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Elemente der Mathematik

Analogies are interesting!

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Let an equilateral triangle ABC be inscribed in a circle. Let P be any point on the minor $\widehat{BC}$ (arc BC) as in Fig. 1. Then it is well-known that $AP = BP + PC$. (This can be easily proved using Ptolemy's theorem, trigonometry or by other means.) Is not this property pretty? Obviously, we do not expect a non equilateral triangle to exhibit such a property. However, in a given triangle ABC it seems reasonable to expect the existence of at least one point P with the above property. Surprisingly, a non equilateral isosceles triangle does *not* possess such a point. Our discussion examines the reason and considers interesting special cases.

Next, we look at the right angled triangle ABC in which $\angle BAC = \pi/2$. If AD is drawn perpendicular to BC (shown in Fig. 2), then it is well-known that $AD^2 = BD \cdot DC$. In fact, if D' is the mid point of BC of the same ΔABC, then also it trivially follows that $AD'^2 = BD' \cdot D'C$ because D' is the center of the circle ABC. The second part of our discussion focuses on the characterization of ΔABC in which at least one point D exists on BC with the property just mentioned. Before we begin the discussion, let us recall the necessary background euclidean geometry results.

Analogien spielen bekanntlich eine wichtige Rolle beim Entdecken neuer Erkenntnisse in der Mathematik. An diese Tatsache knüpft der Autor in der nachfolgenden Arbeit an. Ausgehend von dem bekannten Ergebnis, dass für einen Punkt P auf dem Umkreis eines gleichseitigen Dreiecks ABC zwischen B und C die Beziehung $AP = BP+PC$ besteht, wird in Umkehrung dazu nach notwendigen und hinreichenden Bedingungen an die Seiten eines beliebigen Dreiecks ABC gesucht, so dass ein Punkt P auf dem Umkreis des gegebenen Dreiecks zwischen B und C existiert, der der obigen Gleichung genügt. Ausgangspunkt für das zweite Beispiel ist der Höhensatz im rechtwinkligen Dreieck. Verallgemeinernd dazu wird nach notwendigen und hinreichenden Bedingungen an die Seiten eines beliebigen Dreiecks ABC gefragt, so dass ein Punkt D auf der Seite BC existiert, der die Beziehung $AD^2 = BD \cdot DC$ erfüllt.

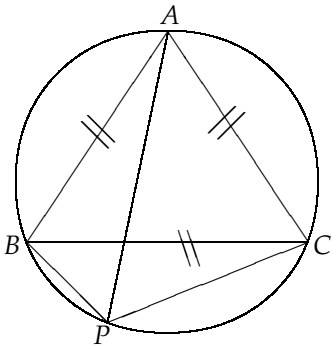

Fig. 1 $AP = BP + PC$

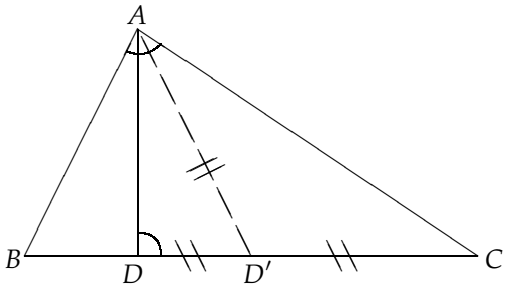

Fig. 2 $AD^2 = BD \cdot DC$ and $AD'^2 = BD' \cdot D'C$

Background material

We use the standard notation throughout our discussion. In ΔABC, a, b, c denote either the sides or the lengths BC, CA, AB and A, B, C denote either the vertices or the measures of $\angle BAC$, $\angle CBA$, $\angle ACB$, respectively. We assume the reader's familiarity with the angle bisector theorem, properties of similar triangles, intersecting chords of a circle and so on. In addition, we need the result that is sometimes called Stewart's theorem. Therefore, we derive it below as an application of the cosine rule. This result is central to our discussion.

Theorem 0 *Let D be a point on the side BC of ΔABC. If $BD/DC = m/n$, then*

$$AD^2 = \frac{mb^2 + nc^2}{m+n} - \frac{mn\,a^2}{(m+n)^2}\,. \qquad (*)$$

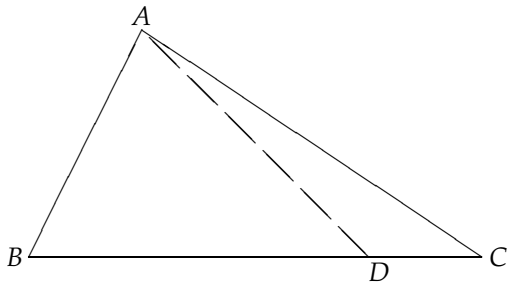

Fig. 3 $\frac{BD}{DC} = \frac{m}{n}$, $AD^2 = ?$

Proof. We refer to Fig. 3. If $BD/DC = m/n$, then by the property of equal ratios $BD/(BD+DC) = m/(m+n)$, so $BD = ma/(m+n)$ and $DC = na/(m+n)$. We now apply the cosine rule for ΔABD. This gives

$$\begin{aligned} AD^2 &= AB^2 + BD^2 - 2 \cdot AB \cdot BD \cdot \cos(\angle ABD) \\ &= c^2 + \frac{m^2 a^2}{(m+n)^2} - 2 \cdot c \cdot \frac{m\,a}{m+n} \cdot \frac{c^2 + a^2 - b^2}{2c\,a}. \end{aligned}$$

The simplification (left to the reader) of the above expression leads to the result given in $(*)$. □

Remark 1 The result $(*)$ is true for any position of the point D on BC that is within BC as in Fig. 3, on BC extended or CB extended. In such an extended case we should observe the usual sign convention, i.e., $BD > 0$, $DC > 0$ if D is as in Fig. 3, $BD > 0$, $DC < 0$ if D is to the right of BC and $BD < 0$, $DC > 0$ if D is to the left of BC.

The analogies

We first discuss the equilateral triangle analogy mentioned in the introduction. Theorem 1 shows that the sides of a ΔABC must be expressible in a certain form for the existence of a point P on the minor $\overset{\frown}{BC}$ with the property $AP = BP + PC$. We will then deduce a number of corollaries. Except for Corollary 6, Theorem 1 and the Corollaries 1 to 5 are true for positive real numbers a, b, c, m and n.

Theorem 1 *Let ΔABC be inscribed in a circle. Then a point P exists on the minor $\overset{\frown}{BC}$ such that $AP = BP + PC$, if and only if the sides have the form $a = mu^2 + nv^2$, $b = u(mu + nv)$, $c = v(mu + nv)$. Here u and v are positive real numbers and m/n is the ratio in which AP divides the side BC.*

Proof. We refer to Fig. 4. From $(*)$ we have

$$AD^2 = \frac{mb^2 + nc^2}{m+n} - \frac{mn\,a^2}{(m+n)^2}. \tag{1}$$

Since the triangles BDP and ADC are similar, we find that

$$\frac{BD}{AD} = \frac{BP}{AC}, \text{ i.e., } \quad BP = \frac{mab}{(m+n)AD}.$$

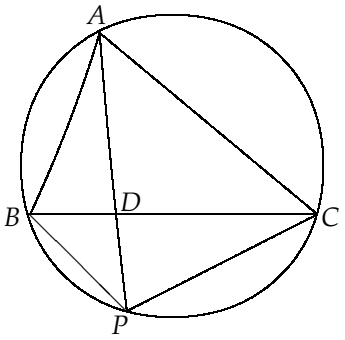

Fig. 4 When $AP = BP + PC$?

Likewise from the similar triangles CDP and ADB we find that

$$PC = \frac{nac}{(m+n)AD}\,.$$

Hence $AP = BP + PC$, if and only if

$$AD \cdot AP = \frac{mab}{m+n} + \frac{nac}{m+n}\,, \qquad \text{or}$$

$$AD^2 + AD \cdot DP = \frac{a(mb+nc)}{(m+n)}\,. \tag{2}$$

In Fig. 4, AP and BC are intersecting chords of the circle. Therefore, $AD \cdot DP = BD \cdot DC = mna^2/(m+n)^2$. Using the preceding result and substituting (1) into (2) gives

$$a(mb+nc) = mb^2 + nc^2.$$

Hence, the sides a, b, c of ΔABC must be expressible as $a = (mb^2 + nc^2)/(mb+nc)$, b, c. We now take $b = u$, $c = v$ as arbitrary positive real numbers and multiply the sides by $(mu + nv)$. This similarity transformation enlarges AP, BP, PC also by the same factor and hence the property $AP = BP + PC$ still holds. This gives the characterization mentioned in Theorem 1. □

We now deduce a number of corollaries from Theorem 1. Corollary 1 gives another but simpler necessary and sufficient condition for ΔABC to have the desired point P.

Corollary 1 *Let ΔABC be inscribed in a circle. A point P exists on the minor $\overset{\frown}{BC}$ such that $AP = BP + PC$, if and only if the sides a, b, c satisfy the inequalities $b > a > c$ or $c > a > b$.*

Proof. Theorem 1 tells us that the desired point P exists if and only if $a = (mb^2 + nc^2)/(mb+nc)$. This is equivalent to $m/n = c(c-a)/b(a-b)$ or $a = b = c$, the latter being excluded here. Since AP intersects BC internally, the ratio $m/n > 0$. This is possible if and only if a, b, c satisfy $b > a > c$ or $c > a > b$ and the proof is complete. □

In fact, Corollary 1 provides greater insight into the subject under discussion as we see below.

Corollary 2 *A non equilateral isosceles triangle ABC does not contain a point P on the minor $\widehat{BC}$ such that $AP = BP + PC$.*

Proof. The hypothesis of Corollary 2 implies exactly one of $a = b$, $a = c$, $b = c$. This is neither $b > a > c$ nor $c > a > b$. Hence, P does not exist. Alternatively, Theorem 1 shows that if one of the three above equalities holds then the other two *must* follow. This forces ΔABC to be equilateral contradicting the hypothesis of Corollary 2. □

Corollary 3 shows the uniqueness of the point P.

Corollary 3 *If the sides of ΔABC satisfy $b > a > c$ or $c > a > b$, then there exists a unique point P on the minor $\widehat{BC}$ such that $AP = BP + PC$.*

Proof. In the proof of Corollary 1 we saw that $m/n = c(c-a)/b(a-b)$. Hence the ratio m/n is uniquely determined by a, b, c. Therefore, the point D on BC and hence P are unique, too. □

Corollary 4 *Suppose that a set of points P_i exists on the minor $\widehat{BC}$ of ΔABC such that $AP_i = BP_i + P_iC$, $i = 1, 2, \ldots$ Then, ΔABC must be equilateral.*

Proof. Since ΔABC is fixed, a, b, c remain constant but the ratio m/n varies as P moves over P_i. In other words, the equation $mb(a-b) = nc(c-a)$ can hold for many pairs of values of m, n, if and only if $a = b$ *and* $c = a$ implying that ΔABC must be equilateral. □

Corollary 5 (Hoppe triangles) *Let AP (Fig. 4) bisect $\angle BAC$ of ΔABC. Then, $AP = BP + PC$, if and only if the sides b, a, c are in arithmetic progression.*

Proof. The angle bisector theorem states that AP, i.e., AD bisects $\angle BAC$, if and only if $m/n = c/b$. Hence, $AP = BP+PC$, if and only if $a = (mb^2+nc^2)/(mb+nc) = \frac{1}{2}(b+c)$. This establishes the corollary. In fact, here $AP = 2BP$ and $AD = \frac{1}{2}\sqrt{3bc}$. □

Remark 2 Dickson in his monumental work, *History of the Theory of Numbers* [1] tells that Hoppe gives a complete discussion of *Heron* triangles (triangles with integer sides and area) in which the sides are in arithmetic progression. To honour Hoppe we call a triangle *Hoppe* triangle if the sides are in arithmetic progression. At this point we mention that there is a vast literature on Heron triangles and the reader is invited to refer to [3, 4, 6, 7] to get a glimpse of the many interesting Heron problems.

Corollary 6 *Suppose the elements a, b, c, m, n of ΔABC are positive integers. If AD is a median of ΔABC, then $AP = BP + PC$, if and only if $AB + AC$ is either a square or twice a square.*

Proof. Since AD is a median it follows that $BD = DC$ and $m = n = 1$. Therefore, $a = u^2 + v^2$, $b = u(u+v)$, $c = v(u+v)$. Now, $AB + AC = (u+v)^2$ is a (perfect) square. However, if u and v are both odd integers, then $\gcd(a, b, c) = 2$. The sides of the reduced (but similar) triangle will have $AB + AC$ twice a square. Conversely, if $AB + AC = w^2$ or $2w^2$, then we can always choose $v = 1$, $u = w - 1$ to get a desired ΔABC. □

An example. To make the argument clearer, suppose $AB + AC = 36 = 6^2$. Then, $u + v = 6$. If $u = 5$, $v = 1$, then $a = 26$, $b = 30$, $c = 6$ and $\gcd(a, b, c) = 2$. If we divide the sides by 2, then the reduced similar ΔABC has sides $a = 13$, $b = 15$, $c = 3$ and $AB + AC = 18 = 2(3)^2$ is twice a square. We remark that under a similarity transformation, the property $AP = BP + PC$ is invariant.

Next, we consider the right angled triangle analogue mentioned in the introduction. Theorem 2 provides a characterization of ΔABC in which there is a point D on the side BC such that $AD^2 = BD \cdot DC$. In other words, AD is the geometric mean between BD and DC.

Theorem 2 *In triangle ABC a point D on the side BC exists with the property $AD^2 = BD \cdot DC$, if and only if the sides a, b, c satisfy the inequalities $\sqrt{2}a \geq b + c > a$.*

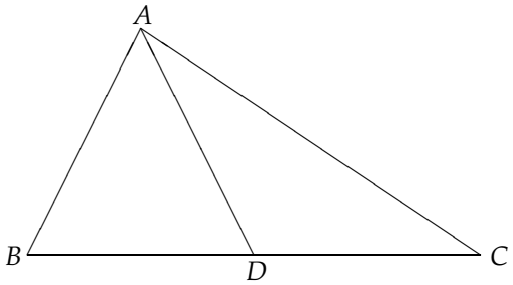

Fig. 5 When is AD the geometric mean of BD and DC?

Proof. We refer to Fig. 5. From $(*)$ the equation $AD^2 = BD \cdot DC$ is equivalent to

$$\frac{mb^2 + nc^2}{(m+n)} - \frac{mn\,a^2}{(m+n)^2} = \frac{mn\,a^2}{(m+n)^2}\,.$$

We rewrite the above equation as a quadratic in m/n. This yields

$$\left(\frac{m}{n}\right)^2 b^2 + (b^2 + c^2 - 2a^2)\left(\frac{m}{n}\right) + c^2 = 0$$

with the solutions

$$\frac{m}{n} = \frac{-b^2 - c^2 + 2a^2 \pm \sqrt{\big((b+c)^2 - 2a^2\big)\big((b-c)^2 - 2a^2\big)}}{2b^2}. \tag{3}$$

By the triangle inequality it follows that $b - c < a$. Hence, $(b-c)^2 - 2a^2 \leq 0$. To get a real root for m/n we must therefore have $(b+c)^2 - 2a^2 \leq 0$. That is $\sqrt{2}\,a \geq b + c > a$, the characterization asserted by the theorem. □

Remark 3 In general, equation (3) yields two roots for $m/n > 0$ and hence two positions for D on BC such that the cevian AD is the geometric mean of BD and DC. D exists uniquely when m/n is a double root. This happens, if and only if $\sqrt{2}\,a = b + c$. Corollary 7 tells more about such triangles.

Corollary 7 *The point D on side BC of ΔABC exists uniquely with the property $AD^2 = BD \cdot DC$, if and only if AD bisects $\angle BAC$.*

Proof. In Remark 3 we saw that D exists uniquely, if and only if $\sqrt{2}\,a = b + c$. In this case, equation (3) shows that $m/n = (-b^2 - c^2 + 2a^2)/2b^2 = c/b$. The angle bisector theorem says that m/n, i.e., $BD/DC = c/b$, if and only if AD bisects $\angle BAC$. This establishes Corollary 7. Moreover, in this case we find that $AD = \sqrt{\frac{1}{2}bc}$. □

Corollary 8 gives the length of the tangent at the vertex A to the circle ABC intercepted by the side BC.

Corollary 8 *Let ΔABC be inscribed in a circle. Suppose the tangent at A meets the side BC (extended) at D. Then, $AD = abc/|c^2 - b^2|$.*

Proof. The tangent-secant property says that $AD^2 = BD \cdot DC$. If $c > b$, then D lies to the right of BC and if $c < b$, then D lies to the left of BC. If $c = b$, then AD is parallel to BC and undefined as a length. In either case, $BD/DC = |c^2/b^2|$. Hence, $BD = c^2a/|c^2 - b^2|$, $DC = b^2a/|c^2 - b^2|$ and the assertion of the corollary follows. □

Conclusion

The reader witnessed an interesting discussion from the consideration of analogies of known results. Euclidean geometry has been studied for thousands of years and yet new results appear on the pages of mathematics journals around the world. We invite the reader to make a new contribution to the following lovely but unsolved problem in natural number solutions (as far as I am aware) on the equilateral triangle:

> Let P be a point in the plane of an equilateral triangle ABC. Assume that $PA = p$, $PB = q$, $PC = r$ are natural numbers. Determine the length s in natural numbers of a side of ΔABC in terms of p, q, r.

This is a famous problem with rich history. Reference [2] gives the details. The numbers p, q, r, s are inter-related by the equation

$$3(p^4 + q^4 + r^4 + s^4) = (p^2 + q^2 + r^2 + s^2)^2 .$$

The remarkable symmetry exhibited by the above equation shows that any three of the four values p, q, r, s may be taken as the distances of the point P and the fourth one gives the side length of the equilateral triangle. The natural number solutions for p, q, r, s are known in the special case when the point P lies on a side or on the circum circle of the equilateral ΔABC. These two instances are covered by setting $p = q + r$ and then solving for s. Another special but new solution is found in [5]. Who is the lucky person who finds the general solution of the above problem/equation in natural numbers?

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