Arthur is a former convict settlement on Tasmania's Tasman Peninsula. It is about 97 kilometres south-east of the state capital, Hobart, where the second Australian convention of 1987 took place. Shula and László were in Hobart with Terry (11 years old) and David (9 years old), while Paola and I were with our two children Irene (9 years old) and Iacopo (4 years old) (the third, Giuseppe, was not yet born and would later blame us for not taking him with us to Australia). We had rented two small flats in the same building, so communication between the two families was daily, so much so that we celebrated my daughter Irene's ninth birthday together in our small flat. During the weekend we rented a large family car with a large boot where a couple of children could take turns. We spent a beautiful day visiting the ruins of the prison houses, documenting escape attempts made many years ago by inmates who ended up being eaten by sharks – stories which disturbed and fascinated the little ones.

On the Ireland tour, the Salce line-up included, in addition to Paola and me, Iacopo and Giuseppe, then aged 15 and 9, while the Fuchs line-up included, in addition to Shula and László, our mutual friend Claudia Metelli. We were in two rented cars and toured most of Ireland. I remember that I got a flat tyre on a rough stretch of the road and that László's help in changing the tyre was providential. I have kept a photo taken by Iacopo as we were changing the tyre, László caught bent over in profile and me, leaning forward, mercilessly taken from the back. The black *Cliffs of Mohere*, the emerald expanses of the Irish meadows, the stops at local pubs (see Figure 2) and the tasty salmon dinners (which only Iacopo detested) stand out in my memory.

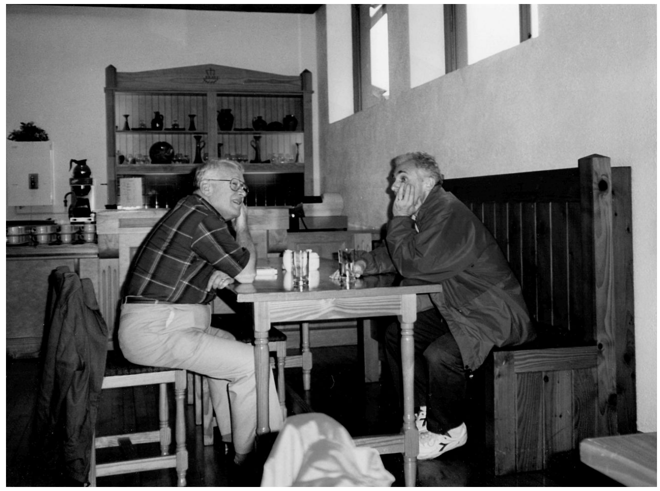

Figure 2. Ireland tour.

And we come to the new century, which began, as already mentioned, with the release of our second book in 2001. From then on, our collaboration slowed down a lot and was episodic, resulting in five works, four to six years apart. One of our works with Jan Trlifaj on *strongly flat* modules came out in 2004 [4], a second with Rüdiger Göbel in 2010 on inverse-direct systems of modules [1], a third with Paolo Zanardo in 2014 on divisibility in cyclically presented modules [5], a fourth with only László and myself as co-authors in 2018 on *almost perfect* rings with zero-divisors [3], and finally our last work on *cellular covers* with Brendan Goldsmith and Lutz Strüngmann in 2024 [2]. Interspersed between these five works, László has published on his own or with other co-authors, after 2001 and to date, another 50 works, many with the Korean Sang Bum Lee, with Rüdiger Göbel, with William Heinzer and Bruce Olberding, and with Kulumani M. Rangaswamy, to name only the most frequent co-authors.

I want to recall how our collaboration came about in the last two works mentioned, the 2018 one on *almost perfect* rings [3] and the 2024 one on *cellular covers* [2].

On the occasion of my retirement, Silvana and Paolo, with the help of Riccardo Colpi and Alberto Tonolo, organised a conference in the department on 30 September 2016, at which László, Alberto Facchini, Brendan Goldsmith, Jan Trlifaj and Peter Vámos gave a lecture. László brought me his latest book *Abelian Groups*, just published by Springer, as a gift. I was admirably astonished at how László had managed, at the age of 90, to produce that comprehensive and weighty book, which updated his two volumes of 1970 and 1973 about 40 years later. The dinner held at the end of that day was attended by many colleagues from the department and many algebraists who had come from various parts of Italy, in addition to the lecturers, and, very welcome, Rangaswamy with his wife Sarah. László, who stayed in Padua with Shula for a few days, came to my studio (see Figure 3) to propose that I help him complete a paper with which he intended to extend the *almost perfect* domains, which I had studied with Silvana in 2002, to rings with zero-divisors. The work was already well advanced, but several finishing touches were needed, which we worked on together. That was the last time we met Shula, who passed away at Easter 2023.

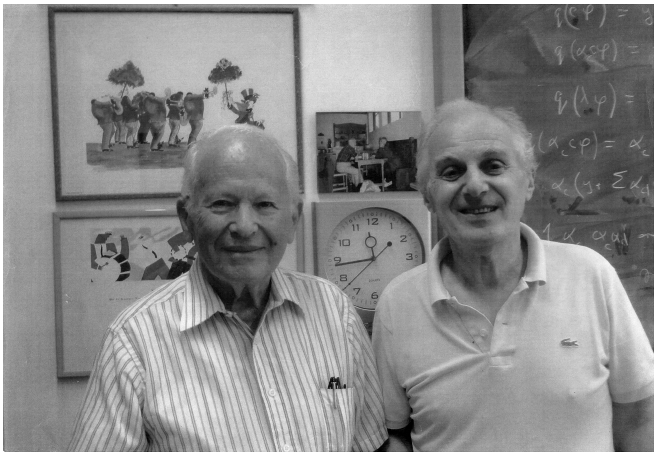

Figure 3. László Fuchs and Luigi Salce in Padova in 2016.

Speaking of celebrations, I am reminded of when two years earlier, in 2014, László turned 90. A gathering of a few colleagues was organised in New Orleans – in non-Tulane premises – with László's family, Shula, Terry and David and their children. Rangaswamy and Bruce Olberding gave two lectures on László's research in commutative algebra, while I spoke about recent results in module theory. At the end, Brendan Goldsmith gave a wonderful talk on the merits and great qualities of László, both as a mathematician and as a friend and point of reference for the large community of algebraists who had worked on Abelian groups and modules. It was an event with few participants, but very intense, which I was also happy to attend with Paola.

The history of the eight-handed work on *cellular covers* is particularly curious. In January 2023, our department organised a conference for the retirement of Alberto Facchini. At the final dinner I was at the table with Paolo and Paola Zanardo, Sylvia and Roger Wiegand, Patrizia Longobardi and Ann and Brendan Goldsmith. Towards the end of the dinner, Brendan came up next to me and suggested that I look at some notes by László, Lutz Strüngmann and him on *cellular covers* for modules on valuation domains. Here is the remark that appears at the end of the introduction to the eight-handed paper published in 2024, *Cellular covers of divisible uniserial modules over valuation domains* [2].

Remark. *This paper has had a long gestation period: it started with an unpublished preliminary note of 2011 in which László Fuchs prompted his co-authors to continue to develop the theory of cellular covers for divisible modules over valuation domains. Unfortunately, the note arrived at a time when Rüdiger Göbel was ill and unable to participate in the project. With his untimely death in July 2014, the note remained untouched until 2019 when two of the current authors* [Brendan and Lutz – ed.] *revived the project and made good progress. They were subsequently joined by Luigi Salce, who succeeded in bringing the project to a partial conclusion. Finally, the ninety-nine-year-old László Fuchs reorganised the material, adding more insights and results, and gave his final touch to the paper. We are delighted to dedicate this work to the memory of our beloved late friend, Rüdiger Göbel, pioneer in our present topic: the algebraic theory of cellular covers, and co-author of many joint papers with each of us.*

The work done with Brendan and Lutz is perhaps not the last of the work with co-authors László and me, as that work left a tail with interesting open issues that we are studying together. We will see what will come out of it.

I go now with my memory to my last meeting with László, which took place, as mentioned earlier, in Budapest on 4–5 June 2024. The occasion was the so-called *Big Five Meeting*, which this special year was called the Big Five Centenary. Traditionally, this event celebrated five great Hungarian mathematicians each ten years, all born in 1924. This year was therefore the centenary, but only László was present, the other four mathematicians having already passed away. As always, the day was organised by the *Hungarian Academy of Sciences* and the *Alfréd Rényi Institute of Mathematics*.[1] During the ceremony, the president of the Academy presented László with the *János Arany Lifetime Achievement Award*, the most prestigious academic honour that the Hungarian Academy of Sciences bestows on Hungarian scientists who do not live in Hungary. During those two days, besides enjoying the excellent Hungarian hospitality, I got to spend some time with László and with Terry and David who had accompanied him from Atlanta. László gave a beautiful thank-you speech at the end of the day, which reminded me of the speeches he had given on several occasions at the conclusion of the many conferences on Abelian groups and modules held over more than 50 years.

[1] https://mta.hu/english/big-five-100-video-of-the-international-conference-celebrating-the-100th-anniversary-of-the-birth-of-five-legendary-mathematicians-113767

To conclude, it seems appropriate to add one last note.

In an article that recently appeared in the EMS Magazine, entitled *Are old mathematicians useless?* [7], Albrecht Pietsch reports the following statements made in G. H. Hardy's famous book *A mathematician's apology* [6].

"I do not know an instance of a major mathematical advance initiated by a man past fifty. (…) A mathematician may still be competent enough at sixty, but it is useless to expect him to have original ideas."

Disagreeing with Hardy, Pietsch comments as follows:

"In the preceding statements Hardy uses two bounds, 'fifty' and 'sixty.' In my opinion, this difference is inessential, since any age limit should be avoided."

In support of his opinion, Albrecht Pietsch brings the example of ten great mathematicians who produced *mathematical advances* and *original ideas* well after *"fifty,"* starting with Henri Poincaré and ending with Yitang Zhang. Of these ten, only Henri Cartan and Eugenio Calabi made it to over one hundred years old and only four were born before 1924. To these examples one can therefore with good reason add László Fuchs.

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Luigi Salce (born in 1946 in Padua) is professor emeritus of algebra of the University of Padua, where he worked since 1984 as full professor of geometry and algebra. He retired in 2016. He is author of more than 100 papers in several areas of algebra, of two books on matrix theory, of a book on Abelian groups and co-author of two monographs on module theory. He gave lectures in several European countries, in the United States, in China, and in Australia.

salce@math.unipd.it

What happens if you don’t want to pay to publish?

Serena Dipierro, Nicola Soave and Enrico Valdinoci

Suppose that you have never paid to publish your papers and you believe that it is wrong to do so.

You may have various reasons to hold this belief. Maybe you think that paying to publish lowers scientific standards and encourages predatory behaviors (of course, paying enough, one can publish whatever). Maybe you think that paying to publish negatively affects other virtuous forms of open access (if paying enough ensures access, why maintaining public repositories and free journals?). Or maybe you are concerned that paying to publish marginalizes researchers who cannot afford to pay, or, even more humiliating, forces them to live out of the charity of the publishers who may discretionally decide to wave the article processing charges in these cases. Or perhaps you gauge other objective facts (such as, an excessive pressure on academics to publish whatever, an overproduction of articles especially in some countries, the forceful pressure made by these countries to place some of their academics in editorial boards of commercial journals, the enormous flow of money that some countries are putting into publishing) and you end up adhering to the conspiracy theory according to which commercial publishers will be free (no matter what restrictions we believe we can impose) to increase the (already gigantic) processing fees as much as they want, because everybody will pay what they ask for, if careers, confirmations, and promotions are at stake.

Maybe you have all this in your mind, but this is not the point: here we do not wish to dive into the (rather obvious) reasons for which paying to publish is incontrovertibly wrong, since others with more experience than us have already thoroughly clarified this, see, e.g., [1, 3].

Here we just want to tell a story. The story of what happens if you do not want to have your article published with a gold-open-access agreement.

That is, suppose that an article gets accepted by a major commercial publisher and that, without you being consulted, without you knowing the costs of it, and without you agreeing, your university stipulated an agreement to let you, and everybody in your institution, publish open-access “for free.”

Given your aversion to the paying-to-publish idea, your consequent action is to ask to opt out of this agreement, believing that this can simply be implemented while completing the online copyright forms.

But things are not so simple. The online form only allows you to choose the gold-open-access option, with the rather direct question “How would you like to pay for open-access?”. The platform does offer you a choice to answer, but the choice is simply between “I’d like to use my institution agreement” or “I have other funds for open access.” At this point, any reasonable person would just go for the first option. If someone else is going to pay, and if not accepting this will produce delay and bureaucracy, then why not, let’s let our institution pay. But sometimes you feel like a romantic weirdo, you think that maybe this is not OK, especially considering that universities often go through financial crises and restructurings, with people struggling to maintain their jobs (or losing their jobs overnight). After all, this open-access feature is not for free: we (our institutions, and therefore ourselves) are handsomely paying for it, rather than investing our resources somewhere else. These are the moments in which you think it is your duty to let your university become aware of its scientific and ethical responsibility. You decide to contact the publishers’ service desk explaining gently, but clearly, that you want to opt out from the gold-open-access option.

Done, problem solved, the service desk kindly provides you with a new link. But the link is still the same, you still have only the gold-open-access option, you can only decide if you want your institution to pay for it or you yourself pay for it in some other ways. This email bouncing goes on for several days, you keep repeating that, as mentioned already on multiple occasions, you don’t want your article to be published via the gold-open-access route, you want to opt out of the gold-open-access option, you won’t pay the article processing fee, you won’t use your research grants to cover it, you don’t want your university to sponsor it (and if all this sounds repetitive to you, believe me, it’s because so is this type of email exchange).

After several days, your request is escalated to a higher level and the publisher wants you to liaise with your institution, which has to approve. Involving your institution may be scary for some people. Suppose that you are under confirmation, or you would like to be promoted, are you sure you want to muck up? Remember all

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the surrounding voices, speaking of task forces and expert panels with fantastic engagement initiatives, for the significant progress towards our strategic objectives. If you need to go for confirmation or promotion, remember what you have been told, just talk to your supervisor about volunteering to serve on a university committee. Don't mess up with decisions that wise people have already taken for you.

But you are a stubborn romantic weirdo. You go on. You contact the People in Charge of this process. You are asked to explain why you want to opt out. Well, you think there are tons of obvious reasons, but still you explain that your papers are already available for free from public repositories, and that you wish that your university spent its resources in more valuable ways, including supporting staff, research, diamond open access, green open access.

The People in Charge are surprised, their understanding is that the publishers provided opt-out options on their platforms (surprisingly, they are not surprised that the way this procedure is implemented makes it so unappealing to be essentially impracticable). You are now being informed that the negotiation with the publishers is not done by the university itself, but by a national council of librarians.

This sounds unexpected, since, given the use of the word "leadership" in your environment, you would have expected your institution playing the role of a leader, negotiating directly, maintaining full responsibility of the process, building virtuous examples for others to follow, without delegating to institutions that one cannot directly influence. But you look at the positive side. The existence of a national institution ideally solves every problem: rather than pursuing pay-to-publish agreements, a national council can invest to create public repositories for disciplines lacking them, and a set of brand-new diamond-open-access journals, led by prominent scholars, with top-class editorial boards.

You receive some information that should make you appreciate the value of the agreement: there is no new money being introduced into the system, it is just repurposing the existing money to facilitate open access (that is, up to yesterday people were complaining about the "exorbitant" fees paid to publishers [2], but now it seems we are relieved that we pay the same fees, for something we can have by other means, in a period in which our resources were so scarce to lose valuable staff).

Then, you are shown why your behavior is detrimental for the open science cause: if you choose not to take advantage of the open-access arrangements, it means that your paper remains behind a paywall and a greater portion of costs are being used to pay for reader access rather than publishing. You don't understand why this is so, since your paper is already available on arXiv and can be freely downloaded by everybody. Also, you start thinking that refusing to publish under this agreement may be the only way to make People in Charge aware of the boundless dangers entailed by these policies.

By the way, you are also notified that the gold-open-access agreement is what the powers-that-be wanted (and a word should be enough for the wise).

Yet, you persist, and try to figure out how much your institution pays for gold/green/diamond-open-access schemes, but this remains a bit foggy, in spite of statistics on library expenditure being publicly available. The lack of transparency in these costs is probably only due to your lack of financial competence; rumors, however, estimate the agreement to be of the order of a dozen million dollars.

You are made aware that many of your colleagues do not self-archive their preprints, yet it seems we don't have any indication about the cause for this.

The publisher then makes another clear point, stating, once again, that, choosing open-access, your article will benefit from greater visibility, which can result in increased readership and citations for your research. The publisher also informs you that if you still don't wish to publish your article open-access, you must request funding from your organization and respond to the email that you have just received by confirming that you don't wish to publish open-access. Your institution will then be informed of your decision, and they can decline your request.

This is not the end of the story yet, but we would spare the reader on how time-consuming it is to go through again a number of internet links that keep you directing to the gold-open-access option, how many emails you still receive promoting the gold-open-access agreement, how worrisome it is to discover that the procedure requires you to accept a (temporary?) agreement that you don't want to agree with, and that a number of bureaucratic operations still have to be performed before ending this tedious story.

In the end, you just hope that someone important enough has now understood how forceful the gold-open-access rhetoric is, appreciating that getting rid of this invasive blueprint is essentially impossible, takes too much time, too much work, and virtuous people who really want to promote free science for everybody happen to be considered as retrograde weirdos.

The agreement certainly stemmed from good intentions, but this story could indicate that universities need to take a step back, overhaul the whole procedure, involve different people, build a different vision and a totally different plan to genuinely deal with open science.

People in Charge should consider that the way this agreement is implemented risks to kill all sorts of virtuous, truly open, behaviors (obviously, if someone pays for our papers to be "openly" published, why bothering with self-archiviation and building free journals).

Our managers should keep in mind that their staff have the scientific stature required for diamond and green open access to prosper. What we need is that our managers stop placing pay-per-publishing models at the same level (or higher) than the brave enterprises for free, and authentically open, science. Rather than sponsoring the commercial gold-open-access journals, you start thinking that your institution should just allocate the same amount of money, rhetoric, and workload:

1. to support the disciplines for which green/diamond open access is a consolidated standard, because they can exemplarily lead more ethical publishing procedures;
2. to "gently" invite people to post their preprints in public repositories: this is important (to have research not hidden by paywalls controlled by commercial corporations) and can be easily implemented, for instance, by linking promotions, awards, and access to funds to preprint archiviation habits, as much as promotions, university awards, and access to funds are presently linked to "students experience" and "industry engagement"; it would also be important to change the rhetoric related to people who don't have the habit of self-archiving their preprint (presently, many consider this as a "different culture," rather than an unacceptable lack of culture which reinforces the dominant position of commercial publishers and, consequently, heavily impacts everybody's resources);
3. to build new, high-quality journals managed by academics, supported by public universities, and completely free for everybody;
4. to use a correct language and appropriate adjectives, distinguishing between the irreconcilable models of publishing after the payment (by individuals, grants, or universities) of publication fees (that is, pay-to-publish or gold open access), the procedure of posting preprints on free public repositories (green open access), and journals completely free for authors and readers (diamond open access) – rather than employing an all-inclusive "open" umbrella;
5. to state clearly that no, it's not OK to pay to publish, and no, not everybody does so.

   But, after all, you are just a stubborn romantic weirdo.

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Serena Dipierro is professor at the University of Western Australia and fellow of the Australian Mathematical Society. She has been awarded the Australian Mathematical Society Medal, the Mahony–Neumann–Room Prize, the Bartolozzi Prize, the Christopher Heyde Medal, the Book Prize of the Italian Mathematical Union, and the Prize of the Bulletin of the Brazilian Mathematical Society.

serena.dipierro@uwa.edu.au

Nicola Soave is professor at the University of Torino. His research interests include partial differential equations and free boundary problems.

nicola.soave@unito.it

Enrico Valdinoci is professor and Australian Laureate Fellow at the University of Western Australia. He has been awarded the James S. W. Wong Prize, the Mahony–Neumann–Room Prize, the Orazio Arena Prize, the Book Prize of the Italian Mathematical Union, the Amerio Prize, and the Prize of the Bulletin of the Brazilian Mathematical Society.

enrico.valdinoci@uwa.edu.au

Veni – research grant for young talent in the Netherlands

EMYA column regularly presented by Vesna Iršič

Solveig van der Vegt and Olivia Muthsam

Amongst the many different lines of research funding that the Dutch Research Council (NWO) offers, the Talent Programme, consisting of the Veni, Vidi and Vici grants, is one of the most prestigious and potentially most impactful for a researcher's career. These personal grants each target researchers at different stages in their academic career, allowing successful applicants to initiate, develop and establish their own innovative line of research and build or expand their laboratory or research group. In this article, we will introduce you further to the Veni grant, which is aimed at early-career researchers who have recently obtained their PhD. We will discuss whom it is for, what it encompasses, and how the application and assessment procedure works. Additionally, two recent Veni laureates in mathematics, Dr. Havva Yoldaş[1] and Dr. Pol van Hoften,[2] share their experiences, tips and tricks for a successful Veni application.

For whom and what for?

The Veni grant is aimed at early-career academics who have shown exceptional academic qualities in their work and offers them the chance to develop themselves as fully independent researchers. The grant is meant for researchers who obtained their PhD no longer than three years prior to the submission deadline and provides successful applicants with the opportunity to spend three years developing their own, independent line of research.

Researchers with and without (permanent) contracts, from within and outside of the Netherlands are encouraged to apply, provided that they have an embedding guarantee from a Dutch university or research institute, i.e., they are guaranteed a place to conduct their project at an institute in the Netherlands if their application is successful.

Applicants apply for a research project that lasts for a maximum of three years in case of full-time research. It is possible to apply for a shorter project, as well as to apply for an extension of at most one year in case of part-time research. The grant primarily covers the salary of the successful applicant, with a little room in the budget as well as for the salary of non-academic staff and materials necessary for the project. It is not allowed to hire academic staff, i.e., a PhD candidate or postdoc, on the budget. The maximum amount that could be applied for in the 2024 round is 320,000 euro, but this maximum may change year to year.

Havva Yoldaş received a Veni grant in 2023. Her project is entitled "Mathematical analysis of metastability in complex biological systems" and started in February 2024. She is currently a tenured assistant professor and a Delft Technology Fellow at the Delft Institute of Applied Mathematics, Delft University of Technology (TU Delft). Her research looks at the metastable behaviour of mathematical models of biological systems. Havva had already successfully applied for an assistant professor position at the TU Delft when she learnt about the Veni grant and decided to apply immediately as it was her last year of eligibility. She feels that taking the time after her PhD to fully develop her research idea was useful, and she would recommend not rushing into applying for this type of grant.

Pol van Hoften received the Veni grant in 2024 for his project "New cases of zeta functions of Shimura varieties." He is currently an assistant professor at the Vrije Universiteit Amsterdam (VU Amsterdam) and is interested in the Langlands programme, in particular in the mod p and p-adic geometry of Shimura varieties. Pol applied after just having moved back to the Netherlands to start a position at the VU Amsterdam. He was still eligible to apply for the Veni, and since grant writing is an important part of being an academic, he started right away with the application process.

Veni (assessment) procedure

An application within the NWO Talent Programme can be submitted within one of the following four domains: Science, Applied and Engineering Sciences (AES), Social Sciences and Humanities (SSH) and Health Research and Development (in Dutch: ZonMW). NWO advises you to consider well in advance which domain

[1] https://www.tudelft.nl/ewi/over-de-faculteit/afdelingen/applied-mathematics/people/dr-h-havva-yoldas
[2] https://polvanhoften2.github.io

is most suitable for your application, as procedures may vary slightly between domains. Most mathematics proposals are submitted to the domain Science. However, if you are in doubt about which domain to apply through, please get in touch with one of the contact persons of the programme as early as possible. They will be able to advise you which domain may best fit your application, but the final choice of the domain is, of course, yours.

Starting in the 2023 round, there is a two-stage assessment throughout the entire Talent Programme. The first phase, the pre-proposal phase, is based primarily on the evaluation of the so-called evidence-based CV, while the second stage focuses on the research proposal.

Pre-proposal phase

Applicants must submit a pre-proposal to be considered for a Veni grant. The pre-proposal includes a narrative description of the applicant's academic profile, up to ten key output items and a short research idea. In the academic profile, applicants can highlight aspects of their career that they consider important, like prior achievements in their scientific work, impact their work has had on their research field or on society, or their research vision, to name just a few options. Applicants are free to shape the narrative in any way that suits their profile. The key output items are a selection of outputs by the applicant which may include scientific publications, but also other works that best show the impact of applicant's work on science and society. The research idea is used to gauge whether the applicant's CV fits the research.

Together with the pre-proposal, it is mandatory to submit an embedding guarantee, signed by the dean. With the embedding guarantee, the university declares that, if granted, the researcher will be given a research position and the opportunity to carry out the research within the institution, using all the facilities necessary for this purpose. The research position can be temporary, but must cover at least the duration of the proposed project. The embedding guarantee will be assessed by NWO and is one of the eligibility criteria. Based on his experience, Pol recommends that securing an embedding guarantee from the university should be the first step you take in the application process.

All eligible pre-proposals are assessed by a committee, and no external referees are involved in this stage. The assessment criteria include aspects like whether the applicant's qualities exceed what is customary in their peer group, the visibility and network of the applicant, their capability for generating innovative ideas, and if the key output items and academic profile align with the proposed research idea. The assessment committee ranks the applications, thereby makes a selection of applicants that can apply in the full-proposal phase, and advices the NWO board, who will then take the final decision. Applicants who receive a negative decision on their pre-proposal may not submit a full proposal.

Havva's first piece of advice to those considering applying: make sure your research idea is fully developed when you apply! Her own proposal was centred around a research idea that she already had and developed further to apply for the Veni. She points out that in the pre-proposal, there is very limited space to describe your research idea, but the time between the decision on the pre-proposal and the deadline for submitting the full proposal is too short to work out the details of the research proposal, as well as write it down clearly. So make sure you already know exactly what your plan is for the full proposal before you submit a pre-proposal.

Another important piece of advice from Pol: before you start writing, find out who your audience is. The committee which assesses the pre-proposal is discipline-specific, while the committee for the full proposal has a much broader composition. Of course, the proposal is also read by expert referees, so one has to strike a careful balance. Additionally, Pol notes, in the pre-proposal phase, don't be too modest!

Havva and Pol both stress that it is important to reach out and find help with preparing your application. Pol reached out to colleagues and reviewed successful proposals from previous years to get a better understanding of what was expected. Additionally, he received a lot of support from colleagues from his group. Their regular feedback was a crucial part of the (pre-)proposal writing process. He also had a number of peers applying to the Veni with whom he regularly discussed the applications.

Havva received significant support during the writing of her proposal from the Innovation & Impact Centre at TU Delft, who helped her improve not only the evidence-based CV part of the application, but also the research proposal itself. She notes that it was useful to have the Innovation & Impact Centre read her proposal, especially because they forced her to think more about the applications of her work. The evidence-based CV is still a fairly new concept in research funding applications and Havva is now paying it forward by sharing her evidence-based CV with colleagues as an example.

Full proposal phase

Applicants receiving a positive decision in the pre-proposal phase may submit a full proposal. The full proposal contains a more extensive description of the research proposal, including the overall aim and key objectives, the research plan, the alignment between the research proposal and the applicant's expertise and the motivation for the choice of host institute. Moreover, the expected impact of the proposed research has to be described. The applicant may choose to focus on achieving scientific impact, societal impact, or a combination of both, as long as they can motivate their choice well. Additionally, a budget plan and a section on data management have to be included in the full proposal, although these two sections are not considered in the committee's assessment.

The full proposal will be assessed on the basis of the following criteria:

1. quality and innovative character of the research proposal (75%),
2. scientific and/or societal impact (25%).

The first criterion includes the assessment of the potential of the proposed research to make an important contribution to the advancement of science, whether the proposed research goes beyond a gradual evolution of the applicant's current research, and whether the proposal aligns with the researcher's expertise. Furthermore, the scientific innovation, the clarity of the proposal and the balance between challenging elements and feasibility are judged.

The second criterion considers the motivation for the type of impact, as well as the means of achieving it. The committee assesses if the proposal includes an appropriate strategy and ambitious vision for the dissemination and/or implementation of results in the own discipline, the broader scientific field, and society if societal impact is considered. Additionally, if the chosen focus is primarily on either scientific or societal impact, it is assessed if proportional attention is given to increasing the opportunities for the other type of impact.

In the full proposal phase for the domains Science and AES, NWO will request input from at least two external referees. These are independent advisers who are experts in the subject of the proposal and will assess the proposal based on the criteria mentioned above. Subsequently, the applicant receives the anonymised referee reports and has the chance to react to the referee's comments in the form of a rebuttal.

The proposal, the referee reports and the rebuttal are then sent to the assessment committee. The committee will make its own assessment based on these documents. Depending on the domain, all applicants who submitted a full proposal or a selection of these applicants will enter the interview stage.

During the interview, the assessment committee has the opportunity to discuss their proposal with the applicant in a Q&A-style session. Following the discussion, the committee draws up a written advice addressed to the relevant decision-making body about the quality and ranking of the proposals. Finally, the relevant decision-making body will assess the procedure followed and the advice from the assessment committee. It will subsequently determine the final qualifications and take a decision about awarding or rejecting the proposals.

Pol admits that the days leading up to the interview were stressful, but that he enjoyed the actual interview. The committee asked interesting questions about the proposal that he liked answering. He prepared for difficult interview questions with professional interview trainers employed by his university, and feels that this was incredibly helpful. He recommends contacting them well in advance and to plan multiple sessions with them. Additionally, it is useful to ask experienced colleagues what kind of questions to expect.

Preparation with colleagues and the Innovation & Impact Centre was also essential for Havva as she worked towards her interview. She stresses the importance of doing mock interviews, especially as the interview panels tend to be broad and multidisciplinary, so you may get unexpected questions. Furthermore, training in scientific storytelling helped her prepare the five-minute presentation with which the interview for the Veni grant starts. Havva now enjoys sitting on the mock panels to help current Veni candidates prepare for their interviews. She hopes this will help them come out of the interview the same way she did: feeling like she had done everything she could, and that whatever the outcome was, she was happy with her performance.

Even after you obtained the grant and are preparing to start the project, there is a lot to learn, says Havva. It takes some know-how to obtain all the right signatures, especially if you have only just joined a new university. Havva ended up delaying the start of the Veni project, so she could finish up ongoing research and fulfil her teaching duties, but is now in full swing.

Pol concludes that he really likes the fact that full proposals are only requested from a small number of applicants because it feels like your full proposal will be taken more seriously this way. He learnt a lot from writing the full proposal, especially about how to craft a narrative that contains almost no jargon but still explains something about your research. He started his project in September 2024.

Information about the Veni programme can be found on the website of the NWO Talent Programme.[3]

Please note that depending on the domain you plan to submit your application, there may be minor deviations from the procedure (e.g., disciplinary or domain-wide/broad committees in the pre-proposal phase, external referees in the full proposal phase or interview selection). Please read the call for proposals and the information provided on the NWO website carefully before submitting a Veni proposal and contact an NWO staff member if you have any questions.

Solveig van der Vegt and Olivia Muthsam are both programme officers on the Mathematics and Computer Science Team at NWO, the Dutch Research Council.

s.vandervegt@nwo.nl, o.muthsam@nwo.nl

[3] https://www.nwo.nl/en/calls/nwo-talent-programme

Thematic Working Group on Mathematics Teacher Knowledge, Beliefs and Identity, TWG20

ERME column regularly presented by Frode Rønning and Andreas Stylianides

In this issue presented by the group leaders Francesca Martignone, Miguel Montes, Miguel Ribeiro, Federica Ferretti, Veronika Hubeňáková, Nadia Kennedy and Jimmy Karlsson

CERME Thematic Working Groups

We continue the initiative of introducing the CERME Thematic Working Groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for people working in pure and applied mathematics. Our aim is to disseminate developments in mathematics education research discussed at CERMEs and enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

In this issue, we introduce the CERME Thematic Working Group on Mathematics Teacher Knowledge, Beliefs and Identity, TWG20.

Introduction

The purpose of this report is to briefly present CERME's Thematic Working Group (TWG) 20 on Mathematics Teacher Knowledge, Beliefs and Identity. At CERME13 (Hungary, 2023) this TWG was led by the authors, inheriting the legacy of more than thirty scholars in the field who have led the group since it started in 2013. We start by briefly outlining the history of CERME TWG20, then present the main topics and ideas addressed in the Group and conclude with some reflections on the future of the Group drawing on the synthesis that has emerged from previous conferences.

TWG20 history

From 1998 to the present, each CERME conference has included one or more TWGs that deal with teacher education and practice. A group at CERME1, named "From a Study of Teaching Practices to Issues in Teacher Education," focused on a large span of issues, from teacher practices to teacher education. Over the years there has been an increasing interest in the latter, leading to a sizable increase in the number of researchers attending this TWG. Due to this interest and considering the different foci of research within teacher education, three new groups have been constituted since CERME9 (2013), building on the original TWG: TWG18 "Mathematics teacher education and professional development"; TWG19 "Mathematics teacher and classroom practices"; and TWG20 "Mathematics teacher knowledge, beliefs, and identity." TWG20 is one of the most popular in CERME: since CERME9, 142 research papers have been presented by members of this group, including researchers from Europe and all other continents; at CERME13 for example, the TWG20 participants came from 16 different countries.

TWG20 topics

Teacher knowledge, beliefs and identity, their relationships, and their impact on professional practice and teacher education are the topics of attention by the TWG20 group. Since Shulman's work [8] and up to more recent research in mathematics education (e.g., [1, 2]), two major domains of content knowledge have been identified and studied: mathematical knowledge and pedagogical content knowledge. Professional knowledge, both mathematical and pedagogical, can have a powerful impact on the quality of mathematics instruction [3]. A key role is also provided by teachers' beliefs about teaching and learning mathematics [6]. In recent decades, there has been a growing interest in mathematics teachers' beliefs [5] and their relationship to teachers' knowledge. These issues are linked to studies on teacher identity. The latter reflects the way in which the teacher feels part of and in possession of agency within the community of teachers, as well as equipped to contribute to its development [9]. Many studies in mathematics education have adopted the conception of identity as based on the narratives that teachers "tell" about themselves as professionals: narratives provide a practical means to make identity accessible and investigable [4, 7].

Over the years, TWG20 participants have discussed and debated various issues related to teacher knowledge, beliefs and identity, ranging from theoretical to methodological perspectives and approaches, while keeping in mind the role and impact of such research on mathematics teacher education, with the goal of improving students' mathematical thinking. While at the beginning of the TWG the focus was mainly on models concerning teacher knowledge, recent CERMEs have seen an increase in contributions concerning the role and impact of such knowledge on mathematics teachers' beliefs and their effect on practice. As such, the relationship between teachers' knowledge, beliefs, and identity is increasingly considered.

TWG20 future

Over the last years, TWG20 discussions have focused mainly on two ideas. First, there is a concern with how context, and therefore, cultural issues, impact on teachers' knowledge, beliefs and identity. Second, the discussions have revolved around the fact that research often implicitly suggests that teachers should have strong specialized knowledge (mathematical and pedagogical), with a coherent belief system, based on a well-developed professional identity. As a consequence, discussions have moved to an inquiry into how much one should and can demand of both pre-service and in-service teachers in order for them to develop the skills and dispositions of teaching for mathematical understanding. These two core ideas have led, in the medium term, to a discussion about what standards are necessary to ensure the mathematical quality of the instruction that teachers provide. To meet this challenge, research focusing on tasks for teacher education is one of the cornerstones of the group's developing agenda, as it provides an opportunity for participants to discuss reachable teacher standards (both in pre-service and in-service teacher education). Also, research on tasks is useful in identifying issues that should be addressed in order to further deepen and clarify the field's understanding of the role of mathematics teachers' knowledge, beliefs, and professional identity for effective mathematics teaching. A topic that has not yet been explored in depth, but which could be the subject of future discussions within the group, is the knowledge and beliefs of mathematics teachers in tertiary education. The research shared in TWG20 on tertiary education has focused mainly on university teachers involved in courses for future mathematics teachers, but, given the nature of mathematics taught in tertiary courses, such as engineering and other science programs, it could be an interesting and fruitful field of study.

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Francesca Martignone is an associate professor of mathematics education at the University of Eastern Piedmont. Her research interests include teacher knowledge, teacher education, and assessment. She was the former leader of CERME TWG20.

francesca.martignone@uniupo.it

Miguel Montes is a senior lecturer of mathematics education at the University of Huelva, focusing on teacher education and problem posing and solving. He currently leads CERME TWG20.

miguel.montes@ddcc.uhu.es

Miguel Ribeiro is an associate professor of mathematics education at the State University of Campinas. He researches teachers' knowledge, professional development, and education tasks. He was a leader of CERME TWG20.

cmribas78@gmail.com

Federica Ferretti is an associate professor of mathematics education at the University of Ferrara. She focuses on assessment, large-scale assessments, and teacher development. She coordinates the Ferrara University Center for Teacher Education.

federica.ferretti@unife.it

Veronika Hubeňáková is an assistant professor at Pavol Jozef Šafárik University in Košice, Slovakia. Her research includes teacher knowledge and future teacher preparation. She co-leads the Slovak National Center for Digital Transformation of Education and also CERME TWG20.

veronika.hubenakova@upjs.sk

Nadia Stoyanova Kennedy is a professor of mathematics education at the New York City College of Technology of the City University of New York (CUNY). Her research centers on the philosophy of mathematics

education, dialogic teaching, and teacher professional identity, with a focus on critical approaches. She is a former co-leader of CERME TWG20.

nkennedy@citytech.cuny.edu

Jimmy Karlsson is a PhD candidate in mathematics education at Karlstad University. He studies students' mathematical knowledge development, focusing on cognitive activation and classroom practices.

jimmy.karlsson@kau.se

Report of the ICME15 survey on challenges and perspectives of mathematics assessment

ICMI column in this issue presented by Kaye Stacey

This article provides a short summary of the work of one of five survey teams commissioned by the organisers of ICME15. The findings were presented at the congress held in July 2024 in Sydney, Australia. Survey Team 1 worked on "Challenges and perspectives of mathematics assessment." The team leader was Kaye Stacey, and the members were Yuriko Yamamoto Baldin (Universidade Federal de São Carlos, Brazil), Kim Koh (University of Calgary, Canada), Ruhama Even (The Weizmann Institute of Science, Israel) and Ross Turner (The Australian Council for Educational Research, Australia).

Assessment has very many forms, with many players involved, and it is conducted for multiple purposes, so this survey could consider only a small part. The chosen focus is the teaching-learning-assessment cycle as handled by classroom teachers at any level of mathematics. The assessment is of mathematical knowledge and the intended audience for the report from the assessment is the teacher and/or the learner with the purpose of informing future teaching or learning. To distinguish assessment from the monitoring of students' understanding that teachers carry out continuously, we require some formality to the process (e.g., by recording results). Summative assessment for credentials or for evaluation of education systems or to provide information for next year's teacher is beyond the scope of this survey.

The survey consisted of five separate components. We begin with formative assessment (FA) which sits right at the heart of classroom teaching. Ruhama Even presented results from a systematic survey (following PRISMA guidelines) of literature published from 2017 to 2023 on teachers' use of assessment for informing short-term instructional decisions. The literature search focused on two lines of inquiry with which this topic is associated in the mathematics education literature. One is formative assessment (FA), which involves three key teaching competencies: (1) eliciting evidence of student learning, (2) interpreting evidence of student learning, and (3) responding or acting on evidence of student learning. The other is the extended approach to teacher noticing (TN), which involves three similar competencies: (a) attending to the details in children's strategies, (b) interpreting children's understandings, and (c) deciding how to respond (or actually responding). Of an initial 2990 papers, 105 papers from 10 top-tier mathematics education sources were found to meet the criteria. About a quarter of them involved international collaboration. Although the similarity of FA and TN competencies suggests a close connection, the review reveals a surprising disconnection. Authors who write about FA rarely link their work to TN and vice versa. The number of papers on TN increased steadily over the period, clearly outnumbering FA. Additionally, the survey identified an important absence: teachers' responses and acting on evidence of students' learning are rarely studied in real classrooms.

Ross Turner provided a sweeping survey of how learning progressions (LPs) are used in assessment. He exemplified this by sharing his experience in working on the mathematics LP developed by the Australian Council of Educational Research (ACER).[1] LPs have attracted considerable recent attention among education researchers, practitioners and policymakers. Research on LPs forms part of a long tradition of scholarship to describe trajectories for teaching and learning mathematics, especially the order in which concepts and skills are learned and mastered as well as common states of partial knowledge along the way. Research also explores how such descriptions of learning can guide and inform education. LPs vary in their scope, from a focus on a single concept or skill, through to attempts to describe learning across the whole domain of mathematics. There are different approaches to their development. For example, different ways of using learner data and expert input and different measurement models are used. There are also different uses intended, from tools to support formative assessment to descriptions to guide curriculum development.

The work at ACER is grounded in measurement principles and leverages extensive experience with described reporting scales for assessments. The ACER mathematics LP describes the key milestones in mathematical learning from the early years through to the end of compulsory schooling. It is designed to assist with curriculum design and syllabus content (including internationally in relation to the United Nations Education 2030 agenda); to help identify the current knowledge of individual learners; to describe and report learning progress of individuals; to guide teachers in the selection or development of suitable teaching and learning resources; and also to support teacher professional learning. Action

[1] https://learning-progression-explorer.acer.org

research is continuing on these fronts. The long-term challenge is for LPs to offer support to teachers to implement their craft in such a way that better treats individual learners at their point of need, rather than as members of a group who 'should' be at a certain point in their learning.

Kim Koh presented the findings of a systematic literature review on assessment of 21st century competencies for mathematics and higher-level mathematics thinking skills and competencies (together labelled MTSC). Curricula from around the world now highlight the importance of supporting students to develop a deeper mathematical understanding with competencies such as critical thinking, creativity, collaboration, and communication, and to help them become competent, confident, and creative users and communicators of mathematics. Teachers are urged to adopt assessments that provide authentic experiences for all learners. Inquiry-based learning and modelling approaches are also deemed to be affording students with the opportunity to develop competencies. Yet, questions remain about how students' competencies are assessed and what form assessments can take.

The review included empirical studies published from 2018 to 2023 focusing on assessment of MTSC for K–12 students and preservice teachers. From an initial 2489 papers, 87 were found to meet the criteria. The review identified the skills and competencies that were assessed, and the tools and strategies used. Approximately half the studies were published in research journals specifically focused on mathematics education. Quantitative, qualitative and mixed methods were well represented. More than half involved students in grades four to nine and more than half involved more than one content area. A third of articles published after the pandemic addressed online assessment. A first outcome of the study was to clarify the construct of MTSC, observing that its components should neither be treated as atomized competencies nor a 'laundry list,' and that there are both cognitive and affective competencies. Since 72% of studies focused on cognitive skills, Koh calls for more research into innovative assessment of students' socio-emotional skills and motivation, considering them simultaneously in classroom-based formative and authentic assessments.

Yuriko Yamamoto Baldin reported on a recent online survey on assessment in teacher education. There were 38 respondents from seven Latin-American countries (Brazil, Ecuador, Mexico, Venezuela, Costa Rica, Peru and Columbia) and from Portugal and Spain. Nearly all the respondents work in initial or continuing teacher education and most undertake mathematics education research. Designing and using assessment that attends to 21st century demands is especially significant to several countries in the region with recent curricular reforms and where COVID-19 changed teaching and learning dynamics. The survey examined changes in national assessment policy, how assessment (especially FA) is treated in teacher education, and guidance on assessment for teachers using active learning, problem-solving, or modelling methodologies. The concerns that most respondents shared will be used as the basis for further study.

Most respondents noted the lack of cohesion between curriculum demands and the reality of classroom practices. They urged better preparation for teachers to understand practices for FA, and to plan assessment instruments. They especially noted the need to link FA to the active teaching and learning methodologies essential for students to achieve 21st century competencies. Respondents generally believed that it was not productive to use external large-scale assessments with uniform ranking criteria that do not take local cultural, political and educational contexts into account. These put pressure on national education policies, often without much analysis of the roots of the difficulties.

Finally, Kaye Stacey provided a brief overview of some of the major changes to assessment within the teaching-learning cycle that are already being implemented using new technologies and some that seem just around the corner. The changed conditions of school and university education imposed by the pandemic greatly accelerated these changes. The work of Michael Obiero Oyengo (Maseno University, Kenya) was one case presented. By using a computer-algebra based assessment tool, he is providing students in extremely large classes with regular and immediate mathematically-detailed feedback on their own work and as many opportunities for practice as they wish. Most students access this with mobile phones. Such formative assessment has never previously been possible and may change the mathematics learning experience of millions of students around the world. Further details relating to all these contributions are available from the author and survey team members.

Emeritus Professor Kaye Stacey was the leader of the five member ICME15 survey team on assessment. In addition to assessment, her interests centre around mathematical thinking and problem-solving, the school curriculum and new technologies. She is author of many articles and books in mathematics education, reporting research and for teacher professional development. She was Foundation Professor of Mathematics Education at the University of Melbourne and holds several awards including ICMI's Emma Castelnuovo Medal.

k.stacey@unimelb.edu.au

Open access status and licences in zbMATH Open

Dariush Ehsani and Olaf Teschke

Since December 2024, zbMATH Open displays open access and licence information not just on a journal level but also for individual articles. We give some examples how the feature can be employed in an analysis of the publication landscape in mathematics.

1 Introduction

For several years, zbMATH has included information about open access policies in its journals database[1]. However, the number of journals with a mixed situation has grown recently, due to the increased presence of hybrid access (e.g., due to APC[2] options, transformative agreements, or as a result of subscribe-to-open policies). Moreover, also for non-OA publications an open version may exist (e.g., on the arXiv). Therefore, it is beneficial to have OA information also available at the document level, not just as journal policies.

Moreover, it becomes increasingly important that publications are not just openly accessible but come along with an open licence such as CC-BY[3]. The licence situation is also quite diverse – e.g., older arXiv submissions generally come just with a non-exclusive distribution licence, while broader CC licences are more frequently available for recent documents. Hence, a lot of effort has been made to collect both OA and licence data in zbMATH Open, enabling document level information.

2 OA and licence information in zbMATH Open

Open access information at zbMATH is collected from a large variety of sources. For journals with a homogeneous OA policy, the information is derived from the zbMATH Open journal database[4]. For hybrid journals, access and licence information is often provided by the publishers. Additional information is gathered from sources like OpenAlex, Unpaywall, the arXiv, or HAL. Especially, the integration of the arXiv (both by the matching of published documents to their arXiv versions, and the inclusion of unpublished maths preprints, see [1]) adds a significant number of open items.

The results are retrievable using the "open access" search and filter option in zbMATH Open. Additionally, an open licence information is displayed if available. This will look as follows:

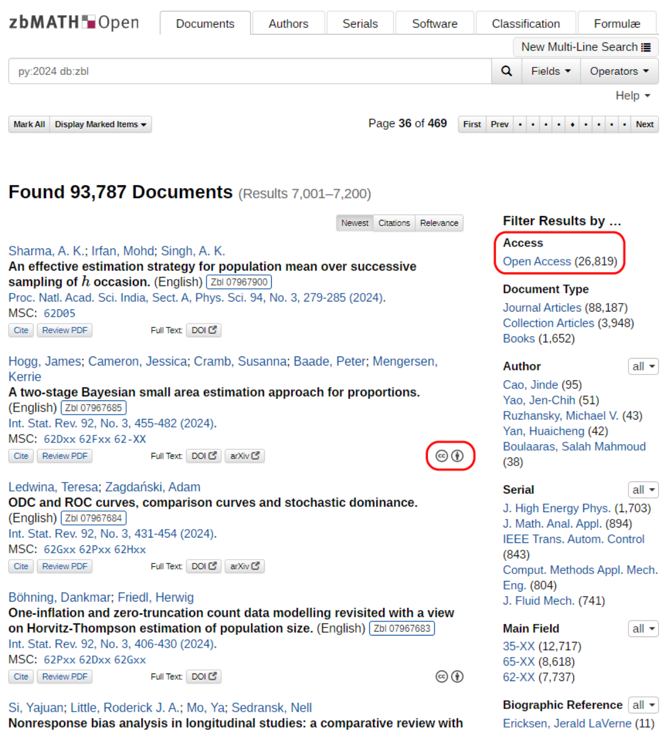

Figure 1. Open access filters and licence information in zbMATH Open.

[1] Currently, https://zbmath.org/serials/?q=st:o lists more than 1000 mathematics journals which employ various general OA policies.

[2] Article processing charge.

[3] Creative Commons Attribution 4.0 International Licence, https://creativecommons.org/licenses/by/4.0/.

[4] For some OA journals without stable document identifiers, it may even happen that there is no direct full-text link to an open full-text article, but one needs to go via the journal homepage.

Moreover, the data are also available via the zbMATH Open API [5–7]. In the next section, we give an application to use them for an update of our investigation [2] on the progress of open access in mathematics.

3 Share of open available mathematics

In [2], we gave a detailed overview of the (then) main forms of open access in mathematics. The most significant developments during the past five years pertain to hybrid open access (propelled also by transformative publish-and-read agreements) and S2O[5] policies. The following diagram displays the share of OA versions of articles indexed in zbMATH Open published in a given year since 1931, for genuine OA documents, existing free versions at the arXiv, and documents freely available by both options.[6]

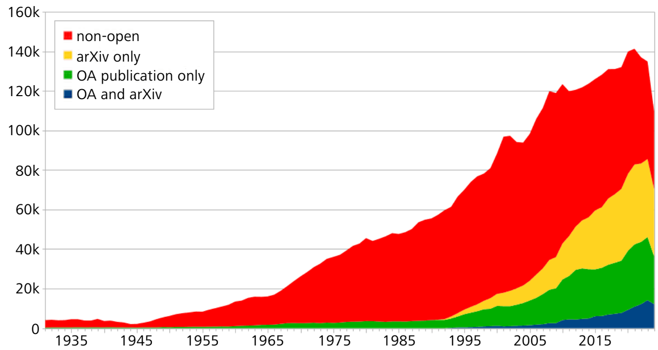

Figure 2. Share of open versions for published documents in the zbMATH Open corpus.

The first observation is that only quite recently (in fact, since 2021) the share of free documents surpassed the 50% mark. While this had been achieved in several core areas of mathematics by arXiv coverage alone for some years (see [3, 4]), this figure indicates that OA policies have been adopted also by mathematical fields that were less present at the arXiv before.

The second observation is that the growing share of open documents is mostly driven by available arXiv versions. In fact, both the share of articles solely available free on the arXiv, and of genuine OA publications with an arXiv version, has increased – but the second one much more significantly: while the share of genuine OA publications with an arXiv version was hovering around 5% ten years ago, it has been tripled since then. On the other hand, the share of genuine OA publications *not* available on arXiv has actually slightly decreased. Initially, that seems quite surprising – does this mean that the implementation of policies like transformative agreements and S2O has actually been offset by an increased engagement of arXiv submissions, to an extent that they did not achieve much to increase the overall OA share?

In fact, zbMATH allows for a closer look via its journal filters and data to explore the stagnating figures of sole OA publications. The journal distribution of this kind of publication has changed significantly: Around 2013, when the share of OA publications not on arXiv peaked at 18%, it was dominated by the relatively new journals with an APC policy ("Gold Open Access"). However, in the course of the next years, in a number of these journals a significant content degradation became visible, perhaps also triggered by the effect that APCs create incentives for quantitative publishing (see [8] for a detailed discussion). This resulted in discontinued indexing of several APC journals in zbMATH Open (in fact, some of these were delisted later also from other services amid concerns of problematic behaviour[7]).

On the other hand, recently, genuine OA articles not at the arXiv appear much more frequently in hybrid journals, often subject to transformative agreements of funding requirements. So indeed there has been an effect of the OA policies, mostly in areas where the usage of the arXiv is not yet widespread. The following diagram indicates the distribution of the four segments with respect to the Mathematical Subject Classification.

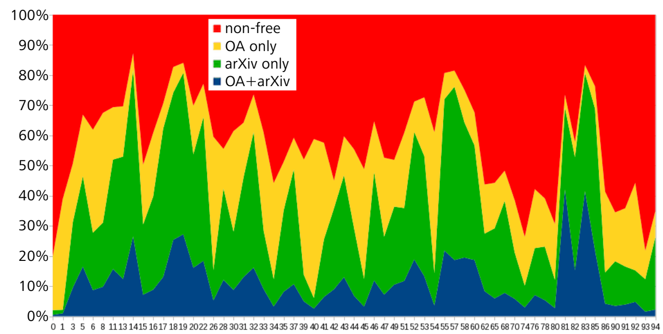

Figure 3. Share of open versions for published documents for main MSC classes.

Overall, this diagram mainly reflects the different degree of arXiv usage of the different communities. While arXiv submissions are standard in most fields, its usage is less widespread for applications of mathematics (except for mathematical physics) and numerical analysis. Moreover, a significant fraction of publications is made open through S2O, transformative agreements, or moving walls. On the other hand, as discussed in [8], one observes still a prevalence

[5] Subscribe to Open

[6] Data are available through the zbMATH Open API https://api.zbmath.org/.

[7] See, e.g., https://retractionwatch.com/2023/03/21/nearly-20-hindawi-journals-delisted-from-leading-index-amid-concerns-of-papermill-activity/.

of non-arXiv OA in a few areas where communities with a more quantitative publication attitude exist (e.g., fractional analysis in 26, 34, 35, fixed point theorems in 47, 54, fuzzy structures in 06, 08, 26, 54, or classes of functions or sequence spaces in 30, 33, 40). The prevalence of Gold OA in only some very specific fields may raise further questions about whether the Gold Open Access approach (either through APCs, or transformative agreements) leads to a desirable allocation of resources.

4 Open licences

Free access to a publication has been the most pressing concern in the past, but the issue of open licences has become increasingly important as well. This is especially true since data have become highly valuable, and the FAIR principles (findability, accessibility, interoperability, reusability) became relevant also for mathematics content. For instance, with the advent of generative AI, there is already a clear tendency that paywalled scientific content is used to create commercial oligopolies of some large AI firms. While open licences may hardly prevent the usage of mathematics content in generative AI services (even though CC-BY-ND or CC-BY-SA licences might not be compatible with how most AI services are currently trained and distributed, experience indicates that the legal requirements are ignored by large companies anyway), they offer at least the option to create smaller, community-driven non-commercial services which might be more tailored to specific needs.

Naturally, these questions have only become important relatively recently, so it is not surprising that only a fraction of the free documents in zbMATH Open come also along with a FAIR licence (e.g., most historical arXiv submissions come with a non-exclusive distribution licences, although also CC licences are now available for some years). Figure 4 shows the current share of various CC licence information collected for documents in zbMATH Open from various sources (publishers, platforms, OpenAlex…), but are currently available for only about 5% of the entries.

5 Conclusion

The entire mathematics community can benefit from the availability of OA documents. zbMATH Open will contribute to the process by collecting information regarding the open access policies of journals as well as individual documents. The mathematics community can contribute by making available OA preprints via a preprint server whenever possible, and choosing an appropriate non-restrictive open access licence (e.g., CC-BY). The granular information provided by zbMATH Open on OA policies and licences, both through its web interface and APIs, offers options for a detailed analysis of the availability of resources, and the evaluation of the different OA strategies.

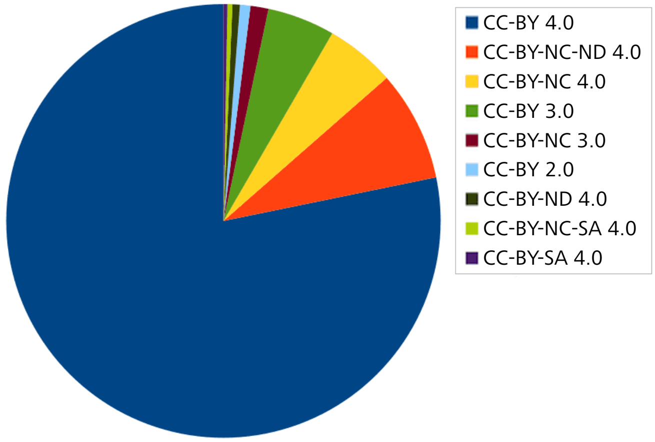

Figure 4. Distribution of known open licences in the zbMATH Open corpus.

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Dariush Ehsani has worked as a mathematician at various universities within the United States and Germany, specializing in the field of several complex variables. In 2020, he joined FIZ Karlsruhe to facilitate the transition of zbMATH to an open access platform, with a special focus on the integration of digital mathematics libraries.

dariush.ehsani@fiz-karlsruhe.de

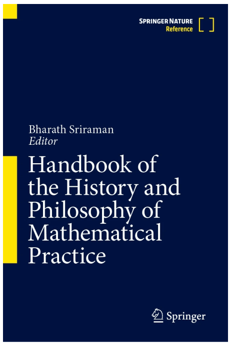
Olaf Teschke studied mathematics at Humboldt University Berlin, and completed his PhD in algebraic geometry there. He moved to FIZ Karlsruhe in 2008, and has been working there since 2009 as the head of the Mathematics Department and managing editor of zbMATH (including a short intermediate term as editor-in-chief). Since 2017, he has been serving as the vice chair and chair of the EMS Committee on Publications and Electronic Dissemination. His main occupation is information infrastructure for mathematics.

olaf.teschke@fiz-karlsruhe.de

Book review

Handbook of the History and Philosophy of Mathematical Practice by Bharath Sriraman (editor)

Reviewed by Nicolas Michel



From the late 1970s to the late 1980s, a motley crew of mathematicians, logicians, and computer scientists expressed frustration towards what had become the dominant trend in the philosophy of mathematics. "Traditional philosophical formulations," Thomas Tymoczko wrote in the preface to an influential anthology published in 1986 [6, p. ix], fail "to articulate the actual experience of mathematicians." An exclusive emphasis on metaphysical, logical, and foundational problems, disconnected from the everyday labor and efforts of mathematicians, had allegedly made much of the recent philosophical production at best an uninteresting and self-centered discourse, at worst an artificial construct with no substantial ties to its purported subject-matter. Although widely diverging in the solutions they brought to this shared diagnosis, philosophical essays such as Imre Lakatos' *Proofs and Refutations* (1976) [4] and Philip Kitcher's *The Nature of Mathematical Knowledge* (1984) [3] sought to root their epistemologies in historical studies, and thereby to explore questions a formal presentation of mathematical theories could not even begin to open, for instance the fallibility and empirical character of proofs or the role of authority and community in the collective shaping of mathematical knowledge.

By the early 2000s, these counter-currents – often referred to as the "maverick tradition"[1] – were progressively assimilated into a shared project by a range of mainstream historians and philosophers of mathematics, who wished to take up Tymoczko's challenge all the while softening his wholesale rejection of logical investigations and general analytical philosophy as useful tools for reflecting on mathematical practice. Consensus-building efforts by Paolo Mancosu, José Ferreirós, Jeremy Gray, and many others slowly enabled the formation of a self-identified collective of scholars interested in what is now commonly called "the philosophy of mathematical practice," for which an association was eventually created in 2008.[2]

Undeniably, the international and interdisciplinary community rallying under this banner nowadays has grown to a consequent size, and it gathers a wide breadth of research interests and methodologies. The research it carries out includes dense micro-historical studies based on the dissection of ancient manuscripts as well as empirical surveys of pedagogical practices; conceptual analyses of themes such as objectivity and pluralism in mathematics as well as sweeping proposals for bringing cognitive sciences into the philosophical arena. What emerged as a counter-reaction to overly systematic and formal views of mathematics is now a multipronged enterprise to study the latter science from all possible angles, so long as it remains rooted in actual practices–be they historically documented, experimentally tested, gathered with the help of sociological surveys, etc. Yet, in many ways, the challenge is now to take stock of the common results achieved by these wildly different methods, and to formulate anew the identity and value of this research programme – if one may call it thus.

Perhaps such was the intention underlying the *Handbook of the History and Philosophy of Mathematical Practice* presently under review. Gathering an astonishing 114 chapters distributed across some 3200 pages and 14 sections (each corresponding to a general philosophical theme, e.g., ontology, proof, signs, etc.), this volume perfectly displays the aforementioned breadth of approaches that has come to characterize the philosophy of mathematical

[1] This label originates from *History and Philosophy of Modern Mathematics*(1988) [1, p. 17]; since then, it has taken on a broader and looser meaning.

[2] See https://www.philmathpractice.org. Two collective volumes which played a structuring role in the emergence of this community are *The Architecture of Modern Mathematics* [2], and *The Philosophy of Mathematical Practice* [5]. The introduction to both of these volumes provides a much more precise genealogy for the tradition collectively labeled "philosophy of mathematical practice" than the space devoted to this review could allow for.

practice. Some of those chapters are written by renowned experts of their respective fields, yet the balance between junior and senior scholars amongst the contributors is to be applauded. At its best, the *Handbook* thus provides illuminating and stimulating entry points into key themes of the philosophy of mathematical practice, with such leading scholars making the effort of leveraging their expertise towards general yet precise expositions.[3] Many other chapters, however, are much closer in form and content to specialized research articles, so that their role in a "handbook" is rather questionable: what they bring to an audience looking to familiarize itself with the general philosophy of mathematical practice remains wholly unclear.[4] More troubling still, several chapters have seemingly little to do with mathematical practice: three consecutive chapters, spanning some 170 pages, thus yield erudite analyses of Plato's philosophy of mathematics that could form the basis of a stand-alone monograph, but make no effort whatsoever to connect with the common goals and projects of a philosophy of mathematical practice, let alone to be in the spirit of a handbook.

It remains somewhat unclear what the overarching purpose and target audience of this book is, something which its very short general introduction does not fully address. It must be noted here that some sections have been more tightly edited than others, and therefore near much closer to what could be expected from a handbook than others – notably, the sections on proof, ontology, semiology, and experimental mathematics are to be commended in this regard.

In the first of these three sections, for instance, mathematical proofs are successively approached from a variety of angles: as epistemic devices that can include visual and diagrammatic elements on top or instead of formal ones (Giardino's chapter), as social goods whose validity ought to be collectively negotiated and assessed (Andersen's chapter), and as historiographical categories that played a pivotal role in the shaping of Eurocentric and exclusionary narratives (Chemla's chapter). Read collectively, these three chapters convincingly place mathematical proofs – a classical locus of the epistemology of mathematics – at the nexus of cognitive, social, and cultural processes, thereby building towards a richer picture of proving practices throughout history all the while maintaining profound connections with key philosophical questions.

[3] To cite but a few, see the chapters by Carter, Ferreirós, and Wagner, each of which could be used in graduate teaching or as a primer in (respectively) experimental mathematics, philosophical outlooks on practices, and mathematical semiology.

[4] Such is the case, for instance, of A. Papadopoulos' obituary for Yuri Ivanovich Manin (pp. 13–34); a moving and interesting tribute which nonetheless seems rather out of place here.

For all this stimulating material, however, there are as many poorly-edited chapters or contributions which fail to uphold historiographical and philosophical standards of rigor and precision, making this volume an extremely unequal sum which cannot reasonably be recommended to newcomers to the field – especially those looking for a concise introduction to this scholarship. It is a volume from which select chapters and sections can be plucked out and used as valuable surveys of state-of-the-art research questions, the sort of things expected from a handbook, but that selection itself requires familiarity with the field and a great deal of effort to wade through thousands of pages. And so, the search for a synoptic and accessible presentation of the exciting perspectives opened by the philosophy of mathematical practice must continue.

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Nicolas Michel is a Simons Postdoctoral Fellow with the Isaac Newton Institute at the University of Cambridge. His research concerns the history and philosophy of mathematics in the 19th and 20th centuries. His work has been published with journals such as *Isis*, *The British Journal for History of Science*, *Science in Context*, and the *Revue d'Histoire des Mathématiques*.

nprm3@cam.ac.uk

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Authors’ right to know the real status of their submissions

Mox Sal Moslehian

Some research journals utilize online systems for scientists to submit articles. Unfortunately, once an article is submitted to some of these journals, the status of the article is shown as “under review.” This can be frustrating for authors, who have the right to know the real status of their papers after spending months conducting research and submitting their results in the form of research articles to journals.

Sometimes editors are very slow to act, either by sending articles to reviewers (referees) late or by ignoring articles that have not been agreed upon for review. In the latter case, some editors even fail to invite new reviewers on time. Adding to the problem, some reviewers do not respond to editors’ invitations and leave articles in an unknown status. Whether it is because reviewers are too busy or they do not respect deadlines, some of them often reply weeks after the deadline. Moreover, some editors neglect to send reminders.

Many researchers have had negative experiences with journals that do not accurately report the status of submissions.

As editor-in-chief of a number of journals, I make sure that all the reviewing processes are very clear to authors. The statuses we use are as follows:

- Submitted to journal
- With editor (for pre-evaluation and reviewer invitation)
- Reviewer invited
- Under review
- Reviews completed
- With editor (for decision-making)

I believe that journals should consider changing their current method of updating authors on the status of their papers. Editors-in-chief are expected to respond to emails from all authors at their earliest convenience. While I understand that editors may not be happy with authors constantly emailing them for updates, it has been proven that most authors are aware of the typical timeline and average time for decisions on papers in their field or specific journals and are willing to wait patiently.

Mox Sal Moslehian is a professor of mathematics and a fellow of The World Academy of Sciences (TWAS). His research focuses on functional analysis and operator theory. He serves as the editor-in-chief of the journals *Banach Journal of Mathematical Analysis*, *Annals of Functional Analysis* and *Advances in Operator Theory*, published by Birkhäuser/Springer.
His homepage is https://prof.um.ac.ir/moslehian/.

moslehian@um.ac.ir

Report from the EMS Executive Committee meeting in Turin, 15–17 November 2024

Enrico Schlitzer

The EMS Executive Committee (EC) usually holds two major meetings each year, plus regular shorter virtual meetings. The second major meeting of 2024 was held from 15 to 17 November at the Department of Mathematics of the University of Turin (Italy). The event was organized by EC member Susanna Terracini, with support from the Italian Mathematical Union (UMI) and the Italian Society for Applied and Industrial Mathematics (SIMAI).

This report focuses on matters of direct interest to EMS members, particularly the Committee's strategic objectives and positions on relevant issues, leaving aside technical and procedural details.

The meeting marked the final attendance of three distinguished Executive Committee (EC) members: *Frédéric Hélein*, *Luis Narváez Macarro* and *Beatrice Pelloni*. Their successors – *María Ángeles García Ferrero*, *Adam Skalski* and *Alain Valette* – participated in the discussions alongside other EC members,[1] *Donatella Donatelli*, the new editor-in-chief of the EMS Magazine, *Elvira Hyvönen* from the EMS Secretariat and *Enrico Schlitzer*, the newly appointed EMS and EMS Press community engagement manager.

Members of the EMS Executive Committee present at the EC Meeting in Turin, along with Donatella Donatelli and Elvira Hyvönen.

Inclusive mathematics, accessible publishing

EMS Press, the publishing house of the European Mathematical Society, was founded with a clear mission: to ensure that experts guide the presentation of scientific results, reclaiming control from large commercial publishers. The company is now a successful publishing house and an example of fair and sustainable Open Access publishing. EMS Press uses a model called Subscribe to Open (S2O): libraries subscribe to journals as before, and when a journal reaches a sufficient number of subscriptions, it becomes open access for that year. This approach has proven successful: as of 2024, all 26 EMS Press journals have been published open access.

André Gaul, managing director of EMS Press, presented an impressive year of achievements: ten new books published in 2024, adding to a catalog of over 250 titles, and a steady publication of 26 journals. The company continued the development of in-house publishing technology and was present at major mathematics and publishing events worldwide. Gaul emphasized that as a community publisher, EMS Press needs ongoing support from the EMS community to strengthen its commitment to open access. The EC expressed interest in opening dialogues with other Open Access initiatives in mathematics. While these initiatives may differ from EMS Press in their specific approach to Open Access, they share fundamental values: being community-driven and opposing commercialism in academic publishing.

Diversity in mathematics

The EC dedicated a significant amount of time to discussing concerns raised by some members about the representation of applied mathematics in EMS events and prizes, and considers it essential to include mathematics in all its diversity. Simultaneously, optimizing every diversity parameter – such as gender, geographical origin, and mathematical fields – is a nearly impossible task. Rather than

[1] https://euromathsoc.org/committee-executive

seeking perfect balance in each individual occasion, the Committee is determined to ensure satisfactory representation across all EMS activities when viewed as a whole.

The EC considers mathematics as a unified field, while recognizing that different research areas may have their specific characteristics and needs. Importantly, the Committee unanimously expressed reservations about classifying research into pure and applied mathematics.

Investing in the EMS's future

The EMS Young Academy (EMYA) continues to grow, with 20 new nominations bringing its membership to 80 early-career mathematicians: their perspective will continue to offer new impulses for the development of the EMS. The EC also reviewed changes to EMS Committees compositions, to be announced in the coming year, and discussed recommendations for events to be supported in 2025 coming from the Meetings Committee, the Committee for Developing Countries, and the European Solidarity Committee.

Ethics and professional conduct

Stefan Jackowski, Chair of the Ethics Committee, presented the development of the *EMS Code of Ethics*, a framework that will include several documents: an update of the current *Code of Practice*, the *Code of Practice for Mathematical Publication*, the *Code of Conduct for EMS Events*, *Advice on Setting up a Welcoming Environment at Mathematical Events*, and the *EMS Statement on Anti-harassment*. This initiative is being developed through the collaborative efforts of three committees: the Ethics Committee, the Committee for Publications and Electronic Dissemination, and the Women in Mathematics Committee (WIM). A discussion followed the presentation, with contributions from André Gaul, *Mikaela Iacobelli* (chair of WIM), and *Apostolos Damialis* (EMS Press editorial director) and *Olaf Teschke* (chair of the Committee on Publications and Electronic Dissemination). The participants addressed both the impact of new technologies on research and publishing, and strategies to maintain a safe, inclusive, and respectful environment across all EMS activities.

The Ethics Committee is also expanding its scope to address a broader range of professional conduct in mathematics. This includes not only research integrity but also practices in publication, teaching, and the practical applications of mathematics.

EMS Magazine and zbMATH Open

Donatella Donatelli, the new editor-in-chief of the EMS Magazine, presented an encouraging overview of the publication's development. The Magazine has successfully maintained a steady output, with issues averaging 70 pages and carefully working to balance different types of contributions. While achieving perfect balance remains challenging, the Magazine tries to represent the entire spectrum of mathematical activity – from research features to educational perspectives, from historical insights to public awareness initiatives. A notable upcoming project is the publication of articles by the fourteen prize winners from the 9th ECM, starting with Felix Klein Prize winner Fabien Casenave in the previous issue (no. 134).

Olaf Teschke, in his role as managing editor of zbMATH Open, reported on the platform's significant developments. Since becoming Open Access, zbMATH Open has seen substantial growth in its user base, particularly from developing countries. Recent improvements include enhanced user interfaces, expanded coverage through preprint integration, and better connectivity with other mathematical resources.

Looking ahead: building a stronger community

A highlight of the meeting was the Sunday morning session featuring presentations from two Italian mathematical societies. *Marco Andreatta*, president of the Italian Mathematical Union (UMI), discussed how the society has evolved to meet contemporary needs, including the creation of specialized groups to address specific mathematical areas and engage with public interest. *Luca Formaggia*, president of the Italian Society for Applied and Industrial Mathematics (SIMAI), outlined his society's activities in organizing congresses, supporting young mathematicians through prizes, and promoting mathematical education through various initiatives, from editorial activities to teacher training programs.

The meeting concluded with a *Strategic Session* where participants reflected on EMS objectives and their practical implementation. The discussion highlighted two main directions for future

From left to right: Susanna Terracini, Marco Andreatta, Luca Formaggia and Jan Philip Solovej.

action: externally, strengthening mathematics' voice at the European level through engagement with policymakers; internally, creating more opportunities for the mathematical community, with particular attention to early-career researchers through networking and professional development initiatives. Throughout the three days, the Executive Committee repeatedly emphasized its commitment to diversity in all forms – geographic, regarding gender, and across mathematical fields.

Enrico Schlitzer is a science communicator and journalist with a background in mathematics (PhD from SISSA, Trieste, Italy). He serves as community engagement manager for the European Mathematical Society and its publishing house EMS Press.

schlitzer@ems.press

New editors appointed

Michele Coti Zelati is a professor of mathematics at Imperial College London. He holds an MSc in mathematics from Politecnico di Milano (2008) and a PhD in mathematics from Indiana University (2014).

His scientific work focuses on analysis and partial differential equations arising from fluid dynamics and kinetic theory. Key research interests include the long-time behavior of solutions, stability, mixing, and random dynamics.

He is the recipient of an ERC Starting Grant (2023–2028) for the project STABLE-CHAOS and holds a Royal Society University Research Fellowship (2019–2027). Michele is also a member of the London Mathematical Society.

His webpage is https://www.ma.imperial.ac.uk/~mcotizel/.

Photo by Rémi Grizard

Bruno Teheux is assistant professor in the Department of Mathematics at the University of Luxembourg. He obtained his PhD from the University of Liège in 2009. Subsequently, he worked at Animath, a non-profit organization in Paris, from 2010 to 2012, before joining the University of Luxembourg as a researcher in 2012.

His research focuses on the algebraic aspects of non-classical logics, particularly modal and substructural logics. His work extends to the study of ordered algebraic structures, including lattices and ordered semigroups.

In Luxembourg, he develops innovative mathematics outreach initiatives that present contemporary mathematical research to general audiences. These programs reach an international audience of several thousand participants annually and emphasize accessibility and inclusion. Notable projects include a data sonification collaboration with composers for the Esch 2022 European Capital of Culture program and the development of two interactive mathematics exhibitions hosted at the Luxembourg Pavilion during World Expo Dubai 2020.

Homepage: http://math.uni.lu/teheux

Valdemar V. Tsanov is an associate professor at the Institute of Mathematics and Informatics of the Bulgarian Academy of Sciences, Department of Analysis, Geometry and Topology. He holds a Maîtrise in mathematics from University of Nantes, France, a Master's degree in mathematics and mathematical physics (2006) from Sofia University, Bulgaria, a PhD in mathematics (2011) from Queen's University, Canada, and a Habilitation in mathematics (2020) from Ruhr-University Bochum, Germany.

His research interests are in geometry, Lie theory and representation theory, with applications to mathematics physics and quantum information theory. His work is concerned with geometry of spaces with group actions, homogeneous spaces, equivariant embeddings, branching laws for representations, constructive aspects of geometric invariant theory. He has applied methods from algebraic geometry, symplectic geometry, and Lie theory to the theory of quantum entanglement. Furthermore, he has worked on a constructive approach to descriptions of momentum map images, with applications to the quantum marginal problem.

He has worked as a teaching fellow and participated in several research projects of the Deutsche Forschungsgemeinschaft at the University of Bochum, the University of Göttingen, and the Jacobs University Bremen. In the period 2016–2018 he held a personal DFG grant at the University of Göttingen with a project on projective geometry, invariants and momentum maps. He has participated in the organization of several conferences and seminars, including the Seminar Sophus Lie.

European Mathematical Society

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Jan Philip Solovej (2023–2026)
University of Copenhagen, Denmark
solovej@math.ku.dk

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jbuescu@gmail.com

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victoria.gould@york.ac.uk

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samuli.siltanen@helsinki.fi

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rakosnik@math.cas.cz

Members

María Ángeles García Ferrero (2025–2028)
Instituto de Ciencias Matemáticas (ICMAT), Madrid, Spain
garciaferrero@icmat.es

Barbara Kaltenbacher (2025–2028)
Universität Klagenfurt, Austria
barbara.kaltenbacher@aau.at

Adam Skalski (2025–2028)
Mathematical Institute of the Polish Academy of Sciences, Warsaw, Poland
skalski@impan.pl

Susanna Terracini (2025–2028)
Università di Torino, Italy
susanna.terracini@unito.it

Alain Valette (2025–2028)
University of Neuchâtel, Switzerland
alain.valette@unine.ch

Community Engagement Manager

Enrico Schlitzer
schlitzer@ems.press

EMS secretariat

Elvira Hyvönen
Department of Mathematics and Statistics
P.O. Box 68
00014 University of Helsinki, Finland
ems-office@helsinki.fi

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- Option to join the EMS as reviewer of zbMATH Open.

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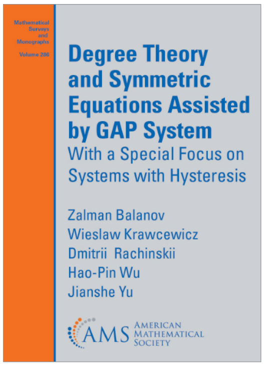

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Zalman Balanov, University of Texas at Dallas et al

Mathematical Surveys and Monographs, Vol. 286

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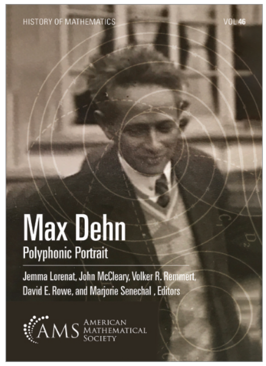

MAX DEHN

Polyphonic Portrait

Edited by Jemma Lorenat, Pitzer College et al

History of Mathematics, Vol. 46

Max Dehn (1878–1952) is known to mathematicians today for his seminal contributions to geometry and topology - Dehn surgery, Dehn twists, the Dehn invariant, etc. He is also remembered as the first mathematician to solve one of Hilbert's famous problems. This book is a collection of essays written by mathematicians and historians of art and science. It treats Dehn's mathematics and its influence, his journeys, and his remarkable engagement in history and the arts.

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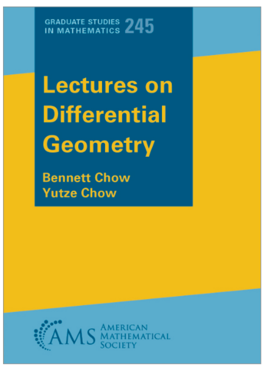

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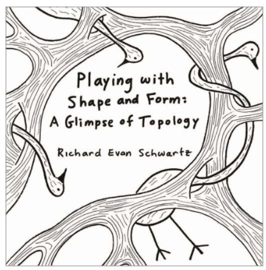

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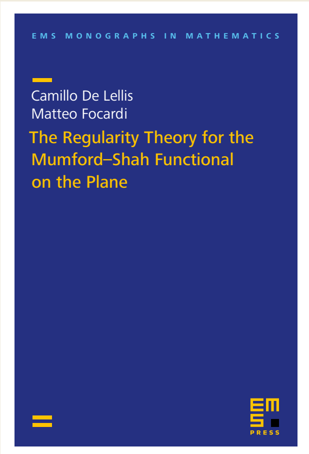

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EMS Monographs in Mathematics
ISBN 978-3-98547-089-1. eISBN 978-3-98547-589-6
February 2025. Hardcover. 266 pages. €69.00*

This book provides a thorough and up-to-date account of the state of art of the regularity theory for minimizers of the Mumford–Shah functional of image segmentation in the 2-dimensional setting. Starting with some classical preliminary results, which settle the issue of existence of minimizing couples (u, K), the structure of the set K is then analyzed by using ε-regularity theorems. Several consequences of the latter are also investigated, in particular leading to different characterizations of the Mumford–Shah conjecture.

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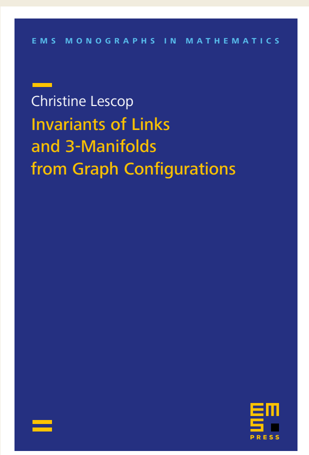

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The book is written for graduate students and more advanced researchers interested in low-dimensional topology and knot theory.

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