On Quartic Surfaces and Sextic Curves with Singularities of type $\tilde{E}_8, T_{2,3,7}, E_{12}$

By

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§0. Introduction

In this article we discuss normal quartic surfaces in P^3 and reduced sextic curves in P^2. Especially we would like to treat the case where they have a simple elliptic singularity $\tilde{E}_8$, a cusp singularity $T_{2,3,7}$, or a unimodular exceptional singularity E_{12}. (Cf. Arnold [1], Saito [19]). We show that when they have such a singularity and other several singularities, the combination of singularities is subject to a certain law explained from the viewpoint of Dynkin graphs. Indeed we will verify the following theorems. Now in this article we assume that every variety is defined over the complex number field C.

Definition 0.1. For a given set of several connected Dynkin graphs, the following procedure is called an *elementary transformation* of it.
(1) Replace each component by the extended Dynkin graph of the corresponding type.
(2) Choose in arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuring from them. (Cf. Bourbaki [3], Dynkin [7])

Note that any Dynkin graph without multiple lines is associated to a rational double point on a surface. (Cf. Artin [2])

Theorem 0.2. *Assume that a normal quartic surface X in the projective space P^3 of dimension 3 has a simple elliptic singularity $\tilde{E}_8$. Then the combination of singularities on X is $\tilde{E}_8$ plus one of the following.*
(I) *a combination of rational double points associated to a set of Dynkin graphs*

which is obtained from the Dynkin graph B_9 by elementary transformations repeated twice *such that the resulting set of Dynkin graphs has no vertex corresponding to a short root.*

(II)   *a combination on rational double points associated to a set of Dynkin graphs obtained from the Dynkin graph E_8 by elementary transformations repeated twice.*

(III)   *another $\tilde{E}_8$.*

   Conversely every combination appearing in the above (I), (II), (III) plus $\tilde{E}_8$ can be realized on a normal quartic surface in $\mathbf{P}^3$ as singularities.

   Remark.   1.   The singularity obtained by contracting a smooth elliptic curve with self-intersection number -1 on a smooth surface is the singularity $\tilde{E}_8$. It has the next normal form of the defining equation (Cf. Saito [19]): $x^2+y(y+z^2)(y+az^2)=0,\ a\neq 0, 1$.

2.   In case (III) two elliptic curves appearing in the resolution of singularities on X are isomorphic. This is Y. Umezu's result. (Cf. Umezu [22])

3.   We can find the notion of the elementary transformation already in Dynkin [7]. However, his elementary transformation is slightly different from ours.

4.   Consider the Dynkin graph B_9.
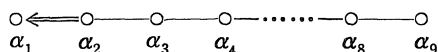
The vertex α_1 corresponds to a short root. Now we consider the following case in particular. We erase α_2, but keep α_1 in the extended Dynkin graph of B_9,
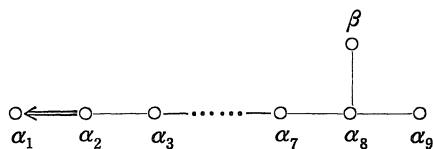
obtaining the graph

(here $\alpha_3, \cdots, \alpha_9$ and β may or may not have been erased.) This is an elementary transformation. Next, we apply another elementary transformation. In the extended Dynkin graph, the new vertex γ joined to α_1 must be regarded as a short root.
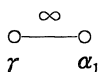
Thus both γ and α_1 have to be erased in the second step of this elementary transformation.

Theorem 0.3. (*resp.* **Theorem 0.4.**) *Consider a normal quartic surface in* $\mathbb{P}^3$ *with a cusp singularity* $T_{2,3,7}$. (*resp. an exceptional singularity* E_{12}) *The combination of singularities on* X *is* $T_{2,3,7}$ (*resp.* E_{12}) *plus one of the following.*
(*I*) *a combination of rational double points associated to a subgraph of the Dynkin graph* D_9. (*resp. a subgraph of the Dynkin graph* A_8.)
(*II*) *a combination of rational double points associated to a proper subgraph of the extended Dynkin graph* $\tilde{E}_8$. (*resp. a subgraph of the Dynkin graph* E_8.)
Conversely every combination in the above (*I*), (*II*) *plus* $T_{2,3,7}$ (*resp.* E_{12}) *can be realized on a normal quartic surface in* $\mathbb{P}^3$ *as singularities.*

Remark. 1. Note that two different objects are called by the same name $\tilde{E}_8$. One is a surface singularity and the other is the extended Dynkin graph.
2. Let D be an irreducible rational curve on a smooth surface S whose singularity is an ordinary double point (resp. an ordinary cusp). We assume that self-intersection number D^2 is -1. The normal isolated singularity obtained by contracting D to a point is $T_{2,3,7}$ (resp. E_{12}) with the following normal form of the local defining equation (Cf. Arnold [1]): $x^2+y^3+z^7+xyz=0$ (resp. $x^2+y^3+z^7+ayz^5=0$, $a\in\mathbb{C}$).
3. (I) is equivalent to saying "a set of graphs with no vertex corresponding to a short root obtained from the Dynkin graph B_9 by one elementary transformation". (resp. "a subgraph of the Dynkin graph B_9 with no vertex corresponding to a short root") In Section 5 we see that the Dynkin graph B_9 is the essential one.
4. Of course we can state (II) in a different way using the word "elementary transformation", too.
5. Indeed, we will see that the number of extensions 2, 1, 0 in Theorem 0.2, Theorem 0.3, Theorem 0.4 respectively is the rank of the fundamental group π_1 of the exceptional curve in the minimal resolution of the singularity $\tilde{E}_8$, $T_{2,3,7}$, E_{12} respectively.

Now recall that if two power series $z^2+f(x,y)$, $z^2+g(x,y)$ with f, $g\in\mathbb{C}\{x,y\}$ can be transformed in $\mathbb{C}^3$ to each other by an analytic coordinate change around the origin, then f and g themselves can also be transformed in $\mathbb{C}^2$ to each other by an analytic coordinate change around the origin. Thus we shall call the singularity defined by $f(x,y)=0$ by the same name as the one defined by $z^2+f(x,y)=0$. Under this convention, we can use such phrase as

"a plane curve singularity of type $\tilde{E}_8$" etc.

Theorem 0.5. (i) *Let B be a reduced sextic curve in the projective space P^2 of dimension 2. Assume that B has a simple elliptic singularity $\tilde{E}_8$. Then the combination of singularities on B is $\tilde{E}_8$ plus one of the following.*
(A) *a combination of rational double points associated to a set of Dynkin graphs obtained from the Dynkin graph $E_8 + A_1$ by elementary transformations repeated twice.*
(B) *either another $\tilde{E}_8$ or another $\tilde{E}_8$ plus one A_1.*

Conversely every combination appearing in the above (A), (B) *plus $\tilde{E}_8$ can be realized on a reduced sextic curve as singularities.*
(ii) *The set of reduced curves with any one of the following combination of sin-guralities has two or more connected components in the space of all sextic curves $P(H^0(P^2, \mathcal{O}_{P^2}(6)))$.*

$\langle 1 \rangle$ $\tilde{E}_8 + A_7$ $\langle 2 \rangle$ $\tilde{E}_8 + 2A_3$ $\langle 3 \rangle$ $\tilde{E}_8 + A_5 + A_1$ $\langle 4 \rangle$ $\tilde{E}_8 + A_3 + 2A_1$
$\langle 5 \rangle$ $\tilde{E}_8 + 4A_1$ $\langle 6 \rangle$ $\tilde{E}_8 + A_7 + A_1$ $\langle 7 \rangle$ $\tilde{E}_8 + 2A_3 + A_1$ $\langle 8 \rangle$ $\tilde{E}_8 + A_5 + 2A_1$
$\langle 9 \rangle$ $\tilde{E}_8 + A_3 + 3A_1$ $\langle 10 \rangle$ $\tilde{E}_8 + 5A_1$

Theorem 0.6. *(resp. Theorem 0.7.)   Consider a reduced sextic plane curve B with a cusp singularity $T_{2,3,7}$. (resp. a unimodular exceptional singularity E_{12}.) Then the combination of singuralities on B is $T_{2,3,7}$ (resp. E_{12}) plus a combination of rational double points associated to a proper subgraph of $\tilde{E}_8 + A_1$ which is not equal to $\tilde{E}_8$. (resp. a subgraph of the Dynkin graph E_8.)*

Conversely such combinations are realized on reduced sextic curves.

The study of projective varieties and their singularities has long history and it has been done from various view-points.   From among them let us pick up some results deeply connected with this article.   In 1934 Du Val found out that combination of singularities on cubic surfaces, plane quartic curves and sextic curves on a singular quadric surface in P^3 can be classified from the view-point of so-called Coxeter groups and root systems of E-type.   (Du Val [6]).   His result was rediscovered by modern mathematicians from a different point of view during 1970's.   (Pinkham [17], Looijenga [12], Mérindol [15], Urabe [24]).   In particular in a paper treating related topics Looijenga has established a Torelli-type theorem for rational surfaces with effective anti-canonical divisors by the mixed Hodge theory and integration of rational 2-forms.   His theorem is a powerful tool to study them.   (Looijenga [12]).   On the other hand Shah classified singularities on quartic surfaces from the view-

point of the geometric invariant theory. (Shah [21]). An example of non-ambient-isotopic sextic curves was given in Zariski [27].

The results in this article will be mainly obtained by developing the above-mentioned Looijenga's method further. Indeed, the fundamental idea in this article is like the following. Firstly we reduce the case of sextic curves to considering branched double coverings over $\mathbb{P}^2$ branching along sextic curves and we show that surfaces under our consideration are rational. Secondly we apply Looijenga's method to construct the moduli space of them. Thirdly we deduce the necessary and sufficient condition for any point on the moduli space to correspond exactly to a quartic surface or a branched double covering along a sextic curve. Lastly we examine closely the action of the Weyl group to the moduli space. With the aid of the theory of Weyl chambers of affine Weyl groups, we get our theorems.

The contents of this article is like the following. Section 1 is the preliminary part. We explain that the study of a sextic curve B is reduced to the study of branched double covering X of $\mathbb{P}^2$ branching along B and that such branched coverings and quartic surfaces are rational surfaces with anti-canonical divisors and ruled surfaces with positive irregularity. From Section 2 to Section 5 we study rational surfaces. In Section 2 we explain a generalized version of Looijenga's Torelli-type theorem. As a result we have an algebraic group Hom (Γ, E) as a moduli space of a certain class of rational surfaces, where Γ is a certain free $\mathbb{Z}$-module with a bilinear form and E is either an elliptic curve with a group law, a multiplicative group $\mathbb{C}^*$, or an additive group $\mathbb{C}$. In addition the relation between our version, theory of integration and the mixed Hodge theory is explained. Section 3 is devoted to studying properties of linear systems on them. Section 4 is the Diophantine theoretic part. We determine the class of the polarization in the Picard group. The action of the Weyl group on Hom (Γ, E) is studied in Section 5.

I would like to express my heartly thanks to my teachers and colleagues. In particular we thank Mr. T. Fukui for pointing out an error in the first version of this article.

Now we guess that our theorem is a small part of a big theorem dominating all quartic surfaces and all sextic curves, of course. There are two reasons we discuss only surfaces with $\tilde{E}_8$, $T_{2,3\ 7}$, E_{12} here. One is that since most of them are rational, they have a rather simple global structure. The other is that the fundamental domain of the Coxeter group introduced in Section 2 is easier to handle than that in other cases. Therefore the next problem should be the next

step of our study.   (Cf. Kato and Naruki [10], Umezu and Urabe [23])

Problem.   Find out the general law explaining which singularities appear on quartic surfaces and sextic curves.

For line bundles L, M and divisors A, B on a smooth surface Z, the intersection number is denoted by $L \cdot M$, $L \cdot A$, or $A \cdot B$ in this article.   Sometimes we write L^2, A^2 instead of $L \cdot L$, $A \cdot A$.   The complete linear system associated to the line bundle L is denoted by $|L|$.   The complete linear system $|\mathcal{O}_Z(A)|$ associated to a divisor A is denoted by $|A|$ for brevity.   If M is a dual line bundle of L, we denote $|M|$ by $|-L|$.

§1.   Preliminaries

In this section we explain that quartic surfaces and branched double coverings of $\boldsymbol{P}^2$ branching along sextic curves are roughly classified into 3 types; K3 surfaces, rational surfaces and ruled surfaces with positive irregularity.

First of all, we consider sextic curves.   Let B be a sextic curve in the 2 dimensional projective space $\boldsymbol{P}^2$.   We introduce the branched double covering X of $\boldsymbol{P}^2$ branching along B.   Let $F(z_0, z_1, z_2)$ be the homogeneous defining polynomial of B.   We give weight 1, 1, and 1 to variables z_0, z_1 and z_2 respectively.   Let z_3 be another variable with weight 3.   Then $z_3^2 - F(z_0, z_1, z_2) = 0$ defines a surface X in the weighted projective space $\boldsymbol{P}(1, 1, 1, 3)$ not passing through the point $(0, 0, 0, 1)$.   Here recall that the quotient of $\boldsymbol{C}^4 - \{(0, 0, 0, 0)\}$ by the following action of $\boldsymbol{C}^* = \boldsymbol{C} - \{0\}$ is $\boldsymbol{P}(1, 1, 1, 3)$.   Action: $t(z_0, z_1, z_2, z_3) = (tz_0, tz_1, tz_2, t^3 z_3)$ where $t \in \boldsymbol{C}^*$ and $(z_0, z_1, z_2, z_3) \in \boldsymbol{C}^4 - \{(0, 0, 0, 0)\}$. $\boldsymbol{P}(1, 1, 1, 3)$ has a unique singular point at $(0, 0, 0, 1)$.   The restriction to X of the projection $\pi \colon \boldsymbol{P}(1, 1, 1, 3) - \{(0, 0, 0, 1)\} \to \boldsymbol{P}^2$, $(z_0, z_1, z_2, z_3) \to (z_0, z_1, z_2)$ defines a finite morphism of degree 2.   We denote it by the same letter $\pi \colon X \to \boldsymbol{P}^2$.   The following lemma is easily checked.   (Cf. Arnold [1])

Lemma 1.1.   *A point $x \in X$ is singular if and only if $\pi(x)$ is a singular point of B.   Moreover the isomorphism class of a surface singularity (X, x) and that of a curve singularity $(B, \pi(x))$ determine each other uniquely.   Thus singular points on X and those on B has one-to-one correspondence.*

Thus the study of B is reduced to that of X.   Note that X is normal if B is reduced, since normality is equivalent to that X has only isolated singular points in our case.   (Cf. Matsumura [12])

The next lemma is easy to verify.

Lemma 1.2. *Let X denote either a quartic surface in $\mathbb{P}^3$ or a branched double covering over $\mathbb{P}^2$ branching along a sextic curve B. Let $\mathcal{O}_X$ denote its structure sheaf.*

(1) *The dualizing sheaf ω_X is a trivial invertible sheaf, i.e., $\omega_X \cong \mathcal{O}_X$.*

(2) $H^1(\mathcal{O}_X) = 0$.

Note that Lemma 1.1 and Lemma 1.2 hold even if the sextic curve is not reduced or the quartic surface is not normal. However, in the sequel we treat only the case of reduced sextic curves and normal quartic surfaces. Let X be as in Lemma 1.2. We assume moreover that X is normal.

Let $\rho: Z \to X$ be the minimal resolution of singularities of normal X. We have the Leray spectral sequence

$$E_2^{p,q} = H^p(R^q \rho_* \mathcal{O}_Z) \Rightarrow H^{p+q}(\mathcal{O}_Z)$$

Thus we have the next lemma since normality implies $R^0 \rho_* \mathcal{O}_Z = \mathcal{O}_X$. Here the geometric genus of a singular point $x \in X$ is defined by $p_g(X, x) = \dim_C (R^1 \rho_* \mathcal{O}_Z)_x$. It is known that $p_g(X, x)$ is well-defined. (Wagreich [26]). Moreover $p_g(X, x) = 0$ if and only if $x \in X$ is either a smooth point or a rational double point. (Artin [2])

Lemma 1.3. $\chi(\mathcal{O}_Z) + \displaystyle\sum_{x \in X:\ \text{singular points}} p_g(X, x) = \chi(\mathcal{O}_X) = 2$

where $\chi(F)$ is the Euler-Poincaré characteristic of the sheaf F.

Note that the minimality of Z implies the next lemma, of which we omit the proof.

Lemma 1.4. *There exists an effective divisor D on Z with $\omega_Z \cong \mathcal{O}_Z(-D)$. Moreover*

$$\mathrm{Supp}\, D = \bigcup_{x \in X;\ \text{singular points with } p_g(X, x) > 0} \rho^{-1}(x)\,.$$

Proposition 1.5. *Let X be either a normal quartic surface in $\mathbb{P}^3$ or a branched double covering over $\mathbb{P}^2$ branching along a reduced sextic curve B. Set $P = \displaystyle\sum_{x \in X:\ \text{singular points}} p_g(X, x)$.*

⟨1⟩ *If $P = 0$, then the minimal resolution Z of X is a K3 surface.*

⟨2⟩ *If $P = 1$, then Z is a rational surface with an anti-canonical effective divisor D.*

⟨3⟩ *If $P \geq 2$, then Z is birationally equivalent to a ruled surface over a smooth irreducible curve of genus $P-1$.*

Proof. If $P = 0$, $\omega_Z \cong \mathcal{O}_Z$ by Lemma 1.4. By the Leray spectral sequence

and by Lemma 1.2 we have $H^1(\mathcal{O}_Z) = 0$ since $R^1\rho_*\mathcal{O}_Z = 0$. Thus Z is a K3 surface.

Assume $P = 1$. By Lemma 1.4 one sees that $\omega^{\otimes m} \cong \mathcal{O}_Z(-mD)$ for an effective divisor $D \neq 0$. In particular the Kodaira dimension $\kappa(Z)$ of Z is $-\infty$. By the theory of classification of surfaces (Cf. Shafarevich [20]) one sees that Z is birationally equivalent to $\boldsymbol{P}^2$ or a ruled surface over a curve with positive genus. On the other hand we have $\chi(\mathcal{O}_Z) = 2 - P$ by Lemma 1.3. Since the Euler-Poincaré characteristic of the structure sheaf is a birational invariant, Z is rational.

In the case where $P \geq 2$, we have $\langle 3 \rangle$ by the same reason.   Q.E.D.

Remark. In Umezu [22] Y. Umezu showed that if X is quartic and if $P \geq 2$, then $P = 2$ or 4 and she gave the classification of quartic surfaces with $P \geq 2$. As for branched coverings, if $P \geq 2$, then $P = 2$ or 3.

We chiefly discuss in this article case $\langle 2 \rangle$ in Proposition 1.5.

Let X be as in Proposition 1.5. Assume further that X has unique $\tilde{E}_8$ singularity plus several rational double points and no other singularities. The minimal resolution Z of X is rational with a non-zero effective anti-canonical divisor D. Moreover in this case D is an irreducible smooth elliptic curve with self-intersection number $D^2 = -1$. If X has $T_{2,3,7}$ instead of $\tilde{E}_8$, then D is an irreducible rational curve whose singularity is one ordinary double point with self-intersection number $D^2 = -1$. If X has E_{12} instead of $\tilde{E}_8$, then D is an irreducible rational curve whose singularity is one ordinary cusp with $D^2 = -1$.

Proposition 1.5. *Assume that Z is a smooth rational surface with an effective irreducible anti-canonical divisor D. If Z is not a relatively minimal model, then Z can be blown-down to $\boldsymbol{P}^2$.*

Proof. The proof is the same as in Looijenga [12], Theorem (1.1). Therefore we omit it here.   Q.E.D.

Lemma 1.6. *A non-zero irreducible anti-canonical effective divisor D on a smooth rational surface Z is either;*

   (a) *an irreducible smooth elliptic curve*

   (b) *an irreducible rational curve whose singularity is one ordinary double point,*

or (c) *an irreducible rational curve whose singularity is one ordinary cusp.*

In particular examples just before Proposition 1.5 exhaust all the possibilities.

Proof. It is an easy consequence of the adjunction formula.   Q.E.D.

§2. A Theorem of Torelli Type

In this section, we would like to explain a theorem of Torelli type for rational surfaces with an effective anti-canonical divisor. Most of the essential ideas of this theorem are due to Looijenga. However the situation we treat here is slightly different from Looijenga's original one. (In Looijenga [12] it is assumed that the anti-canonical divisor D is a cycle of rational curves. In this article we assume that D is either (a), (b) or (c) in Lemma 1.6. Though (b) is equivalent to a cycle of rational curves with length 1, the case (a) and (c) did not be treated in [12].)

Because the proof of the theorem is the same as in [12], we omit it here.

Anyway we would like to begin this section by explaining several notions.— Dynkin graphs, Weyl groups, roots, etc.

Let Z be a smooth rational surface with irreducible effective anti-canonical divisor D. Moreover we assume in this section that the self-intersection number of the dualizing sheaf ω_Z^2 is less than or equal to 6. Set $t=9-\omega_Z^2$. We have $t \geq 3$. Under this assumption, Z is not a minimal model. Thus by Proposition 1.8, we have a sequence

$$(2.1) \quad Z = Z_t \xrightarrow{\sigma_t} Z_{t-1} \xrightarrow{\sigma_{t-1}} \cdots \to Z_2 \xrightarrow{\sigma_2} Z_1 \xrightarrow{\sigma_1} Z_0 = \mathbb{P}^2$$

where each σ_i is a blowing-up of a point $z_i \in Z_{i-1}$. Note that the number of blowing-ups is $t = \omega_{\mathbb{P}^2}^2 - \omega_Z^2$. We denote $D_t = D$, $D_{i-1} = \sigma_i(D_i)$ $(1 \leq i \leq t)$. We have $z_i \in D_{i-1} \subset Z_{i-1}$. We consider the Picard group Pic (Z). Let e_0 be the class of the total inverse image on Z of a line in $Z_0 = \mathbb{P}^2$. Let e_i $(i \geq 1)$ be the class of the total inverse image on Z of the exceptional curve $\sigma_i^{-1}(z_i)$. Elements $e_0, e_1, \cdots, e_t \in$ Pic (Z) define a free $\mathbb{Z}$-basis with the following mutual intersection numbers;

$$e_0^2 = +1, \quad e_i^2 = -1 \, (1 \leq i \leq t), \quad e_i \cdot e_j = 0 \quad (i \neq j).$$

We say that (2.1) is the *blowing-down sequence* along $e_0, e_1, \cdots, e_t$, when each e_i is the above-mentioned class of effective divisors. Here we note that

$$\omega_Z = \mathcal{O}_Z(-D) = -3e_0 + e_1 + \cdots + e_t.$$

Let $P = \mathbb{Z}\varepsilon_0 + \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_t$ be a $\mathbb{Z}$-module with a bilinear form which is isomorphic to Pic (Z) with the intersection form, where $\varepsilon_0, \cdots, \varepsilon_t \in P$ is a basis

with

$$\varepsilon_0^2 = +1\,, \quad \varepsilon_i^2 = -1\ (1 \leq i \leq t)\,, \quad \varepsilon_i \circ \varepsilon_j = 0 \quad (i \neq j)\,.$$

We set $\kappa = -3\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_t$. Let Γ be the orthogonal complement of $\boldsymbol{Z}\kappa$ in P. $\Gamma = \{x \in P \mid x \circ \kappa = 0\}$. The restriction of the bilinear form of P to Γ is described by the following graph.
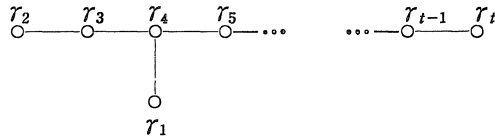
Here we denote $\gamma_1 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, $\gamma_j = \varepsilon_{j-1} - \varepsilon_j$ $(2 \leq j \leq t)$ for simplicity. Vertex $\circ$ corresponding to γ_i indicates a member of a basis of Γ with the self-intersection -2. (It is easily checked that the above $\gamma_1, \gamma_2, \cdots, \gamma_t$ defines a basis of Γ and that $\gamma_i^2 = -2$ if $t \geq 3$.) Two vertices $\overset{\gamma_i \ \ \gamma_j}{\circ\!-\!\!-\!\circ}$ are connected with an edge—if $\gamma_i \circ \gamma_j = 1$ and they are not connected if $\gamma_i \circ \gamma_j = 0$. In particular Γ is isomorphic to the root lattice (Cf. Bourbaki [3]) of type $A_2 + A_1$, A_4, D_5, E_6, E_7 or E_8 according as $t = 3, 4, 5, 6, 7, 8$. If $t \geq 9$, then Γ is not negative-definite.

Let $\gamma \in P$ be an element with $\gamma^2 = -2$. Let $s_\gamma \colon P \to P$ be a linear map defined by $s_\gamma(x) = x + (x \circ \gamma)\gamma$ for $x \in P$. It is easily checked that s_γ is an isomorphism of order 2 preserving the bilinear form. In addition if $\gamma \circ \kappa = 0$, then $s_\gamma(\kappa) = \kappa$. s_γ is called the *reflection* associated to γ. The group generated by $s_{\gamma_1}, \cdots, s_{\gamma_t}$ is called the *Weyl group* of P and it is denoted by W or W_P. (Note that for $w \in W$, $w(\kappa) = \kappa$.) We call any element in $\bigcup_{i=1}^{t} W\gamma_i \ (\subset \Gamma)$ a *root*.

Indeed s_γ defines the reflection with respect to the hyperplane orthogonal to γ i.e., $\{x \in P \otimes \boldsymbol{R} \mid x \circ \gamma = 0\}$ in $P \otimes \boldsymbol{R}$. $(W, \{s_{\gamma_1}, s_{\gamma_2}, \cdots, s_{\gamma_t}\})$ defines a Coxeter system. (Cf. Looijenga [12], Bourbaki [3]). Now let $\gamma \in \Gamma$ be a root. Writing $\gamma = \sum_{i=1}^{t} n_i \gamma_i$ $(n_i \in \boldsymbol{Z})$, then we have either $n_i \geq 0$ for any i or $n_i \leq 0$ for any i. If $n_i \geq 0$ for any i, we say that γ is a *positive root*. Otherwise it is called a *negative root*. Note that this notion depends on the choice of the basis. Let $R_+(\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_t)$ denote the set of positive roots.

For roots in Pic(Z) we can distinguish the following property. A root $r \in \mathrm{Pic}(Z)$ is called a *nodal root* if the restriction of r to D is a trivial line bundle. (This terminology is due to Looijenga.)

Lemma 2.1. *Let $r \in \mathrm{Pic}(Z)$ be a nodal root. Then either r or $-r$ is effective.*

Proof. Assume that $r^2 = -2$, $r|_D \cong \mathcal{O}_D$ and $H^0(-r) = 0$. By the Serre duality we have $H^2(r(-D)) = 0$. Consider the exact sequence

$$0 \to r(-D) \to r \to r|_D \to 0 .$$

One sees that $h^2(r) = 0$ and $H^1(r) \to H^1(r|_D) \cong \mathcal{C}$ is surjective. Thus $h^1(r) > 0$. By the Riemann-Roch formula

$$h^0(r) = (r^2 + D \cdot r)/2 + 1 + h^1(r) > 0 ,$$

i.e., r is effective. Q.E.D.

Let S_+ denote the set of effective nodal roots. $S = S_+ \cup (-S_+)$ is the set of nodal roots. Let W_S be the group generated by $\{s_r \,|\, r \in S\}$. W_S is a subgroup of $W_{\mathrm{Pic}(Z)}$. We call W_S the Weyl group of Z associated to nodal roots.

The following theorem is due to Demazure when $3 \leq t \leq 9$ and it is essentially due to Looijenga when $t \geq 10$. (Demazure [5], Looijenga [12])

Theorem 2.2. *Let Z be a rational surface with an effective irreducible anti-canonical divisor D such that $t = 9 - \omega_Z^2 \geq 3$. Let $e_0, e_1, \cdots, e_t \in \mathrm{Pic}(Z)$ be a basis such that there exists a blowing-down sequence along $e_0, e_1, \cdots, e_t$. Let W be the Weyl group of $\mathrm{Pic}(Z)$ defined depending on $e_0, e_1, \cdots, e_t$ and let $w \in W$. Then there exists a blowing-down sequence along $w(e_0), w(e_1), \cdots, w(e_t)$ if and only if every effective nodal root is a positive root, i.e., $S_+ \subset R_+(w(e_0), w(e_1), \cdots, w(e_t))$. Moreover for any two basis $e_0, e_1, \cdots, e_t \in \mathrm{Pic}(Z)$ and $e'_0, e'_1, \cdots, e'_t \in \mathrm{Pic}(Z)$ such that there exist blowing-down sequences along both of them, there exists an element $w \in W$ with $e'_i = w(e_i)$ for $0 \leq i \leq t$.*

Corollary 2.3. *The set of roots R in $\mathrm{Pic}(Z)$ and the Weyl group W of $\mathrm{Pic}(Z)$ do not depend on the choice of the blowing-down sequence (2.1).*

Note that the *positive cone* $\{x \in P \otimes \mathbb{R} \,|\, x \circ x > 0\}$ in $P \otimes \mathbb{R}$ has two connected components since the signature of the bilinear form of P is $(1, t)$.

Definition 2.4. Let t be an integer with $t \geq 3$. Let E be a one-dimensional algebraic group isomorphic to either a smooth elliptic curve, $\mathcal{C}^* = \mathcal{C} - \{0\}$, or $\mathcal{C}$. We call the following object $\mathcal{Z} = (Z, D, \alpha, \iota)$ a marked rational surface over E of degree $9 - t$.

(1) The first item Z is a smooth rational surface with $\omega_Z^2 = 9 - t$.

(2) The second item D is an effective irreducible anti-canonical divisor on Z which has the following isomorphism ι.

(3) The third one $\alpha: P \to \mathrm{Pic}(Z)$ is a linear isomorphism satisfying the following

conditions (i), (ii), (iii) and (iv), where $P=\mathbf{Z}\varepsilon_0+\mathbf{Z}\varepsilon_1+\cdots+\mathbf{Z}\varepsilon_t$ is an abstract free $\mathbf{Z}$-module with a bilinear form defined by $\varepsilon_0^2=+1$, $\varepsilon_i^2=-1$ $(1\leq i\leq t)$, $\varepsilon_i\cdot\varepsilon_j=0$ $(i\neq j)$.

(i) α preserves the bilinear form, i.e., $x\cdot y=\alpha(x)\cdot\alpha(y)$ for any $x,\ y\in P$.

(ii) $\alpha(\kappa)=\omega_Z$ where $\kappa=-3\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_t$.

(iii) $\alpha(\Pi)=R$ where Π and R are the sets of roots in P and $\mathrm{Pic}(Z)$ respectively.

(iv) $\alpha(\Lambda_+)=C_+$ where Λ_+ (resp. C_+) is a connected component of the positive cone in $P\otimes R$ (resp. $\mathrm{Pic}(Z)\otimes R$) containing ε_0. (resp. e_0)

(4) The fourth one $\iota\colon\mathrm{Pic}^0(D)\to E$ is an isomorphism as algebraic groups, where $\mathrm{Pic}^0(D)$ is the connected component of $\mathrm{Pic}(D)$ containing the zero element.

Definition 2.5. Two marked rational surface over E (Z,D,α,ι) and (Z',D',α',ι') are *isomorphic* if there exists an isomorphism of varieties $f\colon Z\to Z'$ satisfying the following conditions (A), (B), and (C).

(A) $f(D)=D'$.

(B) The composition

$$\mathrm{Pic}\,(Z)\overset{\alpha}{\underset{\sim}{\leftarrow}}P\overset{\alpha'}{\to}\mathrm{Pic}\,(Z')\overset{f^*}{\to}\mathrm{Pic}\,(Z)$$

can be written as a composition of finite reflections corresponding to nodal roots on Z.

(C) The diagram $\mathrm{Pic}^0(D')\overset{f^*}{\longrightarrow}\mathrm{Pic}^0(D)$

$$\iota'\searrow\qquad\swarrow\iota$$

$$E\qquad\qquad\text{is commutative.}$$

Definition 2.6. Let $Q\subset\mathrm{Pic}(Z)$ be the orthogonal complement of $\mathbf{Z}\omega_Z$, i.e., $Q=\{x\in\mathrm{Pic}(Z)\,|\,x\cdot\omega_Z=0\}$. Note that the image of Q by the restriction map $\mathrm{Pic}(Z)\to\mathrm{Pic}(D)$ is contained in $\mathrm{Pic}^0(D)$. The following composition of homomorphisms is called the *characteristic homomorphism* $\varphi_{\mathscr{Z}}$ of $\mathscr{Z}=(Z,D,\alpha,\iota)$.

$$\Gamma\overset{\alpha}{\to}Q\xrightarrow{\text{restriction}}\mathrm{Pic}^0(D)\overset{\iota}{\to}E$$

Here Γ is the orthogonal complement of $\mathbf{Z}\kappa$ in P.

It is easy to check the next lemma.

Lemma 2.7. *The characteristic homomorphism $\varphi_{\mathscr{Z}}$ depends only on the isomorphism class of* (Z,D,α,ι).

Now we can state the main theorem in this section. It gives a powerful tool to study rational surfaces. Even though the situation treated by Looijenga is slightly different from ours, this theorem is due to Looijenga, we think.

Theorem 2.8. (*A theorem of Torelli type.*) *The map induced by associating a marked rational surface (Z, D, α, ι) to its characteristic homomorphism*

$$\frac{\{(Z, D, \alpha, \iota)\colon \text{a marked rational surface over } E \text{ of degree } 9-t\}}{\text{isomorphisms}} \to \mathrm{Hom}\,(\Gamma, E)$$

is bijective.

Next we would like to explain why this theorem is called one of Torelli type. It is explained by the Deligne's mixed Hodge theory. For simplicity we assume that D is an irreducible smooth elliptic curve with $D^2 = -1$. Consider an exact sequence of mixed Hodge structures (Cf. Deligne [4])

$$H^0(D)(-1) \to H^2(Z) \to H^2(Z-D) \to H^1(D)(-1)\,.$$

Note that $F^2 H^2(Z) = 0$ and $F^2(H^1(D)(-1)) = H^0(\omega_D)$. Thus we have that $\dim_C F^2 H^2(Z-D) = 1$. Now by definition $F^2 H^2(Z-D)$ is represented by a logarithmic 2-form ψ on Z with the pole along D, which is unique up to constant multiple. Since this situation is very similar to that of the second cohomology group of K3 surfaces, we can consider the periods of ψ. Here the periods are nothing but the linear mapping

$$H_2(Z-D) \to C;\ \varDelta \to \textstyle\int_\varDelta \psi\,.$$

Note that there is a submodule $\mathrm{Im}\,(H_1(D) \overset{\tau}{\to} H_2(Z-D))$. Since $\int_{\tau(\gamma)} \psi = 2\pi\sqrt{-1}\int_\gamma \mathrm{Res}\,(\psi)$, we have that $C/\mathrm{Im}\,(H_1(D) \to H_2(Z-D) \to C) \cong D$. Let Q be the orthogonal complement of $Z\omega_Z$ in $\mathrm{Pic}\,(Z)$. One sees easily that there exists an exact sequence

$$0 \to H_1(D) \to H_2(Z-D) \to Q \to 0\,.$$

Thus we have an induced group homomorphism $Q \to D$. We can check that this homomorphism is identified with the restriction of the mapping $\mathrm{Pic}\,(Z) \to \mathrm{Pic}\,(D)$. Therefore the characteristic homomorphism φ_Z can be regarded as the periods of $Z-D$. This is the reason why the above theorem is called one of Torelli type.

§3. Properties of Line Bundles

This section is devoted to study properties of line bundles on a smooth

rational surface Z with an effective irreducible anticanonical divisor D. We owe ideas in this section to Saint-Donat [18] very much. However, in [18] two assumptions that the canonical bundle is trivial and that the second cohomology group is an even lattice are frequently used since K3 surfaces are treated. Though our situation in this article is very similar to that in [18], we can use neither assumption. Therefore in this section, though it is slightly lengthy, we carefully give proofs to analogous assertions to those in [18] one after one.

Recall that a line bundle L (resp. a divisor C) on Z is *numerically effective* if for any curve A on Z, the intersection $L \cdot A$ (resp. $C \cdot A$) is non-negative.

Definition 3.1. A line bundle L on Z with the following properties are called a *polarization* of Z.

(1) The self-intersection number L^2 is positive.

(2) L is numerically effective.

(3) The restriction of L to D is a trivial line bundle, i.e., $L|_D \cong \mathcal{O}_D$.

(4) For every exceptional curve of the first kind A, the intersection $L \circ A$ is strictly positive. $(L \cdot A > 0)$

The number L^2 is called the *degree* of L.

Lemma 3.2. (1) *If Z has a polarization, then $t = 9 - \omega_Z^2 \geqq 10$.*

(2) *For any polarization L, $h^1(L) = 1$ and $h^0(L) = (L^2/2) + 2$. Moreover the linear system $|L|$ has no fixed points on D.*

Proof. (1) If $t \leqq 9$, for every element $M \in \mathrm{Pic}(Z)$ with $M \cdot \omega_Z = 0$, $M^2 \leqq 0$ holds. However $L^2 > 0$ and $L \cdot \omega_Z = 0$ for any polarization.

(2) By the Kawamata-Ramanujan vanishing theorem (Kawamata [11]), we have $H^1(L(-D)) = H^2(L(-D)) = 0$. Thus the mapping $H^0(L) \to H^0(L|_D) \cong H^0(\mathcal{O}_D) \cong C$ is surjective, and $h^1(L) = h^1(\mathcal{O}_D) = 1$, $h^2(L) = 0$. Surjectivity implies that $|L|$ has no fixed points on D. On the other hand by the Riemann-Roch formula we have

$$h^0(L) = (L^2 - L \cdot \omega_Z)/2 + \chi(\mathcal{O}_Z) + h^1(L) - h^2(L) = (L^2/2) + 2 . \quad \text{Q.E.D.}$$

If X is a normal quartic surface in $\boldsymbol{P}^3$ and $\rho: Z \to X \subset \boldsymbol{P}^3$ is its minimal resolution of singularities, then $L = \rho^* \mathcal{O}_{\boldsymbol{P}^3}(1)$ is a polarization of degree 4. Similarly for a branched double covering branching along a sextic curve we can define a polarization of degree 2. However note that conversely the polarization L does not necessarily define a generically one-to-one morphism $\varphi_L: Z \to \boldsymbol{P}^N$. The linear system $|L|$ may have fixed components. Even if it has no fixed components, it may have isolated fixed points. Even if it has no fixed points,
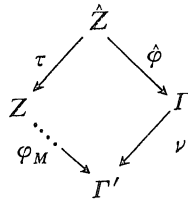
it may define a morphism whose degree is greater than 1.

In this section we give a necessary and sufficient condition in order that L does not define a generically one-to-one morphism in the case $L^2 = 2$ or 4.

Proposition 3.3. *Let M be a line bundle on Z satisfying*

(a) $H^0(M) \neq 0$

(b) *The linear system $|M|$ has no fixed components. And*

(c) *the intersection $M \cdot D$ is zero.*

(1) *If the image of the rational map φ_M associated to M is a curve, then $M^2 = 0$.*

(2) *One of the following* (i), (ii) *holds.*

(i) $M^2 > 0$, *any generic member of $|M|$ is an irreducible curve with arithmetic genus $(M^2/2) + 1$ and $h^1(M) = 1$.*

(ii) $M^2 = 0$ *and there exists a smooth irreducible elliptic curve F and a positive integer k with $M \cong \mathcal{O}_Z(kF)$. Moreover $h^1(M) = k$. Every member of $|M|$ can be written in the form $F_1 + F_2 + \cdots + F_k$, where $F_i \in |F|$.*

Proof. Firstly assume that the image Γ' of the rational map $\varphi_M : Z \cdots \to \mathbb{P}^N$ associated to M is a curve. Let $\nu : \Gamma \to \Gamma'$ be the normalization of Γ'. For a suitable choice of a birational morphism $\tau : \hat{Z} \to Z$, there exists a morphism $\phi : \hat{Z} \to \Gamma$ with $\varphi_M \tau = \nu \phi$.

$$
\begin{array}{ccc}
 & \hat{Z} & \\
\tau \swarrow & & \searrow \phi \\
Z & & \Gamma \\
\varphi_M \searrow & & \swarrow \nu \\
 & \Gamma' &
\end{array}
$$

If the genus of Γ is positive, we have a non-zero global regular 1-form α on Γ. Since $\phi^* \alpha$ defines a non-zero global regular 1-form on $\hat{Z}$, we have $H^0(\Omega_{\hat{Z}}^1) \neq 0$, which contradicts that $\hat{Z}$ is rational. Thus Γ is a smooth rational curve. It implies that for every point $p, p' \in \Gamma$, divisors $\tau(\phi^{-1}(p))$ and $\tau(\phi^{-1}(p'))$ are linearly equivalent. Choose a general point $q \in \Gamma$ and set $F = \tau(\phi^{-1}(q))$. One sees that $M \cong \mathcal{O}_Z(kF)$ for some integer k. If $\dim |F| \geq 2$, then we have a member $F_1 \in |F|$ such that for any point $p \in \Gamma$, $F_1 \neq \tau(\phi^{-1}(p))$. Choose points $q = q_1, q_2, \cdots, q_k \in \Gamma$ such that $\tau(\phi^{-1}(q_1)) + \tau(\phi^{-1}(q_2)) + \cdots + \tau(\phi^{-1}(q_k)) \in |M|$. Since $\tau(\phi^{-1}(q_1)) = F \sim F_1$, we have $G = F_1 + \tau(\phi^{-1}(q_2)) + \cdots + \tau(\phi^{-1}(q_k)) \in |M|$ since $|M|$ is a complete linear system. However, by the choice of F_1 and the definition of ϕ, we have $G \notin |M|$, a contradiction. Therefore we have $\dim |F| = 1$.

We have $kD \cdot F = D \cdot M = 0$ and thus $D \cdot F = 0$. We can conclude that $D \cap F = \phi$. Consider the exact sequence

$$0 \to \mathcal{O}_Z(-F-D) \to \mathcal{O}_Z \to \mathcal{O}_F \oplus \mathcal{O}_D \to 0.$$

It implies $h^1(\mathcal{O}_Z(-F-D)) = 1$. By the Serre duality, we have $h^1(\mathcal{O}_Z(F)) = 1$. Moreover $h^2(\mathcal{O}_Z(F)) = h^0(\mathcal{O}_Z(-F-D)) = 0$. It follows from the Riemann-Roch formula

$$\begin{aligned}
2 = 1 + \dim|F| &= h^0(\mathcal{O}_Z(F)) \\
&= \chi(\mathcal{O}_Z) + (F^2 + F \cdot D)/2 + h^1(\mathcal{O}_Z(F)) = F^2/2 + 2
\end{aligned}$$

that $F^2 = 0$ and $M^2 = k^2 F^2 = 0$. In particular the linear system $|F|$ has no fixed points and F is smooth by the Bertini theorem. By adjunction formula F is an elliptic curve.

Next we would like to compute $h^1(M)$. Let $F_1, F_2, \cdots, F_k \in |F|$ be general members. We can assume that $F_1, \cdots, F_k$ and D are mutually disjoint since $D \cdot F = 0$ and $F^2 = 0$. Using the exact sequence

$$0 \to \mathcal{O}_Z(-F_1 - \cdots - F_k - D) \to \mathcal{O}_Z \to \bigoplus_{i=1}^{k} \mathcal{O}_{F_i} \oplus \mathcal{O}_D \to 0$$

and the Serre duality, one sees that $h^1(\mathcal{O}_Z(F_1 + \cdots + F_k)) = h^1(M) = k$.

Secondly assume that the image of φ_M is not a curve. We have $A^2 \geq 0$ since $|M|$ has no fixed components. If $A^2 = M^2 = 0$, then $|M|$ has no fixed points and the image of the morphism φ_M is a curve. Thus $A^2 = M^2 > 0$. By the Bertini theorem A is irreducible. We have $p_a(A) = A^2/2 + 1$ by the adjunction formula. It follows that $A \cap D = \phi$ from $M \cdot D = A \cdot D = 0$. Thus

$$0 \to \mathcal{O}_Z(-A-D) \to \mathcal{O}_Z \to \mathcal{O}_A \oplus \mathcal{O}_D \to 0$$

is exact and one sees that $h^1(\mathcal{O}_Z(A)) = 1$.   Q.E.D.

Lemma 3.4. *Let C be an effective divisor on Z with* $\operatorname{Supp} C \cap D = \phi$ *and* $h^0(\mathcal{O}_C) = 1$. *Then we have* $h^1(\mathcal{O}_Z(C)) = 1$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_Z(-C-D) \to \mathcal{O}_Z \to \mathcal{O}_C \oplus \mathcal{O}_D \to 0.$$

We have $h^1(\mathcal{O}_Z(-C-D)) = 1$. By the Serre duality we have the conclusion.
                                                                                Q.E.D.

Lemma 3.5. *Let $\varDelta$ be a non-zero effective divisor on Z with $h^0(\mathcal{O}_Z(\varDelta)) = 1$ and* $\operatorname{Supp} \varDelta \cap D = \phi$. *We have $h^1(\mathcal{O}_Z(\varDelta)) \geq 1$ and $\varDelta^2 = -2h^1(\mathcal{O}_Z(\varDelta)) \leq -2$.*

Proof. Consider the sequence

$$0 \to \mathcal{O}_Z(-\Delta-D) \to \mathcal{O}_Z \to \mathcal{O}_\Delta \oplus \mathcal{O}_D \to 0 .$$

By assumption $\mathrm{Supp}\,\Delta \cap D = \phi$, it is exact. We have $h^1(\mathcal{O}_Z(\Delta)) = h^1(\mathcal{O}_Z(-\Delta-D)) = h^0(\mathcal{O}_\Delta) \geq 1$ since $h^0(\mathcal{O}_Z) = h^0(\mathcal{O}_D) = 1$, $h^1(\mathcal{O}_Z) = 0$. Note that $h^2(\mathcal{O}_Z(\Delta)) = h^0(\mathcal{O}_Z(-\Delta-D)) = 0$. By the Riemann-Roch theorem, we have

$$\begin{aligned}
1 = h^0(\mathcal{O}_Z(\Delta)) &= \chi(\mathcal{O}_Z) + (\Delta^2 + D \cdot \Delta)/2 + h^1(\mathcal{O}_Z(\Delta)) - h^2(\mathcal{O}_Z(\Delta)) \\
&= 1 + (\Delta^2/2) + h^1(\mathcal{O}_Z(\Delta)) .
\end{aligned}$$

Q.E.D.

Corollary 3.6. *Let Θ be an irreducible curve on Z with $h^0(\mathcal{O}_Z(\Theta)) = 1$ and $\Theta \cdot D = 0$. We have $\Theta^2 = -2$ and Θ is a smooth rational curve.*

Proof. Since Θ and D are irreducible, $\Theta \cdot D = 0$ implies $\Theta \cap D = \phi$. Obviously $h^0(\mathcal{O}_\Theta) = 1$. Thus by Lemma 3.4 and Lemma 3.5, we obtain $\Theta^2 = -2$. Moreover by the adjunction formula, Θ is smooth and rational. Q.E.D.

Proposition 3.7. *Let L be a polarization on Z. If $|L|$ has a fixed component, then $|L|$ contains a divisor with the following form; $kF + \Gamma$ where F is an irreducible smooth elliptic curve on Z with $F^2 = 0$ and $D \cdot F = 0$, Γ is an irreducible smooth rational curve with $\Gamma^2 = -2$, $\Gamma \cdot D = 0$ and $\Gamma \cdot F = 1$ and k is an integer with $k \geq 2$. The divisor Γ is the fixed part of $|L|$.*

Proof. The proof is slightly complicated. By Lemma 3.2 the linear system $|L|$ is non-empty. Let $C \in |L|$ be a general member. Let Δ be the fixed part of the linear system $|L| = |C|$. We set $C = A + \Delta$ where A is the moving part. By Lemma 3.2 one sees $\mathrm{Supp}\,\Delta \cap D = \phi$ and $\Delta \cdot D = 0$. We also have by Lemma 3.2, (2)

$$h^0(\mathcal{O}_Z(C)) \geq 2 + (C^2/2) \geq 2$$

and thus $A \neq 0$. One may assume that $\mathrm{Supp}\,A \cap D = \phi$. Note that $A^2 \geq 0$ since A is the moving part.
Case 1. $A^2 > 0$.

By Proposition 3.3 any general member of $|A|$ is an irreducible curve with arithmetic genus $(A^2/2) + 1$ and $h^1(\mathcal{O}_Z(A)) = 1$. One has

$$h^0(\mathcal{O}_Z(A)) = \chi(\mathcal{O}_Z) + (A^2 + D \cdot A)/2 + h^1(\mathcal{O}_Z(A)) = (A^2/2) + 2$$

by the Riemann-Roch formula. On the other hand one has also

$$h^0(\mathcal{O}_Z(A+\Delta)) = ((A+\Delta)^2/2) + 2$$

since $h^1(\mathcal{O}_Z(A+\Delta)) = 1$ by Lemma 3.2, (2). It implies that $A^2 = (A+\Delta)^2$

since $h^0(\mathcal{O}_Z(A)) = h^0(\mathcal{O}_Z(A+\varDelta))$. We have $2A \cdot \varDelta + \varDelta^2 = 0$. Now recall that C is numerically effective. Thus

$$0 \leqq C \cdot \varDelta = (A+\varDelta) \cdot \varDelta = -A \cdot \varDelta .$$

However $A \cdot \varDelta \geqq 0$ since A is the moving part of $|C|$. In conclusion we have $A \cdot \varDelta = 0$ and $\varDelta^2 = 0$.

If $\varDelta \neq 0$, then $\varDelta^2 = -2h^1(\mathcal{O}_Z(\varDelta)) < 0$ by Lemma 3.5. Therefore $\varDelta = 0$, i.e., $|C|$ has no fixed components.

Case 2. $A^2 = 0$.

By Proposition 3.3, there exists a smooth irreducible elliptic curve F and a positive integer k with $\mathcal{O}_Z(A) \cong \mathcal{O}_Z(kF)$ and $F \cdot D = 0$. Let $\varDelta_1, \varDelta_2, \cdots, \varDelta_N$ be connected components of $\varDelta$.

We divide the rest of the proof into several lemmas.

Lemma 3.8. *For every i, $F \cdot \varDelta_i > 0$.*

Proof. If for some i, $F \cdot \varDelta_i = 0$, then by Lemma 3.5

$$0 \leqq C \cdot \varDelta_i = (kF + \textstyle\sum \varDelta_j) \cdot \varDelta_i = \varDelta_i^2 = -2h^1(\mathcal{O}_Z(\varDelta_i)) < 0 ,$$

which is a contradiction.   Q.E.D.

Lemma 3.9. $k \geqq 2$.

Proof. If $k = 1$, then by the same reason as in case 1, we have $A \cdot \varDelta = F \cdot \varDelta = 0$. However we have just proved that $F \cdot \varDelta = \sum F \cdot \varDelta_i > 0$, which is a contradiction. Thus $k \geqq 2$.   Q.E.D.

Let $\varGamma_i$ be an irreducible component of $\varDelta_i$ with $F \cdot \varGamma_i > 0$.

Lemma 3.10. $N = 1$.

Proof. Assume $N \geqq 2$. Choose general members $F_1, \cdots, F_k \in |F|$ and set $P = F_1 + \cdots + F_k + \varGamma_1$. $Q = P + \varGamma_2$. Obviously $\operatorname{Supp} P \cap D = \operatorname{Supp} Q \cap D = \phi$ and $h^0(\mathcal{O}_P) = h^0(\mathcal{O}_Q) = 1$. We have $h^1(\mathcal{O}_Z(P)) = h^1(\mathcal{O}_Z(Q)) = 1$ by Lemma 3.4. By the Riemann-Roch formula we have

$$h^0(\mathcal{O}_Z(P)) = (P^2/2) + 2 , \quad h^0(\mathcal{O}_Z(Q)) = (Q^2/2) + 2 .$$

Since $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(Q))$ by definition, it implies that

$$P^2 = Q^2 = (P + \varGamma_2)^2 = P^2 + 2P \cdot \varGamma_2 - 2 .$$

Here note that $\varGamma_2^2 = -2$ by Corollary 3.6. We have

$$1 = P \cdot \Gamma_2 = (kF + \Gamma_1) \cdot \Gamma_2 = kF \cdot \Gamma_2 \geqq k \geqq 2 ,$$

which is a contradiction. Thus $N=1$. Q.E.D.

Set $\varDelta_1 = \varDelta = \sum_{j=1}^{J} a_j \Theta_j$ where Θ_j is a mutually different irreducible curve and a_j is a positive integer. We assume that $\Theta_1 \cdot F > 0$. By Corollary 3.6 every Θ_j is a smooth rational curve with $\Theta_j^2 = -2$ and $D \cdot \Theta_j = 0$.

Lemma 3.11. $F \cdot \Theta_1 = 1$.

Proof. First note that $h^0(\mathcal{O}_Z(kF)) = 1 + k$ by Proposition 3.3 and by the Riemann-Roch formula. Since $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(kF))$ for the divisor P in the proof of Lemma 3.9, we have

$$2k + 2 = (kF + \Theta_1)^2 + 4 = 2kF \cdot \Theta_1 + 2 ,$$

which implies the lemma. Q.E.D.

Lemma 3.12. $F \cdot \Theta_i = 0$ *if* $i \neq 1$.

Proof. Fix an integer i with $i \neq 1$. There exists a subset S of $\{1, 2, \cdots, J\}$ with $1 \in S$, $i \notin S$, such that $\varDelta_S = \sum_{j \in S} \Theta_j$ and $\varDelta_S + \Theta_i$ are connected. Set $P = kF + \varDelta_S$ and $Q = kF + \varDelta_S + \Theta_i$. By the Riemann-Roch formula, we have

$$h^0(\mathcal{O}_Z(P)) = (P^2/2) + 2 , \quad h^0(\mathcal{O}_Z(Q)) = (Q^2/2) + 2 .$$

We have $P^2 = Q^2$ since $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(Q))$. It implies $(kF + \varDelta_S) \cdot \Theta_i = P \cdot \Theta_i = -\Theta_i^2/2 = 1$. By the choice of $\varDelta_S$, we have $\varDelta_S \cdot \Theta_i > 0$. Thus $F \cdot \Theta_i = 0$. Q.E.D.

Lemma 3.13. *Assume that there is a subset S of $\{1, 2, \cdots, J\}$ with $1 \in S$ such that $\varDelta_S = \sum_{j \in S} \Theta_j$ is connected and $k + \varDelta_S \cdot \Theta_1 \geqq 2$. Then $a_1 = 1$.*

Proof. Set $P = kF + \varDelta_S$, $Q = P + \Theta_1$ and $N = \mathcal{O}_Z(Q)|_{\Theta_1}$. Note that $\deg N = (kF + \varDelta_S + \Theta_1) \cdot \Theta_1 = k + \varDelta_S \cdot \Theta_1 - 2 \geqq 0$ by assumption. One sees easily $h^1(\mathcal{O}_Z(P)) = 1$. Consider the exact sequence

$$0 \to \mathcal{O}_Z(P) \to \mathcal{O}_Z(Q) \to N \to 0 .$$

We have $h^1(\mathcal{O}_Z(Q)) \leqq 1$ since $h^1(N) = 0$. Consider the sequence

$$0 \to \mathcal{O}_Z(-Q-D) \to \mathcal{O}_Z \to \mathcal{O}_Q \oplus \mathcal{O}_D \to 0 .$$

It is exact since Supp $Q \cap D = \phi$. Thus $h^1(\mathcal{O}_Z(Q)) = h^1(\mathcal{O}_Z(-Q-D)) = h^0(\mathcal{O}_Q) \geqq 1$. It follows that $h^1(\mathcal{O}_Z(Q)) = 1$. By Riemann-Roch

$$h^0(\mathcal{O}_Z(P)) = P^2/2 + 2 , \quad h^0(\mathcal{O}_Z(Q)) = Q^2/2 + 2 .$$

Assume that $a_1 \geq 2$. Then $h^0(\mathcal{O}_Z(P)) = h^0(\mathcal{O}_Z(Q))$. We have $P^2 = Q^2 = P^2 + 2P \cdot \Theta_1 - 2$. Thus $P \cdot \Theta_1 = 1$.

On the other hand by definition of P and by assumption $P \cdot \Theta_1 = (kF + \Delta_S) \cdot \Theta_1 = k + \Delta_S \cdot \Theta_1 \geq 2$. We get a contradiction.   Q.E.D.

Lemma 3.14. *If $a_1 = 1$, then $F \cdot \Delta = 1$ and $\Delta^2 = -2$.*

Proof. Assume $a_1 = 1$. We write $\Delta = \Theta_1 + \Delta'$. Since $\Delta' \cdot F = 0$ by Lemma 3.12, we have $F \cdot \Delta = F \cdot \Theta_1 = 1$. By Riemann-Roch we have $h^0(\mathcal{O}_Z(kF)) = 1 + k$ and $h^0(\mathcal{O}_Z(kF + \Delta)) = (kF + \Delta)^2/2 + 2$. Since these two numbers are equal, we have $(kF + \Delta)^2 = 2k - 2$. It implies $\Delta^2 = -2$ since $F^2 = 0$ and $F \cdot \Delta = 1$.   Q.E.D.

Lemma 3.15. *If $k \geq 4$, then $\Delta = \Theta_1$.*

Proof. We assume $k \geq 4$. Set $S = \{1\}$. The assumption of Lemma 3.13 is satisfied. Thus we have $a_1 = 1$ and $\Delta^2 = -2$ by Lemma 3.13 and Lemma 3.14. Set $\Delta' = \Delta - \Theta_1$. The divisor Δ' does not contain Θ_1. Assume $\Delta' \neq 0$. Then $\Delta' \cdot \Theta_1 > 0$ since Δ is connected. It follows from the equality

$$-2 = \Delta^2 = (\Theta_1 + \Delta')^2 = -2 + \Delta' \cdot \Theta_1 + \Delta \cdot \Delta'$$

that $\Delta \cdot \Delta' < 0$. However, since C is numerically effective and $F \cdot \Delta' = 0$ by Lemma 3.12, we have that $0 \leq C \cdot \Delta' = (kF + \Delta) \cdot \Delta' = \Delta \cdot \Delta'$, a contradiction. Thus $\Delta' = 0$.   Q.E.D.

Lemma 3.16. *If $k = 3$, then $\Delta = \Theta_1$.*

Proof. We assume $k = 3$. Moreover assume $\Delta' = \Delta - a_1 \Theta_1 \neq 0$. There exists a suffix $i \neq 1$ with $\Theta_i \cdot \Theta_1 \neq 0$. Set $S = \{1, i\}$. Since $k + \Delta_S \cdot \Theta_1 = 3 + \Theta_i \cdot \Theta_1 - 2$, the assumption of Lemma 3.13 is satisfied. Thus we have $a_1 = 1$ and $\Delta^2 = -2$. By the same reasoning as in Lemma 3.15, one obtains a contradiction. Thus $\Delta = a_1 \Theta_1$.

By the same reasoning as in Lemma 3.14 one sees $4 = (3F + \Delta)^2 = (3F + a_1 \Theta_1)^2 = 6a_1 - 2a_1^2$ since $F \cdot \Theta_1 = 1$ and $\Theta_1^2 = -2$. We have $a_1 = 1$ or 2. If $a_1 = 2$, then $C \cdot \Theta_1 = (3F + 2\Theta_1) \cdot \Theta_1 = -1$, that is, C is not numerically effective. We have consequently $\Delta = \Theta_1$.   Q.E.D.

Lemma 3.17. *If $k = 2$, then $\Delta = \Theta_1$.*

Proof. We assume that $k = 2$. Moreover assume that $a_1 = 1$. Set $\Delta' = \Delta - \Theta_1$. We have $\Delta' \cdot \Theta_1 \geq 0$ since Δ' does not contain Θ_1. By Lemma 3.12 we have also $\Delta \cdot \Delta' = (2F + \Delta) \cdot \Delta' = C \cdot \Delta' \geq 0$. On the other hand by Lemma 3.5

$\varDelta^2 \leqq -2$. We have

$$-2 \geqq \varDelta^2 = (\Theta_1 + \varDelta')^2 = -2 + \varDelta' \circ \Theta_1 + \varDelta \circ \varDelta' \geqq -2 \, .$$

It implies that $\varDelta' \circ \Theta_1 = \varDelta \circ \varDelta' = 0$, $\varDelta^2 = -2$. We have $\varDelta'^2 = \varDelta \circ \varDelta' - \Theta_1 \circ \varDelta' = 0$. But $\varDelta'^2 < 0$ if $\varDelta' \neq 0$ by Lemma 3.5. Thus $\varDelta' = 0$.

Next assume that $a_1 \geqq 2$. Since $0 \leqq L \circ \Theta_1 = (2F + a_1 \Theta_1) \circ \Theta_1 + \sum_{i \neq 1} a_i \Theta_i \circ \Theta_1 = 2 - 2a_1 + \sum_{i \neq 1} a_i \Theta_i \circ \Theta_1$ there is an index i with $i \neq 1$, $\Theta_i \circ \Theta_1 > 0$. If there are two indices i, j, $1 \neq i \neq j \neq 1$ with $\Theta_i \circ \Theta_1 > 0$, $\Theta_j \circ \Theta_1 > 0$, setting $S = \{1, i, j\}$ we have $a_1 = 1$ by Lemma 3.13. Thus for some unique index i_2 $\Theta_{i_2} \circ \Theta_1 > 0$. By renumbering if necessary we can assume $i_2 = 2$. We have that $\Theta_2 \circ \Theta_1 = 1$ since $0 > (\Theta_1 + \Theta_2)^2 = -4 + 2\Theta_1 \circ \Theta_2$ by Lemma 3.5. We have the next inequality.

$$\langle 3.1 \rangle \qquad\qquad a_2 - 2a_1 + 2 = L \circ \Theta_1 \geqq 0 \, .$$

In particular $a_2 \geqq 2$. Now since $0 \leqq L \circ \Theta_2 = a_1 - 2a_2 + \sum_{i > 2} a_i \Theta_i \circ \Theta_2$, there is an index $i > 2$ with $\Theta_i \circ \Theta_2 > 0$. Assume that for mutually different three indices $i_\alpha > 2$, $\alpha = 1, 2, 3$, $\Theta_{i_\alpha} \circ \Theta_2 > 0$ holds. Set $P_1 = 2F + \Theta_1 + \Theta_2 + \sum_{\alpha=1}^{3} \Theta_{i_\alpha}$ and $Q_1 = P_1 + \Theta_2$. Since $\mathcal{O}_Z(Q_1)|_{\Theta_2} \cong \mathcal{O}_{\Theta_2}$ and $\Theta_2 \cong P^1$ and since $h^0(\mathcal{O}_Z(P_1)) = h^0(\mathcal{O}_Z(Q_1))$ it follows from the exact sequence

$$0 \to \mathcal{O}_Z(P_1) \to \mathcal{O}_Z(Q_1) \to \mathcal{O}_{\Theta_2} \to 0$$

that $h^1(\mathcal{O}_Z(Q_1)) = 0$. However by the exact sequence

$$0 \to \mathcal{O}_Z(-Q_1 - D) \to \mathcal{O}_Z \to \mathcal{O}_{Q_1} \oplus \mathcal{O}_D \to 0$$

we have $h^1(\mathcal{O}_Z(Q_1)) = h^1(\mathcal{O}_Z(-Q_1 - D)) \geqq 1$, a contradiction. Thus renumbering if necessary we can assume that one of the following two assertions holds for $k = 3$.

$(1)_k$ $\quad \Theta_k \circ \Theta_{k-1} = 1$ and $\Theta_i \circ \Theta_{k-1} = 0$ for $i > k$.

$(2)_k$ $\quad \Theta_k \circ \Theta_{k-1} = \Theta_{k+1} \circ \Theta_{k-1} = 1$ and $\Theta_i \circ \Theta_{k-1} = 0$ for $i > k+1$.

For a moment assume that case $(1)_3$ takes place. Since

$$\langle 3.2 \rangle \qquad\qquad L \circ \Theta_2 = a_1 - 2a_2 + a_3 \geqq 0$$

and by $\langle 3.1 \rangle$, we have $a_3 \geqq 2$. Repeating the similar argument as just the above one sees that we can assume that $(1)_4$ or $(2)_4$ holds. If $(1)_4$ takes place, inequalities

$$\langle 3.k \rangle \qquad\qquad L \circ \Theta_k = a_{k-1} - 2a_k + a_{k+1} \geqq 0$$

$k=2, 3$ and $\langle 3.1 \rangle$ imply that $a_4 \geq 2$ and we can repeat the similar discussion further. Since inequalities $\langle 3.k \rangle$ $1 \leq k \leq K$ implies $a_{K+1} \geq 2$ and since the number of irreducible components of $\varDelta$ is finite, we can consequently assume that $(2)_{K+1}$ takes place for some $K \geq 2$. Set $\sum = \Theta_1 + \Theta_2 + \cdots + \Theta_K$, $P_2 = 2F + \sum + \Theta_{K+1} + \Theta_{K+2}$, and $Q_2 = P_2 + \sum$. We can see easily that $\mathcal{O}_Z(Q_2)|_\Sigma \cong \mathcal{O}_\Sigma$, $h^0(\mathcal{O}_\Sigma) = 1$ and $h^1(\mathcal{O}_\Sigma) = 0$. Now $h^1(\mathcal{O}_Z(P_2)) = 1$ by Lemma 3.4 and $h^0(\mathcal{O}_Z(P_2)) = h^0(\mathcal{O}_Z(Q_2))$ since $\varDelta$ is a sum of $2\sum + \Theta_{K+1} + \Theta_{K+2}$ and some effective divisor. It follows from the exact sequence

$$0 \to \mathcal{O}_Z(P_2) \to \mathcal{O}_Z(Q_2) \to \mathcal{O}_\Sigma \to 0$$

that $h^1(\mathcal{O}_Z(Q_2)) = 0$. On the other hand since the sequence

$$0 \to \mathcal{O}_Z(-Q_2 - D) \to \mathcal{O}_Z \to \mathcal{O}_{Q_2} \oplus \mathcal{O}_D \to 0$$

is exact, we have $h^1(\mathcal{O}_Z(Q_2)) = h^1(\mathcal{O}_Z(-Q_2 - D)) = h^0(\mathcal{O}_{Q_2}) \geq 1$, a contradiction. Thus the case $a_1 \geq 2$ never takes place.   Q.E.D.

The above lemma completes the proof of Proposition 3.7.

Proposition 3.18. *Let C be an effective divisor on Z with $C \cdot D = 0$. Assume that the linear system $|C|$ has no fixed components and that $C^2 = 2$ or 4. Then $|C|$ has no fixed points.*

Proof. Assume that $|C|$ has no fixed components but it has isolated fixed points.

By induction we define a sequence of blowing-ups,

$$\hat{Z} = Z_{(k)} \xrightarrow{\tau_k} Z_{(k-1)} \xrightarrow{\tau_{k-1}} \cdots \to Z_{(2)} \xrightarrow{\tau_2} Z_{(1)} \xrightarrow{\tau_1} Z_{(0)} = Z$$

an integer m_j for $1 \leq j \leq k$ and a line bundle L_j on $Z_{(j)}$ for $0 \leq j \leq k$ as follows. First of all set $Z_{(0)} = Z$ and $L_0 = \mathcal{O}_Z(C)$. Next assume that $Z_{(i)}$, τ_i, m_i, L_i have been constructed for $0 \leq i \leq j-1$. If $|L_{j-1}|$ has no fixed points, then setting $k = j-1$ and $\hat{Z} = Z_{(j-1)}$, we terminate the procedure. If $|L_{j-1}|$ has fixed points, then let $\tau_j : Z_{(j)} \to Z_{(j-1)}$ be the blowing-up of one of the fixed points $z_j \in Z_{(j-1)}$. Set $m_j = \min \{\text{mult}_{z_j}(A) \mid A \in |L_{j-1}|\}$, where $\text{mult}_z(A)$ denotes the multiplicity of the curve A at z. We define $L_j = (\tau_j^* L_{j-1}) \otimes \mathcal{O}_{Z_{(j)}}(-m_j \tau_j^{-1}(z_j))$. We have $L_j^2 \geq 0$ for every j since $|L_j| \neq \phi$ and $|L_j|$ has no fixed components. Since $L_j^2 = L_{j-1}^2 - m_j^2$ this procedure terminates in finite steps.

Set $\hat{L} = L_k$. If $\hat{L}^2 = 0$, then the image of the rational map $\varphi_L : Z \cdots \to \mathbf{P}^N$ associated to the line bundle $L = \mathcal{O}_Z(C)$ has dimension ≤ 1. We have $L^2 = C^2 = 0$

by Proposition 3.3, which contradicts the assumption. Thus $\hat{L}^2 > 0$.

Next we show that $p_a(A) \leq 1$ for any general member A of $|\hat{L}|$.

Case 1. $C^2 = 2$.

Note that $h^1(L) = 1$ by Proposition 3.3. We have $h^0(L) = h^0(\hat{L}) = C^2/2 + 2 = 3$ by Riemann-Roch. We have a morphism $\varphi_{\hat{L}} : \hat{Z} \to \mathbf{P}^2$. On the other hand $\hat{L}^2 = 1$ since $0 < \hat{L}^2 < 2 = C^2$. Thus any general member A of $|L|$ has a morphism of degree 1 to a line in $\mathbf{P}^2$. Thus $p_a(A) = 0$.

Case 2. $C^2 = 4$.

We have a morphism $\varphi_{\hat{L}} : \hat{Z} \to \mathbf{P}^3$ since $h^0(L) = h^0(\hat{L}) = 4$ by Riemann-Roch. Since $0 < \hat{L}^2 < L^2 = C^2 = 4$, one sees that $\varphi_{\hat{L}}$ is a generically one to one morphism whose image is an irreducible cubic surface or an irreducible quadratic surface. Then any general member A of $|\hat{L}|$ has a morphism of degree 1 to either a plane irreducible cubic curve or a plane irreducible quadratic curve. Thus $p_a(A) \leq 1$.

We know $p_a(A) \leq 1$ in any case.

Now let $E_1, \cdots, E_k$ be the total inverse image on $\hat{Z}$ of the curve $\tau_1^{-1}(z_1)$, $\cdots, \tau_k^{-1}(z_k)$. We have

$$\hat{L} = (\tau^* L)(-m_1 E_1 - m_2 E_2 - \cdots - m_k E_k), \quad \omega_{\hat{Z}} = (\tau^* \omega_Z)(E_1 + E_2 + \cdots + E_k)$$

where $\tau = \tau_1 \tau_2 \cdots \tau_k$. Thus we have $\hat{L} \cdot \omega_{\hat{Z}} = C \cdot \omega_Z + \sum m_i = \sum m_i$. By the adjunction formula

$$p_a(A) = (\hat{L}^2 + \omega_{\hat{Z}} \cdot \hat{L})/2 + 1 = (\hat{L}^2/2) + (\sum m_i/2) + 1 \geq 2.$$

We obtain a contradiction. Thus $|C|$ has no fixed points. Q.E.D.
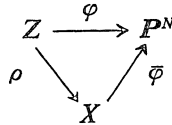
Lemma 3.19. *Let L be a polarization on Z.*

(1) *If an irreducible curve A on Z satisfies $L \cdot A = 0$, then either A coincides with D or it is a smooth rational curve with $A^2 = -2$ and $A \cap D = \phi$.*

(2) *Let $\mathcal{E}$ be the union of irreducible curves A with $L \cdot A = 0$ and $\mathcal{E}_0$ be a connected component of $\mathcal{E}$. Let $A_1, A_2, \cdots, A_k$ be all the irreducible curves contained in $\mathcal{E}_0$. Then the intersection matrix $(A_i \cdot A_j)_{1 \leq i, j \leq k}$ is negative definite.*

(3) *Unless $\mathcal{E}_0 = D$, $\mathcal{E}_0$ is the support of the exceptional curves in the minimal resolution of a rational double point.*

Proof. We can assume that $A \neq D$. Under this assumption we have $A \cdot D \geq 0$. By the Hodge index theorem we have also $A^2 < 0$. By the adjunction formula $0 \leq p_a(A) = (A^2 - A \cdot D)/2 + 1$. We have either $A^2 = -1$ and $A \cdot D = 1$ or $A^2 = -2$ and $A \cdot D = 0$. In any case $p_a(A) = 0$. It is well-known that if $p_a(A) = 0$,

then A is a smooth rational curve. If $A^2=-1$ and $A \cdot D=1$, then A is an exceptional curve of the first kind. Since L is a polarization we have $A \cdot L>0$, which contradicts the choice of A. Thus $A^2=-2$ and $A \cdot D=0$. The last equality implies $A \cap D=\phi$. (2) is an easy consequence of the Hodge index theorem. (3) follows from (1) and (2). (Cf. Artin [2])   Q.E.D.

By the well-known Grauert's theorem, (Cf. Grauert [8]), we can contract all the connected components of $\mathcal{E}$ to isolated normal singular points. Let $\rho: Z \to X$ be the contraction morphism. Here X is a normal surface with a unique singular point with positive geometric genus at $w=\rho(D)$ and several rational double points.

Proposition 3.20. *Assume that a polarization L on Z defines a morphism $\varphi=\varphi_L: Z \to P^N$. Then we have a finite morphism $\bar{\varphi}: X \to P^N$ with $\varphi=\bar{\varphi} \circ \rho$.*

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & P^N \\ {\scriptstyle \rho} \searrow & & \nearrow {\scriptstyle \bar{\varphi}} \\ & X & \end{array}$$

Proof. Set $\rho(\mathcal{E})=S$. Note that $\rho|Z-\mathcal{E}: Z-\mathcal{E} \to X-S$ is an isomorphism. Thus we can define a morphism $\bar{\varphi}=\varphi \circ (\rho|Z-\mathcal{E})^{-1}$. Since $\varphi(\mathcal{E})$ is a set of isolated points and X is normal, we can extend $\bar{\varphi}$ to whole X. Obviously the resulting morphism $X \to P^N$ is proper. Assume that there exists a point $z \in P^N$ such that $\bar{\varphi}^{-1}(z)$ has dimension 1. Let A be an irreducible curve contained in $\bar{\varphi}^{-1}(z)$. Let $\hat{A}$ be the strict inverse image of A by ρ. We have $L \cdot \hat{A}=0$. Thus $\hat{A} \subset \mathcal{E}$ and $\rho(\hat{A})=A$ is a point, which is a contradiction. Thus $\bar{\varphi}$ is a finite morphism.   Q.E.D.

Proposition 3.21. *Assume that a polarization L on Z defines a morphism $\varphi=\varphi_L: Z \to P^3$ of degree 2 whose image is a quadratic surface. We have a smooth irreducible elliptic curve F on Z with $L \cdot F=2$, $F \cap D=\phi$ and $F^2=0$.*

Proof. Case A. Assume the image of φ is a smooth quadratic surface Σ. Let $p: \Sigma \to P^1$ be the composition of an isomorphism $\Sigma \to P^1 \times P^1$ and the projection to a factor $P^1 \times P^1 \to P^1$. Choose a general point $z \in P^1$ and set $G=p^*(z)$ and $F=\varphi^*(G)$. F is irreducible. We have $F \cap D=\phi$ since $\varphi(D)$ and $p\varphi(D)$ are isolated points by assumption $L|_D \cong \mathcal{O}_D$. We have $L \cdot F=2\mathcal{O}_{P^3}(1) \cdot G=2$ and $F^2=2G^2=0$. Obviously the linear system $|F|$ has no fixed components. By Proposition 3.3, one sees that F is a smooth elliptic curve.

Case B. Assume that the image of φ is a quadratic surface $\sum_0$ with a unique singular point $v \in \sum_0$.

Lemma 3.22. *If $\varphi(D) = \{v\}$, then $\varphi^{-1}(v) = D$.*

Proof. Set $\{w\} = \rho(D)$, $w \in X$. Note that $\bar{\varphi}(w) = \{v\}$ by assumption. Let U be a sufficiently small neighbourhood of $v \in U \subset \sum_0$. Let V be the connected component of $\bar{\varphi}^{-1}(U)$ containing w. Let $S \subset V - \{w\}$ be the discriminant of $\bar{\varphi} \mid V - \{w\}$.

Case 1. Assume that the closure of $\bar{\varphi}(S)$ in U does not contain v. By choosing a smaller U, we can assume that $\bar{\varphi} \mid V - \{w\}$ is unramified. Note that $\pi_1(U - \{v\}) \cong \mathbb{Z}/2\mathbb{Z}$ since the A_1-singularity (U, v) is the quotient of $(\mathbb{C}^2, 0)$ by the action of $\mathbb{Z}/2\mathbb{Z}$ defined by $(x, y) \to (-x, -y)$. Thus $\pi_1(V - \{w\})$ is either a trivial group $\{e\}$ or $\mathbb{Z}/2\mathbb{Z}$. If $\pi_1(V - \{w\}) = \{e\}$, then $w \in X$ is a simple point by a Mumford's theorem. (Cf. Mumford [16]). If it is $\mathbb{Z}/2\mathbb{Z}$, $\bar{\varphi} \mid V - \{w\}$ is an isomorphism. Since V and U are normal, it induces an isomorphism $\bar{\varphi} \mid V: V \to U$. Thus $w \in X$ is a A_1-singular point. However by the construction we have $p_g(X, w) \geq 1$. Therefore one sees that our Case 1 never takes place under our assumption.

Case 2. Next we assume that the closure of $\bar{\varphi}(S)$ in U contains v. Since $\bar{\varphi}$ is a finite morphism of degree 2, the set $\{x \in U \mid \sharp \bar{\varphi}^{-1}(x) = 1\}$ coincides with the closure of $\bar{\varphi}(S)$ in U. Thus $\sharp \bar{\varphi}^{-1}(v) = 1$. Here $\sharp$ denotes the number of elements in the set. We have $\{w\} = \bar{\varphi}^{-1}(v)$. It implies $\varphi^{-1}(v) = \rho^{-1}(w)$. Under the assumption of the lemma $D \subset \varphi^{-1}(v)$. However since $\rho^{-1}\rho(D) = D$ by the definition of ρ, we have $\varphi^{-1}(v) = D$. Q.E.D.

Lemma 3.23. *Let G be a general member of the ruling $\mathbb{P}^1$-family of $\sum_0$ and F be the strict inverse image of G by φ. We have $\dim |F| = 1$ and $|F|$ has no fixed components.*

Proof. We define a linear system Λ on Z by $\Lambda = \{\varphi^* P \mid P \text{ is a plane in } \mathbb{P}^3 \text{ with } v \in P\}$. Let Δ be the fixed components of Λ. Obviously we have $\mathrm{Supp} \Delta \subset \varphi^{-1}(v)$. Let P_0 be a general plane in $\mathbb{P}^3$ passing through v. We set $P_0 \cap \sum_0 = G \cup G'$ where G and G' are members of the ruling $\mathbb{P}^1$-family of $\sum_0$. Let F (resp. F') be the strict inverse image of G (resp. G') by φ. We have $F + F' + \Delta \in \Lambda$.

Next we define a 1-dimensional linear system Ξ by

$$\Xi = \{\varphi^* P - F' - \Delta \mid P \text{ is a plane in } \mathbb{P}^3 \text{ with } P \supset G'\} .$$

We have $|F| \supset \Xi$ since $F \in \Xi$. Let $A \in |F|$ be an arbitrary member. $A + F' + \Delta \in |L|$ since $F + F' + \Delta \in |L|$. Thus there is a plane P_1 in P^3 with $A + F' + \Delta = \varphi^* P_1$ because $|L|$ is a complete linear system. P_1 necessarily contains G'. It implies that $A \in \Xi$. Thus $|F| = \Xi$, which concludes the proof.   Q.E.D.

Lemma 3.24. $\varphi(D) \neq \{v\}$.

Proof. Assume that $\varphi(D) = \{v\}$. We will deduce a contradiction.

Let F be the divisor as in Lemma 3.23. By the Riemann-Roch formula and by Lemma 3.23, we have

$$2 = 1 + \dim |F| = (F^2 + F \cdot D)/2 + 1 + h^1(\mathcal{O}_Z(F)) \,.$$

Lemma 3.23 also implies $F^2 \geqq 0$. Since $\varphi^{-1}(v) = D$ by Lemma 3.22, we have $F \cdot D > 0$. Thus if $h^1(\mathcal{O}_Z(F)) \geqq 1$, this equality yields a contradiction.

Now note that $h^2(\mathcal{O}_Z(F-D)) = h^0(\mathcal{O}_Z(-F)) = 0$ by the Serre duality. It implies that the map $H^1(\mathcal{O}_Z(F)) \to H^1(\mathcal{O}_Z(F)|_D) \cong H^1(\mathcal{O}_D) \cong C$ is surjective. Thus $h^1(\mathcal{O}_Z(F)) \geqq 1$ and $\varphi(D) \neq \{v\}$.   Q.E.D.

Now we go back to the proof of Proposition 3.21, Case B. By Lemma 3.24, we can choose a general member G of the ruling P^1-family of Σ_0 with $G \cap \varphi(D) = \phi$. Let F be the strict inverse image of G by φ. We have $\mathcal{O}_Z(F)|_D \cong \mathcal{O}_D$. By Riemann-Roch $2 = (F^2/2) + 1 + h^1(\mathcal{O}_Z(F))$. Since $h^1(\mathcal{O}_Z(F)) \geqq 1$, we have $F^2 = 0$.

The equality $L \cdot F = 2$ is obvious by definition. It concludes the proof of Proposition 3.21.   Q.E.D.

Theorem 3.25. *Let L be a polarization of degree 4 on a rational surface Z with an irreducible effective anti-canonical divisor D. The following conditions are equivalent.*

(1) *The rational map φ_L associated to L defines a birational morphism to a quartic surface in P^3.*

(2) *There exists no element $M \in \mathrm{Pic}(Z)$ with $M^2 = 0$, $M \cdot L = 2$ and $M|_D \cong \mathcal{O}_D$.*

Besides if one of the above equivalent conditions holds, then the induced morphism $\bar\varphi: X \to P^3$ by φ_L is an embedding.

Proof. First we show (2)$\Rightarrow$(1). Assume that $|L|$ has fixed components. By Proposition 3.7 there exists a smooth irreducible elliptic curve F and a smooth irreducible rational curve Γ with $F^2 = 0$, $F \cdot D = 0$, $\Gamma^2 = -2$, $\Gamma \cdot D = 0$, $\Gamma \cdot F = 1$ and $L \cong \mathcal{O}_Z(3F + \Gamma)$. The line bundle $M = \mathcal{O}_Z(F + \Gamma)$ satisfies the conditions in (2). Next assume that $|L|$ has no fixed components. By Pro-

position 3.18 $|L|$ has no fixed points. Thus φ_L is a morphism. By Lemma 3.2 one sees φ_L maps Z to $\mathbf{P}^3$. Since $L^2=4$, the image of φ_L is either a quadratic surface or a quartic surface. Assume moreover that $\operatorname{Im}\varphi_L$ is a quadratic surface. By Proposition 3.21 we have a smooth elliptic curve F on Z with $F^2=0$, $F\cdot D=0$ and $L\cdot F=2$. The line bundle $M=\mathcal{O}_Z(F)$ satisfies (2). Thus (2) implies (1).

Next we show (1)$\Rightarrow$(2). Assume that there is an element $M\in\operatorname{Pic}(Z)$ with $M^2=0$, $M\cdot L=2$, $M|_D\cong\mathcal{O}_D$ and that φ_L is a birational morphism to a quartic surface in $\mathbf{P}^3$. We will deduce a contradiction. By Riemann-Roch we have $h^0(M)+h^2(M)\geqq 1$. If $h^2(M)=h^0(-M+\omega_Z)\neq 0$, we have $(-M+\omega_Z)\cdot L\geqq 0$ since L is numerically effective. However we have $(-M+\omega_Z)\cdot L=-2+0=-2$, a contradiction. Thus $h^2(M)=0$ and $h^0(M)\neq 0$, i.e. M is effective. Let A be an effective divisor with $M\cong\mathcal{O}_Z(A)$. We set

$$A = mD+\sum_{i=1}^{k} n_i A_i+F$$

where $k, m, n_1, \cdots, n_k$ are integers with $k\geqq 0$, $m\geqq 0$, $n_i\geqq 1$ $(1\leqq i\leqq k)$, $A_1, \cdots, A_k$ are mutually different irreducible curves with $A_i\neq D$, $A_i\cdot D>0$ for every i and F is an effective divisor with $\operatorname{Supp}F\cap D=\phi$. Let $\mathcal{E}$ be the union of exceptional curves of $\rho: Z\to X$. Since D is a connected component of $\mathcal{E}$ and since $A_i\cdot D>0$, $\rho(A_i)$ has dimension 1 for every i. Thus $L\cdot A_i>0$ for every i. Since

$$2 = M\cdot L = mD\cdot L+\sum_{i=1}^{k} n_i A_i\circ L+F\cdot L = \sum_{i=1}^{k} n_i A_i\circ L+F\cdot L$$

we have 4 cases.

⟨1⟩ $k=0$.

⟨2⟩ $k\geqq 1$ and $n_i=1$ for every i.

⟨3⟩ $k=1$, $n_1=2$, $m\geqq 1$, $A_1^2\geqq 0$.

⟨4⟩ $k=1$, $n_1=2$, $m\geqq 1$, $A_1^2<0$.

(Note that $k=0$ if and only if $m=0$.)

Now we need two lemmas.

Lemma 3.26. *Consider an effective divisor* $A=mD+\sum_{i=1}^{k} A_i+F$ *satisfying the following conditions.*

(i) $k\geqq 1, m\geqq 1$

(ii) $D\cdot A_i>0$ *for every i and $A_1, \cdots, A_k$ are mutually different irreducible divisors.*

(iii) $\operatorname{Supp}F\cap D=\phi$ *and F is an effective divisor.*

(iv) $\mathcal{O}_Z(A)|_D\cong\mathcal{O}_D$.

Then A is linearly equivalent to an effective divisor containing no D.

Proof. By induction we show that $H^1(\mathcal{O}_Z(\sum_{i=1}^{j} A_i))=0$. If $j=0$, it is trivial. Consider the exact sequence

$$0 \to \mathcal{O}_Z(\sum_{i=1}^{j} A_i) \to \mathcal{O}_Z(\sum_{i=1}^{j+1} A_i) \to \mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)|_{A_{j+1}} \to 0$$

Since $\deg \mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)|_{A_{j+1}} = A_{j+1}^2 + \sum_{i=1}^{j} A_i \cdot A_{j+1} > A_{j+1}^2 - D \cdot A_{j+1} \geq 2p_a(A_{j+1})-2$, we have $H^1(\mathcal{O}_Z(\sum_{i=1}^{j+1} A_i)|_{A_{j+1}})=0$. By the above sequence and by induction hypothesis we have $H^1(\mathcal{O}_Z(\sum_{i=1}^{j+1} A_i))=0$.

Next by induction we show that $H^1(\mathcal{O}_Z(nD + \sum_{i=1}^{k} A_i)) = 0$ for $0 \leq n < m$. We have just shown it when $n=0$. Assume $n \leq m-2$. Set $N = \mathcal{O}_Z((n+1)D + \sum_{i=1}^{k} A_i)|_D$. Since $\deg N = \deg \mathcal{O}_Z(-(m-n-1)D)|_D = -(m-n-1)D^2 > 0$, we have $H^1(N)=0$. By the exact sequence of sheaves

$$0 \to \mathcal{O}_Z(nD + \sum_{i=1}^{k} A_i) \to \mathcal{O}_Z((n+1)D + \sum_{i=1}^{k} A_i) \to N \to 0$$

we have inductively $H^1(\mathcal{O}_Z((n+1)D + \sum_{i=1}^{k} A_i))=0$.

Note that in particular $H^1(\mathcal{O}_Z((m-1)D + \sum_{i=1}^{k} A_i))=0$. It implies that $H^0(\mathcal{O}_Z(A')) \to H^0(\mathcal{O}_Z(A')|_D) \cong H^0(\mathcal{O}_D) = C$ is surjective where $A' = mD + \sum_{i=1}^{k} A_i$. Surjectivity implies that there exists an effective divisor A'' linearly equivalent to A' which contains no D. Since $A \sim A'' + F$, we have the desired result.   Q.E.D.

Lemma 3.27. *Let A be an effective divisor with $\mathcal{O}_Z(A)|_D \cong \mathcal{O}_D$ and with $A^2 \geq 0$. We have $h^0(\mathcal{O}_Z(A)) \geq 2$.*

Proof. Note that $h^2(\mathcal{O}_Z(A-D)) = h^0(\mathcal{O}_Z(-A)) = 0$. It implies that $H^1(\mathcal{O}_Z(A)) \to H^1(\mathcal{O}_Z(A)|_D) \cong H^1(\mathcal{O}_D) \cong C$ is surjective. Thus $h^1(\mathcal{O}_Z(A)) \geq 1$. By Riemann-Roch, we have

$$h^0(\mathcal{O}_Z(A)) = (A^2 + A \cdot D)/2 + 1 + h^1(\mathcal{O}_Z(A)) \geq 2.$$ Q.E.D.

We continue the proof of Theorem 3.25.

Case $\langle 1 \rangle$. In this case Supp $A \cap D = \phi$. Let Δ be the fixed components of the linear system $|A|$. Set $C = A - \Delta$. By Lemma 3.27, we have $C \neq 0$ and $C^2 \geq 0$. We first consider the case $C^2 = 0$. By Proposition 3.3 we have a smooth

irreducible elliptic curve G with $G^2=0$, $G\cap D=\phi$ and an positive integer p with $C\in|pG|$. We have $\mathit{\Delta}\cdot L\geqq0$ and $G\cdot L\geqq0$ since L is numerically effective. Since the condition $G\cdot L=0$ implies $G^2<0$ by the Hodge index theorem, we have moreover $G\cdot L>0$. Now since $2=A\cdot L=pG\cdot L+\mathit{\Delta}\cdot L$, one sees that $G\cdot L=1$ or 2. Secondly we consider the case $C^2>0$. By Proposition 3.3 we can assume that C is an irreducible curve with $p_a(C)=(C^2/2)+1$. Since the condition $C\cdot L=0$ implies $C^2<0$, we have $C\cdot L>0$. Thus it follows from the equality $C\cdot L+\mathit{\Delta}\cdot L=2$ that $C\cdot L=1$ or 2.

Anyway one sees that there exists an irreducible curve C_1 on Z with $p_a(C_1)\geqq1$, $C_1\cap D=\phi$ and $C_1\cdot L=1$ or 2. Since $\varphi:Z\to P^3$ is generically one-to-one, and since $\dim|C_1|\geqq1$, we can assume that $\varphi|_{C_1}:C_1\to P^3$ is a birational morphism. The image of $\varphi|_{C_1}$ is a line or a curve of degree 2 in P^3 since $C_1\cdot L=1$ or 2. Because such curves have arithmetic genus 0, we have $p_a(C_1)\leqq0$, a contradiction.

Case $\langle2\rangle$. This case is reduced to Case $\langle1\rangle$ by Lemma 3.26.

Case $\langle3\rangle$. First we show $H^1(\mathcal{O}_Z(lA_1))=0$ for $l=0,1,2$ by induction. Since Z is rational, the case $l=0$ is trivial. Assume $l\geqq0$ and consider the exact sequence

$$0\to\mathcal{O}_Z(lA_1)\to\mathcal{O}_Z((l+1)A_1)\to\mathcal{O}_Z((l+1)A_1)|_{A_1}\to0.$$

We have $H^1(\mathcal{O}_Z((l+1)A_1)|_{A_1})=0$ because $\deg\mathcal{O}_Z((l+1)A_1)|_{A_1}=(l+1)A_1^2\geqq A_1^2>A_1^2-A_1\cdot D=2p_a(A_1)-2$. By induction hypothesis we have $H^1(\mathcal{O}_Z((l+1)A_1))=0$. Secondly we show $H^1(\mathcal{O}_Z(nD+2A_1))=0$ for $0\leqq n<m$ by induction as well. The case $n=0$ has been verified. Assume $0\leqq n<m-1$ and consider the sequence

$$0\to\mathcal{O}_Z(nD+2A_1)\to\mathcal{O}_Z((n+1)D+2A_1)\to\mathcal{O}_Z((n+1)D+2A_1)|_D\to0.$$

Note that $D^2=\omega_Z^2=9-t<0$ by Lemma 3.2, (1) and that $\mathcal{O}_Z(A)|_D\cong\mathcal{O}_D$. Thus we have $\deg\mathcal{O}_Z((n+1)D+2A_1)|_D=\deg\mathcal{O}_Z(-(m-n-1)D)|_D=-(m-n-1)D^2>0$ and $H^1(\mathcal{O}_Z((n+1)D+2A_1)|_D)=0$. By the last equality and by the induction hypothesis, we have $H^1(\mathcal{O}_Z((n+1)D+2A_1))=0$.

Now in particular $H^1(\mathcal{O}_Z((m-1)D+2A_1))=0$. This implies that $H^0(\mathcal{O}_Z(mD+2A_1))\to H^0(\mathcal{O}_Z(mD+2A_1)|_D)\cong H^0(\mathcal{O}_D)\cong C$ is surjective. Thus there exists a member $A'\in|mD+2A_1|$ which contains no D. We have an effective divisor $A'+F\in|A|$ containing no D.

Case $\langle4\rangle$. This is the last case. Since $A_1^2<0$ and $A_1\cdot D>0$, A_1 is an exceptional curve of the first kind. Since there are on Z at most countably many divisors with the form $mD+2E$ where E is an exceptional curve of the first kind, if $mD+2A_1$ is not contained in the fixed components of $|A|$, then there is a divisor

$A' \in |A|$ with the form in cases $\langle 1 \rangle$, $\langle 2 \rangle$ and $\langle 3 \rangle$.

Assume that $mD+2A_1$ is a part of the fixed components of $|A|$. Since $A=mD+2A_1+F$, we have $h^0(\mathcal{O}_Z(F))=h^0(\mathcal{O}_Z(A))\geqq 2$ by Lemma 3.27. However since for a numerically effective line bundle L, $A \cdot L=2$, $D \cdot L=0$ and $A_1 \cdot L>0$, we have $F \cdot L=0$. It implies that every component of a divisor linearly equivalent to F is an exceptional curve of $\rho: Z \rightarrow X$. Thus $h^0(\mathcal{O}_Z(F))=1$, which is a contradiction. Therefore this case $\langle 4 \rangle$ is reduced to other cases.

Here in all cases we have got a contradiction. Thus (1) implies (2).

It remains to show that $\bar{\varphi}$ is an embedding.

Let Y be the image of $\bar{\varphi}$. By assumption Y is a quartic surface. Assume that Y has the one-dimensional singular locus S. Let H be a general hyperplane in $\mathbf{P}^3$. The intersection $Y \cap H$ has singularities at $S \cap H$. The arithmetic genus of $Y \cap H$ is $(4-1)(4-2)/2=3$. Now let $C \subset Z$ be the strict inverse image of $Y \cap H$. $\varphi|_C: C \rightarrow Y \cap H$ is a birational morphism. We have $p_a(C) \leqq p_a(Y \cap H)=3$ and the equality holds if and only if $\varphi|_C$ is an isomorphism. On the other hand since any general member of $|L|$ is irreducible by Proposition 3.3, we have $C \in |L|$. Moreover C is smooth by the Bertini theorem. Thus $\varphi|_C$ is not an isomorphism and we have $p_a(C)<3$. However by the adjunction formula $p_a(C)=(L^2-D \cdot L)/2+1=3$, which is a contradiction. One sees that the singular locus of Y is 0-dimensional.

Note that every local ring of Y is Cohen-Macaulay of dimension $\leqq 2$ since Y is a hypersurface. The singular locus of Y has codimension $\geqq 2$. Thus by the Serre's criterion of normality (Cf. Matsumura [14]), the local ring $\mathcal{O}_{Y,y}$ is normal for every $y \in Y$. The morphism $X \rightarrow Y$ is a birational finite one to a normal variety and therefore it is an isomorphism.   Q.E.D.

Theorem 3.28. *Let L be a polarization of degree 2 on a rational surface Z with an irreducible effective anti-canonical divisor D. The following conditions are equivalent.*

(1) *The rational map φ_L associated to L defines a surjective morphism of degree 2 to $\mathbf{P}^2$.*

(2) *The linear system $|L|$ has no fixed components.*

(3) *There exists no element $M \in \mathrm{Pic}(Z)$ with $M^2=0$, $M \cdot L=1$ and $M|_D \cong \mathcal{O}_D$.*

Besides if one of the above equivalent conditions holds, then with the induced morphism $\bar{\varphi}: X \rightarrow \mathbf{P}^2$ by φ_L, X has the structure of the branched double covering of $\mathbf{P}^2$ branching along a reduced sextic curve B.

Proof. First we show (3)$\Rightarrow$(2). Assume that $|L|$ has fixed components.

Then $|L|$ contains a divisor $kF+\Gamma$ where k is a positive integer, F is an irreducible smooth elliptic curve with $F^2=0$, $F\cdot D=0$, Γ is an irreducible smooth rational curve with $\Gamma^2=-2$, $\Gamma\cdot D=0$, $\Gamma\cdot F=0$, by Proposition 3.7. Since $(kF+\Gamma)^2=2$, we have $k=2$. Set $M=\mathcal{O}_Z(F+\Gamma)$. This M satisfies the conditions in (3). Thus (3) does not hold.

The implication (2)$\Rightarrow$(1) follows from Proposition 3.18.

Next we show (1)$\Rightarrow$(3). Assume that there exists $M\in\mathrm{Pic}(Z)$ with $M^2=0$, $M\cdot L=1$ and $M|_D\cong\mathcal{O}_D$. We will deduce a contradiction under the assumption that φ_L is a morphism. By the same reason as in the proof of Theorem 3.25 one sees that the linear system $|M|$ is not empty. Let $A\in|M|$ and set

$$A = mD+\sum_{i=1}^{k} n_iA_i+F$$

where k, m, n_1, $\cdots$, n_k are integers with $k\geqq0$, $m\geqq0$ and $n_j\geqq1$ ($1\leqq j\leqq k$), F is an effective divisor with Supp $F\cap D=\phi$, and A_1, $\cdots$, A_k are mutually different irreducible curves with $A_i\neq D$ and $A_i\cdot D>0$ for $1\leqq i\leqq k$. Now we have $A_i\cdot L>0$ for every i by the same reason as in Theorem 3.25. Since

$$1 = M\cdot L = mD\cdot L+\sum_{i=1}^{k} n_iA_i\cdot L+F\cdot L$$

only one of the following two cases takes place.
⟨1⟩ $k=0$
⟨2⟩ $k=1$, $n_1=1$, $L\cdot A_1=1$ and $F\cdot L=0$.
Note that condition $k=0$ is equivalent to that $m=0$ because $0=mD^2+\sum_{i=1}^{k} n_iA_i\cdot D$, $A_i\cdot D\neq0$ and $D^2\neq0$. The case ⟨2⟩ is reduced to ⟨1⟩ by Lemma 3.26. Thus we can assume that $A=F$, namely Supp $A\cap D=\phi$. Let $\varDelta$ be the fixed component of $|A|$ and $C=A-\varDelta$. By Lemma 3.27 $C\neq0$ and $C^2\geqq0$ since it is the moving part. For the moment we assume $C^2=0$. By Proposition 3.3 there is a smooth elliptic curve G with $G\cap D=\phi$ and a positive integer p with $C\in|pG|$. If $G\cdot L=0$, then $G^2<0$ by the Hodge index theorem. By the adjunction formula $p_a(G)=(G^2-G\cdot D)/2+1=(G^2/2)+1\leqq0$, which is a contradiction since G is an elliptic curve. Thus $G\cdot L>0$. We have $p=1$, $G\cdot L=1$ and $\varDelta\cdot L=0$ since $1=M\cdot L=pG\cdot L+\varDelta\cdot L$. Thus $\varphi|_G: G\rightarrow\boldsymbol{P}^2$ is a generically one-to-one morphism and its image is a line in $\boldsymbol{P}^2$. We have $p_a(G)\leqq0$, a contradiction again. Next we treat the case $C^2>0$. By Proposition 3.3, we can assume that C is an irreducible curve with $p_a(C)=(C^2/2)+1\geqq2$. By the same reason as just the above, one has $C\cdot L=1$. Thus $\varphi|_C: C\rightarrow\boldsymbol{P}^2$ is a generically one-to-one morphism

to a line.   We have $p_a(C) \leqq 0$, a contradiction.

Thus conditions (1), (2) and (3) are equivalent.

Now we show the latter half of the theorem. By the Kawamata-Ramanujam vanishing theorem one sees easily that $h^1(mL)=1$ and $h^2(mL)=0$ for any positive integer m. By Riemann-Roch we have $h^0(mL)=m^2+2$. Let u_1, u_2, u_3 be a basis of $H^0(L)$. Let S_m be the subspace of $H^0(mL)$ generated by monomials of u_i's of degree m. Since φ_L is a surjective morphism to P^2, there is no non-zero homogeneous polynomial $P(U_1, U_2, U_3)$ with $P(u_1, u_2, u_3)=0$. Thus $\dim_C S_m=(m+2)(m+1)/2$. One sees that $H^0(L)=S_1$, $H^0(2L)=S_2$ and that there is a non-zero element $w \in H^0(3L)$ such that $H^0(3L)$ is a direct sum of Cw and S_3. Let $\Phi: Z \to P(1, 1, 1, 3)$ be the morphism to the weighted projective space defined by $z \to (u_1(z), u_2(z), u_3(z), w(z))$. Let Y be its image. Note that since u_i's do not vanish simultaneously on Z, the image Y does not contain the point $(0, 0, 0, 1)$. Thus the composition $\pi\Phi$ with the projection $P(1, 1, 1, 3)$ $-\{(0, 0, 0, 1)\} \to P(1, 1, 1)=P^2$ has the meaning and $\pi\Phi=\varphi_L$ by definition. Moreover we can show that $\Phi: Z \to P(1, 1, 1, 3)$ factors through $\rho: Z \to X$ by the same reason as in Proposition 3.20. Let $\bar{\Phi}: X \to Y \subset P(1, 1, 1, 3)$ be the induced morphism.

Lemma 3.29.   *If* $P(u_1, u_2, u_3)+wQ(u_1, u_2, u_3)=0$ *for homogeneous polynomials* $P(U_1, U_2, U_3)$, $Q(U_1, U_2, U_3)$ *with* $\deg P=\deg Q+3$, *then* $P=Q=0$.

Proof.   First assume that P and Q has a common non-constant divisor R. Set $P_1=P/R$ and $Q_1=Q/R$. , They are homogeneous polynomials with $\deg P_1=\deg Q_1+3$. Moreover under the assumption of the lemma we have $P_1(u_1, u_2, u_3)+wQ_1(u_1, u_2, u_3)=0$ since $R(u_1, u_2, u_3) \neq 0$. Thus one sees that one can assume that P and Q has no non-constant common divisor and that one of P and Q is non-zero. Then the polynomial $P(U_1, U_2, U_3)+WQ(U_1, U_2, U_3)$ is irreducible and non-zero. Besides its zero-locus $Y'=\{(a_1, a_2, a_3, b) \in P(1, 1, 1, 3)$ $|P(a_1, a_2, a_3)+bQ(a_1, a_2, a_3)=0\}$ is irreducible. We have $Y=Y'$ since $Y \subset Y'$ by definition. However if $\deg Q>0$, we have $(0, 0, 0, 1) \in Y=Y'$, which is a contradiction. If $\deg Q=0$, $Q \neq 0$, then $w \in S_3$, a contradiction.   Q.E.D.

By the above lemma and by dimensional reasons one sees that $H^0(4L)=S_4+wS_1$, $H^0(5L)=S_5+wS_2$ and $H^0(6L)=S_6+wS_3$. (Here $+$ denotes a direct sum.)   Now since $w^2 \in H^0(6L)$, there are homogeneous polynomial P of degree 6 and Q of degree 3 such that

$$w^2+wQ(u_1\ u_2, u_3)+P(u_1, u_2, u_3) = 0 .$$

By replacing w by $w - Q(u_1, u_2, u_3)/2$, we can assume moreover that $Q = 0$. Here by construction Y agrees with the hypersurface in $\mathbb{P}(1, 1, 1, 3)$ defined by $W^2 - P(U_1, U_2, U_3) = 0$, which is nothing but the branched double covering branching along the sextic curve B; $P(U_1, U_2, U_3) = 0$.

It remains to show that $\bar{\Phi}: X \to Y$ is an isomorphism. Note that every local ring of Y is Cohen-Macaulay since Y is a hypersurface of a smooth manifold $\mathbb{P}(1, 1, 1, 3) - \{(0, 0, 0, 1)\}$. Thus it suffices to show that the singular locus S of Y is 0-dimensional by the same reason as in the proof of Theorem 3.25. It is equivalent to that B is reduced by Lemma 1.1. Now let H be a general line in $\mathbb{P}^2$. The inverse image $\pi^{-1}(H)$ by $\pi: Y \to \mathbb{P}^2$ has singularities at $\pi^{-1}(H) \cap S$. The arithmetic genus of $\pi^{-1}(H)$ is $(\pi^*(H)^2 + \omega_Y \circ \pi^*(H))/2 + 1 = 2$. Let $C \subset Z$ be the strict inverse image of $\pi^{-1}(H)$ by Φ. $\Phi|_C: C \to \pi^{-1}(H)$ is a birational morphism. We have $p_a(C) \leq p_a(\pi^{-1}(H)) = 2$ and the equality holds if and only if $\Phi|_C$ is an isomorphism. However $C \in |L|$ and C is smooth. Thus $p_a(C) \leq 1$ if $\dim S \geq 1$. On the other hand we have $p_a(C) = (L^2 - L \circ D)/2 + 1 = 2$ and thus $\dim S = 0$. $\hfill$ Q.E.D.

Before concluding this section we would like to give one more proposition and a lemma. The next lemma is due to Looijenga. We omit the proof here. (Cf. Looijenga [12])

Lemma 3.30. (Looijenga) *Let A be an irreducible curve on Z with $A \cap D = \phi$ and $A^2 = -2$. Then $\mathcal{O}_Z(A) \in \mathrm{Pic}(Z)$ is an effective nodal root.*

Remark. Since the conditions $\alpha^2 = -2$ and $\alpha \circ \omega_Z = 0$ for $\alpha \in \mathrm{Pic}(Z)$ do not imply that α is a root, this lemma is not a trivial one.

Proposition 3.31. *Let $\tilde{S} \subset \mathrm{Pic}(Z)$ be the set of nodal roots orthogonal to the polarization L. Then $\tilde{S}$ is a root system. Moreover singularities on X are a unique point with positive geometric genus at $w = \rho(D) \in X$ plus combination of rational double points consisted of p_k of A_k-points, q_l of D_l-points, and r_m of E_m-points $(k \geq 1, l \geq 4, m = 6, 7, 8)$ if and only if $\tilde{S}$ is isomorphic to the direct sum of p_k of irreducible root systems of type A_k for every k, q_l of ones of type D_l for every l and r_m of ones of type E_m for $m = 6, 7, 8$. Here $\rho: Z \to X$ is the contraction defined just after Lemma 3.19.*

Proof. Let R be the set of all roots in $\mathrm{Pic}(Z)$. It is obvious by definition that $(\tilde{S} + \tilde{S}) \cap R \subset \tilde{S}$ and $\tilde{S} = -\tilde{S}$. Since the orthogonal complement of L in $\mathrm{Pic}(Z)$ is negative-definite, the former half of the proposition follows from the definition of the root system. (Cf. Bourbaki [3])

Let us proceed to the latter half. Let $\mathcal{E}$ be the union of exceptional curves of $\rho\colon Z{\to}X$. Let $\mathcal{E}'$ be the union of D and the support of effective nodal roots orthogonal to L. In view of Lemma 2.1, it suffices to show that $\mathcal{E}=\mathcal{E}'$.

Let A be an irreducible curve on Z such that $\rho(A)$ is a point. If $A=D$, then $A\subset\mathcal{E}'$ by definition. Assume $A\neq D$. By Lemma 3.19, we have $A^2=-2$ and $A\cap D=\phi$. By Lemma 3.30, we have $A\subset\mathcal{E}'$. Thus $\mathcal{E}\subset\mathcal{E}'$. Conversely let A be an irreducible component of $\mathcal{E}'$. If $A=D$, then $A\subset\mathcal{E}$ by Lemma 3.19. Assume $A\neq D$. There exists an effective divisor $\sum n_i A_i$ $(0<n_i\in\mathbf{Z}$, A_i is an irreducible curve$)$ containing A as a component such that $\mathcal{O}_Z(\sum n_i A_i)\in$ Pic (Z) is a nodal root orthogonal to L. We may assume $A=A_1$. It follows that $A_i\cdot L=0$ for every i from $\sum n_i A_i\cdot L=0$ since L is numerically effective. By Lemma 3.19 we have $A=A_1\subset\mathcal{E}$. Thus $\mathcal{E}=\mathcal{E}'$.   Q.E.D.

Now according to Theorem 3.25 and Theorem 3.28 we can decide whether Z represents a reduced sextic curve or a normal quartic surface by studying the morphism $\mathrm{Pic}(Z){\to}\mathrm{Pic}(D)$. Proposition 3.31 shows that the morphism $\mathrm{Pic}(Z){\to}\mathrm{Pic}(D)$ contains information about singularities on the objects we are considering. Therefore if we had a criterion written with group-theoretic words about $\mathrm{Pic}(Z){\to}\mathrm{Pic}(D)$ by which we could decide $L\in\mathrm{Pic}(Z)$ were a polarization or not, then classification of all singularities of objects under consideration would be accomplished.

In the next section, we show that this is the case when $t=9-\omega_Z^2=10$.

§4. Determination of the Polarization Class (when $t=10$)

In Sections 1, 2 and 3, we assumed only that $t=9-\omega_Z^2\geqq3$. In Section 4 restriction appeared; existence of polarization implies $t\geqq10$. However in this section and the next one, we restrict ourselves to the case $t=10$. There are two reasons to do so. First if $t=10$, we can easily determine all elements $\lambda\in P$ with $\lambda\cdot\kappa=0$ and $\lambda^2=2$ or 4 compared with the case $t\geqq11$. Secondly we have a group-theoretic criterion by which we can decide $L\in\mathrm{Pic}(Z)$ with $L\cdot\omega_Z=0$ and $L^2=2$ or 4 is a polarization or not.

In this section we always assume that $t=10$ (i.e. $\omega_Z^2=-1$) even if we do not mention it.

Proposition 4.1. *Assume that* $\omega_Z^2=-1$. *(i.e.* $t=10$*). An element* $L\in\mathrm{Pic}(Z)$ *with* $L|_D\cong\mathcal{O}_D$ *and* $L^2>0$ *is a polarization if and only if* $L\in V_S\cap C_+$ *where* C_+ *is the connected component of the positive cone* $C=\{x\in\mathrm{Pic}(Z)\otimes\mathbf{R}\,|\,x^2>0\}$ *containing ample line bundles and*

$$V_S = \{x \in \text{Pic}(Z) \otimes \mathbb{R} \mid x \cdot \omega_Z = 0, \ x \cdot r \geqq 0 \ \textit{for any effective nodal}$$
$$\textit{root } r \in \text{Pic}(Z)\} \ .$$

Proof. "Only if" part is trivial since L is numerically effective. To show "if" part, we have to check conditions in Definition 3.1. The conditions (1) and (3) are obvious by assumption. We show (2), i.e., L is numerically effective. It suffices to show that for every irreducible curve A, the inequality $L \cdot A \geqq 0$ holds.

Recall that the positive cone C has just two connected components. One is C_+. The other is $C_- = -C_+$.

If $A^2 > 0$, the restriction to the orthogonal complement $(\mathbb{R}A)^\perp$ of A in $\text{Pic}(Z) \otimes \mathbb{R}$ of the intersection form is negative definite since the intersection form on $\text{Pic}(Z)$ has signature $(1, 10)$. Thus $(\mathbb{R}A)^\perp \cap \bar{C} = \{0\}$. (‾ denotes the closure.) It implies that C_+ lies in a half space bounded by the hyperplane $(\mathbb{R}A)^\perp$. Since both L and any ample line bundle belong to C_+, we have $L \cdot A > 0$. Moreover by a similar argument we have $L \cdot A > 0$ for any curve A with $A^2 = 0$. Here note that we did not use that A is irreducible until now. Assume that $A^2 < 0$. By the adjunction formula, one sees that there are only three cases.

(i) $A = D$.

(ii) $A^2 = -2$ and $A \cap D = \phi$.

(iii) $A^2 = -1$ and $A \cdot D = 1$.

If $A = D$, then $L \cdot D = 0$ by assumption $L|_D \cong \mathcal{O}_D$. In case (ii), $\mathcal{O}_Z(A)$ is an effective nodal root by Lemma 3.30. Thus it follows from the assumption $L \in V_S$ that $A \cdot L = \mathcal{O}_Z(A) \cdot L \geqq 0$. In order to manipulate case (iii), we need the assumption $D^2 = -1$. Set $C = A + D$. We have $C^2 = -1 + 2 - 1 = 0$. Thus by the above argument we have $L \cdot (A + D) = L \cdot A > 0$. We obtain not only numerical effectiveness but also condition (4) in Definition 3.1.   Q.E.D.

Next we determine elements $\lambda \in P = \mathbb{Z}\varepsilon_0 + \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_{10}$ with $\lambda^2 = 2$ or 4 and $\lambda \cdot \kappa = 0$ up to the action of the Weyl group W. Here $\kappa = -3\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_{10}$. Let Γ be the orthogonal complement of $\mathbb{Z}\kappa$ in P. We denote

$$\tilde{U} = \{x \in \Gamma \otimes \mathbb{R} \mid x^2 > 0\}$$
$$\tilde{U}_+ = \{x \in \tilde{U} \mid x \cdot \varepsilon_0 > 0\}$$
$$\tilde{U}_- = \{x \in \tilde{U} \mid x \cdot \varepsilon_0 < 0\} \ .$$

It is easy to see that $\tilde{U}_\pm$ are connected components of $\tilde{U}$ and $\tilde{U} = \tilde{U}_+ \cup \tilde{U}_-$. Moreover we denote

$$\tilde{V} = \{x \in \Gamma \otimes R \,|\, x \cdot \gamma_i \geqq 0 \text{ for } 1 \leqq i \leqq 10\}$$

where $\gamma_1 = \varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, $\gamma_i = \varepsilon_{i-1} - \varepsilon_i$ for $2 \leqq i \leqq 10$. The following lemma is due to Looijenga. (Cf. Looijenga [12])

Lemma 4.2. $\tilde{U}_+ \subset W\tilde{V}$.

The rest of this section is devoted to verify the following.

Proposition 4.3. *Assume $t = 10$. Any element $\lambda \in P$ with $\lambda^2 = 4$ and $\lambda \cdot \kappa = 0$ is conjugate to one of the following elements with respect to the action of W.*

$$\pm (9\varepsilon_0 - 3\varepsilon_1 - 3\varepsilon_2 - 3\varepsilon_3 - 3\varepsilon_4 - 3\varepsilon_5 - 3\varepsilon_6 - 3\varepsilon_7 - 3\varepsilon_8 - 2\varepsilon_9 - \varepsilon_{10})$$
$$\pm (7\varepsilon_0 - 3\varepsilon_1 - 2\varepsilon_2 - 2\varepsilon_3 - 2\varepsilon_4 - 2\varepsilon_5 - 2\varepsilon_6 - 2\varepsilon_7 - 2\varepsilon_8 - 2\varepsilon_9 - 2\varepsilon_{10}) \,.$$

Proposition 4.4. *Assume $t = 10$. Any element $\lambda \in P$ with $\lambda^2 = 2$ and $\lambda \cdot \kappa = 0$ is conjugate to one of the following elements with respect to the action of W.*

$$\pm (6\varepsilon_0 - 2\varepsilon_1 - 2\varepsilon_2 - 2\varepsilon_3 - 2\varepsilon_4 - 2\varepsilon_5 - 2\varepsilon_6 - 2\varepsilon_7 - 2\varepsilon_8 - \varepsilon_9 - \varepsilon_{10}) \,.$$

Proof of Proposition 4.3.

If λ belongs to $\tilde{U}_-$, then obviously $-\lambda$ belongs to $\tilde{U}_+$, $(-\lambda)^2 = 4$ and $(-\lambda) \cdot \kappa = 0$. Besides every element in $\tilde{U}_+$ is conjugate to an element in $\tilde{V}$ by Lemma 4.3. Thus we have only to show that the following system of equalities and inequalities holds for integers $x, y_1, \cdots, y_{10}$ if and only if $(x, y_1, \cdots, y_{10}) = (9, 3, \cdots, 3, 2, 1)$ or $(7, 3, 2, \cdots, 2)$.

$$(4.1) \qquad \begin{cases} x^2 = \sum_{i=1}^{10} y_i^2 + 4 \\ 3x = \sum_{i=1}^{10} y_i \\ x \geqq y_1 + y_2 + y_3 \\ y_1 \geqq y_2 \geqq y_3 \geqq y_4 \geqq y_5 \geqq y_6 \geqq y_7 \geqq y_8 \geqq y_9 \geqq y_{10} \,. \end{cases}$$

We need several steps.
STEP 1.

Lemma 4.5. *If (4.1) holds, then $x \geqq 7$ and $y_i > 0$ for $1 \leqq i \leqq 10$.*

Proof. By the Schwartz inequality we have for $1 \leqq \alpha \leqq 10$ $(3x - y_\alpha)^2 = (\sum_{i \neq \alpha} y_i)^2 \leqq 9(x^2 - y_\alpha^2 - 4)$. Thus $5(y_\alpha - \frac{3}{10}x)^2 - \frac{9}{20}x^2 + 18 \leqq 0$. One sees that $x \neq 0$ and that $y_\alpha > 0$ or < 0 according as $x > 0$ or < 0. Assume $x < 0$. We have $y_{10} < 0$. It implies that $3x \geqq 3(y_1 + y_2 + y_3) \geqq \sum_{j=1}^{9} y_j > \sum_{j=1}^{10} y_j = 3x$, a contradiction.

Therefore $x > 0$ and $y_\alpha > 0$ for $1 \le \alpha \le 10$. Moreover by the Schwartz inequality we have $9x^2 = (\sum y_i)^2 \le 10(\sum y_i^2) = 10(x^2 - 4)$. Thus $x \ge 7$. Q.E.D.

Lemma 4.6. *If* (4.1) *holds and if* $x \le 10$, *then* $(x, y_1, \cdots, y_{10}) = (9, 3, \cdots, 3, 2, 1)$ *or* $(7, 3, 2, \cdots, 2)$.

Proof. We can assume $7 \le x \le 10$ by Lemma 4.5. First assume $x = 7$. By the Schwartz inequality we have $(21 - y_1)^2 \le 9(45 - y_1^2)$. It implies $5y_1^2 - 21y_1 + 18 \le 0$ and thus $0 < y_1 \le 3$. If $y_1 = 3$, then $y_2 + \cdots + y_{10} = 18$ and $y_2^2 + \cdots + y_{10}^2 = 36$. Since $18^2 = 9 \times 36$, the equality in the Schwartz inequality $(\sum_{i \ge 2} y_i)^2 \le 9(\sum_{i \ge 2} y_i^2)$ holds. Thus $y_2 = \cdots = y_{10} = 2$. We have the solution $(7, 3, 2, \cdots, 2)$. If $y_1 \le 2$, then $21 = y_1 + y_2 + \cdots + y_{10} \le 20$, which is a contradiction. Secondly assume $x = 8$. We can show similarly that there is no solution in this case. Thirdly assume $x = 9$. By the Schwartz inequality we have $5y_1^2 - 27y_1 + 18 \le 0$. Thus $0 < y_1 \le 4$. Assume $y_1 = 4$. We have $(23 - y_2)^2 = (y_3 + \cdots + y_{10})^2 \le 8 \sum_{i \ge 3} y_i^2 = 8(61 - y_2^2)$, which implies $y_2 \le 3$. If $y_2 \le 2$, then $23 = \sum_{i \ge 2} y_i \le 18$, a contradiction. Thus $y_2 = 3$. Since $y_1 + y_2 + y_3 \le x = 9$ we have moreover $y_3 \le 2$. We have $20 = \sum_{i \ge 3} y_i \le 16$, a contradiction again. Thus $0 < y_1 \le 3$. Now we assume that k of $\{y_1, y_2, \cdots, y_{10}\}$ are 3, l of them are 2 and m of them are 1. We have $k + l + m = 10$, $3k + 2l + m = 27$ and $9k + 4l + m = 77$. One sees easily that $k = 8$, $l = 1$ and $m = 1$. We have the solution $(9, 3, 3, \cdots, 2, 1)$. Lastly assume $x = 10$. Similarly we see that there is no solution in this case. Q.E.D.

STEP 2.

Next we set

$$x = 3z + \varepsilon, \quad y_i = z + \delta_i \ (1 \le i \le 9), \quad y_{10} = \delta_{10}.$$

Equalities and inequalities (4.1) are equivalent to the next ones.

$$(4.2) \quad \begin{cases} \langle 1 \rangle & \varepsilon \ge \delta_1 + \delta_2 + \delta_3 \\ \langle 2 \rangle & \delta_1 \ge \delta_2 \ge \cdots \ge \delta_9 \\ \langle 3 \rangle & z + \delta_9 \ge \delta_{10} \\ \langle 4 \rangle & \delta_{10} > 0 \\ \langle 5 \rangle & \delta_1 + \delta_2 + \cdots + \delta_{10} = 3\varepsilon \\ \langle 6 \rangle & 2z(\delta_1 + \delta_2 + \cdots + \delta_9) + (\delta_1^2 + \delta_2^2 + \cdots + \delta_9^2) + \delta_{10}^2 \\ & \qquad = 6\varepsilon z + \varepsilon^2 - 4. \end{cases}$$

Lemma 4.7. *If* $\varepsilon, \delta_1, \cdots, \delta_{10}$ *are* 0 *or* ± 1, *then the solution of* (4.2) *is* $z = 3$, $\varepsilon = 0$, $\delta_1 = \delta_2 = \cdots = \delta_8 = 0$, $\delta_9 = -1$, $\delta_{10} = +1$.

Proof. By $\langle 4 \rangle$ we have $\delta_{10}=1$. First assume $\varepsilon=0$. If $\delta_1=1$, then by $\langle 1 \rangle$, $\langle 2 \rangle$ we have only two cases; (a) $\delta_2=0$, $\delta_3=\cdots=\delta_9=-1$, (b) $\delta_2=\delta_3=\cdots=\delta_9=-1$. In both cases $\langle 5 \rangle$ does not hold. If $\delta_1=0$, then by $\langle 2 \rangle$, $\langle 5 \rangle$ $\delta_2=\cdots=\delta_8=0$, $\delta_9=-1$. Substituting them to $\langle 6 \rangle$, we have $z=3$. Thus $\langle 3 \rangle$ is also satisfied. We have the desired solution. If $\delta_1=-1$, by $\langle 2 \rangle$ $\delta_2=\cdots=\delta_9=-1$. They do not satisfy $\langle 5 \rangle$. Secondly assume $\varepsilon=+1$. If $\delta_1\leq 0$, then by $\langle 2 \rangle$, $\langle 5 \rangle$ $1\geq\delta_1+\delta_2+\cdots+\delta_{10}=3$, which is a contradiction. Thus $\delta_1=1$. By $\langle 1 \rangle$, $\langle 2 \rangle$ we have only three cases. (c) $\delta_2=1$, $\delta_3=\cdots=\delta_9=-1$, (d) $\delta_2=\delta_3=0$, δ_4, δ_5, $\cdots$, $\delta_9\leq 0$, (e) $\delta_2=0$, $\delta_3=\cdots=\delta_9=-1$. In any case $\langle 5 \rangle$ does not hold. Thirdly assume $\varepsilon=-1$. If $\delta_1=1$, then $\delta_2=\delta_3=-1$ by $\langle 1 \rangle$. By $\langle 2 \rangle$ we have moreover $\delta_4=\cdots=\delta_9=-1$. In this case $\langle 5 \rangle$ does not hold. If $\delta_1=0$, then there are only two cases by $\langle 1 \rangle$, $\langle 2 \rangle$, (f) $\delta_2=0$, $\delta_3=\cdots=\delta_9=-1$, (g) $\delta_2=\delta_3=\cdots=\delta_9=-1$. Anyway $\langle 5 \rangle$ does not hold. If $\delta_1=-1$, then we have $\delta_2=\cdots=\delta_9=-1$ by $\langle 2 \rangle$ and $\langle 5 \rangle$ does not hold.   Q.E.D.

Lemma 4.8. *Assume one of ε, δ_1, $\cdots$, δ_{10} is ± 2, at most one of them is ± 1 and the rest are 0. Then (4.2) has no solution.*

Proof. First assume $\varepsilon=\pm 2$. By $\langle 4 \rangle$ we have $\delta_{10}=1$. By assumption we have $\delta_1=\cdots=\delta_9=0$. Then $\langle 5 \rangle$ does not hold. Secondly assume $\varepsilon=\pm 1$. By $\langle 4 \rangle$ we have $\delta_{10}=2$. By assumption one sees $\delta_1=\cdots=\delta_9=0$. Then $\langle 5 \rangle$ does not hold. Thirdly assume $\varepsilon=0$. We have 3 cases: (a) $\delta_1=\cdots=\delta_8=0$, $\delta_9=-2$, $\delta_{10}=1$ (b) $\delta_1=\cdots=\delta_8=0$, $\delta_9=-1$, $\delta_{10}=2$ (c) $\delta_1=\cdots=\delta_8=\delta_9=0$, $\delta_{10}=2$. In any case $\langle 5 \rangle$ is not satisfied.   Q.E.D.

By the next lemma we can complete the proof of Proposition 4.3.

Lemma 4.9. *If an integral solution of (4.1) satisfies $x\geq 11$, then there exist integers z, ε, δ_1, $\cdots$, δ_{10} satisfying $x=3z+\varepsilon$, $y_i=z+\delta_i$ $(1\leq i\leq 9)$, $y_{10}=\delta_{10}$, equalities and inequalities (4.2) and $\varepsilon^2+\sum_{i=1}^{10}\delta_i^2\leq 5$.*

Since inequality $\varepsilon^2+\sum\delta_i^2\leq 5$ implies that one of the assumptions in Lemmas 4.7 and 4.8 is satisfied, it follows from Lemmas 4.7, 4.8 and 4.9 that (4.1) has no solution with $x\geq 11$. Thus by Lemma 4.6 we have Proposition 4.3.

                                                    Q.E.D.

STEP 3.

Now we have to show Lemma 4.9. Here we introduce an Euclidean metric $(\ ,\ )$ on $P\otimes R$ by $(\varepsilon_i, \varepsilon_i)=1$ $(0\leq i\leq 10)$ and $(\varepsilon_i, \varepsilon_j)=0$ for $i\neq j$. By this metric we can define the distance $\mathrm{dist}(A, B)$ of two subsets A, $B\subset P\otimes R$. Let

P_i denote the orthogonal complement of the set $\{\kappa, r_1, r_2, \cdots, r_{10}\} - \{r_i\}$ in $P \otimes R$ with respect to the intersection form, i.e., $P_i = \{x \in P \otimes R \mid x \cdot \kappa = 0, x \cdot r_j = 0 \text{ for } 1 \leq j \leq 10, j \neq i\}$. Set $T_c = \{x \in P \otimes R \mid x \cdot \kappa = 0, x \cdot x = c, x \cdot r_i \geq 0 \text{ for } 1 \leq i \leq 10\} \subset \Gamma \otimes R$, $H_g = \{x \in P \otimes R \mid x \cdot \varepsilon_0 \geq g\}$ and $\partial H_g = \{x \in P \otimes R \mid x \cdot \varepsilon_0 = g\}$, where c, g are positive real numbers. We would like to show that $T_4 \cap H_{11}$ lies too near to P_{10} to have lattice points on it. We need further several lemmas.

The following one treats a general situation.

Lemma 4.10. *Let F be a three-dimensional real vector space equipped with an intersection form $\langle \ , \ \rangle$ of signature $(1, 2)$ and with a positively definite inner product $(\ , \ )$. Let L be a line in F passing through the origin. For a positive real number a we set $Q = \{x \in F \mid \langle x, x \rangle = a\}$. Let $E \subset F$ be a two-dimensional linear subspace of F with $E \cap Q \neq \phi$. Then $E \cap Q$ has two connected component each of which is diffeomorphic to R. Let $\varphi\colon R \to E \cap Q$ be a diffeomorphism to one connected component. Then for any closed interval $[b, c] \subset R$ and for every $\lambda \in [b, c]$,*

$$\text{dist}(\varphi(\lambda), L) \leq \max\{\text{dist}(\varphi(b), L), \text{dist}(\varphi(c), L)\}.$$

Proof. Since the restriction of the intersection form $\langle \ , \ \rangle$ to E has signature $(1, 1)$, $E \cap Q$ is a hyperbolic curve. Therefore $E \cap Q$ is diffeomorphic

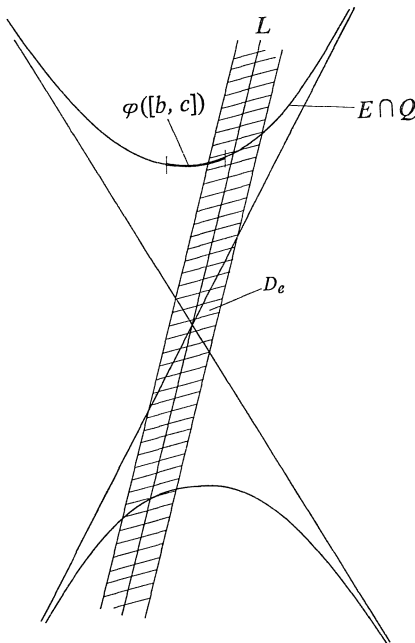Figure 4.1.

to two copies of R. We divide the rest of the proof into two cases.

Case 1. $L \subset E$.

For every non-negative real number $e \in R$, set $D_e = \{x \in E \mid \text{dist } (x, L) \leq e\}$. D_e is a closed connected set bounded by two lines parallel to L. Note that $D_e \cap \varphi([b, c])$ is always connected. Set $d_0 = \text{dist } (\varphi(\lambda), L)$ and assume $d_0 > \max \{\text{dist } (\varphi(b), L), \text{dist } (\varphi(c), L)\}$. There exists a sufficiently small positive real number $\varepsilon > 0$ such that $D_{d_0 - \varepsilon} \ni \varphi(b), \varphi(c)$. Since $D_{d_0 - \varepsilon} \cap \varphi([b, c])$ is connected, $D_{d_0 - \varepsilon} \cap \varphi([b, c]) = \varphi([b, c])$. It implies $\varphi(\lambda) \in D_{d_0 - \varepsilon}$. We have $d_0 = \text{dist } (\varphi(\lambda), L) \leq d_0 - \varepsilon$, a contradiction.

Case 2. $L \not\subset E$.

Similarly we set for non-negative real number $e \in R$, $D_e = \{x \in E \mid \text{dist } (x, L) \leq e\}$. In this case D_e is the interior and the boundary of an oval. Since $D_e \cap \varphi([b, c])$ is always connected, we get the desired inequality by the same reason as in Case 1.   Q.E.D.
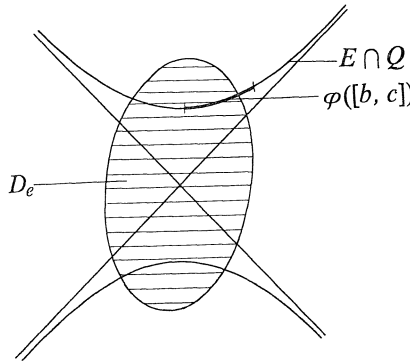
Figure 4.2.

We now return to our case. For every subset $I \subset \{1, 2, 3, \cdots, 10\}$, we set $P_I = (\bigcap_{i \in I^c} F_i) \cap (R\kappa)^{\perp}$ where I^c is the complement of I, F_i is the orthogonal complement of γ_i in $P \otimes R$, and $(R\kappa)^{\perp}$ is the orthogonal complement of κ. Note that $P_{\{i\}} = P_i$. Next we define linear functions $u, v_1, \cdots, v_{10} : P \otimes R \to R$ by $u(x) = x \cdot \varepsilon_0$ and $v_i(x) = x \circ \gamma_i$ for $1 \leq i \leq 10$. By direct calculation we obtain;

Lemma 4.11. $P_i \cap T_4$ *is a unique point for* $1 \leq i \leq 9$ *and we have* $u(x_i) < 11$ *for* $\{x_i\} = P_i \cap T_4$, $1 \leq i \leq 9$. $P_{10} \cap T_4$ *is empty.* (*Indeed* $\max \{u(x_i) \mid 1 \leq i \leq 9\} = u(x_9) = 6\sqrt{2}$)

The next lemma is the key part of this section.

Lemma 4.12. *For every subset* $I \subset \{1, 2, \cdots, 10\}$ *with* $\#I \geq 3$ *and for every*

$x \in P_I \cap T_4 \cap H_{11}$, *there exist a subset* $J \subset I$ *with* $\#J = \#I - 1$ *and a point* $y \in P_J \cap T_4 \cap H_{11}$ *with* $\mathrm{dist}(y, P_{10}) \geqq \mathrm{dist}(x, P_{10})$.

Proof. First note that unless $I = \{10\}$ or $I = \phi$, the restriction of the intersection form of $P \otimes R$ to the space spanned by r_i, $i \in \{1, 2, \cdots, 10\} - I$ is negatively definite. Thus the intersection form has signature $(1, k-1)$ on P_I unless $I = \{10\}$ or $I = \phi$ where $k = \#I$. Assume $k \geqq 3$. One sees easily that $P_I \cap T_4 \cap H_{10} \neq \phi$. Assume that there exists $i \in I$ with $v_i(x) = 0$ for $x \in P_I \cap T_4 \cap H_{11}$. Then $x \in P_{I-\{i\}} \cap T_4 \cap H_{11}$ and setting $J = I - \{i\}$, $y = x$ we get the lemma. Thus in what follows we assume that $v_i(x) \neq 0$ for every $i \in I$. Since $x \in T_4$, we have $v_i(x) > 0$ for $i \in I$. We denote $Q = \{z \in P \otimes R \mid z \cdot z = 4\}$. $P_I \cap Q$ is a quadratic hypersurface spanning P_I. $P_I \cap Q$ has two connected components. Let $(P_I \cap Q)_0$ be the connected component of $P_I \cap Q$ containing x. Set $c_0 = \min \{u(y) \mid y \in (P \cap Q)_0\}$. We have $c_0 > 0$ and $c_0 < 11$ by Lemma 4.11. If $-c_0 < g < c_0$, then $P_I \cap Q \cap \partial H_g = \phi$. If $g = \pm c_0$, then $P_I \cap Q \cap \partial H_g$ is one point. If $|g| > c_0$, then $P_I \cap Q \cap \partial H_g$ is a smooth $(k-2)$-dimensional manifold. In particular $P_I \cap Q \cap \partial H_{u(x)}$ is a smooth $(k-2)$-dimensional manifold. Let S' be the tangent space of $P_I \cap Q \cap \partial H_{u(x)}$ at x. If $0 \in S'$, then $0 \in S' \subset \partial H_{u(x)}$ and $0 = u(0) = u(x) \geqq 11$. It is a contradiction. Thus $0 \notin S'$. Let $\hat{V} = \{z \in P_I \mid v_i(z) \geqq 0 \text{ for } i \in I\}$. $\hat{V}$ is a convex cone in P_I and x belongs to the interior of $\hat{V}$. Since $\dim S' \geqq 1$, S'
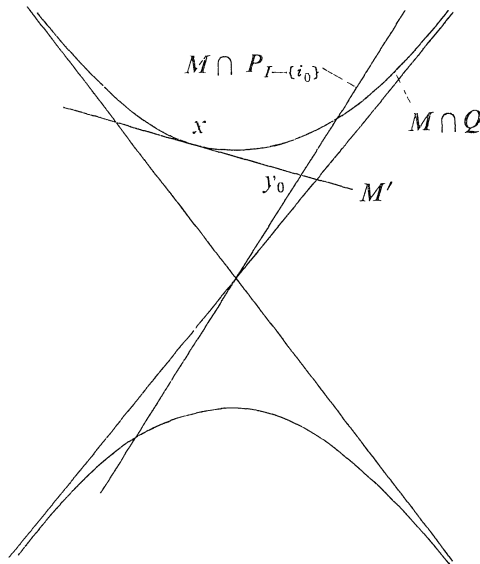

Figure 4.3.

intersects some wall of $\hat{V}$, i.e., $S' \cap (\hat{V} \cap P_{I-\{i_0\}}) \neq \phi$ for some $i_0 \in I$. Note that there exists $y_0 \in S' \cap (\hat{V} \cap P_{I-\{i_0\}})$ with $y_0 \cdot y_0 > 0$. Otherwise $S' \cap (\hat{V} \cap P_{I-\{i_0\}}) \subset P_{10}$ and moreover the tangent space S' of $P_I \cap Q \cap \partial H_{u(x)}$ at x intersects P_{10}, which is impossible. Thus such y_0 always exists. Let M' be the line passing through x and y_0. If $0 \in M'$, then $x \in M' \subset P_{I-\{i_0\}}$ and we have $v_{i_0}(x) = 0$, a contradiction. Let M be the linear span of x, y_0 and 0. Note that $\dim M = 2$. Since $x \in M$ and $x \cdot x = 4$, the restriction of the intersection form to M has signature $(1, 1)$. We have the figure 4.3.

Next we would like to show $(M \cap Q)_0 \subset H_{u(x)}$, i.e., $u(y) \geq u(x)$ for every $y \in (M \cap Q)_0$, where $(M \cap Q)_0$ is the connected component of $M \cap Q$ containing x. If $M \subset \partial H_{u(x)}$, we have nothing to prove. Thus we assume $M \not\subset \partial H_{u(x)}$. $M \cap \partial H_{u(x)}$ is a line containing x and y_0, that is, $M \cap \partial H_{u(x)} = M'$. Recall that M' is the tangent line of $M \cap Q$ at x by definition. Since $M \cap Q$ is a hyperbolic curve, $(M \cap Q)_0$ lies on one side of M'. We have either $u(y) \geq u(x)$ for every $y \in (M \cap Q)_0$ or $0 < u(y) \leq u(x)$ for every $y \in (M \cap Q)_0$. Since obviously $u(y)$ is unbounded on $(M \cap Q)_0$, we have $(M \cap Q)_0 \subset H_{u(x)}$. Now $M \cap P_{I-\{i\}}$ is a line in M passing through the origin for every $i \in I$ since $P_{I-\{i\}} = \mathrm{Ker}\, v_i \cap P_I \ni x$. One sees that $M \cap T_4$ coincides with the closure of the connected component of $M \cap Q - \bigcup_{i \in I} M \cap P_{I-\{i\}}$ containing x. Since $y_0 \in P_{I-\{i_0\}}$ and $y_0 \cdot y_0 > 0$, $M \cap P_{I-\{i_0\}}$ intersects with $(M \cap Q)_0$. It implies that $M \cap T_4$ is a connected closed proper subset of $(M \cap Q)_0$. Thus we have $Y = \partial(M \cap T_4) \cap (\bigcup_{i \in I} M \cap P_{I-\{i\}}) \neq \phi$, where $\partial(M \cap T_4)$ is the boundary of $M \cap T_4$. Pick $y_1 \in Y$. There exists $i_1 \in I$ with $y_1 \in \partial(M \cap T_4) \cap P_{I-\{i_1\}}$. Set $J = I - \{i_1\}$. Then $y_1 \in P_J \cap T_4$ and $y_1 \in (M \cap Q)_0 \subset H_{u(x)} \subset H_{11}$. Moreover by Lemma 4.10, $\mathrm{dist}(y_1, P_{10}) \geq \mathrm{dist}(x, P_{10})$.　　　Q.E.D.

Lemma 4.13. *For every subset $I \subset \{1, 2, 3, \cdots, 10\}$ with $\#I = 2$ and $10 \notin I$, we have $P_I \cap H_{11} \cap T_4 = \phi$.* (see figure 4.4)

Proof. Set $I = \{i, j\}$. Since $i \neq 10$, $j \neq 10$, we have $P_i - \{0\}$, $P_j - \{0\} \subset \{y \in P_I \mid y \cdot y > 0\}$. Thus if $T_4 \cap P_I$ is not empty, it is a compact connected arc contained in a hyperbolic curve. However, for a point y in $P_i \cap T_4$ and $P_j \cap T_4$, $u(y) < 11$ by Lemma 4.11. Thus for every $y \in T_4 \cap P_I$, $u(y) < 11$. It implies $T_4 \cap P_I \cap H_{11} = \phi$. (See figure 4.4).　　　Q.E.D.

Lemma 4.14. *For a subset $I = \{k, 10\}$ with $1 \leq k \leq 9$, the function $P_I \cap T_4 \cap H_{11} \ni x \to \mathrm{dist}(x, P_{10})$ attains its maximal value on the set $P_I \cap T_4 \cap \partial H_{11}$.*

Proof. Since $P_{10} \subset \{y \in P_I \mid y \cdot y = 0\}$ and $P_k - \{0\} \subset \{y \in P_I \mid y \cdot y > 0\}$, $P_I \cap T_4$ is an arc as in the following figure 4.5. Since $u(y_2) < 11$ for $y_2 \in P_k \cap T_4$, y_2 and
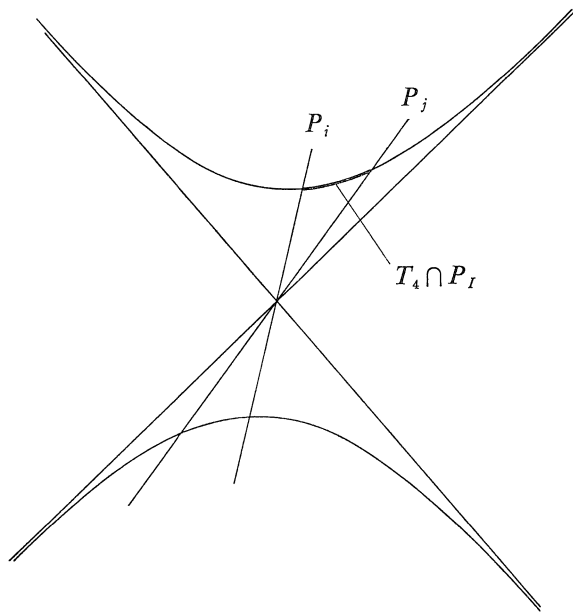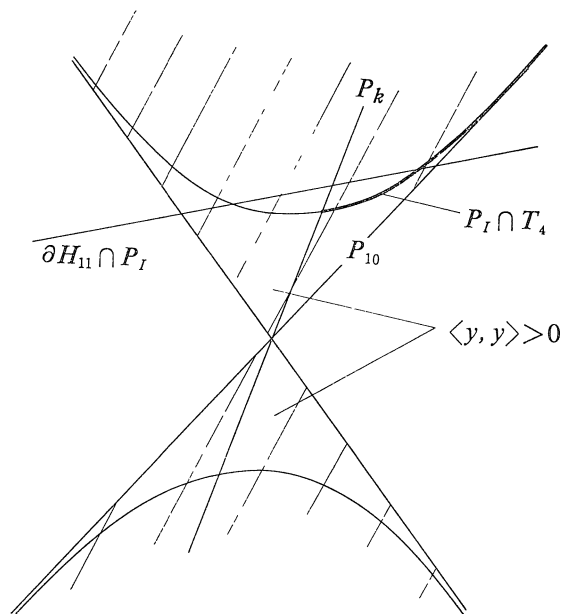
Figure 4.4.

Figure 4.5.

the origin lie on the same side with respect to ∂H_{11}. It implies that there are not two connected components of $T_4 \cap P_I \cap H_{11}$ but there is only one. In view of the fact that $P_{10} \cap T_4$ is the asymptotic line of $T_4 \cap P_I \cap H_{11}$, one sees that the distance to P_{10} attains the maximal value at $T_4 \cap P_I \cap \partial H_{11}$ by Lemma 4.10.

$$\text{Q.E.D.}$$

Lemma 4.15. *The set $T_4 \cap P_{\{k,10\}} \cap \partial H_{11}$ consists of a unique point $\{y_k\}$ for $1 \leq k \leq 9$. Besides we have* dist$(y_k, P_{10}) < 1$ *for $1 \leq k \leq 9$.*

Proof. The former half is trivial. By direct calculation we have

$$\max_k \text{dist}\ (y_k, P_{10}) = \text{dist}(y_9, P_{10}) = \sqrt{70}/9 < 1. \qquad\qquad \text{Q.E.D.}$$

Corollary 4.16. *For every point $x \in T_4 \cap H_{11}$,* dist$(x, P_{10}) < 1$.

Proof of Lemma 4.9. First note that the set $\{z\,(3\varepsilon_0 - \sum_{i=1}^{9} \varepsilon_i)\,|\,z \in \mathbb{Z}\}$ exhausts the lattice points (points whose coordinates are all integers) on P_{10}. The minimum distance of lattice points on P_{10} is $\sqrt{18}$. Thus for every point $x \in P_{10}$ there exists a lattice point $w \in P_{10}$ with dist$(x, w) \leq \sqrt{18}/2$.

Let $y_0 \in T_4 \cap H_{11}$ be an arbitrary lattice point. Let $x_0 \in P_{10}$ be the point on P_{10} which attains the distance between y_0 and P_{10}, i.e., dist$(y_0, P_{10}) =$dist(y_0, x_0). The line passing through x_0 and y_0 is perpendicular to P_{10}. Let $w_0 \in P_{10}$ be the lattice point with dist$(x_0, w_0) \leq \sqrt{18}/2$. By the Pythagorean theorem and by Corollary 4.16 dist$(y_0, w_0)^2 < 18/4 + 1 = 5.5$. Since dist$(y_0, w_0)^2$ is an integer, we have dist$(y_0, w_0)^2 \leq 5$, which is the desired result. $\qquad$ Q.E.D.

By the same method we can also verify Proposition 4.4. Indeed it is easy to check the following lemmas.

Lemma 4.17. *The system of equalities and inequalities*

$$(4.3) \qquad \begin{cases} x^2 = \sum_{i=1}^{10} y_i^2 + 2 \\ 3x = \sum_{i=1}^{10} y_i \\ x \geq y_1 + y_2 + y_3 \\ y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_{10} \end{cases}$$

is satisfied by integers $x, y_1, \cdots, y_{10}$ with $x \leq 10$ if and only if $(x, y_1, \cdots, y_{10}) = (6, 2, 2, \cdots, 2, 1, 1)$.

Lemma 4.18. (1) *For every point $y \in T_2 \cap H_{11}$,* dist$(y, P_{10}) < 1$. (2) *If an*

integral solution of (4.3) *satisfies* $x \geq 11$, *then there exist integers* z, ε, δ_1, $\cdots$, δ_{10} *satisfying*

(4.4)
$$
\begin{cases}
\langle 1 \rangle & \varepsilon \geq \delta_1 + \delta_2 + \delta_3 \\
\langle 2 \rangle & \delta_1 \geq \delta_2 \geq \cdots \geq \delta_9 \\
\langle 3 \rangle & z + \delta_9 \geq \delta_{10} \\
\langle 4 \rangle & \delta_{10} > 0 \\
\langle 5 \rangle & \delta_1 + \delta_2 + \cdots + \delta_{10} = 3\varepsilon \\
\langle 6 \rangle & 2z(\delta_1 + \delta_2 + \cdots + \delta_9) + (\delta_1^2 + \delta_2^2 + \cdots + \delta_9^2) + \delta_{10}^2 \\
& \quad = 6\varepsilon z + \varepsilon^2 - 2 \\
\langle 7 \rangle & \varepsilon^2 + \sum_{i=1}^{10} \delta_i^2 \leq 5
\end{cases}
$$

such that $x = 3z + \varepsilon$, $y_i = z + \delta_i$ $(1 \leq i \leq 9)$, $y_{10} = \delta_{10}$.

Lemma 4.19. *If* ε, δ_1, $\cdots$, δ_{10} *are* 0 *or* ± 1, *then the solution of* (4.4) *is* $z = 2$, $\varepsilon = 0$, $\delta_1 = \cdots = \delta_8 = 0$, $\delta_9 = -1$, $\delta_{10} = 1$.

Lemma 4.20. *Assume that one of* ε, δ_1, $\cdots$, δ_{10} *is* ± 2, *at most one of them is* ± 1, *and the rest are* 0. *Then* (4.4) *has no solution.*

Here we complete the proof of Proposition 4.3 and Proposition 4.4.

§5. The Action of the Weyl Group

In this section we give the proof to the main part of our main theorems.

Let $X \subset P^3$ be a normal quartic surface [resp. Let $\pi: X \to P^2$ be a branched double covering over P^2 branching along a reduced sextic curve B.] with a singularity $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} at $x_0 \in X$. We assume that other singularities on X than $x_0 \in X$ are rational double points. Let $\rho: Z \to X$ be the minimal resolution of singularities. Let $D = \rho^{-1}(x_0)$. Then for a suitably chosen α and ι, $\mathcal{Z} = (Z, D, \alpha, \iota)$ is a marked rational surface of degree -1. (Cf. Lemma 1.4, Proposition 1.5, Definition 2.4.) Moreover by exchanging α by αw with a suitable $w \in W_P$, we can assume that either $\alpha(\lambda_1) = L$ or $\alpha(\lambda_2) = L$ holds, where $\lambda_1 = 7\varepsilon_0 - 3\varepsilon_1 - 2\varepsilon_2 - \cdots - 2\varepsilon_{10}$, $\lambda_2 = 9\varepsilon_0 - 3\varepsilon_1 - \cdots - 3\varepsilon_8 - 2\varepsilon_9 - \varepsilon_{10}$ and $L = \rho^* \mathcal{O}_{P^3}(1)$. (Cf. Proposition 4.3.) [resp. we can assume that $\alpha(\lambda_3) = \rho^* \pi^* \mathcal{O}_{P^3}(1) = L$ holds where $\lambda_3 = 6\varepsilon_0 - 2\varepsilon_1 - 2\varepsilon_2 - \cdots - 2\varepsilon_8 - \varepsilon_9 - \varepsilon_{10}$. (Cf. Proposition 4.4)]. Since the restriction of L to D is trivial, the characteristic homomorphism $\varphi_{\mathcal{Z}}: \Gamma \to E$ satisfies $\varphi_{\mathcal{Z}}(\lambda_i) = 0$ and belongs to the subset $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_i, E)$ of $\mathrm{Hom}(\Gamma, E)$ where $i = 1$ or 2 according as $\alpha(\lambda_1) = L$ or $\alpha(\lambda_2) = L$. [resp. the characteristic homomorphism $\varphi_{\mathcal{Z}}: \Gamma \to E$ satisfies $\varphi_{\mathcal{Z}}(\lambda_3) = 0$ and belongs to the subset $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_3, E)$ of $\mathrm{Hom}(\Gamma, E)$.]

(Cf. Definition 2.6). Furthermore the kernel Ker $\varphi_{\mathscr{Z}}$ contains no element $\mu \in \Gamma$ with $\mu^2=0$ and $\mu \cdot \lambda_i=2$. $(i=1, 2)$ (Cf. Theorem 3.25.) [resp. the kernel Ker $\varphi_{\mathscr{Z}}$ contains no element $\mu \in \Gamma$ with $\mu^2=0$ and $\mu \cdot \lambda_3=1$. (Cf. Theorem 3.28.)]

Conversely for a fixed $i=1$ or 2 choose an element $\varphi \in \mathrm{Hom}(\Gamma, E)$ such that

(1) $\varphi(\lambda_i)=0$ and

(2) Ker φ contains no element μ with $\mu^2=0$ and $\mu \cdot \lambda_i=2$.

[resp. Conversely choose an element $\varphi \in \mathrm{Hom}(\Gamma, E)$ such that

(1) $\varphi(\lambda_3)=0$ and

(2) Ker φ contains no element μ with $\mu^2=0$ and $\mu \cdot \lambda_3=1$.] Then by theorem 2.8 there exists a marked rational surface $\mathscr{Z}=(Z, D, \alpha, \iota)$ with $\varphi=\varphi_{\mathscr{Z}}$. Exchanging α by $w\alpha$ where $w \in W_s$ is an element of the Weyl group associated to nodal roots, we can assume that $\alpha(\lambda_i) \in V_s \cap C_+$ [resp. $\alpha(\lambda_3) \in V_s \cap C_+$] and $\varphi=\varphi_{\mathscr{Z}}$, since $V_s \cap C_+$ is a fundamental domain of W_s. By Proposition 4.1 and since it follows from the above condition that $L|_D \cong \mathcal{O}_D$ for $L=\alpha(\lambda_i) \in \mathrm{Pic}(Z)$ [resp. $L=\alpha(\lambda_3) \in \mathrm{Pic}(Z)$], the line bundle L is a polarization of Z. Moreover by the above condition (2) and by Theorem 3.25, L defines a morphism $\varphi_L: Z \to X \subset \boldsymbol{P}^3$ to a normal quartic surface [resp. Moreover by the above condition (2) and by Theorem 3.28, L defines a morphism $\Phi: Z \to X \subset \boldsymbol{P}(1, 1, 1, 3)$ to a branched double covering over $\boldsymbol{P}^2$ branching along a reduced sextic curve B] with singularity $\tilde{E}_8$, $T_{2,3,7}$, or E_{12} according as E is an elliptic curve, C^* or C.

Note that by Proposition 3.31, singularities on X are described by $\Pi \cap$ Ker$\varphi_{\mathscr{Z}} \cap (\boldsymbol{Z}\lambda_i)^{\perp}$ $(i=1, 2, 3)$ where Π is the set of roots in P and $(\boldsymbol{Z}\lambda_i)^{\perp}$ is the orthogonal complement of λ_i in $\Gamma=(\boldsymbol{Z}\kappa)^{\perp}$.

Thus classification of singularities of surfaces under consideration is reduced to studying the abelian group $\mathrm{Hom}(\Gamma/\boldsymbol{Z}\lambda_i, E)$. $(i=1, 2, 3)$

Let Λ denote the orthogonal complement of $\boldsymbol{Z}\lambda_i$ in Γ. We define a homomorphism

$$u: \Gamma \to \mathrm{Hom}(\Lambda, \boldsymbol{Z}) = \Lambda^*$$

by $u(\alpha)(\xi)=\alpha \cdot \xi$ for $\alpha \in \Gamma$ and $\xi \in \Lambda$. It is easy to see that its kernel is $\Lambda^{\perp}=\boldsymbol{Z}\lambda_i$ and it is surjective since Γ is a unimodular lattice. Thus it induces an isomorphism $\bar{u}: \Gamma/\boldsymbol{Z}\lambda_i \xrightarrow{\sim} \Lambda^*$. In what follows we sometimes consider $\psi \in \mathrm{Hom}(\Lambda^*, E)$ instead of $\varphi \in \mathrm{Hom}(\Gamma, E)$ with $\varphi(\lambda_i)=0$. Since $\bar{u}$ is bijective they are equivalent. Note that the composition $\Lambda \to \Gamma \to \Gamma/\boldsymbol{Z}\lambda_i \xrightarrow{\sim} \Lambda^*$ is injective since $\Lambda \cap \boldsymbol{Z}\lambda_i=\{0\}$. We regard Λ as a subset of Λ^* by this injective mapping. Conversely Λ^* is regarded as a subset of $\Lambda \otimes \boldsymbol{Q}$. We can define a bilinear form

on Λ^* with values in rational numbers by extending that on Λ. For any element $0 \neq \theta \in \Lambda \otimes Q$, the reflection s_θ with respect to the hyperplane orthogonal to θ is defined by $s_\theta(x) = x - \dfrac{2(x \cdot \theta)}{(\theta \cdot \theta)} \theta$ for $x \in \Lambda \otimes Q$. It is an automorphism of order 2 preserving the linear form. (In what follows an affine automorphism of order 2 of an affine space whose set of fixed points has codimension 1 is called a *reflection*.)

Now we would like to give a remark. Let A be an arbitrary abelian group. When a group G acts on Λ we define an action of G on $\operatorname{Hom}(\Lambda, A)$ by $(gF)(\xi) = F(g^{-1}(\xi))$ for $g \in G$, $F \in \operatorname{Hom}(\Lambda, A)$, and $\xi \in \Lambda$. With this definition the inclusion $\Lambda \hookrightarrow \Lambda^*$ is an equivalent homomorphism if the action preserves the bilinear form.

Next we consider the case concerning $\lambda_1 = 7\varepsilon_0 - 3\varepsilon_1 - 2\varepsilon_2 - \cdots - 2\varepsilon_{10}$. Set $\Xi_1 = Zr_1 + Zr_3 + Zr_4 + Zr_5 + Zr_6 + Zr_7 + Zr_8 + Zr_9 + Zr_{10}$. ($r_2$ does not appear.) It is easy to see that the orthogonal complement of $Z\lambda_1$ in Γ is Ξ_1 (i.e., $\Lambda = \Xi_1$) and that Ξ_1 is the root lattice of type D_9.
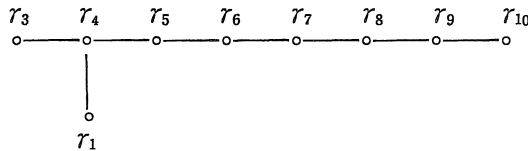
Let W_{Ξ_1} be the group generated by $s_{\gamma_1}, s_{\gamma_3}, \cdots, s_{\gamma_{10}}$. It is the Weyl group of type D_9. W_{Ξ_1} acts on Ξ_1 and Ξ_1^*. Set $\omega_1 = \dfrac{1}{4}r_1 - \dfrac{1}{4}r_3 + \dfrac{1}{2}r_4 + \dfrac{1}{2}r_6 + \dfrac{1}{2}r_8 + \dfrac{1}{2}r_{10}$.

We can check that $\Xi_1^* = \Xi_1 + Z\omega_1$. Set $\theta_1 = \dfrac{1}{2}r_1 - \dfrac{1}{2}r_3$. One can see easily $\theta_1 \in \Xi_1^*$ and $\theta_1^2 = -1$. Moreover $2\theta_1 \cdot \Xi_1^* \subset Z$ since $\theta_1 \cdot \omega_1 = -\dfrac{1}{2}$ and $\theta_1 \cdot \Xi_1 \subset Z$. Note that it implies that the reflection $s_{\theta_1}(x) = x + 2(x \cdot \theta_1)\theta_1$ defines a homomorphism Ξ_1^* to Ξ_1^*. Let G_1 be the subgroup of the orthogonal group of Ξ_1^* generated by $s_{\theta_1}, s_{\gamma_3}, s_{\gamma_4}, s_{\gamma_5}, s_{\gamma_6}, s_{\gamma_7}, s_{\gamma_8}$ and $s_{\gamma_{10}}$. The group G_1 is the Weyl group of type B_9 since the mutual intersection numbers of $\theta_1, r_3, \cdots, r_{10}$ give the following Dynkin graph.
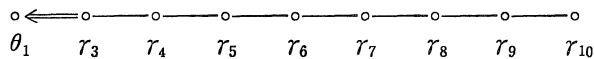
Lemma 5.1. *Every element $\xi \in \Xi_1^*$ with $\xi^2 = -1$ is conjugate to θ_1 with respect to the action of G_1. Moreover every element $\xi \in \Xi_1^*$ with $\xi^2 = -2$ is conjugate to r_3 with respect to the action of W_{Ξ_1}.*

Proof. We first show that every element $\xi \in \varXi_1^*$ with $\xi^2 = -1$ or $\xi^2 = -2$ belongs to the free submodule $\varGamma'$ generated by $\theta_1, r_3, r_4, \cdots, r_{10}$. Otherwise we have an element $y \in \varGamma'$ with $\xi = y + \omega_1$ since $[\varXi_1^* : \varGamma'] = 2$. It is easily checked that the restriction of the intersection form to $\varGamma'$ has values in $\mathbb{Z}$. Thus y^2 and $2y \cdot \omega_1$ are integers since $2\omega_1 \in \varGamma'$. It follows that $\omega_1^2 = \xi^2 - y^2 - 2y \cdot \omega_1$ is an integer. However we have $\omega_1^2 = -9/4$, a contradiction. Secondly we show that every element $\xi \in \varXi_1^*$ with $\xi^2 = -2$ belongs to $\varXi_1$. We may assume that $\xi \in \varGamma'$. Assume moreover that $\xi \notin \varXi_1$. Then we have an element $z \in \varXi_1$ with $\xi = z + \theta_1$ since $[\varGamma' : \varXi_1] = 2$. It follows that $\theta_1^2 = \xi^2 - z^2 - 2\theta_1 \cdot z$ is an even integer. However $\theta_1^2 = -1$, which is a contradiction. Since $\varGamma'$ and $\varXi_1$ are the root lattices of type B_9 and D_9 respectively one obtains the desired claim by the theory of root systems. Q.E.D.

Corollary 5.2. (1) *Every element* $r \in \varXi_1 \subset \varGamma$ *with* $r^2 = -2$ *is a root.* (*Recall that an element* $r \in \varGamma$ *conjugate to some* r_i $(1 \leq i \leq 10)$ *with respect to* W_P *is called a root.*)

(2) *For every element* $\theta \in \varXi_1^*$ *with* $\theta^2 = -1$, *the reflection* s_θ *belongs to* G_1.

(3) *For every element* $\theta \in \varXi_1^*$ *with* $\theta_1^2 = -1$, *we have an element* $\xi \in \varXi_1^*$ *with* $2\xi \cdot \theta = 1$.

(4) *For every element* $\eta \in \varXi_1^*$ *with* $\eta^2 = -2$, *we have an element* $\xi \in \varXi_1^*$ *with* $\xi \cdot \eta = 1$.

Proof. (1) Since $G_1 \subset W_P$ it is obvious.

(2) There is $g \in G_1$ with $\theta = g(\theta_1)$. Thus $s_\theta = g s_{\theta_1} g^{-1} \in G_1$.

(3) Since $2(\omega_1 + r_3) \cdot \theta_1 = 1$, $2g(\omega_1 + r_3) \cdot \theta = 1$ for $\theta = g(\theta_1)$.

(4) We can assume that $\eta = g(r_3)$ for $g \in G_1$. Then $g(r_4)$ has the desired property. Q.E.D.

Let $\varPi_1$ be the set of all elements $\xi \in \varXi_1^*$ with $\xi^2 = -1$ or -2. $\varPi_1$ is the root system of type B_9. $\varXi_1$ is identified with the co-root lattice $Q(\varPi_1^\vee)$, i.e., the free module generated by co-roots. $\varXi_1^*$ is the weight lattice $P(\varPi_1)$. Moreover $\varGamma' = Q(\varPi_1) = P(\varPi_1^\vee)$.

Let us proceed to the case concerning to $\lambda_2 = 9\varepsilon_0 - 3\varepsilon_1 - 3\varepsilon_2 - \cdots - 3\varepsilon_8 - 2\varepsilon_9 - \varepsilon_{10}$. Set $\varXi_2 = \mathbb{Z}r_1 + \mathbb{Z}r_2 + \mathbb{Z}r_3 + \mathbb{Z}r_4 + \mathbb{Z}r_5 + \mathbb{Z}r_6 + \mathbb{Z}r_7 + \mathbb{Z}r_8$ and $\omega_2 = 3\varepsilon_0 - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 - \varepsilon_8 - 2\varepsilon_9 + \varepsilon_{10}$. $\varXi_2$ is the root lattice of type E_8 and it is easy to see that the orthogonal complement $\varLambda$ of $\mathbb{Z}\lambda_2$ in $\varGamma$ is the orthogonal
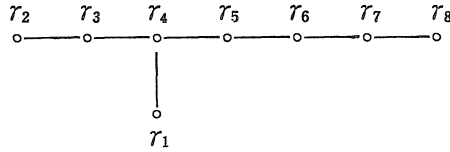
direct sum of $\mathbb{Z}\omega_2$ and $\mathcal{Z}_2$, i.e., $\Lambda = \mathbb{Z}\omega_2 + \mathcal{Z}_2$. Thus we have $\Lambda^* = \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$, since $\omega_2^2 = -4$. Let G_2' be the Weyl group of type E generated by s_{γ_1}, s_{γ_2}, s_{γ_3}, s_{γ_4}, s_{γ_5}, s_{γ_6}, s_{γ_7} and s_{γ_8}. G_2' acts on $\mathbb{Z}\omega_2$ trivially. Let T be a cyclic group of order 2 generated by the reflection $s_{(\omega_2/2)}$ acting on $\Lambda^* = \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$. T acts on $\mathcal{Z}_2^*$ trivially and acts on $\mathbb{Z}(\omega_2/4)$ as the change of the sign; $\alpha \rightarrow -\alpha$. We set $G_2 = T \times G_2'$.

Lemma 5.3. (1) *If* $\theta^2 = -1$ *for* $\theta \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$, *then* $\theta = \pm\omega_2/2$.

(2) *If* $\eta^2 = -2$ *for* $\eta \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$, *then* $\eta \in \mathcal{Z}_2^*$ *and such an element* η *is conjugate to each other with respect to the action of* G_2'.

Proof. (1) Set $\theta = (m\omega_2/4) + \xi$ with $m \in \mathbb{Z}$, $\xi \in \mathcal{Z}_2^*$. We have $-1 = -(m^2/4) + \xi^2$ since $\omega_2^2 = -4$. Since ξ^2 is a negative even integer unless $\xi = 0$, one sees that $m = \pm 2$ and $\xi = 0$.

(2) We set $\eta = (m\omega_2/4) + \xi$ with $m \in \mathbb{Z}$, $\xi \in \mathcal{Z}_2^*$. We have $-2 = -(m^2/4) + \xi^2$. Thus $m = 0$ and $\eta \in \mathcal{Z}_2^*$ since ξ^2 is a non-positive even integer and since $8 = 2 \times 4$ is not a square of any integer. Every element $\eta \in \mathcal{Z}_2^*$ with $\eta^2 = -2$ is conjugate with respect to G_2' since $\mathcal{Z}_2^*$ is the root lattice of type E_8. Q.E.D.

Corollary 5.4. (1) *Every element* $\gamma \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^* \subset \Gamma$ *with* $\gamma^2 = -2$ *is a root.*

(2) *For every element* $\theta \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ *with* $\theta^2 = -1$, *the reflection* s_θ *belongs to* T.

(3) *For every element* $\theta \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ *with* $\theta^2 = -1$, *we have an element* $\xi \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ *with* $2\xi \cdot \theta = 1$.

(4) *For every element* $\eta \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ *with* $\eta^2 = -2$ *we have an element* $\xi \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ *with* $\xi \cdot \eta = 1$.

Let Π_2 be the set of elements $\xi \in \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ with $\xi^2 = -1$ or -2. Π_2 is the root system of type $A_1 + E_8$. The irreducible component of type A_1 is consisted of $\{\pm\omega_2/2\}$ and they are regarded as short roots compared with those in the system of type E_8. Equalities $Q(\Pi_2^\vee) = \mathbb{Z}\omega_2 + \mathcal{Z}_2^*$, $Q(\Pi_2) = P(\Pi_2^\vee) = \mathbb{Z}(\omega_2/2) + \mathcal{Z}_2^*$, $P(\Pi_2) = \mathbb{Z}(\omega_2/4) + \mathcal{Z}_2^*$ holds.

Lemma 5.5. *Assume* $i = 1$ *or* 2. *Let* Λ *be the orthogonal complement of*

$Z\lambda_i$ in Γ. The following conditions are equivalent for $\psi \in \mathrm{Hom}(\Lambda^*, E)$.

(a) There exists an element $\mu \in \Gamma$ with $\mu^2 = 0$, $\mu \cdot \lambda_i = 2$ and $\psi u(\mu) = 0$.

(b) There exists an element $\theta \in \Lambda^*$ with $\theta^2 = -1$ and $\psi(\theta) = 0$.

(c) There exists an element $\theta \in \Lambda^*$ with $\theta^2 = -1$ such that $s_\theta(\psi) = \psi$.

Proof. (a)$\Rightarrow$(b). Recall the definition of u. Since $\Gamma \subset Z(\lambda_i/4) + \Lambda^*$, every element $\alpha \in \Gamma$ can be written uniquely as $\alpha = (m\lambda_i/4) + \alpha'$ with $m \in Z$, $\alpha' \in \Lambda^*$. Then $\alpha' = u(\alpha)$. Thus setting $\theta = u(\mu)$, we have $\mu = (\lambda_i/2) + \theta$ since $\mu \cdot \lambda_i = 2$. We have $\theta^2 = ((\lambda_i/2) - \mu)^2 = 1 - 2 + 0 = -1$ and $\psi(\theta) = \psi u(\mu) = 0$.

(b)$\Rightarrow$(a). Since u is surjective, there is an elemnt $\mu' \in \Gamma$ with $\theta = u(\mu')$. Then there is an integer $m \in Z$ with $\mu' = (m\lambda_i/4) + \theta$. We have $(\mu')^2 = m^2/4 - 1$, which implies that $m = 4n + 2$ for some integer n, since $(\mu')^2$ is an even integer. (Γ is an even lattice.) Set $\mu = \mu' - n\lambda_i$. Then $\mu \in \Gamma$, $\mu^2 = 0$, $\mu \cdot \lambda_i = 2$ and $\psi u(\mu) = 0$.

(b)$\Rightarrow$(c). If (b) is satisfied, then for $x \in \Lambda^*$, $(s_\theta(\psi))(x) = \psi(s_\theta(x)) = \psi(x + 2(x \cdot \theta)\theta) = \psi(x) + 2(x \cdot \theta)\psi(\theta) = \psi(x)$.

(c)$\Rightarrow$(b). Note that there is an element $\xi \in \Lambda^*$ with $2\xi \cdot \theta = 1$. (Corollary 5.2, Corollary 5.4.) If (c) is satisfied, then $\psi(\xi) = \psi s_\theta(\xi) = \psi(\xi) + \psi(\theta)$. Thus $\psi(\theta) = 0$.   Q.E.D.

The above lemma implies that the criterion for whether the marked rational surface can be realized as a quartic surface or not can be interpreted with group-theoretic words.

To help reader's understanding we write down one more lemma.

Lemma 5.6. *For every element $\gamma \in \Lambda$ with $\gamma^2 = -2$, the following conditions are equivalent.*

(a) $\psi u(\gamma) = 0$.

(b) $\psi(\gamma) = 0$.

(c) $s_\gamma(\psi) = \psi$.

Proof. Here we only give the proof of (c)$\Rightarrow$(b). The other parts are trivial. Recall that there is an element $\xi \in \Lambda^*$ with $\xi \cdot \gamma = 1$. (Corollary 5.2, Corollary 5.4) If (c) is satisfied, then $\psi(\xi) = \psi s_\gamma(\xi) = \psi(\xi) + \psi(\gamma)$. Thus $\psi(\gamma) = 0$.   Q.E.D.

Summing up the above results we have the following proposition.

Proposition 5.7. *Assume $i = 1$ or 2. Let Λ be the orthogonal complement of $Z\lambda_i$ in Γ and $u: \Gamma \to \Lambda^*$ be the canonical surjection. Let G_i be the group generated by all reflections s_η corresponding to elements $\eta \in \Lambda^*$ with $\eta^2 = -1$ or -2. The following conditions are equivalent for $\psi \in \mathrm{Hom}(\Lambda^*, E)$.*

(A)   *There exists a marked rational surface* $\mathcal{Z}=(Z, D, \alpha, \iota)$ *over E of degree* -1 *such that*

(i)   *the characteristic homomorphism* $\varphi_{\mathcal{Z}}$ *of* $\mathcal{Z}$ *coincides with* ψu;

(ii)   *the line bundle* $L=\alpha(\lambda_i)$ *defines a generically one-to-one morphism* $\varphi_L\colon Z\to X\subset \mathbb{P}^3$ *to a normal quartic surface X; and*

(iii)   *the combination of singularities on X is a unique* $\tilde{E}_8$, $T_{2,3,7}$, *or* E_{12} (*It depends on whether E is an elliptic curve,* $\mathbb{C}^*$, *or* $\mathbb{C}$.) *plus a combination of rational double points associated to the set of Dynkin graphs* $\sum p_k A_k + \sum q_l D_l + \sum r_m E_m$.

(B)   *The kernel* Ker ψ *contains no element* $\theta\in \Lambda^*$ *with* $\theta^2=-1$ *and the set of elements* $\eta\in \Lambda^*$ *with* $\eta^2=-2$, $\psi(\eta)=0$ *is the root system of type* $\sum p_k A_k + \sum q_l D_l + \sum r_m E_m$.

(C)   *The isotropy group* $I_{G_i}(\psi)=\{g\in G_i \,|\, g(\psi)=\psi\}$ *of* ψ *with respect to* G_i *contains no reflections associated to any element* $\theta\in \Lambda^*$ *with* $\theta^2=-1$ *and moreover the maximal subgroup of* $I_{G_i}(\psi)$ *generated by reflections is the Weyl group of type* $\sum p_k A_k + \sum q_l D_l + \sum r_m E_m$.

Remark.   The group G_1 is the Weyl group of type B_9 and G_2 is the Weyl group of type A_1+E_8. In the latter case the irreducible component of type A_1 corresponds to the elements $\theta\in \Lambda^*$ with $\theta^2=-1$.

Now our classification is reduced to the classification of subgroups of G_i which can be realized as the maximal subgroup generated by reflections of $I_{G_i}(\psi)=\{g\in G_i \,|\, g(\psi)=\psi\}$ for some $\psi\in \mathrm{Hom}(\Lambda^*, E)$.

Definition 5.8.   The following procedure which associates a root system R to its root subsystem R' is called the *elementary transformation* of the root system.

(1)   We divide R into the direct sum of irreducible root system, say $R=\bigoplus_i R_i$.

(2)   We choose a fundamental system of roots for every i, say $\Delta_i\subset R_i$.

(3)   For every i, we choose a proper subset $\tilde{\Delta}_i$ of the union $\Delta_i\cup\{-\eta_i\}$ where η_i is the highest root associated to Δ_i.

(4)   We set $R'=\bigoplus_i R_i'$ where R_i' is the root system generated by $\tilde{\Delta}_i$.

Proposition 5.9.   *When E is an irreducible smooth elliptic curve (resp.* $\mathbb{C}^*$), *the following conditions are equivalent for any subgroup H of the Weyl group* $W=W(R)$ *associated to a fixed root system R. We denote by Q the co-root lattice of R, i.e., the free* $\mathbb{Z}$-*module generated by co-roots* $\{\eta^\vee \,|\, \eta\in R\}$.

(1)   *The group H coincides with the maximal subgroup generated by reflections*

of the isotropy group $I_W(\psi)$ for some $\psi \in Q \otimes E$.

(2) *The group H is generated by a set of reflections $\{s_\eta \mid \eta \in R''\}$ where R'' is a root subsystem of R which is obtained by elementary transformations repeated twice (resp. only once.) from R.*

Proof. Let $\hat{Q}$ be the root lattice of R. The vector space $Q \otimes R$ is regarded as the dual space of $\hat{Q} \otimes R$. We denote the canonical pairing $Q \otimes R \times \hat{Q} \otimes R \to R$ by $\langle \ , \ \rangle$.

We first assume that E is an elliptic curve. We have representation $E = C/Z + Z\tau$ where $\tau \in C$ and Im $\tau > 0$. We fix such repreentations. The covering mapping $\pi: C \to C/Z + Z\tau$ induces the covering mapping $\bar{\pi}: Q \otimes C \to Q \otimes E$. Set $\bar{W} = W \ltimes (Q \oplus Q)$ where $\ltimes$ denotes the semi-direct product with respect to the diagonal action of W to $Q \oplus Q$. (i.e., for $g \in W$, $(\xi', \xi'') \in Q \oplus Q$, $g(\xi', \xi'') = (g\xi', g\xi'')$.) The group $\bar{W}$ acts on $Q \otimes C$ by $(g, \xi', \xi'')(\psi' + \tau\psi'') = (g(\psi') + \xi') + \tau(g(\psi'') + \xi'')$ where $g \in W$, ξ', $\xi'' \in Q$ and ψ', $\psi'' \in Q \otimes R$. We have a canonical isomorphism of isotropy groups. $I_{\bar{W}}(\bar{\psi}) \cong I_W(\bar{\pi}(\bar{\psi}))$ for $\bar{\psi} \in Q \otimes C$. Thus we can consider the action of $\bar{W}$ on $Q \otimes C$ instead of that of W on $Q \otimes E$.

Set $W_a = W \ltimes Q$. The group W_a is the affine Weyl group of R. We have a diagram

$$
\begin{array}{ccc}
\bar{W} & \xrightarrow{\rho_2} & W_a \\
\rho_1 \downarrow & & \downarrow \nu_2 \\
W_a & \xrightarrow{\nu_1} & W
\end{array}
$$

where $\rho_1(g, \xi', \xi'') = (g, \xi')$, $\rho_2(g, \xi', \xi'') = (g, \xi'')$ and $\nu_i(g, \xi') = g$ ($i = 1, 2$). Set $\bar{\psi} = \psi' + \tau\psi''$ with $\psi', \psi'' \in Q \otimes R$. Let $(g, \xi') \in I_{W_a}(\psi')$. We have $g(\psi') + \xi' = \psi'$ and one sees that ξ' is uniquely determined by g and ψ'. Thus the restriction $\nu_1 | I_{W_a}(\psi')$ of ν_1 is injective. Set $J(\psi') = \nu_1(I_{W_a}(\psi'))$. $J(\psi')$ is isomorphic to $I_{W_a}(\psi')$ and $\nu_2^{-1} J(\psi') = J(\psi') \ltimes Q$ is isomorphic to $\rho_1^{-1} I_{W_a}(\psi')$ via ρ_2. We have

(5.1) $$I_{\bar{W}}(\bar{\psi}) = \rho_1^{-1} I_{W_a}(\psi') \cap \rho_2^{-1}(\psi'') \cong I_{J(\psi') \ltimes Q}(\psi'') .$$

We claim here that there is a root subsystem R' of R which is obtained from R by one elementary transformation and $J(\psi')$ is the Weyl group generated by $\{s_\eta \mid \eta \in R'\}$ and that conversely for any root subsystem R' obtained by one elementary transformation from R, there is a point $\psi' \in Q \otimes R$ such that $J(\psi')$ coincides with the Weyl group generated by $\{s_\eta \mid \eta \in R'\}$.

To see this recall that the action of W_a on $Q \otimes R$ has a fundamental domain C_0. C_0 is called a small Weyl chamber. (Cf. Bourbaki [3]). Since every small Weyl chamber is conjugate we can assume that $\psi' \in \bar{C}_0$. ($^-$ denotes the closure.) Now let s_H denote the reflection of $Q \otimes R$ in W_a whose set of fixed points coincides with a hyperplane H. Let M be the set of all hyperplanes H with $s_H \in W_a$. The domain C_0 is a connected component of $Q \otimes R - \bigcup_{H \in M} H$. Set $M_0 = \{H \in M \mid \dim(H \cap \bar{C}_0) = \dim H\}$. M_0 is the set of walls of the small chamber C_0. It is known that for every $H \in M_0$ there is a unique root $\eta \in R$ perpendicular to H and such that $\langle x, \eta \rangle > 0$ for $x \in C_0$. We denote it by $\eta(H)$. Let $R = \bigoplus_i R_i$ be the decomposition into irreducible root systems. Then there is a fundamental system of roots $\Delta_i \subset R_i$ for each i such that the union $\bigcup_i \Delta_i \cup \{-\eta_i\}$ coincides with the set $\{\eta(H) \mid H \in M_0\}$ where η_i is the highest root of R_i associated to Δ_i. Let $M_0(\psi') = \{H \in M_0 \mid \psi' \in H\}$. It is the set of walls of C_0 passing through ψ'. Then it is also known that the isotropy group $I_{W_a}(\psi')$ coincides with the subgroup of W_a generated by $\{s_H \mid H \in M_0(\psi')\}$, the set of reflections corresponding to walls of C_0 passing through ψ'. Since the intersection of all walls of the small Weyl chamber of an irreducible root system is empty, for every i, $(\Delta_i \cup \{-\eta_i\}) \cap \{\eta(H) \mid H \in M_0(\psi')\}$ is a proper subset of $\Delta_i \cup \{-\eta_i\}$. Let R' be the root system generated by $\{\eta(H) \mid H \in M_0(\psi')\}$, the set of roots perpendicular to some wall of C_0 passing through ψ' and in the direction of the inside of C_0. By the construction R' is the one obtained by one elementary transformation from R and $J(\psi')$ is the Weyl group generated by $\{s_\eta \mid \eta \in R'\}$.

Conversely let R' be a root subsystem of $R = \bigoplus_i R_i$ obtained by one elementary transformation from R. Choosing the fundamental system of $\Delta_i \subset R_i$ of the irreducible root system R_i is equal to choosing a Weyl chamber C_i of $W(R_i)$ in $Q_i \otimes R$ where Q_i is the co-root lattice of R_i. Let C_{i0} be the small Weyl chamber contained in C_i and such that $0 \in \bar{C}_{i0}$, which is the fundamental domain of $W_a(R_i) = W(R_i) \ltimes Q_i$. Let $M_{i0} = \{H:$ hyperplane in $Q_i \otimes R \mid s_H \in W_a(R_i),$ $\dim(H \cap \bar{C}_{i0}) = \dim H\}$. M_{i0} is the set of walls of C_{i0}. Then the set $\{\eta(H) \mid H \in M_{i0}\}$ coincides with $\Delta_i \cup \{-\eta_i\}$ where η_i is the highest root. For the specified proper subset $\tilde{\Delta}_i$ of $\Delta_i \cup \{-\eta_i\}$ let ψ'_i be a general point in the intersection $\cap \{H \mid H \in M_{i0}, \eta(H) \in \tilde{\Delta}_i\}$. The isotropy group $I_{W_a(R_i)}(\psi'_i)$ coincides with the Weyl group generated by $\{s_\eta \mid \eta \in R'_i\}$ where R'_i is the root system generated by $\tilde{\Delta}_i$. Let ψ' be the image of $\bigoplus \psi'_i$ by the inclusion $\bigoplus Q_i \otimes R \subset Q \otimes R$. One knows that the isotropy group $I_{W_a}(\psi')$ is the Weyl group generated by $\{s_\eta \mid \eta \in$

$\oplus R'_i = R'\}$.   Thus we have the above claim.

In what follows we assume that $\psi' \in Q \otimes R$ and R' has the relation mentioned in the above claim.

Let Q' be the co-root lattice associated to R'. Then $J(\psi') \ltimes Q'$ is the affine Weyl group associated to R'. Thus applying the above claim to R' one sees that subgroups H of W with the property (2) in Proposition 5.9 coincide with subgroups which can be written as $I_{J(\psi') \ltimes Q'}(\psi'')$ for some ψ', $\psi'' \in Q \otimes R$. Therefore by the equality (5.1) and by the next lemma we conclude that (1) and (2) are equivalent when E is an elliptic curve.

Lemma 5.10. *Any reflection in $I_{J(\psi') \ltimes Q}(\psi'')$ belongs to $I_{J(\psi') \ltimes Q'}(\psi'')$.*
(Note that in general $Q \supsetneqq Q'$.)

Proof. Any reflection in $W \ltimes Q$ can be written as (s_η, ξ) where $\eta \in R$ and $\xi \in Q$. Assume $(s_\eta, \xi) \in I_{J(\psi') \ltimes Q}(\psi'')$. We have $\eta \in R'$ and $\psi'' - \langle \eta, \psi'' \rangle \eta^\vee + \xi = \psi''$. Thus $\xi = \langle \eta, \psi'' \rangle \eta^\vee$. Note that we have an element $w \in P(R)$ such that $\langle w, \eta^\vee \rangle = 1$. One sees that $\langle w, \xi \rangle = \langle \eta, \psi'' \rangle$ is an integer since $P(R)$ is the dual lattice of Q. Thus we have $\xi \in Q'$ and $(s_\eta, \xi) \in J(\psi') \ltimes Q'$.   Q.E.D.

Next assume $E = C^*$. Let $\pi: C \to C^*$ be the covering mapping. It induces the covering mapping $\bar{\pi}: Q \otimes C \to Q \otimes C^*$. If $\bar{\pi}(\bar{\psi}) = \psi$ then $I_{W_a}(\bar{\psi}) \cong I_W(\psi)$, where $W_a = W \ltimes Q$. Thus the problem is reduced to the classification of isotropy groups of the action by W_a to $Q \otimes C$. However note that the answer never changes by replacing C by R since the condition $g(\bar{\psi}) = \bar{\psi}$ for $g \in W_a$, $\bar{\psi} \in Q \otimes C$ is written with an affine equation whose coefficients are all real numbers.

Pick $\chi \in Q \otimes R$. Let C_0 be a small Weyl chamber whose closure contains χ. Then as mentioned above, $I_{W_a}(\chi)$ is the Weyl group generated by reflections associated to walls of C_0 passing through χ and moreover the set of generating reflections corresponds to a root system R' which is obtained by one elementary transformation from R.

We conclude the proof of both cases in Proposition 5.9.

The next proposition deals with the case $E = C$.

Proposition 5.11. *Let $W = W(R)$ be the Weyl group associated to a fixed root system R. Let Q be the co-root lattice of R. Then for any subgroup $H \subset W$, the following conditions are equivalent.*

(1) *For some $\psi \in Q \otimes C$, $H = I_W(\psi)$.*

(2) *For some fundamental system of roots $\Delta \subset R$ and for some subset $\Delta' \subset \Delta$, H*

is the Weyl group generated by $\{s_\eta \mid \eta \in R'\}$ *where* R' *is the root system generated by* Δ'.

Proof. For $g \in W$ and $\psi \in Q \otimes C$, the condition $g(\psi) = \psi$ is described by a linear equation whose coefficients are all real numbers. Therefore we can replace C by $\boldsymbol{R}$. Pick $\chi \in Q \otimes \boldsymbol{R}$. Let C be the Weyl chamber of W such that the closure of C contains χ. Let $\boldsymbol{M}$ be the set of hyperplanes $H \subset Q \otimes \boldsymbol{R}$ such that for some reflection in W its fixed-point-set equals to H. A connected component of $Q \otimes \boldsymbol{R} - \bigcup_{H \in M} H$ is C. Let $\boldsymbol{M}_0$ be the set of walls of C, i.e., $\boldsymbol{M}_0 = \{H \in \boldsymbol{M} \mid \dim H = \dim(H \cap \bar{\boldsymbol{C}})\}$. For $H \in \boldsymbol{M}_0$ we have a unique root $\eta \in R$ perpendicular to H and $\langle x, \eta \rangle > 0$ for $x \in C$. If we denote it by $\eta(H)$, the set $\{\eta(H) \mid H \in \boldsymbol{M}_0\}$ is a fundamental system of roots of R. Moreover it is known that choosing a Weyl chamber C is equivalent to choosing a fundamental system of roots. Set $\Delta' = \{\eta(H) \mid H \in \boldsymbol{M}_0, \chi \in H\}$. Δ' is the set of walls passing through χ. It is also known that $I_W(\chi)$ is the Weyl group generated by reflections $\{s_\eta \mid \eta \in R'\}$, where R' is the root system generated by Δ'. Thus (1) and (2) are equivalent.   Q.E.D.

Now by Proposition 5.7, Remark just after Proposition 5.7, Proposition 5.9 and Proposition 5.11, the main parts of Theorem 0.2, Theorem 0.3 and Theorem 0.4 are obvious.

Recall that the intersection numbers of elements in the union of a fundamental system Δ of an irreducible root system and (-1) times its associated highest root are described by the extended Dynkin graph. Thus the elementary transformation of root systems corresponds to the elementary transformation of the Dynkin graphs. The series (I) in Theorem 0.2, Theorem 0.3 and Theorem 0.4 corresponds to $\lambda_1 = 7\varepsilon_0 - \cdots - 2\varepsilon_{10}$ and the series (II) corresponds to $\lambda_2 = 9\varepsilon_0 - \cdots - \varepsilon_{10}$. However we did not necessarily use the expression containing B_9 or $A_1 + E_8$ in those theorems. We used a simpler expression to say the same contents.

The part left unproved is the following proposition.

Proposition 5.12. (*Umezu* [22]) *Assume that a normal quartic surface X has singularity $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} and that $\sum_{x \in X} p_g(X, x) \geqq 2$. Then X has only 2 singular points and both of them are of type $\tilde{E}_8$. Conversely a normal quartic surface with 2 singular points of type $\tilde{E}_8$ exists.*

However this is Y. Umezu's result.

Let us proceed further to the case of branched double coverings.

In this case it is obvious that the orthogonal complement Λ of $\mathbb{Z}\lambda_3$ is the orthogonal direct sum of $\mathbb{Z}r_{10}$ and $\mathcal{Z}_2=\mathbb{Z}r_1+\mathbb{Z}r_2+\mathbb{Z}r_3+\mathbb{Z}r_4+\mathbb{Z}r_5+\mathbb{Z}r_6+\mathbb{Z}r_7+\mathbb{Z}r_8$. $(\lambda_3=6\varepsilon_0-2\varepsilon_1-2\varepsilon_2-2\varepsilon_3-2\varepsilon_4-2\varepsilon_5-2\varepsilon_6-2\varepsilon_7-2\varepsilon_8-\varepsilon_9-\varepsilon_{10}.)$ $\mathcal{Z}_2$ is the root lattice of type E_8. Let Π_3 be the set of all elements $\xi\in\mathbb{Z}r_{10}+\mathcal{Z}_2$ with $\xi^2=-2$. Π_3 is the root system of type A_1+E_8. The lattice $\mathbb{Z}r_{10}+\mathcal{Z}_2$ is its root lattice and $\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2$ is its weight lattice. Moreover we have that $Q(\Pi_3)=Q(\Pi_3^\vee)=\mathbb{Z}r_{10}+\mathcal{Z}_2$ and $P(\Pi_3)=P(\Pi_3^\vee)=\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2=\Lambda^*$. Thus $\mathrm{Hom}(\Gamma/\mathbb{Z}\lambda_3, E)$ is identified with $\mathrm{Hom}(\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2, E)$. We denote by G_3 the Weyl group generated by $s_{\gamma_1}, s_{\gamma_2}, s_{\gamma_3}, s_{\gamma_4}, s_{\gamma_5}, s_{\gamma_6}, s_{\gamma_7}, s_{\gamma_8}, s_{\gamma_{10}}$. ($s_{\gamma_9}$ does not appear.) The group G_3 acts on $\mathbb{Z}r_{10}+\mathcal{Z}_2$ and $\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2$ and it is of type A_1+E_8.

The next lemma is easily checked.

Lemma 5.13. (1) *Every element* $r\in\mathbb{Z}r_{10}+\mathcal{Z}_2$ *with* $r^2=-2$ *is a root.*
(2) *For every* $r\in\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2$ *with* $r^2=-2$, *we have* $\xi\in\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2$ *with* $r\circ\xi=1$.

Thus Lemma 5.6 holds even when $i=3$.

Lemma 5.14. *The following conditions are equivalent for* $\psi\in\mathrm{Hom}(\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2, E)$.
(a) *There exists an element* $\mu\in\Gamma$ *with* $\mu^2=0$, $\mu\circ\lambda_3=1$ *and* $\psi u(\mu)=0$.
(b) $\pi_1(\psi)=0$ *where* π_1: $\mathrm{Hom}(\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2, E)\to\mathrm{Hom}(\mathbb{Z}(r_{10}/2), E)$ *is the projection.*

Proof. Let $\mu\in\Gamma$ be an element with $\mu^2=0$ and $\mu\circ\lambda_3=1$. Since $\Gamma\subset\mathbb{Z}(\lambda_3/2)+\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2$, we have an integer m and $\xi\in\mathcal{Z}_2$ such that $\mu=(\lambda_3/2)+(mr_{10}/2)+\xi$. (The coefficient of λ_3 is $1/2$ since $\mu\circ\lambda_3=1$.) It yields the equality $0=\mu^2=(1/2)-(m^2/2)+\xi^2$. Thus $m=\pm1$ and $\xi=0$ since ξ^2 is a negative integer unless $\xi=0$. One knows $\mu=(\lambda_3/2)\pm(r_{10}/2)$. Since $u(\mu)=\pm r_{10}/2$, we have the desired equivalence.   Q.E.D.

We have the following proposition.

Proposition 5.15. *The following conditions are equivalent for* $\psi\in$ $\mathrm{Hom}(\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2, E)$. *Let* π_1: $\mathrm{Hom}(\mathbb{Z}(r_{10}/2)+\mathcal{Z}_2, E)\to\mathrm{Hom}(\mathbb{Z}(r_{10}/2), E)$ *be the projection and* G_3 *be the Weyl group of the root lattice* $\mathbb{Z}r_{10}+\mathcal{Z}_2$. *($G_3$ is of*

type A_1+E_8.)

(A) *There exists a marked rational surface $\mathcal{Z}=(Z, D, \alpha, \iota)$ over E of degree -1 such that*

(i) *the characteristic homomorphism $\varphi_{\mathcal{Z}}$ of $\mathcal{Z}$ coincides with ψu;*

(ii) *the line bundle $L=\alpha(\lambda_3)$ defines a generically one-to-one morphism Φ: $Z\to X\subset\mathbb{P}(1, 1, 1, 3)$ to a branched double covering over $\mathbb{P}^2$ branching along a reduced sextic curve B; and*

(iii) *the combination of singularities on X is a unique $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} (It depends on whether E is an elliptic curve, $\mathbb{C}^*$ or $\mathbb{C}$.) plus a combination of rational double points associated to the set of Dynkin graphs $\sum p_k A_k + \sum q_l D_l + \sum r_m E_m$.*

(B) $\pi_1(\psi)\neq 0$ *and the set of elements $\eta\in\mathbb{Z}(r_{10}/2)+\Xi_2$ satisfying $\eta^2=-2$ and $\psi(\eta)=0$ is the root system of type $\sum p_k A_k + \sum q_l D_l + \sum r_m E_m$.*

(C) $\pi_1(\psi)\neq 0$ *and the maximal subgroup generated by reflections of the isotropy group $I_{G_3}(\psi)$ is the Weyl group of type $\sum p_k A_k + \sum q_l D_l + \sum r_m E_m$.*

Lemma 5.16. (1) *Assume that E is an elliptic curve or $\mathbb{C}^*$. If $\pi_1(\psi)=0$ for $\psi\in\mathrm{Hom}\,(\mathbb{Z}(r_{10}/2)+\Xi_2, E)$, then we have another element $\psi'\in\mathrm{Hom}\,(\mathbb{Z}(r_{10}/2)+\Xi_2, E)$ such that $\pi_1(\psi')\neq 0$ and $I_{G_3}(\psi')=I_{G_3}(\psi)$.*

(2) *Assume $E=\mathbb{C}$. Let G_3' be the subgroup of G_3 generated by $s_{\gamma_1}, s_{\gamma_2}, \cdots, s_{\gamma_8}$. If $\pi_1(\psi)\neq 0$ for $\psi\in\mathrm{Hom}\,(\mathbb{Z}(r_{10}/2)+\Xi_2, \mathbb{C})$, then $I_{G_3}(\psi)=I_{G_3'}(\psi)$.*

Proof. Let T be the cyclic group of order 2 generated by $s_{\gamma_{10}}$ and π_2: $\mathrm{Hom}\,(\mathbb{Z}(r_{10}/2)+\Xi_2, E)\to\mathrm{Hom}\,(\Xi_2, E)$ be the projection. Note that the equality $I_{G_3}(\psi)=I_T(\pi_1(\psi))\times I_{G_3'}(\pi_2(\psi))$ holds.

(1) Let $\chi\in\mathrm{Hom}\,(\mathbb{Z}(r_{10}/2)+\Xi_2, E)$ be the element with $\chi(\Xi_2)=0$, $\chi(r_{10})=0$ and $\chi(r_{10}/2)\neq 0$. If E is an elliptic curve or $\mathbb{C}^*$, such χ exists. The element $\psi'=\psi+\chi$ satisfies the above condition.

(2) If $E=\mathbb{C}$, then the condition $\chi(r_{10})=0$ and $\chi(r_{10}/2)=0$ are equivalent. Thus if $\pi_1(\psi)\neq 0$, then $I_T(\pi_1(\psi))$ is the trivial group.   Q.E.D.

The important parts of Theorem 0.5, Theorem 0.6 and Theorem 0.7 follow from Proposition 5.15, Lemma 5.16, Proposition 5.9 and Proposition 5.11.

The parts left unproved are disconnectedness of strata in $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)))$ and the case $\sum p_g(X, x)\geq 2$. As for the case $\sum p_g(X, x)\geq 2$, please see the last remark in this section.

The basis of disconnectedness is the following fact.

Fact 5.17. (Cf. Dynkin [7]) The root system R of type E_8 with the

action of the Weyl group $W(R)$ contains two non-conjugate root subsystems of the following types.

(1) A_7 (2) $2A_3$ (3) A_5+A_1 (4) A_3+2A_1 (5) $4A_1$

Moreover both of non-conjugate ones of each type can be obtained by elementary transformations repeated twice from R.

According to this fact one knows for 10 cases in Theorem 0.5, (ii) there are two root subsystems R_1, R_2 of Π_3 of the same type such that for any automorphism of lattices $\beta\colon P\to P$ satisfying $\beta(\kappa)=\kappa$ and $\beta(\lambda_3)=\lambda_3$, $\beta(R_1)$ never coincides with R_2. Indeed if we have a homomorphism β with $\beta(R_1)=R_2$, then $\beta(R_1\cap\Xi_2)=R_2\cap\Xi_2$ since the root subsystem $\Pi_3\cap\Xi_2$ of Π_3 is the unique one of type E_8. However for type E_8 the Weyl group coincides with the automorphism group. Thus $R_1\cap\Xi_2$ and $R_2\cap\Xi_2$ are conjugate with respect to $W(\Xi_2\cap\Pi_3)$.

Let E be a fixed elliptic curve. By Proposition 5.15, there are two marked rational surfaces of degree -1 over E, $\mathscr{Z}_1=(Z_1, D_1, \alpha_1, \iota_1)$ and $\mathscr{Z}_2=(Z_2, D_2, \alpha_2, \iota_2)$ such that $L_i=\alpha_i(\lambda_3)$ defines a morphism $\varPhi_i\colon Z_i\to X_i$ to a branched double covering $\pi_i\colon X_i\to P^2$ and $\mathrm{Ker}\,\varphi_{\mathscr{Z}_i}\cap\Pi_3=R_i$ $(i=1, 2)$. Thus for any intersection preserving homomorphism $\beta\colon \mathrm{Pic}\,(Z_1)\to\mathrm{Pic}\,(Z_2)$ satisfying $\beta(\omega_{Z_1})=\omega_{Z_2}$ and $\beta(\alpha_1(\lambda_3))=\alpha_2(\lambda_3)$, two root subsystems $\beta\,(\mathrm{Ker}\,(\mathrm{Pic}\,(Z_1)\to\mathrm{Pic}\,(D_1)))\cap\alpha_2(\Pi_3)$ and $\mathrm{Ker}\,(\mathrm{Pic}\,(Z_2)\to\mathrm{Pic}\,(D_2))\cap\alpha_2(\Pi_3)$ never coincide. However if the set of sextic curves with a combination of singularities under consideration is connected, we get a contradiction by the following lemma.

Lemma 5.18. *Let* $\mathfrak{B}\subset U\times P^2$ *be a family of reduced sextic curves over a connected analytic variety* U, *i.e., a subvariety of codimension* 1 *of* $U\times P^2$ *such that for every* $t\in U$, $B_t=\mathfrak{B}\cap\{t\}\times P^2$ *is a reduced sextic plane curve. We assume that* B_t *has a unique* $\tilde{E}_8$ *singular point and other several rational singular points. We assume moreover that the number of each type of rational singular points is independent of* $t\in U$. *Let* t' *and* t'' *be arbitrary points on* U. *We define varieties* X', X'', Z', Z'', D', D'' *and morphisms* π', π'', ρ', ρ'' *as follows. The branched double coverings over* P^2 *with the branch locus* $B'=B_{t'}$ *and* $B''=B_{t''}$ *are* $\pi'\colon X'\to P^2$ *and* $\pi''\colon X''\to P^2$ *respectively. The minimal resolution of singularities are denoted by* $\rho'\colon Z'\to X'$ *and* $\rho''\colon Z''\to X''$. *Let* D' *and* D'' *be the exceptional curves of the simple elliptic singularities in* X' *and* X'' *respectively. We set* $\Pi=\{M\in\mathrm{Pic}\,(Z'')\,|\,M^2=-2, M\circ\omega_{Z''}=0, M\circ\rho''^*\pi''^*\mathcal{O}_{P^2}(1)=0\}$. *Then there is an intersection-form-preserving homomorphism* $\beta\colon \mathrm{Pic}\,(Z')\to\mathrm{Pic}\,(Z'')$ *satisfying* $\beta(\omega_{Z'})=\omega_{Z''}$, $\beta\,(\rho'^*\pi'^*\mathcal{O}_{P^2}(1))=\rho''^*\pi''^*\mathcal{O}_{P^2}(1)$ *and* $\Pi\cap\beta\,(\mathrm{Ker}\,(\mathrm{Pic}$

$(Z') \to \mathrm{Pic}\ (D'))) = \Pi \cap \mathrm{Ker}\ (\mathrm{Pic}\ (Z'') \to \mathrm{Pic}\ (D''))$.

Proof. If U is connected, we can choose finite points $t_1, t_2, \cdots, t_q \in U$ with $t' = t_1$, $t'' = t_q$ and analytic morphisms $f_i: T \to U$, $1 \leq i \leq q$ from the unit disc $T = \{z \in C \mid |z| < 1\}$ such that t_i and t_{i+1} belong to the image $f_i(T)$. Considering the pullback of the family $\mathfrak{B}$ by f_i instead of $\mathfrak{B}$ itself, we can assume that U is the unit disc T without loss of generality.

Let $X_t \subset P(1, 1, 1, 3)$ be the branched double covering along $B_t \subset P^2$. Obviously the set $\mathfrak{X} = \bigcup_{t \in T} \{t\} \times X_t \subset T \times P(1, 1, 1, 3)$ is an analytic variety. Let Z_t be the minimal resolution of singularities of X_t. The set $\mathfrak{Z} = \bigcup_{t \in T} \{t\} \times Z_t$ also has the structure of analytic variety. The relative Picard group $\mathrm{Pic}_{\mathfrak{Z}/T}$ is a constant sheaf over T of free $\mathbb{Z}$-modules equipped bilinear forms. Let $\alpha: P_T \to \mathrm{Pic}_{\mathfrak{Z}/T}$ be an isomorphism from the constant sheaf with values in P. Let β be the composition

$$\mathrm{Pic}(Z') = \mathrm{Pic}(Z_{t'}) \xleftarrow{\sim} (\mathrm{Pic}_{\mathfrak{Z}/T})_{t'} \xrightarrow{\alpha_{t'}} P \xleftarrow{\alpha_{t''}} (\mathrm{Pic}_{\mathfrak{Z}/T})_{t''} \xrightarrow{\sim} \mathrm{Pic}(Z_{t''}) = \mathrm{Pic}(Z').$$

Note that for any $\eta \in \mathrm{Pic}(Z_t)$ with $\eta^2 = -2$ such that η is orthogonal to the dualizing sheaf and the polarization, either η or $-\eta$ is effective if and only if η or $-\eta$ is the class of a exceptional divisor of the resolution of $Z_t \to X_t$. By assumption that the combination of singularities on B_t and thus that on X_t is independent of $t \in T$, one sees that the above β has the desired property.

$$\text{Q.E.D.}$$

Remark. Applying the same method as in Umezu [22], we can also deduce the next proposition. We omit the proof here, since we can find essential parts in [22].

Proposition 5.19. *Assume that the branched double covering X over P^2 branching along a reduced sextic curve has a singularity of type $\tilde{E}_8$, $T_{2,3,7}$ or E_{12} and that $\sum_{x \in X} p_g(X, x) \geq 2$. Then the combination of singularities on X is either $2\tilde{E}_8$ or $2\tilde{E}_8 + A_1$. Conversely the branched double covering along a reduced sextic curve with $2\tilde{E}_8$ singularities and that with $2\tilde{E}_8 + A_1$ exist.*

Recall that the existence of X with given combination of singularities is equivalent to the existence of the sextic curve with the same combination by Lemma 1.1. The next figure gives the example of curves with $2\tilde{E}_8$ and $2\tilde{E}_8 + A_1$.
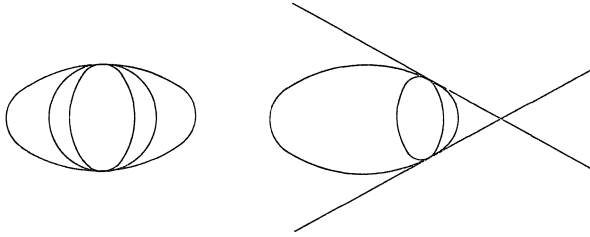
Figure 6.1.

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