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On "M-Functions" Closely Related to the Distribution of L'/L-Values

By

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Abstract

For each global field K, we shall construct and study two basic arithmetic functions, $M_\sigma^{(K)}(z)$ and its Fourier dual $\tilde{M}_\sigma^{(K)}(z)$, on $\mathbb{C}$ parametrized by $\sigma > 1/2$. These functions are closely related to the density measure for the distribution of values on $\mathbb{C}$ of the logarithmic derivatives of L-functions $L(\chi, s)$, where s is fixed, with $Re(s) = \sigma$, and χ runs over a natural infinite family of Dirichlet or Hecke characters on K. Connections with the Bohr-Jessen type value-distribution theories for the logarithms or (not much studied) logarithmic derivatives of $\zeta_K(\sigma+\tau i)$, where σ is fixed and τ varies, will also be briefly discussed.

Contents

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§1. Introduction

1.1 − When the ground field is the rational number field $\mathbb{Q}$, the functions $M_\sigma(z)$ and $\tilde{M}_\sigma(z)$ of $\sigma > 1/2$ and $z \in \mathbb{C}$ that we are going to construct and study can be uniquely characterized by the following properties (i) $\sim$ (iii);

(i) as functions of z, they are Fourier duals of each other,
(ii) $\tilde{M}_\sigma(z)$ is *real analytic in* (σ, z),
(iii) (at least) when $\sigma > 1$, $M_\sigma(z)|dz|$ gives the *density measure* for the distribution of values of the logarithmic derivative

$$L'(\chi, \sigma + \tau i)/L(\chi, \sigma + \tau i) \tag{1.1.1}$$

of L-functions on $\mathbb{C}$. Here, $\tau \in \mathbb{R}$ is also fixed and χ runs over all Dirichlet characters with prime conductors. (The density measure turns out to be independent of τ; for the other notations, see §1.2 below.)

We shall work over any global field K, i.e., either an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. These "M-functions" shall depend on K.

The main purpose of this article is (I) to construct and study these functions, with more weight on the study of $\tilde{M}_\sigma(z)$, which seems to be of independent interest, and (II) to establish the above relation (iii) including other cases of K and some other families of χ (Dirichlet characters, or Hecke Grössencharacters on K); in particular, for some range of σ including some to the left of $\sigma = 1$ in the function field case. The motivation of this work came from our previous study related to $L'(\chi, 1)/L(\chi, 1)$ [2], [3], [4]. The present paper is for the first stage. Even unconditional results for $\sigma = 1$ in the number field case require further substantial work. As for the connections with, and differences from the Bohr-Jessen type value distribution theories, where $\chi = 1$ and τ varies, see §1.5 below.

1.2 − The function $M_\sigma(z)$ to be constructed is real valued, ≥ 0, and belongs to C^∞, while $\tilde{M}_\sigma(z)$ is complex-valued, $|\tilde{M}_\sigma(z)| \leq 1$, and real-analytic. They are the Fourier transforms of each other in the sense that

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|. \tag{1.2.1}$$

Here, $\psi_z : \mathbb{C} \mapsto \mathbb{C}^1$ is the additive character

$$\psi_z(w) = \exp(i \cdot \mathrm{Re}(\bar{z}w)) \tag{1.2.2}$$

parametrized by $z \in \mathbb{C}$, and $|dw|$ denotes the self-dual Haar measure on $\mathbb{C}$ with respect to the self-dual pairing $\psi_z(w)$ of $\mathbb{C}$; namely, $|dw| = (2\pi)^{-1}dxdy$ for $w = x + iy$.

Both are continuous also in σ, and $\tilde{M}_\sigma(z)$ is even real-analytic in σ. They have quite interesting arithmetic and analytic properties. $\tilde{M}_\sigma(z)$ has a convergent *Euler product* expansion each of whose $\wp$-factor can be expressed explicitly in terms of Bessel functions, and correspondingly, $M_\sigma(z)$ has a *convolution Euler product* expansion, each of whose $\wp$-factor being a certain hyperfunction. Also, $\tilde{M}_\sigma(z)$ has an everywhere convergent power series expansion in $z, \bar{z}$ whose coefficients are some arithmetic Dirichlet series in σ, which may be regarded also as a Dirichlet series expansion in $\sigma > 1/2$ whose coefficients are arithmetic polynomials of $z, \bar{z}$. Both decay rapidly as $|z| \mapsto \infty$. Thus, even when $1/2 < \sigma < 1$ in the number field case, where we do not know much about the zeros of $L(\chi, s)$ and hence about the poles of $L'(\chi, s)/L(\chi, s)$, and hence about the distribution of $L'(\chi, s)/L(\chi, s)$ near $z = \infty$, still, the corresponding function $M_\sigma(z)$ can be constructed independently and can be proved to be rapidly decreasing with $|z|$.

1.3 – The symbolical relations among $M_\sigma(z)$, $\tilde{M}_\sigma(z)$ and $L'(\chi, s)/L(\chi, s)$, under optimal circumstances are,

$$M_\sigma(z) = \mathrm{Avg}_\chi \delta_z \left(\frac{L'(\chi, s)}{L(\chi, s)} \right), \quad \tilde{M}_\sigma(z) = \mathrm{Avg}_\chi \psi_z \left(\frac{L'(\chi, s)}{L(\chi, s)} \right), \tag{1.3.1}$$

where Avg_χ means a certain weighted average, ψ_z is the additive character of $\mathbb{C}$ defined above (1.2.2), and $\delta_z(w)|dw|$ is the Dirac delta measure on $\mathbb{C}$ with support at z. In other words, the first formula of (1.3.1) means that

$$\int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| = \mathrm{Avg}_\chi \Phi \left(\frac{L'(\chi, s)}{L(\chi, s)} \right) \tag{1.3.2}$$

holds for any test function Φ on $\mathbb{C}$, and the second formula is its special case where $\Phi = \psi_z$. When $\Phi(w) = P^{(a,b)}(w) = \bar{w}^a \cdot w^b$ (a, b non-negative integers), again under optimal circumstances,

$$\begin{aligned} \mathrm{Avg}_\chi P^{(a,b)} \left(\frac{L'(\chi, s)}{L(\chi, s)} \right) &= \int_{\mathbb{C}} M_\sigma(w) P^{(a,b)}(w)|dw| \\ &= \left(\frac{2}{i} \right)^{a+b} \frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b} \tilde{M}_\sigma(z) \mid_{z=0} = (-1)^{a+b} \mu_\sigma^{(a,b)}, \end{aligned} \tag{1.3.3}$$

where $\mu_\sigma^{(a,b)}$ is a certain Dirichlet series in σ whose coefficients are product of two iterated Λ-functions on K (§3.8). The present work is, though logically independent, in a sense, a continuation of [4] where this value was studied in the special case $K = \mathbb{Q}$ and $s = 1$.

1.4 – Our main results may be summarized as follows. Here and in the following, $N(D)$ for a divisor D denotes the norm of D.

Theorem $\tilde{M}$ (cf §3, Theorems 3–5). **(i)** *For each non-archimedean prime $\wp$ of K, consider the function of $\sigma > 0$ and $z \in \mathbb{C}$ defined by the convergent series*

$$\tilde{M}_{\sigma,\wp}(z) = 1 + \sum_{n=1}^{\infty} \frac{G_n(-\frac{i}{2} z \log N(\wp)) G_n(-\frac{i}{2} \bar{z} \log N(\wp))}{N(\wp)^{2\sigma n}}, \tag{1.4.1}$$

where $i = \sqrt{-1}$ and

$$G_n(w) = \sum_{k=1}^{n} \frac{1}{k!} \binom{n-1}{k-1} w^k. \tag{1.4.2}$$

Then when $\sigma > 1/2$, the Euler product

$$\tilde{M}_\sigma(z) = \prod_{\wp} \tilde{M}_{\sigma,\wp}(z) \tag{1.4.3}$$

converges in the following sense. For any $R > 0$ there exists $y = y(\sigma, R)$ such that $\tilde{M}_{\sigma,\wp}(z)$ for $N(\wp) > y$ has no zeros on $|z| \leq R$ and that their product over all these $\wp$ converges absolutely to a nowhere vanishing function on this disk. This function $\tilde{M}_\sigma(z)$ is real analytic in (σ, z), and as a function of z, belongs to L^p for all $1 \leq p \leq \infty$.

(*As for other expressions, and for the zeros of $\tilde{M}_{\sigma,\wp}(z)$, see* §3.1.)

(ii) *$\tilde{M}_\sigma(z)$ has an everywhere convergent power series expansion*

$$\tilde{M}_\sigma(z) = 1 + \sum_{a,b=1}^{\infty} (-i/2)^{a+b} \mu_\sigma^{(a,b)} \frac{z^a \bar{z}^b}{a! b!}, \tag{1.4.4}$$

and a convergent Dirichlet series expansion on $\sigma > 1/2$

$$\tilde{M}_\sigma(z) = \sum_{D:integral} \frac{\lambda_D(z) \lambda_D(\bar{z})}{N(D)^{2\sigma}}, \tag{1.4.5}$$

with positive real constants $\mu_\sigma^{(a,b)}$ and polynomials $\lambda_D(z)$ defined in §3.8. *Here, D runs over all "integral" divisors of K, i.e., the products of non-negative powers of non-archimedean primes.*

The author's initial definition of $\tilde{M}_\sigma(z)$ was by the (formal) power series (1.4.4), because the only information from [4] on the "would-be" density measure function $M_\sigma(z)$ was (1.3.3). Then the Euler product decomposition was found by a different route (Prop 3.8.11), and then the more natural explanation described below (§1.5) was recognized. Incidentally, our proof of the fact that the series (1.4.4) converges everywhere requires an argument where $z, \bar{z}$ are treated as two independent *complex* variables (§3.5).

Theorem M (§2 Theorem 2, §3 Theorem 3). *There exists a unique continuous function $M_\sigma(z)$ of $\sigma > 1/2$ and z such that*

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|. \tag{1.4.6}$$

It is non-negative real valued, C^∞ in z, and satisfies

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = 1. \tag{1.4.7}$$

As for the connections with the L'/L-values, presently, we focus our attention to the following three families of characters (see §4 for discussions and details);

(Case A) K is either the rational number field $\mathbb{Q}$, an imaginary quadratic field, or a function field of one variable over a finite field given together with an "infinite" prime divisor $\wp_\infty$ of degree 1, which will be considered "archimedean" and excluded from the L and $\tilde{M}$, M -Euler factors. The character χ runs over all Dirichlet characters on K with *prime* conductors normalized by the condition $\chi(\wp_\infty) = 1$;

(Case B) K is a number field having more than one archimedean prime, and χ runs over all "normalized unramified Grössencharacters" of K. (This family forms a free $\mathbb{Z}$-module of rank $[K:\mathbb{Q}] - 1$);

(Case C) $K = \mathbb{Q}$ and χ runs over the characters of the form $N(\wp)^{-\tau i}$. This is the case related to Bohr-Jessen type theories (§1.5).

For each such family, the average Avg_χ will be suitably defined (cf. §4).

Theorem $L{\sim}M$ (§4 Theorem 6, §6 Theorem 7). *Let $\sigma = \mathrm{Re}(s)$, and χ run over one of the above families. Assume $\sigma > 1$ in the number field case, and in the function field case, $\sigma > 3/4$ for* (i)(ii) *and $\sigma > 1/2$ for* (iii). *Then*
(i)

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} M_\sigma(z)\Phi(z)|dz| \tag{1.4.8}$$

holds for any "mild" test function Φ on $\mathbb{C}$.
(ii)

$$\mathrm{Avg}_\chi \psi_z \left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \tilde{M}_\sigma(z), \tag{1.4.9}$$

(iii)

$$\mathrm{Avg}_\chi P^{(a,b)} \left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = (-1)^{a+b}\mu_\sigma^{(a,b)}, \tag{1.4.10}$$

for the polynomials $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a, b \geq 0)$.

We expect that Theorem $L{\sim}M$ should hold for any $\sigma > 1/2$. But even in the function field case where the Weil's Riemann Hypothesis is valid, the above restriction $\sigma > 3/4$ (for (i)(ii)) seems to be the limit of our method (see §§6.3, 6.6, 6.9). On the other hand, we have an optimistic point of view for the possibility of having (unconditional) theory for $\sigma \leq 1$ also in the number field case, because $\psi_z(L'(\chi,s)/L(\chi,s))$ makes sense even at possible zeros of $L(\chi,s)$ (see §6.9(iii)).

1.5 – We shall now explain the main line of construction of the functions $M_\sigma(z)$ and $\tilde{M}_\sigma(z)$. It is based on the Euler sum expansion of L'/L coming from the Euler product expansion of L, and the basic geometric idea goes back to a seminal work of Bohr-Jessen [1]. Fix $s \in \mathbb{C}$, with $\sigma = \mathrm{Re}(s)$.

(**Local constructions**) Let $\sigma > 0$, and P be a finite set of non-archimedean primes of K. Put

$$T_P = \prod_{\wp \in P} \mathbb{C}^1 \tag{1.5.1}$$

(a torus), and let $g_{\sigma,P} : T_P \longmapsto \mathbb{C}$ be defined by

$$g_{\sigma,P}(t) = \sum_{\wp \in P} g_{\sigma,\wp}(t_\wp) = \sum_{\wp \in P} \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma} \tag{1.5.2}$$

$(t = (t_\wp) \in T_P)$. For any abelian character χ on K which is unramified over P, let

$$L_P(\chi,s) = \prod_{\wp \in P} (1 - \chi(\wp)N(\wp)^{-s})^{-1} \tag{1.5.3}$$

be the partial L-function. Then $L'_P(\chi,s)/L_P(\chi,s)$ can be regarded as a special value

$$\frac{L'_P(\chi,s)}{L_P(\chi,s)} = g_{\sigma,P}(\chi_P \cdot N(P)^{-i\cdot\tau}) \tag{1.5.4}$$

of the rational function $g_{\sigma,P}$ on T_P at the point $\chi_P \cdot N(P)^{-i\cdot\tau}$, where $\tau = \mathrm{Im}(s)$ and

$$\chi_P = (\chi(\wp))_\wp, \quad N(P)^{-i\cdot\tau} = (N(\wp)^{-i\cdot\tau})_\wp. \tag{1.5.5}$$

The optimal circumstance is where the family of χ satisfies the following two conditions:

(UniDistr) χ_P is *uniformly distributed* on T_P,

(Conv) $L'_P(\chi,s)/L_P(\chi,s)$ converges to $L'(\chi,s)/L(\chi,s)$ fast enough, ideally, uniformly in χ.

Now, for each family of χ that we shall consider, all but finitely many χ have conductors coprime with P and the above condition (UniDistr) is satisfied (§4, Lemma 4.3.1). Therefore, (1.3.1), with L_P in place of L, must be given by the corresponding integrals

$$M_{\sigma,P}(z) = \int_{T_P} \delta_z(g_{\sigma,P}(t))d^\star t, \quad \tilde{M}_{\sigma,P}(z) = \int_{T_P} \psi_z(g_{\sigma,P}(t))d^\star t. \tag{1.5.6}$$

($d^\star t$: the normalized Haar measure on T_P. Note that the contribution of $\mathrm{Im}(s)$ is "averaged away".) We thus have

$$\int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| = \mathrm{Avg}_\chi \Phi\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right) \tag{1.5.7}$$

for any continuous function Φ on $\mathbb{C}$. Note here that each $M_{\sigma,P}(z)$ is compactly supported. The summation over $\wp \in P$ in (1.5.2) is translated into "the basic product expansions"

$$M_{\sigma,P}(z) = *_{\wp\in P} M_{\sigma,\wp}(z), \quad \tilde{M}_{\sigma,P}(z) = \prod_{\wp\in P} \tilde{M}_{\sigma,\wp}(z), \tag{1.5.8}$$

where $*$ denotes the convolution product. Using the simple fact that each $g_{\sigma,\wp}$ maps $\mathbb{C}^1$ to another small circle, we are able to compute each of $M_{\sigma,\wp}(z)$ and $\tilde{M}_{\sigma,\wp}(z)$ explicitly.

(**Global constructions**) Now let $\sigma > 1/2$, and set $P = P_y = \{\wp; N(\wp) \leq y\}$. Then the point is that when $y \mapsto \infty$, each $M_{\sigma,P}(z)$ (resp. $\tilde{M}_{\sigma,P}(z)$) converges uniformly (and also w.r.t. some other L^p topologies) to a not-everywhere vanishing function $M_\sigma(z)$ (resp. $\tilde{M}_\sigma(z)$). Thus, these are functions obtained from $\delta_z(L'_P(\chi,s)/L_P(\chi,s))$ (resp. $\psi_z(L'_P(\chi,s)/L_P(\chi,s))$) by first fixing P and averaging over an infinite family of characters χ and then letting $y \mapsto \infty$. This way we can enter the region $1/2 < \sigma < 1$ unconditionally. The connection between the global objects, $\delta_z(L'(\chi,s)/L(\chi,s))$ with $M_\sigma(z)$, (resp. $\psi_z(L'(\chi,s)/L(\chi,s))$ with $\tilde{M}_\sigma(z)$), can be made when the condition (Conv) is satisfied at a sufficiently high level. First, when $\sigma > 1$, this convergence is uniform; hence the local relation (1.5.7) directly passes over to the global relation (1.3.2). Secondly, for the family (A) in the function field case, the convergence holds for any $\sigma > 1/2$ but (apparently) not uniformly with respect to χ. In this case, by choosing the intermediate object

$$\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}(\psi_z(L'_P(\chi,s)/L_P(\chi,s))), \tag{1.5.9}$$

where m and $P = P_y$ are related by $y = (\log m)^b$, with a suitable positive constant b, we are able to relate the M's with the L'/L for $\sigma > 3/4$. (If $(\sigma - 1/2)b > 1$ the convergence is fast enough, while if $(1-\sigma)b < 1$ the distribution is quantitatively uniform enough.) This is done after some Fourier analysis of the function $\psi_z(g_{\sigma,P}(t))$ on T_P.

(**Relations with Bohr-Jessen type theories**) Bohr-Jessen studied the distribution of values of $\log\zeta(\sigma+\tau i)$, where $\sigma > 1/2$ is fixed and $\tau \in \mathbb{R}$ varies [1](cf. also [6], [7], [10]). Since $\zeta(\sigma+\tau i) = L(\chi_\tau,\sigma)$, where $\chi_\tau(\wp) = N(\wp)^{-\tau i}$, this is the same as the value distribution of $\log L(\chi_\tau, s)$, where s is fixed and χ_τ runs over this one parameter family of trivial characters (the Case C family). Although we started by studying the "variable χ problem" for the Case A family and found the above construction, as was kindly pointed out by A.Fujii, the basic idea for such construction was essentially the same as in [1], i.e., goes back to Bohr-Jessen. Indeed, [1] uses the Euler sum expansion of $\log\zeta(s)$ and the uniform distribution property of χ_P on T_P, to relate the problem to the distribution of sum of points on the images of $\mathbb{C}^1$ by the mappings $t \mapsto -\log(1-p^{-\sigma}t)$. Then Jessen-Wintner [5] gave more general treatments using probability measure theory including Fourier analysis, and Matsumoto [8], [9] generalized this to the case of any number field K.

In spite of these similarities, there are three major differences.

(I) The directly related "$d\log\zeta(s)$-version" does not seem to have been so seriously studied. This is probably because of the difficulty in this case to get to the left of $\sigma = 1$. There are also differences in local structures; the $d\log$-version is in a sense easier, as the image $g_{\sigma,\wp}(\mathbb{C}^1)$ for this case is a circle; but on the other hand, the center-shifts and the metric twists cause some complications by which we cannot directly apply their theories, e.g. [5]. There are plenty of similarities, but it is still easier to treat this case directly (our §2).

(II) For this one parameter family $\chi = \chi_\tau$, the uniform distribution of χ_P on T_P holds *only when* $K = \mathbb{Q}$. Indeed, $\chi(\wp)$ depends only on $N(\wp)$. So, the $d\log$ analogue of Matsumoto's distribution measure for $K \neq \mathbb{Q}$ is different from ours. In the function field case, the situation is decisively different from the Case A family; all $(\chi_\tau)_P$ lie on a one-dimensional subtorus of T_P, and the image of the global $\zeta'_K(\sigma+\tau i)/\zeta_K(\sigma+\tau i)$ ($\sigma > 1/2$: fixed) is a bounded curve.

(III) The Fourier dual appears in [5], [9], etc., but mainly for auxiliary purpose to prove the convergence of the local measure. Those properties of $\tilde{M}_\sigma(z)$ as in the above stated Theorem $\tilde{M}$ (ii), which are obtained by using analytic functions of three complex variables (s, z_1, z_2) extending $(\sigma, z, \bar{z})$, do not seem to have been known.

We hope that the present approach will shed some light to the variable τ theory (too).

1.6 – We have also left untouched various basic questions related to $M_\sigma(z)$, $\tilde{M}_\sigma(z)$; for example, their zeros, their values at the central points (such as $M_\sigma(0)$, $M_\sigma(\zeta'_K(2\sigma)/\zeta_K(2\sigma))$), determination of the value of

$$\int_{\mathbb{C}} M_\sigma(z)^2 |dz| = \int_{\mathbb{C}} |\tilde{M}_\sigma(z)|^2 |dz|. \tag{1.6.1}$$

We hope to be able to discuss these in the near future, together with more applications.

§2. Constructions of $M_{\sigma,P}(z)$ and $M_\sigma(z)$

2.1 – We fix a global field K. By $\wp$ we shall denote any non-archimedean prime divisor of K, and by P any non-empty finite set of such $\wp$. For $y > 1$, put

$$P_y = \{\wp;\ N(\wp) \leq y\}. \tag{2.1.1}$$

We shall construct, for each P, a function $M_{\sigma,P}(z)$ on $\mathbb{C}$ parametrized by $\sigma > 0$, and then show that $M_{\sigma,P_y}(z)$ converges uniformly, as $y \mapsto \infty$, to a function $M_\sigma(z)$ when $\sigma > 1/2$.

As in §1, $T_P = \prod_{\wp\in P} \mathbb{C}^1$, and $g_{\sigma,P} : T_P \longmapsto \mathbb{C}$ is defined by

$$g_{\sigma,P}(t_P) = \sum_{\wp\in P} g_{\sigma,\wp}(t_\wp), \qquad g_{\sigma,\wp}(t_\wp) = \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma}, \tag{2.1.2}$$

where $t_P = (t_\wp)_{\wp\in P}$.

Theorem 1. *Let $\sigma > 0$. There exists a unique function $M_{\sigma,P}(z)$ of $z \in \mathbb{C}$, which is a hyperfunction (Schwartz distribution) when $|P| = 1$, that satisfies*

$$\int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| = \int_{T_P} \Phi(g_{\sigma,P}(t_P))d^*t_P \tag{2.1.3}$$

for any continuous function $\Phi(w)$ on $\mathbb{C}$, where $|dw| = (2\pi)^{-1}dxdy$ ($w = x+yi$) and d^*t_P is the normalized Haar measure on T_P. It is compactly supported, and satisfies

$$M_{\sigma,P}(z) \geq 0, \quad M_{\sigma,P}(\bar{z}) = M_{\sigma,P}(z), \quad \int_{\mathbb{C}} M_{\sigma,P}(w)|dw| = 1. \tag{2.1.4}$$

Before the proof, we note that each linear fractional function $g_{\sigma,\wp}$ maps the unit circle $\mathbb{C}^1$ to another circle, with center $c_{\sigma,\wp}$ and radius $r_{\sigma,\wp}$ given respectively by

$$c_{\sigma,\wp} = \frac{-\log N(\wp)}{N(\wp)^{2\sigma} - 1}, \quad r_{\sigma,\wp} = \frac{N(\wp)^\sigma \log N(\wp)}{N(\wp)^{2\sigma} - 1}. \tag{2.1.5}$$

If we write $g_{\sigma,\wp}(t_\wp) = c_{\sigma,\wp} + r_{\sigma,\wp} \cdot t'_\wp$, then

$$t'_\wp = \frac{N(\wp)^\sigma t_\wp - 1}{t_\wp - N(\wp)^\sigma}, \quad t_\wp = \frac{N(\wp)^\sigma t'_\wp - 1}{t'_\wp - N(\wp)^\sigma} \tag{2.1.6}$$

(involutive), and $t_\wp \in \mathbb{C}^1$ if and only if $t'_\wp \in \mathbb{C}^1$. The image of the normalized Haar measure $d^*t_\wp = (2\pi i t_\wp)^{-1}dt_\wp$ of $\mathbb{C}^1$ on the $t'_\wp$-unit circle is given by

$$d^*t_\wp = \frac{N(\wp)^{2\sigma} - 1}{|N(\wp)^\sigma - t'_\wp|^2} \cdot d^*t'_\wp, \tag{2.1.7}$$

where $d^*t'_\wp = (2\pi i t'_\wp)^{-1}dt'_\wp$.

Proof of Theorem 1. The uniqueness is obvious. The solution is given explicitly as

$$M_{\sigma,\wp}(c_{\sigma,\wp} + r \cdot e^{i\theta}) = \frac{N(\wp)^{2\sigma} - 1}{|N(\wp)^\sigma - e^{i\theta}|^2} \cdot \frac{\delta(r - r_{\sigma,\wp})}{r} \tag{2.1.8}$$

($r \geq 0$, $\theta \in \mathbb{R}$, $\delta(r)$: the usual 1-dimensional Dirac delta function), and

$$M_{\sigma,P}(z) = *_{\wp \in P} M_{\sigma,\wp}(z), \tag{2.1.9}$$

where $*$ denotes the convolution product with respect to $|dz|$. □

Note that

$$M_{\sigma,P}(\bar{z}) = M_{\sigma,P}(z) = \overline{M_{\sigma,P}(z)}. \tag{2.1.10}$$

It is clear from (2.1.3) that

$$\int_U M_{\sigma,P}(w)|dw| = \mathrm{Vol}(g_{\sigma,P}^{-1}(U)) \tag{2.1.11}$$

for any open set U on $\mathbb{C}$, where Vol denotes the volume with respect to d^*t_P. Therefore, the support of $M_{\sigma,P}(z)$ is exactly the image of $g_{\sigma,P}$:

$$\mathrm{Supp}(M_{\sigma,P}(z)) = \left\{ \sum_{\wp\in P} (c_{\sigma,\wp} + r_{\sigma,\wp}e^{i\theta_\wp}), \quad 0 \le \theta_\wp < 2\pi \right\}; \tag{2.1.12}$$

hence it is contained in the disk with center $c_{\sigma,P}$ and radius $r_{\sigma,P}$ given by

$$c_{\sigma,P} = \sum_{\wp\in P} c_{\sigma,\wp}, \quad r_{\sigma,P} = \sum_{\wp\in P} r_{\sigma,\wp}. \tag{2.1.13}$$

When $|P| = 1$, this support is a circle, and when $|P| > 1$, this can either be an annulus or a disk, depending on P and σ (cf. e.g.[10, §11]).

2.2 – For any P and $\wp \notin P$, one can express the convolution product $M_{\sigma,P\cup\wp} = M_{\sigma,P} * M_{\sigma,\wp}$ explicitly as

$$M_{\sigma,P\cup\wp}(z) = \frac{N(\wp)^{2\sigma} - 1}{2\pi} \int_0^{2\pi} \frac{M_{\sigma,P}(z - c_{\sigma,\wp} - r_{\sigma,\wp}e^{i\theta})}{|N(\wp)^\sigma - e^{i\theta}|^2} d\theta. \tag{2.2.1}$$

So, $M_{\sigma,P\cup\wp}(z)$ is obtained by averaging $M_{\sigma,P}(z)$ over the circle with center $z - c_{\sigma,\wp}$ and radius $r_{\sigma,\wp}$, with respect to the image of $d^*t_\wp$ on this circle.

When $P = \{\wp, \wp'\}$ with $r_{\sigma,\wp} \ge r_{\sigma,\wp'}$, we see easily that $M_{\sigma,P}(z)$ is a (non-negative real valued) function whose support is

$$r_{\sigma,\wp} - r_{\sigma,\wp'} \le |z - c_{\sigma,P}| \le r_{\sigma,\wp} + r_{\sigma,\wp'}. \tag{2.2.2}$$

But $M_{\sigma,\wp\cup\wp'}(z)$ is unbounded near the border of support. When $|P| = 3$, $M_{\sigma,P}(z)$ is bounded, but still discontinuous at the border. We shall see that $M_{\sigma,P}(z)$ gets smoother and smoother as $|P|$ increases.

In fact, as a reflection of the rapid decaying property of its Fourier dual (Cor 3.3.3), we obtain

Proposition 2.2.3. *$M_{\sigma,P}(z)$ belongs to class C^k if $|P| > 2(k+2)$.*

2.3 – Now let (a,b) be any pair of non-negative integers, and consider the derivation

$$D^{(a,b)}=\frac{\partial^{a+b}}{\partial z^a\partial\bar{z}^b}. \tag{2.3.1}$$

If $|P|>2(a+b+2)$, then $M_{\sigma,P}(z)$ belongs to C^{a+b}; hence $D^{(a,b)}$ acts on (2.2.1) and commutes with the integration with respect to the parameter θ. Thus,

$$D^{(a,b)}M_{\sigma,P\cup\wp}(z)=\frac{N(\wp)^{2\sigma}-1}{2\pi}\int_0^{2\pi}\frac{(D^{(a,b)}M_{\sigma,P})(z-c_{\sigma,\wp}-r_{\sigma,\wp}e^{i\theta})}{|N(\wp)^{\sigma}-e^{i\theta}|^2}d\theta \tag{2.3.2}$$

holds whenever $|P|>2(a+b+2)$. In particular,

$$\mathrm{Max}_z|D^{(a,b)}M_{\sigma,P\cup\wp}(z)|\le\mathrm{Max}_z|D^{(a,b)}M_{\sigma,P}(z)|. \tag{2.3.3}$$

Therefore, for each (a,b), there exists a positive constant $\mathbf{m}_\sigma^{(a,b)}$ such that

$$|D^{(a,b)}M_{\sigma,P}(z)|\le\mathbf{m}_\sigma^{(a,b)} \tag{2.3.4}$$

holds for any $P=P_y$ with $|P|>2(a+b+2)$.

2.4 – We shall need the following

Lemma 2.4.1. *Fix $\sigma>0$ and $a,b\ge 0$. Then for any $P=P_y$ with $|P|>2(a+b+4)$ and $\wp\notin P$,*

$$|D^{(a,b)}M_{\sigma,P\cup\wp}(z)-D^{(a,b)}M_{\sigma,P}(z)|\ll\left(\frac{\log N(\wp)}{N(\wp)^{\sigma}}\right)^2, \tag{2.4.2}$$

where $\ll$ is independent of $P,\wp,z$.

The proof is based on (2.3.2) for $(a,b),(a+1,b),(a,b+1),(a+1,b+1)$, and on the following well-known formula in harmonic analysis.

Sublemma 2.4.3. *Let $\triangle=4D^{(1,1)}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ be the Laplacian on $\mathbb{C}=\mathbb{R}^2$ and take any $R>0$. Then for any complex valued function $u(z)$ belonging to class C^2 on a domain $\subset\mathbb{C}$ containing the disk $|z-z_0|\le R$,*

$$\frac{1}{2\pi}\int_0^{2\pi}u(z_0+Re^{i\theta})d\theta-u(z_0)=\frac{1}{2\pi}\int_{|z-z_0|\le R}\log\left(\frac{R}{|z-z_0|}\right)(\triangle u)(z)dxdy. \tag{2.4.4}$$

Corollary 2.4.5. *If* $|\triangle u(z)| \le U$ *on* $|z - z_0| \le R$, *then*

$$\left|\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + Re^{i\theta})d\theta - u(z_0)\right| \le \frac{1}{4}UR^2. \tag{2.4.6}$$

Proof of Lemma 2.4.1. Let us suppress σ from the notation, and write

$$q = N(\wp)^\sigma,\ c = c_{\sigma,\wp},\ r = r_{\sigma,\wp},\ z' = z - c. \tag{2.4.7}$$

Decompose

$$D^{(a,b)}M_{\sigma,P\cup\wp}(z) - D^{(a,b)}M_{\sigma,P}(z) = A + B + C, \tag{2.4.8}$$
$$\begin{cases} A = \frac{q^2-1}{2\pi}\int_0^{2\pi}\left(\frac{1}{|q-e^{i\theta}|^2} - \frac{1}{q^2-1}\right) D^{(a,b)}M_P(z' - re^{i\theta})d\theta, \\ B = \frac{1}{2\pi}\int_0^{2\pi} D^{(a,b)}M_P(z' - re^{i\theta})d\theta - D^{(a,b)}M_P(z'), \\ C = D^{(a,b)}M_P(z') - D^{(a,b)}M_P(z). \end{cases}$$

We shall estimate each of A, B, C.

First, it is clear that

$$C \ll |c|(\mathbf{m}^{(a+1,b)} + \mathbf{m}^{(a,b+1)}) \ll |c| \ll \frac{\log N(\wp)}{N(\wp)^{2\sigma}}. \tag{2.4.9}$$

Secondly, it follows directly from Cor 2.4.5 that

$$B \ll r^2\mathbf{m}^{(a+1,b+1)} \ll r^2 \ll \frac{(\log N(\wp))^2}{N(\wp)^{2\sigma}}. \tag{2.4.10}$$

As for A, decompose it as

$$A = \frac{1}{\pi}\int_0^{2\pi} \frac{q\cos\theta}{|q - e^{i\theta}|^2} D^{(a,b)}M_P(z' - re^{i\theta})d\theta - \frac{1}{\pi}\int_0^{2\pi} \frac{D^{(a,b)}M_P(z' - re^{i\theta})}{|q - e^{i\theta}|^2}d\theta. \tag{2.4.11}$$

Observe now that the absolute value of the second term on the right hand side is bounded by $(q-1)^{-2}\mathbf{m}^{(a,b)} \ll N(\wp)^{-2\sigma}$. As for the first term, this decomposes as

$$\frac{q}{\pi}\int_0^{2\pi}\frac{\cos\theta}{|q - e^{i\theta}|^2}(D^{(a,b)}M_P(z'-re^{i\theta})-D^{(a,b)}M_P(z'))d\theta+2(q^2-1)^{-1}D^{(a,b)}M_P(z'), \tag{2.4.12}$$

because

$$\frac{q}{\pi}\int_0^{2\pi}\frac{\cos\theta d\theta}{|q - e^{i\theta}|^2} = 2(q^2 - 1)^{-1} \qquad (q > 1). \tag{2.4.13}$$

Since the absolute value of the first (resp. the second) term of (2.4.12) is $\ll q^{-1}r(\mathbf{m}^{(a+1,b)}+\mathbf{m}^{(a,b+1)})$ (resp. $q^{-2}\mathbf{m}^{(a,b)}$), we conclude that

$$A \ll q^{-2}+q^{-1}r \ll \frac{\log N(\wp)}{N(\wp)^{2\sigma}}. \tag{2.4.14}$$

Therefore,

$$A+B+C \ll \frac{(\log N(\wp))^2}{N(\wp)^{2\sigma}}. \tag{2.4.15}$$

□

2.5 – Since the sum of the right-hand side of (2.4.2) over all $\wp$ converges when $\sigma > 1/2$, we immediately obtain the first two items (i)(ii) of the following theorem.

Theorem 2. *Let $\sigma > 1/2$, $P = P_y$ and let $y \mapsto \infty$. Then*
(i) *$M_{\sigma,P}(z)$ converges uniformly to a non-negative real valued C^∞-function $M_\sigma(z)$. It satisfies*

$$M_\sigma(\bar{z}) = M_\sigma(z) = \overline{M_\sigma(z)}. \tag{2.5.1}$$

(ii) *Each $D^{(a,b)}M_{\sigma,P}(z)$ converges uniformly to $D^{(a,b)}M_\sigma(z)$ (starting with $|P|$ sufficiently large).*
(iii) *$M_\sigma(z) = O((1+|z|)^{-n})$ for any $n \geq 1$.*
(iv) *The function $M_\sigma(z)$ is not identically zero; in fact,*

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = 1. \tag{2.5.2}$$

Remark 2.5.3. (i) $M_\sigma(z)$ is continuous also in (σ, z) (see Cor 3.11.10).
(ii) When $\sigma > 1, \sum_\wp r_{\sigma,\wp} < \infty$; hence $M_\sigma(z)$ is compactly supported.

Proof of (iii). Fix $\sigma > 1/2$ and $N \geq 0$. We shall first prove the existence of a constant $C = C_{\sigma,N} > 0$ satisfying the inequality

$$\int_{\mathbb{C}} M_{\sigma,P}(z)|z|^{2N}|dz| = \int_{T_P} |g_{\sigma,P}(t_P)|^{2N} d^*t_P \leq C \tag{2.5.4}$$

for any P. Put $\lambda_\wp = \log N(\wp), q_\wp = N(\wp)^\sigma$ ($\wp \in P$), and

$$g^0_{\sigma,\wp}(t_\wp) = -\lambda_\wp q_\wp^{-1} t_\wp, \quad g^0_{\sigma,P}(t_P) = \sum_{\wp\in P} g^0_{\sigma,\wp}(t_\wp). \tag{2.5.5}$$

Recall that $g_{\sigma,\wp}(t_\wp) = \lambda_\wp t_\wp (t_\wp - q_\wp)^{-1}$. It is clear that

$$|g_{\sigma,\wp}(t_\wp) - g^0_{\sigma,\wp}(t_\wp)| \le \lambda_\wp q_\wp^{-1}(q_\wp - 1)^{-1}. \tag{2.5.6}$$

Since the sum of the right hand side of (2.5.6) over all $\wp$ is convergent, $|g_{\sigma,P}(t_P) - g^0_{\sigma,P}(t_P)|$ is bounded independently of P; hence

$$\int_{T_P} |g_{\sigma,P}(t_P) - g^0_{\sigma,P}(t_P)|^{2N} d^*t_P \le B_1, \tag{2.5.7}$$

with some $B_1 = B_1(\sigma, N)$ independent of P. On the other hand,

$$g^0_{\sigma,P}(t_P)^N = \sum_{k_P} \binom{N}{k_P} \prod_{\wp\in P} g^0_{\sigma,\wp}(t_\wp)^{k_\wp}, \tag{2.5.8}$$

where $k_P = (k_\wp)$ runs over all $|P|$ -ples of non-negative integers such that $\sum_{\wp\in P} k_\wp = N$, and $\binom{N}{k_P} = N!/\prod_{\wp\in P}(k_\wp!)$ are the multinomial coefficients. Therefore,

$$\begin{aligned}\int_{T_P} |g^0_{\sigma,P}(t_P)|^{2N} d^*t_P &= \sum_{k_P} \binom{N}{k_P}^2 \prod_{\wp\in P} (-\lambda_\wp q_\wp^{-1})^{2k_\wp} \\ &\le N! \sum_{k_P} \binom{N}{k_P} \prod_{\wp\in P} (\lambda_\wp q_\wp^{-1})^{2k_\wp} = N! \Big(\sum_{\wp\in P} (\lambda_\wp q_\wp^{-1})^2\Big)^N \le B_2,\end{aligned} \tag{2.5.9}$$

with some $B_2 = B_2(\sigma, N)$, because $\sum_{\wp\in P}(\lambda_\wp q_\wp^{-1})^2$ is also convergent. Since x^{2N} is convex on $x > 0$, we obtain from (2.5.7), (2.5.9) that

$$\frac{1}{2^{2N}} \int_{T_P} |g_{\sigma,P}(t_P)|^{2N} d^*t_P \le \frac{1}{2}(B_1 + B_2), \tag{2.5.10}$$

which settles (2.5.4).

Now from (2.5.4) we obtain

$$\int_{\mathbb{C}} M_\sigma(z)|z|^{2N}|dz| \le C, \tag{2.5.11}$$

because we obtain for each fixed $R > 0$ the corresponding inequality for the integral over $|z| \le R$ from (2.5.4) by virtue of (i).

Now let (r, θ) be the polar coordinate system for $\mathbb{C}$. Then $|\partial M_\sigma(z)/\partial r|$ is uniformly bounded from above by $2\mathbf{m}_\sigma^{(1,0)}$ (cf. (2.3.4) and Theorem 2(ii)). So, if $u_0 = M_\sigma(z_0) > 0$, there exists $d_0 \gg u_0$ such that $M_\sigma(z) \ge u_0/2$ holds on the disk neighborhood $|z-z_0| \leq d_0$ of z_0. Since $M_\sigma(z) \mapsto 0$ ($|z| \mapsto \infty$), we may choose $|z_0|$ large enough so that u_0 is small enough and so we may choose $d_0 \ll 1$. Then for each fixed $n \geq 1$,

(2.5.12)
$$\int_{|z-z_0|\leq d_0} M_\sigma(z)|z|^{3n}|dz| \geq (u_0/2)(|z_0|-d_0)^{3n}(\pi d_0^2)/(2\pi) \gg u_0^3|z_0|^{3n};$$

hence by (2.5.11) we obtain $M_\sigma(z_0)|z_0|^n \ll 1$. This settles (iii).

For the proof of (iv), we need some results on the limit of the Fourier transform $\tilde{M}_{\sigma,P}(z)$ of $M_{\sigma,P}(z)$. This will be given in the next §3 (§3.11).

§3. Constructions of $\tilde{M}_{\sigma,P}(z)$ and $\tilde{M}_\sigma(z)$

3.1 – For each non-archimedean prime $\wp$ of K and $\sigma > 0$, $\tilde{M}_{\sigma,\wp}(z)$ is, by definition, the Fourier transform of $M_{\sigma,\wp}(z)$;

$$\tilde{M}_{\sigma,\wp}(z) = \int_{\mathbb{C}} M_{\sigma,\wp}(w)\psi_z(w)|dw|, \tag{3.1.1}$$

where $\psi_z(w) = \exp(i \cdot \mathrm{Re}(\bar{z}w))$ and $|dw|$ is the self-dual measure w.r.t. ψ_z, i.e., $|dw| = (2\pi)^{-1}dxdy$ for $w = x+yi$. Thus, either from (2.1.3) or (2.1.8), it follows directly that

$$\begin{aligned}\tilde{M}_{\sigma,\wp}(z) &= \int_{\mathbb{C}^1} \psi_z(g_{\sigma,\wp}(t_\wp))d^*t_\wp \\ &= \exp(i \cdot c_{\sigma,\wp} \cdot \mathrm{Re}(z)) \cdot H_{\sigma,\wp}(z),\end{aligned} \tag{3.1.2}$$

where

$$H_{\sigma,\wp}(z) = \frac{N(\wp)^{2\sigma}-1}{2\pi} \cdot \int_0^{2\pi} \frac{\exp(ir_{\sigma,\wp}|z|\cos(\theta-\vartheta))}{|N(\wp)^\sigma - \exp(i\theta)|^2}d\theta, \tag{3.1.3}$$

with $\vartheta = \mathrm{Arg}(z)$. Let

$$J_n(x) = \frac{i^{-n}}{2\pi}\int_0^{2\pi} \exp(ix\cos(\theta))\cos(n\theta)d\theta \tag{3.1.4}$$

be the Bessel function of order n. Then

$$H_{\sigma,\wp}(z) = \sum_{n=0}^{\infty} \epsilon_n \left(\frac{i}{N(\wp)^\sigma}\right)^n \cos(n\vartheta)J_n(r_{\sigma,\wp}|z|), \tag{3.1.5}$$

where ϵ_n is the Neumann factor $\epsilon_n = 1 (n = 0), \ = 2 (n \geq 1)$. Indeed,

$$(N(\wp)^{2\sigma} - 1)|N(\wp)^\sigma - \exp(i\theta)|^{-2} = \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) N(\wp)^{-n\sigma}, \tag{3.1.6}$$

and ($\sin(n\theta)$ being an odd function)

(3.1.7)
$$\int_0^{2\pi} \exp(ix\cos(\theta - \vartheta))\cos(n\theta)d\theta = \cos(n\vartheta)\int_0^{2\pi} \exp(ix\cos(\theta))\cos(n\theta)d\theta,$$

from which (3.1.5) follows directly.

Since

$$J_n(x) = \left(\frac{x}{2}\right)^n j_n\left(\left(\frac{x}{2}\right)^2\right), \qquad j_n(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!(n+k)!}, \tag{3.1.8}$$

with an entire function $j_n(z)$ on $\mathbb{C}$, (3.1.5) may be rewritten as an everywhere convergent power series in $z, \bar{z}$;

(3.1.9)
$$H_{\sigma,\wp}(z) = j_0\left(\left(\frac{r_{\sigma,\wp}}{2}\right)^2 z\bar{z}\right) + \sum_{n=1}^{\infty}\left(\frac{ir_{\sigma,\wp}}{2N(\wp)^\sigma}\right)^n (z^n + \bar{z}^n) j_n\left(\left(\frac{r_{\sigma,\wp}}{2}\right)^2 z\bar{z}\right).$$

Clearly, $H_{\sigma,\wp}(z)$, and hence also $\tilde{M}_{\sigma,\wp}(z)$, are real-analytic functions of z, and by their definitions,

$$|\tilde{M}_{\sigma,\wp}(z)| = |H_{\sigma,\wp}(z)| \leq 1. \tag{3.1.10}$$

Remark 3.1.11. The function $\tilde{M}_{\sigma,\wp}(z)$ of z has infinity of zeros on the imaginary axis. To be precise, exactly one simple zero $z = y \cdot i$ with $y \in [m\pi/r_{\sigma,\wp}, (m+1)\pi/r_{\sigma,\wp})$ for each $m \in \mathbb{Z}$ when $|m|$ is large enough. As we shall see (Theorem 4), each zero of $\tilde{M}_\sigma(z)$ comes from that of some of its local $\wp$-factor.

For example, let $K = \mathbb{Q}$. Then for $\sigma = 1$, $\tilde{M}_\sigma(z)$ has zeros only on the imaginary axis $z = yi$, with (approximately) $|y| = 6.35, 6.49, 7.38, 8.59, 11.00, 12.01, 12.17, 13.55, \ldots$, coming from $p = 3, 2, 5, 7, 11, 2, 13, 3, \ldots$, respectively. While for $\sigma = \frac{1}{2} + \frac{1}{20}$, it also has some zeros *off* the imaginary axis, e.g., $z = \pm 1.738 \pm 13.268 \cdot i$ (from $p = 2$), $z = \pm 1.276 \pm 6.418 \cdot i$ (from $p = 3$). Such a phenomenon appears only when $N(\wp)^\sigma$ is small. What does this mean?!

3.2 – For any finite set P of non-archimedean primes of K, define

$$\tilde{M}_{\sigma,P}(z) = \prod_{\wp \in P} \tilde{M}_{\sigma,\wp}(z), \qquad H_{\sigma,P}(z) = \prod_{\wp \in P} H_{\sigma,\wp}(z), \tag{3.2.1}$$

so that $\tilde{M}_{\sigma,P}(z) = e^{ic_{\sigma,P}\mathrm{Re}(z)} H_{\sigma,P}(z)$. Note that

$$\tilde{M}_{\sigma,P}(z) = \int_{\mathbb{C}} M_{\sigma,P}(w)\psi_z(w)|dw| = \int_{T_P} \psi_z(g_{\sigma,P}(t_P))d^*t_P. \tag{3.2.2}$$

The Fourier inversion formula gives

$$M_{\sigma,P}(z) = \int_{\mathbb{C}} \tilde{M}_{\sigma,P}(w)\psi_{-z}(w)|dw|. \tag{3.2.3}$$

These functions $H_{\sigma,P}(z)$, $\tilde{M}_{\sigma,P}(z)$ are also obviously real analytic, satisfy $|H_{\sigma,P}(z)| = |\tilde{M}_{\sigma,P}(z)| \leq 1$, and

$$\begin{aligned} \tilde{M}_{\sigma,P}(0) = H_{\sigma,P}(0) = 1 \qquad &(\textit{all } P), \\ |\tilde{M}_{\sigma,P'}(z)| \leq |\tilde{M}_{\sigma,P}(z)| \leq 1 \qquad &(P \subseteq P'). \end{aligned} \tag{3.2.4}$$

Also, note that

$$\tilde{M}_{\sigma,P}(\bar{z}) = \tilde{M}_{\sigma,P}(z), \quad \tilde{M}_{\sigma,P}(-z) = \overline{\tilde{M}_{\sigma,P}(z)}. \tag{3.2.5}$$

3.3 – We shall show now that

Proposition 3.3.1. *Let $\sigma > 0$ and P be fixed. Then*

$$|\tilde{M}_{\sigma,P}(z)| = O\left((1+|z|)^{-\frac{|P|}{2}}\right). \tag{3.3.2}$$

In particular, $|z|^k \tilde{M}_{\sigma,P}(z)$ belongs to L^1 if $|P| > 2(k+2)$.

Thus the Fourier dual satisfies:

Corollary 3.3.3. *$M_{\sigma,P}(z)$ belongs to class C^k when $|P| > 2(k+2)$.*

To prove Prop 3.3.1, we need the following

Lemma 3.3.4. *There exists an absolute positive constant A such that*

$$x^{\frac{1}{2}}|J_n(x)| < A(n+1)^{\frac{1}{2}} \tag{3.3.5}$$

holds for any non-negative integer n and $x \geq 0$.

It is well-known that $x^{1/2}|J_n(x)|$ is bounded for each n, and also that this bound must depend on n. (In fact, by Cauchy, $n^{1/2}|J_n(n)| \sim n^{1/6}$.) Since the author could not find a suitable reference for a simple explicit bound like (3.3.5), we shall give this a full proof.

We first need:

Sublemma 3.3.6. *$x^{\frac{1}{4}}|J_n(x)|$ for $x \geq 0, n = 0, 1, 2, \ldots$ has a universal upper bound.*

Proof. The Schläfli-Neumann formula for $J_n(x)^2$ ([11, §2.6]) gives

$$J_n(x)^2 = \frac{1}{\pi}\int_0^\pi J_0(2x\sin\theta)\cos(2n\theta)d\theta. \tag{3.3.7}$$

But since $x^{1/2}|J_0(x)| \ll 1$,

$$J_n(x)^2 \ll \int_0^\pi \frac{d\theta}{\sqrt{x\sin\theta}} \ll \frac{1}{\sqrt{x}}. \tag{3.3.8}$$

□

Proof of Lemma 3.3.4. As for the constant A, it suffices that (3.3.5) holds for $n = 0, 1$ and that $2^{-1/4}A$ exceeds the universal upper bound for $x^{1/4}|J_n(x)|$. We shall fix such A and $x \geq 0$, and prove (3.3.5) by induction on $n \geq 2$.

[Case $n^2 \leq x/2$] By the recurrence formula

$$J_n(x) = \frac{2(n-1)}{x}J_{n-1}(x) - J_{n-2}(x) \tag{3.3.9}$$

and the assumptions, we obtain

$$\begin{aligned} x^{\frac{1}{2}}|J_n(x)| &\leq \left(\frac{n-1}{n^2}n^{\frac{1}{2}} + (n-1)^{\frac{1}{2}}\right)A \\ &< n^{\frac{1}{2}}\left(n^{-1} + (1-n^{-1})^{\frac{1}{2}}\right)A < (n+1)^{\frac{1}{2}}A, \end{aligned} \tag{3.3.10}$$

as desired. The last inequality follows from $(1+x)^{1/2} - (1-x)^{1/2} > x$ for $0 < x < 1$, in particular for $x = n^{-1}$ $(n \geq 2)$.

[Case $n^2 > x/2$] In this case, by the sublemma and the assumptions, we obtain

$$x^{\frac{1}{2}}|J_n(x)| \leq x^{\frac{1}{4}}2^{-\frac{1}{4}}A < (2n^2)^{\frac{1}{4}}2^{-\frac{1}{4}}A < A(n+1)^{\frac{1}{2}}, \tag{3.3.11}$$

as desired. This proves Lemma 3.3.4. □

Proof of Proposition 3.3.1. We only need a weaker version $x^{1/2}|J_n(x)| \ll n+1$ of Lemma 3.3.4. By this and (3.1.5), we obtain

$$\begin{aligned} |H_{\sigma,\wp}(z)| &\ll (r_{\sigma,\wp}|z|)^{-1/2} \sum_{n=0}^{\infty} \epsilon_n (n+1) N(\wp)^{-\sigma n} \\ &= (r_{\sigma,\wp}|z|)^{-1/2} \left(2(1-N(\wp)^{-\sigma})^{-2} - 1\right). \end{aligned} \tag{3.3.12}$$

But since $N(\wp)^\sigma r_{\sigma,\wp} \geq \log N(\wp) \geq \log 2$, and $(1-N(\wp)^{-\sigma})^{-2} \leq (1-2^{-\sigma})^{-2}$, this gives

$$|H_{\sigma,\wp}(z)| \ll N(\wp)^{\sigma/2} |z|^{-1/2}. \tag{3.3.13}$$

Since $H_{\sigma,P}(z) = \prod_{\wp \in P} H_{\sigma,\wp}(z)$, the proof is completed. □

3.4 – By Prop 3.3.1, $\tilde{M}_{\sigma,P}(z)$ belongs to L^∞ (i.e., continuous, and for any $\epsilon > 0$ there exists $R > 0$ such that $|\tilde{M}_{\sigma,P}(z)| < \epsilon$ for $|z| > R$), and if $|P| > 4$, $\tilde{M}_{\sigma,P}(z) \in L^1 \cap L^\infty$; hence $\in L^t \quad (1 \leq t \leq \infty)$. The main goal of §3 is to prove the following

Theorem 3. *Let $\sigma > 1/2$. Then*
(i) *When $P = P_y$ and $y \mapsto \infty$, $\tilde{M}_{\sigma,P}(z)$ converges uniformly on $\sigma \geq 1/2 + \epsilon$ and $z \in \mathbb{C}$ to a continuous function $\tilde{M}_\sigma(z)$ of (σ, z).*
(ii) *For each $\sigma > 1/2$, the function $\tilde{M}_\sigma(z)$ of z belongs to L^t for any $1 \leq t \leq \infty$, and the convergence $\tilde{M}_{\sigma,P}(z) \mapsto \tilde{M}_\sigma(z)$ is also L^t-convergence.*
(iii) *$\tilde{M}_\sigma(z)$ is real analytic in (σ, z).*
(iv) *$\tilde{M}_\sigma(z) = O((1+|z|)^{-n})$ for any $n \geq 1$.*
(v) *$M_\sigma(z)$ and $\tilde{M}_\sigma(z)$ are Fourier duals of each other*;

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w) \psi_z(w) |dw|, \qquad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w) \psi_{-z}(w) |dw|. \tag{3.4.1}$$

(vi) *$\tilde{M}_\sigma(z)$ has a power series expansion*

$$\tilde{M}_\sigma(z) = \sum_{a,b=0}^{\infty} (-i/2)^{a+b} \mu_\sigma^{(a,b)} \frac{z^a \bar{z}^b}{a!b!} \qquad (z \in \mathbb{C}), \tag{3.4.2}$$

with the Dirichlet series coefficients

$$\mu_\sigma^{(a,b)} = \sum_{D\, integral} \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2\sigma}} \qquad (\sigma > 1/2). \tag{3.4.3}$$

Here, D runs over all integral ideals (effective divisors) of K, and $\Lambda_k(D)$ is as defined later in §3.8. The expansion (3.4.2) can also be regarded as a Dirichlet series expansion

$$\tilde{M}_\sigma(z) = \sum_{D\ integral} \frac{\lambda_D(z)\lambda_D(\bar{z})}{N(D)^{2\sigma}} \qquad (\sigma > 1/2), \tag{3.4.4}$$

with the polynomial coefficients $\lambda_D(z)\lambda_D(\bar{z})$, where

$$\lambda_D(z) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(D)}{k!} z^k \tag{3.4.5}$$

(*which is actually a finite sum*).

Remark 3.4.6. Clearly, $|\tilde{M}_\sigma(z)| \leq 1$, and

$$\tilde{M}_\sigma(0) = 1. \tag{3.4.7}$$

In particular, $\tilde{M}_\sigma(z)$ does not vanish identically. Finally, note also that

$$\tilde{M}_\sigma(\bar{z}) = \tilde{M}_\sigma(z) = \overline{\tilde{M}_\sigma(-z)}. \tag{3.4.8}$$

The proof of Theorem 3 requires, among other things, a complex analytic treatment (in 3 complex variables s, z_1, z_2). We shall go on to this, and leave the final stage of the proof of Theorem 3 until the end of §3.

3.5 – First, for any $s, u_1, u_2 \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, and a real parameter $q > 1$, define the complex analytic function

$$h_q(s; u_1, u_2) = \sum_{a,b=0}^{\infty} i^{a+b} q^{-s|b-a|} \frac{u_1^a u_2^b}{a!b!}, \tag{3.5.1}$$

of 3 variables s, u_1, u_2, where $i = \sqrt{-1}$. (Note the absolute value $|b-a|$ instead of $b-a$ itself, which makes this function not as simple as a product of two exponential series.) Obviously, this series converges absolutely. Rearrange this with respect to $n = |b-a|$ to get

$$h_q(s; u_1, u_2) = \frac{1}{2} \sum_{n=0}^{\infty} \epsilon_n \left(\frac{i}{q^s}\right)^n (u_1^n + u_2^n) j_n(u_1 u_2), \tag{3.5.2}$$

$\epsilon_n, j_n(x)$ being as in §3.1. It has the following integral expression

$$h_q(s; u_1, u_2) = \int_{\mathbb{C}^1} \exp\left(i\left(\frac{q^s - t}{1 - q^s t} u_1 + \frac{1 - q^s t}{q^s - t} u_2\right)\right) d^*t, \tag{3.5.3}$$

where $d^*t = dt/(2\pi it)$. (Note that the integrand is invariant under $(t, u_1, u_2) \mapsto (t^{-1}, u_2, u_1)$.) Indeed, the right hand side is holomorphic in u_1, u_2, and the Taylor coefficient of each $u_1^a u_2^b$, computed by operating $\partial^{a+b}/\partial u_1^a \partial u_2^b$ under the integral sign, is given by

(3.5.4)
$$\frac{i^{a+b}}{a!b!}\int_{\mathbb{C}^1}\left(\frac{1-q^st}{q^s-t}\right)^{b-a} d^*t = \frac{i^{a+b}}{a!b!}\int_{\mathbb{C}^1}\left(\frac{1-q^st}{q^s-t}\right)^{a-b} d^*t \qquad (\text{by } t \mapsto t^{-1}).$$

Depending on whether $b \geq a$ or $a \geq b$, use the left (resp. right) expression and compute the residue at $t = 0$. This shows that the value of (3.5.4) is $i^{a+b}q^{-|b-a|s}/a!b!$, as desired.

Now let K and P be as before. Set

$$H_{s,\wp}(z_1, z_2) = h_{N(\wp)}\left(s; \frac{r_{s,\wp}}{2}z_1, \frac{r_{s,\wp}}{2}z_2\right), \tag{3.5.5}$$
$$H_{s,P}(z_1, z_2) = \prod_{\wp \in P} H_{s,\wp}(z_1, z_2),$$

where

$$r_{s,\wp} = \frac{N(\wp)^s \log N(\wp)}{N(\wp)^{2s} - 1}. \tag{3.5.6}$$

Note that these are complex analytic functions of s, z_1, z_2 on $\mathrm{Re}(s) > 0$, and

$$H_{\sigma,P}(z) = H_{\sigma,P}(z, \bar{z}) \qquad (\sigma > 0). \tag{3.5.7}$$

3.6 –

Theorem 4. *Fix any $\epsilon > 0$ and $R > 0$. Then the sum*

$$\sum_{\wp} |H_{s,\wp}(z_1, z_2) - 1|, \tag{3.6.1}$$

where $\wp$ runs over all non-archimedean primes of K, converges uniformly on the region $\mathrm{Re}(s) \geq \frac{1}{2} + \epsilon$, $|z_1|, |z_2| \leq R$. In particular, there exists $y = y_{\epsilon,R}$ such that the sum

$$\sum_{N(\wp) > y} \log H_{s,\wp}(z_1, z_2) \tag{3.6.2}$$

converges absolutely and uniformly on this region, and hence the product

$$\prod_{N(\wp) > y} H_{s,\wp}(z_1, z_2) \tag{3.6.3}$$

converges absolutely and uniformly to a nowhere vanishing analytic function on this region.

Proof. The key point is to reduce to the fact that the series

$$\sum_{\wp}(\log N(\wp))^2 N(\wp)^{-2\sigma} \tag{3.6.4}$$

converges uniformly on $\sigma \geq 1/2+\epsilon$. To avoid inessential complication of the notation (to worry about ϵ), we shall fix $\sigma > 1/2$. The uniformity statement for $\sigma \geq 1/2+\epsilon$ should be clear from the argument.

We first claim that if $|z_1|, |z_2| \leq R$, $\sigma > 1/2$ and if $N(\wp)$ is so large as to satisfy

$$Rr_{\sigma,\wp} \leq 2, \tag{3.6.5}$$

then

$$|H_{s,\wp}(z_1,z_2)-1| < \frac{5}{2}(Rr_{\sigma,\wp})^2 + 2Rr_{\sigma,\wp}N(\wp)^{-\sigma}. \tag{3.6.6}$$

In fact, by (3.5.1), (3.5.5), (writing $r = r_{\sigma,\wp}$ and $q = N(\wp)^\sigma$ here),

$$\begin{aligned} |H_{s,\wp}(z_1,z_2)-1| &\leq \sum_{(a,b)\neq(0,0)} q^{-|b-a|}\frac{1}{a!b!}(Rr/2)^{a+b} \\ &= \sum_{k=1}^{\infty}\frac{1}{(k!)^2}(Rr/2)^{2k} + 2\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} q^{-n}\frac{1}{k!(k+n)!}(Rr/2)^{2k+n} \\ &\leq \sum_{k=1}^{\infty}\frac{1}{k!}(Rr/2)^{2k} + 2\left(\sum_{n=1}^{\infty}\frac{1}{n!}(Rr/2q)^n\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}(Rr/2)^{2k}\right) \\ &= \big(\exp((Rr/2)^2)-1\big) + 2\exp((Rr/2)^2)\left(\exp(Rr/2q)-1\right). \end{aligned} \tag{3.6.7}$$

But since $e^{x/2}-1 < x$ for $0 \leq x \leq 2$, and $Rr \leq 2$, we obtain

(3.6.8)

$$|H_{s,\wp}(z_1,z_2)-1| \leq \frac{1}{2}(Rr)^2 + 2\left(1+\frac{1}{2}(Rr)^2\right)(Rr/q) < \frac{5}{2}(Rr)^2 + 2Rr/q,$$

as desired. Since

$$r_{\sigma,\wp}^2 \ll \frac{(\log N(\wp))^2}{N(\wp)^{2\sigma}}, \qquad r_{\sigma,\wp}N(\wp)^{-\sigma} \ll \frac{\log N(\wp)}{N(\wp)^{2\sigma}}, \tag{3.6.9}$$

the series (3.6.1) converges uniformly on this region.

Now let $N(\wp)$ be even so large that $Rr_{\sigma,\wp} < 1/5$. Then (3.6.6) gives

$$|H_{s,\wp}(z_1,z_2)-1| < \frac{1}{2}. \tag{3.6.10}$$

For such s, z_1, z_2, and over such $\wp$ that satisfy $Rr_{\sigma,\wp} < 1/5$, consider the infinite sum

$$\sum_{\wp \text{ as above}} \log H_{s,\wp}(z_1, z_2), \tag{3.6.11}$$

where log takes the principal values. Then, since $|w| \leq 1/2$ implies $|\log(1 + w)| \leq (3/2)|w|$, and hence

$$|\log H_{s,\wp}(z_1, z_2)| \leq \frac{3}{2}|H_{s,\wp}(z_1, z_2) - 1|, \tag{3.6.12}$$

(3.6.11) converges uniformly and absolutely. □

3.7 – For each $\wp$, we define the analytic function $\tilde{M}_{s,\wp}(z_1, z_2)$ of s, z_1, z_2 ($\mathrm{Re}(s) > 0$) by

$$\begin{aligned} \tilde{M}_{s,\wp}(z_1, z_2) &= \exp\left(\frac{i}{2}c_{s,\wp}(z_1 + z_2)\right) H_{s,\wp}(z_1, z_2) \\ &= \int_{\mathbb{C}^1} \exp\left(\frac{i}{2}(z_1 g_{s,\wp}(\bar{t}_\wp) + z_2 g_{s,\wp}(t_\wp))\right) d^* t_\wp, \end{aligned} \tag{3.7.1}$$

where

$$c_{s,\wp} = \frac{-\log N(\wp)}{N(\wp)^{2s} - 1}, \qquad g_{s,\wp}(t_\wp) = \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^s}. \tag{3.7.2}$$

The second equality in (3.7.1) follows directly from (3.5.3).

For $\mathrm{Re}(s) > 1/2$, we also define, in view of Theorem 4, the global functions

$$H_s(z_1, z_2) = \prod_\wp H_{s,\wp}(z_1, z_2), \tag{3.7.3}$$

$$\tilde{M}_s(z_1, z_2) = \prod_\wp \tilde{M}_{s,\wp}(z_1, z_2) = \exp\left(\frac{i}{2} \cdot \frac{\zeta_K'(2s)}{\zeta_K(2s)}(z_1 + z_2)\right) H_s(z_1, z_2), \tag{3.7.4}$$

$\zeta_K(s)$ being the Dedekind zeta function of K. Note here that

$$\sum_\wp c_{s,\wp} = \frac{\zeta_K'(2s)}{\zeta_K(2s)}. \tag{3.7.5}$$

In particular,

$$\tilde{M}_\sigma(z) = \tilde{M}_\sigma(z, \bar{z}) = \exp\left(i\frac{\zeta_K'(2\sigma)}{\zeta_K(2\sigma)}\mathrm{Re}(z)\right) H_\sigma(z), \tag{3.7.6}$$

where

$$H_\sigma(z) = H_\sigma(z, \bar{z}) = \prod_\wp H_{\sigma,\wp}(z). \tag{3.7.7}$$

Theorem 5. *The analytic function $\tilde{M}_s(z_1, z_2)$ has the following power series and Dirichlet series expansions.* (*The notation for their coefficients will be defined in* §3.8.)

$$\tilde{M}_s(z_1, z_2) = \sum_{a,b=0}^{\infty} (-i/2)^{a+b} \mu_s^{(a,b)} \frac{z_1^a z_2^b}{a!b!}, \tag{3.7.8}$$

$$\begin{aligned} \tilde{M}_s(z_1, z_2) &= \sum_{D \ integral} \frac{\lambda_D(z_1)\lambda_D(z_2)}{N(D)^{2s}} \\ &= \prod_\wp \left(\sum_{n=0}^{\infty} \frac{\lambda_{\wp^n}(z_1)\lambda_{\wp^n}(z_2)}{N(\wp)^{2ns}} \right). \end{aligned} \tag{3.7.9}$$

In fact, each $\wp$-factor in (3.7.9) *is equal to $\tilde{M}_{s,\wp}(z_1, z_2)$. Here, D runs over all integral ideals, and $\wp$, all non-archimedean prime divisors of K. The series* (3.7.8) *converges for all $z_1, z_2 \in \mathbb{C}$, and* (3.7.9) *for all s with* $\mathrm{Re}(s) > 1/2$.

3.8 – To define the coefficients in Theorem 5, first, for any integral ideal D of K, set

$$\begin{aligned} \Lambda(D) &= \log N(\wp) && \cdots \textit{if } D = \wp^r,\ r \geq 1, \\ &= 0 && \cdots \textit{otherwise}, \end{aligned} \tag{3.8.1}$$

for a prime divisor $\wp$. Then define $\Lambda_k(D)$ ($k \geq 0$, $k \in \mathbb{Z}$) by

$$\begin{aligned} \Lambda_0(D) &= 1 && \cdots \textit{if } D = (1) \\ &= 0 && \cdots \textit{otherwise}, \end{aligned} \tag{3.8.2}$$

$$\Lambda_k(D) = \sum_{D = D_1 \cdots D_k} \Lambda(D_1) \cdots \Lambda(D_k) \qquad (k \geq 1). \tag{3.8.3}$$

Here, the summation is over all ordered k-ples of integral ideals $(D_1, \cdots D_k)$ whose product is equal to D. (One may assume that each D_i is a prime power, for, otherwise, $\Lambda(D_i) = 0$.) Thus, if $D = \prod_\wp \wp^{n_\wp}$ is the prime factorization of D, then $\Lambda_k(D)$ is the coefficient of $\prod_\wp x_\wp^{n_\wp}$ in the polynomial

$$(3.8.4) \qquad \left(\sum_\wp (\log N(\wp))(x_\wp + \cdots + x_\wp^{n_\wp}) \right)^k,$$

where $x_\wp$ are independent variables. In particular, (put $x_\wp = 1$ for all $\wp$),

$$(3.8.5) \qquad \Lambda_k(D) \le (\log N(D))^k.$$

Also note that

$$(3.8.6) \qquad \Lambda_k(D) = 0 \qquad\qquad if \quad k > \sum_\wp n_\wp.$$

For each D, by (3.8.6), the following $\lambda_D(z)$ is a *polynomial* of z.

$$(3.8.7) \qquad \lambda_D(z) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(D)}{k!} z^k.$$

And for each pair (a, b) of non-negative integers and Re$(s)> 1/2$, define the Dirichlet series

$$(3.8.8) \qquad \mu_s^{(a,b)} = \sum_D \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2s}}.$$

By (3.8.5), this Dirichlet series converges absolutely on Re$(s)> 1/2$.

Remark 3.8.9. Since

$$-\frac{\zeta_K'(s)}{\zeta_K(s)} = \sum_D \frac{\Lambda(D)}{N(D)^s},$$

we have

$$(3.8.10) \qquad \left(-\frac{\zeta_K'(s)}{\zeta_K(s)} \right)^k = \sum_D \frac{\Lambda_k(D)}{N(D)^s} \qquad (k \ge 1).$$

Only when $K = \mathbb{Q}$ in which case $N(D)$ determines D uniquely, this can be used as an alternative definition of $\Lambda_k(D)$.

These arithmetic functions $\Lambda_k(D)$ and $\lambda_D(z)$ enjoy the following properties which are direct consequences of their definitions.

Proposition 3.8.11. (i) *When D, D' are integral ideals with $(D, D') = 1$,*

$$\frac{\Lambda_k(DD')}{k!} = \sum_{\substack{a+b=k \\ a,b\geq 0}} \frac{\Lambda_a(D)\Lambda_b(D')}{a!b!}, \tag{3.8.12}$$

$$\lambda_{DD'}(z) = \lambda_D(z)\lambda_{D'}(z). \tag{3.8.13}$$

(ii) *When $\wp$ is a prime, we have $\Lambda_k(\wp^n) = 0$ $(n < k)$, and*

$$\Lambda_k(\wp^n) = \binom{n-1}{k-1}(\log N(\wp))^k \qquad (n \geq k), \tag{3.8.14}$$

$$\lambda_{\wp^n}(z) = G_n\left(-\frac{i}{2}(\log N(\wp))z\right), \tag{3.8.15}$$

where $G_n(w)$ is the polynomial of w defined by

$$\exp\left(\frac{wt}{1-t}\right) = \sum_{n=0}^{\infty} G_n(w)t^n \qquad (|t| < 1); \tag{3.8.16}$$

namely, $G_0(w) = 1$ and

$$G_n(w) = \sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1} w^k \qquad (n \geq 1). \tag{3.8.17}$$

3.9 – In this subsection, we shall reduce the proof of Theorem 5 to some estimations of $|\lambda_D(z)|$. First, by (3.7.2) and (3.8.16) applied to $w \mapsto (-iz/2)\log N(\wp)$, $t \mapsto t_\wp N(\wp)^{-s}$, and by (3.8.15), we obtain

$$\exp\left(\frac{i}{2}zg_{s,\wp}(t_\wp)\right) = \sum_{n=0}^{\infty} \lambda_{\wp^n}(z)N(\wp)^{-ns}t_\wp^n \qquad (|t_\wp| < N(\wp)^\sigma). \tag{3.9.1}$$

By (3.7.1), $\tilde{M}_{s,\wp}(z_1, z_2)$ is equal to the constant term of the Fourier expansion of

$$\exp\left\{\frac{i}{2}(z_1 g_{s,\wp}(\bar{t}_\wp) + z_2 g_{s,\wp}(t_\wp))\right\} \tag{3.9.2}$$

in $t_\wp \in \mathbb{C}^1$; hence by (3.9.1),

$$\tilde{M}_{s,\wp}(z_1, z_2) = \sum_{n=0}^{\infty} \lambda_{\wp^n}(z_1)\lambda_{\wp^n}(z_2)N(\wp)^{-2ns}. \tag{3.9.3}$$

Among the two statements in Theorem 5, we first pay attention to the second equality (3.7.9). Note that (3.9.3) and Prop 3.8.11 give the *formal* Euler product decomposition. But we must also show that the global Dirichlet series converges on Re$(s)>1/2$. We have already established the absolute convergence of the Euler product as analytic function on this domain, but the absolute convergence of the Dirichlet series on this domain is (at least a priori) a separate matter. We shall use the following estimations of $|\lambda_D(z)|$.

Proposition 3.9.4. (i) *For any* $n \geq 1$,

$$|\lambda_{\wp^n}(z)| < \exp\sqrt{2n|z|\log N(\wp)} \qquad (n \geq 1).$$

(ii) *For any non-trivial integral divisor* $D \neq (1)$,

$$|\lambda_D(z)| < \exp\{(\log N(D))\sqrt{2C_K|z|/(\log\log N(D)+2)}\},$$

where C_K *is a positive constant depending only on* K.

The proof of Prop 3.9.4 will be postponed until §3.10.

Remark 3.9.5. The inequality (3.8.5) leads only to $|\lambda_k(D)| \leq N(D)^{|z|/2}$, from which follows only that (3.7.9) converges on Re$(s) > (1+|z|)/2$.

Proof of Theorem 5 *assuming Proposition* 3.9.4. First, by Prop 3.9.4 (ii), it is clear that for any given $\epsilon > 0$ and $R > 0$, $|\lambda_D(z)| \ll N(D)^\epsilon$ holds for all $|z| \leq R$ if $N(D)$ is sufficiently large. Therefore, (3.7.9) converges absolutely and uniformly in the wider sense on Re$(s) > 1/2$.

Secondly, to prove (3.7.8), fix s with Re$(s) > 1/2$. Since (3.7.9) converges uniformly on $|z_1|$, $|z_2| \leq 1$, we can compute the derivative $(\partial^{a+b}/\partial z_1^a \partial z_2^b)\tilde{M}_s(z_1, z_2)$ at $z_1 = z_2 = 0$ by termwise differentiation. And since

$$\frac{\partial^k}{\partial z^k}\lambda_D(z)\mid_{(0)} = (-i/2)^k \Lambda_k(D), \tag{3.9.6}$$

the Taylor expansion of $\tilde{M}_s(z_1, z_2)$ at 0 is as given by (3.7.8). But $\tilde{M}_s(z_1, z_2)$ being analytic everywhere, this power series must converge everywhere.

Thus, Theorem 5 is reduced to Prop 3.9.4.

3.10 – For the proof of Prop 3.9.4, we need two sublemmas.

Sublemma 3.10.1. *Let*

$$L_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\binom{n}{k} x^k \qquad (n \geq 0). \tag{3.10.2}$$

($L_n(-x)$ is Laguerre's polynomial.) Then

$$L_n(x) < \exp(2\sqrt{nx}) \qquad (x > 0). \tag{3.10.3}$$

Proof. Take any $t > 0$. Then

$$\begin{aligned} L_n(x) &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} t^{-k} \frac{1}{k!}(tx)^k \\ &\leq (1+t^{-1})^n \exp(tx) < \exp(nt^{-1} + tx). \end{aligned} \tag{3.10.4}$$

Take $t = (n/x)^{1/2}$. This gives $L_n(x) < \exp(2\sqrt{nx})$, as desired. □

Sublemma 3.10.5. *For each global field, there exists a positive constant C_K such that*

$$|\mathrm{Supp}(D)| \leq C_K \frac{\log N(D)}{\log\log N(D) + 2} \tag{3.10.6}$$

holds for any integral divisor $D \neq (1)$ of K. Here, Supp(D) *denotes the support of the effective divisor D, i.e., the set of prime factors of D.*

This is well-known (together with that one can take $C_K = 1 + \epsilon$ for $N(D)$ sufficiently large), especially when K is a number field. But for the convenience of readers, we shall provide with a simple unified proof for the general case. The main ingredients are:

(i) Put

$$\psi_0(y) = \sum_{N(\wp)<y} \log N(\wp) + \log y \qquad (y > 1) \tag{3.10.7}$$

($\wp$: always non-archimedean). Then there exist positive constants A_1, A_2 (depending only on K) such that

$$A_1 y < \psi_0(y) < A_2 y \qquad (y \geq 2). \tag{3.10.8}$$

(Indeed, $\lim_{y\mapsto\infty} \psi_0(y)/y = 1$ (resp. $\log q$), for a number field K (resp. a function field over $\mathbf{F}_q$).)

(ii) Let $\pi(y)$ denote, as usual, the number of (non-archimedean) primes with norm $\leq y$. Then there exists a positive constant B (depending only on K) such that

$$\pi(y) < B \cdot y/(\log y + \log A_2 + 2) \qquad (y \geq 2). \tag{3.10.9}$$

(Indeed, $\lim \pi(y)\log y/y = 1$ (resp. $\log q$).)

(iii) The function $x/(\log x + 2)$ is monotone increasing for $x > e^{-1}$; in particular, for $x \geq \log 2$.

Now write $P = \mathrm{Supp}(D)$, $n = |P|$, $D' = \prod_{\wp \in P} \wp$, and choose any set P_0 of non-archimedean primes of K consisting of "the first n primes with the smallest possible norms", namely, such as satisfying $|P_0| = n$ and $N(\wp) \leq N(\wp')$ whenever $\wp \in P_0, \wp' \notin P_0$. Now let $y = \mathrm{Max}_{\wp \in P_0} N(\wp)$, and put $D_0 = \prod_{\wp \in P_0} \wp$. Then since P_0 contains all (non-archimedean) primes $\wp$ with $N(\wp) < y$ (if any) and at least one with $N(\wp) = y$, we have

$$\pi(y) \geq |P_0| = |P|, \quad \log N(D_0) \geq \psi_0(y) \geq \log y \geq \log 2. \tag{3.10.10}$$

Also, obviously, $\log N(D) \geq \log N(D') \geq \log N(D_0)$; hence by (i)–(iii), we obtain

(3.10.11)
$$\frac{\log N(D)}{\log\log N(D) + 2} \geq \frac{\psi_0(y)}{\log \psi_0(y) + 2} > \frac{A_1 \cdot y}{\log(A_2 \cdot y) + 2} > A_1 B^{-1} \pi(y) \geq A_1 B^{-1} |P|;$$

hence (3.10.6) with $C_K = BA_1^{-1}$. (When $N(D)$ is large, each of A_1, B can be chosen to be close to 1 (resp. $\log q$); hence C_K can be chosen to be close to 1.) $\square$

Proof of Proposition 3.9.4. (i) Recall that $\lambda_{\wp^n}(z) = G_n(-\frac{i}{2}(\log N(\wp))z)$. Since $\binom{n-1}{k-1} \leq \binom{n}{k}$, we have $G_n(x) \leq L_n(x)$ for $x \geq 0$. Hence

$$|\lambda_{\wp^n}(z)| \leq L_n\left(\frac{1}{2}\log N(\wp)|z|\right) < \exp\sqrt{2n|z|\log N(\wp)}, \tag{3.10.12}$$

by Sublemma 3.10.1.

(ii) Let $D = \prod_{\wp \in P} \wp^{n_\wp}$, with $n_\wp \geq 1$, $P = \mathrm{Supp}(D)$. Then

(3.10.13)
$$\sum_{\wp \in P} (n_\wp \log N(\wp))^{1/2} \leq \left(|P| \sum_{\wp \in P} n_\wp \log N(\wp)\right)^{1/2} = (|P| \log N(D))^{1/2}.$$

This, combined with (i) and Sublemma 3.10.5 gives

(3.10.14)
$$|\lambda_D(z)| = \prod_{\wp \in P} |\lambda_{\wp^{n_\wp}}(z)| < \exp\{(\log N(D))\,(2C_K|z|/(\log\log N(D) + 2))^{1/2}\},$$

as desired.

This settles the proof of Prop 3.9.4 and hence also that of Theorem 5.

3.11 – Proof of Theorem 3

Proofs of (i)–(iii).

(i)(ii) Since we have proved Theorem 4, it remains to show the uniformity of convergence without restriction on the range of $|z|$ (namely, (ii) for $t = \infty$). This and (ii) follow directly by combining the following three properties of $\tilde{M}_{\sigma,P}(z)$. Here, t is *fixed*, with $1 \le t \le \infty$. (And here, our restriction that σ be *real* is essential.)

(a) If $|P_0| > 4$, then $\tilde{M}_{\sigma,P_0} \in L^t$ (cf. Prop 3.3.1); in particular, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\begin{cases} \int_{|z|\ge R} |\tilde{M}_{\sigma,P_0}(z)|^t |dz| < \epsilon & \cdots if\, t \ne \infty, \\ \mathrm{Sup}_{|z|\ge R} |\tilde{M}_{\sigma,P_0}(z)| < \epsilon & \cdots if\, t = \infty. \end{cases} \tag{3.11.1}$$

(b) $|\tilde{M}_{\sigma,\wp}(z)| \le 1$ for each $\wp$; hence

$$|\tilde{M}_{\sigma,P}(z)| \le |\tilde{M}_{\sigma,P_0}(z)| \qquad \cdots for\ any\ P \supseteq P_0, \tag{3.11.2}$$

and

$$|\tilde{M}_{\sigma}(z) - \tilde{M}_{\sigma,P}(z)|^t = \left| \prod_{\wp \notin P} \tilde{M}_{\sigma,\wp}(z) - 1 \right|^t |\tilde{M}_{\sigma,P}(z)|^t \le 2^t |\tilde{M}_{\sigma,P_0}(z)|^t. \tag{3.11.3}$$

(c) $\tilde{M}_{\sigma,P}(z)$ converges to $\tilde{M}_{\sigma}(z)$ uniformly on $|z| \le R$ for any given $R > 0$. (For a given $\epsilon > 0$, first choose R to validate (a); then apply (3.11.3) to show (say, for $t < \infty$) that the integral of $|\tilde{M}_{\sigma}(z) - \tilde{M}_{\sigma,P}(z)|^t$ over $|z| \ge R$ is small, then choose $P \supseteq P_0$ large enough to make this integral over $|z| \le R$ also small.)

(iii) Obvious by Theorem 4.

(iv) Also obvious by Prop 3.3.1, because $|\tilde{M}_{\sigma}(z)| \le |\tilde{M}_{\sigma,P}(z)|$.

(v) In general, use the symbols $\wedge, \vee$ for

$$f^{\wedge}(z) = \int_{\mathbb{C}} f(w)\psi_z(w)|dw|, \tag{3.11.4}$$

$$g^{\vee}(z) = \int_{\mathbb{C}} g(w)\psi_{-z}(w)|dw|. \tag{3.11.5}$$

Recall that $\tilde{M}_{\sigma,P} = M^{\wedge}_{\sigma,P}$, $M_{\sigma,P} = \tilde{M}^{\vee}_{\sigma,P}$ for each P. Recall also that for each t $(1 \le t \le \infty)$, $\tilde{M}_{\sigma,P}$ (for $|P| > 4$) "L^t-converges" to $\tilde{M}_\sigma$. The case $t = 2$ reflects to that $\tilde{M}^{\vee}_{\sigma,P}$ L^2-converges to $\tilde{M}^{\vee}_\sigma$. But $\tilde{M}_\sigma$ belongs to L^1; hence $\tilde{M}^{\vee}_\sigma$ is continuous. Therefore, $\tilde{M}^{\vee}_\sigma$ must coincide with the L^∞-limit M_σ of $\tilde{M}^{\vee}_{\sigma,P} = M_{\sigma,P}$.

$$\tilde{M}^{\vee}_\sigma(z) = M_\sigma(z). \tag{3.11.6}$$

Now, since $M_{\sigma,P}(z)$ converges uniformly to $M_\sigma(z)$ (Theorem 2), and each $M_{\sigma,P}(z)$ has total volume 1, we have

$$\int_{\mathbb{C}} M_\sigma(z)|dz| \le 1; \tag{3.11.7}$$

hence ($M_\sigma(z)$ being non-negative real valued) $M_\sigma \in L^1$. Therefore, $M^{\wedge}_\sigma$ is continuous. But $M^{\wedge}_\sigma = (\tilde{M}^{\vee}_\sigma)^{\wedge}$ is equal to $\tilde{M}_\sigma$ in L^2, i.e., $M^{\wedge}_\sigma = \tilde{M}_\sigma$ almost everywhere. Both being continuous, we conclude

$$M^{\wedge}_\sigma(z) = \tilde{M}_\sigma(z), \tag{3.11.8}$$

as desired.

(vi) This is a special case of Theorem 5. □

Now, since $\tilde{M}_\sigma = M^{\wedge}_\sigma$, we obtain the expected equality

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = \tilde{M}_\sigma(0) = 1. \tag{3.11.9}$$

.

Corollary 3.11.10. *$M_\sigma(z)$ is continuous in (σ, z).*

Proof. Since $\tilde{M}_\sigma(w)\psi_{-z}(w)$ is continuous in (σ, z, w), the integral

$$\int_{|w|\le R} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw| \tag{3.11.11}$$

is continuous in (σ, z) for each $R > 0$, and as $R \mapsto \infty$, this converges uniformly in the wider sense to $M_\sigma(z)$, because if we choose any P with $|P| = 5$, then

$$|\tilde{M}_\sigma(w)| \le |\tilde{M}_{\sigma,P}(w)| \ll \left(\prod_{\wp\in P} N(\wp)\right)^{\sigma/2} |w|^{-5/2} \tag{3.11.12}$$

by (3.3.13). □

By Theorem 2 (iii) and Theorem 3, we also obtain directly

Corollary 3.11.13. *For each pair of integers $a, b \geq 0$, let $D^{(a,b)} = \partial^{a+b}/\partial z^a \partial \bar{z}^b$, and $P^{(a,b)}(w) = \bar{w}^a w^b$. Then*

$$D^{(a,b)}\tilde{M}_\sigma(z) = \left(\frac{i}{2}\right)^{(a+b)} \int_{\mathbb{C}} M_\sigma(w) P^{(a,b)}(w)\psi_z(w)|dw|. \tag{3.11.14}$$

In particular,

$$(-1)^{(a+b)}\mu_\sigma^{(a,b)} = \int_{\mathbb{C}} M_\sigma(w) P^{(a,b)}(w)|dw|. \tag{3.11.15}$$

(The latter can also be deduced through its "$M_{\sigma,P}$-version".)

Remark 3.11.16. Some readers may wonder why we do not treat the whole content of this section under the complex analytic framework. This is mainly because the basic inequalities $|\tilde{M}_{\sigma,\wp}(z)| \leq 1$ and $|\tilde{M}_{\sigma,\wp}(z)| = O(|z|^{-1/2})$ (Prop 3.3.1) that support (not only our proof but also the validity itself of) the main statements of Theorem 3, do not hold in general unless σ is real.

Remark 3.11.17. By (3.7.6), $H_\sigma(z)$ is the Fourier transform of

$$M_\sigma\left(z + \frac{\zeta_K'(2\sigma)}{\zeta_K(2\sigma)}\right). \tag{3.11.18}$$

Each of $\zeta_K'(2\sigma)/\zeta_K(2\sigma)\,(<0)$ and 0 may be regarded as a "central point" for the density measure $M_\sigma(z)|dz|$ in its own sense. The former by construction, and the latter is the center of gravity of the distribution, because

$$\int_{\mathbb{C}} M_\sigma(z)\cdot z|dz| = -\mu_\sigma^{(1,0)} = 0. \tag{3.11.19}$$

A related open question is "where is the maximal value of $M_\sigma(z)$ attained?" . (When $K = \mathbb{Q}$ and $\sigma = 1$, the approximate values of $M_\sigma(0)$ and $M_\sigma(\zeta'(2)/\zeta(2))$, computed by the Fourier inversion formula ($\tilde{M}_\sigma(z)$ resp. $H_\sigma(z)$ being integrated over $|x|, |y| \leq 10$, the Euler product taken over the first 60 primes), are $2.41, 1.48$, respectively.)

Another, even more interesting question is about the "Plancherel Volume"

$$\int_{\mathbb{C}} M_\sigma(z)^2|dz| = \int_{\mathbb{C}} |\tilde{M}_\sigma(z)|^2|dz|, \tag{3.11.20}$$

which may be interpreted also as the density at $z = 0$ of the distribution of *differences* of two points in the original measure space. The author has noticed (after the submittence of this manuscript) that, in our case, this quantity is very close to the multiplicative inverse of the moment $\mu_\sigma^{(1,1)}$. (The equality holds for the 2-dimensional Gaussian distribution!) A quantitative study of this relation will be left to the future publication.

§4. Connections with $L'(\chi, s)/L(\chi, s)$; (I) Case $\sigma > 1$

4.1 – In general, it is not clear to the author what family of characters χ one should treat, and how one should define the "average" of $\Phi(L'(\chi, s)/L(\chi, s))$ over χ. If C_K denotes the idèle class group of K, and C_K^1 its norm one part, so that C_K^1 is compact (and $C_K/C_K^1 \simeq \mathbf{R}$ or $\mathbf{Z}$), the Pontrjagin dual $(C_K^1)^\wedge$ is discrete, and the "average" of any function of χ over this group depends essentially on the order of counting. For example, if the order is such that the highly ramified characters appear earlier, then the distribution of $L'(\chi, s)/L(\chi, s)$ will be "unreasonably" denser near 0. Such an ordering will not be interesting. At this stage, instead of trying to give a general formulation, we shall focus our attention to the following three typical families (A)(B)(C) of χ.

(A) (i) K is either $\mathbb{Q}$, an imaginary quadratic number field, or a function field over $\mathbb{F}_q$ with an assigned prime divisor $\wp_\infty$ of degree 1 treated as being archimedean, i.e., excluded from the L- and the M-factors. In short, K has exactly *one* archimedean prime.
(ii) χ runs over all Dirichlet characters on K (the non-archimedean part of) whose conductor is a prime divisor, such that $\chi(\wp_\infty) = 1$. (We may or may not impose χ even when $K = \mathbb{Q}$.)

(B) (i) K is a number field having *at least two* archimedean primes.
(ii) χ runs over the free group of rank $N-1$ $(N = [K : \mathbb{Q}])$ consisting of "normalized *unramified Grössencharacters*", defined as follows. First, choose a $\mathbb{Z}$-basis χ_j^0 $(1 \le j \le N-1)$ of the character group of K_∞^1/E_K (an $N-1$ -dimensional torus), where K_∞^* is the infinite part of the idèle group, the superscript 1 means the norm one part, and E_K is the global unit group. Then fix one archimedean prime $\wp_{\infty 1}$ and extend each χ_j^0 uniquely to a character of K_∞^*/E_K by imposing that its $\wp_{\infty 1}$-component is trivial on the image of $\mathbb{R}_+^*$. Then, in view of the exact sequence

$$1 \mapsto K_\infty^*/E_K \mapsto C_K/i(U_K) \mapsto Cl_K \mapsto 1$$

($i(U_K)$: the image of the local unit group on C_K, Cl_K: the ideal class group), extend each χ_j^0 to a character χ_j of $C_K/i(U_K)$ (make one choice).

(C) (The value distribution for $d\log\zeta(\sigma+\tau i)/d\sigma$)
(i) $K = \mathbb{Q}$;
(ii) $\chi_\tau(\wp) = N(\wp)^{-\tau i}$ $(\tau \in \mathbb{R})$.

In each case, the average of a complex valued function $\phi(\chi)$ of χ over the family is defined as follows.

(A) First, for each prime divisor $\mathbf{f}$, we take the usual average of $\phi(\chi)$ over all those χ with the (non-archimedean part of the) conductor $\mathbf{f}$. Then we take the average of this average over all $\mathbf{f}$ with $N(\mathbf{f}) \leq m$;

$$\mathrm{Avg}_{N(\mathbf{f})\leq m}\phi(\chi) = \frac{\sum_{N(\mathbf{f})\leq m}(\sum_{\mathbf{f}_\chi=\mathbf{f}}\phi(\chi))/(\sum_{\mathbf{f}_\chi=\mathbf{f}}1)}{\sum_{N(\mathbf{f})\leq m}1}, \tag{4.1.1}$$

where the summation $\sum_{N(\mathbf{f})\leq m}$ is over all non-archimedean prime divisors $\mathbf{f}$ of K with $N(\mathbf{f}) \leq m$. Finally, we define

$$\mathrm{Avg}_\chi\phi(\chi) = \lim_{m\mapsto\infty}(\mathrm{Avg}_{N(\mathbf{f})\leq m}\phi(\chi)), \tag{4.1.2}$$

whenever the limit exists.

(B) For $\chi^{(m)} = \chi_1^{m_1}\cdots\chi_{N-1}^{m_{N-1}}$ $(m_1, ..., m_{N-1} \in \mathbb{Z})$,

$$\mathrm{Avg}_\chi\phi(\chi^{(m)}) = \lim_{T\mapsto\infty}\frac{1}{(2T+1)^{N-1}}\sum_{\mathrm{Max}|m_j|\leq T}\phi(\chi^{(m)}). \tag{4.1.3}$$

(C)

$$\mathrm{Avg}_\chi\,\phi(\chi_\tau) \;=\; \lim_{T\mapsto\infty}\frac{1}{2T}\int_{-T}^{T}\phi(\chi_\tau)d\tau, \tag{4.1.4}$$

where $\phi(\chi_\tau)$ is any integrable function of τ on $\mathbb{R}$.

In the following, when we state a formula for $\mathrm{Avg}_\chi\Phi(L'(\chi,s)/L(\chi,s))$, it will first mean that it exists. Thus, in Theorem 6 below and in Theorem 7 in §6, the existence of the limit average is, in each case, a part of the theorem. We add here also that our main results will not be affected by minor changes of weights; the present definition is chosen because it is suitable for making direct use of the orthogonality relation for characters.

4.2 − The main purpose of §4 is to prove the following

Theorem 6. *Let $s \in \mathbb{C}$ be fixed, with $\sigma = \mathrm{Re}(s) > 1$. Then in each of Case* A, B, C*,*
(i)

$$\mathrm{Avg}_\chi\Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|$$

holds for any continuous function $\Phi(w)$ on $\mathbb{C}$.

(ii)

$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)=\tilde{M}_\sigma(z),$$

(iii)

$$\mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)=(-1)^{(a+b)}\mu_\sigma^{(a,b)},$$

where $\psi_z(w)=\exp(i\mathrm{Re}(\bar{z}w))$, $P^{(a,b)}(w)=\bar{w}^a w^b$ $(a,b\in\mathbb{Z},a,b\geq 0)$, *and* $\mu_\sigma^{(a,b)}$ *is as in* §3.8.

Corollary 4.2.1. *When* $\mathrm{Re}(s)>1$, *and* k *is an odd positive integer,*

$$\mathrm{Avg}_\chi\left(\mathrm{Re}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)\right)^k\leq 0, \tag{4.2.2}$$

with the equality if and only if $k=1$.

Proof. By (iii), this average is equal to

$$(-2)^{-k}\sum_{a+b=k}\binom{k}{a}\mu_\sigma^{(a,b)}, \tag{4.2.3}$$

but $\mu_\sigma^{(a,b)}\geq 0$ with the equality if and only if $ab=0$. □

4.3 – The first key to the proof of Theorem 6 is the uniformity of distribution of $\{\chi_P\}_\chi$ on T_P for each P.

Lemma 4.3.1. *Let* P *be any finite set of non-archimedean primes of* K, *and set* $T_P=\prod_{\wp\in P}\mathbb{C}^1$. *Let* χ *run over any one of the three families of characters on* K *described in* §4.1 *but for the family* (*A*), *exclude those* (*finitely many*) χ *such that* $\mathbf{f}_\chi\in P$. *For each such* χ, *put* $\chi_P=(\chi(\wp))_{\wp\in P}\in T_P$. *Then* $(\chi_P)_\chi$ *is uniformly distributed on* T_P; *namely, for any continuous function* $\Psi:T_P\mapsto\mathbb{C}$, *we have*

$$\mathrm{Avg}_\chi(\Psi(\chi_P))=\int_{T_P}\Psi(t_P)d^*t_P. \tag{4.3.2}$$

Proof. Let $\mathbb{Z}_P=\prod_{\wp\in P}\mathbb{Z}$, and for $n=(n_\wp)\in\mathbb{Z}_P$ and $t=(t_\wp)\in T_P$, write $t^n=\prod_{\wp\in P}t_\wp^{n_\wp}\in\mathbb{C}^1$ (a dual pairing between T_P and $\mathbb{Z}_P$). By Weyl's criterion for uniform distribution, it suffices to prove (4.3.2) when $\Psi(t)$ is any character $\Psi(t) = t^n$, or what amounts to the same, it suffices to prove

$$\mathrm{Avg}_\chi(\chi_P^n) = 0 \qquad (n \in \mathrm{Z}_P \setminus (0)). \tag{4.3.3}$$

Now we treat three cases (A)(B)(C) separately.

(Case A) To prove (4.3.3), pick any $n = (n_\wp) \in \mathbb{Z}_P \setminus (0)$, and call P^n the divisor defined by $\prod_{\wp \in P} \wp^{n_\wp}$. Note that $P^n \neq (1)$ and that $\chi(P^n) = \chi_P^n$. Now if $\mathbf{f}$ is any non-archimedean prime not contained in P, the orthogonality of characters gives

$$T_{\mathbf{f}} := \sum_{\mathbf{f}_\chi | \mathbf{f}} \chi(P^n) / \sum_{\mathbf{f}_\chi | \mathbf{f}} 1 \ = \begin{cases} 1 \cdots \text{“}\mathrm{P^n} \equiv 1 \ (\mathrm{mod}\ \mathbf{f})\text{”}, \\ 0 \cdots \text{otherwise}, \end{cases} \tag{4.3.4}$$

where for any divisor D of K, “$D \equiv 1(\mathrm{mod}\ \mathbf{f})$” means that D belongs to the common kernel of all χ with $\mathbf{f}_\chi | \mathbf{f}$. But since $P^n \neq (1)$, and the unit group of our field K is finite, there exist at most finitely many $\mathbf{f}$ such that $P^n \equiv 1(\mathrm{mod}\ \mathbf{f})$. Just from this follows (4.3.3) by easy estimations. In fact, put

$$T'_{\mathbf{f}} \ = \sum_{\mathbf{f}_\chi = \mathbf{f}} \chi(P^n) / \sum_{\mathbf{f}_\chi = \mathbf{f}} 1. \tag{4.3.5}$$

Then it is easy to see that $|T'_{\mathbf{f}} - T_{\mathbf{f}}| \ll N(\mathbf{f})^{-1}$, where the implied constant depends only on K; hence

$$\frac{\sum_{N(\mathbf{f}) \le m} (T'_{\mathbf{f}} - T_{\mathbf{f}})}{\sum_{N(\mathbf{f}) \le m} 1} \ll \frac{\sum_{N(\mathbf{f}) \le m} N(\mathbf{f})^{-1}}{\sum_{N(\mathbf{f}) \le m} 1} \ll \frac{(\log m)(\log\log m)}{m}; \tag{4.3.6}$$

hence by (4.3.4),

$$\frac{\sum_{N(\mathbf{f}) \le m} T'_{\mathbf{f}}}{\sum_{N(\mathbf{f}) \le m} 1} \ll \frac{\log m}{m} \#\{\mathbf{f}; P^n \equiv 1(\mathrm{mod}\,\mathbf{f})\} + \frac{(\log m)(\log\log m)}{m}. \tag{4.3.7}$$

Since the limit, as $m \mapsto \infty$, of the left hand side (resp. the right hand side) of (4.3.7) is $\mathrm{Avg}_\chi \chi(P^n)$ (resp. 0), (4.3.3) holds for each $n \in \mathbb{Z}_P \setminus (0)$.

(Case B) Here, the key point is the following

Sublemma 4.3.8. *If D is a non-trivial fractional ideal of K, then $\chi_j(D) \neq 1$ for at least one $1 \le j \le N-1$.*

In fact, suppose on the contrary that $\chi_j(D) = 1$ for all j, and put $D^h = (\alpha)$ (h: the class number, $\alpha \in K$). Then $\chi_j(\alpha_\infty) = 1$ for all j (χ_j being regarded as a character of the idèle class group, α_∞ the infinite component). But this means that α must coincide with some unit; hence $D = 1$, contradiction. $\square$

By applying this to the non-trivial divisor P^n ($n \in Z_P \setminus (0)$), we obtain (4.3.3) easily.

(Case C) This case is known as the Kronecker-Weyl lemma, and is simply based on the $\mathbb{Q}$-linear independence of $(\log N(\wp))_{\wp \in P}$. Thus, this is valid only when $K = \mathbb{Q}$. $\square$

4.4 – *Proof of Theorem* 6. Write $s = \sigma + \tau i$. First, take any finite set P of non-archimedean primes of K. Recall that

$$\frac{L_P'(\chi, s)}{L_P(\chi, s)} = g_{\sigma,P}(N(P)^{-\tau \cdot i}\chi_P) \tag{4.4.1}$$

if $(\mathbf{f}_\chi, P) = 1$. First, let χ run over all characters described in §4.1 such that $(\mathbf{f}_\chi, P) = 1$. Then since $\{\chi_P\}_\chi$ is uniformly distributed on T_P, so is its translate $\{N(P)^{-\tau \cdot i}\chi_P\}_\chi$. Therefore, by Lemma 4.3.1 applied to $\Psi = \Phi \circ g_{\sigma,P}$, we obtain

$$\begin{aligned} \operatorname{Avg}'_\chi \left(\Phi \left(\frac{L_P'(\chi, s)}{L_P(\chi, s)} \right) \right) &= \int_{T_P} \Phi(g_{\sigma,P}(t_P)) d^* t_P \\ &= \int_{\mathbb{C}} M_{\sigma,P}(w) \Phi(w) |dw| \end{aligned} \tag{4.4.2}$$

(cf. Theorem 1). Here, Avg'_χ means that we excluded finitely many χ such that $\mathbf{f}_\chi \in P$. But since this difference does not affect the value of Avg_χ, we obtain

$$\operatorname{Avg}_\chi \left(\Phi \left(\frac{L_P'(\chi, s)}{L_P(\chi, s)} \right) \right) = \int_{\mathbb{C}} M_{\sigma,P}(w) \Phi(w) |dw|. \tag{4.4.3}$$

Now since $\mathrm{Re}(s) > 1$ (and s is fixed), $L_P'(\chi, s)/L_P(\chi, s)$ tends *uniformly* to $L'(\chi, s)/L(\chi, s)$. Indeed, for any χ we have

$$\left| \frac{L'(\chi, s)}{L(\chi, s)} - \frac{L_P'(\chi, s)}{L_P(\chi, s)} \right| \leq \sum_{\wp \notin P} \frac{\log N(\wp)}{N(\wp)^\sigma - 1}, \tag{4.4.4}$$

and the right hand side tends to 0 when $P = P_y$ and $y \mapsto \infty$. Moreover, since $|L'(\chi, s)/L(\chi, s)|$ and $|L_P'(\chi, s)/L_P(\chi, s)|$ are uniformly bounded (by $|\zeta_K'(\sigma)/\zeta_K(\sigma)|$), and $M_\sigma(w)$ is compactly supported (because $\sigma > 1$), the effect of Φ is only within these bounds; hence we may assume Φ to be uniformly continuous. Therefore, $\Phi(L_P'(\chi, s)/L_P(\chi, s))$ tends uniformly to $\Phi(L'(\chi, s)/L(\chi, s))$.

Therefore, the same holds for their averages Avg_χ. And since $M_{\sigma,P}(w)$ tends uniformly to $M_\sigma(w)$ (Theorem 2), we obtain from (4.4.3) by letting $P = P_y, y \mapsto \infty$, the statement (i) of Theorem 6.

The second statement (ii) is a special case of (i). The last formula (iii) is also a special case where $\Phi(w) = P^{(a,b)}(w)$. This is the "easiest" case of Cor 3.11.13 (because now $M_\sigma(z)$ is compactly supported). This completes the proof of Theorem 6. □

§5. Some Fourier Analysis of $\psi_z(g_{\sigma,P}(t))$

5.1 – We come back to the general situation where K is any global field, P is any finite set of non-archimedean primes of K, and $T_P = \prod_{\wp\in P}\mathbb{C}^1$, $\mathbb{Z}_P = \prod_{\wp\in P}\mathbb{Z}$, with the dual pairing

$$t^n = \prod_{\wp\in P} t_\wp^{n_\wp} \in \mathbb{C}^1 \qquad (t=(t_\wp)\in T_P,\ n=(n_\wp)\in\mathbb{Z}_P). \tag{5.1.1}$$

For $\sigma > 0$, put, as before,

$$g_{\sigma,P}(t) = \sum_{\wp\in P} g_{\sigma,\wp}(t_\wp), \qquad g_{\sigma,\wp}(t_\wp) = \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma}. \tag{5.1.2}$$

For $z_1, z_2, w \in \mathbb{C}$, put

$$\psi_{z_1,z_2}(w) = \exp\left(\frac{i}{2}(z_1\bar{w} + z_2 w)\right). \tag{5.1.3}$$

Thus, $\psi_{z_1,z_2} : \mathbb{C} \mapsto \mathbb{C}^\times$ is a quasi-character of the additive group $\mathbb{C}$, which is a character into $\mathbb{C}^1$ when $z_2 = \bar{z}_1$. In our previous notation,

$$\psi_{z,\bar{z}}(w) = \psi_z(w). \tag{5.1.4}$$

We shall study the Fourier expansion of $\psi_{z_1,z_2}(g_{\sigma,P}(t))$, as a preparation for §6. First, we shall prove the following

Proposition 5.1.5. *For each $\sigma > 0$, $z_1, z_2 \in \mathbb{C}$ and P, the function $\psi_{z_1,z_2}(g_{\sigma,P}(t))$ of $t \in T_P$ has an absolutely convergent Fourier expansion*

$$\psi_{z_1,z_2}\left(g_{\sigma,P}(t)\right) = \sum_{n\in\mathbb{Z}_P} A_{\sigma,P}(n;\, z_1, z_2) t^n, \tag{5.1.6}$$

with

$$\begin{aligned} A_{\sigma,P}(n;\, z_1,z_2) &= \int_{T_P} \psi_{z_1,z_2}(g_{\sigma,P}(t)) t^{-n} d^*t \\ &= \sum_{D_2 D_1^{-1} = P^n} \lambda_{D_1}(z_1)\lambda_{D_2}(z_2) N(D_1D_2)^{-\sigma}. \end{aligned} \tag{5.1.7}$$

Here, the last summation is over all integral ideals D_1, D_2 with supports in P such that $D_2 D_1^{-1} = P^n (= \prod_{\wp \in P} \wp^{n_\wp})$, and the polynomial $\lambda_D(z)$ is as defined in §3.8.

Proof. Each side of these formulas being multiplicative, it suffices to prove them when P consists of a single prime $\wp$. So, write $t = t_\wp$. Since $\exp(\frac{i}{2} z \cdot g_{\sigma,\wp}(t))$ is a holomorphic function of t outside the point $t = N(\wp)^\sigma$, its Taylor expansion (cf. (3.9.1))

$$\exp\left(\frac{i}{2} z \cdot g_{\sigma,\wp}(t)\right) = \sum_{n=0}^{\infty} \lambda_{\wp^n}(z) N(\wp)^{-n\sigma} t^n \tag{5.1.8}$$

at $t = 0$ is absolutely convergent on $|t| < N(\wp)^\sigma$. Therefore, $\psi_{z_1,z_2}(g_{\sigma,\wp}(t))$ is the product of two absolutely convergent series, for $\exp(\frac{i}{2} z_2 \cdot g_{\sigma,\wp}(t))$ and for $\exp(\frac{i}{2} z_1 \cdot g_{\sigma,\wp}(\bar{t}))$, on the domain $|t| < N(\wp)^\sigma$. By restricting this to $|t| = 1$, replacing $\bar{t}$ by t^{-1} and rearranging the absolutely convergent double series, we obtain the absolutely convergent series (5.1.6) for $P = \{\wp\}$, with

$$A_{\sigma,\wp}(n; z_1, z_2) = \sum_{\substack{n_1, n_2 \geq 0 \\ n_2 - n_1 = n}} \lambda_{\wp^{n_1}}(z_1) \lambda_{\wp^{n_2}}(z_2) N(\wp)^{-(n_1+n_2)\sigma}. \tag{5.1.9}$$

□

It is clear that

$$A_{\sigma,P}(n; z_1, z_2) = \prod_{\wp \in P} A_{\sigma,\wp}(n_\wp; z_1, z_2) \qquad (n = (n_\wp)), \tag{5.1.10}$$

$$A_{\sigma,P}(-n; z_1, z_2) = A_{\sigma,P}(n; z_2, z_1), \tag{5.1.11}$$

$$A_{\sigma,P}(0; z_1, z_2) = \tilde{M}_{\sigma,P}(z_1, z_2). \tag{5.1.12}$$

(cf. (3.7.1), (5.1.7)). Put

$$A_{\sigma,P}(n, z) = A_{\sigma,P}(n; z, \bar{z}) \qquad (n \in \mathbb{Z}_P,\ z \in \mathbb{C}), \tag{5.1.13}$$

so that

$$A_{\sigma,P}(0, z) = \tilde{M}_{\sigma,P}(z). \tag{5.1.14}$$

Then, clearly,

$$\left| \sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n, z) t^n \right| = |\psi_z(g_{\sigma,P}(t))| = 1 \qquad (t \in T_P), \tag{5.1.15}$$

and the Parseval equality gives also that

$$\sum_{n\in\mathbb{Z}_P} |A_{\sigma,P}(n,z)|^2 = \int_{T_P} |\psi_z(g_{\sigma,P}(t))|^2 d^*t = 1. \tag{5.1.16}$$

On the other hand, the value of the (convergent) series

$$\sum_{n\in\mathbb{Z}_P} |A_{\sigma,P}(n,z)| \tag{5.1.17}$$

grows (unboundedly when $\sigma \le 1$) with P. What we shall actually need is Cor 5.2.18 below which gives an estimation of a sum similar to (5.1.17) for $P = \{\wp\}$. (Remark 5.2.23 says that this is essentially the best possible.)

5.2 – In this subsection, we shall first generalize the formulas given in §3.5 for the function (5.1.12), to the case $n \neq 0$. By (5.1.10), (5.1.11), it suffices to give a formula for $A_{\sigma,\wp}(n_\wp; z_1, z_2)$ when $n_\wp > 0$. Then we apply this to the estimation mentioned above.

Proposition 5.2.1. *Let $n > 0$, and write $q = N(\wp)^\sigma, \lambda = \log N(\wp)$. Then*

(5.2.2)

$$\exp\left(-\frac{i}{2}c_{\sigma,\wp}(z_1+z_2)\right) A_{\sigma,\wp}(n;\, z_1, z_2) = \frac{1}{q^n}\sum_{\nu=1}^{n}\binom{n-1}{\nu-1}\left(\frac{-i\lambda z_2}{2}\right)^{\nu} B^{(\nu)}_{\sigma,\wp}(z_1,z_2),$$

where

$$B^{(\nu)}_{\sigma,\wp}(z_1,z_2) = \sum_{\ell=0}^{\infty}\left(\frac{ir_{\sigma,\wp}z_2}{2q}\right)^{\ell}\binom{\nu+\ell}{\nu} j_{\nu+\ell}\left(\frac{r^2_{\sigma,\wp}z_1z_2}{4}\right), \tag{5.2.3}$$

$j_n(z)$ being as in (3.1.8). *In particular,*

(5.2.4)

$$\exp(-ic_{\sigma,\wp}\mathrm{Re}(z))A_{\sigma,\wp}(n,z) = \frac{1}{q^n}\sum_{\nu=1}^{n}\binom{n-1}{\nu-1}(1-q^2)^{\nu}\left(\frac{i}{q}\right)^{\nu}\left(\frac{\bar z}{|z|}\right)^{\nu} C^{(\nu)}_{\sigma,\wp}(z),$$

with

$$C^{(\nu)}_{\sigma,\wp}(z) = \sum_{\ell=0}^{\infty}\left(\frac{i}{q}\right)^{\ell}\left(\frac{\bar z}{|z|}\right)^{\ell}\binom{\nu+\ell}{\nu} J_{\nu+\ell}(r_{\sigma,\wp}|z|). \tag{5.2.5}$$

Proof. As in §2.1, we change variables,

$$g_{\sigma,\wp}(t) = c_{\sigma,\wp} + r_{\sigma,\wp}t', \tag{5.2.6}$$

$$t' = \frac{1-qt}{q-t}. \tag{5.2.7}$$

Recall that $|t| = 1$ if and only if $|t'| = 1$. As a function of t, t' is holomorphic outside $t = q$; hence $(t')^a(\bar{t}')^b$ $(a, b \geq 0)$ has an absolutely convergent Fourier expansion

$$(t')^a(\bar{t}')^b = \sum_{n\in\mathbb{Z}} \gamma^{(a,b)}(n)t^n \qquad (|t| = 1), \tag{5.2.8}$$

with

$$\gamma^{(a,b)}(n) = \int_{\mathbb{C}^1} \left(\frac{q-t}{1-qt}\right)^{b-a} t^{-n}d^*t = \int_{\mathbb{C}^1} \left(\frac{q-t}{1-qt}\right)^{a-b} t^n d^*t. \tag{5.2.9}$$

By direct residue calculus at $t = q^{-1}$, we obtain $\gamma^{(a,b)}(n) = 0$ if $a \leq b$, and

(5.2.10)
$$\gamma^{(a,b)}(n) = \frac{1}{q^{n+a-b}} \sum_{\nu=1}^{\min(n,a-b)} \binom{n-1}{\nu-1}\binom{a-b}{\nu}(1-q^2)^\nu \qquad (a > b).$$

In fact, when $a > b$, $q^{n+a-b}\gamma^{(a,b)}(n)$ is the coefficient of y^{a-b-1} in

$$(y + 1 - q^2)^{a-b}(y + 1)^{n-1}$$

(y corresponds to $qt - 1$).

Now, each $A_{\sigma,\wp}(n; z_1, z_2)$ is an analytic function of z_1, z_2, and the coefficients of its Taylor expansion at (z)=(0) can be obtained by operating $\partial^{a+b}/\partial z_1^a \partial z_2^b$ under the integration symbol in (5.1.7) and by putting (z)=(0). But since

$$\psi_{z_1,z_2}(g_{\sigma,\wp}(t)) = \exp\left(\frac{i}{2}c_{\sigma,\wp}(z_1 + z_2)\right)\exp\left(\frac{i}{2}r_{\sigma,\wp}(z_2t' + z_1\bar{t}')\right), \tag{5.2.11}$$

we obtain
(5.2.12)
$$A_{\sigma,\wp}(n; z_1, z_2) = \exp\left(\frac{i}{2}c_{\sigma,\wp}(z_1 + z_2)\right) \sum_{a,b=0}^{\infty} \left(\frac{ir_{\sigma,\wp}}{2}\right)^{a+b} \gamma^{(a,b)}(n)\frac{z_2^a z_1^b}{a!b!}.$$

By inserting (5.2.10) into (5.2.12), and by rearranging the series (use $\ell = a - b - \nu$, and note that $r_{\sigma,\wp}(1 - q^2) = -\lambda q$), we obtain the desired formula. □

Corollary 5.2.13. *When* $n \neq 0$,

$$|A_{\sigma,\wp}(n,z)| \leq \frac{1}{N(\wp)^{\sigma|n|}} e^{\frac{1}{2}|c_{\sigma,\wp}z|} \sum_{\nu=1}^{|n|} \frac{1}{\nu!} \binom{|n|-1}{\nu-1} \left(\frac{|z|\log N(\wp)}{2}\right)^{\nu}. \tag{5.2.14}$$

Proof. Since

$$|J_n(w)| \leq \frac{1}{n!}\left(\frac{|w|}{2}\right)^n e^{|\mathrm{Im}(w)|} \tag{5.2.15}$$

(cf. e.g. [11, §3.31]), with the notation of Prop 5.2.1, we obtain

$$\begin{aligned} |C^{(\nu)}_{\sigma,\wp}(z)| \; &\leq \; \sum_{\ell=0}^{\infty} \frac{1}{q^\ell}\binom{\nu+\ell}{\nu}\frac{1}{(\nu+\ell)!}\left(\frac{r_{\sigma,\wp}|z|}{2}\right)^{\nu+\ell} \\ &= \frac{1}{\nu!}\left(\frac{r_{\sigma,\wp}|z|}{2}\right)^{\nu}\exp\left(\frac{r_{\sigma,\wp}|z|}{2q}\right); \end{aligned} \tag{5.2.16}$$

$$|A_{\sigma,\wp}(n,z)| \leq \frac{1}{q^{|n|}}\sum_{\nu=1}^{|n|}\binom{|n|-1}{\nu-1}(q-q^{-1})^{\nu}|C^{(\nu)}_{\sigma,\wp}(z)|. \tag{5.2.17}$$

But since $(q-q^{-1})r_{\sigma,\wp} = \log N(\wp)$ and $q^{-1}r_{\sigma,\wp} = |c_{\sigma,\wp}|$, (5.2.14) follows. □

Corollary 5.2.18. *Put* $q = N(\wp)^\sigma$, $\lambda = \log N(\wp)$, *and assume now that* $\sigma > 1/2$. *Then*

$$\sum_{n\in\mathbb{Z}} |A_{\sigma,\wp}(n,z)|(|n|+1) \; < \; \exp\{2C_0\lambda|z|/(q-1)\}, \tag{5.2.19}$$

where C_0 *is an absolute positive constant.*

Proof. By (5.2.14) (and by $|A_{\sigma,\wp}(0,z)| \leq 1$), the left hand side of (5.2.19) is bounded by

$$1 + 2e^{\frac{1}{2}|c_{\sigma,\wp}z|}\left\{\sum_{n=1}^{\infty}\frac{n+1}{q^n}\sum_{\nu=1}^{n}\frac{1}{\nu!}\binom{n-1}{\nu-1}\left(\frac{\lambda|z|}{2}\right)^{\nu}\right\}. \tag{5.2.20}$$

Now since

$$\sum_{k=0}^{\infty}\binom{\nu+k-1}{k}(\nu+k+1)t^k = t^{-\nu}\frac{d}{dt}\left(t^{\nu+1}(1-t)^{-\nu}\right) = (\nu+1-t)(1-t)^{-\nu-1},$$

for $\nu \geq 1$, the sum in the braces in (5.2.20) may be rewritten (using $k = n - \nu \geq 0$) as

$$\begin{aligned}(5.2.21)\qquad &\sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(\frac{\lambda|z|}{2q}\right)^{\nu}\left\{\sum_{k=0}^{\infty}\binom{\nu+k-1}{k}\frac{(\nu+k+1)}{q^k}\right\}\\ &= \sum_{\nu=1}^{\infty}\frac{1}{\nu!}\left(\frac{\lambda|z|}{2q}\right)^{\nu}(\nu+1-q^{-1})(1-q^{-1})^{-\nu-1}\\ &= \left(1+\frac{\lambda|z|}{2q(1-q^{-1})^2}\right)\exp\left(\frac{\lambda|z|}{2(q-1)}\right)-1\\ &< \exp\left\{\frac{\lambda|z|}{2}\left(\frac{q}{(q-1)^2}+\frac{1}{q-1}\right)\right\}-1.\end{aligned}$$

Since $|c_{\sigma,\wp}| = \lambda(q^2-1)^{-1}$, we obtain

$$\begin{aligned}(5.2.22)\qquad \sum_{n\in\mathbb{Z}} |A_{\sigma,\wp}(n,z)|(|n|+1) &< 2\exp\left\{\lambda|z|\frac{q^2+q-1}{(q-1)(q^2-1)}\right\}-1\\ &\leq \exp\left\{2\lambda|z|\frac{q^2+q-1}{(q-1)(q^2-1)}\right\} < \exp\{6\lambda|z|/(q-1)\}.\end{aligned}$$

(The last inequality is because $q = N(\wp)^{\sigma} > \sqrt{2}$ gives $2q^2 > q+2$.) □

Remark 5.2.23. We can show that if $N(\wp)$ is sufficiently large compared with $|z|$, then

$$(5.2.24)\qquad \sum_{n\in\mathbb{Z}} |A_{\sigma,\wp}(n,z)| \geq |\tilde{M}_{\sigma,\wp}(z)|\exp(\lambda B/(q-1)),$$

where $B > 0$ depends on σ, K and z. Thus, the "core" of Cor 5.2.18 that will be used later cannot be expected to be improved.

Remark 5.2.25. This is just to draw a full circle and not for applications in the present paper. Theorem 5 in §3.7 gives a Dirichlet series expansion for $\tilde{M}_s(z_1,z_2)$. By (5.1.7), we meet a more general Dirichlet series

$$(5.2.26)\qquad \sum_{D_2D_1^{-1}=D_0} \lambda_{D_1}(z_1)\lambda_{D_2}(z_2)N(D_1D_2)^{-s},$$

where D_0 is a given fractional divisor of K, and D_1, D_2 run over all integral divisors of K such that $D_2D_1^{-1} = D_0$. (D_0 corresponds to P^n). This series is also absolutely convergent on $\mathrm{Re}(s) > 1/2$.

§6. Connections with $L'(\chi,s)/L(\chi,s)$; (II) Case $\sigma > 3/4$ (FF)

6.1 – The main goal of §6 is to prove the following

Theorem 7. *Consider the function field case in Case A of* §4. *Then, among the equalities* (i) ∼ (iii) *in Theorem* 6, *namely*
(i)
$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw|,$$
(ii)
$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \tilde{M}_\sigma(z),$$
(iii)
$$\mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = (-1)^{(a+b)}\mu_\sigma^{(a,b)},$$
(iii) *holds for any* $\sigma > 1/2$, *and* (ii), *for* $\sigma > 3/4$. *The equality* (i) *holds if either* (*a*) $\sigma > 3/4$, $\Phi \in L^1 \cap L^\infty$ *and moreover the Fourier transform of* Φ *has compact support, or* (*b*) $\sigma > 5/6$ *and* Φ *is a standard function in the sense of* [12].

In fact, (iii) holds in a stronger form, and the method for proof is quite different from that for (i)(ii). This is the main point of renewal of the previous versions. For this and for related remarks, see §6.8. Remarks alluded to the number field case will be in §6.9.

Conjecture. The theorem holds for any $\sigma > 1/2$ and any continuous function $\Phi(w)$ with compact support.

6.2 – First, we fix some notation. We denote by Cl_K the group of divisor classes of K in degree 0, and $h_K = |Cl_K|$ the class number. For a *prime* divisor $\mathbf{f} \neq \wp_\infty$, denote by $I_\mathbf{f}$ (resp. $I_\mathbf{f}^{(0)}$) the group of divisors of K (resp. those of degree 0) that are coprime with $\mathbf{f}$, and by $G_\mathbf{f}$ the quotient of $I_\mathbf{f}$ by the subgroup generated by $\wp_\infty$ and all principal divisors of the form (α) with $\alpha \equiv 1 \pmod{\mathbf{f}}$. Since $\deg(\wp_\infty) = 1$, $G_\mathbf{f}$ is canonically isomorphic to the quotient of $I_\mathbf{f}^{(0)}$ by the subgroup generated by those (α). Call

(6.2.1) $$i_\mathbf{f} : I_\mathbf{f} \longmapsto G_\mathbf{f},$$

(6.2.2) $$j_\mathbf{f} : G_\mathbf{f} \longmapsto Cl_K,$$

the projections. Since $\mathrm{Ker}(j_{\mathbf{f}})$ is the quotient of the group of principal divisors (α) in $I_{\mathbf{f}}$ modulo those with $\alpha \equiv 1 (\mathrm{mod}\, \mathbf{f})$, we have $\mathrm{Ker}(j_{\mathbf{f}}) \cong \kappa_{\mathbf{f}}^{\times}/\mathbb{F}_q^{\times}$, where $\kappa_{\mathbf{f}}$ denotes the residue field of $\mathbf{f}$. Thus,

$$|G_{\mathbf{f}}| = h_K(N(\mathbf{f}) - 1)/(q-1). \tag{6.2.3}$$

For any finite abelian group G, $\hat{G}$ will denote its character group. Thus, $\hat{j}_{\mathbf{f}}$ embeds $\hat{Cl}_K$ into $\hat{G}_{\mathbf{f}}$. The Dirichlet characters χ of K with conductor $\mathbf{f}$ satisfying $\chi(\wp_\infty) = 1$ are those elements of $\hat{G}_{\mathbf{f}}$ with conductor $\neq (1)$, i.e., those elements of $\hat{G}_{\mathbf{f}} \setminus \hat{j}_{\mathbf{f}}(\hat{C}l_K)$ ($\setminus$: the complement); hence

$$\#\{\chi;\ \mathbf{f}_\chi = \mathbf{f}\} = |G_{\mathbf{f}}| - |Cl_K| = h_K(N(\mathbf{f}) - q)/(q-1). \tag{6.2.4}$$

6.3 – Now let P be any finite set of primes $\neq \wp_\infty$ of K. For a prime $\mathbf{f} \neq \wp_\infty$, consider two averages

$$S'_{\mathbf{f}} = \frac{\sum_{\mathbf{f}_\chi = \mathbf{f}} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{\sum_{\mathbf{f}_\chi = \mathbf{f}} 1} = \frac{\sum_{\chi \in \hat{G}_{\mathbf{f}} \setminus \hat{j}_{\mathbf{f}}(\hat{C}l_K)} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{|G_{\mathbf{f}}| - |Cl_K|}, \tag{6.3.1}$$

$$S_{\mathbf{f}} = \frac{\sum_{\mathbf{f}_\chi | \mathbf{f}} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{\sum_{\mathbf{f}_\chi | \mathbf{f}} 1} = \frac{\sum_{\chi \in \hat{G}_{\mathbf{f}}} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)}{|G_{\mathbf{f}}|}. \tag{6.3.2}$$

Clearly, $|S_{\mathbf{f}}|, |S'_{\mathbf{f}}| \leq 1$. It is also easy to see that

$$|S'_{\mathbf{f}} - S_{\mathbf{f}}| \leq \frac{2(q-1)}{N(\mathbf{f}) - q} \ll \frac{1}{N(\mathbf{f})}. \tag{6.3.3}$$

When $\mathbf{f} \notin P$, $S_{\mathbf{f}}$ can be expressed in terms of the Fourier series

$$\psi_z(g_{\sigma,P}(t_P)) = \sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n;\, z) t_P^n, \tag{6.3.4}$$

(cf. §5; $A_{\sigma,P}(n;\, z) = A_{\sigma,P}(n;\, z, \bar{z})$) as follows. Since

$$\frac{L'_P(\chi,s)}{L_P(\chi,s)} = g_{\sigma,P}\left(\chi_P N(P)^{-\tau i}\right) \tag{6.3.5}$$

($\tau = \mathrm{Im}(s)$), we have

$$\psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right) = \sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n;\, z)(\chi_P N(P)^{-\tau i})^n. \tag{6.3.6}$$

For each $n \in \mathbb{Z}_P$, define the divisor P^n of K by

$$P^n = \prod_{\wp \in P} \wp^{n_\wp}, \tag{6.3.7}$$

where $n = (n_\wp)_{\wp \in P}$. Then $\chi_P^n = \chi(P^n)$, $(N(P)^{-\tau i})^n = N(P^n)^{-\tau i}$; hence the orthogonality relation for characters gives

$$\frac{1}{|G_{\mathbf{f}}|} \sum_{\chi \in \hat{G}_{\mathbf{f}}} \chi_P^n = \begin{cases} 1 \cdots i_{\mathbf{f}}(P^n) = 1, \\ 0 \ldots \mathit{otherwise}. \end{cases} \tag{6.3.8}$$

Therefore,

$$S_{\mathbf{f}} = \sum_{\substack{n \in \mathbb{Z}_P \\ i_{\mathbf{f}}(P^n) = 1}} A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i}, \tag{6.3.9}$$

whenever $\mathbf{f} \notin P$. Now let us estimate the quantity

$$|\mathrm{Avg}_{N(\mathbf{f}_\chi) \le m} \psi_z \left(\frac{L'_P(\chi, s)}{L_P(\chi, s)} \right) - \tilde{M}_{\sigma,P}(z)| \tag{6.3.10}$$

when m is large compared with $|P|$. Let $\pi(x) = \pi_K(x)$ denote the number of prime divisors $\mathbf{f} \neq \wp_\infty$ with $N(\mathbf{f}) \le x$. Then $\pi(x) \sim a_K x / \log x$, with $a_K = 1$ resp. $\log q$ (K a number field resp. a function field over $\mathbb{F}_q$); hence by our definition of Avg (§4.1),

(6.3.11)

$$\begin{aligned}
\mathrm{Avg}_{N(\mathbf{f}_\chi) \le m} \psi_z \left(\frac{L'_P(\chi, s)}{L_P(\chi, s)} \right) &= \frac{\sum_{N(\mathbf{f}) \le m} S'_{\mathbf{f}}}{\sum_{N(\mathbf{f}) \le m} 1} \\
&= \frac{\sum_{N(\mathbf{f}) \le m} S_{\mathbf{f}}}{\pi(m)} + O\left(\frac{\sum_{N(\mathbf{f}) \le m} N(\mathbf{f})^{-1}}{\pi(m)} \right) \\
&= \frac{\sum_{N(\mathbf{f}) \le m,\, \mathbf{f} \notin P \cup \{\wp_\infty\}} S_{\mathbf{f}}}{\pi(m)} \\
&\quad + O\left(\frac{\log m}{m} |P| + \frac{(\log m)(\log\log m)}{m} \right),
\end{aligned}$$

and by (6.3.9) the main term on the last line of (6.3.11) is given by

$$\sum_{n \in \mathbb{Z}_P} \epsilon^{(m)}(n) A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i}, \tag{6.3.12}$$

where

$$\epsilon^{(m)}(n) = \frac{1}{\pi(m)} \#\{\mathbf{f} \notin P \cup (\wp_\infty);\, N(\mathbf{f}) \le m,\, i_{\mathbf{f}}(P^n) = 1\}. \tag{6.3.13}$$

Note that $0 \leq \epsilon^{(m)}(n) \leq 1$, and that if $P = P_y$ with $y < m$, then

$$\epsilon^{(m)}(0) = 1 - \frac{\pi(y)}{\pi(m)}. \tag{6.3.14}$$

Recall (5.1.12):

$$A_{\sigma,P}(0;\, z) = \tilde{M}_{\sigma,P}(z). \tag{6.3.15}$$

Now we are going to let both m and y grow, but with m much faster than y. We shall take

$$y \leq (\log m)^b \tag{6.3.16}$$

(and $y \mapsto \infty$), where b is a positive constant to be specified later which will depend only on σ. For such a case,

$$\epsilon^{(m)}(0) = 1 + O\left(\frac{\log^{b+1} m}{m}\right); \tag{6.3.17}$$

hence (by $|\tilde{M}_{\sigma,P}(z)| \leq 1$),

$$\epsilon^{(m)}(0) A_{\sigma,P}(0;\, z) = \tilde{M}_{\sigma,P}(z) + O\left(\frac{\log^{b+1} m}{m}\right). \tag{6.3.18}$$

Therefore, by (6.3.11), (6.3.12), we obtain the following basic approximation formula for the discrepancy:

(6.3.19)

$$\begin{aligned} \mathrm{Avg}_{N(\mathbf{f}_\chi) \leq m} \psi_z \left(\frac{L'_P(\chi, s)}{L_P(\chi, s)}\right) - \tilde{M}_{\sigma,P}(z) = \sum_{n \in \mathbb{Z}_P \setminus (0)} \epsilon^{(m)}(n) A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i} \\ + O\left(\frac{\log^{b+1} m}{m}\right), \end{aligned}$$

when $P = P_y$ with $y \leq (\log m)^b$.

Now, as for the sum on the right hand side of (6.3.19), one hopes to obtain "the finest estimate" by making full use of the situations that

$$\left| \sum_{n \in \mathbb{Z}_P} A_{\sigma,P}(n;\, z) N(P^n)^{-\tau i} \right| = |\psi_z(g_{\sigma,P}(N(P)^{-\tau i}))| = 1, \tag{6.3.20}$$

and that the "expected average" of $\epsilon^{(m)}(n)$ for $n \in \mathbb{Z}_P$ is so small as

$$\sim \frac{1}{\pi(m)} \sum_{N(\mathbf{f}) \le m} |G_{\mathbf{f}}|^{-1} \ll \frac{1}{m} (\log m)(\log\log m). \tag{6.3.21}$$

But the range of n where such an estimation is crucially needed is quite narrow (too narrow for geometry of numbers), and the oscillation in the sum (6.3.19) is a delicate mixture of arithmetic and analytic quantities. After some months of trials, the author decided that temporarily he should be satisfied with a weaker "$\sigma > 3/4$" result obtainable by a suitable estimation of the sum of *the absolute value* $\epsilon^{(m)}(n)|A_{\sigma,P}(n;\, z)|$ *of each term*. (As for the question whether $\sigma > 3/4$ would be the best possible result obtainable this way, see Remark 6.6.4.)

6.4 – In order to estimate the sum

$$\sum_{n \in \mathbb{Z}_P \setminus (0)} \epsilon^{(m)}(n)|A_{\sigma,P}(n;\, z)|, \tag{6.4.1}$$

we need the following two sublemmas.

Sublemma 6.4.2. *Let D be any divisor of K such that $D \neq (1)$, $\mathrm{Supp}(D) \not\ni \wp_\infty$. Then*

$$\#\{\mathbf{f};\ i_{\mathbf{f}}(D) = 1\} \ll \frac{\log \parallel D \parallel}{\log\log \parallel D \parallel +2}. \tag{6.4.3}$$

Here, the condition on $\mathbf{f}$ *preassumes that it is a prime not contained in* $\mathrm{Supp}D \cup \{\wp_\infty\}$*, and we put* $\parallel D \parallel = \prod_{\wp \in P} N(\wp)^{|n_\wp|}$ *for* $D = \prod_{\wp \in P} \wp^{n_\wp}$.

Proof. In fact, $i_{\mathbf{f}}(D) = 1$ holds only if $D\wp_\infty^{-\deg D} = (\alpha)$ and $\alpha \equiv c \pmod{\mathbf{f}}$ with some $c \in \mathbb{F}_q^\times$. So, the left hand side of (6.4.3) is bounded by the sum of $|\mathrm{Supp}(\alpha - c)_+|$ over all $c \in \mathbb{F}_q^\times$. Here, in general, D_+ (resp. D_-) denotes the numerator (resp. the denominator) of D. But $N((\alpha - c)_+) = N((\alpha - c)_-) = N((\alpha)_-) = N(D_-)$ (resp. $N(D_+)$) for $\deg D \le 0$ (resp. ≥ 0); hence $N((\alpha - c)_+) \le N(D_+ D_-) = \parallel D \parallel$; hence (6.4.3) follows directly from Sublemma 3.10.5. □

Corollary 6.4.4. *For $P = P_y$, $y \le (\log m)^b$, and $n = (n_\wp) \in \mathbb{Z}_P \setminus (0)$,*

$$\epsilon^{(m)}(n) \ll \frac{(\log m)(\log\log m)}{m} \prod_{\wp \in P} (|n_\wp| + 1). \tag{6.4.5}$$

Proof. Since

$$\log \parallel P^n \parallel = \sum_{\wp \in P} |n_\wp| \log N(\wp) < \left(\prod_{\wp \in P} (|n_\wp| + 1) \right) \log y, \tag{6.4.6}$$

we obtain by Sublemma 6.4.2 that

$$\#\{\mathbf{f};\ i_{\mathbf{f}}(P^n) = 1\} \ll (\log\log m) \prod_{\wp \in P} (|n_\wp| + 1); \tag{6.4.7}$$

hence (6.4.5). □

Sublemma 6.4.8. *Let K be any global field and $\sigma > 0$, both fixed. Then*

$$\sum_{N(\wp) \le y} \frac{\log N(\wp)}{N(\wp)^\sigma - 1} \ll \begin{cases} y^{1-\sigma} & \cdots 0 < \sigma < 1 \\ \log y & \cdots \sigma = 1 \\ 1 & \cdots \sigma > 1 \end{cases}$$

Proof. This is well-known, but let us recall the proof. First, note that we may replace $N(\wp)^\sigma - 1$ by $N(\wp)^\sigma$, and that we may assume $y \in \mathbb{Z}$. By partial summation and the (weak) prime number theorem for K,

$$\begin{aligned} \sum_{N(\wp) \le y} \frac{\log N(\wp)}{N(\wp)^\sigma - 1} &\ll \sum_{1<n<y} \left(\frac{\log n}{n^\sigma} - \frac{\log(n+1)}{(n+1)^\sigma} \right) \pi(n) + \frac{\log y}{y^\sigma} \pi(y) \\ \ll \sum_{1<n<y} \left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma} \right) (\log n) \pi(n) + y^{1-\sigma} &\ll \sum_{1<n<y} \frac{1}{(n+1)^\sigma} + y^{1-\sigma}, \end{aligned} \tag{6.4.9}$$

whence the desired estimation. □

Now, by Cor 6.4.4 and Cor 5.2.18, we obtain

$$\begin{aligned} &\sum_{n \in \mathbb{Z}_P \setminus (0)} \epsilon^{(m)}(n) |A_{\sigma,P}(n;\ z)| \\ &\ll \frac{(\log m)(\log\log m)}{m} \prod_{\wp \in P} \left(\sum_{n_\wp \in \mathbb{Z}} |A_{\sigma,\wp}(n_\wp;\ z)| (|n_\wp| + 1) \right) \\ &\ll \frac{(\log m)(\log\log m)}{m} \exp\left\{ 2|z| C_0 \sum_{\wp \in P} \frac{\log N(\wp)}{N(\wp)^\sigma - 1} \right\}. \end{aligned} \tag{6.4.10}$$

Therefore, by (6.3.19) and Sublemma 6.4.8, we obtain, with some positive constant $C_{K,\sigma}$ depending only on K,σ,

(6.4.11)
$$|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)-\tilde{M}_\sigma(z)| \ll \frac{(\log m)(\log\log m)}{m}\exp\{2|z|C_{K,\sigma}y^{1-\sigma}\}$$
$$+|\tilde{M}_{\sigma,P}(z)-\tilde{M}_\sigma(z)|+\frac{\log^{b+1} m}{m}\cdots\sigma<1,$$

and when $\sigma=1$ (resp. $\sigma>1$), $y^{1-\sigma}$ is to be replaced by $\log y$ (resp. 1).

6.5 – In order to estimate the difference

$$|\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)-\tilde{M}_\sigma(z)|, \tag{6.5.1}$$

we need, in addition to (6.4.11), the following lemma for which our assumption that K be a function field is more essential.

Lemma 6.5.2. *Let K be any function field over $\mathbb{F}_q$, and $P=P_y$ $(y>1)$ be the set of prime divisors of K with $N(\wp)\le y$. Let χ be any non-principal Dirichlet character on K and $L(\chi,s)$ (resp. $L_P(\chi,s)$) be the associated L-function (resp. the partial L-function)*

$$L_P(\chi,s)=\prod_{\wp\in P}(1-\chi(\wp)N(\wp)^{-s})^{-1} \qquad (\mathrm{Re}(s)>0). \tag{6.5.3}$$

(*In this lemma, the $\wp_\infty$-factor is not excluded.*) *Let now $\sigma=\mathrm{Re}(s)>1/2$. Then*

$$\left|\frac{L'(\chi,s)}{L(\chi,s)}-\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right| \ll (\log N(\mathbf{f}_\chi)+1)y^{\frac12-\sigma}. \tag{6.5.4}$$

Here, if $\epsilon>0$ is fixed and $\sigma\ge\frac12+\epsilon$, the implied constant depends only on K and ϵ.

Proof. First note that (for $\mathrm{Re}(s)>0$)

$$-\frac{L'_P(\chi,s)}{L_P(\chi,s)}=-\sum_{N(\wp)\le y}\frac{\chi(\wp)\log N(\wp)}{\chi(\wp)-N(\wp)^s}=\sum_{\substack{N(\wp)\le y,\\ k\ge1}}\frac{\chi(\wp)^k\log N(\wp)}{N(\wp)^{ks}}. \tag{6.5.5}$$

Divide the last double sum over $\wp$ and k into two parts

$$A = \sum_{N(\wp^k) \le y}, \qquad B = \sum_{\substack{N(\wp) \le y \\ N(\wp^k) > y}}, \tag{6.5.6}$$

so that

$$\left| \frac{L'(\chi, s)}{L(\chi, s)} - \frac{L'_P(\chi, s)}{L_P(\chi, s)} \right| \le \left| \frac{L'(\chi, s)}{L(\chi, s)} + A \right| + |B|. \tag{6.5.7}$$

(**Estimation of** $|\frac{L'(\chi,s)}{L(\chi,s)} + A|$) By Weil, since χ is non-principal, $L(\chi, s)$ is a polynomial of $u = q^{-s}$ of the form

$$L(\chi, s) = \prod_{\nu=1}^{D_\chi} (1 - \pi_\nu u), \tag{6.5.8}$$

where

$$D_\chi = 2g - 2 + \deg \mathbf{f}_\chi \tag{6.5.9}$$

(g: the genus), and $|\pi_\nu| = q^{1/2}$ for all ν. And since $du = -u(\log q)ds$,

$$\frac{L'(\chi, s)}{L(\chi, s)} = \sum_{\nu=1}^{D_\chi} \frac{\pi_\nu u}{1 - \pi_\nu u} \log q. \tag{6.5.10}$$

On the other hand, as a power series of u which is convergent for $|u| < q^{-1/2}$,

$$\frac{L'(\chi, s)}{L(\chi, s)} = -\left(\sum_{\wp,\, k \ge 1} \chi(\wp)^k (\deg \wp) u^{k \cdot \deg \wp} \right) \log q. \tag{6.5.11}$$

Therefore, if n denotes the integral part of $\log_q y$, then $L'(\chi, s)/L(\chi, s) + A$ is nothing but the "degree $> n$ -part" of (6.5.11); hence that of (6.5.10); hence is given by

$$\sum_{\nu=1}^{D_\chi} \frac{(\pi_\nu u)^{n+1}}{1 - \pi_\nu u} \log q. \tag{6.5.12}$$

But since $|\pi_\nu u| = q^{1/2-\sigma}$ and $D_\chi \ll 1 + \log N(\mathbf{f}_\chi)$, we obtain

$$\left| \frac{L'(\chi, s)}{L(\chi, s)} + A \right| \le D_\chi \frac{y^{\frac{1}{2}-\sigma}}{1 - q^{\frac{1}{2}-\sigma}} (\log q) \ll (\log N(\mathbf{f}_\chi) + 1) y^{\frac{1}{2}-\sigma}. \tag{6.5.13}$$

(**Estimation of** $|B|$) As for $|B|$,

$$|B| \ll y^{\frac{1}{2}-\sigma} \tag{6.5.14}$$

holds, including the number field case (unconditionally). In fact, $|B| \le B_1 + B_2$, with

$$B_1 = \sum_{\substack{N(\wp)\le\sqrt{y}\\ N(\wp)^k>y}} \frac{\log N(\wp)}{N(\wp)^{k\sigma}} \ll y^{-\sigma} \sum_{N(\wp)\le\sqrt{y}} \log N(\wp) \ll y^{\frac{1}{2}-\sigma}, \tag{6.5.15}$$

and (by the argument similar to the proof of Sublemma 6.4.8)

$$\begin{aligned} B_2 = \sum_{\substack{\sqrt{y}<N(\wp)\le y\\ N(\wp)^k>y}} \frac{\log N(\wp)}{N(\wp)^{k\sigma}} &< \sum_{\substack{\sqrt{y}<N(\wp)\\ k\ge 2}} \frac{\log N(\wp)}{N(\wp)^{k\sigma}} \ll \sum_{\sqrt{y}<N(\wp)} \frac{\log N(\wp)}{N(\wp)^{2\sigma}} \\ &\ll \sum_{\sqrt{y}<n} \frac{1}{(n+1)^{2\sigma}} \ll \frac{1}{2\sigma-1} y^{\frac{1}{2}-\sigma}; \end{aligned} \tag{6.5.16}$$

whence (6.5.14). $\square$

Corollary 6.5.17.

$$\left| \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) - \mathrm{Avg}_{N(\mathbf{f}_\chi)\le m} \psi_z\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)\right| \ll (\log m)|z|y^{\frac{1}{2}-\sigma}. \tag{6.5.18}$$

Proof. Since the segment of a circle is shorter than the arc,

$$\begin{aligned} |\psi_z(w') - \psi_z(w)| = |\exp(i\mathrm{Re}(\bar{z}(w'-w))) - 1| &\le |\mathrm{Re}(\bar{z}(w'-w))| \\ &\le |z||w'-w|. \end{aligned} \tag{6.5.19}$$

Therefore, the Corollary follows immediately from Lemma 6.5.2. $\square$

Remark 6.5.20. Note that $(\log m)$ in (6.5.18) comes from $\log N(\mathbf{f}_\chi)$ in (6.5.4). Since, here, we average over χ, it would be possible that the former can be replaced by something smaller. For the effect of such a possible replacement, see Remark 6.6.4 (i).

6.6 – *Proof of Theorem* 7 (ii). We first prove (ii). By (6.4.11) and Cor 6.5.17, we have, for $y = (\log m)^b$, $b > 0$, $\sigma > 1/2$,

(6.6.1)

$$|\mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m}\psi_z\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) - \tilde{M}_\sigma(z)| \ll \frac{(\log m)(\log\log m)}{m}\exp\{2|z|C_{K,\sigma}y^{1-\sigma}\} \\ +|\tilde{M}_{\sigma,P_y}(z) - \tilde{M}_\sigma(z)| + \frac{\log^{b+1} m}{m} \\ +(\log m)|z|y^{\frac{1}{2}-\sigma},$$

where $y^{1-\sigma}$ inside the exponential braces should be replaced by $\log y$ (resp. 1) when $\sigma = 1$ (resp. $\sigma > 1$).

Note that if

$$1 < \left(\sigma - \frac{1}{2}\right) b, \tag{6.6.2}$$

the last term on the right hand side of (6.6.1) tends to 0 as $m \mapsto \infty$, while if

$$(1-\sigma)b < 1, \tag{6.6.3}$$

then the first term has this property (including the case $\sigma \geq 1$). Note finally that the middle terms tend to 0 for any b (cf. Theorem 3(i)). The necessary and sufficient condition for σ to have a solution b satisfying both (6.6.2) and (6.6.3) is that either (i) $\sigma \geq 1$, or (ii) $\sigma < 1$ and $(\sigma - 1/2)/(1-\sigma) > 1$ holds, i.e., simply that $\sigma > 3/4$ holds. Therefore, when $\sigma > 3/4$, (6.6.1) tends to 0 as $(y = (\log m)^b$ (b being as above) and) $m \mapsto \infty$. This proves (ii).

Remarks 6.6.4. (i) If $(\log m)$ in (6.5.18) can be replaced by, say, $(\log m)^{1/2+\epsilon}$ (resp. $(\log m)^\epsilon$) for any $\epsilon > 0$, then (6.6.2) will be replaced by $(\sigma - 1/2)b > 1/2$ (resp. > 0); hence it would imply that Theorem 7(ii) holds for $\sigma > 2/3$ (resp. $\sigma > 1/2$).

(ii) As regards the estimation of the sum (6.4.1), the partial sum over log $\| P^n \| \leq \log m$, as well as that over log $\| P^n \| \geq (\log m)^{b+\epsilon}$, tends to 0 as $m \mapsto \infty$. The crucial part is the "middle" sum

$$\sum_{\log m < \log\|P^n\| < (\log m)^{b+\epsilon}} \varepsilon^{(m)}(n)|A_{\sigma,P}(n,z)|. \tag{6.6.5}$$

For this, even if we replace $\varepsilon^{(m)}(n)$ by $1/m$, it does not tend to 0 unless $b(1-\sigma) < 1$.

6.7 – *Proof of Theorem* 7 (i). First, since M_σ and $\tilde{M}_\sigma = M_\sigma^\wedge$ belong to $L^1 \cap L^\infty$, and since $\overline{M_\sigma(w)} = M_\sigma(w)$, $\overline{\tilde{M}_\sigma(w)} = \tilde{M}_\sigma(-w)$, we have

$$\int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| = \int_{\mathbb{C}} \tilde{M}_\sigma(-z)\Phi^\wedge(z)|dz|. \tag{6.7.1}$$

(Cf. §3.11 for the notation $\wedge$.) And since

$$\Phi(w) = \int_{\mathbb{C}} \Phi^\wedge(z)\psi_{-z}(w)|dz|, \tag{6.7.2}$$

we have

$$\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} \Phi^\wedge(z)\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_{-z}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right)|dz|. \tag{6.7.3}$$

Therefore,

$$\begin{aligned}&\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\Phi\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) - \int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| \\ &= \int_{\mathbb{C}} \Phi^\wedge(z)\left(\mathrm{Avg}_{N(\mathbf{f}_\chi)\le m}\psi_{-z}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) - \tilde{M}_\sigma(-z)\right)|dz|.\end{aligned} \tag{6.7.4}$$

Case (a). Now, when $\sigma > 3/4$, the argument at the beginning of §6.6 shows that (6.6.1) tends to 0 as $m \mapsto \infty$ uniformly on any domain where $|z|$ is bounded; hence on the compact support of $\Phi^\wedge(z)$. This settles Case (a).

Case (b). When $\sigma > 5/6$, so that $2(1-\sigma) < \sigma - \frac{1}{2}$, we can choose $b > 0$ such that $2(1-\sigma) < b^{-1} < \sigma - \frac{1}{2}$. (We may assume $\sigma \le 1$; hence this also implies $1 - \sigma < b^{-1}$.) Then $a := b(1-\sigma) < \frac{1}{2}$.

Now, $\Phi(z)$ is assumed to be a standard function, i.e., the product of a polynomial of x, y and $\exp(-Q(x,y))$, where $z = x + yi$ and $Q(x,y)$ is some positive definite quadratic form (with real coefficients). This implies that $\Phi^\wedge(z)$ is also a standard function; hence

$$|\Phi^\wedge(z)| \ll \exp(-A|z|^2), \tag{6.7.5}$$

with some $A > 0$. But it is easy to see that for any constant $C > 0$,

$$\int_0^\infty \exp(-Ar^2 + 2Cr(\log m)^a)rdr \ll (\log m)^a \exp\left(\frac{C^2}{A}(\log m)^{2a}\right). \tag{6.7.6}$$

By (6.6.1), this shows that the absolute value of the right hand side of (6.7.4) is bounded by

$$\frac{(\log m)^{a+2}}{m}\exp\left(\frac{C_{K,\sigma}^2}{A}(\log m)^{2a}\right) + o\,(1)\,. \tag{6.7.7}$$

(Details: The contribution of the first term on the right hand side of (6.6.1) gives the main term of (6.7.7), and by Theorem 3 (i) and by (6.7.5), the remaining terms from the right hand side of (6.6.1) give $o(1)$.)
But since $2a < 1$, the main term must also tend to 0 as $m \mapsto \infty$. This proves (i).

6.8 – *Proof of Theorem* 7 (iii). For each prime divisor $\mathbf{f} \neq \wp_\infty$ of K, put

(6.8.1)
$$A_{\mathbf{f},s}^{(a,b)} = \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} P^{(a,b)}\left(-\frac{L'(\chi,s)}{L(\chi,s)}\right) = (-1)^{a+b}\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} P^{(a,b)}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right),$$

where Avg denotes the usual arithmetic mean. We shall prove the following stronger version of (iii).

$$\lim_{N(\mathbf{f})\mapsto\infty} A_{\mathbf{f},s}^{(a,b)} = \mu_\sigma^{(a,b)} \qquad (\sigma = \mathrm{Re}(s) > 1/2). \tag{6.8.2}$$

For this purpose, we shall use, for each $\chi \in \hat{G}_{\mathbf{f}}$ and $y > 1$, the auxiliary sum

$$\phi(\chi, s, y) = \sum_{N(D)\leq y} \frac{\chi(D)\Lambda(D)}{N(D)^s}. \tag{6.8.3}$$

(In this §6.8, when we write D, it will mean that D is integral and coprime with $\mathbf{f}\wp_\infty$.) When $\chi \neq \chi_0$, (6.5.13) gives

$$\left|\frac{L'(\chi,s)}{L(\chi,s)} + \phi(\chi,s,y)\right| \ll (\log N(\mathbf{f}_\chi)+1)y^{1/2-\sigma} \quad (\chi \neq \chi_0), \tag{6.8.4}$$

where the implied constant depends only on K, σ. (The difference coming from $\wp_\infty$ is neglisible. The term "+1" on the right hand side of (6.8.4) is necessary just to include the case $\mathbf{f}_\chi = (1)$, the non-principal unramified characters.)

Now, in proving (6.8.2), we may assume $(a,b) \neq (0,0)$. For each such (a,b), choose and fix a real number β satisfying

$$0 < \beta < (a+b)^{-1} \tag{6.8.5}$$

(e.g., $\beta = (2(a+b))^{-1}$), and set

$$y_{\mathbf{f}} = y_{\mathbf{f}}^{(a,b)} = N(\mathbf{f})^\beta, \tag{6.8.6}$$

$$B_{\mathbf{f},s}^{(a,b)} = \mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}} P^{(a,b)}(\phi(\chi,s,y_{\mathbf{f}})). \tag{6.8.7}$$

We shall first prove

$$\lim_{N(\mathbf{f})\mapsto\infty} B_{\mathbf{f},s}^{(a,b)} = \mu_\sigma^{(a,b)}, \tag{6.8.8}$$

and then

$$\lim_{N(\mathbf{f})\mapsto\infty} (A_{\mathbf{f},s}^{(a,b)} - B_{\mathbf{f},s}^{(a,b)}) = 0, \tag{6.8.9}$$

which together will give (6.8.2).

Proof of (6.8.8). Recall the definition of $\Lambda_k(D)$ (§3.8). For each D (integral, coprime with $\wp_\infty$) and $y > 1$, define also the "partial $\Lambda_k(D)$" by

$$\Lambda_k(D, y) = \sum_{\substack{D=D_1\ldots D_k \\ N(D_\nu)\le y\ (1\le\nu\le k)}} \Lambda(D_1)\ldots\Lambda(D_k) \tag{6.8.10}$$

for $k \geq 1$, and by $\Lambda_0(D, y) = 1$ (resp. 0) for $D = (1)$ (resp. $D \neq (1)$). Clearly,

$$0 \le \Lambda_k(D, y) \le \Lambda_k(D), \tag{6.8.11}$$

$$\begin{aligned} \Lambda_k(D, y) &= \Lambda_k(D) && \cdots \mathit{if}\ N(D) \le y \\ &= 0 && \cdots \mathit{if}\ N(D) > y^k. \end{aligned} \tag{6.8.12}$$

Since $\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}}(\chi(D)\bar{\chi}(D')) = 1$ (resp. $= 0$) for $i_{\mathbf{f}}(D) = i_{\mathbf{f}}(D')$ (resp. $\neq$), we obtain directly from the definition of $\phi(\chi, s, y)$ that

$$\mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}} P^{(a,b)}(\phi(\chi, s, y)) = \sum_{c\in G_{\mathbf{f}}} \ell_{\bar{s}}^{(a)}(c, y)\ell_s^{(b)}(c, y), \tag{6.8.13}$$

where

$$\ell_s^{(k)}(c, y) = \sum_{i_{\mathbf{f}}(D)=c} \frac{\Lambda_k(D, y)}{N(D)^s} \qquad \cdots k \geq 0. \tag{6.8.14}$$

Note that the sum in (6.8.14) contains only those terms with $N(D) \leq y^k$ (by (6.8.12)).

Now, when y is as small as $y_{\mathbf{f}} = N(\mathbf{f})^\beta$, the sum (6.8.14) is very simple. To see this, we first make some distinction among the elements of $G_{\mathbf{f}}$. Let us call $c \in G_{\mathbf{f}}$ *small* when $c = i_{\mathbf{f}}(D)$ with some *integral* divisor D satisfying

$$(D, \wp_\infty) = 1,\ N(D) < N(\mathbf{f}). \tag{6.8.15}$$

Each c cannot contain two such D, D'; for if so with $N(D) \leq N(D')$, then $D' = D\wp_\infty^\nu(\alpha)$ with $\alpha \equiv 1(\mathrm{mod}\,\mathbf{f})$, $\alpha \neq 1$, $\nu \geq 0$. This means $(\alpha) \geq (D\wp_\infty^\nu)^{-1}$ and $(\alpha - 1) \geq \mathbf{f}(D\wp_\infty^\nu)^{-1}$; hence $N(\mathbf{f}) \leq N(D\wp_\infty^\nu) = N(D')$, a contradiction.

Thus, for each $\mathbf{f}$, the small elements c in $G_\mathbf{f}$ correspond *bijectively* with the integral divisors D satisfying (6.8.15), via $c = i_\mathbf{f}(D)$. We shall write as $D = D_c$. (When $d = \deg(\mathbf{f}) \geq 2g$, the number of small elements of $G_\mathbf{f}$ is given by $h_K(q^{d-g} - 1)/(q-1)$, while $|G_\mathbf{f}| = h_K(q^d - 1)/(q-1)$.)

Now take $y = y_\mathbf{f}$ in (6.8.14), for $k = a$ or b. Then since $\beta < (a+b)^{-1}$, we have $y_\mathbf{f}^k = N(\mathbf{f})^{\beta k} < N(\mathbf{f})$. Therefore, any D giving a non-zero term on the right hand side of (6.8.14) must satisfy $N(D) < N(\mathbf{f})$. Therefore, if $\ell_s^k(c, y) \neq 0$, then c must be small, and the sum (6.8.14) consists just of one term for $D = D_c$. Therefore, (6.8.13) for $y = y_\mathbf{f}$ gives

(6.8.16)
$$B_{\mathbf{f},s}^{(a,b)} = \sum_{c \in G_\mathbf{f},\, small} \frac{\Lambda_a(D_c, y_\mathbf{f})\Lambda_b(D_c, y_\mathbf{f})}{N(D_c)^{\bar{s}} N(D_c)^s} = \sum_{N(D) < N(\mathbf{f})} \frac{\Lambda_a(D, y_\mathbf{f})\Lambda_b(D, y_\mathbf{f})}{N(D)^{2\sigma}}.$$

Therefore, by (6.8.11), (6.8.12) we obtain

$$\sum_{N(D) < N(\mathbf{f})^\beta} \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2\sigma}} \leq B_{\mathbf{f},s}^{(a,b)} \leq \sum_{N(D) < N(\mathbf{f})} \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2\sigma}}. \tag{6.8.17}$$

Now, $\mu_\sigma^{(a,b)}$ being as given by (3.8.8), this settles (6.8.8).

Proof of (6.8.9). Note first that Sublemma 6.4.8 gives, for $y > 1$,

$$|\phi(\chi, s, y)| \leq \sum_{N(\wp) \leq y} \frac{\log N(\wp)}{N(\wp)^\sigma - 1} \ll \begin{cases} y^{1-\sigma} & \cdots 1/2 < \sigma < 1 \\ \log y & \cdots \sigma = 1 \\ 1 & \cdots \sigma > 1. \end{cases} \tag{6.8.18}$$

(This includes the case $\chi = \chi_0$). Secondly, by putting $y = (\log N(\mathbf{f}_\chi))^2$ in (6.8.4), we obtain for each $\chi \neq \chi_0$,

$$-\frac{L'(\chi, s)}{L(\chi, s)} = \phi(\chi, s, (\log N(\mathbf{f}_\chi))^2) + O((\log N(\mathbf{f}_\chi) + 1)^{2(1-\sigma)}); \tag{6.8.19}$$

hence by (6.8.18),

$$\left|\frac{L'(\chi, s)}{L(\chi, s)}\right| \ll \begin{cases} (\log N(\mathbf{f}_\chi) + 1)^{2(1-\sigma)} & \cdots 1/2 < \sigma < 1 \\ \log\log N(\mathbf{f}_\chi) & \cdots \sigma = 1 \\ 1 & \cdots \sigma > 1 \end{cases} \tag{6.8.20}$$

for any $\chi \neq \chi_0$. Thirdly, we use the elementary inequality

$$|P^{(a,b)}(z+w) - P^{(a,b)}(z)| \leq (a+b)|w|(|z|+|w|)^{a+b-1} \qquad (z, w \in \mathbb{C}). \tag{6.8.21}$$

This follows directly from $|(z+w)^n - z^n| \leq n|w|(|z|+|w|)^{n-1}$ (the binomial expansions reduce the latter to the obvious inequality $\binom{n}{\nu} \leq n\binom{n-1}{\nu-1}$ $(1 \leq \nu \leq n)$).

Now by applying (6.8.21) to $z = -L'(\chi,s)/L(\chi,s)$, $z+w = \phi(\chi, s, y_{\mathbf{f}})$, and by using (6.8.4), (6.8.20), we obtain directly

$$\begin{aligned} &\left|\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} P^{(a,b)}\left(-\frac{L'(\chi,s)}{L(\chi,s)}\right) - \mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} P^{(a,b)}\left(\phi(\chi,s,y_{\mathbf{f}})\right)\right| \\ &\qquad\qquad \ll_{a,b} (\log N(\mathbf{f}))^c N(\mathbf{f})^{(1/2-\sigma)\beta}, \end{aligned} \tag{6.8.22}$$

where c is a positive constant depending only on σ, a, b, and $\ll$ depends also on a, b.

On the other hand, by using $|G_{\mathbf{f}}| \approx |G_{\mathbf{f}}| - h_K \approx N(\mathbf{f})$, and by (6.8.18), we obtain

$$\begin{aligned} &\left|\mathrm{Avg}_{\mathbf{f}_\chi=\mathbf{f}} P^{(a,b)}\left(\phi(\chi,s,y_{\mathbf{f}})\right) - \mathrm{Avg}_{\chi\in\hat{G}_{\mathbf{f}}} P^{(a,b)}\left(\phi(\chi,s,y_{\mathbf{f}})\right)\right| \\ &\qquad\qquad \ll_{a,b} N(\mathbf{f})^{-1} \times \begin{cases} N(\mathbf{f})^{(1-\sigma)(a+b)\beta}, \\ (\log N(\mathbf{f}))^{a+b} \\ 1, \end{cases} \end{aligned} \tag{6.8.23}$$

for $1/2 < \sigma < 1, \sigma = 1, \sigma > 1$, respectively. Note that $(1-\sigma)(a+b)\beta - 1 < -\sigma < -1/2 < 0$. By (6.8.22), (6.8.23), we obtain (6.8.9). This completes the proof of Theorem 7(iii).

Remark 6.8.24. The above method generalizes and refines a method used in [4] (§4). (The case of $K = \mathbb{Q}, s = 1$ under (GRH)).

The assertion (iii) is closely related to the following "(z_1, z_2)-version" of (ii):

(ii)' *When s is fixed and $m \mapsto \infty$, the analytic function*

$$\mathrm{Avg}_{N(\mathbf{f}_\chi)\leq m} \psi_{z_1,z_2}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) \tag{6.8.25}$$

of $z_1, z_2 \in \mathbb{C}$ tends to $\tilde{M}_\sigma(z_1, z_2)$ uniformly on some neighborhood U of $(0,0)$.

Of course the basic analytic formula underlying this connection is

$$\psi_{z_1,z_2}(w) = \sum_{a,b\geq 0} \left(\frac{i}{2}\right)^{a+b} P^{(a,b)}(w) \frac{z_1^a z_2^b}{a!b!}. \tag{6.8.26}$$

From (ii)' to (iii) (ii)' would also imply the convergence of the higher derivative $D^{(a,b)} = \partial^{a+b}/\partial z_1^a \partial z_2^b$ of (6.8.25) to $D^{(a,b)}\tilde{M}_\sigma(z_1, z_2)$; hence by putting $(z_1, z_2) = (0, 0)$ and using Theorem 5 we would obtain (iii). This was how (iii) was induced in the previous version of this paper (for $\sigma > 3/4$). But the proof of (ii)' in the previous version of this article was incomplete, and this motivated the author to find another proof as given above (applicable to all $\sigma > 1/2$).

From (iii) to (ii)'? If the dependence of the speed of convergence of (6.8.2) on (a, b) can be shown to be so mild that $|A_{\mathbf{f},s}^{(a,b)} - \mu_\sigma^{(a,b)}|$ is universally bounded by $(c_1 \log N(\mathbf{f}))^{a+b} N(\mathbf{f})^{-c_2}$, with some constants $c_1, c_2 > 0$ depending only on K, s, then this would immediately imply (ii)' (for sufficiently small U). But we are not able to prove this, and (ii)' remains open.

6.9 – Remarks alluded to the number field case The first remark here is on the relation with the Generalized Riemann Hypothesis (GRH), the second is for the case $s = 1$, and the third, the more optimistic point of view, is on the interpretation of the natural value of $\psi_z(L'(\chi, s)/L(\chi, s))$ at possible poles of $L(\chi, s)$.

(i) In §6, we have restricted our attention solely to the function field case. This restriction was necessary only for the validity of Lemma 6.5.2. Its validity in the number field case would imply the holomorphy of $L'(\chi, s)/L(\chi, s)$ on $\mathrm{Re}(s) > 1/2$ and hence the (GRH) for $L(\chi, s)$ ($\chi \neq \chi_0$). It is not clear to the author whether, conversely, (GRH) implies Lemma 6.5.2 or something similar (except when $s = 1$; see below).

(ii) Theorem 6 for $s = 1$ (the number field case) is already delicate, but *this is valid at least under* (GRH). This is because, under this situation, (6.5.4) remains valid when the right hand side is replaced by

$$(\log N(\mathbf{f}_\chi))(\log y) y^{-\frac{1}{2}} + (\log y)^2 y^{-\frac{1}{2}} + (\log N(\mathbf{f}_\chi))^2 y^{-1} \tag{6.9.1}$$

(with the implied constant depending only on K); therefore, (6.6.1) tends to 0 as long as $b > 2$.

(iii) The following remark seems more noteworthy. Suppose that $s = \rho$ is a zero of $L(\chi, s)$, so that

$$(6.9.2) \qquad \frac{L'(\chi,s)}{L(\chi,s)} = \frac{k}{s-\rho} + \beta_0 + \beta_1(s-\rho) + \cdots$$

where k is the order of vanishing of $L(\chi, s)$ at ρ. Then for each fixed $z \neq 0$ in $\mathbb{C}$,

$$(6.9.3) \qquad \lim_{r\mapsto 0} \frac{1}{2\pi}\int_0^{2\pi} \psi_z\left(\frac{L'(\chi,\rho+re^{i\theta})}{L(\chi,\rho+re^{i\theta})}\right) d\theta = \lim_{r\mapsto 0} \psi_z(\beta_0) J_0\left(\frac{k|z|}{r}\right) = 0,$$

which implies that the natural interpretation of the oscillating value of $\psi_z(L'(\chi,s)/L(\chi,s))$ at ρ is 0 (for $z \neq 0$). Hence for any s and z, it makes sense to define $\psi_z(L'(\chi,s)/L(\chi,s))$, and hence also their average over χ with $N(\mathbf{f}_\chi) \leq m$. So, if we can show that this has a limit depending only on σ and that this limit is real analytic in σ, then this will lead to Theorem 7 (ii) for all $\sigma > 1/2$ even in the number field case.

As a closely related fact, we add here that

$$(6.9.4) \qquad \lim_{\sigma\mapsto 1/2} \tilde{M}_\sigma(z) = \begin{cases} 1 \ldots\ldots z = 0, \\ 0 \ldots\ldots z \neq 0. \end{cases}$$

In fact, when $z \neq 0$ (fixed),

$$(6.9.5) \qquad \log|\tilde{M}_\sigma(z)| = -(|z|/4)^2\left(\sigma - \frac{1}{2}\right)^{-2} + k\log\left(\sigma - \frac{1}{2}\right) + O_z(1)$$

for σ sufficiently near 1/2, where $k = \sum_\wp k_\wp$, and each $k_\wp$ is defined as follows. If $\tilde{M}_{1/2,\wp}(z) \neq 0$, then it is 0; otherwise, it is the order of vanishing of the function $\tilde{M}_{\sigma,\wp}(z)$ of σ at $\sigma = 1/2$. (It is easy to see that $k_\wp = 0$ for almost all $\wp$.) For example, when $K = \mathbb{Q}$, $z = yi$, $y = 9.394...$ is a zero of $\tilde{M}_{1/2,2}(z)$, and $k = k_2 = 1$ for this value of z. More details related to the behavior of $M_\sigma(z)$, $\tilde{M}_\sigma(z)$ near $\sigma = 1/2$, ∞ will be left to future treatments.

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