# $L^2$-Betti Numbers of Infinite Configuration Spaces

By

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## Abstract

The space $\Gamma_X$ of all locally finite configurations in a infinite covering $X$ of a compact Riemannian manifold is considered. The de Rham complex of square-integrable differential forms over $\Gamma_X$, equipped with the Poisson measure, and the corresponding de Rham cohomology and the spaces of harmonic forms are studied. A natural von Neumann algebra containing the projection onto the space of harmonic forms is constructed. Explicit formulae for the corresponding trace are obtained. A regularized index of the Dirac operator associated with the de Rham differential on the configuration space of an infinite covering is considered.

## Contents

## §1.   Introduction

Let $\Gamma_X$ denote the space of all locally finite subsets (configurations) in a complete, connected, oriented Riemannian manifold $X$ of infinite volume with a lower bounded curvature. In this paper, we study the de Rham complex of square-integrable differential forms over the configuration space $\Gamma_X$ equipped with the Poisson measure, in the case where $X$ is an infinite covering of a compact manifold.

The growing interest in geometry and analysis on the configuration spaces can be explained by the fact that these naturally appear in different problems of statistical mechanics and quantum physics. In [8], [9], [10], an approach to the configuration spaces as infinite-dimensional manifolds was initiated. This approach was motivated by the theory of representations of diffeomorphism groups, see [31], [51], [32] (these references as well as [10], [12] also contain discussion of relations with quantum physics). We refer the reader to [11], [12], [48], [38] and references therein for further discussion of analysis on the configuration spaces and applications.

Stochastic differential geometry of infinite-dimensional manifolds, in particular, their (stochastic) cohomologies and related questions (Hodge–de Rham Laplacians and harmonic forms, Hodge decomposition), has also been a very active topic of research in recent years. It turns out that many important examples of infinite-dimensional non flat spaces (loop spaces, product manifolds, configuration spaces) are naturally equipped with finite measures (Brownian bridge, Gibbs measures, Poisson measures). The geometry of these measures is related in a nontrivial way with the differential geometry of the underlying

spaces themselves, and plays therefore a significant role in their study. More-over, in many cases the absence of a proper smooth manifold structure makes it more natural to work with $L^2$-objects (such as functions and sections, etc.) on these infinite-dimensional spaces, rather than to define infinite dimensional analogs of the smooth finite dimensional objects.

Thus, the concept of an $L^2$-cohomology has an important meaning in this framework. The study of $L^2$-cohomologies for finite-dimensional mani-folds, initiated in [18], is a subject of many works (whose different aspects are treated in e.g. [26], [23], [29], see also the review papers [42], [40]). As for the infinite-dimensional case, loop spaces have been most studied [33], [36], [28], [37], the last two papers containing also a review of the subject. Hypersur-faces in the Wiener space were considered in [35]. The de Rham complex on infinite product manifolds with Gibbs measures (which appear in connection with problems of classical statistical mechanics) was constructed in [1], [2] (see also [19] for the case of the infinite-dimensional torus). We should also men-tion the papers [49], [15], [16], [17], [7], where the case of a flat Hilbert state space is considered (the $L^2$-cohomological structure turns out to be nontriv-ial even in this case due to the existence of interesting measures on such a space).

In [3], [4], the authors started studying differential forms and the cor-responding Laplacians (of Bochner and de Rham type) over the configura-tion space $\Gamma_X$. The main result of [4] is a description of the space $\mathbf{K}^{(*)}$ of square-integrable (with respect to the Poisson measure) harmonic forms over $\Gamma_X$:

$$(1) \qquad \mathbf{K}^{(*)} \simeq \mathcal{A}_{sym}(\mathcal{K}^{(1)}, \ldots, \mathcal{K}^{(d)}),$$

where $\mathcal{A}_{sym}(\mathcal{K}^{(1)}, \ldots, \mathcal{K}^{(d)})$ is a super commutative Hilbert tensor algebra generated by the spaces $\mathcal{K}^{(m)} = \mathcal{K}^{(m)}(X) := \operatorname{Ker} H_X^{(m)}$, $H_X^{(m)}$ denoting the Hodge–de Rham Laplacian in the $L^2$-space of $m$-forms on $X$, $m = 1, \ldots, d$, $d = \dim X - 1$. In other words, $\mathbf{K}^{(n)}$ is described in terms of symmetric and antisymmetric tensor products of the spaces $\mathcal{K}^{(m)}(X)$ (a version of the Künneth formula). The spaces $\mathbf{K}^{(n)}$ appear to be finite-dimensional, provided so are all the $\mathcal{K}^{(m)}(X)$ spaces. Their dimensions are given by the following formula:

$$(2) \qquad \dim \mathbf{K}^{(n)} = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2 \ldots \\ s_1 + 2s_2 + \cdots + ds_d = n}} \beta_1^{(s_1)} \cdots \beta_d^{(s_d)},$$

where

$$(3) \qquad \beta_m^{(s)} := \begin{cases} \dbinom{\beta_m}{s}, \ m = 1, 3, \ldots \\ \dbinom{\beta_m + s - 1}{s}, \ m = 2, 4, \ldots \end{cases}$$

$s \neq 0$, and $\beta_m^{(0)} := 1$. Here $\beta_m := \dim \mathcal{K}^m(X)$, $m = 1, \ldots, d$.

The finiteness of $\beta_m$ is however a rare phenomenon in the geometry of non-compact manifolds. An important example of a manifold $X$ with infinite dimensional spaces $\mathcal{K}^{(m)}(X)$ is given by an infinite cover of a compact Riemannian manifold (say $M$). In this case, an infinite discrete group $G$ acts by isometries on $X$ and consequently on the spaces of differential forms over $X$. The projection $P_m$ onto the space $\mathcal{K}^{(m)}(X)$ of harmonic forms commutes with the action of $G$ and thus belongs to the commutant of this action which is a von Neumann algebra (of $II_\infty$ type under certain conditions on $G$). The corresponding von Neumann trace of $P_m$ gives a regularized dimension of the space $\mathcal{K}^{(m)}(X)$ and is called the $L^2$-Betti number $b_m$ of $X$ (or $M$). $L^2$-Betti numbers were introduced in [18] and have been studied by many authors (see [40], [42] and references given there).

It is natural to ask whether this approach can be extended to configuration spaces over infinite coverings. In particular, is formula (3) valid in this case (with $\beta_m$ replaced by $b_m$)? In the present paper, we construct a von Neumann algebra containing the projection $\mathbf{P}^{(n)}$ onto $\mathbf{K}^{(n)}$ and compute its von Neumann trace $\mathbf{b}_n$ . The result is different from (2) and is given by the following exponential formula:

$$(4) \qquad \mathbf{b}_n = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + 2s_2 + \cdots + ds_d = n}} \frac{(b_1)^{s_1}}{s_1!} \cdots \frac{(b_d)^{s_d}}{s_d!}.$$

The structure of the paper is as follows. In Section 2 we give (following [3], [4]) a description of the de Rham complex over $\Gamma_X$ and the spaces of harmonic forms.

Section 3 is independent of the theory of configuration spaces and plays the central technical role in the paper. We study the following problem which can be formulated in quite a general form. Let us consider a $d$-dimensional complex Hilbert space $\mathcal{H} = \mathbb{C}^d$. It is easy to compute the dimensions of the symmetric and antisymmetric $n$-th tensor powers $\mathcal{H}^{\widehat{\otimes} n}$ and $\mathcal{H}^{\wedge n}$ of $\mathcal{H}$ respectively. We

have obviously

$$\dim \mathcal{H}^{\widehat{\otimes} n} = \binom{d+n-1}{n} = \frac{d(d+1)\ldots(d+n-1)}{n!},$$

(5)

$$\dim \mathcal{H}^{\wedge n} = \binom{d}{n} = \frac{d(d-1)\ldots(d-n+1)}{n!}.$$

Let $\mathcal{H}$ be a subspace of some Hilbert space $\mathcal{X}$, which may in general be infinite dimensional. Then $\mathcal{H}^{\widehat{\otimes} n}$ and $\mathcal{H}^{\wedge n}$ are subspaces of $\mathcal{X}^{\otimes n}$. Let

(6)
$$P : \mathcal{X} \to \mathcal{H},$$

(7)
$$P_s^{(n)} : \mathcal{X}^{\otimes n} \to \mathcal{H}^{\widehat{\otimes} n}$$

and

(8)
$$P_a^{(n)} : \mathcal{X}^{\otimes n} \to \mathcal{H}^{\wedge n}$$

be the corresponding orthogonal projections. Then we have $\operatorname{Tr} P = d$, and formulae (5) can be rewritten in the form

$$\operatorname{Tr} P_s^{(n)} = \frac{\operatorname{Tr} P (\operatorname{Tr} P + 1) \ldots (\operatorname{Tr} P + n - 1)}{n!},$$

(9)

$$\operatorname{Tr} P_a^{(n)} = \frac{\operatorname{Tr} P (\operatorname{Tr} P - 1) \ldots (\operatorname{Tr} P - n + 1)}{n!}.$$

Let now $\mathcal{H}$ be infinite dimensional. Then $\operatorname{Tr} P = \infty$ and formulae (9) have no sense. Let us assume that the projection $P$ has a finite trace as an element of some von Neumann algebra $\mathcal{A}$ (different from the algebra $\mathcal{B}(\mathcal{X})$ of all bounded operators in $\mathcal{X}$), equipped with trace $\operatorname{Tr}_{\mathcal{A}}$. It is interesting to ask whether analogues of formulae (9) involving $\operatorname{Tr}_{\mathcal{A}} P$ hold. The answer seems to be strongly dependent on the structure of the von Neumann algebra $\mathcal{A}$ the projection $P$ belongs to.

In Section 3, we discuss the situation where $\mathcal{X} = L^2 \Omega^{(m)}(X)$ is the space of square-integrable $m$-forms on $X$ and $\mathcal{H} = \mathcal{K}^{(m)}(X)$ (see above). We introduce a natural von Neumann algebra $\mathcal{A}^{(n)}$ containing the operators $P_s^{(n)}$ and $P_a^{(n)}$ and state our main result: finiteness of the corresponding traces of $P_s^{(n)}$ and $P_a^{(n)}$ and explicit formulae for them. We show by a finite dimensional approximation

that our formulae for the traces of $P_s^{(n)}$ and $P_a^{(n)}$ are compatible with formulae (9).

In the case $n = 2$, the results of Section 3 were proved in [25]. A different approach, based on the general theory of factors, has been used in [24].

In Section 4, we apply the constructions of Section 3 and introduce $L^2$-Betti numbers of configuration spaces over infinite covers. We prove formula (4) and apply it to computing of a regularized index of the Dirac operator associated with the de Rham differential of the configuration space.

Let us remark that the spaces of finite configurations, which unlike $\Gamma_X$ possess a natural manifold structure, have been actively studied by geometers and topologists, see e.g. [21], [30] and references given therein. The relationship between these works and our $L^2$-theory, which is relevant for the spaces of finite configurations too [24], is not clear yet.

The situation changes dramatically if the Poisson measure $\pi$ is replaced by a different measure (for instance a Gibbs measure). From the physical point of view, this describes a passage from a system of particles without interaction (free gas) to an interacting particle system, see [11] and references within. For a wide class of measures, including Gibbs measures of Ruelle type and Gibbs measures in low activity-high temperature regime, the de Rham complex has been introduced and studied in [5]. The structure of the corresponding Laplacian is much more complicated in this case, and the spaces of harmonic forms have not been studied yet.

## §2.    De Rham Complex Over a Configuration Space

The aim of this section is to recall some definitions and known facts concerning the differential structure of a configuration space and differential forms over it. For more details and proofs, we refer the reader to [10], [3], [4].

### §2.1.    Differential forms over a configuration space

Let $X$ be a complete connected, oriented, $C^\infty$ Riemannian manifold of infinite volume with a lower bounded curvature.

The configuration space $\Gamma_X$ over $X$ is defined as the set of all locally finite subsets (configurations) in $X$:

$$(10) \qquad \Gamma_X := \left\{ \gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X \right\}.$$

Here, $|A|$ denotes the cardinality of a set $A$.

We can identify any $\gamma \in \Gamma_X$ with the positive, integer-valued Radon measure

$$(11) \qquad \sum_{x \in \gamma} \varepsilon_x \subset \mathcal{M}(X),$$

where $\varepsilon_x$ is the Dirac measure with mass at $x$, $\sum_{x \in \varnothing} \varepsilon_x :=$ zero measure, and $\mathcal{M}(X)$ denotes the set of all positive Radon measures on the Borel $\sigma$-algebra $\mathcal{B}(X)$. The space $\Gamma_X$ is endowed with the relative topology as a subset of the space $\mathcal{M}(X)$ with the vague topology, i.e., the weakest topology on $\Gamma_X$ with respect to which all maps

$$(12) \qquad \Gamma_X \ni \gamma \mapsto \langle f, \gamma \rangle := \int_X f(x)\,\gamma(dx) \equiv \sum_{x \in \gamma} f(x)$$

are continuous. Here, $f \in C_0(X)(:= $ the set of all continuous functions on $X$ with compact support). Let $\mathcal{B}(\Gamma_X)$ denote the corresponding Borel $\sigma$-algebra.

Following [51], [10], we define the tangent space to $\Gamma_X$ at a point $\gamma$ as the Hilbert space

$$(13) \qquad T_\gamma \Gamma_X = \bigoplus_{x \in \gamma} T_x X.$$

The scalar product and the norm in $T_\gamma \Gamma_X$ will be denoted by $\langle \cdot, \cdot \rangle_\gamma$ and $\|\cdot\|_\gamma$, respectively. Thus, each $V(\gamma) \in T_\gamma \Gamma_X$ has the form $V(\gamma) = (V(\gamma)_x)_{x \in \gamma}$, where $V(\gamma)_x \in T_x X$, and

$$(14) \qquad \|V(\gamma)\|_\gamma^2 = \sum_{x \in \gamma} \langle V(\gamma)_x, V(\gamma)_x \rangle_x,$$

where $\langle \cdot, \cdot \rangle_x$ is the inner product in $T_x X$. The sections of the bundle $T\Gamma_X$ will be called vector fields or first order differential forms on $\Gamma_X$. The sections of the bundles $\wedge^n(T\Gamma_X)$, $n \in \mathbb{N}$, with fibers

$$(15) \qquad \wedge^n(T_\gamma \Gamma_X) := \wedge^n \left( \bigoplus_{x \in \gamma} T_x X \right),$$

where $\wedge^n(\mathcal{H})$ (or $\mathcal{H}^{\wedge n}$) stands for the $n$-th antisymmetric tensor power of a Hilbert space $\mathcal{H}$, will be called differential forms of order $n$. Thus, under a differential form $W$ of order $n$ over $\Gamma_X$, we will understand a mapping

$$(16) \qquad \Gamma_X \ni \gamma \mapsto W(\gamma) \in \wedge^n(T_\gamma \Gamma_X).$$

We will now recall how to introduce a covariant derivative of a differential form $W \colon \Gamma_X \to \wedge^n(T\Gamma_X)$.

Let $\gamma \in \Gamma_X$ and $x \in \gamma$. By $\mathcal{O}_{\gamma,x}$ we will denote an arbitrary open neighborhood of $x$ in $X$ such that $\mathcal{O}_{\gamma,x} \cap (\gamma \setminus \{x\}) = \varnothing$. We define the mapping

$$(17) \quad \mathcal{O}_{\gamma,x} \ni y \mapsto W_x(\gamma, y) := W(\gamma_y) \in \wedge^n(T_{\gamma_y}\Gamma_X), \qquad \gamma_y := (\gamma \setminus \{x\}) \cup \{y\}.$$

This is a section of the Hilbert bundle

$$(18) \qquad\qquad \wedge^n(T_{\gamma_y}\Gamma_X) \mapsto y \in \mathcal{O}_{\gamma,x}.$$

The Levi–Civita connection on $TX$ generates in a natural way a connection on this bundle. We denote by $\nabla^X_{\gamma,x}$ the corresponding covariant derivative and use the notation

$$(19) \qquad \nabla^X_x W(\gamma) := \nabla^X_{\gamma,x} W_x(\gamma, x) \in T_x X \otimes (\wedge^n(T_\gamma \Gamma_X))$$

if the section $W_x(\gamma, \cdot)$ is differentiable at $x$.

We say that the form $W$ is differentiable at a point $\gamma$ if for each $x \in \gamma$ the section $W_x(\gamma, \cdot)$ is differentiable at $x$, and

$$(20) \qquad \nabla^\Gamma W(\gamma) := (\nabla^X_x W(\gamma))_{x \in \gamma} \in T_\gamma \Gamma_X \otimes (\wedge^n(T_\gamma \Gamma_X)).$$

The mapping

$$(21) \qquad \Gamma_X \ni \gamma \mapsto \nabla^\Gamma W(\gamma) := (\nabla^X_x W(\gamma))_{x \in \gamma} \in T_\gamma \Gamma_X \otimes (\wedge^n(T_\gamma \Gamma_X))$$

will be called the covariant gradient of the form $W$.

Analogously, one can introduce higher order derivatives of a differential form $W$, the $m$th derivative $(\nabla^\Gamma)^{(m)} W(\gamma) \in (T_\gamma \Gamma_X)^{\otimes m} \otimes (\wedge^n(T_\gamma \Gamma_X))$.

Let us note that, for any $\eta \subset \gamma$, the space $\wedge^n(T_\eta \Gamma_X)$ can be identified in a natural way with a subspace of $\wedge^n(T_\gamma \Gamma_X)$. In this sense, we will use the expression $W(\gamma) = W(\eta)$ without additional explanations.

A form $W \colon \Gamma_X \to \wedge^n(T\Gamma_X)$ is called local if there exists a compact $\Lambda = \Lambda(W)$ in $X$ such that $W(\gamma) = W(\gamma_\Lambda)$ for each $\gamma \in \Gamma_X$.

Let $\mathcal{F}\Omega^n$ denote the set of all local, infinitely differentiable forms $W \colon \Gamma_X \to \wedge^n(T\Gamma_X)$ which together with all their derivatives are polynomially bounded, i.e., for each $W \in \mathcal{F}\Omega^n$ and each $m \in \mathbb{Z}_+$, there exists a function $\varphi \in C_0(X)$ and $k \in \mathbb{N}$ such that

$$(22) \quad \|(\nabla^{(m)} W)(\gamma)\|_{(T_\gamma \Gamma_X)^{\otimes m} \otimes (\wedge^n(T_\gamma \Gamma_X))} \leq \langle \varphi^{\otimes k}, \gamma^{\otimes k} \rangle \qquad \text{for all } \gamma \in \Gamma_X,$$

where $\nabla^{(0)} W := W$.

Our next goal is to give a description of the space of $n$-forms that are square-integrable with respect to the Poisson measure.

Let $dx$ denote the volume measure on $X$, and let $\pi$ denote the Poisson measure on $\Gamma_X$ with intensity $dx$. This measure is characterized by its Laplace transform

$$(23) \qquad \int_{\Gamma_X} e^{\langle f, \gamma \rangle} \pi(d\gamma) = \exp\left[ \int_X (e^{f(x)} - 1) dx \right], \qquad f \in C_0(X).$$

If $F : \Gamma_X \to \mathbb{R}$ is integrable with respect to $\pi$ and local, i.e., $F(\gamma) = F(\gamma_\Lambda)$ for some compact $\Lambda \subset X$, then one has

$$(24) \qquad \int_{\Gamma_X} F(\gamma) \pi(d\gamma) = e^{-\operatorname{vol}(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} F(\{x_1, \ldots, x_n\}) dx_1 \cdots dx_n.$$

We define on the set $\mathcal{F}\Omega^n$ the $L^2$-scalar product with respect to the Poisson measure:

$$(25) \qquad (W_1, W_2)_{L^2_\pi \Omega^n} := \int_{\Gamma_X} \langle W_1(\gamma), W_2(\gamma) \rangle_{\wedge^n (T_\gamma \Gamma_X)} \, \pi(d\gamma).$$

The integral on the right hand side of (25) is finite, since the Poisson measure has all moments finite. Moreover, $(W, W)_{L^2_\pi \Omega^n} > 0$ if $W$ is not identically zero. Hence, we can define a Hilbert space $L^2(\Gamma_X \to \wedge^n(T\Gamma_X); \pi)$ as the completion of $\mathcal{F}\Omega^n$ with respect to the norm generated by the scalar product (25). We denote by $L^2_\pi \Omega^n$ the complexification of $L^2(\Gamma_X \to \wedge^n(T\Gamma_X); \pi)$.

We will now give an isomorphic description of the space $L^2_\pi \Omega^n$ via the space $L^2_\pi \Omega^0 := L^2(\Gamma_X \to \mathbb{C}; \pi)$ and spaces $L^2 \Omega^n(X^m)$ of square-integrable complex forms on $X^m := \overbrace{X \times \ldots \times X}^{m}$, $m = 1, \ldots, n$.

For a finite configuration $\eta = \{x_1, \ldots, x_m\}$ we set

$$(26) \qquad \mathbb{T}^{(n)}_\eta X^m := \bigoplus_{\substack{1 \le k_1, \ldots, k_m \le d \\ k_1 + \cdots + k_m = n}} (T_{x_1} X)^{\wedge k_1} \wedge \cdots \wedge (T_{x_m} X)^{\wedge k_m}.$$

By virtue of (15), we have

$$(27) \qquad \wedge^n(T_\gamma \Gamma_X) = \bigoplus_{m=1}^{n} \bigoplus_{\substack{\eta \subset \gamma \\ |\eta| = m}} \mathbb{T}^{(n)}_\eta X^m.$$

For $W \in \mathcal{F}\Omega^n$, we denote by $W_m(\gamma; \eta)$ the projection of $W(\gamma) \in \wedge^n(T_\gamma \Gamma_X)$ onto the subspace $\mathbb{T}^{(n)}_\eta X^m$.

**Proposition 1** ([4]).     *Setting, for $W \in L_\pi^2 \Omega^n$,*

$$(28) \quad (I^n W)(\gamma, x_1, \ldots, x_m) := (m!)^{-1/2} W_m(\gamma \cup \{x_1, \ldots, x_m\}, \{x_1, \ldots, x_m\}),$$

*$m = 1, \ldots, n$, one gets the isometry*

$$(29) \qquad\qquad I^n : L_\pi^2 \Omega^n \to L_\pi^2 \Omega^0 \bigotimes \left[ \bigoplus_{m=1}^n L^2 \Omega^n(X^m) \right].$$

*Remark* 1.     Actually, formula (28) makes direct sense only for $(x_1, \ldots, x_m) \in \widetilde{X}^m$, where

$$(30) \qquad\qquad \widetilde{X}^m := \{ (x_1, \ldots, x_m) \in X^m \mid x_i \neq x_j \text{ if } i \neq j \}.$$

However, since the set $X^m \setminus \widetilde{X}^m$ is of zero $dx_1 \cdots dx_m$ measure, (28) can be interpreted to hold as it stands, for all $(x_1, \ldots, x_m) \in X^m$.

*Remark* 2.     The corresponding statement in [4] is more refined (the description of the image of $I^n$ is given). In order to avoid unnecessary technical details, we do not give the exact formulation there.

## §2.2.   Exterior differentiation

We define the linear operators

$$(31) \qquad\qquad \mathbf{d}_n \colon \mathcal{F}\Omega^n \to \mathcal{F}\Omega^{n+1}, \qquad n \in \mathbb{N},$$

by

$$(32) \qquad\qquad (\mathbf{d}_n W)(\gamma) := (n+1)^{1/2} \operatorname{AS}_{n+1}(\nabla^\Gamma W(\gamma)),$$

where

$$(33) \qquad\qquad \operatorname{AS}_{n+1} \colon (T_\gamma \Gamma_X)^{\otimes(n+1)} \to \wedge^{n+1}(T_\gamma \Gamma_X)$$

is the antisymmetrization operator.

Let us now consider $\mathbf{d}_n$ as an operator acting from the space $L_\pi^2 \Omega^n$ into $L_\pi^2 \Omega^{n+1}$. We denote by $\mathbf{d}_n^*$ the adjoint operator of $\mathbf{d}_n$.

**Proposition 2.**     1. *$\mathbf{d}_n^*$ is a densely defined operator from $L_\pi^2 \Omega^{n+1}$ into $L_\pi^2 \Omega^n$ with domain containing $\mathcal{F}\Omega^{n+1}$.*
2. *The operator $\mathbf{d}_n : L_\pi^2 \Omega^n \to L_\pi^2 \Omega^{n+1}$ is closable.*

*Proof.* The proof has been given in [3], [4]. $\square$

We denote by $\bar{\mathbf{d}}_n$ the closure of $\mathbf{d}_n$. The space $Z^n := \operatorname{Ker} \bar{\mathbf{d}}_n$ is then a closed subspace of $L^2_\pi \Omega^n$. Let $B^n$ denote the closure in $L^2_\pi \Omega^n$ of the subspace $\operatorname{Im} \mathbf{d}_{n-1}$ (of course, $B^n =$ the closure of $\operatorname{Im} \bar{\mathbf{d}}_{n-1}$).

We have obviously $\mathbf{d}_n \mathbf{d}_{n-1} = 0$, which implies

$$(34) \qquad \operatorname{Im} \mathbf{d}_{n-1} \subset \operatorname{Ker} \mathbf{d}_n \subset Z^n.$$

Hence $B^n \subset Z^n$ and

$$(35) \qquad \bar{\mathbf{d}}_n \bar{\mathbf{d}}_{n-1} = 0.$$

Thus, we have the infinite complex

$$(36) \qquad \cdots \xrightarrow{\mathbf{d}_{n-1}} \mathcal{F}\Omega^n \xrightarrow{\mathbf{d}_n} \mathcal{F}\Omega^{n+1} \xrightarrow{\mathbf{d}_{n+1}} \cdots,$$

and the associated Hilbert complex

$$(37) \qquad \cdots \xrightarrow{\bar{\mathbf{d}}_{n-1}} L^2_\pi \Omega^n \xrightarrow{\bar{\mathbf{d}}_n} L^2_\pi \Omega^{n+1} \xrightarrow{\bar{\mathbf{d}}_{n+1}} \cdots.$$

The homology of the complex (37) will be called the (reduced) $L^2$-cohomology of $\Gamma_X$. We set in a standard way

$$(38) \qquad \mathcal{H}^n_\pi = Z^n / B^n, \qquad n \in \mathbb{N},$$

and call $\mathcal{H}^n_\pi$ the $n$-th $L^2$-cohomology space of $\Gamma_X$.

### §2.3. Hodge–de Rham Laplacian of the Poisson measure

For $n \in \mathbb{N}$, we define a bilinear form $\mathcal{E}^{(n)}_\pi$ on $L^2_\pi \Omega^n$ by

$$(39) \quad \mathcal{E}^{(n)}_\pi(W_1, W_2) := \int_{\Gamma_X} \big[ \langle \mathbf{d}_n W_1(\gamma), \mathbf{d}_n W_2(\gamma) \rangle_{\wedge^{n+1}(T\gamma\Gamma_X)}$$
$$+ \langle \mathbf{d}^*_{n-1} W_1(\gamma), \mathbf{d}^*_{n-1} W_2(\gamma) \rangle_{\wedge^{n-1}(T\gamma\Gamma_X)} \big] \, \pi(d\gamma),$$

where $W_1, W_2 \in \operatorname{Dom} \mathcal{E}^{(n)}_\pi := \mathcal{F}\Omega^n$. The function under the sign of integral in (39) is polynomially bounded, so that the integral exists.

**Theorem 1.** 1. *For any $W_1, W_2 \in \mathcal{F}\Omega^n$, we have*

$$(40) \qquad \mathcal{E}^{(n)}_\pi(W_1, W_2) = \int_{\Gamma_X} \langle \mathbf{H}^{(n)} W_1(\gamma), W_2(\gamma) \rangle_{\wedge^n(T\Gamma_X)} \, \pi(d\gamma).$$

*Here, $\mathbf{H}^{(n)} = \mathbf{d}_{n-1} \mathbf{d}^*_{n-1} + \mathbf{d}^*_n \mathbf{d}_n$ is an operator in the space $L^2_\pi \Omega^n$ with domain $\operatorname{Dom} \mathbf{H}^{(n)} := \mathcal{F}\Omega^n$.*
*2. $\mathcal{F}\Omega^n$ is a core for $\mathbf{H}^{(n)}$.*

*Proof.*   See [3], [4].                                                    □

From Theorem 1 we conclude that the bilinear form $\mathcal{E}_\pi^{(n)}$ is closable in the space $L_\pi^2\Omega^n$. The generator of its closure (being actually the Friedrichs extension of the operator $\mathbf{H}^{(n)}$, for which we preserve the same notation) will be called the Hodge–de Rham Laplacian on $\Gamma_X$ (corresponding to the Poisson measure $\pi$).

## §2.4.   Harmonic forms and $L^2$-cohomologies

The aim of this section is to study the structure of the spaces $\mathcal{H}_\pi^n$ of $L^2$-cohomologies of $\Gamma_X$. Let for a Hilbert space $\mathcal{S}$,

$$(41) \qquad \mathcal{S}^{\overset{k}{\diamond}s} := \begin{cases} \mathcal{S}^{\widehat{\otimes}s}, \ k \text{ is even} \\ \mathcal{S}^{\wedge s}, \ k \text{ is odd} \end{cases}.$$

$s \geq 1$. We will use the convention $\mathcal{S}^{\overset{k}{\diamond}s} = \mathbb{C}^1, \ s = 0$.

**Theorem 2.**   1) *Let $H_{X^m}^{(n)}$ be the Hodge-de Rham Laplacian in $L^2\Omega^n$ $(X^m)$. Then:*

$$(42) \qquad I^n\mathbf{H}^{(n)} = \left( \mathbf{H}^{(0)} \otimes \mathbf{1} + \mathbf{1} \otimes \left( \bigoplus_{m=1}^n H_{X^m}^{(n)} \right) \right) I^n,$$

*where $I^n$ is the isometry given by (28).*
*2) The isometry $I^n$ generates the unitary isomorphism of Hilbert spaces*

$$(43) \qquad \mathbf{K}^{(n)} := \mathrm{Ker}\,\mathbf{H}^{(n)} \simeq \bigoplus_{\substack{s_1,\ldots,s_d=0,1,2\ldots \\ s_1+2s_2+\cdots+ds_d=n}} (\mathcal{K}^1(X))^{\overset{1}{\diamond}s_1} \otimes \cdots \otimes (\mathcal{K}^d(X))^{\overset{d}{\diamond}s_d},$$

*where $\mathcal{K}^m(X) := \mathrm{Ker}\,H_X^{(m)}, \ m = 1, 2, \ldots, d, \ d = \dim X - 1$.*

*Proof.*   See [4].                                                    □

*Remark* 3.   More precisely,

$$(44) \qquad I^n(\mathbf{K}^{(n)}) = \left( \mathrm{Ker}\,\mathbf{H}^{(0)} \otimes \mathrm{Ker} \bigoplus_{m=1}^n H_{X^m}^{(n)} \right) I^n.$$

It is proved in [10] that $\mathrm{Ker}\,\mathbf{H}^{(0)}$ consists of constant functions, i.e. $\mathrm{Ker}\,\mathbf{H}^{(0)} \simeq \mathbb{C}^1$, implies (43).

*Remark* 4.  Formula (43) also holds for spaces of finite configurations, see [24]. In fact,

$$\mathcal{H}_\pi^n \simeq Har^n(B_X^{(n)}), \tag{45}$$

where $Har^n(B_X^{(n)})$ is the space of square-integrable harmonic $n$-forms on the space $B_X^{(n)}$ of configurations of no more than $n$ points. Let us remark that $B_X^{(p)} = \bigcup_{k=0}^p \widetilde{X^k}/S_k$, where $\widetilde{X^k}$ is defined by (30), and (43) is in this case a symmetric version of the Künneth formula.

We see from (43) that all spaces $\mathbf{K}^{(n)}$, $n \in \mathbb{N}$, are finite dimensional provided the spaces $\mathcal{K}^m(X)$, $m = 1, \dots, d$ are so. In this case, it is easy to compute the dimension of $\mathbf{K}^{(n)}$. Indeed, for a finite dimensional space $\mathcal{S}$ we have obviously

$$\dim\left(\mathcal{S}^{\widehat{\otimes}s}\right) = \binom{\dim \mathcal{S} + s - 1}{s} \tag{46}$$

and

$$\dim\left(\mathcal{S}^{\wedge s}\right) = \binom{\dim \mathcal{S}}{s}. \tag{47}$$

Thus we have the following formula:

$$\dim \mathbf{K}^{(n)} = \sum_{\substack{s_1, \dots, s_d = 0, 1, 2 \dots \\ s_1 + 2s_2 + \cdots + ds_d = n}} \beta_1^{(s_1)} \cdots \beta_d^{(s_d)}, \tag{48}$$

where

$$\beta_m^{(s)} := \begin{cases} \binom{\beta_m}{s}, & m = 1, 3, \dots \\ \binom{\beta_m + s - 1}{s}, & m = 2, 4, \dots \end{cases} \tag{49}$$

$s \neq 0$, and $\beta_m^{(0)} := 1$. Here and $\beta_m := \dim \mathcal{K}^m(X)$, $m = 1, \dots, d$.

*Remark* 5.  Standard arguments of the theory of operators in Hilbert spaces (see e.g. [14, Proposition A.1] and [4]) show that

$$\begin{aligned} L_\pi^2 \Omega^n &= \mathbf{K}^{(n)} \oplus B^n \oplus \overline{\operatorname{Im} d^*}, \\ L_\pi^2 \Omega^n &= Z^n \oplus \overline{\operatorname{Im} d^*}. \end{aligned} \tag{50}$$

Thus we have the natural isomorphism of the Hilbert spaces

$$\mathcal{H}_\pi^n \simeq \mathbf{K}^{(n)}. \tag{51}$$

## §3. Von Neumann Dimensions of Symmetric and Antisymmetric Tensor Powers

### §3.1. Setting: Von Neumann algebras associated with infinite coverings of compact manifolds

Let us describe the framework introduced by M. Atiyah in his theory of $L^2$-Betti numbers, which we will use during the rest of the paper. For a detailed exposition, see [18] and e.g. [40]. We refer to [22], [50] for general notions of the theory of von Neumann algebras.

We assume that there exists an infinite discrete group $G$ acting freely on $X$ by isometries and that $M = X/G$ is a compact Riemannian manifold. That is,

$$(52) \qquad G \longrightarrow X \longrightarrow M$$

is a Galois (normal) cover of $M$.

Throughout this section, we fix $p = 1, \dots, d$, $d = \dim X - 1$, and use the following general notations:

$\mathcal{X} := L^2\Omega^p(X)$ - the space of square-integrable $p$-forms on $X$;

$\mathcal{M} := L^2\Omega^p(M)$ - the space of square-integrable $p$-forms on $M$;

$\mathcal{H} := \mathcal{K}^p(X)$ ( $= \mathrm{Ker}\, H_X^{(p)}$ ) - the space of square-integrable harmonic $p$-forms on $X$.

For a Hilbert space $\mathcal{P}$, we denote by $\mathcal{B}(\mathcal{P})$ the space of bounded linear operators in $\mathcal{P}$.

For a von Neumann algebra $\mathcal{S}$, we denote by $\mathrm{Tr}_{\mathcal{S}}$ a semifinite faithful normal trace on $\mathcal{S}$.

The action of $G$ in $X$ generates in the natural way the action of $G$ in $\mathcal{X}$ which we denote

$$(53) \qquad G \ni g \mapsto T_g \in \mathcal{B}(\mathcal{X}).$$

Let $\mathcal{A}$ be the commutant of this action,

$$(54) \qquad \mathcal{A} = \{T_g\}'_{g \in G} \subset \mathcal{B}(\mathcal{X}).$$

It is clear that the space $\mathcal{X}$ can be described in the following way:

$$(55) \qquad \mathcal{X} = \mathcal{M} \otimes l^2(G),$$

with the group action obtaining the form

$$(56) \qquad T_g = id \otimes L_g,$$

$g \in G$, where $L_g$, $g \in G$, are operators of the left regular representation of $G$. Then

$$(57) \qquad \mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{M}) \otimes \mathcal{B}(l^2(G))$$

and

$$(58) \qquad \mathcal{A} = \mathcal{B}(\mathcal{M}) \otimes \mathcal{R}(G),$$

where $\mathcal{R}(G)$ is the von Neumann algebra generated by the right regular representation of $G$.

In what follows, we assume that $G$ is an ICC group, that is,

$$(59) \qquad \textit{all non-trivial classes of conjugate elements are infinite.}$$

This ensures that $\mathcal{R}(G)$ is a $II_1$-factor (see e.g. [41]). Thus $\mathcal{A}$ is a $II_\infty$-factor.

Let us consider the orthogonal projection

$$(60) \qquad P : \mathcal{X} \to \mathcal{H}$$

and its integral kernel

$$(61) \qquad k(x, y) \in \mathcal{B}(T_x^p X, T_y^p X).$$

Then, because of the $G$-invariance of the Laplacian $H_X^{(p)}$, we have $P \in \mathcal{A}$. It was shown in [18] that

$$(62) \qquad \mathrm{Tr}_{\mathcal{A}} P = \int_M \mathrm{tr}\, k(m, m)\, dm,$$

where tr is the usual matrix trace and $dm$ is the Riemannian volume on $M$. Let us remark that, because of $G$-invariance, $k(m, m)$ is a well-defined function on $M$. Moreover, it is known that $H_X^{(p)}$ is elliptic regular, which implies that the kernel $k$ is smooth. Thus

$$(63) \qquad b_p := \mathrm{Tr}_{\mathcal{A}}\, P < \infty.$$

The numbers $b_p$, $p = 0, 1, \dots, d$, are called the $L^2$-Betti numbers of $X$ (or $M$) associated with $G$. The following is known:

1)

$$(64) \qquad \sum_{p=0}^{d} (-1)^p b_p = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$ ([18]);

2) $L^2$-Betti numbers are homotopy invariants of $M$ ([26]).

## §3.2.　Permutations in tensor powers of von Neumann algebras

Let us consider tensor products $\mathcal{X}^{\otimes n} := \mathcal{X} \otimes \ldots \otimes \mathcal{X}$ and $\mathcal{H}^{\otimes n} := \mathcal{H} \otimes \ldots \otimes \mathcal{H}$ of $n$ copies of the spaces $\mathcal{X}$ and $\mathcal{H}$ respectively. Obviously

$$(65) \qquad P^{\otimes n} := P \otimes \ldots \otimes P : \mathcal{X}^{\otimes n} \to \mathcal{H}^{\otimes n},$$

is the orthogonal projection. We have

$$(66) \qquad P^{\otimes n} \in \mathcal{A}^{\otimes n} = \left\{ T_{(g_1,\ldots,g_n)} \right\}'_{(g_1,\ldots,g_n) \in G^n},$$

the commutant of the action $T_{(g_1,\ldots,g_n)} := T_{g_1} \otimes \ldots \otimes T_{g_n}$ of the product group $G^n = G \times \ldots \times G$ in $\mathcal{X}^{\otimes n}$, and

$$(67) \qquad \mathrm{Tr}_{\mathcal{A}^{\otimes n}}\left( P^{\otimes n} \right) = \left( \mathrm{Tr}_{\mathcal{A}} P \right)^n.$$

Next, we consider the symmetric and anti-symmetric tensor powers

$$(68) \qquad \mathcal{X}_s^n := \mathcal{X}^{\widehat{\otimes} n}, \ \mathcal{H}_s^n := \mathcal{H}^{\widehat{\otimes} n}$$

and

$$(69) \qquad \mathcal{X}_a^n := \mathcal{X}^{\wedge n}, \ \mathcal{H}_a^n := \mathcal{H}^{\wedge n}$$

respectively, and the corresponding orthogonal projections

$$P_s := \frac{\sum_{\sigma \in S_n} U_\sigma}{2} : \mathcal{X}^{\otimes n} \to \mathcal{X}^{\widehat{\otimes} n},$$

$$(70)$$

$$P_a := \frac{\sum_{\sigma \in S_n} sign(\sigma) U_\sigma}{2} : \mathcal{X}^{\otimes n} \to \mathcal{X}^{\wedge n},$$

where $S_n$ is the symmetric group of order $n$, and for any $\sigma \in S_n$, $U_\sigma : \mathcal{X}^{\otimes n} \to \mathcal{X}^{\otimes n}$ is the corresponding permutation operator. We have

$$(71) \qquad \left[ P_s, P^{\otimes n} \right] = \left[ P_a, P^{\otimes n} \right] = 0.$$

Thus the operators

$$(72) \qquad P_s^{(n)} := P_s P^{\otimes n} = P^{\otimes n} P_s : \mathcal{X}^{\otimes n} \to \mathcal{H}^{\widehat{\otimes} n}$$

and

$$(73) \qquad P_a^{(n)} := P_a P^{\otimes n} = P^{\otimes n} P_a : \mathcal{X}^{\otimes n} \to \mathcal{H}^{\wedge n}$$

are orthogonal projections.

It is clear that $U_\sigma$, $\sigma \neq e$, does not commute with the action $T_{(g_1,\ldots,g_n)}$ of $G^n$ and thus neither $U_\sigma$ nor $P_s$ or $P_a$ belong to $\mathcal{A}^{\otimes n}$. Thus, the von Neumann algebra

$$(74) \qquad \mathcal{A}^{(n)} := \left\{ \mathcal{A}^{\otimes n}, (U_\sigma)_{\sigma \in S_n} \right\}''$$

generated by $\mathcal{A}^{\otimes n}$ and $(U_\sigma)_{\sigma \in S_n}$ does not coincide with $\mathcal{A}^{\otimes n}$.

Now we can formulate the main result of this section.

**Theorem 3.** 1. $\mathcal{A}^{(n)}$ is a $II_\infty$ factor.

2.

$$(75) \qquad \mathrm{Tr}_{\mathcal{A}^{(n)}} P_s^{(n)} = \mathrm{Tr}_{\mathcal{A}^{(n)}} P_a^{(n)} = \frac{(\mathrm{Tr}_{\mathcal{A}} P)^n}{n!}$$

where $\mathrm{Tr}_{\mathcal{A}^{(n)}}$ is the unique trace on $\mathcal{A}^{(n)}$ such that $\mathrm{Tr}_{\mathcal{A}^{(n)}} B = \mathrm{Tr}_{\mathcal{A}^{\otimes n}} B$ whenever $B \in \mathcal{A}^{\otimes n}$.

We prove this theorem in Section 3.4 using techniques developed below.

*Remark* 6. It is not clear whether the von Neumann algebra $\mathcal{A}^{(n)}$ is the minimal von Neumann algebra containing $\mathcal{A}^{\otimes n}$ and $P_s$ or $P_a$. It will however be shown in Section 3.5 that $\{\mathcal{A}^{\otimes n}, P_s\}''$ and $\{\mathcal{A}^{\otimes n}, P_a\}''$ are factors. Thus they are subfactors of $\mathcal{A}^{(n)}$ and

$$(76) \qquad \begin{aligned} \mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_s\}''} P_s^{(n)} &= \mathrm{Tr}_{\mathcal{A}^{(n)}} P_s^{(n)}, \\ \mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_a\}''} P_a^{(n)} &= \mathrm{Tr}_{\mathcal{A}^{(n)}} P_a^{(n)}, \end{aligned}$$

where $\mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_s\}''}$ and $\mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_a\}''}$ are the unique traces on $\{\mathcal{A}^{\otimes n}, P_s\}''$ and $\{\mathcal{A}^{\otimes n}, P_a\}''$ respectively such that their restrictions to $\mathcal{A}^{\otimes n}$ coincide with $\mathrm{Tr}_{\mathcal{A}^{\otimes n}}$.

In order to give an explicit description of $\mathcal{A}^{(n)}$, we first remark that

$$(77) \qquad U_\sigma = U_\sigma^M U_\sigma^G,$$

where

$$(78) \qquad U_\sigma^M : \mathcal{M}^{\otimes n} \to \mathcal{M}^{\otimes n},$$

and

$$(79) \qquad U_\sigma^G : l^2(G)^{\otimes n} \to l^2(G)^{\otimes n}$$

are the corresponding permutation operators in $\mathcal{M}^{\otimes n}$ and $l^2(G)^{\otimes n}$ respectively (cf. (55)).

Let us introduce the von Neumann algebra

$$(80) \qquad \mathcal{R}^{(n)} = \left\{ \mathcal{R}(G)^{\otimes n}, \left( U_\sigma^G \right)_{\sigma \in S_n} \right\}''$$

generated by the right regular representation of $G^n$ and the operators $U_\sigma^G$.

**Lemma 1.**   *The following decomposition formula holds*:

$$(81) \qquad \mathcal{A}^{(n)} = \mathcal{B}(\mathcal{M})^{\otimes n} \otimes \mathcal{R}^{(n)}$$

*Proof.*   This follows from (58), (77) and the obvious fact that $U_\sigma^M \in \mathcal{B}(\mathcal{M})^{\otimes n} = \mathcal{B}\left( \mathcal{M}^{\otimes n} \right)$. $\qquad\qquad\square$

### §3.3.   Tensor powers of the regular representation, and their extensions by the symmetric group

Our next goal is to investigate the structure of the von Neumann algebra $\mathcal{R}^{(n)}$.

Let $\mathcal{R}(G^n) = \mathcal{R}(G)^{\otimes n}$ be the von Neumann algebra generated by the right regular representation

$$(82) \qquad G^n \ni y \mapsto R_y \in \mathcal{B}\left( L^2(G^n) \right)$$

of $G^n$.

**Lemma 2.**   *The following commutation relation holds for any $y \in G^n$ and $\sigma \in S_n$*:

$$(83) \qquad R_y U_\sigma^G = U_\sigma^G R_{\sigma(y)}.$$

*Proof.*   We have obviously

$$(84) \qquad \left( R_y U_\sigma^G f \right)(x) = R_y f(\sigma(x)) = f(\sigma(xy)).$$

On the other hand,

$$(85) \qquad \begin{aligned} \left( U_\sigma^G R_{\sigma(y)} f \right)(x) &= U_\sigma^G f(x\sigma(y)) = \\ &= f(\sigma(x)\sigma(y)) = f(\sigma(xy)). \end{aligned}$$

$\qquad\qquad\qquad\qquad\square$

**Corollary 1.** *For any* $y_1, \ldots, y_n \in G^n$ *and* $\sigma_1, \ldots, \sigma_n \in S_n$, *there exists* $y \in G^n$ *and* $\sigma \in S_n$ *such that*

$$(86) \qquad R_{y_1} U_{\sigma_1}^G R_{y_2} \ldots U_{\sigma_{n-1}}^G R_{y_n} U_{\sigma_n}^G = \begin{cases} R_y, & n \text{ is even} \\ R_y U_\sigma^G, & n \text{ is odd} \end{cases}$$

*(because* $\left(U_\sigma^G\right)^2 = id$*).*

In what follows, we fix a natural basis $\{(g_1, \ldots, g_n)\}_{g_1, \ldots, g_n \in G}$ in $l^2(G)^{\otimes n}$. We denote by $(A)_{x,y}$ the matrix elements of an operator $A \in \mathcal{B}\left(l^2(G)^{\otimes n}\right)$ in this basis. We have

$$(87) \qquad \left(U_\sigma^G\right)_{x,y} = \begin{cases} 1, & x = \sigma(y) \\ 0, & x \neq \sigma(y) \end{cases}$$

for any $\sigma \in S_n$.

The following two lemmas are crucial for our purposes.

**Lemma 3.** *Let* $R_\sigma \in \mathcal{R}(G^n)$, $\sigma \in S_n$ *be such that*

$$(88) \qquad \sum_{\sigma \in S_n} R_\sigma U_\sigma^G = 0.$$

*Then* $R_\sigma = 0$ *for any* $\sigma \in S_n$.

*Proof.* Any $R \in \mathcal{R}(G^n)$ commutes with the left action of $G^n$ on itself. Thus by [41] its matrix elements satisfy the equality

$$(89) \qquad (R)_{x,y} = (R)_{zx,zy}$$

for any $x, y, z \in G \times G$. It is easy to see that

$$(90) \qquad \left(RU_\sigma^G\right)_{x,y} = \left(RU_\sigma^G\right)_{zx,\sigma(z)y}$$

Rewriting (88) in the form

$$(91) \qquad \sum_{\sigma \in S_n} \left(R_\sigma U_\sigma^G\right)_{x,y} = 0$$

and setting $z = y^{-1}$ and $\xi = y^{-1}x$ we see from (90) that

$$(92) \qquad \sum_{\sigma \in S_n} \left(R_\sigma U_\sigma^G\right)_{\xi, \sigma(y^{-1})y} = 0.$$

Let $S^{(y)} \subset S_n$ be the stationary subgroup of $y$. Then (92) obtains the form

$$(93) \qquad (R)_{\xi,e} = \sum_{\sigma \notin S^{(y)}} \left( R_\sigma U_\sigma^G \right)_{\xi, \sigma(y^{-1})y},$$

where $R := -\sum_{\sigma \in S^{(y)}} R_\sigma U_\sigma^G$.

Let us set $y = (g, e, ...., e)$. Then $S^{(y)} \equiv S^{(n-1)}$, the subgroup consisting of all permutations which act only on $n-1$ last components in $G^n$ and is obviously independent of $g$. For any $\sigma \neq S^{(n-1)}$ we have

$$(94) \qquad \sigma(y^{-1})y \equiv g^{(\sigma)} = (\overbrace{g, e, \dots, e, g^{-1}}^{k}, e, \dots, e)$$

for some $k = 2, \dots, n$. Let us define the set

$$(95) \qquad G^{(\sigma)} = \left\{ g^{(\sigma)} : g \in G \right\}.$$

Obviously, $g_1^{(\sigma)} = g_2^{(\sigma)}$ implies that $g_1 = g_2$. Thus $G^{(\sigma)} \subset G^n$.

Next, (93) implies that

$$(96) \qquad (R)_{\xi,e} = \sum_{\sigma \notin S^{(n-1)}} \left( R_\sigma U_\sigma^G \right)_{\xi, g^{(\sigma)}}.$$

Thus

$$\sum_{g \in G} (R)_{\xi,e}^2 \leq \sum_{g \in G} \left( \sum_{\sigma \notin S^{(n-1)}} \left( R_\sigma U_\sigma^G \right)_{\xi, g^{(\sigma)}} \right)^2$$

$$\leq 2 \sum_{g \in G} \sum_{\sigma \notin S^{(n-1)}} \left( R_\sigma U_\sigma^G \right)_{\xi, g^{(\sigma)}}^2$$

$$(97) \qquad \leq \sum_{\sigma \notin S^{(n-1)}} \sum_{\eta \in G^{(\sigma)}} \left( R_\sigma U_\sigma^G \right)_{\xi, \eta}^2$$

$$\leq \sum_{\sigma \notin S^{(n-1)}} \sum_{\eta \in G^n} \left( R_k U_{\sigma_k}^G \right)_{\xi, \eta}^2$$

$$= (n! - (n-1)!) \sum_{\eta \in G^n} \left( R_k U_{\sigma_k}^G \right)_{\xi, \eta}^2 < \infty$$

because $R_\sigma U_\sigma^G \in \mathcal{B}(L^2(G^n))$. This implies that

$$(98) \qquad (R)_{x,y} = (R)_{\xi,e} = 0$$

for any $x, y \in G^n$. Thus

$$(99) \qquad \sum_{\sigma \in S^{(n-1)}} R_\sigma U_\sigma^G = 0.$$

The statement of the lemma follows by an obvious induction argument. $\qquad \square$

**Corollary 2.** *The set*

$$(100) \qquad R^{(n)} := \left\{ \left( R_{g_1} \otimes \ldots \otimes R_{g_n} \right) U_\sigma^G \right\}_{g_1, \ldots, g_n \in G, \sigma \in S_n}$$

*is a basis of the linear space $\mathcal{R}^{(n)}$.*

*Proof.* It is clear that $R^{(n)}$ is a total in $\mathcal{R}^{(n)}$ set. (88) shows that $R^{(n)}$ is linearly independent. $\qquad \square$

Let $\mathcal{L}(G^n)$ be the von Neumann algebra generated by the left regular representation of $G^n$.

**Lemma 4.** *Let $R_1, R_2, \ldots, R_m \in \mathcal{R}(G^n)$, $\sigma_1, \ldots, \sigma_m \in S_n$, and*

$$(101) \qquad R_1 U_{\sigma_1}^G + R_2 U_{\sigma_2}^G + \cdots + R_m U_{\sigma_m}^G = L$$

*for some $L \in \mathcal{L}(G^n)$. Then $L = \alpha I$ for some constant $\alpha$.*

*Proof.* $R \in \mathcal{R}(G^n)$ implies that

$$(102) \qquad \left( R U_\sigma^G \right)_{x,y} = \left( R U_\sigma^G \right)_{zx, \sigma(z)y}$$

for any $x, y, z \in G^n$ and $\sigma \in S^n$ (cf. (90)). In particular, for $\eta_g = (g, g)$, $g \in G$, we have

$$(103) \qquad \left( R U_\sigma^G \right)_{x,y} = \left( R U_\sigma^G \right)_{\eta_g x, \eta_g y}$$

Consequently we obtain

$$(104) \qquad \sum_{k=1}^m (R_k U_{\sigma_k}^G)_{x,y} = \sum_{k=1}^m (R_k U_{\sigma_k}^G)_{\eta_g x, \eta_g y}.$$

Similarly, for $L \in \mathcal{L}(G^n)$ we have

$$(105) \qquad (L)_{x,y} = (L)_{xz, yz}$$

for any $x, y, z \in G^n$. Equality (101) together with (104) implies that

$$(106) \qquad (L)_{\eta_g x \eta_g^{-1}, \eta_g y \eta_g^{-1}} = (L)_{x,y}.$$

In particular,

$$(107) \qquad (L)_{e, \eta y \eta^{-1}} = (L)_{e,y}.$$

Now, for any $y = (g_1, \ldots, g_n) \neq e$ the set

$$(108) \qquad G_y = \{\eta_g y \eta_g^{-1}, \ g \in G\}$$

is infinite because of condition (59). But $L \in \mathcal{B}\left(l^2(G)^{\otimes n}\right)$. Thus

$$(109) \qquad \sum_{x \in G_y} (L)_{e,y}^2 = \sum_{x \in G_y} (L)_{e,x}^2$$
$$\leq \sum_{\eta \in G^n} (L)_{e,\eta}^2 < \infty$$

and

$$(110) \qquad (L)_{e,y} = 0$$

for any $y \in G^n$, $y \neq e$. By (105) we have

$$(111) \qquad (L)_{x,y} = \begin{cases} \alpha, \ x = y \\ 0, \ x \neq y \end{cases}$$

for some constant $\alpha$, or $L = \alpha I$. $\qquad\qquad\qquad\qquad \square$

Let us consider the cross-product $W^*(\mathcal{R}(G)^{\otimes n}, S_n)$ of the von Neumann algebra $\mathcal{R}(G)^{\otimes n}$ and the group $S_n$, where $S_n$ acts on $\mathcal{R}(G)^{\otimes n}$ by permutations. The standard representation of $W^*(\mathcal{R}(G)^{\otimes n}, S_n)$ (see e.g. [22], [50]) is the von Neumann algebra

$$(112) \qquad \mathcal{P} = \{\pi(A), \ \pi(\sigma): \ A \in \mathcal{R}(G)^{\otimes n}, \sigma \in S_n\}$$

generated by block-operator $n! \times n!$ matrices:

$$(113) \qquad \pi(A) = \begin{bmatrix} A & & & \\ & \ldots & & 0 \\ & & \sigma_k(A) & \\ & 0 & & \ldots \\ & & & & \sigma_{n!}(A) \end{bmatrix},$$

and

$$(114) \qquad \pi(\sigma) = [(\sigma)_{kj} I]_{k,j=1}^{n!}$$

acting in $L^2(G^n)^{\oplus n!}$, where $I$ is the identity operator in $L^2(G^n)$ and $[(\sigma)_{kj}]_{k,j=1}^{n!}$ is the matrix of the permutation $\sigma$ as an element of the regular representation of $S_n$. Thus any element of $\mathcal{P}$ is a matrix $[B_{kj}]_{k,j=1}^{n!}$ with matrix elements $B_{kj} \in \mathcal{R}(G)^{\otimes n}$. $\mathcal{P}$ possesses the canonical trace $\mathrm{Tr}_{\mathcal{P}}$ defined by the formula

$$(115) \qquad \mathrm{Tr}_{\mathcal{P}}\, [B_{kj}]_{k,j=1}^{n!} = \frac{1}{n!} \sum_{k=1}^{n!} \mathrm{Tr}_{\mathcal{R}(G)^{\otimes n}} B_{kk}.$$

In particular,

$$(116) \qquad \mathrm{Tr}_{\mathcal{P}}(\pi(\sigma)) = 0.$$

We have the following result.

**Theorem 4.** 1. *The von Neumann algebras $\mathcal{R}^{(n)}$ and $\mathcal{P}$ are isomorphic.*
2. *$\mathcal{R}^{(n)}$ is a $II_1$-factor.*

*Proof.* 1. It is clear that operators

$$(117) \qquad \pi(A) = \begin{bmatrix} A & & & & \\ & \ddots & & & 0 \\ & & \sigma_k(A) & & \\ & 0 & & \ddots & \\ & & & & \sigma_{n!}(A) \end{bmatrix},$$

and $\pi(A)\pi(\sigma)$, where $A = R_{g_1} \otimes \ldots \otimes R_{g_n}$, $g_1, \ldots, g_n \in G$, $\sigma \in S_n$, form a basis (say, $\Pi$) of the linear space $\mathcal{P}$. Let us define a map

$$(118) \qquad \widetilde{\pi} : R^{(n)} \to \Pi$$

setting

$$(119) \qquad \widetilde{\pi}(A) = \pi(A), \ \ \widetilde{\pi}(AU_\sigma^G) = \pi(A)\pi(\sigma).$$

It is easy to see that the map $\widetilde{\pi}$ can be extended to a homeomorphism of the Banach spaces $\mathcal{R}^{(n)}$ and $\mathcal{P}$. The comparison of the commutation relations for the elements of $R^{(n)}$ and $\Pi$ respectively shows that $\widetilde{\pi}$ is an isomorphism of von Neumann algebras.

2. It follows from the definition of $\mathcal{R}^{(n)}$ (formula (80)) that $\mathcal{R}^{(n)} \supset \mathcal{R}(G^n)$ and thus $\left(\mathcal{R}^{(n)}\right)' \subset \mathcal{L}(G^n)$. By Lemma 4

$$\tag{120} \mathcal{R}^{(n)} \cap \mathcal{L}(G^n) = \{\alpha I, \ \alpha \in \mathbb{C}\}.$$

Thus $\mathcal{R}^{(n)}$ is a factor. It possesses a trace $\mathrm{Tr}_{\mathcal{R}^{(n)}}$ generated by $\mathrm{Tr}_{\mathcal{P}}$. Thus it is a $II_1$-factor. $\qquad \square$

**Corollary 3.** $\mathrm{Tr}_{\mathcal{R}^{(n)}}$ *is the unique finite faithful normalized trace on* $\mathcal{R}^{(n)}$. *It is compatible with the trace of the von Neumann algebra* $\mathcal{R}(G)^{\otimes n}$ *in the sense that*

$$\tag{121} \mathrm{Tr}_{\mathcal{R}^{(n)}} B = \mathrm{Tr}_{\mathcal{R}(G)^{\otimes n}} B$$

*if* $B \in \mathcal{R}(G)^{\otimes n}$. *It follows from* (116) *that*

$$\tag{122} \mathrm{Tr}_{\mathcal{R}^{(n)}}(U_\sigma^G) = 0$$

*and moreover*

$$\tag{123} \mathrm{Tr}_{\mathcal{R}^{(n)}}((R_{g_1} \otimes \ldots \otimes R_{g_n}) U_\sigma^G) = 0$$

*for any* $g_1, \ldots, g_n \in G$ *and* $\sigma \in S_n$.

*Remark* 7. We have

$$\tag{124} \mathcal{P} \simeq \mathcal{R}(G^{(n)}),$$

where $G^{(n)}$ is the cross-product of the groups $G^n$ and $S_n$. This gives an indirect proof of the fact that $\mathcal{R}^{(n)}$ is a factor (because $G^{(n)}$ obviously satisfies condition (59) providing $G$ does so).

### §3.4.   Dimensions of symmetric subspaces

In this section, we return to the study of the von Neumann algebra $\mathcal{A}^{(n)}$. We need the following general fact.

**Lemma 5.** *Let* $\mathcal{M}$ *be a* $II_\infty$ *factor, that is,*

$$\tag{125} \mathcal{M} = \mathcal{B}(\mathcal{K}) \otimes \mathcal{N},$$

*where* $\mathcal{K}$ *is a Hilbert space and* $\mathcal{N}$ *is a* $II_1$-factor. *Let* $B \in \mathcal{B}(\mathcal{K})$ *and* $N \in \mathcal{N}$ *be such that* $\mathrm{Tr}_{\mathcal{N}} N = 0$. *Then*

$$\tag{126} \mathrm{Tr}_{\mathcal{M}}(B \otimes N) = 0.$$

*Proof.* Consider a matrix representation $\{B_{kj}\}$ of $B$. Then $B \otimes N$ can be represented as a block-operator matrix with elements $B_{kj}N$. By construction of the trace in $\mathcal{M}$ (see e.g. [22], [50]), $\text{Tr}_{\mathcal{M}}(B \otimes N) = \sum_k \text{Tr}_{\mathcal{N}}(B_{kk}N) = \sum_k B_{kk}\text{Tr}_{\mathcal{N}}N = 0$. $\qquad\square$

*Proof of Theorem* 3. We have shown that

$$\mathcal{A}^{(n)} = \mathcal{B}(\mathcal{M})^{\otimes n} \otimes \mathcal{R}^{(n)}. \tag{127}$$

Moreover $\mathcal{R}^{(n)}$ is a $II_1$ factor by Theorem 4. Thus $\mathcal{A}^{(n)}$ is a $II_\infty$ factor and possesses a unique trace $\text{Tr}_{\mathcal{A}^{(n)}}$ such that $\text{Tr}_{\mathcal{A}^{(n)}}B = \text{Tr}_{\mathcal{A}^{\otimes n}}B$ whenever $B \in \mathcal{A}^{\otimes n}$.

It remains to prove that

$$\text{Tr}_{\mathcal{A}^{(n)}}P_s^{(n)} = \text{Tr}_{\mathcal{A}^{(n)}}P_a^{(n)} = \frac{(\text{Tr}_{\mathcal{A}}P)^n}{n!}, \tag{128}$$

or equivalently

$$\text{Tr}_{\mathcal{A}^{(n)}}P^{\otimes n}U_\sigma = 0 \tag{129}$$

for any $\sigma \in S_n$.

We have $P \in \mathcal{B}(\mathcal{M}) \otimes \mathcal{R}(G)$. Thus $P$ can be represented as a weakly convergent series

$$P = \sum_{g \in G} p_g \otimes R_g, \tag{130}$$

where $p_g \in \mathcal{B}(\mathcal{M})$. Then

$$P^{\otimes n} = \sum_{g_1,\dots,g_n \in G} p_{g_1} \otimes \dots \otimes p_{g_n} \otimes R_{g_1} \otimes \dots \otimes R_{g_n} \tag{131}$$

and

$$P^{\otimes n}U_\sigma = P^{\otimes n}U_\sigma^M U_\sigma^G$$
$$\tag{132}$$
$$= \sum_{g_1,\dots,g_n \in G} \left[(p_{g_1} \otimes \dots \otimes p_{g_n})U_\sigma^M\right] \otimes \left[(R_{g_1} \otimes \dots \otimes R_{g_n})U_\sigma^G\right].$$

where $(p_{g_1} \otimes \dots \otimes p_{g_n})U_\sigma^M \in \mathcal{B}(\mathcal{M})^{\otimes n}$ and $(R_{g_1} \otimes \dots \otimes R_{g_n})U_\sigma^G \in \mathcal{R}^{(n)}$.

For any $g_1, \dots, g_n \in G$

$$\text{Tr}_{\mathcal{R}^{(n)}}\left[(R_{g_1} \otimes \dots \otimes R_{g_n})U_\sigma^G\right] = 0 \tag{133}$$

by (123). It follows from the lemma above that

$$(134) \qquad \mathrm{Tr}_{\mathcal{A}^{(n)}} \left( \left[ (p_{g_1} \otimes \ldots \otimes p_{g_n}) U_\sigma^M \right] \otimes \left[ (R_{g_1} \otimes \ldots \otimes R_{g_n}) U_\sigma^G \right] \right) = 0.$$

Thus

$$(135) \quad \mathrm{Tr}_{\mathcal{A}^{(n)}} P^{\otimes n} U_\sigma =$$
$$= \sum_{g_1, \ldots, g_n \in G} \mathrm{Tr}_{\mathcal{A}^{(n)}} \left( \left[ (p_{g_1} \otimes \ldots \otimes p_{g_n}) U_\sigma^M \right] \otimes \left[ (R_{g_1} \otimes \ldots \otimes R_{g_n}) U_\sigma^G \right] \right)$$
$$= 0.$$

### §3.5.    Uniqueness of the dimension

As has already been noticed, the minimal von Neumann algebras $\{\mathcal{A}^{\otimes n}, P_s\}''$ and $\{\mathcal{A}^{\otimes n}, P_a\}''$ containing $P_s$ and $P_a$ respectively are subalgebras of $\mathcal{A}^{(n)}$, but it is not clear whether they in general coincide with it.

**Lemma 6.**    $\{\mathcal{A}^{\otimes n}, P_s\}''$ and $\{\mathcal{A}^{\otimes n}, P_a\}''$ are factors.

*Proof.*    We have $\mathcal{A}^{\otimes n} \subset \{\mathcal{A}^{\otimes n}, P_s\}'' \subset \mathcal{A}^{(n)}$. Thus

$$(136)$$
$$\{\mathcal{A}^{\otimes n}, P_s\}'' \cap \{\mathcal{A}^{\otimes n}, P_s\}' \subset \mathcal{A}^{(n)} \cap \{\mathcal{A}^{\otimes n}\}'$$
$$\subset \left( \mathcal{B}(\mathcal{M}) \otimes \mathcal{R}^{(n)} \right) \cap (\mathcal{B}(\mathcal{M}) \otimes \mathcal{R}(G^n))'$$
$$\subset \left( \mathcal{B}(\mathcal{M}) \otimes \mathcal{R}^{(n)} \right) \cap \left( \mathbb{C}^1 \otimes \mathcal{L}(G^n) \right)$$
$$= \mathbb{C}^1 \otimes \left( \mathcal{R}^{(n)} \cap \mathcal{L}(G^n) \right) = \mathbb{C}^1$$

because $\mathcal{R}^{(n)} \cap \mathcal{L}(G^n) = \mathbb{C}^1$ by Lemma 4.    □

Thus $\{\mathcal{A}^{\otimes n}, P_s\}''$ and $\{\mathcal{A}^{\otimes n}, P_a\}''$ possess unique traces $\mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_s\}''}$ and $\mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_a\}''}$ respectively such that their restrictions to $\mathcal{A}^{\otimes n}$ coincide with $\mathrm{Tr}_{\mathcal{A}^{\otimes n}}$. Then obviously

$$(137) \qquad \begin{aligned} \mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_s\}''} P_s^{(n)} &= \mathrm{Tr}_{\mathcal{A}^{(n)}} P_s^{(n)}, \\ \mathrm{Tr}_{\{\mathcal{A}^{\otimes n}, P_a\}''} P_a^{(n)} &= \mathrm{Tr}_{\mathcal{A}^{(n)}} P_a^{(n)}. \end{aligned}$$

## §3.6.  Finite dimensional approximation

We shall discuss the relationship between formulae (9) and (75). The first question is whether they are consistent. In some cases, positive answer can be given by a finite dimensional approximation of the space $\mathcal{X}$. The following result is known.

**Theorem 5** ([39], [40]).  *Suppose that the group $G$ (see (52)) is residually finite and let*

$$(138) \qquad \ldots G_{m+1} \subset G_m \subset \ldots \subset G_1 \subset G$$

*be a nested sequence of normal subgroups of $G$ of finite index such that $\cap_m G_m = \{e\}$. Then we have*

$$(139) \qquad \operatorname{Tr}_{\mathcal{A}} P = \lim_{m \to \infty} \frac{\operatorname{Tr} P_m}{|G/G_m|},$$

*where $G/G_m \to X_m \to M$ is the corresponding sequence of finite covers, which "approximate" $X$, and $P_m$ is the corresponding orthogonal projection onto the space $\mathcal{K}^p(X_m)$ of square-integrable harmonic p-forms on $X_m$ ($\mathcal{K}^p(X_m)$ is finite-dimensional because of the compactness of $X_m$).*

Let us consider the spaces $\mathcal{K}^p(X_m)^{\widehat{\otimes} n}$ and $\mathcal{K}^p(X_m)^{\wedge n}$ and let $(P_m)_s^{(n)}$ and $(P_m)_a^{(n)}$ be the corresponding orthogonal projections. We remark that $\operatorname{Tr}(P_m)_s^{(n)}$ and $\operatorname{Tr}(P_m)_a^{(n)}$ can be computed according to the formulae (9). The following is an easy adaptation of the above theorem to our case.

**Theorem 6.**  *Under the conditions of Theorem 5, we have*

$$(140) \qquad \operatorname{Tr}_{\mathcal{A}^{(n)}} P_s^{(n)} = \lim_{m \to \infty} \frac{\operatorname{Tr}(P_m)_s^{(n)}}{|G/G_m|^n}$$

*and*

$$(141) \qquad \operatorname{Tr}_{\mathcal{A}^{(n)}} P_a^{(n)} = \lim_{m \to \infty} \frac{\operatorname{Tr}(P_m)_a^{(n)}}{|G/G_m|^n}.$$

*Proof.* We have, according to (9),

$$
(142) \quad
\begin{aligned}
&\lim_{m \to \infty} \frac{\operatorname{Tr}(P_m)_s^{(n)}}{|G/G_m|^n} \\
&= \lim_{m \to \infty} \left( \frac{1}{n!} \frac{\operatorname{Tr} P_m}{|G/G_m|} \frac{\operatorname{Tr} P_m + 1}{|G/G_m|} \cdots \frac{\operatorname{Tr} P_m + n - 1}{|G/G_m|} \right) \\
&= \frac{(\operatorname{Tr}_{\mathcal{A}} P)^n}{n!}
\end{aligned}
$$

because $|G/G_m| \to \infty$ and $\frac{\operatorname{Tr} P_m}{|G/G_m|} \to \operatorname{Tr}_{\mathcal{A}} P, \ m \to \infty$. Formula (141) can be obtained by similar arguments. $\qquad\square$

## §4.    $L^2$-Betti Numbers of Configuration Spaces of Infinite Coverings

In order to extend the notion of $L^2$-Betti numbers to the case of $\Gamma_X$, where $X$ is as in Section 3 (see (52)), we need to construct a natural von Neumann algebra containing the orthogonal projection

$$(143) \qquad\qquad \mathbf{P}^{(n)} : L^2_\pi \Omega^n \to \mathcal{H}^n_\pi$$

and compute its von Neumann trace.

*Remark* 8.    There are two group actions which we can try to use. One is the diagonal action of $G$ on $\Gamma_X$:

$$(144) \qquad\qquad g\{\ldots x, y, z, \ldots\} = \{\ldots gx, gy, gz, \ldots\},$$

where $g \in G$, $\{\ldots x, y, z, \ldots\} \in \Gamma_X$. This action commutes with $\mathbf{H}^{(n)}$ and thus $\mathbf{P}^{(n)}$ belongs to it commutant. However, the factor-space $\Gamma_X/G$ is very big (certainly not compact) and the corresponding trace of $\mathbf{P}^{(n)}$ is therefore either zero or infinite.

On the other hand, and we can try to employ the natural actions of the product groups $G^m$ on $X^m$ in the right-hand side of (29). The corresponding commutant $B^{(n)}$ has the form $\mathcal{B}(L^2_\pi \Omega^0) \otimes A^{(n)}$, where

$$(145) \qquad\qquad A^{(n)} := \prod_{\substack{s_1, \ldots, s_d = 0, 1, 2 \ldots \\ s_1 + 2s_2 + \cdots + ds_d = n}} \mathcal{A}_1^{\otimes s_1} \otimes \ldots \otimes \mathcal{A}_d^{\otimes s_d},$$

and $\mathcal{A}_p, \ p = 1, \ldots, d$, is the commutant of the action of $G$ on $L^2 \Omega^p(X)$. The results of the previous section give us the possibility to construct the extension of $B^{(n)}$, which contain $\mathbf{P}^{(n)}$.

In what follows, we will use the notation $\mathcal{A}_p^{(m)}$ for the von Neumann algebra $\mathcal{A}^{(m)}$ associated with the space $\mathcal{X} = L^2 \Omega^p(X)$, $p = 1, \ldots, d$, $d = \dim X - 1$. and set $\mathcal{P}_p^{(m)} := P_s^{(m)}$, if $p$ is even, and $\mathcal{P}_p^{(m)} := P_a^{(m)}$, if $p$ is odd (cf. formulae (72)–(74)). Thus,

$$(146) \qquad\qquad \mathcal{P}_p^{(m)} : \left(L^2 \Omega^p(X)\right)^{\otimes m} \to \left(\mathcal{K}^p(X)\right)^{\overset{p}{\otimes} m},$$

$m = 1, 2, \ldots$, where $\overset{p}{\diamond} \, m = \begin{cases} \widehat{\otimes} m \,, & p \text{ is even} \\ \wedge m \,, & p \text{ is odd} \end{cases}$. We set $\mathcal{A}_p^{(0)} = \mathbb{C}^1$, $\mathcal{P}_p^{(0)}$
$= id$. Obviously, $\mathcal{P}_p^{(m)} \in \mathcal{A}_p^{(m)}$.

Let us introduce the von Neumann algebra

$$(147) \qquad \mathbf{A}^{(n)} = \prod_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + 2s_2 + \cdots + d s_d = n}} \mathcal{A}_1^{(s_1)} \otimes \ldots \otimes \mathcal{A}_d^{(s_d)}.$$

Since all algebras $\mathcal{A}_p^{(s_p)}$, $s_p \neq 0$, are $II_\infty$-factors, so is $\mathbf{A}^{(n)}$, with the trace given by the product of the traces in $\mathcal{A}_p^{(s_p)}$.

Let

$$(148) \qquad \mathcal{P}^{(n)} = I^n \mathbf{P}^{(n)} \left( I^n \right)^{-1},$$

where

$$(149) \qquad I^n : L_\pi^2 \Omega^n \to L_\pi^2(\Gamma_X) \bigotimes \left[ \bigoplus_{m=1}^n L^2 \Omega^n(X^m) \right]$$

is the isometry defined by (28).

**Theorem 7.**

$$(150) \qquad \mathcal{P}^{(n)} \in \mathbf{B}^{(n)} := \mathcal{B}(L_\pi^2 \Omega^0) \otimes \mathbf{A}^{(n)}$$

and

$$(151) \qquad \mathrm{Tr}_{\mathbf{B}^{(n)}} \mathcal{P}^{(n)} = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + 2s_2 + \cdots + d s_d = n}} \frac{(b_1)^{s_1}}{s_1!} \cdots \frac{(b_d)^{s_d}}{s_d!},$$

where $b_1, \ldots, b_d$ are the $L^2$-Betti numbers of $X$.

*Proof.* Formula (42) implies that $\mathcal{P}^{(n)} = P' \otimes P''$, where

$$(152) \qquad P' : L_\pi^2 \Omega^0 \to \mathrm{Ker} \, \mathbf{H}^{(0)}$$

and

$$(153) \qquad P'' : \bigoplus_{m=1}^n L^2 \Omega^n(X^m) \to \mathrm{Ker} \bigoplus_{m=1}^n H_{X^m}^{(n)}$$

are the corresponding orthogonal projections. Moreover, according to formula (43) we have

$$(154) \qquad P'' = \sum_{\substack{s_1,\ldots,s_d=0,1,2,\ldots \\ s_1+2s_2+\cdots+ds_d=n}} \mathcal{P}_1^{(s_1)} \otimes \ldots \otimes \mathcal{P}_d^{(s_d)} \in \mathbf{A}^{(n)}.$$

Thus

$$(155) \qquad \mathcal{P}^{(n)} \in \mathcal{B}(L_\pi^2 \Omega^0) \otimes \mathbf{A}^{(n)} (= \mathbf{B}^{(n)})$$

and

$$(156) \qquad \mathrm{Tr}_{\mathbf{B}^{(n)}} \mathcal{P}^{(n)} = \mathrm{Tr}\, P' \cdot \mathrm{Tr}_{\mathbf{A}^{(n)}} P''.$$

The result follows now from Theorem 3 and the fact that $P'$ is just a 1-dimensional projection. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We will use the notation $\mathbf{b}_n := \mathrm{Tr}_{\mathbf{B}^{(n)}} \mathcal{P}^{(n)}$ and call $\mathbf{b}_n$ the $n$-th $L^2$ Betti number of $\Gamma_X$.

**Example 1.** Let $X = \mathbb{H}^d$, the hyperbolic space of dimension $d$. It is known that the only non-zero $L^2$-Betti number of $\mathbb{H}^d$ is $b_{d/2}$ (provided $d$ is even). Then

$$(157) \qquad \mathbf{b}_n = \begin{cases} \frac{\left(b_{d/2}\right)^k}{k!}, & n = \frac{kd}{2},\ k \in \mathbb{N} \\ 0, & n \neq \frac{kd}{2},\ k \in \mathbb{N} \end{cases}.$$

The precise value of $b_{d/2}$ depends on the choice of the corresponding group $G$ of isometries of $\mathbb{H}^d$, see [13].

Let us introduce a regularized index $\mathrm{ind}_{\Gamma_X}(\mathbf{d} + \mathbf{d}^*)$ of the Dirac operator associated with the de Rham differential of the configuration space setting

$$(158) \qquad \mathrm{ind}_{\Gamma_X}(\mathbf{d} + \mathbf{d}^*) = \sum_{k=0}^{\infty} (-1)^k \mathbf{b}_k.$$

We will use the convention $\mathbf{b}_0 = 1$.

**Theorem 8.** *The series on the right hand side of* (158) *converges absolutely, and*

$$(159) \qquad \mathrm{ind}_{\Gamma_X}(\mathbf{d} + \mathbf{d}^*) = e^{\chi(M)},$$

*where $\chi(M)$ is the Euler characteristic of $M$.*

*Proof.* We have

(160)

$$\text{ind}_{\Gamma_X}\,(\mathbf{d}+\mathbf{d}^*) = \sum_{n=0}^{\infty}(-1)^n\,\mathbf{b}_n$$

$$= \sum_{n=0}^{\infty}(-1)^n \sum_{\substack{s_1,\dots,s_d=0,1,2,\dots \\ s_1+2s_2+\cdots+ds_d=n}} \frac{(b_1)^{s_1}}{s_1!}\cdots\frac{(b_d)^{s_d}}{s_d!}$$

$$= \sum_{\substack{s_1,\dots,s_d=0,1,2,\dots \\ s_1+2s_2+\cdots+ds_d=n}} (-1)^{s_1}\frac{(b_1)^{s_1}}{s_1!}\cdots(-1)^{ds_d}\frac{(b_d)^{s_d}}{s_d!}$$

$$= \sum_{s_1=0}^{\infty} \frac{[(-1)\,b_1]^{s_1}}{s_1!}\cdots\sum_{s_d=0}^{\infty}\frac{\left[(-1)^d\,b_d\right]^{s_d}}{s_d!}$$

$$= e^{-b_1}\cdot\,\ldots\,\cdot e^{(-1)^d b_d} = e^{\sum_{k=0}^{d}(-1)^k b_k}$$

because $b_0 = 0$. It is known that $\sum_{k=0}^{d}(-1)^k b_k = \chi(M)$ [18], which implies (159). To prove the absolute convergence of the series (158) let us remark that

(161)
$$\sum_{k=0}^{\infty}\mathbf{b}_k = e^{\sum_{k=0}^{d} b_k}$$

(similarly to (160)).                                                          □

**Corollary 4.**   $\mathbf{b}_k \to 0,\ k \to \infty.$

*Proof.*   This follows immediately from the convergence of $\sum_{k=0}^{\infty}\mathbf{b}_k$.   □

**Corollary 5.**   *The $L^2$-cohomology of $\Gamma_X$ is infinite provided $\chi(M) \neq 0$.*

*Proof.*   If $\chi(M) \neq 0$ then $\mathbf{b}_k \neq 0$ for some $k \geq 1$ (recall that $\mathbf{b}_0 = 1$). Thus $\text{Ker}\,\mathbf{H}^{(k)}$ is infinite dimensional.   □

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