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EMS Magazine

Claude Sabbah
Michèle Audin (1954–2025): a posthumous interview

Raffaella Mulas
Side by side: interview with Sylvie Benzoni-Gavage and Heather Harrington

Dorin Andrica
Solved and unsolved problems

EUROPEAN MATHEMATICAL SOCIETY

EMS Magazine 140

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The cover illustration is a portrait by A. B. Araújo of the late Michèle Audin, whose posthumous interview is featured in the present issue.

Photo by Jim Høyer, University of Copenhagen.

I am entering the last half year of my presidency, and a new president will be elected in Lisbon by the EMS Council (June 26–28, 2026). I am looking forward to working with the new president-elect and the new EC to secure a smooth transition. I am proud to be leaving a healthy and thriving society to the next president.

I am particularly happy that EMS has finally succeeded in forming our two new policy groups, the internal and the external policy groups. You can find their composition on the EMS's webpage. I am convinced that they will play an important role in unifying the European mathematical community from within and in strengthening the perception of the importance of mathematics throughout Europe. We have also formed the Scientific Committee of the 10ECM which is to be held in Bologna in 2028. You can also see this committee on the EMS's webpage. It is an excellent committee, and I am confident that it will create a fantastic program for the 2028 ECM. I am particularly indebted to Professor András Stipsicz for having accepted the important task of chairing the committee.

The big news of the last few months was the announcement of the solution to the Erdős unit distance conjecture by an OpenAI model. While I had used LLMs for various tasks I clearly was not on top of the developments enough to see this coming and I must admit that it was an enormous surprise to me that it has already happened. Since then, I have already heard of several other exciting mathematical results obtained with the use of AI. This will of course mean that we have to rethink many aspects of mathematics both as researchers and as educators. In education, publishing, research evaluations, hiring and prize committees AI will surely cause many headaches, and, of course, it does so already. There are good reasons for being concerned and pointing out the threats as done in the new Leiden Declaration on Artificial Intelligence and Mathematics.[1] I personally prefer to look at these developments with an open and positive mind. The use of AI will become an integral part of the toolbox that mathematicians use for research, and I believe we must learn to embrace it. It should strengthen and not undermine mathematics as a truly human activity. I am looking forward to seeing where this will go over the next years. While we should probably have done this long ago EMS will address these issues both in our committees and within our new policy groups. At the meeting of presidents held in Lisbon in July in connection with the council we had a session on the use of AI in research evaluations.

In a few weeks I will be going to New York for the general assembly of the IMU to present the activities of the EMS. The venue of the 2030 ICM will be selected at the general assembly, and I am excited to learn where it will be. I am looking forward to continuing from New York to Philadelphia to take part in the 2026 ICM. I am sure there will be a lot of wonderful science and surely also interesting discussions about AI. The EMS will have a shared booth with EMS Press and zbMATH Open, and we hope to showcase all our activities to the greater mathematical community.

For now, I wish you all a great summer break and a good start to the new academic year.

Jan Philip Solovej
President of the EMS

[1] https://leidendeclaration.ai

Dear readers of the EMS Magazine,

As usual in this issue of the Magazine you will find a wide range of articles on diverse topics, along with our regular columns on discussions, societies, education.

In celebration of the International Women in Mathematics Day – May 12, we are pleased to feature two special contributions. The first is a posthumous interview to Michèle Audin (1954–2025) by Claude Sabbah; the text draws on Michèle Audin's personal notes and is structured as an imagined interview. In addition, in the *Side by side* interview series, Raffaella Mulas speaks with Sylvie Benzoni-Gavage (professor at Claude Bernard University Lyon 1 and former director of the Henri Poincaré Institute) and Heather Harrington (professor at Oxford and director of the Max Planck Institute of Molecular Cell Biology and Genetics).

I am very glad to announce that a new editor has joined the Editorial Board since the publication of the previous issue: Dorian Andrica, who will take care of the *Problem corner* section. A short biographical note about him can be found at the end of the issue. I am also grateful to Dorin Andrica for accepting to devote part of his time to the Magazine and to a section that has been absent from recent issues. In this regard, I am pleased to report that the current issue once again includes several pages devoted to solved and unsolved problems. We warmly invite readers to contribute open problems, research problems, and proposed solutions.

Finally, you may have noticed that issues 135 and 139 of the EMS Magazine contained different, and sometimes contrasting, views on topics of interest to the history of mathematics community. I would like to emphasize that the EMS Magazine is open to a diversity of opinions and perspectives, which we are committed to publishing in a transparent and balanced manner. At the same time, the Magazine is not intended to serve as a forum for extended exchanges of replies and counter-replies. I would also like to stress that all those who contribute to the Magazine (members of the editorial board, authors of articles, book reviewers) are researchers whom we hold in high regard for their expertise and the robustness of their scientific work.

Donatella Donatelli
Editor-in-chief

Big mathematics – reflections on the history of the Classification of Finite Simple Groups, 1950s to 1980s

Volker Remmert and Rebecca Waldecker

The Classification of Finite Simple Groups (CFSG) is a highlight of 20th-century mathematics, both with respect to its mathematical content and to the complex process of proving the result. From a historical perspective, it offers an excellent opportunity to focus on more general developments in the history of 20th-century mathematics, such as changing perceptions of what a mathematical proof is, the character and the many contexts of mathematics as an intergenerational and international collaborative enterprise, the roles that trust and consensus play within this enterprise, and the impact of Cold War research policies on the CFSG/pure mathematics. We consider the CFSG as (possibly) the first instant of what we tentatively call "big mathematics" in our project. In this article we report on work in progress, first sketching out our project and then explaining what we mean by "big mathematics."

1 The project: history of the CFSG

Scope

We begin with Hans Zassenhaus (1912–1991), a student of Emil Artin, who briefly had a chair at Hamburg University before permanently leaving Germany in 1948. After an academic peregrination (Glasgow, Montreal, Notre Dame), he worked at Ohio State University from 1964. In his 1947 inaugural lecture in Hamburg, Zassenhaus presented several mathematical areas of great interest to him, starting with groups. He explicitly called for a classification of finite simple groups:

> The most interesting problem, to me, seems to be the search for all finite simple groups. We are still far away from a solution to this problem. [37, p. 5]

Zassenhaus was not the first mathematician to wish for a classification of finite simple groups, and he was not the last one to press the question, lamenting a lack of substantial progress. Richard Brauer (1901–1977), a German mathematician who lost his position in 1933, emigrated and from 1952 held a chair in Harvard, took the question up in his plenary talk at the ICM in Amsterdam in 1954:

> The theory of groups of finite order has been rather in a state of stagnation in recent years. This has certainly not been due to a lack of unsolved problems. As in the theory of numbers, it is easier to ask questions in the theory of groups than to answer them. If I present here some investigations on groups of finite order, it is with the hope of raising new interest in this field. [7, p. 209]

New interest rose, indeed, and around 1980 the community agreed that the CFSG had been accomplished. We state the final CFSG here exactly as it appears in [17, p. 6]:

Classification Theorem. *Every finite simple group is cyclic of prime order, an alternating group,*[1] *a finite simple group of Lie type,*[2] *or one of the twenty-six sporadic*[3] *finite simple groups.*

This statement, in its simplicity and elegance, might seem surprising, and the short list of possibilities for all finite simple groups is misleading in the sense that it does not reveal the complexity and scope of what is going on here. In the words of Michael Aschbacher (1944–), one of the main contributors to the proof of the CFSG theorem:

> The hypothesis of the theorem is simple and easily understood by anyone who has taken a decent undergraduate course in abstract algebra. The conclusion also appears at first glance to be at least moderately simple. However, when one looks more closely, one finds that it takes some effort and sophistication to define many of the examples. [4, p. 2403]

[1] We add for clarity that the alternating groups of degree 1, 2 and 4 are not simple, but that otherwise a finite group is simple if and only if it occurs in the list given in this theorem.

[2] Without giving a definition, we want to mention that a group of Lie type is not the same as a Lie group.

[3] A term coined by William Burnside in 1911 for finite simple groups that do not belong to an infinite series.

And what about the proof? According to [17, p. 3], we were originally dealing with:

> […] somewhere between 10,000 and 15,000 journal pages, spread across some 500 separate articles by more than 100 mathematicians, almost all written between 1950 and the early 1980s. Moreover, it was not until the 1970s that a global strategy was developed for attacking the complete classification problem. In addition, new simple groups were being discovered throughout the entire period […] so that it was not even possible to state the full theorem in precise form […]. Under such circumstances it is not surprising that the existing proof has an organizational structure that is rather inefficient and that its evolution was somewhat haphazard, including some duplications and false starts.

This illustrates that the CFSG is a piece of mathematics of enormous scope, both in terms of the depth and breadth of the mathematics involved and in terms of the duration of the project, the number of people involved and resources used. Towards the end, in the 1970s, there was a strategy for proving the CFSG, but still it is difficult to fully grasp the amount of communication and coordination that was needed. As we will see in more detail below, this leads to several layers of unsurveyability and many interesting questions from the perspective of the history of mathematics as well as the history of science.

Sources

There are easily available printed sources to study the history of the CFSG, such as survey articles by some of the main contributors (e.g., Aschbacher, Solomon), obituaries of key contributors (e.g., Feit, Gorenstein, Suzuki), research articles contributing to the proof, and conference proceedings. Moreover, there are also transcripts of interviews conducted by Joseph Gallian with some of the main contributors. These are available in the Archives of American Mathematics at the Briscoe Center for American History, Austin, Texas, and we have conducted many interviews ourselves over the last two years. Moreover, we have begun to collect personal papers of contributors to the CFSG. Where we have permission, such original material will be made available over the course of our research project. Given the large number of contributors, it is a challenge to decide who are the most important ones, and by what criteria. Such decisions depend, at least partly, on the timeline of the CFSG.

Timeline

Daniel Gorenstein (1923–1992), who played a crucial role in the CFSG and is often referred to as its "CEO" or "architect," as well as other contributors tend to put emphasis on a period of about 30 years. However, there are arguments for both a shorter and a longer time frame of this story. One might argue that just a few years in the late 1960s and 1970s are the crucial time period, starting with John Thompson's (1931–) so-called *N*-group paper ([34], first article of three), Daniel Gorenstein's programme to tackle the classification presented in Chicago in June 1972 [15, pp. 178–193], a number of crucial publications by Michael Aschbacher and others, and with a feeling just a few years later that the work was mostly done. Ron Solomon (1948–) remarked that "we could hear the bell toll" in 1976 at the Duluth conference [30, p. 340], and Jon Alperin (1937–2025) expressed the feeling "that the ball game is over" in a letter of 1976 to Walter Feit (1930–2004) (Figure 1).

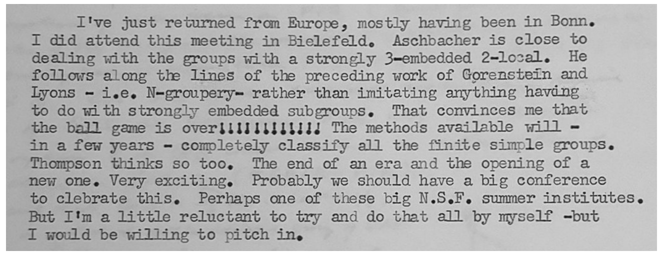
I've just returned from Europe, mostly having been in Bonn. I did attend this meeting in Bielefeld. Aschbacher is close to dealing with the groups with a strongly 3-embedded 2-local. He follows along the lines of the preceding work of Gorenstein and Lyons - i.e. N-groupery- rather than imitating anything having to do with strongly embedded subgroups. That convinces me that the ball game is over!!!!!!!!!!!! The methods available will - in a few years - completely classify all the finite simple groups. Thompson thinks so too. The end of an era and the opening of a new one. Very exciting. Probably we should have a big conference to clebrate this. Perhaps one of these big N.S.F. summer institutes. But I'm a little reluctant to try and do that all by myself -but I would be willing to pitch in.

Figure 1. Jon Alperin to Walter Feit, 27 June 1976, from the Walter Feit Papers, Dolph Briscoe Center for American History, The University of Texas at Austin (4RM121).

However, our story might well begin with the first explicit call for a classification by Otto Hölder (1859–1937), who noted in 1892, that it "would be of the greatest interest, if a survey of all simple groups could be given whose number of operations is finite" [19, p. 55]. If we then take the last relevant publications for the so-called first-generation proof as the end point [35], the classification project would have spanned 114 years. One might begin even earlier with the Mathieu groups discovered in the 1860s and 1870s as highly transitive subgroups of the symmetric groups of degree 11, 12, 22, 23 and 24, respectively, counting among the 26 sporadic groups [28]. We will, however, follow [30] and consider [19] the beginning of the CFSG project. Towards the end, we can see how Gorenstein's strategy from the 1970s was successful and that the techniques did in fact lead to a proof of the CFSG theorem. It took until 2004 for all major cases to be completed and published. To shed some light on this delay, we briefly explain some background.

The analysis of finite simple groups along their local structure includes a case distinction along a parameter called $e(G)$,[4] which captures an aspect of the interplay between the prime 2 and odd primes dividing the group order. The case where $e(G) = 2$, famously called the "quasithin case" of the CFSG, had been partially treated, but without publication, and it was noticed that deep and difficult

[4] $e(G)$ denotes the maximum p-rank, for odd primes p, of 2-local subgroups of a finite group G.

work was necessary to close this gap. This work has been done by Aschbacher and Smith [5] long after the CFSG had been announced to be completed.

People

What is a contribution to the proof of the CFSG theorem? Does it matter whether a result was proven specifically with the CFSG in mind or just out of general interest, moving the theory forward? The definition of a mathematical or strategic contribution will influence whom we consider a major contributor, along with the definition of the timeline. With this in mind, we recall a few names: Jon Alperin, Michael Aschbacher, Richard Brauer, William Burnside, Walter Feit, Daniel Gorenstein, Stephen Smith, Ron Solomon, and John Thompson (in alphabetical order, concentrating on the 20th century and on direct contributions). Some more appear if we look at milestones on the way. Brauer and Fowler proved that, if H is a finite group with a central involution, then there are only finitely many possibilities for a finite simple group G with involution centraliser H [9]. Starting with an involution centraliser of structure $C_2 \times \mathrm{Alt}_5$, this approach led Zvonimir Janko (1932–2022) to the discovery of a new sporadic simple group, J_1, in 1965 [20]. The 1950s and 1960s brought new ideas and techniques. Some of them came from work of Michio Suzuki (1926–1998), a Japanese mathematician who for many years worked at the University of Illinois at Urbana-Champaign and discovered a new series of finite simple groups of Lie type as well as a sporadic simple group. His articles were influential in terms of concepts and strategies for the characterisation of groups with given properties. In the early 1960s, we see the fundamental work by Feit and Thompson, proving that groups of odd order are soluble. In other words, every non-abelian finite simple group contains an involution, and this is one of the reasons why the prime 2 plays this very special role among the prime divisors of group orders for finite simple groups.

Around the same time, we see that the general activity grows, with more young mathematicians entering the field and the culture changing by a rising number of conferences and special programmes. This is true for several, maybe most areas of mathematics and of science in general, but we mention it because the CFSG problem is so difficult and has so many sub-problems that need to be solved, that it is hard to imagine how the actual theorem could have been proven without the conferences and special programmes that brought a substantial number of researchers together, allowed them to discuss their progress, new methods and remaining open problems and that eventually led to a strategy towards a complete proof.

New sporadic groups were found in the late 1960s and the 1970s by John Conway (1937–2020), Bernd Fischer (1936–2020), and some of their students, to name just a few. We would have to list many more people if we took individual results and techniques into account (e.g., Helmut Bender, George Glauberman) or computational work (e.g., Charles Sims). At some point, it became necessary to collect the knowledge that was there and to create some kind of plan for how to proceed further. This is why we identify Daniel Gorenstein's programme from 1972 as a milestone, next to and building on John Thompson's N-group paper series from 1968 to 1973.

Once there was a list of finite simple groups that was thought of as (almost) complete, it was natural to begin with a minimal counterexample, i.e., a finite simple group G that is not on the list of known finite simple groups and of minimal order with this property. Then, Gorenstein suggested a case distinction with several layers, and he explained where existing publications already covered some of the cases, where cases were still wide open, where he had an idea about how to proceed further, and where he expected difficulties.

We have been told in interviews (and this is supported by self-historicising articles) that at the time, in 1972, this looked like a good, but completely unrealistic plan. Still, the mere existence of a strategy and evidence towards its feasibility made a difference in the research community, and the division into cases facilitated splitting up the work still to be done. We would like to better understand the dynamics in the research community exactly around that time, 1972–1976, because of the astonishing activity during these few years and the surprisingly quick progress in Gorenstein's programme. For such a complex project with hundreds of contributions, it is difficult to imagine how, without such a programme, the community would ever have felt certain that they were done classifying. (After all, with an incomplete list, the strategy has to make sure that any unknown finite simple group is discovered, or at least evidence for it, in the thorough case-by-case analysis. This was covered by Gorenstein's approach.)

Therefore, next to the strictly mathematical side and all the theorems that were proven treating cases of Gorenstein's programme, we are interested in the organisational side of the CFSG as well. According to the community, Gorenstein played a fundamental role in assigning tasks and keeping track of progress. After presenting his programme in 1972, Gorenstein became the key authority, trusted by the community (Figure 2). The many names that he has been given by the community point to his being seen as the coordinator of the project, at least for the final stage: CEO, godfather, architect of the CFSG. Jon Alperin explained in 1982 that, in 1968, Gorenstein had not yet been "captain of the team," the role that he would only grow into later, when he, in Alperin's words "was the one that was the chief optimist" [14, pp. 23, 15]. In our view such a complex, technically difficult piece of mathematical research needs, at some point, a person with energy and charisma, drive and chutzpah, motivating colleagues to tackle the difficult problems, to keep surveying the field, while also having expertise themselves. For a young academic entering the research area of finite simple groups, the whole project was probably overwhelming, but Gorenstein's strategy made it possible to pick up a case, work on that and

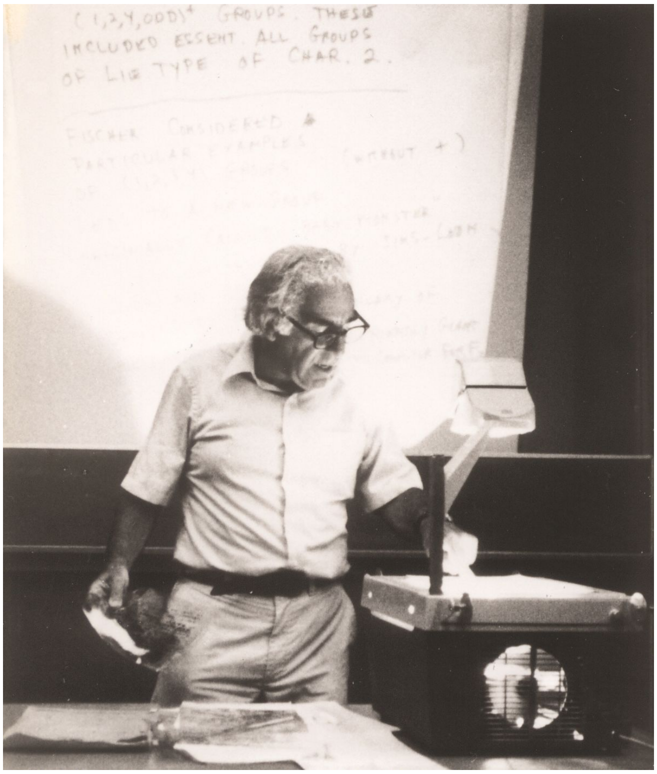

Figure 2. Daniel Gorenstein (1923–1992). (Photo: Konrad Jacobs; source: Archives of the Mathematisches Forschungsinstitut Oberwolfach)

contribute in this way. His energy and charisma helped to attract young academics and secure funding, and towards the end it certainly contributed to "communal energy," moving quickly through the remaining cases. We hope that, in our ongoing research, we can find more sources showing how exactly Gorenstein and others divided up the work and how communication on the progress worked, besides what we know about the key conferences.

Also, for clarity, we do not think that the CFSG would have been impossible without Gorenstein. There are many places in the CFSG where, if one person would not have proven a theorem or found a new group, then probably someone else would have done it, maybe a few years later. But there have been key contributions, key ideas, and their absence would have changed the course of the whole CFSG project dramatically. Maybe someone else would have stepped into Gorenstein's role, maybe a group of people, maybe decades would have gone by with more and more cases being solved, but without an end in sight. Maybe the internet and digital collaboration tools would eventually have led to some kind of coordination, and the CFSG would have happened some decades later. But even then, even with the actual work being done, we do not have a proof without a strategy for how the proof is organised and publications that follow this strategy. This needs at least one person with the whole picture in mind, and we have much evidence that Gorenstein was such a person for the CFSG from the 1970s to his death in 1992.

Geographic spread

Our definition of the timeline and of a contribution to the CFSG will also influence where we think the work on the CFSG took place. Some locations are easy to identify. Chicago, because of the significant special programme in finite group theory in 1960/61, Rutgers and Princeton, again because of special programmes, Duluth, Durham and Santa Cruz because of famous conferences, Cambridge (UK) because of computing facilities and the influence of John Conway, John Thompson and their students, Oberwolfach because of the Mathematical Research Institute, Bielefeld because of Bernd Fischer and his students. The list goes on, and we identify several countries (e.g., France, Germany, Japan, UK, Australia) as well as, within the U.S., several universities with great distances between them.

One way to illustrate the geographic spread of the CFSG community would be to look at printed contributions between 1950 and 1985. Here a preliminary literature review shows a substantial number of CFSG-relevant publications from the USSR during that time. This was surprising to us because we could hardly find any evidence of references to CFSG work being done across the Iron Curtain. Except for the ICM in 1966 in Moscow and individual reports of personal correspondence between colleagues, we do not have much evidence of an exchange of research results in that area or of actual collaboration. We have started to investigate this further.

Teamwork

Getting mathematicians together in thematic (international) meetings was relatively new in the 1950s and 1960s, even though topology set a precedent from the mid-1930s [3]. From the 1950s such in-person meetings became increasingly instrumental for teamwork oriented research such as the CFSG turned out to be. The Oberwolfach Institute with its thematic workshops is a very good case in point – Gerhard Pazderski, a participant of one of these workshops in 1960 who at the time worked in East Germany, remembers vividly a series of lectures given by John Thompson about an early stage of his work on the odd order theorem. These lectures allowed the audience to witness an outstanding development in the theory of finite groups long before publication. Another prominent example is the Group Theory Year in Chicago 1960/61 (Figure 3), organised by Adrian A. Albert (1905–1972), where according to Solomon, the "first real signs appeared of the evolving sociology of the classification effort as a team project" [30, p. 330].

In 1960/61, the Chicago team included many of the central figures of the classification project, such as Jon Alperin, Richard Brauer, Walter Feit, Daniel Gorenstein, Graham Higman, Michio

PREFACE

This document is being issued at the end of a SPECIAL YEAR PROGRAM ON THE THEORY OF GROUPS at the University of Chicago from October 1, 1960 to June 30, 1961. It contains summaries of papers presented at the weekly seminar which was one of the principal features of the program.

Since John Thompson does not plan to submit his paper on Groups of odd order $p^a q^b$ for publication in the near future it is presented here in greater detail than the other papers. The great majority of the summaries are of results to appear within a year.

The mathematical community is grateful to the National Science Foundation, the Office of Scientific Research of the Air Force, and the Esso Foundation for their support of the program.

James Boen

July 1, 1961

Figure 3. Preface of [6].

- 3 -

BUDGET:

	AFOSR	IAS	TOTAL
Salaries for 10-12 mathematicians	$100,000	$15,000	$115,000
Travel within the United States for scientific meetings or consultation, etc.			
12 trips @ $100	1,200		1,200
4 trips @ $200	800		800
Travel cost of participants from abroad not having Fulbright or other outside travel support		3,000	3,000
Publication: Page cost of publication of approximately 250 pages @ $10/page	2,500		2,500
Indirect cost: (based on current 29 per cent overhead rate)			
29 per cent of $100,000	29,000		29,000
29 per cent of $15,000		4,350	4,350
Total	$133,500	$22,350	$155,850

FACULTY SPONSOR:

Armand Borel, Professor

APPROVED FOR
THE INSTITUTE FOR ADVANCED STUDY:

Carl Kaysen, Director

Date May 23, 1967

Figure 4. IAS Princeton, (successful) grant proposal for the Group Theory Year 1968/69 to the Air Force Office of Scientific Research, 1967. (IAS archives, AFSOR 68-1468)

Suzuki, and John Thompson. In 1982, Alperin assessed the Chicago Group Theory Year as "still one of the best one-year-long programs I think there's ever been, in terms of what happened" [14, p. 6]. At the same time Gorenstein described the CFSG as an "unprecedented thirty-year team effort" [16, p. 58]. From this, as well as from recent interviews with some of those involved, it is clear that teamwork was in the DNA of the classification community from the late 1950s on, if not earlier.

This brings us to the question of logistics and funding.

Resources

Even now, organising big (or medium-sized) conferences takes a lot of time and money. It is hard for us to imagine how much more time-consuming and expensive it was in the 1960s and 1970s. The availability of public transport and the political situation often created unexpected obstacles, and long-distance travelling (airfares) was more cost intensive than today. The context of the Cold War offered new possibilities for science funding on a large scale, also for pure mathematics. New funding opportunities facilitated bringing ambitious plans into reality. For the Chicago Group Theory Year 1960/61 mentioned above, we have information about the sponsors, including the National Science Foundation, the Air Force Office of Scientific Research and the Esso Foundation, but not about the actual budget. However, for the Group Theory Year at the IAS Princeton in 1968/69, organised by Armand Borel (1923–2003) who at the time had a strong interest in the CFSG, we have a grant proposal (Figure 4) and know that it was sponsored by the Air Force Office of Scientific Research, a regular patron of the IAS at the time, with 133,500 USD (ca. 1.3 million USD in 2025).[5] Among the people invited from the CFSG community were Jon Alperin, Richard Brauer, George Glauberman, Daniel Gorenstein, Zvonimir Janko, Michio Suzuki, and John Thompson.

Budgets for the conferences the CFSG community attended and organised were much smaller, and some were sponsored by the AMS, supported by an NSF funding scheme. However, it was in trying to get an overview of the relevant and specialised conferences for the CFSG that we began to understand the scope of the resources necessary to facilitate the exchange of knowledge and to keep the CFSG team together. The budget for the crucial conference in Duluth in 1976, covered by the NSF, ran up to 8,700 USD (ca. 50,000 USD in 2025).[6] Considering the number of conferences with at least a partial focus on the CFSG or programmes like the Group Theory Years mentioned, this adds up to a substantial overall budget – at least for research in pure mathematics. We would like to specify this much more, following up on grant numbers (if they are published), searching in university archives (if documentation of a special programme has been kept) or in personal papers of conference organisers (if we have access and if they have kept the relevant documents). We do not claim that the CFSG would not

[5] Cf. https://www.bls.gov/data/inflation_calculator.htm.

[6] NSF to Joseph Gallian, 28 May 1976 (Joseph Gallian has kindly made this information available to us).

have been possible without the funding for the conferences and special programmes. However, given the large number of people involved, in particular in the 1970s, opportunities for face-to-face communication make a huge difference. Without them, the information would have reached much fewer people, and the process would have taken much longer.

Another perspective on the resources needed are the cost of computing time and the availability of computing facilities in the 1960s and 1970s. For the above-mentioned Janko group J_1, for instance, a group of order only 175,560, a computer was used to find two explicit matrices that generate the group as a matrix group. It was a recurring theme that evidence for a new sporadic simple group was found, and its existence was proven later, often supported by computer calculations. Another example illustrates this: the Monster group, the largest of the sporadic groups, of order $2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$.[7] Charles C. Sims (1937–2017), one of the central players in the chase for sporadic groups, when looking back on his efforts to construct sporadic groups in 1978, also commented on the role of computers and the ensuing costs. With respect to the Monster group, he explained:

> These rough calculations indicate that the construction of the monster is technically feasible but not yet economically justifiable. I doubt that anyone would argue that the existence or non-existence of the monster is of such importance to the mathematical community that it warrants devoting the resources of a major computer center for the better part of a year to settle the matter. However, it should be possible to refine the techniques sufficiently so that the question can be answered at a price which can be justified. [29, p. 135]

A computer-free construction of the Monster group was later given by Robert Griess [18]. We would like to better understand the role of computers/computing facilities in the original proof of the CFSG theorem and how this was discussed within the community. After all, depending on how a computer is used, it changes the nature of proof of a mathematical result and therefore the culture of the discipline.

Coming back to the costs of computing time, Richard Brauer, also engaging with the question whether there were "finitely many or infinitely many sporadic groups," expresses concern about the cost of computational resources in 1976 (published after his death in 1979):

> If they succeed in finding an upper bound for the order of sporadic groups, then the whole problem can be handled by a computer, though the costs of the project may very well be prohibitive. [8, p. 22]

These examples give an idea of the resources that were relevant in the classification project, and we have already seen some of the sources of funding. In the context of the Cold War there was a wide array of funding opportunities. Of course, we would expect that in the U.S., pure mathematics would have been supported by the NSF and the AMS as well as private and industrial funding agencies. In the UK we see funding from the Science Research Council, and research visits were paid for by the German Research Foundation in (West) Germany. We still have to systematically investigate funding and budgets of the major CFSG conferences in Australia, Europe, the UK, etc. However, the scope of military funding that went into pure mathematics in general and the CFSG in particular is striking (e.g., in the U.S., the Air Force Office for Scientific Research, the Office for Naval Research, and the NSA), and we intend to specify this in much more detail and find out if the amount of funding for the CFSG was in any way unusual, compared to projects in other disciplines. It fits into this context that, in some of our interviews, it became clear that mathematicians, even when trained in areas that seem far away from applications, were deemed a crucial resource in both the civil and military arenas. This is supported by advertisements such as the in the Notices of the AMS shown in Figure 5.

2 Our framework: big mathematics

As we have seen, the CFSG is of enormous scope, and so is its historical analysis. To be better able to tackle it, we use the analytical framework of what we tentatively call "big mathematics," to be laid out in this section.

From a historical perspective, the history of the CFSG has not yet been systematically studied (but see [32]), even though it offers an excellent opportunity to focus on more general developments in the history of 20th-century mathematics, such as:

(1) changing perceptions of what a mathematical proof might be;
(2) the character of mathematics as an intergenerational and international collaborative enterprise;
(3) the impact of the Cold War on science and mathematics.

(1) The CFSG raises specific questions about the nature of proof in mathematics, mainly with respect to two epistemologically different types of surveyability. On the one hand, the surveyability of mathematical proofs in a sense very specific to the CFSG has been viewed critically, because the proof is scattered in a huge number of papers, each (in principle) individually checked or verified by mathematicians, but difficult or impossible to understand as a whole. On the other hand, the use of computers in proofs was often seen as entailing a certain kind of opacity in allowing access to the content of the proof, i.e., that the proof might not be verifiable in detail by individual mathematicians, a problem discussed since

[7] We display the group order because we want to make the magnitude explicit, also for computer calculations. To this day, it is a challenge to do computer calculations in the Monster group.

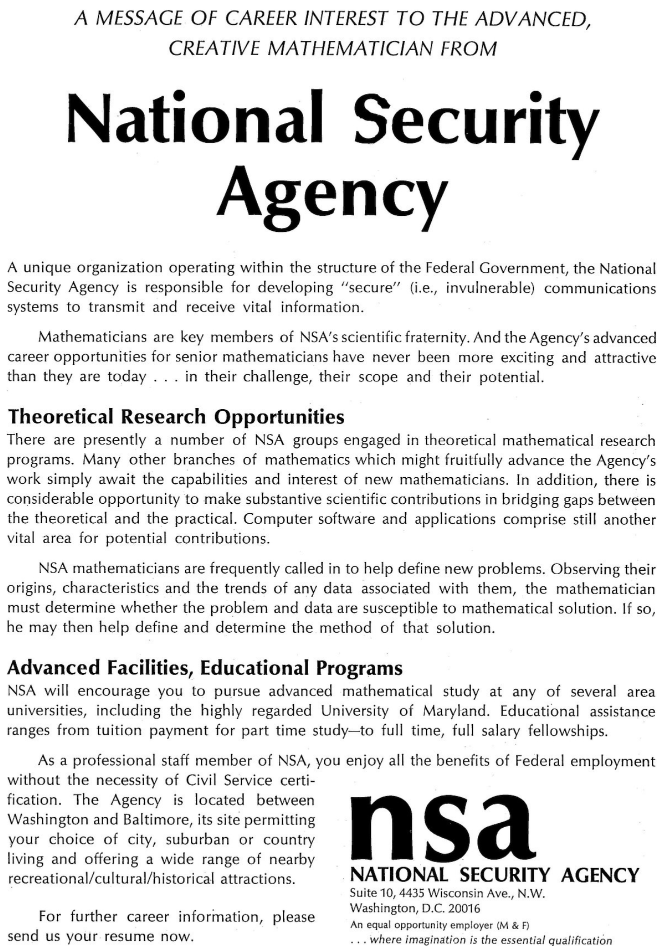

Figure 5. NSA advertisement in the *Notices of the AMS* **13**, p. 179 (1966).

1976 in the context of the proof of the four-colour theorem [22]. These aspects are deeply intertwined with the roles that trust and consensus played within the community.

(2) The CFSG is exemplary with respect to the character and the many contexts of mathematics as an intergenerational and international collaborative enterprise. Apart from international cooperation in the CFSG community, specific aspects such as the importance of German émigré mathematicians, e.g., Richard Brauer and Reinhold Baer (1902–1979) stand out. The example of Baer, who had left Germany in 1933, but returned to a professorship in Frankfurt in 1956, is interesting from an additional perspective. While he might not be seen as a contributor to the CFSG himself, he influenced a generation of mathematicians some of whom became contributors to the CFSG (for example Helmut Bender, Bernd Fischer, and Dieter Held).

(3) Finally, as we have explained, the history of the CFSG can be studied as a key example of the impact of Cold War politics on research in general and on mathematical research in particular, namely via new possibilities of funding through both civil and military agencies. With respect to historical detail about what fields in mathematics profited from this and how, this is a rather unexplored territory, and we have a feeling that it is crucial for the classification project. In this context, the concept of "big mathematics" can be useful.

The classification effort took decades, and more than 100 mathematicians were "enrolled" (many of them via PhD theses) [15, p. 44]. The sheer size of the group active in the CFSG between the 1950s and the 1980s is without precedent in the history of mathematics, and mathematician E. Brian Davies characterised the CFSG as "industrial scale pure mathematics" [11, p. 1353], while George Szpiro, journalist and mathematician, even compared the CFSG to the genome project [33, p. 94]. In this vein we want to tentatively consider the phenomenon of the CFSG as "big mathematics," in a sense adapted from characterizations of "big science" [10, 13]. Big mathematics would be a project that is:

(1) geographically widespread;
(2) teamwork oriented;
(3) resource intensive;
(4) relying on large equipment;
(5) at some point necessitating a central strategy and coordination.

Before we briefly discuss these characteristics, a little caveat is in order. Standard examples usually referred to when discussing big science include Stanford's Linear Accelerator (SLAC), CERN, or Los Alamos. Clearly, when proposing "big mathematics" as an analytical framework to study the history of the CFSG, we understand we need to scale down in terms of "big." Also, we are well aware that the proposed concept of big mathematics needs to be open to adaptation (cf. ideas developed in [1, 23]). We do not intend to use the framework of big mathematics to claim that the CFSG is the only instance of mathematical research possibly meeting these criteria. On the contrary, once we have worked with the concept, and clarified some of its strengths and weaknesses, we hope that it can be used to analyse other mathematical research projects that might meet all, or some, of the criteria. We chose the analytical framework of big mathematics because it helps us to study and analyse the history of the CFSG in its various contexts. We are thinking of the role of Cold War Science/Mathematics, the importance of German émigré mathematicians in the CFSG and, in general, the impact of politics on research in pure mathematics. During the Cold War, we see this in new possibilities of funding of research in general [12, 23, 31] and of mathematical research in particular [2, 27].

(1) The CFSG as a geographically widespread enterprise
Based on the material that we have about the people involved, their affiliations and the locations of conferences and special programmes, our approach shows that the U.S. and some European countries were in centre stage of the CFSG for a substantial amount of the work. As explained above, in the context of the time period we focus on and given the possibilities for travelling, we consider this to be a wide geographic spread.

(2) Teamwork in the CFSG

In the first half of the 20th century scientific research would have been predominantly conceived of as being individualistic, and surely this held for (pure) mathematics. Carl Ludwig Siegel made scathing remarks on teamwork in mathematics after having left Princeton. However, quite naturally, by the mid-20th-century teamwork was an essential ingredient of big science, and teamwork also became a characteristic feature of the CFSG. This can be very prominently seen at the Group Theory Year in Chicago 1960/61, and the proof of the CFSG in its entirety is clearly a product of teamwork. This is exactly in line with contemporary developments in the sciences (e.g., [21, 36] and [25, Chapter 6]).

(3) The CFSG as a resource intensive project

We already explained what "resource intensive" means in our context and that we do not intend to compare this to the resources of big science projects in physics or the earth sciences. Compared to building and operating a large hadron collider, most science funding can hardly be assessed as resource intensive. However, when compared with the costs of average activities in mathematics from the 1950s to 1970s – say library access, office space, pencils, pens and paper, typewriters and ribbons, chalk and blackboards –, then the budgets of some of the major conferences and year-long events in the CFSG are rather amazing. Jon Alperin put it succinctly when interviewed by Joe Gallian in 1982: "Of course, the amount of money is peanuts compared to outside mathematics. […] We're talking about a few million dollars maybe" [14, p. 151]. An open problem in our project is how to quantify these costs based on reliable data, often difficult to come by, and to identify the funding sources. Detailed information and historical work on the scope of military patronage/funding of mathematics in general is scarce (however cf. [24, 26, 27]) and this also pertains to the CFSG.

(4) Large equipment in the CFSG

In terms of large equipment, we have already mentioned computing facilities in the 1960s and 1970s. However, as we are today surrounded by powerful, small and relatively inexpensive computing devices, we tend to forget the sheer size and expense of powerful computers in the 1960s and 1970s. When proving the four-colour theorem in the mid-1970s, Kenneth Appel and Wolfgang Haken used "1,200 hours of time on 3 different computers," and in particular, after they had been granted permission, they used the "IBM 370-168 that had just been installed at the University of Illinois Administrative Data Processing Unit" [22, p. 134]. The same model was used in CERN, and a photo taken in 1977 gives an idea of why large equipment is an adequate description (Figure 6).[8]

[8] Cf. the wonderful photos of an IBM 370-168 being delivered to Newcastle University: https://blogs.ncl.ac.uk/cs-history/2019/09/08/a-moment-in-time-the-370-arrives/.

Figure 6. CERN, IBM 370-168, installed in 1977. (© CERN)

(5) Central strategy and coordination of the CFSG

In our discussion of the people involved, we have mentioned the specific role of Daniel Gorenstein. We have already spent a substantial amount of time and effort to critically discuss his impact, by looking at interviews, asking contributors and searching for evidence in personal correspondence and publications. While the signs are often subtle, we have as of now only evidence confirming his role as coordinator and chief enthusiast, and we intend to clarify and explain the nuances of this specific aspect of the CFSG in much more detail in the future.

Conclusion

We have argued that, at least tentatively, big mathematics can be used as an analytical framework to grasp and describe some of the specific and singular aspects of the CFSG, when adequately downscaled from big science. To analyse the CFSG as a major achievement of 20th-century mathematics from a historical perspective is a rather ambitious project, and we are only at the beginning. We are therefore aware that this article raises many more questions than it can answer. But, as we have sketched out, such a project has a multitude of fascinating dimensions, shedding light on the evolvement of the classification project as such as well as the various relevant contexts framing mathematics in the second half of the 20th century. We hope that big mathematics might be a useful framework to investigate other extensive mathematical projects, such as the Bourbaki group or the proof of Fermat's Last Theorem. Our project abounds with open questions. To tackle them, additional source material is needed (such as grant proposals and responses, final reports, material on the planning of specific conferences, correspondence with the main players, etc.).

Therefore, we close with an invitation: We are very grateful for all hints towards such material.

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[9] https://doi.org/10.14760/978-3-00-082798-3

Michèle Audin (1954–2025): a posthumous interview

Claude Sabbah

This text consists of personal notes by Michèle, organized as a mock interview.

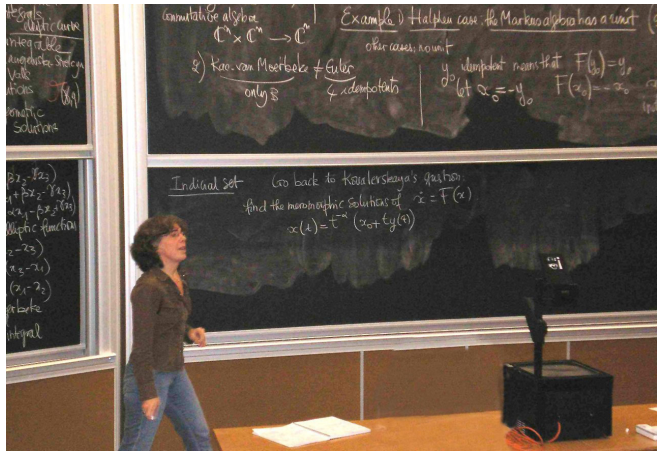
Claude Sabbah: *How did your mathematical life start?*

Michèle Audin: The family balance leaned toward mathematics: on the one hand, my mother, a teacher of this subject, was omnipresent, at home, at the high school where I was a student and where she taught; and, on the other hand, no less present was the figure of my late father (a brilliant young mathematician, my mother said, murdered at the dawn of his life as a researcher). I have always loved mathematics, and at that time, it was almost exclusively thanks to my mother.

The happiest moment was when she had to start teaching what is known as "modern mathematics," an extraordinary discovery that she shared with me: Venn diagrams and Lewis Carroll's book [15], the solutions to the differential equation $y'' = -ky$ (that of the spring), they have the same structure as the set of vectors in the plane. At the same time she always showed me circles, triangles, problems of geometric loci to which their new name of "set of points" had taken nothing away from their charm. This is how I developed my taste for mathematics: recognizing and using all the structure present, but not neglecting the phenomena. Thus, for years, I thought that Feuerbach's theorem (Euler's circle is tangent to the inscribed and circumscribed circles) was the most beautiful achievement possible in mathematics. This is undoubtedly the main reason that prompted me, many years later, to write the book [4].

CS: *How did you integrate into the French mathematical community as a woman?*

MA: In the late 1970s, I started attending the topology seminar at Orsay as a doctoral student. If someone asked a question, the answer was often, "That's trivial!" That is what I call the "Orsay style," which one of my foreign colleagues described as arrogance. But during a year in Geneva, I learned mathematics and I also learned that anyone—including me—can ask a question and get an answer.



Michèle Audin in action at the CIRM, 2006.

Another example: later, in the late 1990s, I was talking about quantum cohomology at the Bourbaki seminar. During the presentation, a famous mathematician in the second row opened his newspaper, unfolded it, and read it. That is also the "Bourbaki style."

To return to my first steps, the first mathematician to invite me to speak at a seminar was a female mathematician, Paulette Libermann. After my first presentation at a conference abroad (Oberwolfach), one of my colleagues asked a (French) mathematician to write a letter of recommendation for me. He refused, saying, "People will think I slept with her."

As a candidate for a professorship, I had dinner with several local bigwigs. One of them said to me, "I don't know if we're mature enough to hire a woman." I still wonder how his maturity issues concerned me. Nevertheless, I was appointed professor in Strasbourg in 1987. That same year, I helped found the association *femmes & mathématiques* (women and mathematics).

CS: *What were your main areas of research?*

MA: In the early 1980s, I worked in algebraic topology, on applications of algebraic topology to symplectic geometry and in symplectic geometry. At the time, there was an article by a Russian mathematician, Viktor Vassiliev, who was working on enumerative geometry of singularities; there is a relationship between singularities in general and Lagrangian submanifolds, and it was while reading this article that I became interested in this topic. I attended the ICM congress in Warsaw in 1983 for the sole purpose of meeting Russian mathematicians, and then Vladimir Arnold invited me to Moscow in 1986. My presentation at the Arnold seminar was the most difficult (and one of the most stimulating) among the things I have done in my entire professional life. My work then focused more specifically on integrable systems, rational curves and quantum cohomology, the relationships between these two theories (quantum cohomology and integrable systems) via Frobenius manifolds, integrability issues, and the topology of Lagrangian submanifolds.

CS: *You wrote, or contributed to, many books, didn't you?*

MA: I took advantage of maternity leave to write my first book [2], which I completely rewrote in 2004: better written, much more material, new proofs. However, before that, there was my doctoral thesis [1]. I remember that, at a reception at Cambridge University Press, after two glasses of a robust Australian red wine, I managed to refuse to write a book on *the life of a mathematician*. But I found myself offering to write one about spinning tops, which I did [3]! I also tried to explain, geometrically, the Galois criterion of non-integrability of Morales and Ramis in [5], where there is a simplified presentation and an original example that is particularly educational. Finally, I co-authored at least one book that is not very funny ([13], on Floer homology), but my co-author and I had a lot of fun writing it!

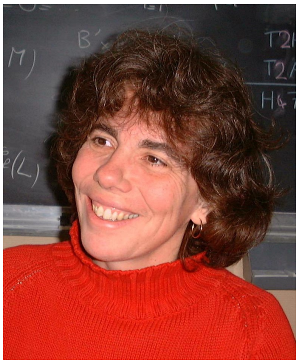
I also contributed, together with Jacques Lafontaine, to the publication of the multi-author book (including myself) [14], which remains a reference on the subject of pseudo-holomorphic curves. Moreover, several mini-courses that I have given have been published at various places.

Last, there is also the geometry book [4] that I mentioned at the beginning, which has been very useful to students preparing for the mathematics teaching certification exam.

CS: *And then there is that book* [9] *about Sofia Kovalevskaya.*

MA: I wrote only one of my research articles in collaboration (with Robert Silhol). It was a great idea, which we tested using Kovalevskaya's spinning top as an example. Then, thanks to Jean-François Peyret, a theater director, and his 2005 play *Le cas de Sophie K*, I felt free to also take an interest in the non-mathematical aspects of Sofia Kovalevskaya's life. Why is it so difficult to say that she was very good at mathematics? The most surprising thing about her reputation is the harm done to her by her "friend" Anne-Charlotte Leffler, who described her as an unhappy woman—at the very moment when she was happiest—in a biography that is still published alongside Sofia's own childhood memories. I had a very hard time finding a publisher for this book on Sofia Kovalevskaya, which did not fit into any category. I see it as a manifesto for the consideration of mathematics, history, and literature as a whole. For example, I love Italo Calvino's *Cosmicomics*, and I could not resist writing one featuring Sofia Kovalevskaya studying Saturn's rings. Reintegrating mathematics into culture is a matter of democracy.

Sofia Kovalevskaya.



Michèle Audin.

CS: *Let us continue with your interest in history, and first of all the history of mathematics.*

MA: Sofia received the Bordin Prize from the Academy of Sciences, and I became interested in the prizes awarded by this Academy, particularly the Grand Prize [6]. In 1918, it was Gaston Julia who received the prize, not Pierre Fatou, and there were also questions of priority with Paul Montel. I had questions about Julia: a war veteran who was seriously wounded in World War I and a renowned mathematician, but who maintained a lively correspondence with Nazi officer Helmut Hasse during World War II, which I wanted to publish, but the French side did not give permission… So I wrote a novel [11]! Research in the archives allowed me to discover Julia's first drawing of a Julia set! In the very first archive box I opened, I found a note by mathematician Jacques Feldbau, published under the name Ehresmann alone, because the Academy of Sciences had struck his name as author. I wrote an article showing how French scientists prevented their Jewish colleagues and compatriots from publishing during the German occupation, and I wrote the book [7]. I also re-edited the "Julia seminar," the predecessor to the Bourbaki seminar, which could not find a publisher [12]. I had access to first-hand documents because I helped catalog Henri Cartan's archives after his death, which moreover enabled me to edit the Cartan–Weil correspondence [8] and to write an almost finished biography of Élie Cartan (Henri's father).

CS: *Now let us talk about your interest in history.*

MA: I am not a historian, I have no degree in history. I am interested in history, but as a writer. I am a writer, but I am quite rigorous, so I like my fiction to be grounded in reality. I started my blog on the Commune[1] because I wanted to write a novel set during that period, so I learned a lot, too much even for a book. So I had the idea of starting this blog, which freed me up, and I was able to write the novel.

I was particularly interested in the Commune because, for once, just for that one time, lots of anonymous people, people who had no history, as they say, people whose opinions were never sought, spoke out and almost took power for two months. It so happens that, in terms of the history of mathematics, I was interested in the interwar period and World War II, which are somewhat delicate periods, not to mention the Algerian War. It is very complicated to do research on these periods because there are rights holders, and because we do not have access to the archives. Whereas for the Commune, there are still archives that no one has looked at, and we have the right to look at all the archives. There are so many missing persons, so many unknowns, that there is still a lot to be done.

I then edited and annotated many texts and correspondences from this period and even earlier, as I did for the Cartan–Weil correspondence and the Julia Seminar.

CS: *So you are a writer. Tell us about it.*

MA: After the book on Sofia, Oulipo invited me to become a member. They are a group of writers, somewhat modeled on the Bourbaki group, who are interested in potential literature, that is, who use forms or constraints to write. For example, I wrote a fairly long text following Pascal's theorem (six points on a conic…); the six points are characters; when three points are aligned, it represents a relationship between the characters. I also used the sextine, invented in the thirteenth century, which is a particular permutation of six letters (spiral permutation) or other permutations (of eleven letters in the novel [11]). These constraints help with writing but are not visible. Did joining Oulipo allow me to become a writer? My answer is: No, and Yes. The first real literary texts I published were included in the book about Sofia. But there is no doubt that I felt more legitimate writing about something other than mathematics after joining Oulipo. Memory is one of my areas of interest, the memory of those whom history overlooks. Life is easier for writers than for certified historians, since nothing prevents us, within the framework of extremely rigorous work, from conveying the emotion of our respect and love for the characters in our books.

CS: *Finally, a few words about your book* [10] *(A Brief Life).*

MA: As you may know, the Audin affair concerns the disappearance and murder by the French army of a young mathematician, anticolonialist activist, and communist during the "Battle of Algiers" in June 1957, Maurice Audin. The Audin affair is also, and above all, the campaign waged for more than 61 years by my mother, Josette Audin, and those who supported her, notably Laurent Schwartz and, more recently, Cédric Villani, to uncover the truth about Maurice Audin's death and disappearance, but above all to fight against torture in Algeria at that time and the system that gave rise to it. Official recognition was finally granted by President Macron on September 13, 2018.

While the search for the exact circumstances and perpetrators of Maurice Audin's murder has been essential for my family throughout all these years, and they remain unknown, it is instead my father's life, traces of which have not disappeared, that I tell in this book. For this book, I organized the data into categories (places, family, mathematics, etc.—count to six), with each of the six chapters containing elements from these six categories, which are permuted in a spiral pattern. This helped me in my attempt to cover everything and be exhaustive.

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Side by side: interview with Sylvie Benzoni-Gavage and Heather Harrington

Raffaella Mulas

In this interview series for the EMS Magazine, each article brings together two mathematicians who share something – such as a research area, a passion, or a professional context – but differ in other ways, as for instance age, gender identity, background, or geographical location. Sitting "side by side," they reflect on their personal and professional journeys, their views, and the worlds they live in.

The series is conducted by Raffaella Mulas and was born also thanks to the ideas and support of Apostolos Damialis (editorial director of EMS Press) and Donatella Donatelli (editor-in-chief of the EMS Magazine). By bringing together both common ground and contrast, side by side *aims to reveal the richness of mathematical lives.*

In this article, Raffaella Mulas interviews Sylvie Benzoni-Gavage (professor at Claude Bernard University Lyon 1 and former director of the Henri Poincaré Institute) and Heather Harrington (professor at Oxford and director of the Max Planck Institute of Molecular Cell Biology and Genetics).

Sylvie, Raffaella and Heather on Zoom.

Before I join the Zoom call, I am very curious about where this conversation will lead us. I have known Heather for a few years, while Sylvie is someone I know through the enthusiastic words of colleagues.

Raffaella Mulas: *Thank you both very much for being part of this interview! To start, do you already know each other? I noticed that your collaboration distance is four, and that you are both applied mathematicians whose work is driven by real-world phenomena. If I understand correctly, Sylvie's research interests might be summarized as nonlinear PDEs applied to wave propagation and fluid dynamics, while Heather's focus is on algebraic, geometric, and topological methods applied to biological systems. So maybe your research areas are still quite far apart?*

Sylvie Benzoni-Gavage: We didn't know each other before. Well, four is a really big collaboration distance!

Heather Harrington: I actually looked into your work before the interview, Sylvie, and PDEs were probably the closest I ever came to your direction. I took several PDE courses and even did a research project – so, I really tried! – but in the end it didn't quite speak to me, and I went in a different direction.

[RM]: *One thing you do have in common is your experience as directors of major research institutes. Sylvie, you were director of the Camille Jordan Institute from 2016 to 2017, and director of the Henri Poincaré Institute (IHP) from 2018 to 2024. Heather, in 2023 you became director of the Max Planck Institute of Molecular Cell Biology and Genetics, where you introduced mathematics as a new discipline. What aspects of this role do you find most meaningful, and what do you find most challenging?*

SBG: The part I liked most at the IHP was the expansion project of the institute. It had been launched when I arrived, but nothing was done yet, so everything had to be built, in particular the design of the Maison Poincaré museum. That was a very exciting part of the job. I also enjoyed the scientific side: evaluating proposals, and shaping the program of the institute. What I didn't enjoy as much was the administrative work and managing people. It was challenging, and sometimes frustrating. I did my best, but I'm also happy not to be doing that anymore.

[RM]: *What about you, Heather?*

Heather. (Photo by Z. Goriely, courtesy of Heather Harrington)

HH: In some ways, I'll echo what Sylvie said. One of the most meaningful aspects for me was arriving at the institute as the only mathematician, knowing that we would build something new, that this space would eventually have many more mathematicians, and mathematics will be a core discipline. That was incredible. At the same time, I quickly realized how different the culture is between mathematics and the life sciences. Even things like hiring work differently, and these differences can be surprisingly challenging if one is not aware of the differences between the fields. More broadly, introducing and staffing a new discipline means that every decision comes with administrative responsibilities, and people management becomes a central part of the job.

[RM]: *And Sylvie, could you say a bit more about the math museum you designed? Did working with people from different backgrounds lead to any unexpected challenges in this case?*

SBG: It did! The Maison Poincaré is about mathematics and its applications, but the team was very diverse. There were mathematicians, of course, but also science communicators, journalists, and people from companies supporting the project. We also had input from people with expertise in architecture but no background in mathematics. So, we had to constantly negotiate what a "mathematics museum" should be. At one point, someone insisted: "There should be parabolas!", because this was their image of mathematics.

HH: That's really interesting, because I experienced something similar. When setting up the department here, we worked with an architect, and our main request was very simple: we wanted large blackboards. But that required a lot of discussions, and I realized that creating a math environment while taking into account what other people think math is really fascinating. It's funny because,

Sylvie. (Photo by S. Benzoni, courtesy of Sylvie Benzoni-Gavage)

in the end, mathematicians are very simple: we just want time to do math, to collaborate, and to be able to think through things. Blackboards are indispensable!

[RM]: *I can totally relate, as I ended up building my own blackboards at home! Now, going back in time, how did you first discover your passion for mathematics, and how did your path lead you to where you are today?*

SBG: Although my family would laugh if they heard this, I'm not sure I would call it a passion. I can say that I have liked mathematics for almost all my life, and at school, it quickly became my favorite subject, so I followed that path. Up to my early twenties, I was actually planning to become a teacher, and only then did I discover that research in mathematics exists.

[RM]: *And what is missing for making it a passion?*

SBG: I can become really obsessed about it, but sometimes I need distance and I want to think about something else.

[RM]: *I think you can still call it a passion! How about you, Heather?*

HH: For a long time, I wanted to become a medical doctor. I liked mathematics, but I didn't know research in mathematics existed

Sylvie glazing ceramic.
(Photo by S. Benzoni, courtesy of Sylvie Benzoni-Gavage)

either. It was really through taking more courses during my pre-med studies that I became interested. Later, I discovered that you could combine mathematics with biology and biomedicine, which I hadn't thought possible before, and that changed everything. For a long time, I still thought I would eventually go to medical school, but I just kept going in mathematics, and I never left.

[RM]: *And how do you feel about that plan now?*

HH: I still work closely with clinicians, so I feel connected to medicine in a different way. Also, mathematics offers flexibility that I really value. And it has a unique pace, as you can move slowly and quickly at the same time. I don't know how to explain it, but I like it. I like where I am now.

[RM]: *That's beautiful. Sylvie, you mentioned that you need breaks from mathematics and sometimes want to think about something else. What is this "something else?"*

SBG: I think I don't have a passion as such. I just like to do different things, and it varies over time. Some of them are still connected to mathematics: when we were designing the museum, I started making mathematical objects, and I enjoy that very much. In fact, I can even become a bit obsessed with it. I also like being outside, gardening, and doing a bit of sport.

I can't help but feel that what she calls "not a passion" might, in fact, be a passion.

[RM]: *How about you, Heather? What recharges you outside mathematics?*

HH: I also really like being outside and walking. It helps to change perspective and clear my mind. I think many mathematicians feel that way. I used to play the piano, and now that my son is playing instruments, I've started playing piano again to accompany him. This morning, for example, I was playing before starting my day.

[RM]: *That's really nice! Now, Heather, knowing you, I feel that you often have very thoughtful advice to offer, so let me ask both of you: what advice would you give to young researchers starting their careers today?*

SBG: It's a difficult question, because the situation is changing a lot, especially with AI. But one thing I would say is: don't do like me. When I was younger, I often refrained from talking to people. I thought I was shy, and I stayed in that mindset. Looking back, I think I should have asked more questions and reached out more. It would have helped me build more collaborations.

[RM]: *That's interesting. But it worked for you anyway!*

SBG: Yes, but it was tough at the beginning.

[RM]: *Heather, you are quite the opposite of shy, I would say.*

HH: It's interesting that you say that, Raffaella, because I was also very quiet at the beginning. Then my PhD supervisor told me: *you must learn how to interrupt.* I really struggled with that at first, but then it changed things for me.

Going back to your question, I think one key piece of advice is to develop critical thinking, which is one of the superpowers of mathematicians. And as things change, especially with AI, that becomes even more important.

Another thing is to build a community. You need people around you who know your work and with whom you can exchange ideas and develop your thinking.

And finally, it's essential to make time to read. It's easy to get overwhelmed, but staying connected to the literature is crucial.

Heather baking her son's birthday cake.
(Picture courtesy of Heather Harrington)

Heather paying respect to René Descartes in Paris.
(Picture courtesy of Heather Harrington)

SBG: Heather, I wish I had the advice that your supervisor gave you! It would have changed things for me. And regarding your point about building a community: maybe young people could ask, how do you make this community? How do you come to belong to one?

I love that, at this point, they interview each other.

HH: Reach out! Email people. Connect with them through workshops, schools, and collaborations, especially early on in your career.

[RM]: *And by interrupting people! Now, you have both shared a few anecdotes already. Would you like to share another story that has had a lasting impact on you, or shaped the way you work? Something uplifting or challenging that has stayed with you over the years.*

SBG: I can think of something which is, in some sense, an uplifting story. At the beginning, my career looked very simple: after my PhD, I was hired by the CNRS as a full-time researcher. But I hadn't planned that path, I wasn't very happy, and for years, I was trying to find something else. But in France, if you want to progress, you are often expected to move, and for personal reasons I didn't want to. So I felt a bit stuck.

Then, I saw an interesting position. I thought I had no chance, but I applied anyway. At the time, applications had to be sent by regular mail. I went to the post office two minutes before closing, on the day of the deadline, and I got the position. In fact, it's the one I still have now.

This taught me that sometimes it's worth trying, even when you believe your chances are very small. And now that I sit on hiring committees, I'm very aware of how much people's lives can depend on very small details. When choosing between candidates, it often comes down to tiny differences, and yet those decisions can change everything.

HH: That's very true. There's also a lot of luck involved. Many strong candidates apply for the same positions, and very small things can make a difference.

[RM]: *How about you, Heather? What has had a lasting impact on you?*

HH: I think what has shaped me most is being surrounded by inspiring and supportive people: mentors, colleagues, peers. Not every day is a good day, and not every week is a good week, so having that kind of support system really matters.

[RM]: *That's so true! Time is running out, unfortunately. Shall we wrap it up here?*

HH: No, Raffaella – I would have expected one more question about our research!

Sylvie woodturning.
(Photo by C. Naessens, courtesy of Sylvie Benzoni-Gavage)

Sylvie gardening. (Photo by S. Benzoni, courtesy of Sylvie Benzoni-Gavage)

[RM]: *Haha, fair enough! What developments in your field are you most excited about right now?*

SBG: Well, I'm working on old problems. And in some sense, I like that: they've been around for decades and are still not solved, which makes them very exciting. But it's also not very comfortable, because you think: if nobody has solved them, how could I find something new?

Also, if I compare our fields, Heather, I appreciate the fact that I don't feel pressure related to people's lives. I find your area very interesting, but I wouldn't like to work on applications to medicine or biology. Physics is fine for me!

HH: Broadly, I'm very interested in the combining of different types of data, and how to think about that geometrically or topologically.

One of the biggest challenges in science right now is that we are generating data at a massive scale, and even for a single patient, you can have many different types of measurements, taken over time, with very different interpretations. So the question is: how do you extract meaningful information from such data?

Making sense of this – using for instance geometry, topology, algebra, and statistics to capture structure across scales or uncover relationships beyond the reach of other methods – is extremely exciting. It requires bridging different areas of mathematics, and connecting mathematics with fields like biology or medicine. You need a common language, as these are deeply collaborative problems, and they require a melting pot of expertise.

Heather looks much happier now. Since we have already gone over time, I sneak in one last question.

[RM]: *Looking back at your PhD years, is there something specific that shaped you, mathematically or personally?*

HH: It was the first time I worked with experimental data, and that was extremely challenging. I also met my husband, who was another PhD student at the time, and that changed my life.

At the same time, my supervisor became terminally ill, which was a very difficult experience. It forced me to become extremely independent. I think some people experience that kind of isolation later in their career – for example, when they first become faculty – but for me, it came much earlier, during the final years of my PhD. In some sense, that experience also influenced how I approach

Heather cycling. (Picture courtesy of Heather Harrington) my work now: learning to look for structure and insight even in situations that feel complex or uncertain.

[RM]: *I'm sorry to hear that, and thank you for sharing it. How about you, Sylvie?*

SBG: My PhD was quite different from the typical one. I was working on a topic connected to industry, in collaboration with a public company in the oil production sector, while at the same time having an academic advisor. So I was moving between two rather different worlds; I had practical reports expected by the company, and more theoretical work with my advisor.

Everything also moved quite quickly. After less than two years, my advisor told me I could finish my thesis within a few months.

[RM]: *Thank you both for your time, for sharing so many insights about your journeys, and for your openness.*

Raffaella Mulas is a discrete mathematician but not a discreet one, as she loves listening to and sharing people's stories. She is an assistant professor at VU Amsterdam, where her work is supported by a VENI grant from the Dutch Research Council (NWO). She is also an elected member of EMYA and the founder of *Nora – Center for Science Communication*.

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Under the hood of a carbon footprint calculator

Indira Chatterji, Ariadna Fossas Tenas and Elise Raphael

We explain the basic linear algebra formulas used in carbon footprints computations.

Have you computed your carbon footprint lately? Wondered where all those numbers came from? Two main methods for calculating carbon footprints are the *input-output* method and the *life-cycle analysis* method, also known as *cradle-to-grave*. Both aim to quantify the emissions of a product or service, but the methods are fundamentally different.

Input-output is an economic approach developed in [2], widely accepted since the 1960s, but only recently adapted to environmental reporting. It uses national or regional data on industries and their interdependencies. It relies on aggregated data about economic sectors (e.g., agriculture, manufacturing) and their emissions, and we shall see in this note (Definition 2), the formula estimating emissions associated with production, supply chains, and services. This is the commonly accepted method for large-scale, economy-wide assessments, is not data-intensive, and relies on publicly available economic data. It will lack detail on specific products and miss the emissions from smaller, niche activities, but offers the warranty that all emissions are taken into account (Proposition 1).

Life-cycle analysis estimates a specific product's environmental impact over its entire life cycle, from raw material extraction to disposal. However, the under or over counting of secondary emissions is only estimated, and the total complexity makes a consistency check impossible to actually carry out.

A concrete question. To estimate the carbon footprint of a T-shirt, for instance, one needs the footprint of the cotton all the way from production of the raw material up to the stitching of the T-shirt, the washing during its lifespan and the disposal of the unusable T-shirt. But to estimate the cotton footprint only, one needs the transport of the water, the production of the pesticides to protect the crop, the fertilizers, all those requiring tractors and fuel, the fuel itself requiring transport and fuel, thus creating loops that are difficult to extricate. And once having estimated all the emissions, how do we know that all the emissions of all the objects that we so compute, actually sum up to all the emissions that we actually produce? The input-output method takes the problem from the other end: it takes *all* the measured emissions, and then attributes them to different sectors, and then computes the carbon footprint of goods from the textile industry according to its direct emissions, combined with all the indirect emissions, as we now explain.

Basic input-output. This method models a closed economy with *n economical sectors*, $S = (S_1, \dots, S_n)$, and comes in the following form:

	S_1	S_2	$\dots$	S_n	D	T
S_1	c_{11}	c_{12}	$\dots$	c_{1n}	d_1	t_1
S_2	c_{21}	c_{22}	$\dots$	c_{2n}	d_2	t_2
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
S_n	c_{n1}	c_{n2}	$\dots$	c_{nn}	d_n	t_n
V	v_1	v_2	$\dots$	v_n		
T	t_1	t_2	$\dots$	t_n		

The $n \times n$ *transaction matrix* is $C = (c_{ij}) \in M_n(\mathbf{R}_+)$ (positive coefficients only), where the entry c_{ij} is the number representing the economical transactions from sector S_i to S_j. The diagonal entries c_{ii} then represent the internal gross revenue of the sector. The *demand vector* $D = (d_1, \dots, d_n)$ expresses the money spent by the population on each sector: d_i is the amount bought from sector S_i, for $i = 1, \dots, n$. The *added value* vector is given by $V = (v_1, \dots, v_n)$ and the *total* vector $T = (t_1, \dots, t_n)$ satisfies the relations

$$t_i = \sum_{j=1}^{n} c_{ij} + d_i = v_i + \sum_{j=1}^{n} c_{ji} \qquad \text{for } i = 1, \dots, n.$$

The value t_i represents the *total output* of the sector S_i, which is also the *total input* because the system is a closed circuit.

Definition 1. The *technical coefficients matrix* $A = (a_{ij})$ is given by

$$a_{ij} = \frac{c_{ij}}{t_j},$$

so that $A = CD_T^{-1}$ where D_T denotes the diagonal matrix with entries $t_1, \dots, t_n$ on the diagonal.

Recall that the scalar product of two vectors $X = (x_1, \dots, x_n)$ and $Y^t = (y_1, \dots, y_n)$ is given by

$$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = XY,$$

where X is a *row vector*, i.e., a $1 \times n$ matrix, and Y is a *column vector*, i.e., an $n \times 1$ matrix and XY is the usual matrix multiplication, so it is a real number.

Footprints. Economic activities have all sorts of footprints that are correlated to the income and the cost of things. To compute the footprint, we need to have an *emission* vector $E = (e_1, \dots, e_n)$, where e_i stands for the direct emission of sector S_i. It could be millions of tons of CO_2, measured or estimated directly above the industries of the sector, or square feet measured by the occupancy of the industries, or tons of plastic trash estimated by the waste management authorities/actors, etc. Then $|E| = e_1 + \cdots + e_n$ is the total footprint of the economic system. The *direct intensity* is given by $F = (f_1, \dots, f_n)$, with

$$f_i = \frac{e_i}{t_i},$$

where e_i is the ith component of the emission vector E, and t_i the one of the total vector T, that is the total output of the sector S_i. In other words, f_i is the print per unit of money. Now, if we compute the inner product

$$\langle D, F \rangle = \sum_{i=1}^{n} d_i f_i,$$

it represents the footprint directly attributed to the consumers, but it does not take into account all the intermediate exchanges that the sectors had in order to create the product. If you buy goods for k dollars to industry S_i, then kf_i indicates the direct footprint of the purchase, but not the total footprint of the goods. The latter has to take into account the exchanges the sector S_i had with the other ones.

Definition 2. The *total intensity vector* $X = (x_1, \dots, x_n)$ giving the total footprint of the sector S_i by money amount of final consumer demand, is given by

$$X = F + FA + FA^2 + \cdots = F(I - A)^{-1},$$

where I denotes the $n \times n$ identity matrix and A the technical coefficients matrix from Definition 1.

Notice that since most matrices are invertible, we will assume that this one is.

Proposition 1. *The total footprint is completely attributed to the consumers. Namely, with the above notations, $\langle X, D \rangle = |E|$.*

Proof. First notice that the equations $t_i = \sum_{j=1}^{n} c_{ij} + d_i$ for $i = 1, \dots, n$ can be recast as

$$D = T - C\mathbf{1} = T - AD_T\mathbf{1} = T - AT = (I - A)T,$$

where $\mathbf{1} = (1, \dots, 1)$, T is the total vector and D is the demand vector, both viewed as column vectors. We compute:

$$\langle X, D \rangle = \langle F(I - A)^{-1}, (I - A)T \rangle = FT = \sum_{i=1}^{n} \frac{e_i}{t_i} t_i = |E|. \qquad \blacksquare$$

Waste. In a closed economy, in order to take into account waste disposal, one needs a sector S_i that runs the waste management. That sector has a large footprint since incinerators produce large quantities of CO_2 (of the order of 1 kg CO_2 per kg of waste incinerated[1]), and is eventually attributed to the consumers through waste management taxes (in France the weight of waste is estimated at half a ton per person and per year). The model also does not discuss demand and waste, assuming that everything is bought and nothing gets wasted.

Corporations. The carbon footprint inventory of a specific firm or corporation is separated into its direct emissions or *scope one*, and its indirect emissions. The latter are separated as *scope two*, consisting of the emissions from acquired electricity, heat, and cooling, and the rest is *scope three*, computed using databases of life cycle analysis.[2]

Inverses. If B is a matrix ε-close to A (say, in terms of the coefficients), then the distance between $(I - A)^{-1}$ and $(I - B)^{-1}$ could a priori be as big as we want, for instance if the matrices both have an eigenvalue close enough to 1. Since all the values are estimated, the model is probably not very stable in an optimization exercise.

Added value. The consumer footprint does not take into account the added value V, which is what the companies make, and the model attributes the footprint solely to the consumers, through the vector D. We could define the *total systemic vector*

$$Y = F(I - A^t)^{-1}$$

and check that $\langle Y, V \rangle = |E|$. The vector Y distributes the total footprint to the added value of the various sectors. That attribution

[1] Municipal waste statistics: https://ec.europa.eu/eurostat/statistics-explained/?title=Municipal_waste_statistics.

[2] Greenhouse Gas Protocol: https://ghgprotocol.org/standards-guidance.

seems more natural than the consumer one, as it links the footprint to the benefits of the companies.[3]

In concrete terms. The tables are quite available, see, e.g., for Switzerland,[4] but a measure of environmental impact for each sector appearing in the table is harder to obtain, especially as it needs to be computed/collected in a homogeneous manner over the different sectors in order to make sense.

Acknowledgements. This note is based on the article by J. Kitzes [1], the talk by J. Steinberger at the Climathics Workshop in 2022,[5] and the third author's talk for the *Commission Romande de Mathématique* in 2024.

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[3] See also the 2021 opinion piece: https://www.nytimes.com/2021/08/31/opinion/climate-change-carbon-neutral.html.

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[5] https://www.la-matrice.ch/climathics-workshop

In memoriam Klaus Krickeberg (1929–2025)

Helga Zeile, Bernhelm Booß-Bavnbek, Klaus Dietz, Nguyễn Xuân Xanh and Hans Zessin

On 27 November 2025 the distinguished mathematician and outstanding teacher Professor Klaus Krickeberg passed away at the age of 96. His broad scientific interests concerned the fields of analysis, martingales, stochastic geometry and point processes, and moreover epidemiology and medical statistics. To him mathematics was a reflection of reality, in which he developed gradually from the abstract to the concrete. Geometrical insight is visible in all his work.

At a meeting of the Oberwolfach Committees 2016. (© Gerd Fischer, source: Archives of the Mathematisches Forschungsinstitut Oberwolfach)

1 Klaus Krickeberg's life

Klaus Krickeberg left a lasting impression not only in his scientific work but also on everyone who knew him: family, friends, students, and colleagues. Science was a defining part of his life, but it never stood alone. He cared deeply about books, languages, history, and politics as well. He was versatile, modest, often humorous, and always ready to help. Above all, he was authentic, whether he was speaking at a stochastics congress, discussing healthcare in Vietnam, enjoying a meal with friends, or repairing his summer house in the Auvergne.

His life spanned ninety-six years. There were decades of political upheavals, wars, and global events of profound significance, as well as technological turning points. He lived through all of this with a kind of inner compass. From an early age, he showed an unusual clarity of mind and the talents that later shaped his work in research and teaching.

Born in 1929 in Ludwigslust, Klaus grew up in a medically oriented family. His father was a radiologist and his mother a medical technical assistant. Three years later his brother Dieter was born. When Klaus was four, Adolf Hitler and the Nazi Party came to power. At home, the family rejected the regime, but the children learned early that political criticism could not be voiced openly.

His intelligence became evident early on. After only three years of primary school in Berlin, he skipped the fourth year and entered the Collège français, a secondary school founded in 1689 for the children of Huguenot refugees, among them his maternal ancestors. Classes were taught in French, and this is where his lifelong connection to France began. He later taught for twenty-five years at the René Descartes University in Paris and, until two years before his death, spent his summers at his house in the Auvergne.

Even during the Nazi era, the Collège maintained a relatively open spirit, quite different from the surrounding dictatorship. As air raids on Berlin intensified, the pupils were evacuated to Züllichau, today Sulechów in Poland. There Klaus repaired a radio so that his teacher and classmates could listen to the BBC. They gathered in the teacher's room to hear news from what they called the free world.

In early 1945, as the war was ending, Klaus left Züllichau with his headmaster's permission and travelled with his brother to Ingolstadt, where his mother's hospital had been relocated. He narrowly escaped conscription, having turned sixteen on 1 March 1945. When American troops arrived, he knew enough English to point to the building and say, "That is a hospital." He did not learn languages for display. He learned them because he wanted to use them.

School resumed in Berlin later that year. Klaus completed his secondary education in 1946 and began university studies that autumn. At first, he chose physics, but he soon realised that mathematics was what truly interested him. The lectures of Erhard Schmidt impressed him deeply. Years later he still described them as "the most wonderful lectures I have ever heard, delivered entirely from memory." After that, his path was clear.

The post-war years were difficult. There was hunger and cold, and life was not easy. Nevertheless, he completed his studies in 1951, earned his doctorate at twenty-three, and soon after completed his habilitation. By twenty-nine, he had become one of the youngest professors of his time. His first professorship was in Würzburg. Later he taught for many years as a full professor in Heidelberg.

His work took him to many countries. He spent extended periods in the United States, first on a Fulbright scholarship, and later an academic year in Aarhus. He did not like idleness. He worked wherever he happened to be, on trains and ships, in waiting rooms and hotel rooms. And if he was not working, he was learning languages. Before moving to Denmark, he learned Danish. Before travelling to Cuba with his French colleague Didier Dacunha-Castelle, he learned Spanish so that he could lecture in Havana and speak with local mathematicians. For conferences in the Soviet Union, he learned Russian. For him, language was simply the way to meet people directly.

This openness led to many encounters. He met and collaborated with Kolmogorov. In Latin America, members of an indigenous community once offered him a ride on their truck, surprised that a European treated them without distance. In Cuba, he sat on the beach with Fidel Castro and spoke with him. In 1973, he was in Chile shortly before the coup of 11 September, at a time when the tension could already be felt.

Political questions mattered to him. Around 1970, he met the French mathematician Laurent Schwartz, who had written an appeal against the American war in Vietnam. Klaus was the only West German mathematician to sign it. From then on, Vietnam became increasingly important to him.

Through one of his Heidelberg students, Xanh, in 1974 he travelled by train to Vietnam to teach mathematics in Hanoi. After that, he returned year after year to support Vietnamese scientists. Over time, his commitment expanded beyond mathematics. In the final decades of his life, improving healthcare and health education in Vietnam became a central focus. He worked to strengthen epidemiological teaching, co-authored books with Vietnamese colleagues, gave lectures, and organised workshops.

His private life was equally full. He married twice and had five children, later grandchildren and great-grandchildren. The last thirty years of his life he spent with his partner Helga, who accompanied him on his journeys to Vietnam and supported his work.

Work was his life. It stayed central to him until the very end. Each morning at home began at his desk. Those hours were reserved for concentration. He worked slowly and carefully and finished what he had set out to do. In his final months, when failing eyesight and hearing made reading and writing impossible, he felt the loss deeply. "It is terrible that I can no longer work" he said.

2 The Bielefeld plan of mathematisation of other disciplines

When Krickeberg accepted a position in Bielefeld in 1971, some of his colleagues and friends, his students and staff may have wondered why such a productive and renowned scientist would leave a leading university in mathematics and natural sciences such as Heidelberg for what then still was a small academic centre such as Bielefeld.

The answer can be found in a 1966 memorandum commissioned by the state government and drafted by sociologist Helmut Schelsky jointly with former Minister of Education Paul Mikat, which henceforth served as the official basis for the establishment of the new University of Bielefeld. In the *structural features* of the new university, the authors emphasised the focus on promoting the mathematisation of the individual sciences. This distinct feature of Bielefeld University was continued year after year in academic committees in Bielefeld and repeatedly reaffirmed by the state government: "One of the university's focal points is mathematisation as a method in the sciences, i.e., reach into practical applications of mathematics to non-mathematical problems."[1]

Krickeberg's ideas, on the other hand, were always very down to earth and concrete; he needed a continuum of competencies:

(I) Firstly, he needed sufficient personal in mathematics to provide advice and assistance to researchers in other disciplines and to teach the necessary basic courses in applied mathematical areas (such as probability theory, statistics, dynamical systems, basis types of partial differential equations, shape and pattern formation, frequency analysis, symmetries, selected algorithms) both within the curricula for mathematics and the curricula of the individual subjects.

(II) He also needed mathematicians who were willing to thoroughly familiarise themselves with areas of application in order to find applications for deep-rooted and perhaps esoteric seeming purely mathematical results.

(III) Finally, regarding hiring, e.g., in biology, chemistry, economics, sociology, psychology, linguistics, and history, Krickeberg suggested that applicants with substantial experience with mathematical tools like stochastic methods, modelling of dynamical systems, algebraic analysis of symmetries and algorithms, or numerical simulation especially should feel invited.

All of this was already included in general terms in the structural characteristics and had been promised to Krickeberg. Research and teaching in mathematisation were to be achieved through research funding and, in particular, personal policy strictly oriented towards this goal. Mikat and Schelsky had based their approach on their experience with US research funding.

At the same time, however, they had also advocated in the structural characteristics for extreme preservation of the Humboldtian European tradition of strong autonomy for university committees.

[1] Cf. Grußwort of K. P. Grotemeyer in [1].

This was realised, for example, with a close focus on academic excellence and international recognition in narrow and narrowest circles. Hence nothing came of the targeted focus on mathematisation. Nothing was realised. It all remained on patient paper.[2] After only three years, Krickeberg therefore accepted a call to Paris.[3]

3 Scientific work and its application

Mathematics

The subject of his doctoral thesis in 1952 were the Gauß–Stokes integral theorems, published in [4]. Part I was motivated by the desire *to obtain the classical Gauß–Green formula in* $\mathbb{R}^d$ *under conditions that are necessary, too.*[4] Part II contains *the first definition and study of locally Lipschitzian manifolds, which were to be investigated in the sequel by several topologists.* Stokes' theorem is proved for differential forms on such manifolds under minimal assumptions. Part III is purely historical. *It traces the ideas underlying the proofs of the various forms of the Gauß–Stokes theorems, starting with early papers by Lagrange (1760), Gauß (1813), Green (1828), Ostrogradsky (1831 and 1838), and Stokes (1854).*

Krickeberg took up again the Gauß–Green theorem in the influential paper [6], but *under a very different angle, viz. via distributions in the sense of Laurent Schwartz. As a main tool it employed the decomposition of measures that was to play once more a decisive role in the solution given in* [9] *of a problem formulated by Rollo Davidson.*

In the 1950s, Krickeberg also spent a year in Illinois in order to work with Doob. *I still vividly remember my first meeting with Doob in his office in the University at Urbana, Ill. When I presented my results to him he said 'Oh, that's interesting' and suggested that I publish them in the Transactions of the American Mathematical Society* [5]*.* The work in question contains the result which now is called *Krickeberg decomposition* in martingale theory.

Paper [8] was motivated by a problem stated by Eberhard Hopf in Chapter 17 of his classical book on ergodic theory. *While the so-called baker's transformation in the unit square was well understood, and in particular known to be mixing, the nature of the corresponding transformation in the entire plane was hardly elucidated and the problem of its mixing properties was open.* The main contribution of this paper is twofold: *Firstly, it was recognized that the natural setting for defining mixing properties of transformations (endomorphisms) of measure spaces with* infinite *total measure are* topological *measure spaces, not pure (abstract) measure spaces, and the endomorphisms in question are to be not only measure preserving but also almost everywhere continuous.* The second main contribution of [8] is *the construction of isomorphisms between a subset Y of the plane endowed with (infinite) Lebesgue measure and the measure space* (X,m) *derived from a suitable Markov chain as above, such that, for example, the baker's transformation in Y corresponds to the shift T in X. By applying known results from the theory of Markov chains to T, one finally obtains a precise description of the type of mixing including the ergodic index of the baker's transformation.*

In the 1970s, Krickeberg became intrigued with stochastic geometry[5] and its integral-geometric foundations,[6] and more generally, point processes and the statistical analysis of such processes, in particular oriented hyperplanes in d-dimensional space $\mathbb{R}^d$. Together with R. Ambartzumian, Davidson, D. G. Kendall, Matheron and Papangelou, he is one of the founders and a recognized master of these new fields. (Cf. [3, 16].)

The problems considered arise if one does not specify the distribution of the underlying process in advance, but requires instead certain geometrical properties, in particular invariance properties, and then investigates their implications on the structure of the distribution. This development was essentially initiated by Rollo Davidson's thesis.

Krickeberg's main achievements, which were tremendously influential, can be found in [9, 10]. In [9] he used the method of decomposition of measures, used already in [6], to solve a problem posed by Rollo Davidson about the correlation measure of a second-order line process in the plane.

Davidson has also stated a conjecture to the effect that every non-degenerate strictly stationary second-order line process in the plane is a Cox (doubly stochastic) process. Kallenberg constructed a counterexample, but Davidson showed that *for every second-order line process with stationary correlation measure which has a density, there exists a Cox process with the same correlation measure.* This was extended by Krickeberg in [10] to higher-order point processes in a fairly general space X and an equally general transformation group G acting in X: *Let k be a positive integer. Then for any point process in X with a locally finite G-invariant k-th moment measure there exists a Cox process that has the same k-th moment measure.*

[2] At least, Krickeberg brought together a multidisciplinary mathematisation commission, which was intended as an advisory body for tenders and appointments and, in a series of hearings on the role of mathematics from the perspectives of the individual sciences, qualified for this task, even though it was not involved in a single personal decision. (See [1].)

[3] https://www.math.uni-bielefeld.de/chronik/mathematisierung.php, last section with Krickeberg's own recount (accessed 6 February 2026).

[4] The texts in italics are citations from Krickeberg [13].

[5] We cite from Santaló's book *Integral Geometry and Geometric Probability* [16]: "In a symposium on integral geometry and geometric probability held at Oberwolfach (Germany) in June 1969, D. G. Kendall, K. Krickeberg, and R. E. Miles suggested the term *stochastic geometry* to indicate precisely those contents of geometry and group theory that are in a sense related to stochastic processes."

[6] Rooted in the approach of Blaschke and Santaló to integral geometry.

The statistical analysis of point processes was treated in the book [11].

Mathematical teaching

Krickeberg's teaching reached a wide audience by means of locally published notes written for his lectures in Chile, Cuba or Vietnam and always in the local language. Here he was able to recast complicated and unarranged material in an accessible and elegant form, thereby setting clear lines for future research.

But also by means of his three books on probability and statistics (cf. [7, 12, 15]). The first, written in 1963 in Heidelberg, was one of the first books on probability theory in Germany after the Second World War and highly appreciated as a modern and rigorous text for students in mathematics of only two hundred pages. The second book, written in Bielefeld, is an introduction to elementary probability and statistics; as such, it was the first propedeutic book in Germany which then lead to similar changes of the probability curriculum in other German mathematical faculties. The third, written in French language at the university René Descartes in Paris, develops elementary statistical methods for students of mathematics or physics having a certain mathematical culture.

Epidemiology and public health

The following information about Krickeberg's achievements in public health and epidemiology and his ties to Vietnam are partly based on his unpublished talk at the Kallenberg Workshop in 2013.

The contacts with two Vietnamese physicians changed his scientific interests completely. In the summer of 1978 he went to Hanoi to work with the Hygiene Institute. He presented a memorandum on how mathematical work in the institute might be organised. In winter 1979–1980 he gave a course on sample survey methods at the General Statistical Office, probably the first course on that subject ever given in Vietnam.

He got a grant from the French Ministry of Foreign Affairs for a program over ten years on mathematical methods in health in Vietnam. It started in 1981 and consisted of two parts:

(1) Seminars for members of the health system on all mathematical aspects of their work. They were taught by some French colleagues and himself and took place in all institutes of the hygiene and epidemiology network, from Hanoi down to Ho Chi Minh City.
(2) His work with a group of four mathematicians, selected by NIHE (National Institute of Hygiene and Epidemiology) for the purpose of "mathematisation" of public health, especially of statistical approach. It dealt with concrete and urgent practical problems as they arose and at these occasions he pointed out to them the underlying general ideas, mainly from epidemiology.

He told himself: "First you have to learn that stuff yourself." Therefore, he offered to his students of applied mathematics in Paris a course on epidemiology in the wide sense, including clinical epidemiology and in particular clinical trials. It comprised six hours a week of theory and practice during the whole academic year. There was a lot of statistics in it but also for example discrete modelling in population genetics and modelling via differential equations of the evolution of epidemics, both deterministically and stochastically. He taught this course more than a dozen times until his retirement in 1998. It turned out that the students who had followed it successfully were very much sought after in the French health system. They readily found positions in big hospitals, health administrations and medical and public health research institutes.

He made up his own definitions of public health and epidemiology: *public health* is the entirety of theoretical and practical activities that are related to health and deal with populations as a whole but not specifically with their individual members. *Epidemiology* is the science of the distribution of diseases and of similar health-related features in human populations and of the factors that influence such distributions.

This shows, in the first place, that epidemiology is to a large extent statistics, even fairly advanced mathematical statistics. In the second place the two definitions imply that epidemiology is part of public health.

Krickeberg became a UNICEF advisor in Vietnam and Cambodia until 1987 for various topics in primary health care. It was a most interesting experience because it meant much fieldwork, that is, visits to village health stations and families.

In 1994 the German "Society for Technical Cooperation" (GTZ) hired him for its program on family planning in Vietnam. There was again some non-trivial mathematical statistics in it, namely sample survey methods and demography. Finally, he worked in 1999 and again in 2004 in the European–Vietnamese "Health Systems Development Program" on "Health Information Systems." All of this practical work taught him that rigorous thinking following the ways of mathematics is of utmost importance, even in seemingly simple fieldwork.

In 2005 Krickeberg went again to Vietnam, for the first time as a tourist, and he visited his old friend Hoàng Thny Nguyên who had retired from the directorship of the National Institute of Hygiene and Epidemiology. He told him that one of the smaller medical universities in the North-eastern region of Vietnam, Thai Binh University of Medicine and Pharmacy, wanted help with the teaching of public health to its students.

He started a program, which in the course of time came to cover all medical universities or faculties of Vietnam except the two big centres Hanoi and Ho Chi Minh City. It was funded by the German foundation "Else Kröner-Fresenius-Stiftung." In the first phase it focused on the teaching of public health to medical students during their six years of basic studies. In Vietnam this basic medical curriculum includes in fact courses on most topics of public health, which is very reasonable because physicians in the countryside or in small cities also have to do a lot of work in public health, for example managing a village health station, promoting health education, prevention including classical hygiene.

In the first phase of their program they widened the horizon of the lecturers and made them work more independently and more rigorously. They did it in two ways. Firstly, they ran yearly workshops in Vietnam where they did not teach the participants any particular matter but had some of them present their own work. Above all, they discussed and worked together on some topic, for example on the structure of a curriculum or on new teaching methods such as "population-side teaching." Excursions to small health facilities and rural families were always included.

Secondly, they wrote new texts on many topics together with Vietnamese lecturers. They always started with a critical analysis of the existing text. In the new text, examples, applications and case studies were mostly taken from the Vietnamese health system; foreign situations were being compared with the corresponding Vietnamese ones. All books are bilingual: Vietnamese and English version appearing in a single volume. In order to accommodate all of these books and to make it clear that they form a coherent body of texts, Krickeberg founded a book series in the Medical Publishing House of Hanoi; it is called "Basic Texts in Public Health."

The second edition on epidemiology, key to public health [14], appeared in 2019, which Krickeberg considered his best book. From the back cover: "The organization of the topics by didactic aspects makes the book ideal for teaching. All examples and case studies are situated in a single country, namely Vietnam; this provides a particularly vivid picture of the role of epidemiology in shaping the health of a population. It can easily be adapted to other developing or transitioning countries."

Krickeberg, together with Booß-Bavnbek, published an article about Covid-19 [2]. Their main point is the critical need for systematic mathematical modelling to combat the Covid-19 pandemic. The authors argue that, in the context of Covid-19, it is essential to take into account spatial inhomogeneities (hot spots, super-spreaders and the functioning of local health authorities) in order to understand chains of infection, make predictions and guide political decisions. They base their analysis on principles of classical medical statistics and descriptive epidemiology. Their arguments utilize demographic data and established mathematical structures for infection dynamics to evaluate prevention strategies and vaccine efficacy.

4 Honours received by Klaus Krickeberg

Klaus Krickeberg became a fellow of the Institute of Mathematical Sciences in 1968. He was elected into the International Statistical Institute (ISI) in 1971, was member of its Council 1985–1989 and chairman of its committee for the development of statistics in developing countries 1987–1991. He was president of the Bernoulli Society for Mathematical Statistics and Probability 1977–1979, contributed to the establishment of regional committees in Latin America and East Asia and was chairman of the program committee of the first world congress of this society in Tashkent in 1986. In 1983, he was elected into the German National Academy of Sciences Leopoldina.

Klaus Krickeberg received an honorary doctorate (Dr. rer. soc. oec. h.c.) from the Faculty of Social and Economic Sciences at the University of Vienna in 1990. This honour recognized his outstanding scientific contributions as a mathematician, particularly in the fields of probability theory and statistics, as well as his interdisciplinary work. He was elected as a fellow of The World Academy of Sciences (TWAS) in 1994.

He has received several distinctions for his activities in Vietnam. In the year 2009 it was the Medal of the Ministry of Health for Contributions to the Health of the Population. Then, in 2011 the National University of Sciences of Ho Chi Minh City bestowed on him its honorary doctorate for his work in mathematics, and in 2015 the Thai Binh University of Medicine and Pharmacy named him honorary professor. In 2018, he received the Friendship Order, the highest order bestowed by the Vietnamese government on foreign individuals by the President of Vietnam for "many positive essential contributions to the development of the Vietnamese health sector."

There was also a distinction that he received already in the early 1980s. One morning a house messenger who arrived at the same time as he by bicycle at the gate of the National Institute of Hygiene and Epidemiology in Hanoi said to him in Vietnamese "Krickeberg is a Vietnamese man."

At his funeral service, leading Vietnamese authorities, including the Vietnamese Consulate General, expressed their condolences and honoured him as a bridge figure in German–Vietnamese friendship. Klaus Krickeberg's work in Vietnam was much more than an advisory activity; it has been a lifelong mission to use statistical science as a tool for social justice and health.

The International Statistical Institute (ISI) dedicated a detailed "In memoriam" article to him, emphasising his worldwide importance for mathematical statistics.

5 Krickeberg, a humanist

Galen, a famous Greek physician during the Roman era, believed that a good doctor is a philosopher. On the other hand, according to Pythagoras, a mathematician is also a philosopher. Hence, a mathematician could also contribute to "healing" people. Krickeberg was such a person. His applied mathematics could help prevent disease, saving thousands of lives. Mathematics is the central point in sciences, theoretically and practically, to understand natural and social phenomena and bring them under control, and in the case of public healthcare, to bring the spread of infectious diseases under

Klaus Dietz (left), Klaus Krickeberg, Nguyễn Xuân Xanh, Hans Zessin; photographed by Helga Zeile. This group came together in Da Lat (Vietnam) on 1 March 2009 for Klaus Krickeberg's 80th birthday.

control. While doing this, Krickeberg's activities lay the foundation for statistical mathematics in Vietnam and to improve education in this area. He has uncountable students during his fifty years of work in Vietnam. He is a great teacher.

To improve health education, Krickeberg created a publishing group of young active Vietnamese colleagues, mathematicians, and well-trained doctors, to work out a very ambitious series of textbooks with state-of-art scientific knowledge replacing outdated ones, from mathematics and statistics, epidemiology, health education to population science and public health, environmental health and nutrition, for the purpose of teaching. That would be a great program to modernize the contents of public health completely with great impact for the future. That would be a "revolution" in knowledge and education in public health for Vietnam, showing how to come to terms with *esprit de science*. That shows the competence and authority of Krickeberg.

Krickeberg did this with the ideal and vision of Renaissance humanists: namely to create knowledge and pass it on to future generations, with a high sense of social responsibility, to help people live with health, dignity and social progress.

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The illusion of scientific authority: from misleading propaganda to dormant realities

Mohammad Sal Moslehian

This critique examines perception of the scientific authority, driven by the misuse of citation metrics. It identifies how individuals produce low-quality papers in predatory journals and increase their citations by forming self-made cliques. This culture of quantitativism exerts an adverse influence on the ecosystem of science and technology. Authentic scientific authority is established through original, impactful contributions recognized by global peers under rigorous qualitative criteria.

1 The illusion of scientific authority

For some time, the concept of "scientific authority" has garnered attention in certain countries, primarily through the use of scientometric indicators. However, in some cases, incomplete and superficial interpretations of these metrics have led to the creation of questionable rankings, such as the designation of "top one or two percent highly cited scientists." While some reputable scholars may be included in these rankings, a significant number are not even recognized nationally as authoritative figures or experts in their respective fields.

In articles [3–5], a comprehensive analysis has been conducted on the misuse of quantitative indicators based on citation counts; see also [1, 2, 6]. These works emphasize that on the periphery of science, certain domains exist where, due to the simplicity of topics and methodologies, groups of researchers, either individually or within self-formed rings, engage in mass production of low-quality papers (predominantly in predatory journals) and the development of artificial topics, the so-called pseudoscience. While benefiting from advantages such as grant funding and academic promotions, these individuals harbor the illusion of participating in genuine fields of knowledge through repeated intra-ring citations, unaware that they are far from the scientific mainstream. This is akin to someone releasing an arrow into the darkness to strike a wall, then drawing a circle around it and proclaiming a bullseye.

This situation seems to be rooted in a specific arrogance, intellectual isolation, and a lack of scientific collaboration with leading scholars from developed countries.

This focus on quantitative measures has had a negative impact on the science and technology community, dissuading those dedicated to qualitative research and guiding university faculty and research institute members towards unscientific and at times unethical paths. Undoubtedly, the quantitative regulations from the ministries of science, particularly those governing faculty promotion and institutional rankings, play a largely negative role. They foster an environment that resembles a pointless competition to maximize publication and citation counts.

Nevertheless, it should not be overlooked that citations and references to scientific works serve as fundamental bridges for scientific communication and the integrated structure of knowledge. While scientometrics provides valuable insights, in particular, when we are dealing with big data, its application should be limited so as to preserve the importance of human judgment in final scholarly evaluation. Our critiques should never become tools in the hands of the indolent or knowledge-deficient to suppress genuine scientific endeavor, often highly productive, under the pretext of criticizing quantitativism, thereby considering the important contributions of true scholars as worthless.

A high volume of publications or citations is not necessarily indicative of poor-quality research. In fact, several Fields medalists, as can be seen by using zbMATH, have authored hundreds of papers and received significant citation counts. On the other hand, over a long period—let's say a decade—a researcher with consistently few publications and citations (by mathematical standards) cannot be reasonably considered a scientific authority.

In reputable universities, quantitative metrics are not highly favored; the main criteria for evaluation are the quality of works and scientific achievements. This assessment is primarily conducted by distinguished scholars and specialists from various parts of the world, drawing on collective wisdom. While this method is not completely free from human biases and prejudices, the use of precise mechanisms and the establishment of peer-review panels consisting of acknowledged experts within strict academic standards significantly reduce these possible drawbacks.

It is evident that leading universities, institutions, and scientific societies base their decisions, such as awarding prizes, allocating grants, and selecting top researchers, not solely on quantitative scientometric indicators, but also on rigorous qualitative evaluation. These processes adhere to principles of professional ethics and meritocracy, as seen in the award systems of the American Mathematical Society and the National Science Foundation in the USA, as well as in the grant mechanisms of European science foundations.

2 Key indicators for assessing scientific authority

Scientific authority refers to the esteemed position attained by an individual through the production of original and influential knowledge that impacts scientific trends or the development of cutting-edge technologies, such that top-tier specialists at the international level consistently refer to their works.

Examining the achievements of Fields medalists and winners of highly prestigious awards highlights the following criteria for assessing scientific authority:

- *Quality of publications:* Demonstrating a consistent record of publishing original and high-impact research in reputable journals.
- *Qualitative citation impact:* Receiving citations (commensurate with the subfield) from renowned international specialists in top-tier journals.
- *Scientific reputation and credibility:* Gaining recognition and credibility within the international specialist community.
- *Tangible and broad effectiveness:* Having a track record of playing a significant role in advancing the global frontiers of knowledge or technological development.

In conclusion, recalling the adage that "ignorance is a scourge more powerful than evil," I would like to emphasize that these arguments and discussions should be continually articulated and clarified. It is only through this persistent engagement that scientific policymakers in developing countries can be influenced to establish regulations founded upon quality rather than quantity. This will enable the gradual reform of the scientific culture.

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The Lie–Størmer Center

Hans Munthe-Kaas and Cordian Riener

The Lie–Størmer Center is a Norwegian research center in mathematics, hosted at UiT – The Arctic University of Norway in partnership with the University of Bergen. The Center's name honors two Norwegian mathematicians whose research bridged abstract theory and practical computation. It carries that spirit forward by serving as a research environment, a training ground, and a point of contact between mathematical sciences and society.

A house of mathematics

Norway has a remarkable mathematical tradition. The works of Niels Henrik Abel, Sophus Lie, and Carl Størmer belong to the canon of the discipline. That tradition continues: Norwegian universities are home to an active and internationally recognized community of researchers, and Norway hosts the Abel Prize, one of the field's most distinguished honors. What has been missing is a dedicated center – a house of mathematics – that brings researchers together across institutions and across the divide that opened in the twentieth century between pure and applied mathematics. That divide has narrowed in recent decades, as abstract structures have proven indispensable to computation, and computational problems have driven new mathematical theory. A center built on that convergence, serving the training of new generations and acting as a point of contact between the mathematical sciences and society, is what Norway has lacked. The Lie–Størmer Center[1] was founded to be that place.

Two Norwegians who bridged the divide

Sophus Lie (1842–1899) did not set out to found a new branch of mathematics. He was concerned with differential equations: understanding why certain methods succeed while others fail. The answer lay in symmetry. By studying the transformation groups that leave an equation invariant, he transformed a collection of ad hoc techniques into a unified theory. Lie groups now underpin general relativity, quantum field theory, and particle physics; with the rise of modern computation, they have become equally central to numerical analysis.

Carl Størmer (1874–1957) began as a number theorist, but his interest in the aurora borealis led him to problems in mathematical physics. Inspired by Kristian Birkeland's experiments on the northern lights, he spent years computing the trajectories of charged particles in the Earth's magnetic field – thousands of calculations performed by hand. The numerical method he developed to integrate the equations of motion became one of the most widely used schemes in physics and engineering and remains so today. Reinterpreted through the lens of Lie group integration, it connects directly to the research program of the Center that now bears his name.

The logo of the Lie–Størmer Center captures this connection: it is inspired by Størmer's charged particle trajectories, computed using modern Lie group methods, joining the two namesakes in a single image. The Center opened in January 2024 at UiT Tromsø and is based on the view that the dialogue between mathematical structure and computational method remains one of the most productive frontiers in contemporary mathematics.

Three pillars

The Lie–Størmer Center is organized around three pillars that are distinct in character but closely connected in practice.

Research is the central pillar. The Center brings together mathematicians working across algebra, geometry, topology, and computational mathematics, fields that, for much of the twentieth century, developed in relative isolation. They are united by the view that abstract mathematical structures and computational methods are mutually reinforcing: new algebraic and geometric ideas reshape the design of algorithms, while computational problems reveal structures that pure mathematics had not previously had reason to study. One example from recent decades is the Hopf-algebraic structures discovered by Alain Connes and Dirk Kreimer in quantum field theory, which turned out to govern the composition of numerical integration methods, linking renormalization in physics to the algebra underlying Runge–Kutta schemes. Such

[1] https://lie-stormer.no

Participants at the Fall School on Geometry in Cryptography and Communication, held in Skibotn, Norway, in 2025.
(Photo: Gabriella Óturai, UiT The Arctic University of Norway)

Marcus du Sautoy giving a public lecture during the "Celebration of Mathematics" event in Tromsø, Norway.
(Photo: Jørn Berger-Nyvoll, UiT The Arctic University of Norway)

unexpected connections are precisely what the Center seeks to uncover and develop. The research is organized into six interlinked themes, spanning algebraic and combinatorial structures, topological invariants, symbolic computation, categorical reasoning, structure-preserving algorithms, and quantum computation, forming a continuous loop between mathematical foundations and computational practice.

Education is the second pillar. The Center hosts Mathesis, Norway's national doctoral school for the mathematical foundations of computation, funded by the Research Council of Norway. Mathesis connects PhD students across Norwegian universities through advanced courses, masterclasses, and research schools, with the aim of building fluency in both pure and computational mathematics. The Lie–Størmer Masterclass runs in a hybrid format, combining weekly online lectures with intensive in-person sessions, and brings research-level topics to students in Bergen, Oslo, Stavanger, Tromsø, and Trondheim that are not typically part of standard curricula. In 2024, the program expanded internationally through a partnership with Centre International de Mathématiques Pures et Appliquées (CIMPA): fellows from developing countries joined the masterclass and spent research periods in Tromsø, taking part in the scientific life of the Center. A further collaboration with the Indian National Centre for Mathematics is underway, supported by the Indo-Norwegian Cooperation Programme in Higher Education.

Outreach is the third pillar. It is rooted in the view that mathematics is a shared intellectual heritage. The Center organizes public lectures, open days for school students and teachers, and science communication through digital media. The opening of the Center in January 2024 set the tone: alongside the scientific program, it included a public lecture by Simon Singh, whose books on Fermat's Last Theorem, cryptography, and the hidden mathematics of popular culture have brought rigorous ideas to audiences well beyond the university. Also this year, a "celebration of mathematics" featuring Marcus du Sautoy brought the beauty of mathematics to the general public.

Research: six themes, one conversation

The Center's scientific program is structured around six research themes, each addressing a fundamental question about the relationship between mathematical structure and computation.

Algebraic and Combinatorial Structures addresses how algebraic symmetries encode the laws of computation and dynamics. It develops the algebraic backbone of the Center's research – Lie algebras, Hopf algebras, quantum groups, and operads – and their connections to combinatorics, topology, and geometry.

Geometric and Topological Invariants addresses how geometry and topology reveal hidden order in computation and data. It develops tools for identifying invariants that remain stable under deformation or noise, with applications ranging from dynamical systems to topological data analysis.

Symbolic and Homological Computation addresses how algebraic and topological structures can be made computationally effective. Working with polynomials, ideals, complexes, and representation-theoretic data, it develops algorithms that exploit symmetry and categorical structure to reach computations previously out of reach.

Categorical Dualities and Diagrammatic Reasoning examines the universal language connecting structure and computation. Categorical dualities, between algebra and geometry, syntax and semantics, and classical and quantum theories, provide the conceptual framework for structure-aware algorithms.

The first recipients of the Lie–Størmer–CIMPA Fellowship during their stay in Tromsø in 2024. Hans Munthe-Kaas and Cordian Riener together with fellows Yadav Rohit and Ngwongwo Ignatius.
(Photo: Lluïsa Puig Moner, UiT The Arctic University of Norway)

Structure-Preserving Algorithms and Learning addresses how mathematical structure can guide computation, learning, and inference. It develops algorithms that respect the symmetries and geometric constraints of the problems they address, including Lie-group-based numerical integration and geometrically informed architectures for machine learning.

Quantum and Noncommutative Computation explores the structural possibilities and limits of quantum computation. Drawing on representation theory, tensor categories, and quantum topology, it explores the algebra underlying quantum information and closes the loop by uncovering new mathematical structures through the demands of computation itself.

An international research environment

The Lie–Størmer Center is a member of ERCOM, a network of European mathematical research centers. The scientific life of the Center is built around schools, colloquia, and workshops that regularly bring the international mathematical community to Tromsø, complemented by a visitor program for extended research stays. In collaboration with CIMPA, the Center hosts two fellows from developing countries each year for one-month visits. It also runs an annual open call for proposals to organize thematic research weeks. Each autumn, the Center issues an open call to organize a research week, inviting proposals from the broader mathematical community. The Center also publishes a book series with Springer, titled Fundamental Structures in Computational and Pure Mathematics.

Funding and context

The Lie–Størmer Center grew out of a collaboration between the mathematics communities at UiT and the University of Bergen that began in 2021, supported by the national project "Pure Mathematics in Norway" and by grants from the Research Council of Norway and the European Research Council. Early support from the Trond Mohn Foundation enabled the Center to establish its organizational framework and begin its scientific activities.

The scientific advisory committee, consisting of Felipe Cucker (City University of Hong Kong), Jesús María Sanz-Serna (Universidad Carlos III de Madrid), and Roland Speicher (Universität des Saarlandes), meets annually to provide an independent assessment of the Center's research direction and development.

Hans Munthe-Kaas is co-director of the Lie–Størmer Center and professor at the University of Bergen and UiT. A leading researcher in computational mathematics and Lie group methods for numerical integration, he has served as chair of the Abel Prize Committee, editor-in-chief of *Foundations of Computational Mathematics*, and president of the Scientific Council of CIMPA.

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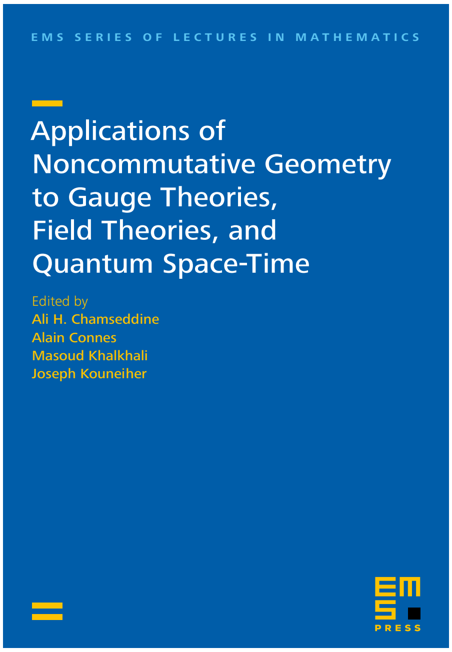

Applications of Noncommutative Geometry to Gauge Theories, Field Theories, and Quantum Space-Time

Edited by
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Noncommutative geometry has emerged over the past decades as a powerful conceptual and technical framework for extending the methods of geometry and topology to contexts where the classical notion of space-time, described in terms of smooth manifolds, is no longer adequate. Motivated both by internal developments in mathematics and by profound questions arising in modern theoretical physics, noncommutative geometry provides a unifying language in which algebraic, analytic, and geometric ideas are deeply intertwined. In particular, it offers new perspectives on gauge theories, quantum field theories, and the structure of space-time at the Planck scale, where quantum and gravitational effects are expected to become inseparable.

The central idea of noncommutative geometry is to replace the algebra of functions on a space by a noncommutative algebra, thereby encoding geometric information in purely algebraic and spectral terms. This paradigm shift allows one to generalize notions such as metrics, connections, curvature, and topological invariants to settings far beyond classical geometry. Spectral triples, cyclic cohomology, and operator-algebraic methods play a fundamental role in this approach, providing tools that are well adapted to the analysis of quantum systems and field-theoretic models.

The present volume explores this framework through a series of contributions that address, from complementary viewpoints, the application of noncommutative geometry to gauge theories, quantum field theories, and models of quantum space-time. Each contribution combines mathematical rigor with physical motivation, emphasizing both conceptual clarity and technical depth.

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Digitisation of scanned reviews from zbMATH Open

Bela Gipp, Ivan Pluzhnikov, Moritz Schubotz and Olaf Teschke

This note reports on the digitisation of scanned reviews from zbMATH volumes 1–529. We describe the procedure, techniques, and achievements, and release the dataset of images, corresponding text, and structured information for the community.

1 Introduction

In an earlier column [1], we discussed the challenge and possible approaches to convert scanned mathematical content into LaTeX, with a focus on the more than 250,000 pages of reviews from the volumes 1–529 of *Zentralblatt für Mathematik und ihre Grenzgebiete*. This corpus plays a specific role since it bridges the gap between the reviews of the *Jahrbuch über die Fortschritte der Mathematik* from 1868 to 1942, whose 181,613 reviews were mostly (OCR-supported) manually retyped in a project 1998–2002 [10], and the digitally-born zbMATH reviews since 1984. Though the scans were human-readable, they do not serve the requirements of a modern information ecosystem: they are not easily searchable, analysable, or processable; in particular, they cannot be automatically integrated to or interlinked with other services.

Especially, the transition of zbMATH into an open service [4, 5] makes it desirable to have these resources available as open resources. Even just within the service, features like the recently introduced entity linking [3] would benefit much from retrodigitisation.

2 LaTeX transition and review assignments

The digitisation of the scanned zbMATH volumes involved two essential components: Pure LaTeX transformation and review assignment through proper segmentation. During the last years, AI tools have improved significantly. For the translation of scans into LaTeX, Mathpix[1] has evolved into the state-of-the-art tool. As already indicated in [1], it performs quite well even for relatively low-quality scans, and has still improved much in the last years. Tests confirmed its capability; in fact, Mathpix was able to convert 99.1% of the pages into further processable LaTeX, outperforming other tools significantly.

However, the task did not stop here: Distributed among the 254,444 pages are 823,280 entries – mostly with single reviews, but also both title-only indexed items and joint reviews for multiple documents. Several short reviews may appear on one page, as well as long multi-page-reviews, especially for books. This creates a complex $n : m$-assignment task. To make things worse, the typesetting style (including placement of metadata and reviewer signatures) changes frequently with publication years. Experiments with traditional procedural algorithms for the first volumes indicated that it may require more than five person years of coding to assign the raw OCR output reliably – which would not just create costs of almost the magnitude of manual typing, but also postpone a release significantly.

3 LLM-based document splitting

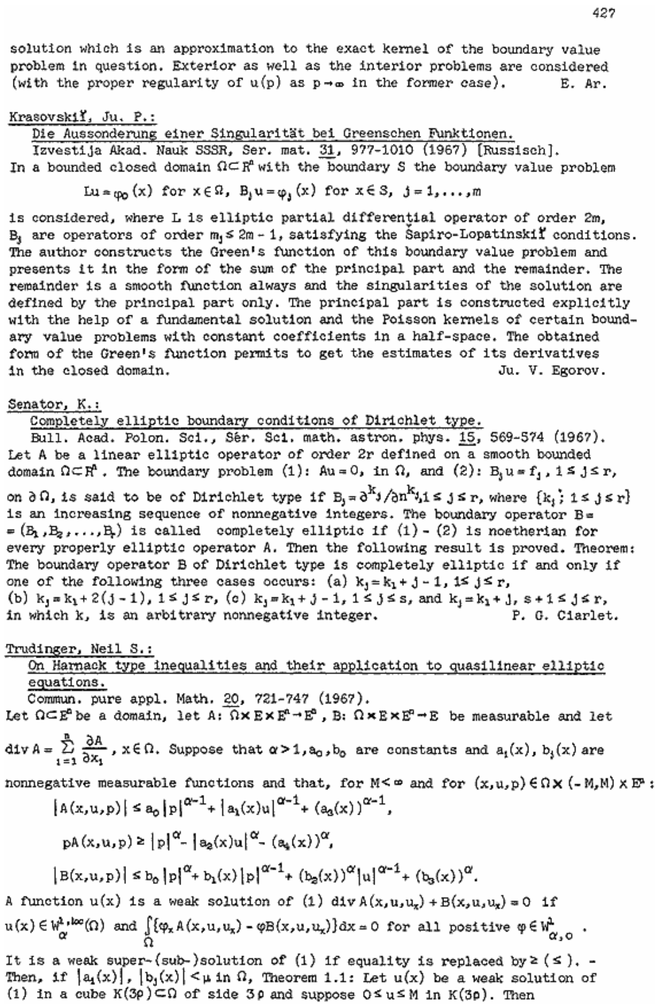
Fortunately, modern artificial intelligence methods offer more efficient alternatives. Large language models (LLMs) have evolved as powerful and flexible tools for such tasks.

The core pipeline processed the OCR rows sequentially, constructing LLM prompts that including the current and next document titles and sources, along with OCR-derived text. These prompts were passed to a generative LLM to extract the correct document boundaries. Special handling was applied for final documents and exclusions.

Due to hardware constraints, three instruction-tuned open-source LLMs were selected based on performance benchmarks: CalmExperiment-7B-slerp, Maxine-7B, and Mistral-Passthrough-10B. These models were further improved via fine-tuning using the LoRa (low-rank adaptation) optimisation technique [11].

To further improve accuracy, a majority voting ensemble was introduced: if two models agree, their output is selected; otherwise, the best-performing model (CalmExperiment) is used. This approach reduces individual model errors through consensus.

[1] https://mathpix.com


427

solution which is an approximation to the exact kernel of the boundary value problem in question. Exterior as well as the interior problems are considered (with the proper regularity of $u(p)$ as $p \to \infty$ in the former case). E. Ar.

Krasovskiĭ, Ju. P.:
Die Aussonderung einer Singularität bei Greenschen Funktionen.
Izvestija Akad. Nauk SSSR, Ser. mat. 31, 977-1010 (1967) [Russisch].
In a bounded closed domain $\Omega \subset R^n$ with the boundary S the boundary value problem

$$Lu = \varphi_0(x) \text{ for } x \in \Omega, \quad B_j u = \varphi_j(x) \text{ for } x \in S, \quad j = 1, \dots, m$$

is considered, where L is elliptic partial differential operator of order $2m$, B_j are operators of order $m_j \le 2m-1$, satisfying the Šapiro-Lopatinskiĭ conditions. The author constructs the Green's function of this boundary value problem and presents it in the form of the sum of the principal part and the remainder. The remainder is a smooth function always and the singularities of the solution are defined by the principal part only. The principal part is constructed explicitly with the help of a fundamental solution and the Poisson kernels of certain boundary value problems with constant coefficients in a half-space. The obtained form of the Green's function permits to get the estimates of its derivatives in the closed domain. Ju. V. Egorov.

Senator, K.:
Completely elliptic boundary conditions of Dirichlet type.
Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 15, 569-574 (1967).
Let A be a linear elliptic operator of order $2r$ defined on a smooth bounded domain $\Omega \subset R^n$. The boundary problem (1): $Au = 0$, in Ω, and (2): $B_j u = f_j$, $1 \le j \le r$, on $\partial\Omega$, is said to be of Dirichlet type if $B_j = \partial^{k_j}/\partial n^{k_j}$, $1 \le j \le r$, where $\{k_j; 1 \le j \le r\}$ is an increasing sequence of nonnegative integers. The boundary operator $B = (B_1, B_2, \dots, B_r)$ is called completely elliptic if (1) - (2) is noetherian for every properly elliptic operator A. Then the following result is proved. Theorem: The boundary operator B of Dirichlet type is completely elliptic if and only if one of the following three cases occurs: (a) $k_j = k_1 + j - 1$, $1 \le j \le r$, (b) $k_j = k_1 + 2(j-1)$, $1 \le j \le r$, (c) $k_j = k_1 + j - 1$, $1 \le j \le s$, and $k_j = k_1 + j$, $s+1 \le j \le r$, in which k_1 is an arbitrary nonnegative integer. P. G. Ciarlet.

Trudinger, Neil S.:
On Harnack type inequalities and their application to quasilinear elliptic equations.
Commun. pure appl. Math. 20, 721-747 (1967).
Let $\Omega \subset E^n$ be a domain, let $A: \Omega \times E \times E^n \to E^n$, $B: \Omega \times E \times E^n \to E$ be measurable and let $\operatorname{div} A = \sum_{i=1}^{n} \frac{\partial A}{\partial x_i}$, $x \in \Omega$. Suppose that $\alpha > 1$, a_0, b_0 are constants and $a_i(x)$, $b_j(x)$ are nonnegative measurable functions and that, for $M < \infty$ and for $(x,u,p) \in \Omega \times (-M,M) \times E^n$:

$$|A(x,u,p)| \le a_0 |p|^{\alpha-1} + |a_1(x)u|^{\alpha-1} + (a_3(x))^{\alpha-1},$$

$$pA(x,u,p) \ge |p|^{\alpha} - |a_2(x)u|^{\alpha} - (a_4(x))^{\alpha},$$

$$|B(x,u,p)| \le b_0 |p|^{\alpha} + b_1(x)|p|^{\alpha-1} + (b_2(x))^{\alpha}|u|^{\alpha-1} + (b_3(x))^{\alpha}.$$

A function $u(x)$ is a weak solution of (1) $\operatorname{div} A(x,u,u_x) + B(x,u,u_x) = 0$ if $u(x) \in W_\alpha^{1,loc}(\Omega)$ and $\int_\Omega \{\varphi_x A(x,u,u_x) - \varphi B(x,u,u_x)\} dx = 0$ for all positive $\varphi \in W^1_{\alpha,0}$.
It is a weak super-(sub-)solution of (1) if equality is replaced by $\ge$ ($\le$). - Then, if $|a_i(x)|$, $|b_j(x)| < \mu$ in Ω, Theorem 1.1: Let $u(x)$ be a weak solution of (1) in a cube $K(3\rho) \subset \Omega$ of side 3ρ and suppose $0 \le u \le M$ in $K(3\rho)$. Then

A typical scan from Vol. 153.

Several baselines were evaluated, including regular expressions, a computer vision model (DIT), and ChatGPT-4o (with and without few-shot prompting). Results showed that non-fine-tuned LLMs failed completely, while fine-tuned models performed strongly with an accuracy of 93–97% [8]. The Majority Voting approach achieved the highest accuracy of 97.5%, significantly outperforming all baselines.

4 Error analysis

The analysis of the output identified four main error sources: OCR formatting issues, ambiguity in similar titles, hallucinations in LLM outputs, and confusion between metadata and content. Attempts to predict errors using OCR confidence scores and token probabilities were unsuccessful, indicating these are not reliable indicators.

To map extracted documents back to their original LaTeX sources, exact string matching (with preprocessing to handle OCR spacing issues) was used to determine start and end indices. This method achieved high accuracy (98.4%) on 90.6% of a test dataset. However, this also indicated the need to exclude a significant fraction of about 10% of hallucinated output.

Finally, the approach was scaled to the full OCR output of the more than 250,000 pages, successfully assigning 662,720 (about 89%) of the reviews to the correct metadata. Beyond this, the approach also corrected OCR errors and removed metadata misplaced by the OCR in the review texts.

5 Processing the data into zbMATH Open

At the beginning of 2026, these reviews had been processed into the zbMATH Open production system, and made internally visible to the editors. In the following weeks, further systematic errors have been identified and corrected. This included about 10k formal LaTeX errors, but also semantic errors mostly resulting from poor print quality (e.g., it turned out that 1, l, and i were hardly distinguishable in some volumes). So far, these corrections are far from being completed, the main remaining tasks being the extraction and identification of reviewer signatures and in-text citations.[2] Both are especially affected by the varying style specifics and could not yet be automatically identified with sufficient precision; but the rapid technological developments will likely make this much easier in the near future. Likewise, the documents which had to be excluded due to LLM hallucinations can be likely addressed by newer versions soon.

However, overall the corpus converged to a state deemed as sufficient to be searched and displayed. They have been publicly released since March 2026, with a remark indicating their status and a feedback option for possible corrections. Gradually, they are checked and receive eventually a flag of an editorial approval (so far, in the course of the testing, this has been done for 31,152 documents). Only then, they will be also available through the zbMATH Open APIs [2, 9], via our standard CC-BY-SA 4.0 licence. However, for research purposes, the full data of the automated processing are already available – the Mathpix results at [7] and the data after the LLM split at [6].

6 Conclusions

Transforming more than 250,000 scanned pages with a high accuracy into LaTeX demonstrates a useful specific application of now standard AI technology tailored to a longstanding challenge.

[2] It might be noted that both issues also still mainly remain unsolved for the manually typed JFM volumes!

Trudinger, N. S.

On Harnack type inequalities and their application to quasilinear elliptic equations. (English)

Zbl 0153.42703

Commun. Pure Appl. Math. 20, 721-747 (1967).

Let $\Omega \subset E^n$ be a domain, let $A : \bar{\Omega} \times E \times E^n \to E^n, B : \Omega \times E \times E^n \to E$ be measurable and let $\operatorname{div} A = \sum_{1=1}^n \frac{\partial A}{\partial x_1}, x \in \Omega$. Suppose that $\alpha > 1, a_0, b_0$ are constants and $a_1(x), b_1(x)$ are nonnegative measurable functions and that, for $M < \infty$ and for $(x, u, p) \in \Omega \times (-M, M) \times E^n$:

$$\begin{aligned} &|A(x,u,p)| \leq a_0|p|^{\alpha-1} + |a_1(x)u|^{\alpha-1} + (a_3(x))^{\alpha-1}, \\ &pA(x,u,p) \geq |p|^\alpha - |a_2(x)u|^\alpha - (a_4(x))^\alpha, \\ &|B(x,u,p)| \leq b_0|p|^\alpha + b_1(x)|p|^{\alpha-1} + (b_2(x))^\alpha |u|^{\alpha-1} + (b_3(x))^\alpha. \end{aligned}$$

A function $u(x)$ is a weak solution of (1) $\operatorname{div} A(x,u,u_x) + B(x,u,u_x) = 0$ if $u(x) \in W_\alpha^2, loc(\Omega)$ and $\int_\Omega \{\varphi_x A(x,u,u_x) - \varphi B(x,u,u_x)\} dx = 0$ for all positive $\varphi \in W_{\alpha,0}^2$. It is a weak super-(sub-) solution of (1) if equality is replaced by $\geq (s)$. Then, if $|a_1(x)|, |b_y(x)| < \mu$ in Ω, Theorem 1.1: Let $u(x)$ be a weak solution of (1) in a cube $K(3p) \subset \Omega$ of side $3p$ and suppose $0 \leq u \leq M$ in $K(3p)$. Then

$$\begin{aligned} &\text{ess } \max u(x) \leq C \text{ ess } \min u(x), C = C(\alpha, n, a_0, b_0 M, \mu\rho). \\ &K(\rho) \quad K(\rho) \end{aligned}$$

Theorem 1.2: Let $u(x)$ be a weak super-solution of (1) in K and $0 \leq u < M$ in $K(3p)$. Then $p^{-n/\gamma} \|u(x)\|_{\gamma, K(2p)} \leq \mathrm{C} \min_{K(p)} u(x)$ for $\gamma < n(\alpha-1)/(n-\alpha)$ when $\alpha \leq n$, and $\gamma \leq \infty$ when $\alpha > n$. Theorem 1.3: Let $u(x)$ be a weak sub-solution of (1) in $K, 0 \leq u < M$ in $K(3p)$. Then ess $\max u(x) \leq Cp^{-n/\gamma} \|u(x)\|_{\gamma,K}(2_p)$. These results are now applied to obtain the Hölder estimate in Theorem 2.2: Let $u(x)$ be a locally bounded weak solution of (1) in Ω. Then if $u(x)$ is locally Holder continuous in Ω (up to a set of measure zero) and if $K = K(p_0) \subset \Omega$ then

$$\operatorname*{osc}_K u \leq c \left(\frac{\rho}{\rho_0}\right)^\delta \left\{\sup_{\mathrm{K}} |u| + \mu\rho(\mu\rho)^{\alpha-1}\right\}$$

for all $p < p_0$, where $\delta > 0$ and $C = C(\alpha, n, a_0, b_0, \sup_K u)$. Theorem 2.3 embodies a similar result for a type of quasi-elliptic equation. Next, for the uniformly elliptic equation $a^{1j}(x, u_x) u_{x_1 x_j} = 0$ with $a^{1j}(x,p) \in C^1(\Omega \times E^n)$ and

$$(1 + |p|) a_p^{19}(x,p) + |a_x^{19}(x,p)| \leq \lambda^{-1} \text{ for } (x,p) \in \Omega \times E^n$$

we have the following solution of the Dirichlet problem. Theorem 3.1: Let $\partial\Omega$ satisfy an exterior sphere condition and let $\varphi(x) \in C^\circ(\partial\Omega)$. Then there is a $u \in C^\circ(\bar{\Omega}) \cap C^2(\Omega)$ such that $a^{1j}(x, u_x) u_{x_1 x_j} = 0, u = \varphi$ in $\partial\Omega$. The paper concludes with some estimates of the solution of (1) at the boundary of Ω and with generalizations of the preceding when the functions a_1, b_1 are no longer bounded but subject only to certain integrability conditions.

This review text is based on a scanned copy of the printed version. It was converted to LaTeX using MathPix and a specifically developed LLM to assign the text parts to the metadata. It may contain errors, misassignments, or gaps; in particular, the reviewer signature has not yet been extracted reliably in general. If you notice any errors, please report them directly to our editorial team via the Contact Form.

Digitised review in zbMATH Open.

Compared to previous approaches, it obtained the goal with significantly reduced costs: Less than 4% of projected costs of traditional OCR/retyping approaches, consisting mainly of licence costs and support for the second author's thesis project at the University of Göttingen.[3] To the best of our knowledge, this might be the largest project of this type so far; and although the technology is already very advanced, human oversight could still provide valuable feedback and corrections.

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[3] These extra costs could actually be covered due to reviewers donating their compensation to the further development of zbMATH Open, to whom we are grateful!

[5] K. Hulek and O. Teschke, The transition of zbMATH towards an open information platform for mathematics II: A two-year progress report. *Eur. Math. Soc. Mag.* **125**, 44–47 (2022)

[6] I. Pluzhnikov, A. Satpute and M. Schubotz, Document texts extraction of zbMATH Open volumes (1–529). URL https://doi.org/10.5281/zenodo.14496349 (2024)

[7] I. Pluzhnikov, A. Satpute and M. Schubotz, Retrodigitalization of zbMATH Open volumes (1–529): Mathpix 2024 OCR results. URL https://doi.org/10.5281/zenodo.14253145 (2024)

[8] I. Pluzhnikov, A. Satpute, M. Schubotz, O. Teschke and B. Gipp, LLM-supported document separation for printed reviews from zbMATH Open. arXiv:2604.00554v1 (2026)

[9] M. Schubotz and O. Teschke, zbMATH Open: towards standardized machine interfaces to expose bibliographic metadata. *Eur. Math. Soc. Mag.* **119**, 50–53 (2021)

[10] O. Teschke, The "Jahrbuch über die Fortschritte der Mathematik" as a part of zbMATH Open. *Eur. Math. Soc. Mag.* **122**, 62–64 (2021)

[11] E. Hu, Y. Shen, P. Wallis, Z. Allen-Zhu, Y. Li, S. Wang, L. Wang and W. Chen, LoRa: low-rank adaptation of large language models. arXiv:2106.09685v2 (2021)

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Mathematics education's most prestigious awards

ICMI column in this issue presented by Alan H. Schoenfeld and Kaye Stacey

The ICMI awards for mathematics and mathematics education

Much as the International Mathematics Union (IMU) offers awards for distinguished contributions to various aspects of the mathematical enterprise,[1] the International Commission on Mathematical Instruction (ICMI) awards three prizes recognizing outstanding achievement in different aspects of mathematics education. And, just as the IMU awards are awarded every four years at the opening ceremony of the International Congress of Mathematicians (ICM), the ICMI awards are presented every four years at the opening ceremony of the International Congress on Mathematical Education (ICME).[2] The three awards are:

- The Felix Klein Award, which honors lifetime achievement in mathematics education. As explained below, awardees are typically senior scholars who have made multiple contributions, in breadth and depth, that forward the enterprise. In this, the Klein Award parallels the IMU's Chern Medal.
- The Hans Freudenthal Award, which honors a major cumulative program of research. Such work is frequently but not always done by a younger scholar. In this, the Freudenthal Award parallels IMU's Fields Medal.
- The Emma Castelnuovo Award, which recognizes outstanding achievements in the practice of mathematics education. Such work might involve a program of instructional design, textbook development, assessment, models of teacher preparation, or other work that advances the state of practice. In this, the Castelnuovo Award perhaps most parallels IMU's Gauss Prize.

Awardees receive certificates that highlight their contributions to mathematics education and medals that commemorate them.

The three awards encapsulate varied aspects of mathematics education as they have evolved chronologically. Klein, of course, was the ultimate mathematician, who developed a strong interest in mathematics education and advocated modernization of the curriculum. His name alone highlights the importance of mathematics at the center of mathematics education. Freudenthal was a mathematician who also studied student thinking and learning, and applications of these ideas as a foundation for theorizing instruction. This opened up possibilities of mathematics education as an empirical discipline. Castelnuovo opened the space of instruction to building on children's thoughts and ideas, helping students to mathematize real-world phenomena and to build on their ideas and intuitions. As such, these three awards symbolize the breadth and scope of the theory and practice of mathematics education.

In the following sections there is some detail about the people after whom the awards are named and the criteria for each of the awards. There is some information about the committee processes involved in the selection of the awardees, and information that may assist in developing a nomination for the 2028 awards. The ICMI website gives further information about all aspects of the awards and official instructions for making a nomination. Our aim for this article, as the chairs of the award committees, is to set the awards in context and encourage nominations for 2028.

The Felix Klein Award, for lifetime academic achievement in mathematics education

Felix Klein (1849–1925) needs no introduction to a mathematical audience. Klein had a broad theoretical sweep. He was celebrated for revolutionizing geometry via group theory in his 1872 'Erlangen Program,' in which geometries were classified by their symmetry groups. More broadly, he was known for a wide range of mathematical syntheses, e.g., advanced function theory, specifically the study of Riemann surfaces, elliptic modular functions, and automorphic functions, for which he created a unified view. Klein was collaborative. He established Göttingen as a premier mathematical research center, fostering collaborations between pure and applied mathematicians.

Klein was a strong supporter of Göttingen's admitting women to study. He supervised Grace Chisholm Young's PhD thesis in mathematics. Young was the first woman to receive a doctorate in any field in Germany. And, Klein had an abiding interest

[1] https://www.mathunion.org/imu-awards/imu-awards-prizes-and-special-lecture

[2] https://www.mathunion.org/icmi/icme/icme-international-congress-mathematical-education

Felix Klein.

in mathematics instruction. He pioneered the idea of examining 'elementary mathematics through an advanced viewpoint,' an enrichment of elementary mathematical ideas for teachers – a concept that is embodied today by ICMI's Klein Project.[3] Klein was an advocate for connecting mathematical research with industrial applications and for reforming school curricula to treat mathematics as a connected whole. Presumably in response to these achievements and efforts, in 1908 Klein was elected the first president of the International Commission on Mathematical Instruction (ICMI).

The Felix Klein Award is awarded for lifetime academic achievement in mathematics education. This award acknowledges distinguished scholars who have shaped the field of mathematics education over the course of their careers. It honors those candidates who have made substantial research contributions, typically along multiple lines of inquiry, and have also introduced new issues, ideas, perspectives, and critical reflections. Additional considerations include leadership roles, mentoring, and peer recognition, as well as the actual or potential relationship between the research conducted and the improvement of mathematics education at large, through connections between research and practice. As you can see from the list of awardees that follows, each of the awardees made many and varied contributions. It should also be apparent that there is no simple template for the award: different awardees made different kinds of contributions.

The Felix Klein Award, first granted in 2003, was at first biennial, and later quadrennial. The complete list of awardees to date is: Guy Brousseau (2003), Ubiratan D'Ambrosio (2005), Jeremy Kilpatrick (2007), Gilah Leder (2009), Alan Schoenfeld (2011), Michèle Artigue (2013), Alan Bishop (2015), Deborah Ball (2017), Tommy Dreyfus (2019), and Ferdinando Arzarello (2024).

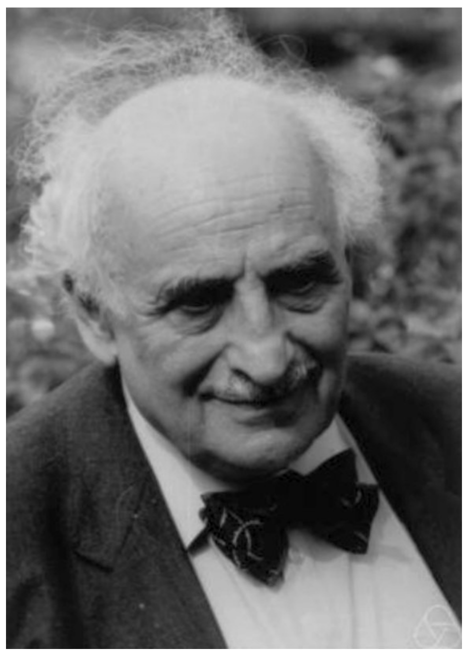
[3] https://www.mathunion.org/icmi/projects/klein-project

The Hans Freudenthal Award, for a major program of research on mathematics education

Hans Freudenthal (1905–1990) was a renowned German-Dutch mathematician, educator, and philosopher. Freudenthal made significant contributions to Lie group theory, geometry, and the foundations of mathematics. His major contributions to mathematics education center on his reconceptualization of mathematics as a human activity and his development of Realistic Mathematics Education (RME), a theory that has shaped curricula, research, and international assessment beginning in the Netherlands but now influential in many countries.

Freudenthal argued that mathematics should not be presented as a finished system of abstract structures but as something people do to organize and understand reality. He maintained that students 'learn mathematics by mathematizing,' that is, by structuring real or imagined situations with progressively more formal mathematical tools. Teaching should begin with students' experiences and gradually work towards mathematical theory. This view opposed concept-attainment approaches in which learners are confronted mainly with ready-made definitions and formal exercises and engage in applications later.

Realistic Mathematics Education was developed from the 1970s onward in Utrecht. RME treats learning mathematics as guided reinvention: students are supported to reinvent central mathematical ideas for themselves along trajectories that echo historical and didactical development. Characteristic principles include



Hans Freudenthal. (© Konrad Jacobs, Erlangen; Archives of the Mathematisches Forschungsinstitut Oberwolfach) mathematizing meaningful situations, progressive formalization from informal strategies to symbols, and a continual back-and-forth between reality-based models and more formal structures. This approach directly informed the notion of 'mathematical literacy' that underpins the Program for International Student Assessment (PISA) assessments, a major international assessment program sponsored by OECD.

Freudenthal was widely influential. He founded the Institute for the Development of Mathematical Education (IOWO) at Utrecht University in 1971. IOWO, renamed the Freudenthal Institute after his death, has been a major R&D center for both curriculum and assessment, in mathematics and science. He was also the founding editor of the journal *Educational Studies in Mathematics*, still one of the top journals in mathematics education. He also served as president of the International Commission on Mathematical Instruction (ICMI) from 1967 to 1970 and organized the first International Congress on Mathematical Education (ICME), which was held in Lyon, France, in 1969. The first International Congress of Mathematicians was held more than seventy years earlier, in 1897.

Freudenthal's work exemplifies a major, focused program of research and development – the pursuit of an important idea in multiple ways, with implications and applications across a range of areas. Accordingly, the Hans Freudenthal Award acknowledges the outstanding contributions of an individual's theoretically robust and coherent research program, which has had a clear impact on our research community. It honors a scholar who has initiated a new research program and has brought it to maturation over the past decade or more. Freudenthal awardees should also be researchers whose work is ongoing and who can be expected to continue contributing to the field. In brief, the criteria for this award are depth, novelty, sustainability, and impact of the research program. As is the case for the Klein Award, there is no simple template for achieving this honor: different awardees have made different kinds of contributions.

The Freudenthal Award, first granted in 2003, was at first biennial, and later quadrennial. The complete list of awardees to date is: Celia Hoyles (2003), Paul Cobb (2005), Anna Sfard (2007), Yves Chevallard (2009), Luis Radford (2011), Frederick Leung (2013), Jill Adler (2015), Terezinha Nunes (2017), Gert Schubring (2019), and Ole Skovsmose (2024).

The Emma Castelnuovo Award, for outstanding achievements in the practice of mathematics education

This is the most recently established ICMI award. Whereas mathematics is a unique creation of human thought, a universal discipline reaching back to the beginning of civilization and central to modern life, mathematics education has a different character. At heart, it has a social purpose, so that more people learn more mathematics with more success and more satisfaction. The Castelnuovo Award addresses this ambitious goal, which is fundamental to the existence of ICMI.

Emma Castelnuovo. (Photo by Raimondo Bolleta, courtesy International Mathematical Union / International Commission on Mathematical Instruction)

The award was established on the 100th birthday of Emma Castelnuovo (1913–2014). Castelnuovo was an Italian mathematician from a famous Italian mathematical family. Her father, Guido Castelnuovo, was a foundational figure in the early years of ICMI. Emma Castelnuovo's career in academic mathematics was thwarted by antisemitic laws in Italy at the time of the Second World War. After the war, she became a very influential mathematics teacher. An example of the approaches she used is given by an article[4] she contributed to the first volume of *Educational Studies in Mathematics*, on her teaching of affine functions in early secondary school. The article is richly illustrated with images of models of geometric figures used to show transformations (e.g., of a square to a parallelogram) and the associated algebra. Although this was the era of 'new math,' hers was not an abstract approach, but designed so that students would experience mathematics in action. Castelnuovo committed herself to research and development of learning materials in mathematics education, aiming for students to use observation and visualization, to puzzle over unexpected outcomes and eventually to arrive at previously unknown results,

[4] E. Castelnuovo, Les transformations affines dans le 1er cycle de l'école secondaire. *Educ. Stud. Math.* **1**, 274–288 (1969)

without being told. She deeply influenced mathematics education in Italy and Spain.

The first Emma Castelnuovo medal was awarded to Hugh Burkhardt and Malcolm Swan in 2016. Their joint work at the Shell Centre for Mathematical Education at Nottingham University had resulted in educational principles and extremely well-designed and imaginative teaching and assessment materials used around the world. The second award went to the National Council of Teachers of Mathematics (NCTM), the mathematics teachers' association for the USA and Canada, for its influential curriculum leadership, its journals for teachers, its conferences, its promotion of research and its advocacy. Clearly the NCTM has long been the most influential body in the practice of mathematics education in the English-speaking world. But making the award to an organization raised an issue for the ICMI Executive Committee. Was the award intended for organizations or even governments, or like the other awards, for individuals or small groups of colleagues who have worked together over time? The criteria for the award were subsequently altered to clarify the rules. The 2024 medal was awarded to Kaye Stacey, a researcher and teacher educator who gave special attention to translating her research findings so that they had an impact on practice.

Given the huge range of individuals or small groups involved in the practice of mathematics education (e.g., teachers and teacher educators, curriculum designers, software developers, assessment leaders, charismatic speakers), the diversity of nominations for the Castelnuovo Award is even greater than for the other awards. The criteria[5] are written to be as inclusive of all these roles as possible. For this award, the essential feature is to demonstrate a positive influence on mathematics learning for schools, districts, regions, or countries. A benefit of the award is that it showcases models and exemplars of inspirational practices.

Committee processes

As explained below – the processes are highly confidential – there is only so much we can say about the processes used by the committees. The goal is for each selection process to be as fair as we can make it, while at the same time insulating committee members from pressure as much as possible. Thus, with the exception of the committee chairs, the names of committee members are kept secret while they are serving their terms. The committee member selection process is intended to be as broadly representative as possible. The ICME President is copied on all correspondence of each committee, which is permanently archived – and embargoed for 50 years! That's prudent, given that the correspondence concerns detailed discussions of the merits of individual candidates. However, committee chairs write reports to the ICME Executive Committee providing details regarding committee processes and describing issues that need resolution. Significant issues that arise during deliberations are brought to the ICMI Executive Committee. For example, we have already been asked during this award cycle whether it would be possible to nominate for the Freudenthal Award a pair of researchers who collaborated intensively on the work that comprised the substance of the nomination. The Executive Committee was sympathetic to the idea but determined that its implementation should not take place until the beginning of the next award cycle, rather than possibly placing some candidates at a disadvantage because the announcement was not made at the beginning of this cycle.

What we can say is that the actual process is taken with great seriousness by the committees, with many rounds of discussions. The process begins with ICMI's attempt to publicize the awards widely – hence this article! – to insure, as best we can, that worthy candidates worldwide are encouraged. As you will see from the nomination requirements below, the nomination packages are quite substantial.

Even when considering a single award, there is a wide range of factors that one can consider. For the Klein Award, for example, how broad and deep are the candidate's contributions? How much does a candidate's theoretical impact matter? What about the candidate's methodological contributions? Impact on practice through editorship or through development of materials? Through leadership? Are these contributions truly international? Then, there is the question of how one weighs some of these contributions against others. If candidate A is stronger in some arenas and candidate B is stronger in others, how do you decide between them? Each field of candidates brings new challenges, because these are not abstract questions – their resolution depends very much on the particulars of the work. Similarly, definitions are not fixed: over the course of someone's career, a theoretical idea that was decidedly outside the mainstream may evolve to the point where it is considered central, or the very nature of what is considered 'theory' may evolve. (To give two examples deliberately chosen from outside our purview, we might think about how Piaget's ideas about constructivism have been regarded over the years, and the controversial decision by the Nobel Prize committee to award the 2016 prize in literature to Bob Dylan.) There are analogous issues in attempting to calibrate the importance of varied lines of work and their impact in both the Freudenthal and Castelnuovo awards.

Please nominate suitable candidates!

The awards process best serves the mathematics education community if it is broadly representative and international. We encourage you to nominate candidates who merit consideration. Creating the nomination package is a lot of work, but we promise that all

[5] https://www.mathunion.org/icmi/awards/emma-castelnuovo-award

nominations will get very serious attention. Moreover, although only one nomination is chosen to win each award, the very fact of having been nominated is very heartening to the nominee. With only one award of each medal every four years, it is often the case that excellent candidates miss out. Repeat applications are encouraged.

Making a nomination

For official details about the nomination process, see the call for the ICMI awards.[6] In addition, nominators might like to consider the following informal advice. First, it is probably easiest if a nomination is prepared by a very small group of reasonably senior mathematics educators. This shares the work, provides multiple perspectives on the nominee's achievements, and brings a good range of contacts who may give additional support. Typically, nominators tell the candidates that they plan to nominate them, especially because nominees usually need to share information with them. Second, the nominating group needs to identify a few strong themes around which coherent arguments for the award can be presented. It is likely that the nominee has made contributions to multiple aspects of mathematics education, but not everything they have done needs to be part of the application. Third, there is an opportunity to have letters of support from others – choose them well, so that they clearly support the main aspects of the candidate's suitability for the award.

[6] https://www.mathunion.org/icmi/awards/call-2028-icmi-awards

Key dates

The 2028 awards will be conferred at the ICME-16 opening ceremonies in Prague, Czech Republic on July 9, 2028. Nominations for the awards are due by November 30, 2026. All nominations for the 2028 awards must be sent to the chair of the appropriate committee using the email addresses listed below.

- Felix Klein Award, Committee chair: Alan Schoenfeld, chair@klein28.mathunion.org
- Hans Freudenthal Award, Committee chair: Alan Schoenfeld, chair@freudenthal28.mathunion.org
- Emma Castelnuovo Award, Committee chair: Kaye Stacey, chair@castelnuovo28.mathunion.org

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Emeritus Professor Kaye Stacey chairs the selection committee for the 2028 Emma Castelnuovo Award. She won the award in 2024. She was holding the Foundation Chair of Mathematics Education at the University of Melbourne. Her broad research interests include mathematical thinking and problem-solving, assessment, the school curriculum and new technologies. She is author of many articles and books in mathematics education reporting research and promoting the practical application of research findings.

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Convenors' reflections on purpose and future directions

ERME column regularly presented by Andreas Stylianides and Florian Schacht

Andreas Stylianides and Florian Schacht

CERME Thematic Working Groups

We continue the initiative of introducing the CERME Thematic Working Groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for people working in pure and applied mathematics. In this issue, we reflect on the purpose of this column and its future directions.

ERME column: convenors' reflections on purpose and future directions

In this contribution, we reflect on the purpose of the ERME column and outline its future directions, while also situating it within its history since 2017. Since then, the column in the *EMS Magazine* has offered a window into the work of Thematic Working Groups (TWGs) within the European mathematics education community. ERME, the European Society for Research in Mathematics Education, brings together researchers across Europe through its flagship CERME congresses, topic conferences, and other activities. The column provides a way to share this work more widely with the broader mathematical community.

Over the years, contributions have highlighted themes as diverse as algebraic thinking, modelling, affect, and teacher education, alongside occasional feature pieces marking significant moments, such as ERME's 20th anniversary. Looking back across this record, we see the column as both a chronicle of ERME's research landscape and a bridge to the wider EMS readership. It documents not only the scope of ongoing work within TWGs, but also how these groups reflect broader concerns in mathematics education: diversity and inclusion, digital tools including AI, the pandemic's impact on our teaching and research practices, and so on.

We use this issue as an opportunity to step back and reflect on the broader role of the column. Beyond serving as a platform for individual TWG contributions, the column is intended to represent the *ongoing vitality of ERME as a community* and to highlight its *relevance to the wider mathematical sciences*. By 'vitality' we mean the diversity of research questions that are pursued across TWGs, the openness to new methods and theoretical perspectives, and the active involvement of early career as well as experienced researchers. By 'relevance' we mean the ways in which mathematics education research addresses issues with consequences for the wider mathematical sciences and for society more broadly: how mathematics is taught and learned in schools and universities, which directly shapes students' preparedness for further study; how teachers are prepared and supported, which influences the quality and sustainability of mathematics teaching; how technology, including AI, is reshaping learning environments; and how inclusion and equity are pursued, which affects who has access to mathematical pathways and careers. In this sense, the column offers readers a perspective on how mathematics education research both contributes to and is shaped by broader developments in mathematics, education policy, and society.

Almost all TWGs have already contributed to the column once. In the coming years, we will therefore begin to invite groups who contributed earlier to return with updates on their work. This will allow readers to see how research trajectories within ERME have evolved over time, while also giving a sense of the continuity and renewal that characterises our community. Alongside such updates, we also plan to include occasional cross-cutting reflections on themes that span multiple TWGs. In this way, the column can continue to capture not only the diversity of research in mathematics education but also its ongoing development and coherence as a collective endeavour.

To support readers in exploring earlier work, we provide in Table 1 a list of ERME contributions already published in the EMS Magazine, together with links to the online archive. We hope this overview encourages readers to (re)visit contributions that resonate with their own interests.

Table 1. Contributions from ERME to the EMS Magazine.

Year	Issue	ERME contribution
2017	105	TWG14 – University Mathematics Education
	106	TWG15 and TWG16 – Teaching and Learning Mathematics with Technology and Other Resources
2018	107	TWG01 – Argumentation and Proof
	108	TWG12 – History in Mathematics Education
	109	TWG09 – Mathematics and Language
	110	S. Prediger and E. Cohors-Fresenborg – ERME's 20th anniversary
2019	111	TWG06 – Applications and Modelling
	112	TWG03 – Algebraic Thinking
	113	TWG21 – Assessment in Mathematics Education
	114	TWG18 – Mathematics Teacher Education and Professional Development
2020	115	J. Cooper – ERME topic conferences
	116	TWG05 – Probability and Statistics Education
	117	TWG08 – Affect and the Teaching and Learning of Mathematics
	118	P. N. Zaragoza and J. Cooper – Report from INDRUM
2021	119	TWG02 – Arithmetic and Number Systems
	120	V. Durand-Guerrier, R. Hochmuth, E. Nardi and C. Winsløw – Report on the book *Research and Development in University Mathematics Education*
	121	TWG10 – Social, cultural and political aspects of mathematics education
	122	M. Artigue – The Covid-19 pandemic: challenging times for the mathematics education community
2022	123	TWG24 – Representations in Mathematics Teaching and Learning
	125	TWG04 – Geometry Teaching and Learning
2023	127	TWG13 – Early Years Mathematics
	128	TWG23 – Implementation of Research Findings in Mathematics Education
	130	TWG22 – Curricular Resources and Task Design in Mathematics Education
2024	131	TWG19 – Mathematics Teaching and Teacher Practice(s)
	133	TWG27 – The Professional Practices, Preparation and Support of Mathematics Teacher Educators
	134	TWG17 – Theoretical Perspectives and Approaches in Mathematics Education
2025	135	TWG20 – Mathematics Teacher Knowledge, Beliefs and Identity
	137	TWG11 – Teaching and Learning of Discrete and Computational Mathematics
	138	TWG25 – Teaching and Learning of Calculus
2026	139	TWG07 – Mathematics for Work, Society and Personal Development: Lifelong Learning

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Florian Schacht is professor of mathematics education at the University of Duisburg-Essen and the academic director of the Zentrum für Lehrkräftebildung at the same university. He is a member of the Board of the European Society for Research in Mathematics Education and, since 2025, serves as the president of the German Society for Research in Mathematics Education.

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Solved and unsolved problems

Dorin Andrica

Starting with the June issue, this column is structured into three parts as follows: olympiad problems, undergraduate problems, and open and research problems. Each part is intended to address a different level of mathematical interest and to encourage broad participation from students, teachers, and researchers alike.

We warmly invite you to submit your solutions, as well as new and original problems that may be of interest to the mathematical community. Contributions that promote creativity, insight, and elegance are especially appreciated.

We also encourage you to contribute research and open problems from any area of mathematics for inclusion in the third part of the section. Submissions that highlight new directions, conjectures, or challenging questions are particularly welcome, as they can stimulate further investigation and collaboration.

For all submissions, questions, and correspondence, please use the following email address: dandrica@math.ubbcluj.ro.

The deadline for submitting solutions to the proposed problems is *one month* from the publication of the Magazine issue.

We look forward to your contributions and to fostering a vibrant and engaging mathematical exchange.

I Proposed problems

I.1 Olympiad problems

O1

The positive real numbers a, b, c, d satisfy $abcd = 1$ and

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+a^2) = 2026.$$

Find the maximum possible value of $(ab+cd)(ad+bc)$.

Titu Andreescu (AwesomeMath, Dallas, USA) and Gabriel Dospinescu (École normale supérieure, Lyon, France)

O2

Given a triangle ABC. Construct equilateral triangles BCD, CAE, and ABF externally on the sides of triangle ABC. Let K and X be the circumcenters of triangles AEF and BCD, respectively. Prove that line XK passes through the nine-point center of triangle ABC.

Nikolaos Dergiades (Thessaloniki, Greece) and Quang Hung Tran (High School for Gifted Students, Vietnam National University, Hanoi, Vietnam)

I.2 Undergraduate problems

U1

Consider the folium of Descartes defined by the plane curve

$$Fo\colon x^3+y^3 = 3axy.$$

Suppose that from a point P in the plane, four tangents can be drawn to Fo, touching the curve at points A, B, C, and D. Prove that the origin $O(0,0)$ is the centroid of the quadrilateral $ABCD$.

Nikolaos Dergiades (Thessaloniki, Greece) and Quang Hung Tran (High School for Gifted Students, Vietnam National University, Hanoi, Vietnam)

U2

For $a \geq 0$, let $\mathcal{F}_a$ denote the set of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that satisfy:

(i) $$\frac{f(x)-f(y)}{x-y} \geq a, \quad \text{for all } x, y \in \mathbb{R},\ x \neq y;$$

(ii) $$\int_{f(x)}^{f(x)+1} f(t)\,dt = af(x) + \frac{a}{2}, \quad \text{for all } x \in \mathbb{R}.$$

(a) Give an example of a non-constant function in $\mathcal{F}_0$.
(b) If $a > 0$, determine the set $\mathcal{F}_a$.

Dan-Ștefan Marinescu (Hunedoara, Romania)

II Solutions to the proposed problems

We present detailed solutions to problems 284–289, originally proposed in the EMS Magazine **131** (2024), offering not only complete answers but also insights into the methods and ideas underlying each problem.

In the next issue, we will turn our attention to the newly proposed problems O1–O2 and U1–U2 stated above. These problems vary in difficulty and style, encouraging both creative thinking and technical proficiency. We invite readers to engage actively with these challenges and to submit their own solutions, particularly those that offer novel perspectives or unexpected arguments.

As always, our goal is not only to present correct solutions but also to foster a deeper appreciation for problem-solving as a creative and exploratory process.

284

Let V be a finite set and $\tilde{V} \subset V$. For a finite path $\mathbf{w} = (w_0, w_1, \dots, w_\eta)$ on V, we let $n_0 = 0$, and for $i \geq 1$, we let

$$n_i = \begin{cases} \max\{n_{i-1} \leq n \leq \eta : w_n = w_{n_{i-1}}\} + 1, & \\ & \text{if } w_{n_{i-1}} \in \tilde{V} \text{ and } w_{n_{i-1}} \neq w_\eta, \\ n_{i-1} + 1, & \text{if } w_{n_{i-1}} \notin \tilde{V} \text{ and } n_{i-1} \neq \eta. \end{cases}$$

We call $\mathfrak{L}_{\tilde{V}}(\mathbf{w}) := (w_{n_0}, w_{n_1}, \dots, w_{n_\iota})$, where n_ι is the last index found by the algorithm, the partial loop erasure (PLE) of $\mathbf{w}$.

Let $(X_n, \mathbb{P}_x)$ be a discrete-time irreducible Markov chain on a finite set V, and let $\sigma_A = \min\{n \geq 0 : X_n \in A\}$ and $X|_{[0,\sigma_A]} = (X_0, X_1, \dots, X_{\sigma_A})$. Prove that

$$\mathfrak{L}_V(X|_{[0,\sigma_A]}) \overset{(d)}{=} \mathfrak{L}_V \circ \mathfrak{L}_{\tilde{V}}(X|_{[0,\sigma_A]}),$$

where $\overset{(d)}{=}$ means that both sides have the same law.

Shiping Cao (Department of Mathematics, University of Washington, Seattle, USA)

Solution by the proposer

The question is a simplified version of Theorem 1 of my recent paper [1]. First, we prove that

$$\mathfrak{L}_{V_2}(X|_{[0,\sigma_A]}) \overset{(d)}{=} \mathfrak{L}_{V_2} \circ \mathfrak{L}_{V_1}(X|_{[0,\sigma_A]}) \tag{1}$$

if $V_1 \subset V_2 \subset V$ and $V_2 \setminus V_1 = \{b\}$ for some $\{b\} \notin A$.

We define a random path L as follows: if $\mathfrak{L}_{V_1}(X|_{[0,\sigma_A]})$ does not hit b, we simply let $L = \mathfrak{L}_{V_1}(X|_{[0,\sigma_A]})$; if $\mathfrak{L}_{V_2}(X|_{[0,\sigma_A]}) = (w_{n_0}, w_{n_1}, \dots, w_{n_\iota})$ hits b with $w_{n_j} = b$, we erase loops partially with respect to V_1 in chronological order before n_j and then erase loops partially with respect to V_1 in reverse order after n_j to get L.

For each $\mathbf{w} = (w_0, w_1, \dots, w_\iota)$, we can show that

$$\mathbb{P}_x(\mathfrak{L}_{V_1}(X|_{[0,\sigma_A]}) = \mathbf{w}) = \mathbb{P}_x(L = \mathbf{w}). \tag{2}$$

If $\mathbf{w}$ does not hit b, clearly (2) holds. If $\mathbf{w} = (w_0, w_1, \dots, w_\iota)$ hits b with $w_j = b$, we let $i_0, i_1, \dots, i_\eta$ be all indices corresponding to points contained in $V_1 \setminus A$ (ordered in chronological order), and let $i_0, i_1, \dots, i_k$ be those smaller than j. Then

$$\begin{aligned} \mathbb{P}_x(\mathfrak{L}_{V_1}(X|_{[0,\sigma_A]}) = \mathbf{w}) &= F(w_{i_0}, \dots, w_{i_\eta}) P(w_0, w_1) \cdots P(w_{\iota-1}, w_\iota), \\ \mathbb{P}_x(L = \mathbf{w}) &= F(w_{i_0}, \dots, w_{i_k}, w_{i_\eta}, \dots, w_{i_{k+1}}) \\ &\quad \times P(w_0, w_1) \cdots P(w_{\iota-1}, w_\iota). \end{aligned}$$

Here

$$F(y_0, \dots, y_n) := G_A(y_0, y_0) \cdots G_{A \cup \{y_0, \dots, y_{n-1}\}}(y_n, y_n),$$

and

$$G_B(x, y) = \sum_{m=0}^{\infty} \mathbb{P}_x(X_m = y; \sigma_B > m).$$

By Lawler [2, Proposition 3.3],

$$F(y_0, \dots, y_n) = F(y_{\pi(0)}, \dots, y_{\pi(n)})$$

for any permutation π. Hence (1) follows.

Iterating gives

$$\mathfrak{L}_{V_m} \circ \cdots \circ \mathfrak{L}_{V_1}(X|_{[0,\sigma_A]}) \overset{(d)}{=} \mathfrak{L}_{V_m}(X|_{[0,\sigma_A]}). \tag{3}$$

The remainder follows by conditioning on the path outside $\tilde{V}$ and applying (3).

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285

The following democratic process is implemented in a population of N sheep during a Yes/No referendum. At the initial time $t = 0$, m sheep are pro-Yes, and $N - m$ sheep are pro-No. At each time $t = 1, 2, \dots$, a sheep is randomly chosen (independently of what happened before) and bleeps its opinion. Instantly, a sheep from the opposite camp (if at least one remains) switches its opinion. The process stops when a consensus is reached.

We denote by $P_{m,N}$ the probability that "Yes" wins when starting with m pro-Yes sheep in the population. Of course, $P_{0,N} = 0$, $P_{N,N} = 1$, and due to the symmetry of the problem $P_{N,2N} = 1/2$.

Question. Find a simple expression for $P_{m,N}$. (*In particular, this expression should allow you to estimate very easily the evolution of this probability when starting from a very small majority, i.e., the estimation of $P_{N+a_N,2N}$ for different sequences $1 \ll a_N \ll N$.*)

Lucas Gerin (CMAP, École Polytechnique, Palaiseau, France)

Solution by the proposer

The starting point is, of course, to condition on the opinion of the first chosen sheep. This yields the recurrence equation

$$P_{m,N} = \frac{m}{N} P_{m+1,N} + \frac{N-m}{N} P_{m-1,N},$$

for every $1 \le m \le N-1$. By playing around with small values of m, N, one can guess that $P_{m,N}$ actually coincides with

$$Q_{m,N} := \sum_{\ell=0}^{m-1} \binom{N-1}{\ell} 2^{-N+1},$$

with the convention $\sum_{\ell=0}^{-1} \cdots = 0$. To prove this we first check that $Q_{0,N} = 0$ and $Q_{N,N} = 1$, as required. Then

$$\begin{aligned} &\frac{m}{N} Q_{m-1,N} + \frac{N-m}{N} Q_{m+1,N} \\ &\quad= \frac{m}{N} \sum_{\ell=0}^{m} \binom{N-1}{\ell} 2^{-N+1} + \frac{N-m}{N} \sum_{\ell=0}^{m-2} \binom{N-1}{\ell} 2^{-N+1} \\ &\quad= \Big(\frac{m}{N} + \frac{N-m}{N}\Big) \sum_{\ell=0}^{m-1} \binom{N-1}{\ell} 2^{-N+1} \\ &\qquad+ \frac{m}{N} \binom{N-1}{m} 2^{-N+1} - \frac{N-m}{N} \binom{N-1}{m-1} 2^{-N+1} \\ &\quad= Q_{m,N}. \end{aligned}$$

Thus, $m \mapsto Q_{m,N}$ satisfies the same recurrence of order two as $m \mapsto P_{m,N}$, with the two same boundary values.

It is then convenient for asymptotics to write this expression in the form

$$P_{m,N} = \mathbb{P}\Big(\mathrm{Binomial}\Big(N-1, \frac{1}{2}\Big) \le m-1\Big).$$

(I do not see a direct probabilistic explanation for the appearance of the binomial distribution here.) The central limit theorem implies that $P_{N+a_N,2N} \to 1/2$ as long as $a_N = o(\sqrt{N})$. More interestingly, Hoeffding's inequality implies a sharp threshold for $m \mapsto P_{m,2N}$ around $N \pm \sqrt{N}$: for every integer $a_N \ge 0$

$$P_{N+a_N,2N} \ge 1 - \exp(-a_N^2/N).$$

(Of course by symmetry $P_{N-a_N,2N} \le \exp(-a_N^2/N)$.)

Bibliographical comments

One can find a very detailed study of this model in "P. Flajolet and T. Huillet, Analytic combinatorics of the Mabinogion urn. *Discrete Math. Theor. Comput. Sci.*, Proc. vol. **AI**, 549–572 (2008)," even if the above explicit formula for $P_{m,N}$ does not appear exactly in that form.

286

Let $a \in \mathbb{R}$ and consider the transition probability density $P(x;t|y)$ from a point $y \ge 0$ after a time $t > 0$ of a Brownian motion on the non-negative real line with reflecting boundary conditions, observing that it is the weak solution of the initial value problem with Robin boundary condition:

(1) $\partial_t P(x;t|y) = a\partial_x P(x;t|y) + \frac{1}{2}\partial_x^2 P(x;t|y)$ for all $x, y \ge 0$ and $t > 0$;

(2) $[2aP(x;t|y) + \partial_x P(x;t|y)]_{x=0} = 0$ for all $t > 0$, where the derivative at the origin is the one-sided right derivative;

(3) $\lim_{t\to 0} \int_0^\infty f(x)P(x;t|y)dx = f(y)$ for all bounded continuous functions f and $y \ge 0$.

Prove that the solution is given by

$$P(x;t|y) = \frac{1}{2}\partial_x\Big(\mathrm{erfc}\Big[\frac{y-x-at}{\sqrt{2t}}\Big] - e^{-2ax}\,\mathrm{erfc}\Big[\frac{y+x-at}{\sqrt{2t}}\Big]\Big)$$

for any $x, y \ge 0$ and $t > 0$, where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-y^2}dy$ is complementary error function and ∂_x is the partial derivative with respect to x.

Mario Kieburg (School of Mathematics and Statistics, University of Melbourne, Parkville, Australia)

Solution by the proposer

This problem has been solved in [1, p. 49]. The slightly more complicated model with two barriers has been studied in [2]. We solve the problem with analytical techniques, though there are other ways to derive the statement. The boundary condition (2) is a regularity condition, guaranteeing that the density stays normalized for any $t > 0$. This condition originates from the assumption of a reflection at the origin of the Brownian motion. The condition (3) is the initial value.

For this goal, we define the function

$$\begin{aligned} F(x;t;y) = {}& \frac{1}{\sqrt{t}} \exp\Big[-\frac{a^2 t}{2} - ax\Big] \\ &\times \Big(\Big[\frac{y-x}{t} - a\Big]\exp\Big[-\frac{(y-x)^2}{2t}\Big] \\ &\quad+ \Big[\frac{y+x}{t} + a\Big]\exp\Big[-\frac{(y+x)^2}{2t}\Big]\Big), \end{aligned}$$

which is related to the presumed solution as

$$P(x;t|y) = \frac{1}{\sqrt{2\pi}}\int_y^\infty F(x;t;\tilde{y})e^{a\tilde{y}}\,d\tilde{y}. \tag{1}$$

(1) One can readily show

$$\partial_t F(x;t;y) = a\partial_x F(x;t;y) + \frac{1}{2}\partial_x^2 F(x;t;y)$$

for all $x, y \ge 0$ and $t > 0$. As the integrand is analytic in $x \in \mathbb{R}$ and $t > 0$, and the integral (1) is uniformly convergent on compact sets, we may interchange derivatives and integration. Hence, $P(x;t|y)$ satisfies the heat equation with drift.

(2) Similarly, one can show that F satisfies

$$[aF(x;t;y) + \partial_x F(x;t;y)]_{x=0} = 0,$$

so $P(x;t|y)$ satisfies this Robin boundary condition.

(3) For the initial condition, let f be bounded and continuous with $M = \sup_{x\geq 0} |f(x)|$. Consider

$$\begin{aligned}\int_0^\infty f(x)P(x;t|y)\,dx &= \frac{1}{\sqrt{2\pi t}}\int_0^\infty f(x)e^{-(y-x-at)^2/2t}\,dx\\ &\quad+\frac{1}{\sqrt{2\pi t}}\int_0^\infty f(x)e^{-(y+x-at)^2/2t-2ax}\,dx\\ &\quad+a\int_0^\infty f(x)e^{-2ax}\operatorname{erfc}\Big(\frac{y+x-at}{\sqrt{2t}}\Big)\,dx.\end{aligned}$$

After substitution,

$$\begin{aligned}&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(\sqrt{t}x+y-at)\Theta(\sqrt{t}x+y-at)e^{-x^2/2}\,dx\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(\sqrt{t}x-y+at)e^{-2a(\sqrt{t}x-y+at)}\\ &\qquad\times\Theta(\sqrt{t}x-y+at)e^{-x^2/2}\,dx\\ &\quad+a\int_0^\infty f(x)e^{-2ax}\operatorname{erfc}\Big(\frac{y+x-at}{\sqrt{2t}}\Big)\,dx.\end{aligned}$$

The first two integrands are dominated by $Me^{-x^2/2}$ and converge pointwise:

$$\lim_{t\to 0} = \begin{cases} f(y)e^{-x^2/2}, & y>0,\\ f(0)e^{-x^2/2}\Theta(x), & y=0.\end{cases}$$

The third term can be bounded and vanishes as $t\to 0$ (or cancels if $y=0$). Hence,

$$\lim_{t\to 0}\int_0^\infty f(x)P(x;t|y)\,dx = f(y).$$

References

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[2] O. Kella and W. Stadje, A Brownian motion with two reflecting barriers and Markov-modulated speed. *J. Appl. Probab.* **41**, 1237–1242 (2004)

Open problem

Consider Brownian motion with drift on Herm(N). The transition density is

$$P(H;t|H^{(0)}) = (2\pi t)^{-N/2}(\pi t)^{-N(N-1)/2}\exp\Big[-\frac{\operatorname{tr}(H-H^{(0)}+At)^2}{2t}\Big].$$

For $A=0$, this induces Dyson Brownian motion [1]:

$$p^{(\mathrm{DB})}(E;t|E^{(0)}) = \det\Big[\frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{(E_a-E_b^{(0)})^2}{2t}\Big)\Big]\prod_{a<b}\frac{E_b-E_a}{E_b^{(0)}-E_a^{(0)}}.$$

The Karlin–McGregor formula gives

$$p^{(\mathrm{B})}(E;t|E^{(0)}) = \det\Big[\frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{(E_a-E_b^{(0)})^2}{2t}\Big)\Big].$$

(i) For drift $A=aI$, one obtains

$$\begin{aligned}&p(E;t|E^{(0)})\\ &\quad= \det\Big[\frac{1}{\sqrt{2\pi t}}\exp\Big(-\frac{(E_a-E_b^{(0)}+at)^2}{2t}\Big)\Big]\prod_{a<b}\frac{E_b-E_a}{E_b^{(0)}-E_a^{(0)}}.\end{aligned}$$

(ii) Introducing walls (e.g., positivity constraints on eigenvalues) leads to largely open problems, especially with drift.

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287

We place n different pairs of socks in a tumble dryer. When the dryer has thoroughly mixed the socks, they are taken out one by one; if a sock is the partner of one of the socks on the sorting table, both are removed, otherwise it is put on the table until its partner emerges from the dryer. For $k\in\{1,\dots,2n\}$, let X_k be the number of socks on the table when the kth sock is taken from the dryer. With $X_0=0$, the random path $(X_0,\dots,X_{2n})$ is a Dyck path. Recall that a path $(x_0,\dots,x_{2n})$ is called a *Dyck path* if $|x_i-x_{i+1}|=1$, $x_i\geq 0$ for $1\in\{1,\dots,2n-1\}$ and $x_0=x_{2n}=0$.

The number of Dyck paths of a given length is given by the Catalan numbers, but not all Dyck paths come up with the same probability in the sock sorting process. For example, for $n=2$ there are two Dyck paths, $(0,1,0,1,0)$ and $(0,1,2,1,0)$. The former has probability $\frac{1}{3}$ and the latter $\frac{2}{3}$. The problem is now, given a Dyck path $(x_0,\dots,x_{2n})$ of arbitrary length, to find a formula for the probability

$$\mathbb{P}(X_0=x_0,\,X_1=x_1,\,\dots,\,X_{2n}=x_{2n}).$$

Peter Mörters (University of Cologne, Germany)

Solution by the proposer

The suggested solution is taken from [1]. A Dyck path $(x_0,\dots,x_{2n})$ has n upward and n downward steps. Given such a path we construct the vector $(k_1,\dots,k_n)\in\mathbb{N}^n$ representing the heights of the path from which each downward step is taken. Formally, define $\ell_0=0$ and, for $j=1,\dots,n$, inductively $\ell_j=\min\{i>\ell_{j-1}: x_{i+1}=x_i-1\}$ and set $k_j=x_{\ell_j}$. The vector $(k_1,\dots,k_n)$ satisfies

$$k_{i+1}\geq k_i-1,\quad\text{for}\quad 1\leq i<n,\quad\text{and}\quad k_n=1. \tag{$*$}$$

Conversely, given $(k_1, k_2, \dots, k_n)$ satisfying $(*)$ there is a unique associated Dyck path given by $x_i = i$ if $i \le k_1$ and

$$x_{k_j+2j-1+i} = k_j - 1 + i \quad \text{if} \quad k_j - 1 + i \le k_{j+1} \quad \text{for} \quad j \in \{1, \dots, n\}.$$

In other words, there is a bijection between the set of Dyck paths of length $2n$ and the set of vectors of length n satisfying $(*)$. We now show that if $(k_1(x), \dots, k_n(x))$ denotes the vector associated with the Dyck path $x = (x_0, \dots, x_{2n})$, then

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots, X_{2n} = x_{2n}) = 2^n \frac{n! \prod_{i=1}^n k_i(x)}{(2n)!}.$$

To this end we look at the probability space Ω_n consisting of all permutations of $2n$ distinguishable socks. In our model each of the $(2n)!$ elements is equally likely. If we write the set of socks as

$$\{1, \dots, n\} \times \{0, 1\},$$

with 0, 1 indicating whether it is the left, resp. right, sock, then $\omega = (\omega_1, \dots, \omega_{2n}) \in \Omega_n$ are the socks in the order drawn from the dryer. Write $\omega_m = (s_m, p_m) \in \{1, \dots, n\} \times \{0, 1\}$ where s_m denotes the type of the sock found in the mth draw and p_m indicates whether it is the left or right partner. We set $s_0 = 0$ and $x_0 = 0$ and define

$$x_m = \begin{cases} x_{m-1} + 1 & \text{if } s_m \notin \{1, \dots, s_{m-1}\}, \\ x_{m-1} - 1 & \text{if } s_m \in \{1, \dots, s_{m-1}\}, \end{cases}$$

and observe that $x(\omega) = (x_1, \dots, x_{2n})$ is the Dyck path associated with ω. Our formula is proved if we show that for given $(k_1, k_2, \dots, k_n) \in \mathbb{N}^n$ satisfying $(*)$ there are exactly $2^n n! \prod_{i=1}^n k_i$ different elements $\omega \in \Omega_n$ with $k_i(x(\omega)) = k_i$ for all $i \in \{1, \dots, n\}$.

As above let $(\ell_1, \dots, \ell_n)$ be the ordered indices at which the Dyck path associated with $(k_1, k_2, \dots, k_n)$ steps down. We partition the set of indices $I := \{1, \dots, 2n\}$ into the sets

$$M := \{\ell_1 + 1, \dots, \ell_n + 1\} \quad \text{and} \quad N := I \setminus M.$$

At the indices in N the first instance of a sock type is drawn. There are $n!$ ways to allocate the n types of socks to these n indices and 2^n choices whether the left or right sock is the first. Once this choice is made we look at the indices in M in order. At each of these indices a pair of socks is completed. As there are k_i socks of different type on the table at time ℓ_i, there are exactly k_i choices for ω_{ℓ_i+1}. Altogether, we find exactly $2^n n! \prod_{i=1}^n k_i$ different elements $\omega \in \Omega_n$ with $k_i(x(\omega)) = k_i$ for all $i \in \{1, \dots, n\}$.

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288

In a university hospital, there are 9 major medical departments performing surgery. The cardiology unit performs 21% of all surgeries conducted hospital-wide, with 5% of total cardiac surgeries occurring at least twice a day. A total of 30 patients are assigned to surgery. Assuming that surgeries are equally likely on any given day of the year, if two or more surgeries occur on the same day, what is the probability that they are assigned to the cardiology unit?

Gaetano Valenza (Bioengineering and Robotics Research Center "E. Piaggio" & Department of Information Engineering, School of Engineering, University of Pisa, Italy)

Solution by the proposer

Let us define the events:

C: The surgery is in the cardiology unit.

T: At least two surgeries occur on the same day.

To solve the problem, we can apply Bayes's theorem:

$$P(C \cap C \mid T) = \frac{P(T \mid C \cap C) \cdot P(C \cap C)}{P(T)}$$

where $P(C \cap C)$ is the probability that two surgeries are assigned to the cardiology unit. We already have $P(C) = 0.21$ (probability of cardiology surgery) and $P(T \mid C \cap C) = 0.05$ (probability of at least two surgeries a day given they are performed in the cardiology unit).

Assuming that the two cardiac surgeries may occur independently,

$$P(C \cap C) = P(C) \cdot P(C) = 0.21 \times 0.21 = 0.0441.$$

Given the number of patients $n = 30$, let us calculate $P(T)$ using the Birthday Paradox rule:

$$\begin{aligned} P(T) &= 1 - \frac{365 \times \cdots \times (365 - n + 1)}{365^n} \\ &= 1 - \frac{1}{365^n}\Big(\frac{365!}{(365-n)!}\Big) = 0.7063, \\ P(C \cap C \mid T) &= \frac{0.05 \times 0.0441}{0.7063} = 0.0031. \end{aligned}$$

289

It is the king's birthday, and he decides to please the republican sentiments in the population and release some prisoners who are held for their republican views. The prisoners all have equally excellent rational skills and are placed in separate castles, so that they are unable to speak and communicate with each other. The king decides to challenge the prisoners and gives each of them a coin. Within a quarter of an hour, each of them, say n prisoners, has to decide whether to toss the coin or not. If at least one coin is tossed and all coins being tossed show head, all n prisoners are released, otherwise none of them are. The king believes the prisoners have a slim chance of being released. But is that so? What is the probability that they all are released? The prisoners are told the number n.

Carsten Wiuf (Department of Mathematical Sciences, University of Copenhagen, Denmark)

Solution by the proposer

If they could communicate with each other, they could simply decide that one of them should toss the coin and the others not. Hence, they would have 50% of being released. However, this is not a possibility. Since all n prisoners have excellent rationale skills, they instantly know they should control the number of prisoners tossing a coin. Let x be the probability that prisoner $i = 1, \dots, n$, tosses the coin, which might depend on n and is to be determined. Then, the chance of being released is

$$\begin{aligned} p_n(x) &= \sum_{k=1}^{n} \left(\frac{1}{2}\right)^k \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \binom{n}{k} x^k (1-x)^{n-k} - (1-x)^n \\ &= \left(1 - \frac{1}{2}x\right)^n - (1-x)^n. \end{aligned}$$

If $n = 1$, then $x = 1$ (the prisoner should toss the coin). For $n \geq 2$, by differentiation, we find the optimal $x \in [0, 1]$:

$$x^* = \frac{2^{1/(n-1)} - 1}{2^{1/(n-1)} - 0.5} \approx \frac{1}{n-1} 2 \log(2),$$

the latter approximation for large n. For $n = 1$, $x^* = 1$ and $p_1(x^*) = 0.5$. For $n = 2$, $x^* = 2/3$ and $p_2(x^*) = 1/3$. Generally, $p_n(x^*)$ is decreasing in n and

$$p_n(x^*) > \frac{1}{4}, \quad p_n(x^*) \approx \frac{1}{4} \quad \text{for large } n.$$

III Open and research problems

Cyclotomic Diophantine norm equations

by Preda Mihăilescu

We discuss variants of the cyclotomic norm equation $\frac{x^p + y^p}{x+y} = p^e z^q$, with coprimes integers x, y, z and $p > 3$, a prime.

III.1 Introduction

The reader familiar with the amazing diversity and richness of criteria developed until 1990, in the attempt to prove that the Fermat's conjecture (also: FLT), purporting that the equation

$$x^p + y^p + z^p = 0; \quad x, y, z \in \mathbb{Z}; \quad xyz \neq 0, \quad p > 2, \tag{1}$$

the most famous special case of the family of Diophantine equations that we shall discuss, has no solutions, can observe that it long resembled a vast and barren desert, sporadically punctuated by rare and isolated oasis—exceptional, innovative, and surprising conditions that would necessarily have to hold if the conjecture were false. These conditions were so intricate and specific that their fulfillment appeared highly implausible. Nevertheless, none of these investigative approaches succeeded in reaching the definitive resolution—the metaphorical "ocean" beyond the desert—where the ultimate truth of the conjecture would be established. This description is particularly apt in the context of the *First Case* (also FLTI), which assumes that the prime $p \nmid xyz$. Ribenboim's extensive and illuminating survey [17] captures the breadth of attempts made, documenting the numerous avenues pursued and the audacity of those efforts. Lamé's failed proof from 1841, which was using the assumption that the integers of the pth cyclotomic field are a factorial ring, was disproved by Kummer's counterexample of the field $\mathbb{Q}[\zeta_{23}]$, and this led the way to the discovery of Dedekind rings and classical algebraic number theory, which in turn gave raise to the above-mentioned multitude of dead-end attempts.

Before being archived to the room of solved conjectures, Fermat's question motivated one more major progress in mathematics: The proof of the Tanyama–Shimura conjecture by Wiles [22] and then completed by Wiles and Taylor [20]—by now the only major result in global Langlands theory. Wiles's solitary, now legendary effort was triggered by the observation of Hellegouarch, proved in detail by Frey, to the effect that a solution to (1) implies the existence of an elliptic curve

$$Y^2 = X(X - a^p)(X + b^p),$$

later called a *Frey curve*, with *particular* properties. The connection of this curve to the Tanyama–Shimura conjecture depended on Serre's *epsilon conjecture*, which was proved by Ken Ribet to be true, in the mid-eighties, thus yielding the *Ribet descent theorem*. It states that the existence of the Frey curve above together with the conjecture imply the presence of a modular form of level 2—which however was known not to exist. Wiles, who had proved together with Mazur the *main conjecture of abelian Iwasawa theory*, using Galois representations in a way echoing the proof of the same Ribet for the *converse of Herbrandt's theorem*, took the challenge and eventually accomplished his famous result. One may also consider the consequence from the perspective of Hilbert's famous *bon mot*, regarding (1) as a famous *mathematical curiosity*; nobody will be able to say, to which extent the curiosity itself or the challenge of the major conjecture of Tanyama–Shimura motivated Wiles's work. Fact is that the proof of Fermat's Last Theorem offered *mathematics*, after the discovery of Dedekind rings more than a century before, a new milestone of theoretical mathematics that opened up new horizons.

The proof of Iwasawa's main conjecture had already provided in 1984 the instance of a proof of an intrinsic cyclotomic question, by reaching out to methods from the algebraic geometry of elliptic curves. At the beginning of the following decade however, the work of Thaine and Kolyvagin then succeeded to prove the main conjecture using new ideas in cyclotomy itself. As we shall see in more detail below, the Fermat equation has an inherent cyclotomic nature; there is however to this day no alternative proof known, that would use solely cyclotomic methods.

Another classical Diophantine equation, a *binary exponential one*—as opposed to (1) which is a *ternary* equation, since it involves three distinct bases—is the Catalan equation

$$x^p - y^q = 1, \text{ with } p, q \text{ being distinct odd primes and } x, y \in \mathbb{Z}. \quad (2)$$

The equation has the solution $3^2 - 2^3 = 1$, and Catalan stipulated that this should be the only non-trivial (i.e., with $xy \neq 0$) solution. In this case, the effective bounds proved by Baker for linear forms in logarithms in 1973 led to Tijedeman's ineffective, and shortly after, to Langevin's effective proof of the fact that (2) has at most finitely many solutions. In this case, the powerful, general result of Baker was useful in leading to a long sequence of improvements on upper bounds on the exponents p, q for which a solution might exist; during a quarter of a century of continuous sharpenings—see also [2] for details about this mathematical advances—the bound reached by Mignotte in 2000 was $\min(p, q) < 1.6 \cdot 10^{11}$. In lack of some powerful general result like Ribet's descent in the case of (1), Baker's theory left an open gap of lower bounds on the exponents, necessary for completing the proof of Catalan's conjecture. And this gap was incompletely covered by some cyclotomic algebraic arguments, akin to classical results about (1). The unexpected success of an improved algebraic condition I proposed in 1999—presently called the *double Wieferich criterion*—and which raised the verified lower bound for p, q by a factor of 100, let to the hope that cyclotomy could provide stronger results on the Catalan equation. Upon my question, whether Frey curves could be helpful in this case, Ken Ribet generously replied, explaining that the presence of two distinct exponents made the descent argument impossible, so the answer was negative. This suggested that more research on the cyclotomic track might not be a mad idea—and eventually led to the first proof of the Catalan conjecture, which used arguments of class field theory as a bootstrap for precise and context-specific Diophantine approximation using binomial series. Essentially, algebra helped move the approximation step from the generic setting of Baker theory to one tailored around the properties of the equation (2). This change of strategy leads us to the core question around which this paper is organized: *In the more general context of cyclotomic norm equations, how can one move the balance from proofs using deep structural results, that may not be cyclotomic in nature, to alternatives based on cyclotomy alone, and possibly using dedicated Diophantine approximation?*

In the year 2021, the complete work of S. Mochizuki on inter-universal Teichmüller theory (IUTT) was published in PRIMS [14], after a long-lasting review by a collective of international experts: as announced, it contained an ineffective proof of an abc-inequality. One year later, a team of five—including Mochizuki—proved an effective version of the same inequality and wished to apply it for a proof of FLT and possibly Catalan. Although this should be a simple application of such inequalities, which easily prove upper bounds for the solutions, even in this case, the explicit constant in the proof made the existing—quite substantial—known lower bounds for solutions of FLT, such as featured in [17], insufficient for crossing the upper bound that the abc-inequality offered. I had thus the pleasant opportunity to use methods described in the next chapter, and provide improved lower bounds, which had been useless for more than a decade since I knew them [12, 13]. The paper [16] could thus be published, including a complete proof of FLT, and I followed my ignited interest for the old problem, which led also to the present note.[1] I mention in the next chapter a general framework of the kind of attempts made, and possible improvements.

[1] I must mention that I am very well aware of the unfortunate controversy about Mochizuki's work, and do not write as any kind of expert that would address the technical issues, but as a mere spectator and member of the mathematical community. One that regrets the interruption of communication that happened immediately after Scholze and Stix's publication of the short note [18], that happened unfortunately after they had offered the press the opportunity to drag the issue in the muddy waters of journalistic polarization. Interruption that continued despite the not less than five letters sent by Mochizuki over around five years, inviting for an open peer reviewed debate [15]. Sadly, the muddy waters of journalism spread into the mathematical community, and I experienced personally being interrupted during a lecture, for the mere fact of mentioning the publication of [14], by a violent voice from the audience urging me to "stop with this": the incident could only be curbed by waiting for the colleague to leave the room. I can only hope that, better sooner than later, the mathematical substance will get to speak first, leading to a final universal understanding of strength, possible weaknesses or even flaws of IUTT, thus allowing the entire community to profit of what it can offer, and improve what may be improved.

III.2 The cyclotomic setting

In this note we consider the following Diophantine norm equation:

$$\begin{gathered}\frac{x^p+y^p}{x+y}=p^e z^q; \quad (x,y,z)\in(\mathbb{Z}^*)^3, \quad (x,y,z)=1,\\ \text{and } \ p,q \ \text{ are odd primes—not necessarily distinct,}\\ \text{with } \ p>3; \ e\in\{0,1\}.\end{gathered} \tag{3}$$

We will refer to the case when $e=0$ as the first case and, when $e=1$, as the second case of equation (3). The relation between the value of e and x,y,z is explained below. The connection to (1) will become apparent below, from the classical formulas of Barlow and Abel. The term *strong Fermat* (SFLT) has been coined by Georges Gras and Roland Quême [4] for the case $p=q$ of equation (3). We thereby refer to (3) as the *strong Fermat–Catalan* equation. Note that for $p=3$, (3) has infinitely many solutions that stem from norms of qth powers in the Eisenstein integers: if $(x,y)\in\mathbb{Z}^2$, where ρ is a complex third root of unity and $\xi^q=x+y\rho$ for some $\xi\in\mathbb{Z}[\rho]$, then $\frac{x^3+y^3}{x+y}=z^q$, with $z=\xi\cdot\bar{\xi}\in\mathbb{Z}^*$. The condition $p>3$ is thus necessary.

Andrew Beal investigated solutions of the more general equation

$$x^p+y^q+z^r=0 \tag{4}$$

with prime exponents p,q,r and integers x,y,z and found, by astutely leading computer searches, that the equation does have solutions when $(x,y,z)>1$. He then formulated the conjecture that (4) has no solutions in mutually coprime (x,y,z) and exponents $p,q,r\geq 3$. This bears the name of *Beal conjecture*. The slightly more general equation, in which the exponents are allowed to be 2 or 4, is called *super-Fermat equation*, and a list of sporadic solutions with some even exponents is known and conjectured to be complete. I refer to [1] for a survey of solutions and known special cases of (4); this equation is however beyond the framework of this paper. The Fermat–Catalan equation

$$x^p+y^p=z^q \tag{5}$$

—with odd prime exponents $p\neq q$—appears though as a particular case of (4).

Classical results, notations and prerequisites

A classical fact, often attributed to Euler, states that for coprime integers x,y and $n\in 2\mathbb{N}+1$, $n>1$, the greatest common divisor $d=(\frac{x^n+y^n}{x+y},x+y)$ divides n. This can be easily verified by using the substitution $s=x+y$ and by introducing it in the fraction $\frac{x^n+y^n}{x+y}=\frac{x^n+(s-x)^n}{s}$: the claim follows from $(x,y)=1$. In our case, $n=p$ and if $d=1$, then $p\nmid z$, while for $d=p$, the same substitution implies that $v_p(\frac{x^p+y^p}{x+y})=1$, hence the introduction of p^e in (3).

Using the methods of Barlow and Abel, we find the generalization of their classical factorization [17, lecture IV]: There exist integers u and v, $(u,v)=1$ and $p\nmid uv$, such that

$$x+y=\frac{u^q}{p^e} \quad \text{and} \quad \frac{x^p+y^p}{x+y}=p^e v^p.$$

We see that having solutions to the equation (3) is a necessary—but not sufficient—condition for the Fermat–Catalan equation (5) to have solutions.

We let $P=\{0,1,\ldots,p-1\}, P^*=P\setminus\{0\}$ be the minimal positive representatives for $\mathbb{F}_p$ and $\mathbb{F}_p^\times$, respectively; ζ will be a primitive pth root of unity and $\mathbb{K}=\mathbb{Q}[\zeta]$ the pth cyclotomic field, with Galois group $G=\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. The automorphisms $\sigma_c\in G$ are given by $\zeta\mapsto\zeta^c$, for $c\in P^*$. Complex conjugation acting on $\mathbb{K}$ is denoted by $\jmath=\sigma_{p-1}=\sigma^{(p-1)/2}$.

Characteristic numbers and characteristic ideals

In the cyclotomic field $\mathbb{K}$, the norm decomposes as:

$$\frac{x^p+y^p}{p^e(x+y)}=\prod_{c\in P^*}\frac{y+\zeta^c x}{(1-\zeta^c)^e}.$$

This leads naturally to the following definition.

Definition 1. We define

$$\alpha=\frac{y+\zeta x}{(1-\zeta)^e},$$

as the *characteristic number* of equation (3) and

$$\mathfrak{A}=(\alpha,z)\subset\mathbb{Z}[\zeta],$$

as the *characteristic ideal* of the same equation.

One notices that $\mathbf{N}_{\mathbb{K}/\mathbb{Q}}(\alpha)=z^q$, so the characteristic number encodes the existence of a non-trivial solution to (5). By consequence of the definition, so does the characteristic ideal $\mathfrak{A}$. The characteristic numbers and ideals enjoy the following properties, which open the gate to applications of class field theory:

(C1) The characteristic number α is integral.

(C2) The Galois group G acts on the characteristic number, giving rise to pairwise coprime integral elements; that is, for $1\leq c<d\leq p-1$,

$$(\sigma_c(\alpha),\sigma_d(\alpha))=(1).$$

(C3) $\mathfrak{A}$ is related to the *characteristic number* α by the relations:

$$\mathfrak{A}^q=(\alpha), \quad \mathbf{N}(\mathfrak{A})=(z).$$

(C4) The characteristic number satisfies:

$$\frac{\alpha}{\bar{\alpha}}=\upsilon\cdot\frac{1+\zeta(x/y)}{1+\bar{\zeta}(x/y)}, \quad \text{with} \quad \upsilon=\begin{cases}1 & \text{if } e=0,\\ -\bar{\zeta} & \text{for } e=1,\end{cases}$$

and if $e = 0$ we have

$$\alpha' = \zeta^u \alpha = (x+y)\cdot(1+O(\lambda^2));$$

$$u \equiv \begin{cases} \frac{x}{x+y} \bmod p & \text{if } e=0, \\ 0 & \text{if } e=1. \end{cases} \tag{6}$$

The fact that the characteristic ideal has order dividing q suggest using the Stickelberger ideal for producing algebraic numbers that encode the existence of solutions to (5).

The Stickelberger element $\vartheta = \frac{1}{p}\sum_{c=1}^{p-1} c\sigma_c^{-1} \in \frac{1}{p}\mathbb{Z}[G]$ generates the Stickelberger ideal in the group ring of G over the rational integers, by intersecting its principal ideal with $\mathbb{Z}[G]$, according to

$$I = \vartheta\mathbb{Z}[G] \cap \mathbb{Z}[G].$$

There exists a base for I^-, made of $(p-1)/2$ elements, called *Fueter elements*, e.g., [10], which are

$$\psi_n = \vartheta(1+\sigma_n - \sigma_{n+1}) = \sum_{c \in S_n} n_c \sigma_c^{-1} \in \mathbb{Z}_{\geq 0}[G],$$
$$\text{for } n \in \Big\{1, 2, \dots, \frac{p-1}{2}\Big\}, \tag{7}$$
$$\text{with } n_c = \left(\left[\frac{(n+1)c}{p}\right] - \left[\frac{nc}{p}\right]\right) \text{ and } n_c + n_{p-c} = 1,$$

where the support $S_n \subset \{1, 2, \dots, p-1\}$, satisfies[2] $S_n \cup (p - S_n) = P^*$ and is deduced from the definition of ψ_n.

It is known that I annihilates the class group $C(\mathbb{K})$—see for instance [21, § 15.1]—and more precisely, for $\mathfrak{Q} \subset \mathbb{Z}[\zeta]$ and $t \in I$, the ideal $\mathfrak{Q}^t$ is principal, generated by a *Jacobi number* $\mathfrak{q}(t) \in \mathbb{Z}[\zeta]$, that verifies $\mathfrak{q}(t) \equiv 1 \bmod (1-\zeta)^2$—see [5]. Moreover, the elements $t \in I$ verify $t + \jmath t = w(t)\cdot \mathbf{N}_{\mathbb{K}/\mathbb{Q}}$, for a $w(t) \in \mathbb{Z}$, called the *relative weight of* t.

Diophantine approximation using binomial power series

By applying this to the characteristic ideal, we may define an injective map $\beta : I \to \mathbb{Z}[\zeta]$ given by

$$(\beta(t)) = \mathfrak{A}^t; \quad \beta(t) \equiv 1 \bmod (1-\zeta)^2;$$
$$\beta \cdot \bar{\beta} = z^{w(t)}; \quad \beta(t)^q = \zeta^{ut}\cdot \alpha^t.$$

The exponent ut is determined by the condition $\beta(t) \equiv 1 \bmod (1-\zeta)^2$ and (6). This leads to the crucial relation:

$$\left(\frac{\beta(t)}{\overline{\beta(t)}}\right)^q = (-1)^{et}\cdot\zeta^{2u(t)+et}\cdot\left(\frac{1+\zeta(x/y)}{1+\bar{\zeta}x/y}\right)^t. \tag{8}$$

[2] The set $p - S_n$ designates naturally $\{p - r : r \in S_n\}$.

By defining for $t = \sum_{c\in P^*} n_c\sigma_c^{-1} \in I$, the formal power series

$$\begin{aligned} F_t(T) &= (1+\zeta T)^{t(1-\jmath)/q} \\ &= \prod_{c\in P^*}\left((1+\zeta^{1/c}T)^{n_c/q}\cdot(1+\zeta^{-1/c}T)^{-n_c/q}\right) \\ &=: 1 + \sum_{n>0} a_n(t)T^n \in \mathbb{K}[[T]], \end{aligned}$$

we conclude from (8) that in every topology in which $F_t(x/y)$ converges, we have

$$\frac{\beta(t)}{\overline{\beta(t)}} = \xi\cdot(-1)^{et}\cdot\zeta^{(2u(t)+et)/q}\cdot F_t(x/y),$$

where ξ is some qth root of unity in the respective field. The formal power series is a product of binomial power series and its coefficients are *well-behaved* and understood—see for instance [9, 11]. The root of unity ξ is the bottleneck, since it behaves randomly under Galois action, with the exception of complex conjugation.

The generic idea is the following: For some subset $J \subset I$—for instance the G-orbit of some well-chosen $t \in I$, or a larger set—one considers linear forms

$$\Delta(J, \vec{\ell}) = \sum_{t\in J}\ell(t)\beta(t) \in \mathbb{Z}[\zeta],$$

where the vector $\vec{\ell} \in (\mathbb{Z}[\zeta])^{|J|}$ is chosen in order to satisfy the conditions of the following strategy.

Strategy 1.

(L1) The infinity-norm $\|\vec{\ell}\| := \max(|\ell(t)|; t \in J) \leq \Lambda \in \mathbb{R}$.

(L2) One proves that $\Delta(t, J) \neq 0$.

(L3) The values of $\ell(t)$ are such that the linear combination $\sum_{t\in J}\ell(t)a_n(t) = 0$ for $n \leq M$, and the bounds M and Λ are such that it is possible to estimate $|\Delta(t, \vec{\ell})|$ and this estimate contradicts the vanishing of the first M terms in the power series development.

(L4) Concretely, in the complex topology, the linear combination of power series will be smaller than 1 in absolute value in all embeddings, while by (L2), $\mathbf{N}(\Delta(t, J))$ is a non-vanishing integer. In local completions, the converse happens: the vanishing of the leading terms of the power series implies that $|\Delta(t, J)|$ is globally *large*—$O(x^M)$—while the choice of Λ and estimates for $|\beta(t)| = z^{w(t)/2}$ are used to prove that $|\Delta(t, J)|$ is globally smaller.

The original proof of Catalan's conjecture used a network of additional properties of solutions to (2); these allowed to extend the β-map to annihilators of the real part of $C(\mathbb{K})$. This way, the dependency of undetermined roots of unity vanishes, and the proof emerged from the investigation of one single series; no linear combinations were necessary. In a few later applications of mine and a few other authors, the generic approach described above was followed more closely, *albeit only in the global case and* *for binary equations*. The roots of unity were treated simply as additional unknowns—which of course increases the complexity of the linear algebra and the bounding strategies. Recently, together with my doctoral student Chang Liu, we completed a new proof of the same conjecture using only annihilation by the Stickelberger ideal [8]. When treating the roots of unity as Galois independent indeterminates—which they are—the consequence is an increased number of linear conditions needed for obtaining the same order of vanishing M in (L3); this in return increases the bound Λ in (L1). The result in [8] is noteworthy, since we succeeded to decrease Λ by use of short vectors in lattices, and nevertheless assure the non-vanishing condition (L2), which is reputed as very difficult when using variants of Siegel's box principle.

All the above results use the global variant of Strategy 1. The local strategy requires semi-local power series development, in all the completions at primes $\mathfrak{R}|x$ in $\mathbb{K}$. One thus works in the *"x-idéles"*[3]

$$\mathbb{K}_x := \prod_{\mathfrak{R}|x} \mathbb{K}_{\mathfrak{R}}.$$

While in this case the unknown roots of unity are acted upon by the idelic lift of the Galois group G, the number of independent roots depends on the number of such prime ideals $\mathfrak{R}$, which is not known to obey any bound. The diversity of fascinating alternative power series and relations, already in the case of the Fermat equation, invites to pursued attempts for searching some combination that succeeds to meet all the requirements. However, the final bottleneck remains in all cases the fact that the size of the ideal I/pI limits the size of J, but it is comparable, possibly identical, to the number of various local values that the roots ξ can have. Unlike the global case, where it was possible to control roots of unity either by directly working in $\mathbb{R}$ when possible, or by treating them as unknowns—which have the advantage of having the smallest possible norm, being units—there is no way to simplify the contribution of the *local* roots of unity. In fact, the very assumption that they would vanish for an entire G-orbit Gt, $t \in I$, suffices for obtaining upper bounds on $|x|$ in (3), which imply FLT.

These challenges and the lack of any successful application of the local strategy would seem not to recommend it for further study. On the other hand, the map β produces a large variety of algebraic numbers with both arithmetic specificity and size anchored in the very values of the purported solution of (3)—whose existence one tries to do infirm—and it encrypts by equivalence the existence of these solutions. There remain reasons to expect that some new ideas, and a better combination of Strategy 1 with arithmetic properties and some ideas connecting applications of the Bombieri–Vaaler theorem could lead to the success which is missed by a fraction, in the crude version of the strategy.

[3] The ring used is not formally equal to the x-idéles, but conserves their arithmetic and Galois properties.

An appealing reason for facing these challenges lays in the fact that—at least in the present crude version—the proposed approach does not see deep distinction between the Fermat and the Fermat–Catalan cases, and the equation (3) is one about which very little is known presently.

III.3 Cyclotomic conjectures and properties implying FLT

The hypothesis of Kummer, stating that p does not divide the class number of the real pth cyclotomic field, fascinated Harry Schultz Vandiver. He published in the first half of the 20th century a long series of criteria for FLT, with the apparent expectation to reduce the proof of the conjecture to the hypothesis, which bared for a long time the name of *Vandiver conjecture*; until Lang discovered a correspondence of Kummer and Dedekind, in which the first referred precisely to that hypothesis, so the name became *the Kummer–Vandiver conjecture*. It was proved in the times of Vandiver, that FLTII follows from this conjecture and an additional condition. Vandiver himself published a paper which promised to reduce the truth of the First Case to a mere consequence of the same conjecture. The proof was however incorrect, and still by the time Ribenboim wrote his *13 Lectures* [17], the statement was considered to be irrecoverably false. However, Sitaraman uncovered from Vandiver's writings and proved with more transparent methods in [19], an implicit additional condition that Vandiver had not stated, and which together with the Kummer–Vandiver conjecture implies the truth of FLTI. There is thus today a program in three points—see also [1]—for proving FLT as a consequence of deeper theoretical conjectures or possible facts of cyclotomy. The assumptions to prove are the following:

Strategy 2. *Assumption A:* The Kummer–Vandiver conjecture is true, so p does not divide the class number of the real p-cyclotomic field $\mathbb{K}^+$.

Assumption B: Assume that the exponent of the p-Sylow subgroup $(C(\mathbb{K}))_p$ is p, whereby the p-part of the class group of the pth cyclotomic field is annihilated by p.

Assumption C: Assume that all the units $\delta \in \mathbb{Z}[\zeta_p + \bar{\zeta}_p]^\times$ for which there exists an algebraic integer $\rho \in \mathbb{Z}[\zeta_p + \bar{\zeta}_p]$ such that $\delta \equiv \rho^{p^2} \bmod (p(1-\zeta))^2 \cdot \mathbb{Z}[\zeta_p + \bar{\zeta}_p]$ are global pth powers.

Assumption D: Assume that the relative class number of the p-cyclotomic field verifies

$$|(C(\mathbb{K}))_p^-| < \sqrt{p} - 1.$$

The first three assumptions imply FLT, but these results would have wider theoretical consequences than merely such proof. Their application to Diophantine equations however would be restricted to this and possibly few more strongly related exponential equations: the core restriction is that all exponents in an equation to which these facts can be applied, must be p! It is probable that ideas and facts from Iwasawa theory could indicate useful approaches for the proof of these assumptions, but discussing more details goes beyond the scope of this brief note.

The Assumption D is of particular interest not only because it implies FLTI, by the result of Eichler [3]. But in spite of the fact that Washington's heuristics—see [6, pp. 261–265]—suggest that $|(C(\mathbb{K}))_p^-| = O(\log(p)/\log\log(p))$, the only upper bound known for the p-part is derived from the very accurate upper bounds given by Lepistö in [7] for the entire minus part of the class group, assuming that it is all limited to the p-Sylow subgroup. This is of course a crude assumption, and we conclude this note by suggesting a combinatorial approach that may provide some improvements. If they succeed in proving less or more than Assumption D depends on the accuracy of the combinatorics. The idea is the following: Iwasawa proves—see [21, Theorem 6.21]—that

$$|(\mathbb{Z}[G]/I)_p^-| = h_p^- := |(C(\mathbb{K}))_p^-|.$$

Lower bounds for $|(I/pI)^-|$ thus provide upper bounds for h_p^-. Let

$$\mathcal{F} = \Big\{\psi_n : n \in P^*,\ n \leq \frac{p-1}{2}\Big\} \subset I, \quad \text{and} \quad \overline{\mathcal{F}} = \{G\psi : \psi \in \mathcal{F}\}$$

be the Fueter elements in (7) and their G-orbits. For $0 < m < p$, the set

$$\mathbf{F}_m = \Big\{\sum_j c_j t_j \ :\ 0 \leq c_j;\ t_j \in \overline{\mathcal{F}} \text{ and } \textstyle\sum_j c_j \leq m\Big\} \subset I/pI.$$

The combinatorial problem consists in estimating the number of mutually distinct elements in $\mathbb{Z}[G]$ that lie in $\mathbf{F}_{p-1}$. Since we allow for entire G-orbits in $\overline{\mathcal{F}}$, the linear combinations in $\mathbf{F}$ will contain multiple representations of the same element in $\mathbb{Z}[G]$ modulo the norm, and estimates become hard. Chang Liu computed $|\mathbf{F}_2| = \frac{(p-1)^2((p-2)^2+27)}{72}$ in [8]. One expects that the general result will be of the form $|\mathbf{F}_m| = \frac{h_m(p)}{D_m}$, with $h_m \in \mathbb{Z}[X]$, a polynomial of degree $2m$ and $D_m \in \mathbb{N}$. The size of D_m however grows unpredictably. But even predicting some recursively provable upper bounds for D_m and lower bounds for $h_m(p)$ may provide useful results.

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New editor appointed

The scientific areas of interest of Dorin Andrica are differential topology (critical point theory with applications, Morse theory with applications), differential geometry, number theory, mathematics education, and mathematics in competitions and Olympiads. He has published numerous books, monographs, and conference volumes, over 200 scientific articles, and has participated in many national and international conferences and symposia. His work is cited in numerous scientific papers, monographs, and doctoral theses. He is a member of the editorial boards of several specialized journals.

It is worth noting that he has published 12 books with prestigious international publishers in English, Arabic, Portuguese, Japanese, and Korean. He is the author of over 1,000 original problems proposed in various journals in Romania and abroad. He ranks 85th among contributors to *Gazeta Matematică* since its founding (over 130 years of continuous publication), based on the number of problems and articles published.

Among the many awards and distinctions received by Professor Dorin Andrica from Babeș–Bolyai University are: the Scientific Excellence Diploma (2002 and 2005), the Institutional Innovation Diploma (2003), the Comenius Award (2009), the Teaching Excellence Diploma (2016). Also, he received the SSMR Diploma "110 years of continuous publication of Gazeta Matematică" (2005), and the SSMR Excellence Diploma (2017).

He has carried out numerous teaching and research visits at prestigious universities abroad, including King Saud University, Riyadh, Saudi Arabia (January 2010–June 2012), and Columbus State University, Columbus, GA, USA (Mildred Miller Fort Foundation Visiting Scholar position, August–December 2013).

The inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$, for every natural number $n \geq 1$, concerning the sequence of prime numbers, is known as "Andrica's conjecture." It is one of the difficult open problems of contemporary mathematics and is mentioned in numerous papers, books, and online sources.

His continuous presence since 1982, in the central committees of the National Mathematics Olympiads represents a landmark for the community of Romanian teachers dedicated to high-performance school mathematics. He also served as senior coordinator at the 40th International Mathematical Olympiad (Bucharest, 1999), at the 50th International Mathematical Olympiad (Cluj Napoca, 2018), and at two Balkan Mathematical Olympiads (Bucharest 1986, Iași 2005).

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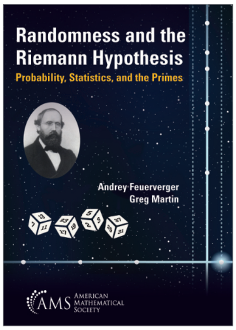

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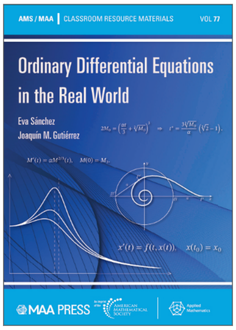

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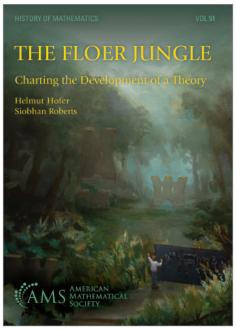

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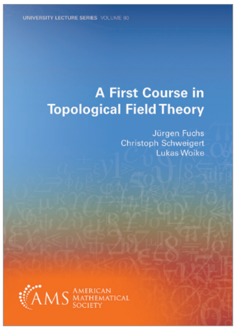

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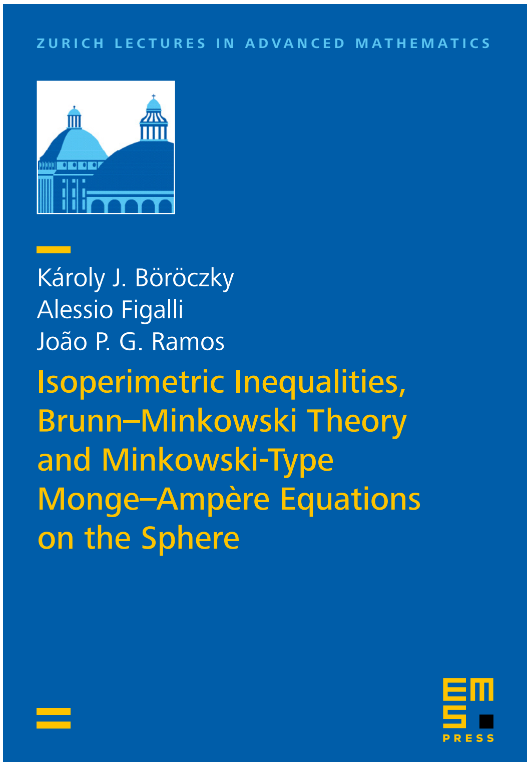

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