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Strong density for higher order Sobolev spaces into compact manifolds

Abstract. Given a compact manifold N^n, an integer $k \in \mathbb{N}_*$ and an exponent $1 \leq p < \infty$, we prove that the class $C^\infty(\overline{Q}^m; N^n)$ of smooth maps on the cube with values into N^n is dense with respect to the strong topology in the Sobolev space $W^{k,p}(Q^m; N^n)$ when the homotopy group $\pi_{\lfloor kp \rfloor}(N^n)$ of order $\lfloor kp \rfloor$ is trivial. We also prove the density of maps that are smooth except for a set of dimension $m - \lfloor kp \rfloor - 1$, without any restriction on the homotopy group of N^n.

1. Introduction

There are two natural approaches to define Sobolev maps with values in a compact manifold. More precisely, let N^n be a compact connected smooth manifold of dimension n imbedded in $\mathbb{R}^\nu$ for some $\nu \geq 1$ [42, 43], $k \in \mathbb{N}_*$ and $1 \leq p < \infty$. One can first define $W^{k,p}(Q^m; N^n)$ as the set

$$\{u \in W^{k,p}(Q^m; \mathbb{R}^\nu) : u \in N^n \text{ a.e.}\},$$

where $Q^m \subset \mathbb{R}^m$ is the open unit cube. The other possibility is to define $H^{k,p}(Q^m; N^n)$ as the completion of the class of smooth maps $C^\infty(\overline{Q}^m; N^n)$ with respect to the Sobolev metric

$$d_{k,p}(u, v) = \|u - v\|_{L^p(Q^m)} + \sum_{i=1}^{k} \|D^i u - D^i v\|_{L^p(Q^m)}.$$

These spaces are the natural framework for the study of harmonic maps [23,34,38,49], biharmonic maps [15,39,47,51,52] and polyharmonic maps [1,21,24,25,33] with values into manifolds. They also arise in some physical models [8, 35]. For instance, maps into

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the sphere, the projective space and other manifolds appear in liquid cristal models [2, 9, 40, 41].

In contrast with the real-valued case [16, 36], these spaces may be different. For instance, $H^{1,p}(Q^2; \mathbb{S}^1) = W^{1,p}(Q^2; \mathbb{S}^1)$ if and only if $p \geq 2$ [4, Theorem 3]. The goal of this paper is to determine when $H^{k,p}(Q^m; N^n) = W^{k,p}(Q^m; N^n)$.

This always happens when $kp \geq m$ [48, Section 4, Proposition], as $W^{k,p}(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ is imbedded into the space $\mathrm{VMO}(\mathbb{R}^m)$ of functions with vanishing mean oscillation [13, Example 1, Eq. (7)]. The main result of this paper completely solves the problem in the case $kp < m$. It is remarkable that such an analytical question has a purely topological answer:

Theorem 1. *If $kp < m$, then $H^{k,p}(Q^m; N^n) = W^{k,p}(Q^m; N^n)$ if and only if $\pi_{\lfloor kp \rfloor}(N^n) = \{0\}$.*

We denote by $\lfloor kp \rfloor$ the integral part of kp and by $\pi_{\lfloor kp \rfloor}(N^n)$ the $\lfloor kp \rfloor$th homotopy group of N^n; the topological condition $\pi_{\lfloor kp \rfloor}(N^n) = \{0\}$ means that every continuous map $f : \mathbb{S}^{\lfloor kp \rfloor} \to N^n$ on the $\lfloor kp \rfloor$-dimensional sphere is homotopic to a constant map. The necessity of this assumption has been known for some time [4, Theorem 2], [17, Theorem 3], [48, Section 4, Example], [37, Theorem 4.4].

The case $k = 1$ of Theorem 1 is the main result of Bethuel's seminal work [3, Theorem 1] (see also [27, 28]). The case $k \geq 2$ cannot be handled by merely adapting Bethuel's tools due to the rigidity of $W^{k,p}$ and requires new ideas. A typical issue one faces when dealing with two maps in $W^{k,p}$ is that they cannot be glued together under the sole assumption that their traces coincide. Results concerning strong density of smooth maps in higher order Sobolev maps have been known in some cases where N^n is a sphere [37, Theorem 5], [11, Theorem 4], [17, Theorem 2].

In the case $\pi_{\lfloor kp \rfloor}(N^n) \neq \{0\}$, we prove that $W^{k,p}(Q^m; N^n)$ is the completion of a set of maps that are smooth outside a small singular set. For this purpose, given i in $\{0, \ldots, m - 1\}$ we denote by $R_i(Q^m; N^n)$ the set of maps $u : \overline{Q}^m \to N^n$ which are smooth on $\overline{Q}^m \setminus T$, where T is a finite union of i-dimensional planes, and such that for every $j \in \mathbb{N}_*$ and $x \in \overline{Q}^m \setminus T$,

$$|D^j u(x)| \leq \frac{C}{(\mathrm{dist}(x, T))^j}$$

for some constant $C \geq 0$ depending on u and j.

Theorem 2. *If $kp < m$ and $\pi_{\lfloor kp \rfloor}(N^n) \neq \{0\}$, then $W^{k,p}(Q^m; N^n)$ is the completion of $R_i(Q^m; N^n)$ with respect to the Sobolev metric $d_{k,p}$ if and only if $i = m - \lfloor kp \rfloor - 1$.*

This result was known for an arbitrary manifold N^n only in the case $k = 1$ [3, Theorem 2] (see also [28, Theorem 1.3]). It is a fundamental tool in the study of the weak density of smooth maps in Sobolev spaces and in the study of topological singularities of Sobolev maps [5, 22, 23, 29, 31, 45]. Counterparts of Theorems 1 and 2 for fractional Sobolev spaces $W^{s,p}(Q^m; N^n)$ such that $0 < s < 1$ have been investigated by Brezis and Mironescu [12].

We explain the strategy of the proofs of Theorems 1 and 2 under the additional assumption $kp > m - 1$. Given a decomposition of Q^m into cubes of size $\eta > 0$, we distinguish between *good cubes* and *bad cubes*—a notion reminiscent of [3]—as follows:

for a map $u \in W^{k,p}(Q^m; N^n)$ and a cube σ_η^m in Q^m of radius $\eta > 0$, σ_η^m is a *good cube* if

$$\frac{1}{\eta^{m-kp}} \int_{\sigma_\eta^m} |Du|^{kp} \lesssim 1,$$

which means that u does not oscillate too much in σ_η^m; otherwise σ_η^m is a *bad cube*. The main steps in the proof of Theorem 2 are the following:

Opening: We construct a map u_η^{op} which is continuous on a neighborhood of the $m-1$ dimensional faces of the bad cubes, and equal to u elsewhere. This map, which takes its values in N^n, is close to u with respect to the $W^{k,p}$ distance because there are not too many bad cubes. Since $kp > m-1$, $W^{k,p}$ maps are continuous on faces of dimension $m-1$.

Adaptive smoothing: By convolution with a smooth kernel, we then construct a smooth map $u_\eta^{\mathrm{sm}} \in W^{k,p}(Q^m; \mathbb{R}^\nu)$. The scale of convolution is chosen to be of the order of η on the good cubes, and close to zero in a neighborhood of the faces of the bad cubes. On the union of these sets, we are thus ensuring that u_η^{sm} takes its values in a small neighborhood of N^n.

Thickening: We propagate diffeomorphically the values of u_η^{sm} near the faces of the bad cubes to the interior of these cubes. The resulting map u_η^{th} coincides with u_η^{sm} on the good cubes and near the faces of the bad cubes, is close to u with respect to the $W^{k,p}$ distance and takes its values in a neighborhood of N^n. This construction creates at most one singularity at the center of each bad cube.

The map obtained by projecting u_η^{th} from a neighborhood of N^n into N^n itself belongs to the class $R_0(Q^m; N^n)$ and converges strongly to u with respect to the Sobolev distance $d_{k,p}$ as $\eta \to 0$. This argument works regardless of the $\lfloor kp \rfloor$th homotopy group of N^n; see Theorem 3 in Section 5 below.

The sketch of the proof we have announced in [6] for $k = 2$ and $2p > m-1$ is based on the strategy above but was organized differently following [46] (see also [20]). The opening technique was introduced by Brezis and Li [10] in their study of homotopy classes of $W^{1,p}(Q^m; N^n)$.

The proof of Theorem 1 in the case $kp \geq m-1$ relies on the fact that $R_0(Q^m; N^n)$ is strongly dense in $W^{k,p}(Q^m; N^n)$ with respect to the Sobolev distance $d_{k,p}$. The approximation of a map $u \in R_0(Q^m; N^n)$ by a map in $C^\infty(\overline{Q}^m; N^n)$ in this case goes as follows:

Continuous extension property: By the assumption on the homotopy group of N^n, for any $\mu < 1$ there exists a smooth map u_μ^{ex} with values into N^n which coincides with u outside a neighborhood of radius $\mu\eta$ of the singular set of u. As a drawback, u_μ^{ex} may be far from u with respect to the $W^{k,p}$ distance.

Shrinking: We propagate diffeomorphically the values of u_μ^{ex} in the neighborhood of radius $\mu\eta$ of each singularity of u into a smaller neighborhood of radius $\tau\mu\eta$ for $\tau < 1$. Since $kp < m$, we obtain a map $u_{\tau,\mu}^{\mathrm{sh}}$ which is still smooth but now close to u with respect to the $W^{k,p}$ distance. This construction is reminiscent of thickening but does not create singularities.

The smooth map $u_{\tau,\mu}^{\text{sh}}$ converges strongly to u with respect to the $W^{k,p}$ distance as $\tau \to 0$ and $\mu \to 0$. The role of this continuous extension property in the case of $W^{1,p}$ approximation of maps u with higher dimensional singularities has been clarified by Hang and Lin [28].

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2. Opening

For $a \in \mathbb{R}^m$ and $r > 0$, we denote by $Q_r^m(a)$ the cube of *radius* r with center a; by the radius of the cube we mean half of the length of its edges. When $a = 0$, we abbreviate $Q_r^m = Q_r^m(0)$.

Definition 2.1. A family $\mathcal{S}^m$ of closed cubes is a *cubication* of $A \subset \mathbb{R}^m$ if all cubes have the same radius, $\bigcup_{\sigma^m \in \mathcal{S}^m} \sigma^m = A$ and for any $\sigma_1^m, \sigma_2^m \in \mathcal{S}^m$ which are not disjoint, $\sigma_1^m \cap \sigma_2^m$ is a common face of dimension $i \in \{0, \ldots, m\}$. The *radius* of a cubication is the radius of any of its cubes.

Definition 2.2. Given a cubication $\mathcal{S}^m$ of $A \subset \mathbb{R}^m$ and $\ell \in \{0, \ldots, m\}$, the *skeleton* of dimension ℓ is the set $\mathcal{S}^\ell$ of all ℓ-dimensional faces of all cubes in $\mathcal{S}^m$. A *subskeleton* of dimension ℓ of $\mathcal{S}^m$ is a subset of $\mathcal{S}^\ell$.

Given a skeleton $\mathcal{S}^\ell$, we denote by S^ℓ the union of all elements of $\mathcal{S}^\ell$,

$$S^\ell = \bigcup_{\sigma^\ell \in \mathcal{S}^\ell} \sigma^\ell.$$

For a given map $u \in W^{k,p}(U^m; \mathbb{R}^\nu)$ on some subskeleton $\mathcal{U}^m$ and for any ℓ in $\{0, \ldots, m-1\}$, we are going to construct a map $u \circ \Phi \in W^{k,p}(U^m; \mathbb{R}^\nu)$ which is constant along the normals to U^ℓ in a neighborhood of U^ℓ. In this region, the map $u \circ \Phi$ will thus be essentially a $W^{k,p}$ map of ℓ variables. Hence, if $kp > \ell$, then $u \circ \Phi$ will be continuous there, whereas in the critical case $\ell = kp$, the map $u \circ \Phi$ need not be continuous but will have vanishing mean oscillation. In this construction the map Φ depends on u and is never injective. This idea of opening a map has been inspired by a similar construction of Brezis and Li [10].

Given a map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$, we denote by $\text{Supp}\,\Phi$ the *geometric support* of Φ, the closure of the set $\{x \in \mathbb{R}^m : \Phi(x) \neq x\}$. This should not be confused with the analytic support $\text{supp}\,\varphi$ of a function $\varphi : \mathbb{R}^m \to \mathbb{R}$, which is the closure of the set $\{x \in \mathbb{R}^m : \varphi(x) \neq 0\}$.

Proposition 2.1. *Let $\ell \in \{0, \ldots, m - 1\}$, $\eta > 0$, $0 < \rho < 1/2$, and $\mathcal{U}^\ell$ be a subskeleton of $\mathbb{R}^m$ of radius η. Then, for every $u \in W^{k,p}(U^\ell + Q^m_{2\rho\eta}; \mathbb{R}^\nu)$, there exists a smooth map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that*

(i) *for every $i \in \{0, \ldots, \ell\}$ and every $\sigma^i \in \mathcal{U}^i$, Φ is constant on each $m-i$-dimensional cube of radius $\rho\eta$ which is orthogonal to σ^i,*

(ii) *$\text{Supp}\,\Phi \subset U^\ell + Q^m_{2\rho\eta}$ and $\Phi(U^\ell + Q^m_{2\rho\eta}) \subset U^\ell + Q^m_{2\rho\eta}$,*

(iii) *$u \circ \Phi \in W^{k,p}(U^\ell + Q^m_{2\rho\eta}; \mathbb{R}^\nu)$, and for every $j \in \{1, \ldots, k\}$,*

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(U^\ell + Q^m_{2\rho\eta})} \leq C \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(U^\ell + Q^m_{2\rho\eta})},$$

for some constant $C > 0$ depending on m, k, p and ρ,

(iv) *for every $\sigma^\ell \in \mathcal{U}^\ell$ and every $j \in \{1, \ldots, k\}$,*

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(\sigma^\ell + Q^m_{2\rho\eta})} \leq C' \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(\sigma^\ell + Q^m_{2\rho\eta})},$$

for some constant $C' > 0$ depending on m, k, p and ρ.

In the case of $W^{2,p}$ maps, the quantity $\|D(u \circ \Phi)\|_{L^p}$ can be estimated in terms of $\|Du\|_{L^p}$; hence there is no explicit dependence on η. However, concerning the second order term, estimate in (iii) reads

$$\|D^2(u \circ \Phi)\|_{L^p(U^\ell + Q^m_{2\rho\eta})} \leq C \|D^2 u\|_{L^p(U^\ell + Q^m_{2\rho\eta})} + \frac{C}{\eta} \|Du\|_{L^p(U^\ell + Q^m_{2\rho\eta})}.$$

The factor $1/\eta$ which comes naturally from a scaling argument is one of the differences with respect to the opening of $W^{1,p}$ maps. In the proof of Theorem 1, we shall use the Gagliardo–Nirenberg interpolation inequality to deal with this extra term.

Since the map u in the statement is defined almost everywhere, the map $u \circ \Phi$ need not be well-defined by standard composition of maps. By $u \circ \Phi$, we mean a map v in $W^{k,p}$ such that there exists a sequence of smooth maps $(u_n)_{n \in \mathbb{N}}$ converging to u in $W^{k,p}$ such that $(u_n \circ \Phi)_{n \in \mathbb{N}}$ converges to v in $W^{k,p}$. By pointwise convergence, this map $u \circ \Phi$ inherits several properties of Φ and u. For instance, if Φ is constant in a neighborhood of some point a, then so is $u \circ \Phi$. One can show that under some assumptions on Φ which are satisfied in all the cases that we consider, $u \circ \Phi$ does not depend on the sequence $(u_n)_{n \in \mathbb{N}}$, but we shall not make use of this fact. The only property we shall need from $u \circ \Phi$ is that its essential range is contained in the essential range of u; this is actually the case in view of Lemma 2.3(ii) below. In particular, *if u is a map with values into the manifold N^n, then $u \circ \Phi$ is also a map with values into N^n.*

The following proposition is the main tool in the proof of Proposition 2.1.

Proposition 2.2. *Let $\ell \in \{0, \ldots, m-1\}$, $\eta > 0$, $0 < \underline{\rho} < \overline{\rho}$ and $A \subset \mathbb{R}^\ell$ be an open set. For every $u \in W^{k,p}(A \times Q_{\overline{\rho}\eta}^{m-\ell}; \mathbb{R}^\nu)$, there exists a smooth map $\zeta : \mathbb{R}^{m-\ell} \to \mathbb{R}^{m-\ell}$ such that*

(i) *ζ is constant in $Q_{\underline{\rho}\eta}^{m-\ell}$,*

(ii) *$\operatorname{Supp}\zeta \subset Q_{\overline{\rho}\eta}^{m-\ell}$ and $\zeta(Q_{\overline{\rho}\eta}^{m-\ell}) \subset Q_{\overline{\rho}\eta}^{m-\ell}$,*

(iii) *if $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is defined for every $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by $\Phi(x) = (x', \zeta(x''))$, then $u \circ \Phi \in W^{k,p}(A \times Q_{\overline{\rho}\eta}^{m-\ell}; \mathbb{R}^\nu)$, and for every $j \in \{1, \ldots, k\}$,*

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(A \times Q_{\overline{\rho}\eta}^{m-\ell})} \leq C \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(A \times Q_{\overline{\rho}\eta}^{m-\ell})},$$

for some constant $C > 0$ depending on m, k, p, $\underline{\rho}$ and $\overline{\rho}$.

The proof of Proposition 2.2 is based on a Fubini type argument, which gives some flexibility in the choice of ζ. In particular, given finitely many measurable subsets $A_1, \ldots, A_s$ of A, the map ζ can be chosen so that in addition, for every $r \in \{1, \ldots, s\}$ and every $j \in \{1, \ldots, k\}$,

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(A_r \times Q_{\overline{\rho}\eta}^{m-\ell})} \leq C \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(A_r \times Q_{\overline{\rho}\eta}^{m-\ell})}.$$

Granting this proposition and the observation that follows it, we prove the main result of the section:

Proof of Proposition 2.1. We first take a finite sequence $(\rho_i)_{0 \leq i \leq \ell}$ such that

$$\rho = \rho_\ell < \cdots < \rho_i < \cdots < \rho_0 < 2\rho.$$

We construct by induction on $i \in \{0, \ldots, \ell\}$ a map $\Phi^i : \mathbb{R}^m \to \mathbb{R}^m$ such that

(a) for every $r \in \{0, \ldots, i\}$ and every $\sigma^r \in \mathcal{U}^r$, Φ^i is constant on the $m-r$-dimensional cubes of radius $\rho_i\eta$ which are orthogonal to σ^r,

(b) $\operatorname{Supp}\Phi^i \subset U^i + Q_{2\rho\eta}^m$ and $\Phi^i(U^i + Q_{2\rho\eta}^m) \subset U^i + Q_{2\rho\eta}^m$,

(c) $u \circ \Phi^i \in W^{k,p}(U^\ell + Q_{2\rho\eta}^m; \mathbb{R}^\nu)$,

(d) for every $\sigma^i \in \mathcal{U}^i$ and every $j \in \{1, \ldots, k\}$,

$$\eta^j \|D^j(u \circ \Phi^i)\|_{L^p(\sigma^i + Q_{2\rho\eta}^m)} \leq C \sum_{\alpha=1}^{j} \eta^\alpha \|D^\alpha u\|_{L^p(\sigma^i + Q_{2\rho\eta}^m)},$$

for some constant $C > 0$ depending on m, k, p and ρ.

The map Φ^ℓ will satisfy the conclusion of the proposition.

If $i = 0$, then $\mathcal{U}^0$ consists of all vertices of cubes in $\mathcal{U}^m$. To construct Φ^0, we apply Proposition 2.2 to the map u around each $\sigma^0 \in \mathcal{U}^0$ with parameters $\rho_0 < 2\rho$ and $\ell = 0$: in this case, the set $A \times Q_{\overline{\rho}\eta}^{m-\ell}$ in Proposition 2.2 is simply $Q_{2\rho}^m$. This gives a map Φ^0 such that for every $\sigma^0 \in \mathcal{U}^0$, Φ^0 is constant on $\sigma^0 + Q_{\rho_0\eta}^m$ and $\Phi^0 = \operatorname{Id}$ outside $U^0 + Q_{2\rho\eta}^m$.

Moreover, $u \circ \Phi^0 \in W^{k,p}(U^\ell + Q_{2\rho\eta}^m; \mathbb{R}^\nu)$ and for every $\sigma^0 \in \mathcal{U}^0$ and every j in $\{1, \ldots, k\}$,

$$\eta^j \|D^j(u \circ \Phi^i)\|_{L^p(\sigma^0 + Q_{2\rho\eta}^m)} \le C \sum_{\alpha=1}^{j} \eta^\alpha \|D^\alpha u\|_{L^p(\sigma^0 + Q_{2\rho\eta}^m)}.$$

Assume that the maps $\Phi^0, \ldots, \Phi^{i-1}$ have been constructed. To define Φ^i, we first apply Proposition 2.2, for each $\sigma^i \in \mathcal{U}^i$, to the map $u \circ \Phi^{i-1}$ with $A = \sigma^i$ and parameters $\rho_i < \rho_{i-1}$. This gives a smooth map $\Phi_{\sigma^i} : \mathbb{R}^m \to \mathbb{R}^m$ such that Φ_{σ^i} is constant on each $m - i$-dimensional cube of radius $\rho_i \eta$ which is orthogonal to σ^i.

Let $\Phi^i : \mathbb{R}^m \to \mathbb{R}^m$ be defined for $x \in \mathbb{R}^m$ by

$$\Phi^i(x) = \begin{cases} \Phi^{i-1}(\Phi_{\sigma^i}(x)) & \text{if } x \in \sigma^i + Q_{\rho_{i-1}\eta}^m \text{ where } \sigma^i \in \mathcal{U}^i, \\ \Phi^{i-1}(x) & \text{otherwise.} \end{cases}$$

We first explain why Φ^i is well-defined. For this purpose, let

$$x \in (\sigma_1^i + Q_{\rho_{i-1}\eta}^m) \cap (\sigma_2^i + Q_{\rho_{i-1}\eta}^m)$$

for some $\sigma_1^i \in \mathcal{U}^i$ and $\sigma_2^i \in \mathcal{U}^i$ such that $\sigma_1^i \ne \sigma_2^i$. In particular, σ_1^i and σ_2^i are not disjoint, and there exists a smallest dimension $r \in \{0, \ldots, i-1\}$ such that

$$x \in \tau^r + Q_{\rho_{i-1}\eta}^m \quad \text{and} \quad \tau^r \subset \sigma_1^i \cap \sigma_2^i$$

for some $\tau^r \in \mathcal{U}^r$. By the formula for Φ_{σ^i} given in Proposition 2.2, the points x, $\Phi_{\sigma_1^i}(x)$ and $\Phi_{\sigma_2^i}(x)$ belong to the same $m - r$-dimensional cube of radius $\rho_{i-1}\eta$ which is orthogonal to τ^r. Since by induction hypothesis Φ^{i-1} is constant on each $m - r$-dimensional cube of radius $\rho_{i-1}\eta$ which is orthogonal to τ^r, we have

$$\Phi^{i-1}(x) = \Phi^{i-1}(\Phi_{\sigma_1^i}(x)) = \Phi^{i-1}(\Phi_{\sigma_2^i}(x)).$$

This implies that Φ^i is well-defined. Moreover, Φ^i is smooth and has properties (a)–(c).

We prove the estimates given by (d). If $e_1, \ldots, e_m$ is an orthonormal basis of $\mathbb{R}^m$ compatible with the skeleton $\mathcal{U}^\ell$, then by abuse of notation we denote by $\sigma^i \times Q_{\alpha\eta}^{m-i}$ the parallelepiped given by

$$\left\{ x + \sum_{s=1}^{m-i} t_s e_{r_s} : x \in \sigma^i \text{ and } |t_s| \le \alpha\eta \right\},$$

where $e_{r_1}, \ldots, e_{r_{m-i}}$ are orthogonal to σ^i. Note that for every $\sigma^i \in \mathcal{U}^i$,

$$\sigma^i + Q_{2\rho\eta}^m = (\sigma^i \times Q_{2\rho\eta}^{m-i}) \cup (\partial\sigma^i + Q_{2\rho\eta}^m),$$

where $\partial\sigma^i$ denotes the $i - 1$-dimensional skeleton of σ^i. By property (iii) of Proposition 2.2,

$$\int_{\sigma^i \times Q_{\rho_{i-1}\eta}^{m-i}} \eta^{jp} |D^j(u \circ \Phi^{i-1} \circ \Phi_{\sigma^i})|^p \le C_1 \sum_{\alpha=1}^{j} \int_{\sigma^i \times Q_{\rho_{i-1}\eta}^{m-i}} \eta^{\alpha p} |D^\alpha(u \circ \Phi^{i-1})|^p,$$

and then, since $\Phi^i = \Phi^{i-1} \circ \Phi_{\sigma^i}$ on $(\sigma^i \times Q_{2\rho\eta}^{m-i}) \setminus (\partial\sigma^i + Q_{2\rho\eta}^m)$ and since the geometric

support Supp Φ_{σ^i} is contained in $\sigma^i \times Q_{\rho_{i-1}\eta}^{m-i}$, we have

$$\int_{(\sigma^i \times Q_{2\rho\eta}^{m-i}) \setminus (\partial \sigma^i + Q_{2\rho\eta}^m)} \eta^{jp} |D^j (u \circ \Phi^i)|^p \leq C_1 \sum_{\alpha=1}^{j} \int_{\sigma^i \times Q_{2\rho\eta}^{m-i}} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p. \quad (2.1)$$

We claim that the maps Φ_{σ^i} can be so chosen that moreover, for every $j \in \{1, \ldots, k\}$,

$$\int_{\partial \sigma^i + Q_{2\rho\eta}^m} \eta^{jp} |D^j (u \circ \Phi^i)|^p \leq C_2 \sum_{\alpha=1}^{j} \int_{\partial \sigma^i + Q_{2\rho\eta}^m} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p. \quad (2.2)$$

Indeed, by the remark following Proposition 2.2, for every $\sigma^i \in \mathcal{U}^i$ we may further require that Φ_{σ^i} satisfies, for every $i-1$-dimensional cube $\tau^{i-1} \subset \partial \sigma^i$ and every $j \in \{1, \ldots, k\}$,

$$\int_{[(\tau^{i-1} + Q_{2\rho\eta}^m) \cap \sigma^i] \times Q_{\rho_{i-1}\eta}^m} \eta^{jp} |D^j (u \circ \Phi^{i-1} \circ \Phi_{\sigma^i})|^p$$

$$\leq C_3 \sum_{\alpha=1}^{j} \int_{[(\tau^{i-1} + Q_{2\rho\eta}^m) \cap \sigma^i] \times Q_{\rho_{i-1}\eta}^m} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p.$$

Next, given $\tau^{i-1} \subset \partial \sigma^i$, denote by $\sigma_1^i, \ldots, \sigma_\theta^i$ the i-dimensional cubes in $\mathcal{U}^i$ containing τ^{i-1} in their boundaries. In this case,

$$\tau^{i-1} + Q_{2\rho\eta}^m \subset \Big(\bigcup_{\beta=1}^{\theta} [(\tau^{i-1} + Q_{2\rho\eta}^m) \cap \sigma_\beta^i] \times Q_{\rho_{i-1}\eta}^m \Big) \cup \{x \in \mathbb{R}^m : \Phi^i(x) = \Phi^{i-1}(x)\}.$$

Since for every $\beta \in \{1, \ldots, \theta\}$, $\Phi^i = \Phi^{i-1} \circ \Phi_{\sigma_\beta^i}$ on $[(\tau^{i-1} + Q_{2\rho\eta}^m) \cap \sigma_\beta^i] \times Q_{\rho_{i-1}\eta}^m$, by the previous estimate on each cube σ_β^i and by additivity of integral, we get

$$\int_{\tau^{i-1} + Q_{2\rho\eta}^m} \eta^{jp} |D^j (u \circ \Phi^i)|^p \leq C_4 \sum_{\alpha=1}^{j} \int_{\tau^{i-1} + Q_{2\rho\eta}^m} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p.$$

Summing both sides of this inequality over the $i-1$-dimensional cubes $\tau^{i-1} \subset \partial \sigma^i$, we deduce estimate (2.2) as we claimed.

By additivity of integral and by estimates (2.1) and (2.2), we then obtain

$$\int_{\sigma^i + Q_{2\rho\eta}^m} \eta^{jp} |D^j (u \circ \Phi^i)|^p \leq C_5 \sum_{\alpha=1}^{j} \int_{\sigma^i + Q_{2\rho\eta}^m} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p.$$

Since by induction hypothesis Φ^{i-1} coincides with the identity map outside $U^{i-1} + Q_{2\rho\eta}^m$, for every $\alpha \in \{1, \ldots, j\}$ we have

$$\int_{\sigma^i + Q_{2\rho\eta}^m} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p$$

$$= \int_{\partial \sigma^i + Q_{2\rho\eta}^m} \eta^{\alpha p} |D^\alpha (u \circ \Phi^{i-1})|^p + \int_{(\sigma^i + Q_{2\rho\eta}^m) \setminus (\partial \sigma^i + Q_{2\rho\eta}^m)} \eta^{\alpha p} |D^\alpha u|^p.$$

By induction hypothesis, for every $i-1$-dimensional face τ^{i-1} of $\partial\sigma^i$,

$$\int_{\tau^{i-1}+Q^m_{2\rho\eta}} \eta^{\alpha p}|D^\alpha(u\circ\Phi^{i-1})|^p \le C_6 \sum_{\beta=1}^{\alpha} \int_{\tau^{i-1}+Q^m_{2\rho\eta}} \eta^{\beta p}|D^\beta u|^p.$$

Since the number of overlaps of the sets $\tau^{i-1}+Q^m_{2\rho\eta}$ is bounded from above by a constant only depending on m, by additivity of integral we have

$$\int_{\partial\sigma^i+Q^m_{2\rho\eta}} \eta^{\alpha p}|D^\alpha(u\circ\Phi^{i-1})|^p \le C_7 \sum_{\beta=1}^{\alpha} \int_{\partial\sigma^i+Q^m_{2\rho\eta}} \eta^{\beta p}|D^\beta u|^p.$$

Therefore,

$$\int_{\sigma^i+Q^m_{2\rho\eta}} \eta^{jp}|D^j(u\circ\Phi^i)|^p \le C_8 \sum_{\alpha=1}^{j} \int_{\sigma^i+Q^m_{2\rho\eta}} \eta^{\alpha p}|D^\alpha u|^p.$$

The map Φ^ℓ satisfies properties (i)–(iv). The estimate of property (iii) is a consequence of (iv) and the additivity of integral. □

We proceed to prove Proposition 2.2 by making precise the meaning of $u\circ\Phi$ in the statement.

Given a continuous function $\Psi:U\times V\to W$ and $z\in V$, we denote by $\Psi_z:U\to W$ the map defined for every $x\in U$ by $\Psi_z(x)=\Psi(x,z)$. For every measurable function $g:W\to\mathbb{R}$, the composition $g\circ\Psi_z$ is well-defined and gives a measurable function defined on W for every z.

Lemma 2.3. *Let $U,W\subset\mathbb{R}^m$ and $V\subset\mathbb{R}^l$ be measurable sets and let $\Psi:U\times V\to W$ be a continuous map such that for every measurable function $g:W\to\mathbb{R}$,*

$$\int_V \|g\circ\Psi_z\|_{L^1(U)}\,dz \le C\|g\|_{L^1(W)}.$$

If $u\in L^p(W;\mathbb{R}^\nu)$ and $(u_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions converging to u in $L^p(W;\mathbb{R}^\nu)$, then there exists a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ such that for almost every $z\in V$:

(i) *the sequence $(u_{n_i}\circ\Psi_z)_{i\in\mathbb{N}}$ converges in $L^p(U;\mathbb{R}^\nu)$ to a function which we denote by $u\circ\Psi_z$,*

(ii) *the essential range of $u\circ\Psi_z$ is contained in the essential range of u.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in W converging to u in $L^p(W;\mathbb{R}^\nu)$. Given a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers, let $(u_{n_i})_{i\in\mathbb{N}}$ be a subsequence such that for every $i\in\mathbb{N}$,

$$\|u_{n_{i+1}}-u_{n_i}\|_{L^p(W)} \le \varepsilon_i.$$

By the assumption on Ψ,

$$\int_V \|u_{n_{i+1}}\circ\Psi_z - u_{n_i}\circ\Psi_z\|^p_{L^p(U)}\,dz \le C\|u_{n_{i+1}}-u_{n_i}\|^p_{L^p(W)} \le C\varepsilon_i^p.$$

Given a summable sequence $(\alpha_n)_{n\in\mathbb{N}}$ of positive numbers, let

$$Y_i = \{z\in V : \|u_{n_{i+1}}\circ\Psi_z - u_{n_i}\circ\Psi_z\|_{L^p(U)} > \alpha_i\}.$$

Then, for every $t \in \mathbb{N}$ and every $z \notin \bigcup_{i=t}^{\infty} Y_i$, the sequence $(u_{n_i} \circ \Psi_z)_{i \in \mathbb{N}}$ is a Cauchy sequence in $L^p(U; \mathbb{R}^\nu)$.

By the Chebyshev inequality,

$$\alpha_i^p |Y_i| \leq \int_{Y_i} \|u_{n_{i+1}} \circ \Psi_z - u_{n_i} \circ \Psi_z\|_{L^p(U)}^p \, dz \leq C \varepsilon_i^p.$$

Hence, for every $t \in \mathbb{N}$,

$$\left| \bigcup_{i=t}^{\infty} Y_i \right| \leq C \sum_{i=t}^{\infty} (\varepsilon_i / \alpha_i)^p.$$

Taking $(\varepsilon_n)_{n \in \mathbb{N}}$ and $(\alpha_n)_{n \in \mathbb{N}}$ such that both $\sum_{i=0}^{\infty} \alpha_i$ and $\sum_{i=0}^{\infty} (\varepsilon_i / \alpha_i)^p$ converge, the set $E = \bigcap_{t=0}^{\infty} \bigcup_{i=t}^{\infty} Y_i$ is negligible and for every $z \in V \setminus E$, $(u_{n_i} \circ \Psi_z)_{i \in \mathbb{N}}$ is a Cauchy sequence in $L^p(U; \mathbb{R}^\nu)$. This proves assertion (i).

It suffices to prove assertion (ii) when W has finite Lebesgue measure. For every $z \in V \setminus E$, we denote by $u \circ \Psi_z$ the limit in $L^p(U; \mathbb{R}^\nu)$ of the sequence $(u_{n_i} \circ \Psi_z)_{i \in \mathbb{N}}$.

Let $\theta : \mathbb{R}^\nu \to \mathbb{R}$ be a continuous function such that $\theta^{-1}(0)$ is equal to the essential range of u and $0 \leq \theta \leq 1$ in $\mathbb{R}^\nu$. For every $i \in \mathbb{N}$,

$$\int_V \|\theta \circ (u_{n_i} \circ \Psi_z)\|_{L^1(U)} \, dz \leq C \|\theta \circ u_{n_i}\|_{L^1(W)}.$$

By Fatou's lemma,

$$\int_V \|\theta \circ (u \circ \Psi_z)\|_{L^1(U)} \, dz \leq \liminf_{i \to \infty} \int_V \|\theta \circ (u_{n_i} \circ \Psi_z)\|_{L^1(U)} \, dz.$$

Since W has finite Lebesgue measure and θ is bounded, as i tends to infinity we get

$$\int_V \|\theta \circ (u \circ \Psi_z)\|_{L^1(U)} \, dz \leq C \|\theta \circ u\|_{L^1(W)} = 0.$$

Therefore, for almost every $z \in V$, $\|\theta \circ (u \circ \Psi_z)\|_{L^1(U)} = 0$, whence the essential range of $u \circ \Psi_z$ is contained in the essential range of u.   □

From the previous lemma, we can prove the following property for maps in $W^{k,p}$:

Lemma 2.4. *Let $U, W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^l$ be open sets and let $\Psi : U \times V \to W$ be a smooth map such that for every measurable function $g : W \to \mathbb{R}$,*

$$\int_V \|g \circ \Psi_z\|_{L^1(U)} \, dz \leq C \|g\|_{L^1(W)}.$$

If $u \in W^{k,p}(W; \mathbb{R}^\nu)$ and if $(u_n)_{n \in \mathbb{N}}$ is a sequence of smooth functions converging to u in $W^{k,p}(W; \mathbb{R}^\nu)$, then there exists a subsequence $(u_{n_i})_{i \in \mathbb{N}}$ such that for almost every $z \in V$ the sequence $(u_{n_i} \circ \Psi_z)_{i \in \mathbb{N}}$ converges to $u \circ \Psi_z$ in $W^{k,p}(U; \mathbb{R}^\nu)$, and for every $j \in \{1, \dots, k\}$,

$$\int_V \|D^j(u \circ \Psi_z)\|_{L^p(U)} \, dz \leq C' |V|^{1-1/p} \sum_{i=1}^{j} \|D^i u\|_{L^p(W)},$$

for some constant $C' > 0$ depending on m, p, k, C and $\max_{1 \leq j \leq k} \sup_{z \in V} \|D^j \Psi_z\|_{L^\infty(U)}$.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of smooth functions in $W^{k,p}(W;\mathbb{R}^\nu)$ converging to u in $W^{k,p}(W;\mathbb{R}^\nu)$. By the previous lemma, there exists a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ such that for almost every $z \in V$, $(u_{n_i} \circ \Psi_z)_{i\in\mathbb{N}}$ converges to $u \circ \Psi_z$ in L^p, and for every j in $\{1,\dots,k\}$, $((D^j u_{n_i}) \circ \Psi_z)_{i\in\mathbb{N}}$ converges to $(D^j u) \circ \Psi_z$ in L^p.

For every $v \in C^\infty(W;\mathbb{R}^\nu)$, for every $z \in V$ and for each $j \in \{1,\dots,k\}$,

$$|D^j(v \circ \Psi_z)(x)| \le C_1 \sum_{i=1}^{j} \sum_{\substack{1\le t_1\le\cdots\le t_i \\ t_1+\cdots+t_i=j}} |D^i v(\Psi_z(x))|\,|D^{t_1}\Psi_z(x)|\cdots|D^{t_i}\Psi_z(x)|$$

$$\le C_2 \sum_{i=1}^{j} |D^i v(\Psi_z(x))|,$$

whence

$$\|D^j(v \circ \Psi_z)\|^p_{L^p(U)} \le C_3 \sum_{i=1}^{j} \big\||D^i v|^p \circ \Psi_z\big\|_{L^1(U)}.$$

This implies that for almost every $z \in V$, $(u_{n_i} \circ \Psi_z)_{i\in\mathbb{N}}$ is a Cauchy sequence in $W^{k,p}(U;\mathbb{R}^\nu)$, thus $(u_{n_i} \circ \Psi_z)_{i\in\mathbb{N}}$ converges to $u \circ \Psi_z$ in $W^{k,p}(U;\mathbb{R}^\nu)$. Moreover, integrating the above estimate with respect to z and using the assumption on Ψ we get

$$\int_V \|D^j(v \circ \Psi_z)\|^p_{L^p(U)}\,dz \le C_3 \sum_{i=1}^{j} \int_V \big\||D^i v|^p \circ \Psi_z\big\|_{L^1(U)}\,dz$$

$$\le C_4 \sum_{i=1}^{j} \big\||D^i v|^p\big\|_{L^1(W)} = C_4 \sum_{i=1}^{j} \|D^i v\|^p_{L^p(W)}.$$

Thus, by Hölder's inequality,

$$\int_V \|D^j(v \circ \Psi_z)\|_{L^p(U)}\,dz \le |V|^{1-1/p}\Big(C_4 \sum_{i=1}^{j} \|D^i v\|^p_{L^p(W)}\Big)^{1/p}$$

$$\le C_5 |V|^{1-1/p} \sum_{i=1}^{j} \|D^i v\|_{L^p(W)}.$$

We obtain the desired estimate by taking $v = u_{n_i}$ and letting n_i tend to infinity. $\qquad\square$

We now show that the functional estimate in Lemmas 2.3 and 2.4 is satisfied for maps Ψ of the form $\Psi(x,z) = \zeta(x+z) - z$. The strategy is based on an averaging device due to Federer and Fleming [18] and adapted by Hardt, Kinderlehrer and Lin [30] in the context of Sobolev maps. It relies on the following lemma:

Lemma 2.5. *Let $U, V, W \subset \mathbb{R}^l$ be measurable sets and let $\zeta : U + V \to \mathbb{R}^l$ be a continuous map such that $\zeta(x+z) - z \in W$ for all $x \in U$ and $z \in V$. Then, for every measurable function $g : W \to \mathbb{R}$,*

$$\int_V \Big(\int_U |g(\zeta(x+z) - z)|\,dx\Big)\,dz \le |U+V| \int_W |g(x)|\,dx.$$

Proof. Let $\xi : (U + V) \times V \to \mathbb{R}^l$ be defined by

$$\xi(x, z) = \zeta(x + z) - z.$$

By Fubini's theorem,

$$\int_V \left(\int_U |(g \circ \xi)(x, z)| \, dx \right) dz = \int_U \left(\int_V |g(\zeta(x + z) - z)| \, dz \right) dx.$$

Applying the change of variables $\tilde{z} = x + z$ in the variable z and Fubini's theorem,

$$\int_V \left(\int_U |(g \circ \xi)(x, z)| \, dx \right) dz = \int_U \left(\int_{x+V} |g(\zeta(\tilde{z}) + x - \tilde{z})| \, d\tilde{z} \right) dx$$

$$= \int_{U+V} \left(\int_{(\tilde{z}-V)\cap U} |g(\zeta(\tilde{z}) + x - \tilde{z})| \, dx \right) d\tilde{z}.$$

We now apply the change of variables $\tilde{x} = \zeta(\tilde{z}) + x - \tilde{z}$ in the variable x, and use the assumption on W to conclude

$$\int_V \left(\int_U |(g \circ \xi)(x, z)| \, dx \right) dz = \int_{U+V} \left(\int_{\zeta(\tilde{z}) - (V \cap (\tilde{z} - U))} |g(\tilde{x})| \, d\tilde{x} \right) d\tilde{z}$$

$$\leq \int_{U+V} \left(\int_W |g(\tilde{x})| \, d\tilde{x} \right) d\tilde{z} = |U + V| \int_W |g(\tilde{x})| \, d\tilde{x}.$$

This gives the desired estimate. $\qquad\square$

Proof of Proposition 2.2. By scaling, it suffices to establish the result when $\eta = 1$. We fix $\hat{\rho}$ such that $2\hat{\rho} < \overline{\rho} - \rho$.

Let $\tilde{\zeta} : \mathbb{R}^{m-\ell} \to \mathbb{R}^{m-\ell}$ be the smooth map defined by

$$\tilde{\zeta}(y) = (1 - \varphi(y))y,$$

where $\varphi : \mathbb{R}^{m-\ell} \to [0, 1]$ is a smooth function such that

- $\varphi(y) = 1$ for $y \in Q_{\rho+\hat{\rho}}^{m-\ell}$,
- $\varphi(y) = 0$ for $y \in \mathbb{R}^{m-\ell} \setminus Q_{\overline{\rho}-\hat{\rho}}^{m-\ell}$.

For any $z \in Q_{\hat{\rho}}^{m-\ell}$, the function $\zeta : \mathbb{R}^{m-\ell} \to \mathbb{R}^{m-\ell}$ defined for $x'' \in \mathbb{R}^{m-\ell}$ by

$$\zeta(x'') = \tilde{\zeta}(x'' + z) - z$$

satisfies properties (i)–(ii).

We claim that for some $z \in Q_{\hat{\rho}}^{m-\ell}$, the function $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ defined for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by

$$\Phi(x) = (x', \zeta(x''))$$

satisfies property (iii).

For this purpose, let $\Psi : \mathbb{R}^m \times Q_{\hat{\rho}}^{m-\ell} \to \mathbb{R}^m$ be defined for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ and $z \in Q_{\hat{\rho}}^{m-\ell}$ by

$$\Psi(x, z) = (x', \tilde{\zeta}(x'' + z) - z).$$

For every measurable function $f : A \times Q_{\bar{\rho}}^{m-\ell} \to \mathbb{R}$, by Fubini's theorem we have

$$\int_{Q_{\hat{\rho}}^{m-\ell}} \|f \circ \Psi_z\|_{L^1(A \times Q_{\bar{\rho}}^{m-\ell})} \, dz = \int_A \left[\int_{Q_{\hat{\rho}}^{m-\ell}} \left(\int_{Q_{\bar{\rho}}^{m-\ell}} \left| f(x', \tilde{\zeta}(x'' + z) - z) \right| dx'' \right) dz \right] dx'.$$

Given $x' \in A$, we apply Lemma 2.5 with $U = Q_{\bar{\rho}}^{m-\ell}$, $V = Q_{\hat{\rho}}^{m-\ell}$, $W = Q_{\bar{\rho}}^{m-\ell}$, and $\tilde{\zeta}$. We deduce that

$$\int_{Q_{\hat{\rho}}^{m-\ell}} \left(\int_{Q_{\bar{\rho}}^{m-\ell}} \left| f(x', \tilde{\zeta}(x'' + z) - z) \right| dx'' \right) dz \leq C_1 \int_{Q_{\bar{\rho}}^{m-\ell}} |f(x', x'')| \, dx''.$$

Thus,

$$\int_{Q_{\hat{\rho}}^{m-\ell}} \|f \circ \Psi_z\|_{L^1(A \times Q_{\bar{\rho}}^{m-\ell})} \, dz \leq C_1 \|f\|_{L^1(A \times Q_{\bar{\rho}}^{m-\ell})}.$$

By Lemma 2.4, for almost every $z \in Q_{\hat{\rho}}^{m-\ell}$, $u \circ \Psi_z \in W^{k,p}(A \times Q_{\bar{\rho}}^{m-\ell}; \mathbb{R}^\nu)$, and for every $j \in \{1, \ldots, k\}$,

$$\int_{Q_{\hat{\rho}}^{m-\ell}} \|D^j(u \circ \Psi_z)\|_{L^p(A \times Q_{\bar{\rho}}^{m-\ell})} \, dz \leq C_2 \sum_{i=1}^{j} \|D^i u\|_{L^p(A \times Q_{\bar{\rho}}^{m-\ell})}.$$

We may thus find some $z \in Q_{\hat{\rho}}^{m-\ell}$ such that $u \circ \Psi_z \in W^{k,p}(A \times Q_{\bar{\rho}}^{m-\ell}; \mathbb{R}^\nu)$ and for every $j \in \{1, \ldots, k\}$,

$$\|D^j(u \circ \Psi_z)\|_{L^p(A \times Q_{\bar{\rho}}^{m-\ell})} \leq C_3 \sum_{i=1}^{j} \|D^i u\|_{L^p(A \times Q_{\bar{\rho}}^{m-\ell})}.$$

The function ζ defined in terms of this point z satisfies the required properties. $\qquad\square$

Addendum 1 to Proposition 2.1. *Let $\mathcal{K}^m$ be a cubication containing $\mathcal{U}^m$ and let $q \geq 1$. If $u \in W^{1,q}(K^m + Q_{2\rho\eta}^m; \mathbb{R}^\nu)$, then the map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ can be chosen with the additional property that $u \circ \Phi \in W^{1,q}(K^m + Q_{2\rho\eta}^m; \mathbb{R}^\nu)$ and for every $\sigma^m \in \mathcal{K}^m$,*

$$\|D(u \circ \Phi)\|_{L^q(\sigma^m + Q_{2\rho\eta}^m)} \leq C'' \|Du\|_{L^q(\sigma^m + Q_{2\rho\eta}^m)},$$

for some constant $C'' > 0$ depending on m, q and ρ.

Proof. Since $u \in W^{1,q}(U^\ell + Q_{2\rho\eta}^m; \mathbb{R}^\nu)$, we may apply Proposition 2.1 with $k = 1$ and $p = q$ to obtain a map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that $u \circ \Phi \in W^{1,q}(U^\ell + Q_{2\rho\eta}^m; \mathbb{R}^\nu)$ and for every $\sigma^\ell \in \mathcal{U}^\ell$,

$$\|D(u \circ \Phi)\|_{L^q(\sigma^\ell + Q_{2\rho\eta}^m)} \leq C \|Du\|_{L^q(\sigma^\ell + Q_{2\rho\eta}^m)}.$$

Since the choice of the point z in the proof of Proposition 2.2 can be done in a set of positive measure, we may do so keeping the properties we already have for $W^{k,p}$.

For every $\sigma^m \in \mathcal{K}^m$, if $\sigma^{m,\ell}$ denotes the skeleton of dimension ℓ of σ^m, then by additivity of integral,

$$\|D(u \circ \Phi)\|_{L^q((\sigma^{m,\ell} \cap U^\ell) + Q_{2\rho\eta}^m)} \leq C \|Du\|_{L^q((\sigma^{m,\ell} \cap U^\ell) + Q_{2\rho\eta}^m)}.$$

Since Φ coincides with the identity map in $(\sigma^m + Q^m_{2\rho\eta}) \setminus ((\sigma^{m,\ell} \cap U^\ell) + Q^m_{2\rho\eta})$,

$$\|D(u \circ \Phi)\|_{L^q(\sigma^m + Q^m_{2\rho\eta})} \leq C \|Du\|_{L^q(\sigma^m + Q^m_{2\rho\eta})}.$$

This concludes the proof.   □

Addendum 2 to Proposition 2.1. *Let $\mathcal{K}^m$ be a cubication containing $\mathcal{U}^m$. If u is in $W^{1,kp}(K^m + Q^m_{2\rho\eta}; \mathbb{R}^\nu)$, then the map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ given by Proposition 2.1 and Addendum 1 above with $q = kp$ satisfies*

$$\lim_{r \to 0} \sup_{Q^m_r(a) \subset U^\ell + Q^m_{\rho\eta}} \frac{r^{\ell/(kp)-1}}{|Q^m_r|^2} \int_{Q^m_r(a)} \int_{Q^m_r(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, dx \, dy = 0$$

and for every $\sigma^m \in \mathcal{U}^m$ and every $a \in \sigma^m$ such that $Q^m_r(a) \subset U^\ell + Q^m_{\rho\eta}$,

$$\frac{1}{|Q^m_r|^2} \int_{Q^m_r(a)} \int_{Q^m_r(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, dx \, dy \leq \frac{C''' r^{1-\ell/(kp)}}{\eta^{(m-\ell)/kp}} \|Du\|_{L^{kp}(\sigma^m + Q^m_{2\rho\eta})},$$

for some constant $C''' > 0$ depending on m, kp and ρ.

If $kp \geq \ell$, then the limit above implies that $u \circ \Phi$ belongs to the space VMO$(U^\ell + Q^m_{\rho\eta}; \mathbb{R}^\nu)$ of functions of vanishing mean oscillation and the estimate yields an estimate on the BMO seminorm on the domain $U^\ell + Q^m_{\rho\eta}$ as defined by Jones [32]. If $kp > \ell > 0$, then the estimate implies that $u \circ \Phi \in C^{0,1-\ell/(kp)}(U^\ell + Q^m_{\rho\eta}; \mathbb{R}^\nu)$ with an upper bound on the $C^{0,1-\ell/(kp)}$ seminorm of $u \circ \Phi$ [14]. The estimates of this addendum are not really useful when $kp < \ell$ since in this case $\lim_{r \to 0} r^{1-\ell/(kp)} = \infty$.

Proof of Addendum 2. Fix $Q^m_r(a) \subset U^\ell + Q^m_{\rho\eta}$. Then $a \in U^\ell + Q^m_{\rho\eta-r}$. Hence there exists an ℓ-dimensional face $\tau^\ell \in \mathcal{U}^\ell$ such that $Q^m_r(a) \subset \tau^\ell + Q^m_{\rho\eta}$. Without loss of generality, we may assume that $\tau^\ell = Q^\ell_\eta \times \{0^{m-\ell}\} \subset \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$. From Proposition 2.1(i), the map Φ is constant on the $m - \ell$-dimensional cubes of radius $\rho\eta$ which are orthogonal to $Q^\ell_{(1+\rho)\eta} \times \{0^{m-\ell}\}$. Write $Q^m_r(a) = Q^\ell_r(a') \times Q^{m-\ell}_r(a'')$; then $u \circ \Phi$ only depends on the first ℓ-dimensional variables in $Q^m_r(a)$. Let $v : Q^\ell_{(1+\rho)\eta} \to \mathbb{R}^\nu$ be defined by

$$v(x') = (u \circ \Phi)(x', a'').$$

By Addendum 1 above with $q = kp$, $u \circ \Phi \in W^{1,kp}(Q^\ell_{(1+\rho)\eta} \times Q^{m-\ell}_{\rho\eta}; \mathbb{R}^\nu)$, whence $v \in W^{1,kp}(Q^\ell_{(1+\rho)\eta}; \mathbb{R}^\nu)$. Note that

$$\frac{1}{|Q^m_r|^2} \int_{Q^m_r(a)} \int_{Q^m_r(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, dx \, dy$$
$$= \frac{1}{|Q^\ell_r|^2} \int_{Q^\ell_r(a')} \int_{Q^\ell_r(a')} |v(x') - v(y')| \, dx' \, dy'.$$

By the Poincaré–Wirtinger inequality,

$$\frac{1}{|Q^\ell_r|^2} \int_{Q^\ell_r(a')} \int_{Q^\ell_r(a')} |v(x') - v(y')| \, dx' \, dy' \leq C_1 r^{1-\ell/(kp)} \|Dv\|_{L^{kp}(Q^\ell_r(a'))}.$$

Thus,

$$\frac{1}{|Q_r^m|^2} \int_{Q_r^m(a)} \int_{Q_r^m(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, dx \, dy \le C_1 r^{1-\ell/(kp)} \|Dv\|_{L^{kp}(Q_r^\ell(a'))},$$

and this implies the first part of the conclusion.

To get the estimate of the oscillation of $u \circ \Phi$ in terms of $\|D(u \circ \Phi)\|_{L^{kp}}$, note that

$$\|D(u \circ \Phi)\|_{L^{kp}(Q_r^\ell(a') \times Q_{\rho\eta}^{m-\ell}(a''))} = (2\rho\eta)^{(m-\ell)/(kp)} \|Dv\|_{L^{kp}(Q_r^\ell(a'))}.$$

This implies for any $\sigma^m \in \mathcal{U}^m$ such that $\tau^\ell \subset \sigma^m$,

$$\|Dv\|_{L^{kp}(Q_r^\ell)} = \frac{1}{(2\rho\eta)^{(m-\ell)/(kp)}} \|D(u \circ \Phi)\|_{L^{kp}(Q_r^\ell \times Q_{\rho\eta}^{m-\ell})}$$

$$\le \frac{1}{(2\rho\eta)^{(m-\ell)/(kp)}} \|D(u \circ \Phi)\|_{L^{kp}(\sigma^\ell + Q_{\rho\eta}^m)} \le \frac{1}{(2\rho\eta)^{(m-\ell)/(kp)}} \|D(u \circ \Phi)\|_{L^{kp}(\sigma^m + Q_{\rho\eta}^m)}.$$

Thus,

$$\frac{1}{|Q_r^m|^2} \int_{Q_r^m(a)} \int_{Q_r^m(a)} |u \circ \Phi(x) - u \circ \Phi(y)| \, dx \, dy \le \frac{C_2 r^{1-\ell/(kp)}}{(\rho\eta)^{(m-\ell)/(kp)}} \|D(u \circ \Phi)\|_{L^{kp}(\sigma^m + Q_{\rho\eta}^m)}.$$

By Addendum 1 above,

$$\|D(u \circ \Phi)\|_{L^{kp}(\sigma^m + Q_{\rho\eta}^m)} \le C_3 \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}.$$

This proves the estimate that we claimed. $\qquad\square$

3. Adaptive smoothing

Given $u \in W^{k,p}(\Omega; \mathbb{R}^\nu)$, we would like to consider a convolution of u with a parameter which may depend on the point where we compute the convolution itself. The main reason is that we want to choose the convolution parameter by taking into account the mean oscillation of u: we choose a large parameter where u does not oscillate too much and a small parameter elsewhere.

For this purpose, consider a function $u \in L^1(\Omega; \mathbb{R}^\nu)$. Let φ be a *mollifier*, in other words,

$$\varphi \in C_c^\infty(B_1^m), \quad \varphi \ge 0 \quad \text{in } B_1^m \quad \text{and} \quad \int_{B_1^m} \varphi = 1.$$

For every $s \ge 0$ and every $x \in \Omega$ such that $d(x, \partial\Omega) \ge s$, we may consider the convolution

$$(\varphi_s * u)(x) = \int_{B_1^m} \varphi(z) u(x + sz) \, dz.$$

We should keep in mind that with this definition,

$$(\varphi_0 * u)(x) = \int_{B_1^m} \varphi(z) \, dz \, u(x) = u(x).$$

This way of writing convolution has the advantage that we may treat the cases $s = 0$ and $s > 0$ using the same formula.

We now introduce a nonconstant parameter in the convolution given by a nonnegative function $\psi \in C^\infty(\Omega)$. The convolution

$$\varphi_\psi * u : \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \psi(x)\} \to \mathbb{R}^\nu$$

is well-defined and if $\psi(a) > 0$ and $|D\psi(a)| < 1$ at some point $a \in \Omega$, then by a change of variable in the integral the map $\varphi_\psi * u$ is smooth in a neighborhood of a.

Proposition 3.1. *Let $\varphi \in C_c^\infty(B_1^m)$ be a mollifier and let $\psi \in C^\infty(\Omega)$ be a nonnegative function such that $\|D\psi\|_{L^\infty(\Omega)} < 1$. Then, for every $u \in L^p(\Omega; \mathbb{R}^\nu)$ and every open set $\omega \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \psi(x)\}$, we have $\varphi_\psi * u \in L^p(\omega; \mathbb{R}^\nu)$,*

$$\|\varphi_\psi * u\|_{L^p(\omega)} \leq \frac{1}{(1 - \|D\psi\|_{L^\infty(\omega)})^{1/p}} \|u\|_{L^p(\Omega)},$$

and

$$\|\varphi_\psi * u - u\|_{L^p(\omega)} \leq \sup_{v \in B_1^m} \|\tau_{\psi v} u - u\|_{L^p(\omega)}, \quad \text{where} \quad \tau_{\psi v} u(x) = u(x + \psi(x)v).$$

For $p > 1$, it is possible to obtain an estimate for $\|\varphi_\psi * u\|_{L^p(\omega)}$ without any dependence on ψ by the theory of the Hardy–Littlewood maximal function [50]; this approach fails for $p = 1$.

In the context of the proposition above, one can prove in a standard way the following statement: given $u \in L^p(\Omega; \mathbb{R}^\nu)$, $0 \leq \beta < 1$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any nonnegative function $\psi \in C^\infty(\Omega)$ satisfying $\|\psi\|_{L^\infty(\Omega)} \leq \delta$ and $\|D\psi\|_{L^\infty(\Omega)} \leq \beta$, and for every open set $\omega \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \psi(x)\}$,

$$\sup_{v \in B_1^m} \|\tau_{\psi v} u - u\|_{L^p(\omega)} \leq \varepsilon.$$

We may pursue these estimates for maps in $W^{k,p}(\Omega; \mathbb{R}^\nu)$:

Proposition 3.2. *Let $\varphi \in C_c^\infty(B_1^m)$ be a mollifier and let $\psi \in C^\infty(\Omega)$ be a nonnegative function such that $\|D\psi\|_{L^\infty(\Omega)} < 1$. For every $k \in \mathbb{N}_*$, every $u \in W^{k,p}(\Omega; \mathbb{R}^\nu)$ and every open set $\omega \subset \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geq \psi(x)\}$, we have $\varphi_\psi * u \in W^{k,p}(\omega; \mathbb{R}^\nu)$ and for every $j \in \{1, \ldots, k\}$,*

$$\eta^j \|D^j(\varphi_\psi * u)\|_{L^p(\omega)} \leq \frac{C}{(1 - \|D\psi\|_{L^\infty(\omega)})^{1/p}} \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(\Omega)},$$

and

$$\eta^j \|D^j(\varphi_\psi * u) - D^j u\|_{L^p(\omega)}$$
$$\leq \sup_{v \in B_1^m} \eta^j \|\tau_{\psi v}(D^j u) - D^j u\|_{L^p(\omega)} + \frac{C'}{(1 - \|D\psi\|_{L^\infty(\omega)})^{1/p}} \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(A)},$$

for some constants $C > 0$ and $C' > 0$ depending on m, k and p, where

$$A = \bigcup_{x \in \omega \cap \text{supp } D\psi} B^m_{\psi(x)}(x)$$

and $\eta > 0$ is such that for every $j \in \{2, \dots, k\}$,

$$\eta^j \| D^j \psi \|_{L^\infty(\omega)} \le \eta.$$

Proof. We only prove the second estimate. We assume for simplicity that $u \in C^\infty(\Omega; \mathbb{R}^\nu)$. For every $x \in \omega$,

$$(\varphi_\psi * u)(x) - u(x) = \int_{B^m_1} \varphi(z)[u(x + \psi(x)z) - u(x)] \, dz.$$

For every $j \in \{1, \dots, k\}$, by the chain rule for higher order derivatives we have

$$
\begin{aligned}
&|D^j(\varphi_\psi * u)(x) - D^j u(x)| \\
&\le \int_{B^m_1} \varphi(z) \big| D^j u(x + \psi(x)z) \circ (\text{Id} + D\psi(x) \otimes z)^j - D^j u(x) \big| \, dz \\
&\quad + C_1 \sum_{i=1}^{j-1} \sum_{\substack{\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j \\ \alpha_1 + \alpha_2 + \cdots + \alpha_j = i}} (1 + |D\psi(x)|)^{\alpha_1} |D^2\psi(x)|^{\alpha_2} \cdots |D^j\psi(x)|^{\alpha_j} \\
&\hspace{6cm} \times \int_{B^m_1} \varphi(z) |D^i u(x + \psi(x)z)| \, dz.
\end{aligned}
$$

Since $\| D\psi \|_{L^\infty(\Omega)} \le 1$, for every $z \in B^m_1$,

$$|(\text{Id} + D\psi(x) \otimes z)^j - \text{Id}| \le C_2 |D\psi(x)|,$$

and we have

$$
\begin{aligned}
&|D^j(\varphi_\psi * u)(x) - D^j u(x)| \\
&\le \int_{B^m_1} \varphi(z) |D^j u(x + \psi(x)z) - D^j u(x)| \, dz + C_2 |D\psi(x)| \int_{B^m_1} \varphi(z) |D^j u(x + \psi(x)z)| \, dz \\
&\quad + C_1 \sum_{i=1}^{j-1} \sum_{\substack{\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j = j \\ \alpha_1 + \alpha_2 + \cdots + \alpha_j = i}} (1 + |D\psi(x)|)^{\alpha_1} |D^2\psi(x)|^{\alpha_2} \cdots |D^j\psi(x)|^{\alpha_j} \\
&\hspace{6cm} \times \int_{B^m_1} \varphi(z) |D^i u(x + \psi(x)z)| \, dz.
\end{aligned}
$$

The second and the third terms on the right hand side are supported on supp $D\psi$ since $\alpha_s \ne 0$ for some $s > 1$. Moreover, by the choice of η,

$$
\begin{aligned}
(1 + |D\psi(x)|)^{\alpha_1} |D^2\psi(x)|^{\alpha_2} \cdots |D^j\psi(x)|^{\alpha_j} &\le (1 + 1)^{\alpha_1} \left(\frac{\eta}{\eta^2}\right)^{\alpha_2} \cdots \left(\frac{\eta}{\eta^j}\right)^{\alpha_j} \\
&= 2^{\alpha_1} \frac{\eta^{\alpha_1 + \alpha_2 + \cdots + \alpha_j}}{\eta^{\alpha_1 + 2\alpha_2 + \cdots + j\alpha_j}} = 2^{\alpha_1} \frac{\eta^i}{\eta^j} \le 2^j \frac{\eta^i}{\eta^j}.
\end{aligned}
$$

Therefore,

$$|D^j(\varphi_\psi * u)(x) - D^j u(x)| \le \int_{B_1^m} \varphi(z)|D^j u(x + \psi(x)z) - D^j u(x)| \, dz$$

$$+ C_3 \sum_{i=1}^j \frac{\eta^i}{\eta^j} \chi_{\text{supp}\, D\psi}(x) \int_{B_1^m} \varphi(z)|D^i u(x + \psi(x)z)| \, dz.$$

By the Minkowski inequality,

$$\left(\int_\omega \left(\int_{B_1^m} \varphi(z)|D^j u(x + \psi(x)z) - D^j u(x)| \, dz \right)^p dx \right)^{1/p}$$

$$\le \int_{B_1^m} \left(\int_\omega |D^j u(x + \psi(x)z) - D^j u(x)|^p \, dx \right)^{1/p} \varphi(z) \, dz$$

$$\le \sup_{v \in B_1^m} \|\tau_{\psi v}(D^j u) - D^j u\|_{L^p(\omega)} \int_{B_1^m} \varphi(z) \, dz = \sup_{v \in B_1^m} \|\tau_{\psi v}(D^j u) - D^j u\|_{L^p(\omega)},$$

and for every $i \in \{1, \ldots, j\}$ we also have

$$\left(\int_{\omega \cap \text{supp}\, D\psi} \left(\int_{B_1^m} \varphi(z)|D^i u(x + \psi(x)z)| \, dz \right)^p dx \right)^{1/p}$$

$$\le \int_{B_1^m} \varphi(z) \left(\int_{\omega \cap \text{supp}\, D\psi} |D^i u(x + \psi(x)z)|^p \, dx \right)^{1/p} dz.$$

Using the change of variable $y = x + \psi(x)z$ with respect to the variable x, we deduce by definition of A that

$$\left(\int_{\omega \cap \text{supp}\, D\psi} \left(\int_{B_1^m} \varphi(z)|D^i u(x + \psi(x)z)| \, dz \right)^p dx \right)^{1/p}$$

$$\le \int_{B_1^m} \varphi(z) \left(\frac{1}{1 - \|D\psi\|_{L^\infty(\omega)}} \int_A |D^i u(y)|^p \, dy \right)^{1/p} dz$$

$$= \frac{1}{(1 - \|D\psi\|_{L^\infty(\omega)})^{1/p}} \|D^i u\|_{L^p(A)}.$$

This gives the desired estimate for $u \in C^\infty(\Omega; \mathbb{R}^\nu)$. The case of functions in $W^{k,p}(\Omega; \mathbb{R}^\nu)$ follows by density. $\qquad\square$

4. Thickening

Given a map $u \in W^{k,p}(U^m; \mathbb{R}^\nu)$ which behaves nicely near the skeleton U^ℓ, we would like to construct a map $u \circ \Phi$ that does not depend on the values of u away from U^ℓ. The price to pay is that the map $u \circ \Phi$ will be singular on the dual skeleton T^{ℓ^*}; these singularities will however be mild enough to allow $u \circ \Phi$ to be in $R_{\ell^*}(U^m; \mathbb{R}^\nu)$ and to satisfy

$W^{k,p}$ estimates for $kp < \ell + 1$. The thickening construction is related to homogenization of functions on cubes that are used in the study of density problems for $k = 1$ [3, 4, 28].

The precise meaning of dual skeleton we use is the following:

Definition 4.1. Given $\ell \in \{0, \ldots, m - 1\}$ and the ℓ-dimensional skeleton $\mathcal{S}^\ell$ of a cubication $\mathcal{S}^m$, the *dual skeleton* $\mathcal{T}^{\ell^*}$ of $\mathcal{S}^\ell$ is the skeleton of dimension $\ell^* = m - \ell - 1$ composed of all cubes of the form $\sigma^{\ell^*} + x - a$, where $\sigma^{\ell^*} \in \mathcal{S}^{\ell^*}$, a is the center and x the vertex of a cube of $\mathcal{S}^m$.

The integer ℓ^* gives the greatest dimension such that $S^\ell \cap T^{\ell^*} = \emptyset$.

The proposition below provides the main properties of the map Φ:

Proposition 4.1. *Let $\ell \in \{0, \ldots, m - 1\}$, $\eta > 0$, $0 < \rho < 1$, $\mathcal{S}^m$ be a cubication of $\mathbb{R}^m$ of radius η, $\mathcal{U}^m$ be a subskeleton of $\mathcal{S}^m$ and $\mathcal{T}^{\ell^*}$ be the dual skeleton of $\mathcal{U}^\ell$. There exists a smooth map $\Phi : \mathbb{R}^m \setminus T^{\ell^*} \to \mathbb{R}^m$ such that*

(i) Φ *is injective,*
(ii) $\Phi(\sigma^m \setminus T^{\ell^*}) \subset \sigma^m \setminus T^{\ell^*}$ *for every $\sigma^m \in \mathcal{S}^m$,*
(iii) *Supp $\Phi \subset U^m + Q^m_{\rho\eta}$ and $\Phi(U^m \setminus T^{\ell^*}) \subset U^\ell + Q^m_{\rho\eta}$,*
(iv) *for every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m \setminus T^{\ell^*}$,*

$$|D^j \Phi(x)| \leq \frac{C\eta}{(\text{dist}(x, T^{\ell^*}))^j},$$

for some constant $C > 0$ depending on j, m and ρ,
(v) *for every $0 < \beta < \ell + 1$, every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m \setminus T^{\ell^*}$,*

$$\eta^{j-1}|D^j \Phi(x)| \leq C'(\text{jac } \Phi(x))^{j/\beta},$$

for some constant $C' > 0$ depending on β, j, m and ρ.

This proposition gives $W^{k,p}$ bounds on $u \circ \Phi$ for every $W^{k,p}$ function u. The proposition and the corollary below will be applied in the proof of Theorem 2 with $\ell = \lfloor kp \rfloor$.

Corollary 4.2. *Let $\Phi : \mathbb{R}^m \setminus T^{\ell^*} \to \mathbb{R}^m$ be the map given by Proposition 4.1. If $\ell + 1 > kp$, then for every $u \in W^{k,p}(U^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$, $u \circ \Phi \in W^{k,p}(U^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$, and for every $j \in \{1, \ldots, k\}$,*

$$\eta^j \|D^j(u \circ \Phi)\|_{L^p(U^m + Q^m_{\rho\eta})} \leq C'' \sum_{i=1}^j \eta^i \|D^i u\|_{L^p(U^m + Q^m_{\rho\eta})},$$

for some constant $C'' > 0$ depending on m, k, p and ρ.

Proof. We first establish the estimate for a map u in $C^\infty(U^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$. By the chain rule for higher order derivatives, for every $j \in \{1, \ldots, k\}$ and every $x \in U^m \setminus T^{\ell^*}$,

$$|D^j(u \circ \Phi)(x)|^p \leq C_1 \sum_{i=1}^j \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} |D^i u(\Phi(x))|^p |D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p.$$

Let $0 < \beta < \ell + 1$. If $1 \le t_1 \le \cdots \le t_i$ and $t_1 + \cdots + t_i = j$, then by property (v) of Proposition 4.1,

$$|D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p \le C_2 \frac{(\text{jac }\Phi(x))^{t_1 p/\beta}}{\eta^{(t_1-1)p}} \cdots \frac{(\text{jac }\Phi(x))^{t_i p/\beta}}{\eta^{(t_i-1)p}}$$

$$= C_2 \frac{(\text{jac }\Phi(x))^{jp/\beta}}{\eta^{(j-i)p}}.$$

Since $kp < \ell + 1$, we may take $\beta = jp$. Thus,

$$|D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p \le C_2 \frac{\text{jac }\Phi(x)}{\eta^{(j-i)p}},$$

and this implies

$$\eta^{jp}|D^j(u \circ \Phi)(x)|^p \le C_3 \sum_{i=1}^{j} \eta^{ip}|D^i u(\Phi(x))|^p \text{ jac }\Phi(x).$$

Since Φ is injective and $\text{Supp }\Phi \subset U^m + Q^m_{\rho\eta}$, we have $\Phi((U^m + Q^m_{\rho\eta}) \setminus T^{\ell *}) \subset U^m + Q^m_{\rho\eta}$. Thus, by the change of variable formula,

$$\int_{(U^m+Q^m_{\rho\eta})\setminus T^{\ell *}} \eta^{jp}|D^j(u \circ \Phi)|^p \le C_3 \sum_{i=1}^{j} \int_{(U^m+Q^m_{\rho\eta})\setminus T^{\ell *}} \eta^{ip}|(D^i u) \circ \Phi|^p \text{ jac }\Phi$$

$$\le C_3 \sum_{i=1}^{j} \int_{U^m+Q^m_{\rho\eta}} \eta^{ip}|D^i u|^p$$

and $u \circ \Phi \in W^{k,p}((U^m + Q^m_{\rho\eta}) \setminus T^{\ell *}; \mathbb{R}^\nu)$. Since $\ell > 0$, the dimension of the skeleton $T^{\ell *}$ is strictly less than $m - 1$. Thus, $u \circ \Phi \in W^{k,p}(U^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$. By density of smooth maps in $W^{k,p}(U^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$, we deduce that for every $u \in W^{k,p}(U^m + Q^m_{\rho\eta}; \mathbb{R}^\nu)$, the function $u \circ \Phi$ also belongs to this space and satisfies the estimate above. $\square$

We describe the construction of the map Φ given by Proposition 4.1 in the case of only one ℓ-dimensional cube:

Proposition 4.3. *Let $\ell \in \{1, \ldots, m\}$, $\eta > 0$, $0 < \underline{\rho} < \rho < \overline{\rho} < 1$ and $T = \{0^\ell\} \times Q^{m-\ell}_{\rho\eta}$. There exists a smooth function $\lambda : \mathbb{R}^m \setminus T \to [1, \infty)$ such that if $\Phi : \mathbb{R}^m \setminus T \to \mathbb{R}^m$ is defined for $x = (x', x'') \in (\mathbb{R}^\ell \times \mathbb{R}^{m-\ell}) \setminus T$ by $\Phi(x) = (\lambda(x)x', x'')$, then*

(i) *Φ is injective,*
(ii) *$\text{Supp }\Phi \subset Q^\ell_{(1-\rho)\eta} \times Q^{m-\ell}_{\rho\eta}$,*
(iii) *$\Phi((Q^\ell_{(1-\rho)\eta} \times Q^{m-\ell}_{\underline{\rho}\eta}) \setminus T) \subset (Q^\ell_{(1-\rho)\eta} \setminus Q^\ell_{(1-\overline{\rho})\eta}) \times Q^{m-\ell}_{\rho\eta}$,*
(iv) *for every $j \in \mathbb{N}_*$ and every $x = (x', x'') \in (Q^\ell_{(1-\rho)\eta} \times Q^{m-\ell}_{\rho\eta}) \setminus T$,*

$$|D^j\Phi(x)| \le C\eta/|x'|^j,$$

for some constant $C > 0$ depending on j, m, $\underline{\rho}$, ρ and $\overline{\rho}$,

(v) *for every $0 < \beta < \ell$, every $j \in \mathbb{N}_*$ and every $x \in (Q^{\ell}_{(1-\rho)\eta} \times Q^{m-\ell}_{\rho\eta}) \setminus T$,*

$$\eta^{j-1}|D^j \Phi(x)| \leq C'(\operatorname{jac} \Phi(x))^{j/\beta},$$

for some constant $C' > 0$ depending on β, j, m, $\underline{\rho}$, ρ and $\overline{\rho}$.

Granting Proposition 4.3, we prove Proposition 4.1.

Proof of Proposition 4.1. We first introduce finite sequences $(\rho_i)_{\ell \leq i \leq m}$ and $(\tau_i)_{\ell \leq i \leq m}$ such that

$$0 < \rho_m < \tau_{m-1} < \rho_{m-1} < \cdots < \rho_{\ell+1} < \tau_\ell < \rho_\ell = \rho.$$

For $i = m$, we take $\Phi_m = \operatorname{Id}$. Using downward induction, we shall define for every $i \in \{\ell, \dots, m-1\}$ smooth maps $\Phi_i : \mathbb{R}^m \setminus T^{i^*} \to \mathbb{R}^m$ such that

(a) Φ_i is injective,
(b) $\Phi_i(\sigma^m \setminus T^r) \subset \sigma^m \setminus T^r$ for all $\sigma^m \in \mathcal{S}^m$ and $r \in \{i^*, \dots, m-1\}$,
(c) $\operatorname{Supp} \Phi_i \subset U^m + Q^m_{\rho_i \eta}$,
(d) $\Phi_i(U^m \setminus T^{i^*}) \subset U^i + Q^m_{\rho_i \eta}$,
(e) for every $x \in \mathbb{R}^m \setminus T^{i^*}$ and every $r \in \{i^*, \dots, m-2\}$,

$$\operatorname{dist}(\Phi_i(x), T^r)\operatorname{dist}(x, T^{r+1}) = \operatorname{dist}(\Phi_i(x), T^{r+1})\operatorname{dist}(x, T^r),$$

(f) for every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m \setminus T^{i^*}$,

$$|D^j \Phi_i(x)| \leq \frac{C\eta}{(\operatorname{dist}(x, T^{i^*}))^j},$$

for some constant $C > 0$ depending on j, m and ρ,
(g) for every $0 < \beta < i + 1$, every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m \setminus T^{i^*}$,

$$\eta^{j-1}|D^j \Phi_i(x)| \leq C'(\operatorname{jac} \Phi_i(x))^{j/\beta},$$

for some constant $C' > 0$ depending on β, j, m and ρ.

The map Φ_ℓ will satisfy the conclusion of the proposition.

Let $i \in \{\ell+1, \dots, m\}$ and let Θ_i be the map obtained in Proposition 4.3 with parameters $\underline{\rho} = \rho_i$, $\rho = \tau_{i-1}$, $\overline{\rho} = \rho_{i-1}$ and $\ell = i$. Given $\sigma^i \in \mathcal{U}^i$, we may identify σ^i with $Q^i_\eta \times \{0^{m-i}\}$ and $T^{(i-1)^*} \cap (\sigma^i + Q^m_{\tau_{i-1}\eta})$ with $\{0^i\} \times Q^{m-i}_{\tau_{i-1}\eta}$. The map Θ_i induces by isometry a map which we shall denote by Θ_{σ^i}.

Let $\Psi_i : \mathbb{R}^m \setminus T^{(i-1)^*} \to \mathbb{R}^m$ be defined for every $x \in \mathbb{R}^m \setminus T^{(i-1)^*}$ by

$$\Psi_i(x) := \begin{cases} \Theta_{\sigma^i}(x) & \text{if } x \in \sigma^i + Q^m_{\tau_{i-1}\eta} \text{ for some } \sigma^i \in \mathcal{U}^i, \\ x & \text{otherwise.} \end{cases}$$

We first explain why Ψ_i is well-defined. Since Θ_{σ^i} coincides with the identity map on $\partial\sigma^i + Q^m_{\tau_{i-1}\eta}$, for every $\sigma^i_1, \sigma^i_2 \in \mathcal{U}^i$, if $x \in (\sigma^i_1 + Q^m_{\tau_{i-1}\eta}) \cap (\sigma^i_2 + Q^m_{\tau_{i-1}\eta})$ and $\sigma^i_1 \neq \sigma^i_2$, then

$$\Theta_{\sigma^i_1}(x) = x = \Theta_{\sigma^i_2}(x).$$

One also verifies directly that Ψ_i is smooth on $\mathbb{R}^m \setminus T^{(i-1)^*}$.

Assuming that Φ_i has been defined satisfying properties (a)–(g), we let $\Phi_{i-1} = \Psi_i \circ \Phi_i$. The map Φ_{i-1} is well-defined on $\mathbb{R}^m \setminus T^{(i-1)^*}$ since $\Phi_i(\mathbb{R}^m \setminus T^{(i-1)^*}) \subset \mathbb{R}^m \setminus T^{(i-1)^*}$.

We now check that Φ_{i-1} satisfies all required properties.

Proof of (a). The map Φ_{i-1} is injective since Ψ_i and Φ_i are injective. □

Proof of (b). For every $r \in \{(i-1)^*, \ldots, m-1\}$ and every $\sigma^m \in \mathcal{S}^m$, by induction hypothesis we have $\Phi_i(\sigma^m \setminus T^r) \subset \sigma^m \setminus T^r$. Moreover, for any $\sigma^m \in \mathcal{S}^m$ and $\tilde{\sigma}^i \in \mathcal{U}^i$, the formula for Θ_i implies that $\Theta_{\tilde{\sigma}^i}(\sigma^m \setminus T^r) \subset \sigma^m \setminus T^r$. □

Proof of (c). By induction hypothesis Φ_i coincides with the identity map outside $U^m + Q^m_{\rho_i \eta}$. By construction, Ψ_i coincides with the identity map outside $U^m + Q^m_{\tau_{i-1}\eta}$ (see Proposition 4.3(ii)). Since $\rho_i < \tau_{i-1} < \rho_{i-1}$, we deduce $\operatorname{Supp}\Phi_{i-1} \subset U^m + Q^m_{\rho_{i-1}\eta}$. □

Proof of (d). By induction hypothesis (property (d))

$$\Phi_i(U^m \setminus T^{i^*}) \subset U^i + Q^m_{\rho_i \eta}$$

and (property (b))

$$\Phi_i(\mathbb{R}^m \setminus T^{(i-1)^*}) \subset \mathbb{R}^m \setminus T^{(i-1)^*}.$$

Since $T^{(i-1)^*} \supset T^{i^*}$, we have

$$\Phi_i(U^m \setminus T^{(i-1)^*}) \subset (U^i + Q^m_{\rho_i \eta}) \setminus T^{(i-1)^*}.$$

By construction of Θ_i (see Proposition 4.3(iii)), for every $\sigma^i \in \mathcal{U}^i$,

$$\Theta_{\sigma^i}((\sigma^i + Q^m_{\rho_i \eta}) \setminus T^{(i-1)^*}) \subset \partial\sigma^i + Q^m_{\rho_{i-1}\eta}.$$

Taking the union over all faces $\sigma^i \in \mathcal{U}^i$, we get

$$\Psi_i((U^i + Q^m_{\rho_i \eta}) \setminus T^{(i-1)^*}) \subset U^{i-1} + Q^m_{\rho_{i-1}\eta}.$$

Combining the information for Φ_i and Ψ_i, we obtain

$$\Phi_{i-1}(U^m \setminus T^{(i-1)^*}) \subset U^{i-1} + Q^m_{\rho_{i-1}\eta}. \qquad \square$$

Proof of (e). Let $r \in \{(i-1)^*, \ldots, m-2\}$ and $x \in \mathbb{R}^m \setminus T^{(i-1)^*}$. If $\Phi_{i-1}(x) = \Phi_i(x)$, then the conclusion follows by induction. If $\Phi_{i-1}(x) \neq \Phi_i(x)$, then there exists $\sigma^i \in \mathcal{U}^i$ such that $\Phi_i(x) \in \sigma^i + Q^m_{\tau_{i-1}\eta}$ and $\Phi_{i-1}(x) = \Theta_{\sigma^i}(\Phi_i(x))$. Since $\Phi_i(x) \in \operatorname{Supp}\Psi_i$,

$$\Phi_i(x) \in (\sigma^i + Q^m_{\tau_{i-1}\eta}) \setminus (\partial\sigma^i + Q^m_{\tau_{i-1}\eta}).$$

Up to an isometry, we may assume that $\sigma^i = Q^i_\eta \times \{0^{m-i}\}$. For every $0 < \lambda < 1$ and every $y = (y', y'') \in Q^i_{(1-\lambda)\eta} \times Q^{m-i}_{\lambda\eta}$,

$$\operatorname{dist}(y, T^r) = \operatorname{dist}\big((y', 0), T^r \cap (Q^i_{(1-\lambda)\eta} \times \{0^{m-i}\})\big).$$

In view of the formula for Θ_i, we deduce that for every $y \in (\sigma^i + Q^m_{\tau_{i-1}\eta}) \setminus (\partial\sigma^i \times Q^m_{\tau_{i-1}\eta})$,

$$\operatorname{dist}(\Theta_{\sigma^i}(y), T^r)\operatorname{dist}(y, T^{r+1}) = \operatorname{dist}(\Theta_{\sigma^i}(y), T^{r+1})\operatorname{dist}(y, T^r);$$

this identity is reminiscent of Thales' intercept theorem from Euclidean geometry. By induction hypothesis, we then get

$$\begin{aligned}
\operatorname{dist}(\Phi_{i-1}(x), T^r)\operatorname{dist}(x, T^{r+1}) &= \operatorname{dist}(\Theta_{\sigma^i}(\Phi_i(x)), T^r)\operatorname{dist}(x, T^{r+1}) \\
&= \operatorname{dist}(\Theta_{\sigma^i}(\Phi_i(x)), T^{r+1})\operatorname{dist}(x, T^r) \\
&= \operatorname{dist}(\Phi_{i-1}(x)), T^{r+1})\operatorname{dist}(x, T^r). \qquad \square
\end{aligned}$$

Proof of (f). Let $x \in \mathbb{R}^m \setminus T^{(i-1)^*}$. If Ψ_i coincides with the identity map in a neighborhood of $\Phi_i(x)$, then $D^j\Phi_{i-1}(x) = D^j\Phi_i(x)$ and the conclusion follows from the induction hypothesis and the fact that $T^{(i-1)^*} \supset T^{i^*}$.

If Ψ_i does not coincide with the identity map in a neighborhood of $\Phi_i(x)$, then there exists $\sigma^i \in \mathcal{U}^i$ such that

$$\Phi_i(x) \in (\sigma^i + Q^m_{\tau_{i-1}\eta}) \setminus (\partial\sigma^i + Q^m_{\tau_{i-1}\eta})$$

and $\Phi_{i-1}(x) = \Theta_{\sigma_i}(\Phi_i(x))$. By the chain rule for higher order derivatives,

$$|D^j\Phi_{i-1}(x)| \le C_1 \sum_{r=1}^{j} \sum_{\substack{1 \le t_1 \le \cdots \le t_r \\ t_1 + \cdots + t_r = j}} |D^r\Theta_{\sigma_i}(\Phi_i(x))|\,|D^{t_1}\Phi_i(x)|\cdots|D^{t_r}\Phi_i(x)|.$$

By construction of Θ_i (see Proposition 4.3(iv)), for any $y = (y', y'') \in (Q^i_{(1-\tau_{i-1})\eta} \times Q^{m-i}_{\tau_{i-1}\eta}) \setminus (\{0^i\} \times Q^{m-i}_{\tau_{i-1}\eta})$ we have

$$|D^r\Theta_i(y)| \le C_2\eta/|y'|^r.$$

This implies

$$|D^r\Theta_{\sigma^i}(\Phi_i(x))| \le \frac{C_2\eta}{(\operatorname{dist}(\Phi_i(x), T^{(i-1)^*}))^r}.$$

By the induction hypothesis, for every $1 \le t_1 \le \cdots \le t_r$ such that $t_1 + \cdots + t_r = j$,

$$|D^{t_1}\Phi_i(x)|\cdots|D^{t_r}\Phi_i(x)| \le C_3 \frac{\eta}{(\operatorname{dist}(x, T^{i^*}))^{t_1}} \cdots \frac{\eta}{(\operatorname{dist}(x, T^{i^*}))^{t_r}} = C_3 \frac{\eta^r}{(\operatorname{dist}(x, T^{i^*}))^j}.$$

Thus,

$$|D^j\Phi_{i-1}(x)| \le C_4 \sum_{r=1}^{j} \frac{\eta^{r+1}}{(\operatorname{dist}(\Phi_i(x), T^{(i-1)^*}))^r(\operatorname{dist}(x, T^{i^*}))^j}.$$

We recall that by property (f),

$$\operatorname{dist}(\Phi_i(x), T^{(i-1)^*})\operatorname{dist}(x, T^{i^*}) = \operatorname{dist}(x, T^{(i-1)^*})\operatorname{dist}(\Phi_i(x), T^{i^*}).$$

Since $\Phi_i(x) \in (\sigma^i + Q^m_{\tau_{i-1}\eta}) \setminus (\partial\sigma^i + Q^m_{\tau_{i-1}\eta})$,

$$\text{dist}(\Phi_i(x), T^{i^*}) \geq (1 - \tau_{i-1})\eta \geq (1 - \rho)\eta.$$

Thus,

$$(\text{dist}(\Phi_i(x), T^{(i-1)^*}))^r (\text{dist}(x, T^{i^*}))^j$$
$$= (\text{dist}(x, T^{(i-1)^*}) \, \text{dist}(\Phi_i(x), T^{i^*}))^r (\text{dist}(x, T^{i^*}))^{j-r}$$
$$\geq (\text{dist}(x, T^{(i-1)^*}))^r ((1 - \rho)\eta)^r (\text{dist}(x, T^{i^*}))^{j-r}.$$

Since $T^{i^*} \subset T^{(i-1)^*}$, we conclude that

$$|D^j \Phi_{i-1}(x)| \leq C_5 \frac{\eta}{(\text{dist}(x, T^{(i-1)^*}))^j}. \qquad \Box$$

Proof of (g). Let $j \in \mathbb{N}_*$ and let $x \in \mathbb{R}^m \setminus T^{(i-1)^*}$. If Ψ_i coincides with the identity map in a in a neighborhood of $\Phi_i(x)$, then $D^j \Phi_{i-1}(x) = D^j \Phi_i(x)$ and jac $\Phi_{i-1}(x) =$ jac $\Phi_i(x)$. The conclusion then follows from the induction hypothesis.

Assume that Ψ_i does not coincide with the identity map in a neighborhood of $\Phi_i(x)$. Let $0 < \beta < i$ and $r \in \{0, \ldots, j\}$. By induction hypothesis, if $1 \leq t_1 \leq \cdots \leq t_r$ and $t_1 + \cdots + t_r = j$, then

$$|D^{t_1} \Phi_i(x)| \cdots |D^{t_r} \Phi_i(x)| \leq C_1 \frac{(\text{jac } \Phi_i(x))^{t_1/\beta}}{\eta^{t_1-1}} \cdots \frac{(\text{jac } \Phi_i(x))^{t_r/\beta}}{\eta^{t_r-1}} = C_1 \frac{(\text{jac } \Phi_i(x))^{j/\beta}}{\eta^{j-r}}.$$

Let $\sigma^i \in \mathcal{U}^i$ be such that

$$\Phi_i(x) \in (\sigma^i + Q^m_{\tau_{i-1}\eta}) \setminus (\partial\sigma^i + Q^m_{\tau_{i-1}\eta})$$

and $\Phi_{i-1}(x) = \Theta_{\sigma_i} \circ \Phi_i(x)$. By construction of Θ_i (see Proposition 4.3(v)), for any $y \in (Q^i_{(1-\tau_{i-1})\eta} \times Q^{m-i}_{\tau_{i-1}\eta}) \setminus (\{0^i\} \times Q^{m-i}_{\tau_{i-1}\eta})$ we have

$$\eta^{r-1} |D^r \Theta_i(y)| \leq C_2 (\text{jac } \Theta_i(y))^{r/\beta \frac{r}{j}} = C_2 (\text{jac } \Theta_i(y))^{j/\beta}.$$

Thus,

$$|D^r \Theta_{\sigma^i}(\Phi_i(x))| \, |D^{t_1} \Phi_i(x)| \cdots |D^{t_r} \Phi_i(x)| \leq C_3 \frac{(\text{jac } \Theta_{\sigma^i}(\Phi_i(x)))^{j/\beta}}{\eta^{r-1}} \frac{(\text{jac } \Phi_i(x))^{j/\beta}}{\eta^{j-r}}$$
$$= \frac{C_3}{\eta^{j-1}} (\text{jac } \Phi_{i-1}(x))^{j/\beta}.$$

Therefore, by the chain rule for higher order derivatives,

$$|D^j \Phi_{i-1}(x)| \leq \frac{C_4}{\eta^{j-1}} (\text{jac } \Phi_{i-1}(x))^{j/\beta}. \qquad \Box$$

By downward induction, we conclude that properties (a)–(g) hold for every $i \in \{\ell, \ldots, m\}$. In particular, Φ_ℓ satisfies properties (i)–(v) of Proposition 4.1. $\qquad \Box$

We establish a couple of lemmas in order to prove Proposition 4.3:

Lemma 4.4. *Let $\ell \in \{1, \ldots, m\}$, $\eta > 0$, $0 < \underline{\rho} < \rho < \overline{\rho} < 1$ and $0 < \kappa < 1 - \overline{\rho}$. There exists a smooth function $\lambda : \mathbb{R}^m \to [1, \infty)$ such that if $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is defined for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by $\Phi(x) = (\lambda(x)x', x'')$, then*

(i) *Φ is a diffeomorphism,*
(ii) *$\operatorname{Supp} \Phi \subset Q^\ell_{(1-\rho)\eta} \times Q^{m-\ell}_{\rho\eta}$,*
(iii) *$\Phi((Q^\ell_\eta \setminus Q^\ell_{\kappa\eta}) \times Q^{m-\ell}_{\underline{\rho}\eta}) \subset (Q^\ell_\eta \setminus Q^\ell_{(1-\overline{\rho})\eta}) \times Q^{m-\ell}_{\underline{\rho}\eta}$,*
(iv) *for every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,*

$$\eta^{j-1}|D^j \Phi(x)| \leq C,$$

   for some constant $C > 0$ depending on j, m, $\underline{\rho}$, ρ, $\overline{\rho}$ and κ,
(v) *for every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,*

$$C' \leq \operatorname{jac} \Phi(x) \leq C'',$$

   for some constants $C', C'' > 0$ depending on m, $\underline{\rho}$, ρ, $\overline{\rho}$ and κ.

Proof. By scaling, we may assume that $\eta = 1$. Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth function such that

- ψ is nonincreasing on $\mathbb{R}_+$ and nondecreasing on $\mathbb{R}_-$,
- $\psi(t) = 1$ for $|t| \leq 1 - \overline{\rho}$,
- $\psi(t) = 0$ for $|t| \geq 1 - \rho$.

Let $\theta : \mathbb{R} \to [0, 1]$ be a smooth function such that

- $\theta(t) = 1$ for $|t| \leq \underline{\rho}$,
- $\theta(t) = 0$ for $|t| \geq \rho$.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be the function defined for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ by

$$\varphi(x) = \prod_{i=1}^\ell \psi(x_i) \prod_{i=\ell+1}^m \theta(x_i).$$

Thus,

- $\varphi(x) = 0$ for every $x \in \mathbb{R}^m \setminus (Q^\ell_{1-\rho} \times Q^{m-\ell}_\rho)$,
- $\varphi(x) = 1$ for every $x \in Q^\ell_{1-\overline{\rho}} \times Q^{m-\ell}_{\underline{\rho}}$.

We shall define the map Φ in terms of its inverse Ψ: let $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ be the function defined for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by

$$\Psi(x) = \big((1 - \alpha\varphi(x))x', x''\big),$$

where $\alpha \in \mathbb{R}$. In particular,

- $\Psi(x) = x$ for every $x \in \mathbb{R}^m \setminus (Q^\ell_{1-\rho} \times Q^{m-\ell}_\rho)$,
- $\Psi(x) = ((1-\alpha)x', x'')$ for every $x = (x', x'') \in Q^\ell_{1-\overline{\rho}} \times Q^{m-\ell}_{\underline{\rho}}$.

In view of this second property, taking $\alpha = 1 - \kappa/(1 - \overline{\rho})$, we deduce that Ψ is a bijection between $Q^\ell_{1-\overline{\rho}} \times Q^{m-\ell}_{\underline{\rho}}$ and $Q^\ell_\kappa \times Q^{m-\ell}_{\underline{\rho}}$.

We now prove that Ψ is injective. If $x, y \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ satisfy $\Psi(x) = \Psi(y)$, then $y'' = x''$ and $y' = t x'$ for some $t > 0$. Since $\alpha \in (0, 1)$, the function

$$g : [0, \infty) \ni s \mapsto s(1 - \alpha\varphi(sx', x''))$$

is the product of an increasing function with a nondecreasing positive function. Thus, g is increasing, whence Ψ is injective. Since $g(0) = 0$ and $\lim_{t\to\infty} g(t) = \infty$, by the intermediate value theorem, $g([0, \infty)) = [0, \infty)$. Thus, Ψ is surjective. Therefore, the map Ψ is a bijection.

We claim that for every $x \in \mathbb{R}^m$, $D\Psi(x)$ is invertible. Indeed, for every $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ and for every $v = (v', v'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$,

$$D\Psi(x)[v] = \big((1 - \alpha\varphi(x))v' - \alpha D\varphi(x)[v]x', v''\big).$$

The Jacobian of Ψ can be computed as the determinant of a nilpotent perturbation of a diagonal linear map to be

$$\mathrm{jac}\,\Psi(x) = (1 - \alpha\varphi(x))^{\ell-1}\big(1 - \alpha\varphi(x) - \alpha D\varphi(x)[(x', 0)]\big).$$

Since ψ is nonincreasing on $\mathbb{R}_+$ and nondecreasing on $\mathbb{R}_-$, $D\varphi(x)[(x', 0)] \leq 0$. Thus,

$$\mathrm{jac}\,\Psi(x) \geq (1 - \alpha\varphi(x))^\ell \geq (1 - \alpha)^\ell > 0.$$

The map $\Phi = \Psi^{-1}$ satisfies all the desired properties. $\square$

Lemma 4.5. *Let $\ell \in \{1, \ldots, m\}$, $\eta > 0$, $0 < \underline{\rho} < \rho < \overline{\rho} < 1$ and $T = \{0^\ell\} \times Q_{\rho\eta}^{m-\ell}$. There exists a smooth function $\lambda : \mathbb{R}^m \setminus T \to \overline{[1, \infty)}$ such that if $\Phi : \mathbb{R}^m \setminus T \to \mathbb{R}^m$ is defined for $x = (x', x'') \in (\mathbb{R}^\ell \times \mathbb{R}^{m-\ell}) \setminus T$ by $\Phi(x) = (\lambda(x)x', x'')$, then*

  (i) Φ *is injective,*
 (ii) $\mathrm{Supp}\,\Phi \subset B_{(1-\rho)\eta}^\ell \times Q_{\rho\eta}^{m-\ell}$,
(iii) $\Phi((B_{(1-\rho)\eta}^\ell \times Q_{\underline{\rho}\eta}^{m-\ell}) \setminus T) \subset (B_{(1-\rho)\eta}^\ell \setminus B_{(1-\overline{\rho})\eta}^\ell) \times Q_{\rho\eta}^{m-\ell}$,
 (iv) *for every $j \in \mathbb{N}_*$ and every $x = (x', x'') \in (B_{(1-\rho)\eta}^\ell \times Q_{\rho\eta}^{m-\ell}) \setminus T$,*

$$|D^j\Phi(x)| \leq C\eta/|x'|^j,$$

   for some constant $C > 0$ depending on j, m, $\underline{\rho}$, ρ and $\overline{\rho}$,
  (v) *for every $0 < \beta < \ell$, every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m \setminus T$,*

$$\eta^{j-1}|D^j\Phi(x)| \leq C'(\mathrm{jac}\,\Phi(x))^{j/\beta},$$

   for some constant $C' > 0$ depending on β, j, m, $\underline{\rho}$, ρ and $\overline{\rho}$.

Proof. By scaling, we may assume that $\eta = 1$. Given $b > 0$, let $\varphi : (0, \infty) \to [1, \infty)$ be a smooth function such that

- $\varphi(s) = \dfrac{1 - \overline{\rho}}{s}\left(1 + \dfrac{b}{\ln(1/s)}\right)$ for $0 < s \leq 1 - \overline{\rho}$,
- $\varphi(s) = 1$ for $s \geq 1 - \rho$,
- the function $(0, \infty) \ni s \mapsto s\varphi(s)$ is increasing.

This is possible for any $b > 0$ such that

$$(1 - \overline{\rho})\left(1 + \frac{b}{\ln \frac{1}{1-\overline{\rho}}}\right) < 1 - \rho.$$

Let $\theta : \mathbb{R}^{m-\ell} \to [0, 1]$ be a smooth function such that

- $\theta(y) = 0$ for $y \in Q_{\underline{\rho}}^{m-\ell}$,
- $\theta(y) = 1$ for $y \in \mathbb{R}^{m-\ell} \setminus Q_\rho^{m-\ell}$.

We now introduce, for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$,

$$\zeta(x) = \sqrt{|x'|^2 + \theta(x'')^2}.$$

Let $\lambda : \mathbb{R}^m \setminus T \to \mathbb{R}$ be the function defined for $x = (x', x'') \in \mathbb{R}^m \setminus T$ by $\lambda(x) = \varphi(\zeta(x))$. Since $\zeta \neq 0$ in $\mathbb{R}^m \setminus T$, the function λ is well-defined and smooth. In addition, $\lambda \geq 1$.

We now check that the map Φ defined in the statement has all the required properties.

Proof of (i). In order to check that Φ is injective, we first observe that if $x = (x', x'')$, $y = (y', y'') \in B_1^\ell \times Q_\rho^{m-\ell}$ and $\Phi(x) = \Phi(y)$, then $x'' = y''$, and there exists $t > 0$ such that $y' = tx'$. The conclusion follows from the fact that the function

$$h : [0, \infty) \ni s \mapsto s\varphi\left(\sqrt{s^2 + \theta(x'')^2}\right)$$

is increasing. □

Proof of (ii). For every $x = (x', x'') \in (\mathbb{R}^\ell \times \mathbb{R}^{m-\ell}) \setminus T$, if $x' \notin B_{1-\rho}^\ell$ or if $x'' \notin Q_\rho^{m-\ell}$, then $\zeta(x) \geq 1 - \rho$. Thus, $\lambda(x) = \varphi(\zeta(x)) = 1$ and $\Phi(x) = x$. We then have $\mathrm{Supp}\,\Phi \subset B_{1-\rho}^\ell \times Q_\rho^{m-\ell}$. □

Proof of (iii). We first observe that since the function $(0, \infty) \ni s \mapsto s\varphi(s)$ is increasing and $\lim_{s \to 0} s\varphi(s) = 1 - \overline{\rho}$, for every $s > 0$,

$$s\varphi(s) \geq 1 - \overline{\rho}.$$

Since for every $x = (x', x'') \in (B_{1-\rho}^\ell \times Q_{\underline{\rho}}^{m-\ell}) \setminus T$ we have $\zeta(x) = |x'|$, we deduce that

$$|\lambda(x)x'| = \varphi(|x'|)|x'| \geq 1 - \overline{\rho}.$$

On the other hand, since the function h defined above is increasing,

$$|\lambda(x)x'| = h(|x'|) \leq h(1 - \rho) = 1 - \rho.$$

We conclude that $\lambda(x)x' \in B_{1-\rho}^\ell \setminus B_{1-\overline{\rho}}^\ell$. □

Proof of (iv). By the chain rule,

$$|D^j \lambda(x)| \leq C_1 \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} |\varphi^{(i)}(\zeta(x))| \, |D^{t_1}\zeta(x)| \cdots |D^{t_i}\zeta(x)|.$$

For every $i \in \mathbb{N}_*$ and every $s > 0$,

$$|\varphi^{(i)}(s)| \leq C_2/s^{i+1}$$

and for every $x \in (B_1^\ell \times \mathbb{R}^{m-\ell}) \setminus T$,

$$|D^i \zeta(x)| \le C_3/\zeta(x)^{i-1}.$$

Thus, for every $1 \le t_1 \le \cdots \le t_i$ such that $t_1 + \cdots + t_i = j$,

$$|D^{t_1}\zeta(x)| \cdots |D^{t_i}\zeta(x)| \le \frac{C_4}{\zeta(x)^{t_1-1} \cdots \zeta(x)^{t_i-1}} = \frac{C_4}{\zeta(x)^{j-i}}.$$

By the chain rule,

$$|D^j \lambda(x)| \le C_5 \sum_{i=1}^{j} \frac{1}{\zeta(x)^{i+1}} \frac{1}{\zeta(x)^{j-i}} = \frac{C_5 j}{\zeta(x)^{j+1}}.$$

Hence, by the Leibniz rule, for any $x \in (B_1^\ell \times \mathbb{R}^{m-\ell}) \setminus T$,

$$|D^j \Phi(x)| \le C_6/\zeta(x)^j. \tag{4.1}$$

Since $\zeta(x) \ge |x'|$, the conclusion follows. $\square$

Proof of (v). For every $x = (x', x'') \in (\mathbb{R}^\ell \times \mathbb{R}^{m-\ell}) \setminus T$ and $v = (v', v'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$,

$$D\Phi(x)[v] = \left(\varphi(\zeta(x))v' + \varphi^{(1)}(\zeta(x)) \frac{x' \cdot v' + \theta(x'')D\theta(x'')[v'']}{\zeta(x)} x', v'' \right).$$

The Jacobian can be computed as the determinant of a nilpotent perturbation of a diagonal linear map to be

$$\text{jac } \Phi(x) = \varphi(\zeta(x))^{\ell-1} \left(\varphi(\zeta(x)) + \varphi^{(1)}(\zeta(x)) \frac{|x'|^2}{\zeta(x)} \right)$$

$$= \varphi(\zeta(x))^{\ell-1} \left(\varphi(\zeta(x)) \left(1 - \frac{|x'|^2}{\zeta(x)^2} \right) + \left(\varphi^{(1)}(\zeta(x))\zeta(x) + \varphi(\zeta(x)) \right) \frac{|x'|^2}{\zeta(x)^2} \right).$$

Since for every $s > 0$,
$$s\varphi^{(1)}(s) + \varphi(s) = (s\varphi(s))^{(1)} \ge 0,$$

and since there exists $c_1 > 0$ such that for every $s > 0$,

$$\varphi(s) \ge c_1/s,$$

we have

$$\text{jac } \Phi(x) \ge \varphi(\zeta(x))^\ell \left(1 - \frac{|x'|^2}{\zeta(x)^2} \right) \ge \frac{c_2}{\zeta(x)^\ell} \left(1 - \frac{|x'|^2}{\zeta(x)^2} \right).$$

If $|x'| \le \theta(x'')$, then $\zeta(x) \ge \sqrt{2}|x'|$ and we get

$$\text{jac } \Phi(x) \ge c_3/\zeta(x)^\ell.$$

On the other hand, by direct inspection, for every $\alpha < 1$, there exists a constant $c_4 > 0$ depending on α such that for every $s > 0$,

$$s\varphi^{(1)}(s) + \varphi(s) \ge c_4/s^\alpha.$$

Thus,

$$\operatorname{jac} \Phi(x) \geq \varphi(\zeta(x))^{\ell-1}\big(\varphi^{(1)}(\zeta(x))\zeta(x) + \varphi(\zeta(x))\big)\frac{|x'|^2}{\zeta(x)^2} \geq \frac{c_5}{\zeta(x)^{\ell-1+\alpha}}\frac{|x'|^2}{\zeta(x)^2}.$$

If $|x'| > \theta(x'')$, then $\zeta(x) \leq \sqrt{2}|x'|$ and we get

$$\operatorname{jac} \Phi(x) \geq c_6/\zeta(x)^{\ell-1+\alpha}.$$

In both cases, we deduce that for every $\beta < \ell$ and every $x \in \mathbb{R}^m \setminus T$,

$$\operatorname{jac} \Phi(x) \geq c_7/\zeta(x)^{\beta}.$$

Thus, by estimate (4.1) in the proof of property (iv) above, when $x \in (B^{\ell}_{1-\rho} \times Q^{m-\ell}_{\rho}) \setminus T$,

$$|D^j\Phi(x)| \leq \frac{C_5}{\zeta(x)^j} \leq \frac{C_5}{(c_7)^{j/\beta}}(\operatorname{jac} \Phi(x))^{j/\beta}. \qquad \square$$

The proof of Lemma 4.5 is complete. $\qquad \square$

Proof of Proposition 4.3. Define Φ to be the composition of the map Φ_1 given by Lemma 4.4 with any parameter $\kappa \leq (1 - \overline{\rho})/\sqrt{\ell}$ together with the map Φ_2 given by Lemma 4.5; more precisely, $\Phi = \Phi_1 \circ \Phi_2$. Then Φ is injective and $\operatorname{Supp} \Phi \subset Q^{\ell}_{(1-\rho)\eta} \times Q^{m-\ell}_{\rho\eta}$. Moreover, the choice of κ implies that $Q^{\ell}_{\kappa\eta} \subset B^{\ell}_{(1-\overline{\rho})\eta}$. Hence,

$$\Phi\big((Q^{\ell}_{(1-\rho)\eta} \times Q^{m-\ell}_{\underline{\rho}\eta}) \setminus T\big) \subset (Q^{\ell}_{(1-\rho)\eta} \setminus Q^{\ell}_{(1-\overline{\rho})\eta}) \times Q^{m-l}_{\underline{\rho}\eta}.$$

By the chain rule for higher order derivatives and by the estimate of the derivatives of Φ_1 (Lemma 4.4(iv)),

$$|D^j\Phi(x)| \leq C_1 \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} |D^i\Phi_1(\Phi_2(x))|\,|D^{t_1}\Phi_2(x)| \cdots |D^{t_i}\Phi_2(x)|$$

$$\leq C_2 \sum_{i=1}^{j} \sum_{\substack{1 \leq t_1 \leq \cdots \leq t_i \\ t_1 + \cdots + t_i = j}} \frac{|D^{t_1}\Phi_2(x)| \cdots |D^{t_i}\Phi_2(x)|}{\eta^{i-1}}.$$

The estimate for $D^j\Phi$ is a consequence of the estimates of the derivatives of Φ_2 (see Lemma 4.5(iv)). The estimate for $\operatorname{jac} \Phi$ is a consequence of the estimate for $\operatorname{jac} \Phi_2$ given by Lemma 4.5(v) and the lower bound for $\operatorname{jac} \Phi_1$ given by Lemma 4.4(v). $\qquad \square$

5. Density of the class $R_{m-\lfloor kp \rfloor-1}(Q^m; N^n)$

In this section, we prove that the class $R_{m-\lfloor kp \rfloor-1}(Q^m; N^n)$ is dense in $W^{k,p}(Q^m; N^n)$ regardless of the topology of the manifold N^n.

Theorem 3. *If $kp < m$, then $R_{m-\lfloor kp \rfloor -1}(Q^m; N^n)$ is strongly dense in $W^{k,p}(Q^m; N^n)$.*

This result implies the "if" part of Theorem 2.

Proof of Theorem 3. First observe that if $u \in W^{k,p}(Q^m; N^n)$, then the restrictions to Q^m of the maps $u_\gamma \in W^{k,p}(Q^m_{1+2\gamma}; N^n)$ defined for $x \in Q^m_{1+2\gamma}$ by $u_\gamma(x) = u(x/(1+2\gamma))$ converge strongly to u in $W^{k,p}(Q^m; N^n)$ when γ tends to 0. We can thus assume from the beginning that $u \in W^{k,p}(Q^m_{1+2\gamma}; N^n)$. We apply successively the opening, smoothing and thickening constructions to this map u.

We divide the proof into four parts:

Part 1. Construction of a map $u_\eta^{\mathrm{th}} \in W^{k,p}(Q^m_{1+\gamma}; \mathbb{R}^\nu) \cap C^\infty(Q^m_{1+\gamma} \setminus T_\eta^{\ell^*}; \mathbb{R}^\nu)$ such that for every $j \in \{1, \ldots, k\}$,

$$\eta^j \|D^j u_\eta^{\mathrm{th}} - D^j u\|_{L^p(Q^m_{1+\gamma})}$$

$$\leq \sup_{v \in B_1^m} \eta^j \|\tau_{\psi_\eta v}(D^j u) - D^j u\|_{L^p(Q^m_{1+\gamma})} + C \sum_{i=1}^j \eta^i \|D^i u\|_{L^p(U_\eta^m + Q^m_{2\rho\eta})},$$

where $\mathcal{U}_\eta^m$ is a subskeleton of $Q^m_{1+\gamma}$ and $\mathcal{T}_\eta^{\ell^*}$ is the dual skeleton of $\mathcal{U}_\eta^\ell$.

Using the terminology presented in the Introduction, the subskeleton $\mathcal{U}_\eta^m$ will be chosen to be the set of all bad cubes together with the set of good cubes which intersect some bad cube. The precise choice of $\mathcal{U}_\eta^m$ will be made in Part 2.

Let $\mathcal{K}_\eta^m$ be a cubication of $Q^m_{1+\gamma}$ of radius $0 < \eta \leq \gamma$ and let $\mathcal{U}_\eta^m$ be a subskeleton of $\mathcal{K}_\eta^m$. Let $0 < \rho < 1/2$; thus,

$$2\rho\eta \leq \gamma.$$

Given $\ell \in \{0, \ldots, m-1\}$, we begin by opening the map u in a neighborhood of U_η^ℓ. More precisely, let $\Phi^{\mathrm{op}} : \mathbb{R}^m \to \mathbb{R}^m$ be the smooth map given by Proposition 2.1 and consider the map

$$u_\eta^{\mathrm{op}} = u \circ \Phi^{\mathrm{op}}.$$

In particular, $u_\eta^{\mathrm{op}} \in W^{k,p}(Q^m_{1+2\gamma}; N^n)$ and $u_\eta^{\mathrm{op}} = u$ in the complement of $U_\eta^\ell + Q^m_{2\rho\eta}$. For every $j \in \{1, \ldots, k\}$,

$$\eta^j \|D^j u_\eta^{\mathrm{op}} - D^j u\|_{L^p(Q^m_{1+2\gamma})} = \eta^j \|D^j u_\eta^{\mathrm{op}} - D^j u\|_{L^p(U_\eta^\ell + Q^m_{2\rho\eta})}$$

$$\leq \eta^j \|D^j u_\eta^{\mathrm{op}}\|_{L^p(U_\eta^\ell + Q^m_{2\rho\eta})} + \eta^j \|D^j u\|_{L^p(U_\eta^\ell + Q^m_{2\rho\eta})}$$

$$\leq C_1 \sum_{i=1}^j \eta^i \|D^i u\|_{L^p(U_\eta^\ell + Q^m_{2\rho\eta})}. \tag{5.1}$$

We next consider a smooth function $\psi_\eta \in C^\infty(Q^m_{1+2\gamma})$ such that

$$\boxed{0 < \psi_\eta \leq \rho\eta.}$$

Given a mollifier $\varphi \in C_c^\infty(B_1^m)$, let for every $x \in Q^m_{1+\gamma+\rho\eta}$,

$$u_\eta^{\mathrm{sm}}(x) = (\varphi_{\psi_\eta(x)} * u_\eta^{\mathrm{op}})(x).$$

Since $0 < \psi_\eta \leq \rho\eta$, the map $u_\eta^{\mathrm{sm}} : Q_{1+\gamma+\rho\eta}^m \to \mathbb{R}^\nu$ is well-defined and smooth. If

$$\boxed{\|D\psi_\eta\|_{L^\infty(Q_{1+2\gamma}^m)} \leq \beta}$$

for some $\beta < 1$ and if for every $i \in \{2, \dots, k\}$,

$$\boxed{\eta^i \|D^i \psi_\eta\|_{L^\infty(Q_{1+2\gamma}^m)} \leq \eta,}$$

then by Proposition 3.2 with $\omega = Q_{1+\gamma}^m$, for every $j \in \{1, \dots, k\}$ we have

$$\eta^j \|D^j u_\eta^{\mathrm{sm}} - D^j u_\eta^{\mathrm{op}}\|_{L^p(Q_{1+\gamma}^m)}$$

$$\leq \sup_{v \in B_1^m} \eta^j \|\tau_{\psi_\eta v}(D^j u_\eta^{\mathrm{op}}) - D^j u_\eta^{\mathrm{op}}\|_{L^p(Q_{1+\gamma}^m)} + C_2 \sum_{i=1}^{j} \eta^i \|D^i u_\eta^{\mathrm{op}}\|_{L^p(A)},$$

where $A = \bigcup_{x \in Q_{1+\gamma}^m \cap \operatorname{supp} D\psi_\eta} B_{\psi_\eta(x)}^m(x)$. For every $v \in B_1^m$,

$$\eta^j \|\tau_{\psi_\eta v}(D^j u_\eta^{\mathrm{op}}) - D^j u_\eta^{\mathrm{op}}\|_{L^p(Q_{1+\gamma}^m)}$$

$$\leq \eta^j \|\tau_{\psi_\eta v}(D^j u_\eta^{\mathrm{op}}) - \tau_{\psi_\eta v}(D^j u)\|_{L^p(Q_{1+\gamma}^m)}$$

$$+ \eta^j \|\tau_{\psi_\eta v}(D^j u) - D^j u\|_{L^p(Q_{1+\gamma}^m)} + \eta^j \|D^j u_\eta^{\mathrm{op}} - D^j u\|_{L^p(Q_{1+\gamma}^m)}$$

and, by the change of variable formula,

$$\|\tau_{\psi_\eta v}(D^j u_\eta^{\mathrm{op}}) - \tau_{\psi_\eta v}(D^j u)\|_{L^p(Q_{1+\gamma}^m)} \leq C_3 \|D^j u_\eta^{\mathrm{op}} - D^j u\|_{L^p(Q_{1+2\gamma}^m)}.$$

If we further assume that

$$\boxed{\operatorname{supp} D\psi_\eta \subset U_\eta^m,}$$

then since $\psi_\eta \leq \rho\eta$, we have $A \subset U_\eta^m + Q_{\rho\eta}^m$. By Proposition 2.1, we then have

$$\sum_{i=1}^{j} \eta^i \|D^i u_\eta^{\mathrm{op}}\|_{L^p(A)} \leq \sum_{i=1}^{j} \eta^i \|D^i u_\eta^{\mathrm{op}}\|_{L^p(U_\eta^m + Q_{\rho\eta}^m)} \leq C_4 \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(U_\eta^m + Q_{2\rho\eta}^m)}.$$

Thus, for every $j \in \{1, \dots, k\}$,

$$\eta^j \|D^j u_\eta^{\mathrm{sm}} - D^j u_\eta^{\mathrm{op}}\|_{L^p(Q_{1+\gamma}^m)}$$

$$\leq \sup_{v \in B_1^m} \eta^j \|\tau_{\psi_\eta v}(D^j u) - D^j u\|_{L^p(Q_{1+\gamma}^m)} + C_5 \sum_{i=1}^{j} \eta^i \|D^i u\|_{L^p(U_\eta^m + Q_{2\rho\eta}^m)}. \quad (5.2)$$

Given $0 < \underline{\rho} < \rho$, we apply thickening to the map u_η^{sm} in a neighborhood of U_η^ℓ of size $\underline{\rho}\eta$. More precisely, denote by $\Phi^{\mathrm{th}} : \mathbb{R}^m \to \mathbb{R}^m$ the smooth map given by Proposition 4.1 with the parameter $\underline{\rho}$ and let

$$u_\eta^{\mathrm{th}} = u_\eta^{\mathrm{sm}} \circ \Phi^{\mathrm{th}}.$$

Then, $u_\eta^{\mathrm{th}} = u_\eta^{\mathrm{sm}}$ in the complement of $U_\eta^m + Q_{\underline{\rho}\eta}^m$. Assume in addition that

$$\boxed{\ell + 1 > kp;}$$

then by Corollary 4.2, $u_\eta^{\mathrm{th}} \in W^{k,p}(K_\eta^m; \mathbb{R}^\nu)$ and for every $j \in \{1, \dots, k\}$,

$$
\begin{aligned}
\eta^j \|D^j u_\eta^{\mathrm{th}} - D^j u_\eta^{\mathrm{sm}}\|_{L^p(K_\eta^m)} &\le \eta^j \|D^j u_\eta^{\mathrm{th}} - D^j u_\eta^{\mathrm{sm}}\|_{L^p(U_\eta^m + Q_{\underline{\rho}\eta}^m)} \\
&\le \eta^j \|D^j u_\eta^{\mathrm{th}}\|_{L^p(U_\eta^m + Q_{\underline{\rho}\eta}^m)} + \eta^j \|D^j u_\eta^{\mathrm{sm}}\|_{L^p(U_\eta^m + Q_{\underline{\rho}\eta}^m)} \\
&\le C_6 \sum_{i=1}^j \eta^i \|D^i u_\eta^{\mathrm{sm}}\|_{L^p(U_\eta^m + Q_{\underline{\rho}\eta}^m)}.
\end{aligned}
$$

Thus, by Propositions 3.2 and 2.1,

$$
\begin{aligned}
\eta^j \|D^j u_\eta^{\mathrm{th}} - D^j u_\eta^{\mathrm{sm}}\|_{L^p(K_\eta^m)} &\le C_7 \sum_{i=1}^j \eta^i \|D^i u_\eta^{\mathrm{op}}\|_{L^p(U_\eta^m + Q_{(\underline{\rho}+\rho)\eta}^m)} \\
&\le C_8 \sum_{i=1}^j \eta^i \|D^i u\|_{L^p(U_\eta^m + Q_{2\rho\eta}^m)}.
\end{aligned} \tag{5.3}
$$

By the triangle inequality, we deduce from (5.1)–(5.3) that for every $j \in \{1, \dots, k\}$,

$$
\begin{aligned}
&\eta^j \|D^j u_\eta^{\mathrm{th}} - D^j u\|_{L^p(K_\eta^m)} \\
&\qquad \le \sup_{v \in B_1^m} \eta^j \|\tau_{\psi_\eta v}(D^j u) - D^j u\|_{L^p(Q_{1+\gamma}^m)} + C \sum_{i=1}^j \eta^i \|D^i u\|_{L^p(U_\eta^m + Q_{2\rho\eta}^m)}.
\end{aligned}
$$

This gives the estimate we claimed since $K_\eta^m = Q_{1+\gamma}^m$. We observe that u_η^{th} is smooth except on $(U_\eta^m + Q_{\underline{\rho}\eta}^m) \cap T_\eta^{\ell*}$ where $T_\eta^{\ell*}$ is the dual skeleton corresponding to the cubication $\mathcal{K}_\eta^m$. $\qquad\square$

The map u_η^{th} need not have its values on the manifold N^n, so we need to estimate the distance between the image of u_η^{th} and N^n.

Part 2. The directed Hausdorff distance from the image of the map u_η^{th} to the manifold N^n satisfies the estimate

$$
\begin{aligned}
\mathrm{Dist}_{N^n}\,(u_\eta^{\mathrm{th}}(K_\eta^m \setminus T_\eta^{\ell*})) \le \max\Bigg\{ &\max_{\sigma^m \in \mathcal{K}_\eta^m \setminus \mathcal{E}_\eta^m} \frac{C'}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}, \\
&\sup_{x \in U_\eta^\ell + Q_{\underline{\rho}\eta}^m} \frac{C''}{|Q_s^m|^2} \int_{Q_s^m(x)} \int_{Q_s^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)|\, dy\, dz \Bigg\},
\end{aligned}
$$

where the *directed Hausdorff distance* from a set $S \subset \mathbb{R}^\nu$ to N^n is

$$
\mathrm{Dist}_{N^n}\,(S) = \sup\{\mathrm{dist}(x, N^n) : x \in S\},
$$

$\mathcal{E}_\eta^m$ is a subskeleton of $\mathcal{U}_\eta^m$, and $0 < s < \eta$.

The subskeleton $\mathcal{E}_\eta^m$ *will be chosen at the end of Part 2 as the set of bad cubes and* $\mathcal{K}_\eta^m \setminus \mathcal{E}_\eta^m$ *will be the set of good cubes. This estimate implies that for every* $\eta > 0$ *sufficiently small, the image of* u_η^{th} *is contained in a small tubular neighborhood of* N^n.

We first observe that by Proposition 4.1(ii), $\Phi^{\text{th}}(K_\eta^m \setminus (T^{\ell^*} \cup U_\eta^m)) \subset K_\eta^m \setminus U_\eta^m$, while by Proposition 4.1(iii), $\Phi^{\text{th}}(U_\eta^m \setminus T^{\ell^*}) \subset U_\eta^\ell + Q_{\underline{\rho}\eta}^m$. Hence,

$$\Phi^{\text{th}}(K_\eta^m \setminus T_\eta^{\ell^*}) \subset (K_\eta^m \setminus U_\eta^m) \cup (U_\eta^\ell + Q_{\underline{\rho}\eta}^m).$$

In terms of the directed Hausdorff distance we have

$$\text{Dist}_{N^n} (u_\eta^{\text{th}}(K_\eta^m \setminus T_\eta^{\ell^*})) \le \text{Dist}_{N^n} \left(u_\eta^{\text{sm}}((K_\eta^m \setminus U_\eta^m) \cup (U_\eta^\ell + Q_{\underline{\rho}\eta}^m)) \right).$$

Since the image of the map u_η^{op} obtained by opening u is contained in N^n (see Lemma 2.3), for every $x \in K_\eta^m$ we have

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \le \frac{1}{|Q_{\psi_\eta(x)}^m|} \int_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\text{sm}}(x) - u_\eta^{\text{op}}(z)| \, dz.$$

On the other hand, since u_η^{sm} is the convolution of u_η^{op} with a mollifier,

$$|u_\eta^{\text{sm}}(x) - u_\eta^{\text{op}}(z)| \le \frac{1}{\psi_\eta(x)^m} \int_{B_{\psi_\eta(x)}^m(x)} \varphi\left(\frac{x - y}{\psi_\eta(x)}\right) |u_\eta^{\text{op}}(y) - u_\eta^{\text{op}}(z)| \, dy$$

$$\le \frac{C_1}{|Q_{\psi_\eta(x)}^m|} \int_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\text{op}}(y) - u_\eta^{\text{op}}(z)| \, dy.$$

Thus,

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \le \frac{C_1}{|Q_{\psi_\eta(x)}^m|^2} \int_{Q_{\psi_\eta(x)}^m(x)} \int_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\text{op}}(y) - u_\eta^{\text{op}}(z)| \, dy \, dz. \qquad (5.4)$$

Since N^n is a compact subset of $\mathbb{R}^\nu$, u is bounded. By the Gagliardo–Nirenberg interpolation inequality (see [19, 44]), $Du \in L^{kp}(Q_{1+2\gamma}^m)$. By the Poincaré–Wirtinger inequality,

$$\frac{1}{|Q_{\psi_\eta(x)}^m|^2} \int_{Q_{\psi_\eta(x)}^m(x)} \int_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\text{op}}(y) - u_\eta^{\text{op}}(z)| \, dy \, dz$$

$$\le \frac{C_2}{\psi_\eta(x)^{m/(kp)-1}} \|Du_\eta^{\text{op}}\|_{L^{kp}(Q_{\psi_\eta(x)}^m(x))}.$$

Since $\psi_\eta \le \rho\eta$, if $\sigma^m \in \mathcal{K}_\eta^m$ is such that $x \in \sigma^m$, then $Q_{\psi_\eta(x)}^m(x) \subset \sigma^m + Q_{\rho\eta}^m$. Hence,

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \le \frac{C_3}{\psi_\eta(x)^{m/(kp)-1}} \|Du_\eta^{\text{op}}\|_{L^{kp}(Q_{\psi_\eta(x)}^m(x))}$$

$$\le \frac{C_3}{\psi_\eta(x)^{m/(kp)-1}} \|Du_\eta^{\text{op}}\|_{L^{kp}(\sigma^m + Q_{\rho\eta}^m)}.$$

Thus, by Addendum 1 to Proposition 2.1,

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \le \frac{C_4}{\psi_\eta(x)^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}.$$

We rewrite this estimate for every $x \in K_\eta^m$ as

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \leq \left(\frac{\eta}{\psi_\eta(x)}\right)^{m/(kp)-1} \frac{C_4}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}. \qquad (5.5)$$

If $x \in (U_\eta^\ell + Q_{\underline{\rho}\eta}^m) \cap U_\eta^m$, then $x \in \sigma^m$ for some cube $\sigma^m \in \mathcal{U}_\eta^m$. If

$$\boxed{\psi_\eta(x) \leq (\rho - \underline{\rho})\eta,}$$

then $Q_{\psi_\eta(x)}^m(x) \subset U_\eta^\ell + Q_{\rho\eta}^m$. By Addendum 2 to Proposition 2.1, we have

$$\frac{1}{|Q_{\psi_\eta(x)}^m|^2} \int_{Q_{\psi_\eta(x)}^m(x)} \int_{Q_{\psi_\eta(x)}^m(x)} |u_\eta^{\text{op}}(y) - u_\eta^{\text{op}}(z)| \, dy \, dz$$

$$\leq (\psi_\eta(x))^{1-\ell/(kp)} \frac{C_5}{\eta^{(m-\ell)/(kp)}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}.$$

Therefore,

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \leq (\psi_\eta(x))^{1-\ell/(kp)} \frac{C_5}{\eta^{(m-\ell)/kp}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}.$$

We rewrite this estimate for every $x \in (U_\eta^\ell + Q_{\underline{\rho}\eta}^m) \cap U_\eta^m$ as

$$\text{dist}(u_\eta^{\text{sm}}(x), N^n) \leq \left(\frac{\psi_\eta(x)}{\eta}\right)^{1-\ell/(kp)} \frac{C_5}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}. \qquad (5.6)$$

We now describe the function ψ_η we shall take. Given two parameters $0 < s < t$ and a function $\zeta \in C^\infty(Q_{1+2\gamma}^m)$, we define

$$\psi_\eta = t\zeta + s(1 - \zeta).$$

More precisely, let $\mathcal{E}_\eta^m$ be a subskeleton of $\mathcal{U}_\eta^m$ such that

$$\boxed{E_\eta^m \subset \text{int } U_\eta^m}$$

in the relative topology of $Q_{1+\gamma}^m$. Since $\text{dist}(E_\eta^m, K_\eta^m \setminus U_\eta^m) \geq \eta$, we take a function $\zeta \in C^\infty(K_\eta^m)$ such that

(i) $0 \leq \zeta \leq 1$ in K_η^m,
(ii) $\zeta = 1$ in $K_\eta^m \setminus U_\eta^m$,
(iii) $\zeta = 0$ in E_η^m,
(iv) for every $j \in \{1, \ldots, k\}$, $\eta^j \|D^j \zeta\|_{L^\infty} \leq \tilde{C}$, for some constant $\tilde{C} > 0$ depending only on m.

Thus, $\operatorname{supp} D\psi_\eta \subset U_\eta^m$ and

$$\eta^j \|D^j \psi_\eta\|_{L^\infty} \le \tilde{C}t.$$

In order to apply Proposition 3.2 and to have $\psi_\eta \le (\rho - \underline{\rho})\eta$, we choose

$$t = \min\{\kappa/\tilde{C}, \rho - \underline{\rho}\}\eta,$$

for some fixed number $0 < \kappa < 1$.

Since $\psi_\eta = t$ in $K_\eta^m \setminus U_\eta^m$ and $t \ge c\eta$ for some constant $c > 0$ independent of η, from (5.5) we have

$$\operatorname{Dist}_{N^n}(u_\eta^{\mathrm{sm}}(K_\eta^m \setminus U_\eta^m)) \le \max_{\sigma^m \in \mathcal{K}_\eta^m \setminus \mathcal{U}_\eta^m} \frac{C_6}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}.$$

Since $\psi_\eta = s$ in E_η^m, from (5.4) we have

$$\operatorname{Dist}_{N^n}\left(u_\eta^{\mathrm{sm}}((U_\eta^\ell + Q_{\underline{\rho}\eta}^m) \cap E_\eta^m)\right) \le \sup_{x \in U_\eta^\ell + Q_{\underline{\rho}\eta}^m} \frac{C_1}{|Q_s^m|^2} \int_{Q_s^m(x)} \int_{Q_s^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \, dy \, dz.$$

Finally, if

$$\boxed{\ell \le kp,}$$

then by (5.6) and by the estimate $\psi_\eta(x) \le t = C_7 \eta$, we get

$$\operatorname{Dist}_{N^n}\left(u_\eta^{\mathrm{sm}}((U_\eta^\ell + Q_{\underline{\rho}\eta}^m) \cap (U_\eta^m \setminus E_\eta^m))\right) \le \max_{\sigma^m \in \mathcal{U}_\eta^m \setminus \mathcal{E}_\eta^m} \frac{C_8}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}.$$

Since we have already required that $\ell + 1 > kp$, we are thus led to take

$$\boxed{\ell = \lfloor kp \rfloor.}$$

We deduce that

$$\operatorname{Dist}_{N^n}(u_\eta^{\mathrm{th}}(K_\eta^m \setminus T_\eta^{\ell^*})) \le \max\left\{ \max_{\sigma^m \in \mathcal{K}_\eta^m \setminus \mathcal{E}_\eta^m} \frac{C'}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)}, \right.$$

$$\left. \sup_{x \in U_\eta^\ell + Q_{\underline{\rho}\eta}^m} \frac{C''}{|Q_s^m|^2} \int_{Q_s^m(x)} \int_{Q_s^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \, dy \, dz \right\}.$$

This gives the estimate we claimed.

The nearest point projection Π onto N^n is well-defined and smooth on a tubular neighborhood of N^n of radius $\iota > 0$. We now choose the subskeleton $\mathcal{E}_\eta^m$ used in the definition of ζ and ψ_η as the set of cubes $\sigma^m \in \mathcal{K}_\eta^m$ such that

$$\frac{C'}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)} > \iota.$$

Thus,

$$\max_{\sigma^m \in \mathcal{K}_\eta^m \setminus \mathcal{E}_\eta^m} \frac{C'}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\sigma^m + Q_{2\rho\eta}^m)} \le \iota.$$

We then take the subskeleton $\mathcal{U}_\eta^m$ used in the constructions of opening and thickening as the set of cubes $\sigma^m \in \mathcal{K}_\eta^m$ which intersect some cube in $\mathcal{E}_\eta^m$; in particular int $E_\eta^m \subset U_\eta^m$ in the relative topology of $Q_{1+\gamma}^m$.

In view of the uniform limit of Addendum 2 to Proposition 2.1, since $\ell \leq kp$, for every $s > 0$ small enough,

$$\sup_{x \in U_\eta^\ell + Q_{\rho\eta}^m} \frac{C''}{|Q_s^m|^2} \int_{Q_s^m(x)} \int_{Q_s^m(x)} |u_\eta^{\mathrm{op}}(y) - u_\eta^{\mathrm{op}}(z)| \, dy \, dz \leq \iota.$$

Therefore, $u_\eta^{\mathrm{th}}(K_\eta^m \setminus T_\eta^{\ell*})$ is contained in a tubular neighborhood of N^n of radius ι.   □

Part 3. The maps $\Pi \circ u_\eta^{\mathrm{th}}$ converge to u in $W^{k,p}(Q_1^m; N^n)$ as η tends to 0.

Using the estimate from Part 1, we show that for every $j \in \{1, \ldots, k\}$,

$$\lim_{\eta \to 0} \|D^j u_\eta^{\mathrm{th}} - D^j u\|_{L^p(Q_{1+\gamma}^m)} = 0.$$

By continuity of the translation operator in L^p (see the remark following Proposition 3.1),

$$\lim_{\eta \to 0} \sup_{v \in B_1^m} \|\tau_{\psi_\eta v}(D^j u) - D^j u\|_{L^p(Q_{1+\gamma}^m)} = 0. \tag{5.7}$$

We now need to show that

$$\lim_{\eta \to 0} \sum_{i=1}^j \eta^{i-j} \|D^i u\|_{L^p(U_\eta^m + Q_{2\rho\eta}^m)} = 0.$$

By the Gagliardo–Nirenberg interpolation inequality, for every $i \in \{1, \ldots, k-1\}$, $D^i u \in L^{kp/i}(Q_{1+2\gamma}^m)$. By Hölder's inequality, for every $i \in \{1, \ldots, k\}$ we have

$$\eta^{i-j} \|D^i u\|_{L^p(U_\eta^m + Q_{2\rho\eta}^m)} \leq \eta^{i-j} |U_\eta^m + Q_{2\rho\eta}^m|^{(k-i)/(kp)} \|D^i u\|_{L^{kp/i}(U_\eta^m + Q_{2\rho\eta}^m)}$$

$$= \eta^{k-j} \left(\frac{|U_\eta^m + Q_{2\rho\eta}^m|}{\eta^{kp}} \right)^{(k-i)/(kp)} \|D^i u\|_{L^{kp/i}(U_\eta^m + Q_{2\rho\eta}^m)}.$$

From this estimate, we need that $|U_\eta^m + Q_{2\rho\eta}^m| = O(\eta^{kp})$ as $\eta \to 0$. We observe that $|U_\eta^m + Q_{2\rho\eta}^m|$ satisfies the following estimate in terms of the number of elements $\#\mathcal{U}_\eta^m$ of the subskeleton $\mathcal{U}_\eta^m$:

$$|U_\eta^m + Q_{2\rho\eta}^m| \leq 2^m (\eta + 2\rho\eta)^m (\#\mathcal{U}_\eta^m) = C_1 \eta^m (\#\mathcal{U}_\eta^m).$$

Note that for every cube $\sigma^m \in \mathcal{U}_\eta^m$, if $\tau^m \in \mathcal{E}_\eta^m$ intersects σ^m, then $\tau^m + Q_{2\rho\eta}^m \subset \sigma^m + Q_{2(1+\rho)\eta}^m$. Denoting σ^m by $Q_\eta^m(a)$, we have $\tau^m + Q_{2\rho\eta}^m \subset Q_{\alpha\eta}^m(a)$, where $\alpha = 3 + 2\rho$, whence

$$\tau^m + Q_{2\rho\eta}^m \subset Q_{\alpha\eta}^m(a) \cap Q_{1+2\gamma}^m.$$

By the definition of $\mathcal{E}_\eta^m$,

$$\iota < \frac{C'}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(\tau^m + Q_{2\rho\eta}^m)} \leq \frac{C'}{\eta^{m/(kp)-1}} \|Du\|_{L^{kp}(Q_{\alpha\eta}^m(a) \cap Q_{1+2\gamma}^m)}.$$

Thus, for every $Q_\eta^m(a) \in \mathcal{U}_\eta^m$,

$$1 < \frac{C_2}{\eta^{m-kp}} \int_{Q_{\alpha\eta}^m(a) \cap Q_{1+2\gamma}^m} |Du|^{kp}.$$

Since the cubes $Q_{\alpha\eta}^m(a)$ intersect each other finitely many times and the number of overlaps only depends on α and on the dimension m,

$$\#\mathcal{U}_\eta^m \le \frac{C_2}{\eta^{m-kp}} \sum_{Q_\eta^m(a) \in \mathcal{U}_\eta^m} \int_{Q_{\alpha\eta}^m(a) \cap Q_{1+2\gamma}^m} |Du|^{kp} \le \frac{C_3}{\eta^{m-kp}} \int_{Q_{1+2\gamma}^m} |Du|^{kp}.$$

We deduce that

$$\left| U_\eta^m + Q_{2\rho\eta}^m \right| \le C_4 \eta^m \frac{1}{\eta^{m-kp}} \int_{Q_{1+2\gamma}^m} |Du|^{kp} = C_4 \eta^{kp} \int_{Q_{1+2\gamma}^m} |Du|^{kp}.$$

This means that

$$\limsup_{\eta \to 0} \frac{\left| U_\eta^m + Q_{2\rho\eta}^m \right|}{\eta^{kp}} < \infty.$$

Hence, by Lebesgue's dominated convergence theorem,

$$\lim_{\eta \to 0} \|D^i u\|_{L^{kp/i}(U_\eta^m + Q_{2\rho\eta}^m)} = 0.$$

In view of (5.7) and the estimate from Part 1, we have $\lim_{\eta \to 0} \|D^j u_\eta^{\text{th}} - D^j u\|_{L^p(Q_{1+\gamma}^m)} = 0$.

Recall that $u_\eta^{\text{th}} = u_\eta^{\text{sm}}$ in the complement of $U_\eta^m + Q_{\rho\eta}^m$. Since $u_\eta^{\text{sm}} \to u$ in measure and $|U_\eta^m + Q_{\rho\eta}^m| \to 0$ as $\eta \to 0$, $u_\eta^{\text{th}} \to u$ in measure as $\eta \to 0$. Hence, u_η^{th} converges to u in $L^p(Q_{1+\gamma}^m)$ and

$$\lim_{\eta \to 0} \|u_\eta^{\text{th}} - u\|_{W^{k,p}(Q_{1+\gamma}^m)} = 0.$$

Therefore,

$$\lim_{\eta \to 0} \|\Pi \circ u_\eta^{\text{th}} - u\|_{W^{k,p}(Q_{1+\gamma}^m)} = 0.$$

This gives the conclusion of this part. $\qquad\square$

Part 4. The map $\Pi \circ u_\eta^{\text{th}}$ belongs to the class $R_{\ell*}(Q^m; N^n)$.

It suffices to prove the pointwise estimates of $D^j(\Pi \circ u_\eta^{\text{th}})$. Since $\Pi \circ u_\eta^{\text{th}} = (\Pi \circ u_\eta^{\text{sm}}) \circ \Phi^{\text{th}}$ and the map $\Pi \circ u_\eta^{\text{sm}}$ is smooth in K_η^m, by the chain rule for higher order derivatives,

$$|D^j(\Pi \circ u_\eta^{\text{th}})| \le C_5 \sum_{i=1}^{j} \sum_{\substack{1 \le \alpha_1 \le \cdots \le \alpha_i \\ \alpha_1 + \cdots + \alpha_i = j}} |D^i(\Pi \circ u_\eta^{\text{sm}})| |D^{\alpha_1} \Phi^{\text{th}}| \cdots |D^{\alpha_i} \Phi^{\text{th}}|$$

$$\le C_6 \sum_{i=1}^{j} \sum_{\substack{1 \le \alpha_1 \le \cdots \le \alpha_i \\ \alpha_1 + \cdots + \alpha_i = j}} |D^{\alpha_1} \Phi^{\text{th}}| \cdots |D^{\alpha_i} \Phi^{\text{th}}|.$$

By Proposition 4.1(iv), for $x \in K_\eta^m \setminus T_\eta^{\ell*}$ we have

$$|D^j(\Pi \circ u_\eta^{\text{th}})(x)| \le C_7 \sum_{i=1}^{j} \sum_{\substack{1 \le \alpha_1 \le \cdots \le \alpha_i \\ \alpha_1 + \cdots + \alpha_i = j}} \frac{\eta}{(\text{dist}(x, T_\eta^{\ell*}))^{\alpha_1}} \cdots \frac{\eta}{(\text{dist}(x, T_\eta^{\ell*}))^{\alpha_i}}$$

$$\le \frac{C_8}{(\text{dist}(x, T_\eta^{\ell*}))^j}.$$

This concludes the proof of the theorem.   □

6. Proof of Theorem 2

Let $kp < m$. It is a consequence of Theorem 3 that $R_{m-\lfloor kp \rfloor -1}(Q^m; N^n)$ is dense in $W^{k,p}(Q^m; N^n)$. In this section, we prove that if $\pi_{\lfloor kp \rfloor}(N^n) \neq \{0\}$ and if $i \in \{0, \ldots, m-1\}$ is such that

(a) $R_i(Q^m; N^n) \subset W^{k,p}(Q^m; N^n)$,
(b) $R_i(Q^m; N^n)$ is dense in $W^{k,p}(Q^m; N^n)$,

then $i = m - \lfloor kp \rfloor - 1$.

We first prove that $i < m - \lfloor kp \rfloor$. For this purpose, let $\gamma : \mathbb{R} \to N^n$ be a geodesic in N^n. Given $i \ge m - \lfloor kp \rfloor$, the map $u : \overline{Q}^m \to N^n$ defined for $x = (x', x'') \in \overline{Q}^{m-i} \times \overline{Q}^i$ by

$$u(x) = \gamma(\log |x'|)$$

belongs to $R_i(Q^m; N^n)$. Taking γ parametrized by arc-length, we have

$$|Du(x)| = 1/|x'|.$$

Since $i \ge m - \lfloor kp \rfloor$, it follows that $Du \notin L^{\lfloor kp \rfloor}(Q^m)$. By the Gagliardo–Nirenberg interpolation inequality, we deduce that $R_i(Q^m; N^n) \not\subset W^{k,p}(Q^m; N^n)$.

We now prove that $i > m - \lfloor kp \rfloor - 2$. Given a smooth map $\varphi : \mathbb{S}^{\lfloor kp \rfloor} \to N^n$, we define $u : \overline{Q}^m \to N^n$ for $x = (x', x'') \in \overline{Q}^{\lfloor kp \rfloor +1} \times \overline{Q}^{m-\lfloor kp \rfloor -1}$ by

$$u(x) = \varphi(x'/|x'|).$$

Then $u \in W^{k,p}(Q^m; N^n)$. Given $i \in \{0, \ldots, m-\lfloor kp \rfloor -2\}$, assume for contradiction that there exists a sequence $(u_j)_{j\in\mathbb{N}}$ in $R_i(Q^m; N^n)$ converging to u in $W^{k,p}(Q^m; N^n)$. Passing to a subsequence if necessary, for almost every $x'' \in Q^{m-\lfloor kp \rfloor -1}$ and for almost every $\rho \in (0,1)$, the sequence $(u_j|_{\mathbb{S}_\rho^{\lfloor kp \rfloor} \times \{x''\}})_{j\in\mathbb{N}}$ converges to $u|_{\mathbb{S}_\rho^{\lfloor kp \rfloor} \times \{x''\}}$ in $W^{k,p}(\mathbb{S}_\rho^{\lfloor kp \rfloor}; N^n)$, whence in $\text{BMO}(\mathbb{S}_\rho^{\lfloor kp \rfloor}; N^n)$.

For every $j \in \mathbb{N}$, denote by T_j a finite union of i-dimensional planes such that $u_j \in C^\infty(\overline{Q}^m \setminus T_j; N^n)$. Since $i \le m - \lfloor kp \rfloor - 2$, for every $(x'', \rho) \in Q^{m-\lfloor kp \rfloor -1} \times (0,1)$ such that $\mathbb{S}_\rho^{\lfloor kp \rfloor} \times \{x''\} \subset \overline{Q}^m \setminus T_j$, there exist $a \in \overline{Q}^m \setminus T_j$ and a continuous map $h : [0,1] \times (\mathbb{S}_\rho^{\lfloor kp \rfloor} \times \{x''\}) \to \overline{Q}^m \setminus T_j$ such that for every $y \in \mathbb{S}_\rho^{\lfloor kp \rfloor} \times \{x''\}$, $h(0, y) = y$ and $h(1, y) = a$. This implies that $u_j|_{\mathbb{S}_\rho^{\lfloor kp \rfloor} \times \{x''\}}$ is homotopic to a constant.

We recall that homotopy classes are preserved under BMO convergence:

Claim. *Let $(v_j)_{j\in\mathbb{N}}$ be a sequence in $C^0(\mathbb{S}^{\lfloor kp\rfloor}; N^n)$ which converges to some $v \in C^0(\mathbb{S}^{\lfloor kp\rfloor}; N^n)$ in $\mathrm{BMO}(\mathbb{S}^{\lfloor kp\rfloor}; N^n)$. Then, for every $j \in \mathbb{N}$ sufficiently large, v_j is homotopic to v in $C^0(\mathbb{S}^{\lfloor kp\rfloor}; N^n)$.*

This claim is essentially [13, Lemma A.19], but we present a proof for the convenience of the reader.

Proof of the Claim. For every $\epsilon > 0$, we consider the map $v_\epsilon : \mathbb{S}^{\lfloor kp\rfloor} \to N^n$ defined for $x \in \mathbb{S}^{\lfloor kp\rfloor}$ by

$$v_\epsilon(x) = \frac{1}{|D_\epsilon^{\lfloor kp\rfloor}(x)|} \int_{D_\epsilon^{\lfloor kp\rfloor}(x)} v$$

where $D_\epsilon^{\lfloor kp\rfloor}(x) = \mathbb{S}^{\lfloor kp\rfloor} \cap Q_\epsilon^{\lfloor kp\rfloor+1}(x)$. Accordingly, for every $j \in \mathbb{N}$ we define $v_{j,\epsilon}$, with v replaced by v_j.

The nearest point projection Π is well-defined and smooth on a tubular neighborhood of N^n of radius $\iota > 0$. Since v_ϵ converges uniformly to v as ϵ tends to 0, there exists $\epsilon_1 > 0$ such that for every $0 < \epsilon \le \epsilon_1$, $\Pi \circ v_\epsilon$ is well-defined and homotopic to v.

Next, for every $j \in \mathbb{N}$ and every $x \in \mathbb{S}^{\lfloor kp\rfloor}$, since $v_j(x) \in N^n$,

$$\mathrm{dist}(v_{j,\epsilon}(x), N^n) \le \frac{1}{|D_\epsilon^{\lfloor kp\rfloor}(x)|} \int_{D_\epsilon^{\lfloor kp\rfloor}(x)} \left| v_j(y) - \frac{1}{|D_\epsilon^{\lfloor kp\rfloor}(x)|} \int_{D_\epsilon^{\lfloor kp\rfloor}(x)} v_j \right| dy$$

$$\le \|v_j - v\|_{\mathrm{BMO}(\mathbb{S}^{\lfloor kp\rfloor})} + 2 \sup_{y \in D_\epsilon^{\lfloor kp\rfloor}(x)} |v(y) - v(x)|.$$

Since the sequence $(v_j)_{j\in\mathbb{N}}$ converges to v in $\mathrm{BMO}(\mathbb{S}^{\lfloor kp\rfloor})$ and v is uniformly continuous, there exist $J \in \mathbb{N}$ and $\epsilon_2 > 0$ such that for every $j \ge J$ and every $0 < \epsilon \le \epsilon_2$,

$$\mathrm{dist}(v_{j,\epsilon}(x), N^n) \le \iota.$$

In particular, $\Pi \circ v_{j,\epsilon}$ is well-defined, and the continuous extension of the function $(0, 1] \ni t \mapsto \Pi \circ v_{j,t\epsilon}$ gives a homotopy between $\Pi \circ v_{j,\epsilon}$ and v_j.

Finally, for every $\epsilon > 0$ the sequence $(v_{j,\epsilon})_{j\in\mathbb{N}}$ converges uniformly to v_ϵ. For $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$ and $j \ge J$, the functions $\Pi \circ v_{j,\epsilon}$ are well-defined and converge uniformly to $\Pi \circ v_\epsilon$ as j tends to infinity. Thus, there exists $\overline{J} \ge J$ such that for every $j \in \mathbb{N}$ with $j \ge \overline{J}$, $\Pi \circ v_{j,\epsilon}$ is homotopic to $\Pi \circ v_\epsilon$. By transitivity of the homotopy relation, we conclude that for every such j, v_j is homotopic to v. $\square$

We deduce from the claim that $u|_{\mathbb{S}_\rho^{\lfloor kp\rfloor} \times \{x''\}}$ is homotopic to a constant, whence $\varphi : \mathbb{S}^{\lfloor kp\rfloor} \to N^n$ is homotopic to a constant. Since $\pi_{\lfloor kp\rfloor}(N^n) \ne \{0\}$ and $\varphi : \mathbb{S}^{\lfloor kp\rfloor} \to N^n$ is an arbitrary smooth function, we get a contradiction, proving Theorem 2. $\square$

7. Continuous extension property

From Theorem 3 we are able to approximate a map by another map which is smooth except on a dual skeleton of dimension $\lfloor kp\rfloor^*$. We would like to modify our approximation near this singular set in order to obtain a smooth map. An important tool will be:

Proposition 7.1. *Let $\mathcal{K}^m$ be a skeleton of radius $\eta > 0$, $\ell \in \{0, \dots, m-1\}$, $\mathcal{T}^{\ell^*}$ be the dual skeleton of $\mathcal{K}^\ell$ and let $u \in C^\infty(K^m \setminus T^{\ell^*}; N^n)$. If there exists $f \in C^0(K^m; N^n)$ such that $f|_{K^\ell} = u|_{K^\ell}$, then for every $0 < \mu < 1$ there exists $v \in C^\infty(K^m; N^n)$ such that $v = u$ on $K^m \setminus (T^{\ell^*} + Q^m_{\mu\eta})$.*

In the proof of Proposition 7.1, we shall rely on the fact that K^ℓ is a homotopy retract of $K^m \setminus T^{\ell^*}$, that is, there exists a continuous retraction of $K^m \setminus T^{\ell^*}$ onto K^ℓ which is homotopic to the identity map on $K^m \setminus T^{\ell^*}$:

Fact 7.2. *There exists a continuous homotopy $H_\ell : [0, 1] \times (K^m \setminus T^{\ell^*}) \to K^m \setminus T^{\ell^*}$ such that*

(i)   *$H_\ell(0, x) = x$ for every $x \in K^m \setminus T^{\ell^*}$,*
(ii) *$H_\ell(1, x) \in K^\ell$ for every $x \in K^m \setminus T^{\ell^*}$,*
(iii) *$H_\ell(1, x) = x$ for every $x \in K^\ell$.*

Proof of Proposition 7.1. Given $0 < \underline{\delta} < \delta < \overline{\delta} < \mu$, let $\varphi : K^m \to [0, 1]$ be a continuous function such that

- $\varphi(x) = 0$ for every $x \in K^m \setminus (T^{\ell^*} + Q^m_{\overline{\delta}\eta})$,
- $\varphi(x) = 1$ for every $x \in \partial(T^{\ell^*} + Q^m_{\delta\eta})$,
- $\varphi(x) = 0$ for every $x \in T^{\ell^*} + Q^m_{\underline{\delta}\eta}$.

We define $w : K^m \to N^n$ by

$$
w(x) = \begin{cases} (u \circ H_\ell)(\varphi(x), x) & \text{if } x \in K^m \setminus (T^{\ell^*} + Q^m_{\underline{\delta}\eta}), \\ (f \circ H_\ell)(\varphi(x), x) & \text{if } x \in (T^{\ell^*} + Q^m_{\underline{\delta}\eta}) \setminus T^{\ell^*}, \\ f(x) & \text{if } x \in T^{\ell^*}. \end{cases}
$$

By Fact 7.2(i)&(ii), w is well-defined and continuous on K^m, and $w = u$ on $K^m \setminus (T^{\ell^*} + Q^m_{\overline{\delta}\eta})$. Let $\overline{w} : \mathbb{R}^m \to \mathbb{R}^\nu$ be a continuous extension of w. Given a mollifier $\varphi \in C_c^\infty(B_1^m)$, there exists a nonnegative function $\psi \in C^\infty(\mathbb{R}^m)$ such that for any $\iota > 0$,

- $\operatorname{supp} \psi \subset T^{\ell^*} + Q^m_{\mu\eta}$,
- $\psi > 0$ in a neighborhood of $T^{\ell^*} + Q^m_{\delta\eta}$,
- $\|\varphi_\psi * \overline{w} - \overline{w}\|_{L^\infty(\mathbb{R}^m)} \leq \iota$.

If the nearest point projection Π onto N^n is well-defined and smooth on a tubular neighborhood of N^n of radius $\iota > 0$, then the map $\Pi \circ (\varphi_\psi * \overline{w})$ restricted to K^m satisfies all the required properties. $\square$

The natural question that arises is whether a continuous extension of $u|_{K^\ell}$ to K^m exists. This property depends on the skeleton $\mathcal{K}^m$ and on the manifold N^n.

Proposition 7.3. *Let $\mathcal{K}^m$ be a skeleton of radius $\eta > 0$ and $\ell \in \{0, \dots, m-1\}$. If K^m is a cube and if $\pi_\ell(N^n) = \{0\}$, then for every $u \in C^0(K^\ell; N^n)$ there exists $f \in C^0(K^m; N^n)$ such that $f|_{K^\ell} = u$.*

We will use the fact that it is always possible to find a continuous extension, regardless of N^n, by losing one dimension. This property has been introduced as the ℓ *extension property* by Hang and Lin [28, Definition 2.3].

Proposition 7.4. *Let $\mathcal{K}^m$ be a skeleton of radius $\eta > 0$ and $\ell \in \{0, \ldots, m-1\}$. If K^m is a cube, then for every $u \in C^0(K^{\ell+1}; N^n)$ there exists $g \in C^0(K^m; N^n)$ such that $g|_{K^\ell} = u|_{K^\ell}$.*

In the proof of Proposition 7.4, we shall assume that if K^m is a cube, then the identity map on K^ℓ is homotopic to a constant with respect to $K^{\ell+1}$:

Fact 7.5. *If K^m is a cube, then there exists a continuous homotopy $G_\ell : [0, 1] \times K^\ell \to K^{\ell+1}$ such that*

(i) $G_\ell(0, x) = x$ *for every* $x \in K^\ell$,
(ii) *there exists* $a \in K^\ell$ *such that* $G_\ell(1, x) = a$ *for every* $x \in K^\ell$.

Proof of Proposition 7.4. Let $\varphi : K^m \to [0, 1]$ be a continuous function such that

- $\varphi(x) = 0$ for every $x \in K^\ell$,
- $\varphi(x) = 1$ for every $x \in T^{\ell^*}$.

We define $g : K^m \to N^n$ by

$$g(x) = \begin{cases} u\big(G_\ell(\varphi(x), H_\ell(1, x))\big) & \text{if } x \in K^m \setminus T^{\ell^*}, \\ u(a) & \text{if } x \in T^{\ell^*}. \end{cases}$$

where $H_\ell : [0, 1] \times K^m \setminus T^{\ell^*} \to K^m \setminus T^{\ell^*}$ is the homotopy retraction of Fact 7.2. The map g is continuous and by Fact 7.2(iii) we have $g(x) = u(x)$ for every $x \in K^\ell$. $\square$

Proof of Proposition 7.3. Let $u \in C^0(K^\ell; N^n)$. Since $\pi_\ell(N^n) = \{0\}$, for every $\sigma^{\ell+1} \in \mathcal{K}^{\ell+1}$, the restriction $u|_{\partial\sigma^{\ell+1}}$ has a continuous extension $u_{\sigma^{\ell+1}}$ to $\sigma^{\ell+1}$. Let $v : K^{\ell+1} \to N^n$ be the map defined for every $x \in K^{\ell+1}$ by $v(x) = u_{\sigma^{\ell+1}}(x)$, where $\sigma^{\ell+1} \in \mathcal{K}^{\ell+1}$ is such that $x \in \sigma^{\ell+1}$. The map v is well-defined and continuous; moreover, $v|_{K^\ell} = u$. By Proposition 7.4 applied to v, there exists $f : K^m \to N^n$ such that $f|_{K^\ell} = v|_{K^\ell}$; hence f is a continuous extension of u to K^m. $\square$

8. Shrinking

Given a map $u \in W^{k,p}(K^m; \mathbb{R}^\nu)$ whose energy is controlled outside a neighborhood of the dual skeleton T^{ℓ^*}, we construct for $\tau > 0$ a map $u \circ \Phi$ whose energy will be controlled on the whole K^m when τ is small enough. This shrinking construction is very similar to the thickening construction. In both cases, the dimension of the dual skeleton T^{ℓ^*} must satisfy $\ell^* < m - kp$, or equivalently $l + 1 > kp$. The main differences are that shrinking only acts in a neighborhood of the dual skeleton T^{ℓ^*} and does not create singularities. Shrinking can be thought of as desingularized thickening and requires more careful estimates.

Just as for thickening, we begin by constructing the diffeomorphism Φ regardless of u:

Proposition 8.1. *Let $\ell \in \{0, \dots, m-1\}$, $\eta > 0$, $0 < \mu < 1/2$, $0 < \tau < 1/2$, $\mathcal{S}^m$ be a cubication of $\mathbb{R}^m$ of radius η and $\mathcal{T}^{\ell^*}$ be the dual skeleton of $\mathcal{S}^\ell$. There exists a smooth map $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ such that*

 (i) *Φ is injective,*
 (ii) *$\Phi(\sigma^m) \subset \sigma^m$ for every $\sigma^m \in \mathcal{S}^m$,*
 (iii) *$\operatorname{Supp} \Phi \subset T^{\ell^*} + Q_{2\mu\eta}^m$ and $\Phi(T^{\ell^*} + Q_{\tau\mu\eta}^m) \supset T^{\ell^*} + Q_{\mu\eta}^m$,*
 (iv) *for every $0 < \beta < \ell + 1$, every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,*

$$(\mu\eta)^{j-1} |D^j \Phi(x)| \le C (\operatorname{jac} \Phi(x))^{j/\beta},$$

 for some constant $C > 0$ depending on β, j and m,
 (v) *for every $0 < \beta < \ell + 1$, every $j \in \mathbb{N}_*$ and every $x \in \Phi^{-1}(T^{\ell^*} + Q_{\mu\eta}^m)$,*

$$(\mu\eta)^{j-1} |D^j \Phi(x)| \le C' \tau^{j(\frac{\ell+1}{\beta} - 1)} (\operatorname{jac} \Phi(x))^{j/\beta},$$

 for some constant $C' > 0$ depending on β, j and m.

As a consequence of the estimates of Proposition 8.1, we have the following $W^{k,p}$ estimates that will be applied in the proof of Theorem 1 with $\ell = \lfloor kp \rfloor$.

Corollary 8.2. *Let $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ be the map given by Proposition 8.1 and let $\mathcal{K}^m$ be a subskeleton of $\mathcal{S}^m$. If $\ell + 1 > kp$, then for every $u \in W^{k,p}(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m); \mathbb{R}^\nu)$, $u \circ \Phi \in W^{k,p}(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m); \mathbb{R}^\nu)$, and for every $j \in \{1, \dots, k\}$,*

$$(\mu\eta)^j \|D^j(u \circ \Phi)\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))} \le C'' \sum_{i=1}^{j} (\mu\eta)^i \|D^i u\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m) \setminus (T^{\ell^*} + Q_{\mu\eta}^m))}$$

$$+ C'' \tau^{(\ell+1-kp)/p} \sum_{i=1}^{j} (\mu\eta)^i \|D^i u\|_{L^p(K^m \cap (T^{\ell^*} + Q_{\mu\eta}^m))},$$

for some constant $C'' > 0$ depending on m, k and p.

Proof. We first establish the estimate for a map u in $C^\infty(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m); \mathbb{R}^\nu)$. By the chain rule for higher order derivatives, for every $j \in \{1, \dots, k\}$ and every $x \in K^m$,

$$|D^j(u \circ \Phi)(x)|^p \le C_1 \sum_{i=1}^{j} \sum_{\substack{1 \le t_1 \le \dots \le t_i \\ t_1 + \dots + t_i = j}} |D^i u(\Phi(x))|^p |D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p.$$

As in the proof of Corollary 4.2, if $1 \le t_1 \le \dots \le t_i$ and $t_1 + \dots + t_i = j$, then for every $x \in K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m)$,

$$|D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p \le C_2 \frac{\operatorname{jac} \Phi(x)}{\eta^{(j-i)p}},$$

and this implies

$$\eta^{jp}|D^j(u \circ \Phi)(x)|^p \le C_3 \sum_{i=1}^{j} \eta^{ip}|D^i u(\Phi(x))|^p \operatorname{jac} \Phi(x).$$

Let $\sigma^m \in \mathcal{K}^m$. Since Φ is injective, by the change of variable formula,

$$\int_{\Phi^{-1}(\sigma^m \cap (T^{\ell*}+Q^m_{2\mu\eta}) \setminus (T^{\ell*}+Q^m_{\mu\eta}))} (\mu\eta)^{jp}|D^j(u \circ \Phi)|^p$$

$$\le C_3 \sum_{i=1}^{j} \int_{\Phi^{-1}(\sigma^m \cap (T^{\ell*}+Q^m_{2\mu\eta}) \setminus (T^{\ell*}+Q^m_{\mu\eta}))} (\mu\eta)^{ip}|(D^i u) \circ \Phi|^p \operatorname{jac} \Phi$$

$$\le C_3 \sum_{i=1}^{j} \int_{\sigma^m \cap (T^{\ell*}+Q^m_{2\mu\eta}) \setminus (T^{\ell*}+Q^m_{\mu\eta})} (\mu\eta)^{ip}|D^i u|^p.$$

Let $0 < \beta < \ell + 1$. If $1 \le t_1 \le \cdots \le t_i$ and $t_1 + \cdots + t_i = j$, then by Proposition 8.1(v), for every $x \in \Phi^{-1}(K^m \cap (T^{\ell*} + Q^m_{\mu\eta}))$ we have

$$|D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p$$

$$\le C_4 \tau^{t_1 p(\frac{\ell+1}{\beta}-1)} \frac{(\operatorname{jac}\Phi(x))^{t_1 p/\beta}}{(\mu\eta)^{(t_1-1)p}} \cdots \tau^{t_i p(\frac{\ell+1}{\beta}-1)} \frac{(\operatorname{jac}\Phi(x))^{t_i p/\beta}}{(\mu\eta)^{(t_i-1)p}}$$

$$= C_4 \tau^{jp(\frac{\ell+1}{\beta}-1)} \frac{(\operatorname{jac}\Phi(x))^{jp/\beta}}{(\mu\eta)^{(j-i)p}}.$$

Taking $\beta = jp$, we have

$$|D^{t_1}\Phi(x)|^p \cdots |D^{t_i}\Phi(x)|^p \le C_4 \tau^{\ell+1-jp} \frac{\operatorname{jac}\Phi(x)}{(\mu\eta)^{(j-i)p}},$$

and this implies

$$(\mu\eta)^{jp}|D^j(u \circ \Phi)(x)|^p \le C_5 \tau^{\ell+1-jp} \sum_{i=1}^{j} (\mu\eta)^{ip}|D^i u(\Phi(x))|^p \operatorname{jac}\Phi(x).$$

Since Φ is injective, by the change of variable formula,

$$\int_{\Phi^{-1}(\sigma^m \cap (T^{\ell*}+Q^m_{\mu\eta}))} (\mu\eta)^{jp}|D^j(u \circ \Phi)|^p$$

$$\le C_5 \tau^{\ell+1-jp} \sum_{i=1}^{j} \int_{\Phi^{-1}(\sigma^m \cap (T^{\ell*}+Q^m_{\mu\eta}))} (\mu\eta)^{ip}|(D^i u) \circ \Phi|^p \operatorname{jac}\Phi$$

$$= C_5 \tau^{\ell+1-jp} \sum_{i=1}^{j} \int_{\sigma^m \cap (T^{\ell*}+Q^m_{\mu\eta})} (\mu\eta)^{ip}|D^i u|^p.$$

Since $\sigma^m \cap (T^{\ell^*} + Q^m_{2\mu\eta}) \subset \Phi^{-1}(\sigma^m \cap (T^{\ell^*} + Q^m_{2\mu\eta}))$, by additivity of integral we have

$$\int_{\sigma^m \cap (T^{\ell^*} + Q^m_{2\mu\eta})} (\mu\eta)^{jp} |D^j(u \circ \Phi)|^p \le C_3 \sum_{i=1}^j \int_{\sigma^m \cap (T^{\ell^*} + Q^m_{2\mu\eta}) \setminus (T^{\ell^*} + Q^m_{\mu\eta})} (\mu\eta)^{ip} |D^i u|^p$$
$$+ C_5 \tau^{\ell+1-jp} \sum_{i=1}^j \int_{\sigma^m \cap (T^{\ell^*} + Q^m_{\mu\eta})} (\mu\eta)^{ip} |D^i u|^p.$$

We may take the union over all faces $\sigma^m \in \mathcal{K}^m$ and deduce the estimate for smooth maps. By density of smooth maps in $W^{k,p}(K^m \cap (T^{\ell^*} + Q^m_{2\mu\eta}); \mathbb{R}^\nu)$, we conclude that for every u in $W^{k,p}(K^m \cap (T^{\ell^*} + Q^m_{2\mu\eta}); \mathbb{R}^\nu)$, the function $u \circ \Phi$ also belongs to this space and satisfies the estimate above.   □

We first describe the construction of the map Φ in the case of only one ℓ-dimensional cube.

Proposition 8.3. *Let* $\ell \in \{1, \ldots, m\}$, $\eta > 0$, $0 < \underline{\mu} < \mu < \overline{\mu} < 1$ *and* $0 < \tau < \mu/\underline{\mu}$. *There exists a smooth function* $\lambda : \mathbb{R}^m \to [1, \infty)$ *such that if* $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ *is defined for* $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ *by* $\Phi(x) = (\lambda(x)x', x'')$, *then*

(i) Φ *is injective,*
(ii) $\mathrm{Supp}\,\Phi \subset Q^\ell_{\mu\eta} \times Q^{m-\ell}_{(1-\mu)\eta}$,
(iii) $\Phi(Q^\ell_{\tau\mu\eta} \times Q^{m-\ell}_{(1-\overline{\mu})\eta}) \supset Q^\ell_{\underline{\mu}\eta} \times Q^{m-\ell}_{(1-\overline{\mu})\eta}$,
(iv) *for every* $0 < \beta < \ell$, *every* $j \in \mathbb{N}_*$ *and every* $x \in \mathbb{R}^m$,

$$(\mu\eta)^{j-1} |D^j \Phi(x)| \le C (\mathrm{jac}\,\Phi(x))^{j/\beta},$$

for some constant $C > 0$ *depending on* β, j, m, $\mu/\underline{\mu}$ *and* $\overline{\mu}/\mu$,
(v) *for every* $\beta > 0$, *every* $j \in \mathbb{N}_*$ *and every* $x \in Q^\ell_{\tau\mu\eta} \times Q^{m-\ell}_{(1-\overline{\mu})\eta}$,

$$(\mu\eta)^{j-1} |D^j \Phi(x)| \le C' \tau^{j(\ell/\beta-1)} (\mathrm{jac}\,\Phi(x))^{j/\beta},$$

for some constant $C' > 0$ *depending on* β, j, m, $\mu/\underline{\mu}$ *and* $\overline{\mu}/\mu$.

We postpone the proof of Proposition 8.3 and we proceed to establish Proposition 8.1.

Proof of Proposition 8.1. We first introduce finite sequences $(\mu_i)_{\ell \le i \le m}$ and $(\nu_i)_{\ell \le i \le m}$ such that

$$0 < \mu_\ell = \mu < \nu_{\ell+1} < \mu_{\ell+1} < \cdots < \mu_{m-1} < \nu_m < \mu_m \le 2\mu.$$

Let $\Phi_m = \mathrm{Id}$. Using downward induction, we shall define maps $\Phi_i : \mathbb{R}^m \to \mathbb{R}^m$ for $i \in \{\ell, \ldots, m-1\}$ so that Φ_i satisfies the following properties:

(a) Φ_i is injective,
(b) $\Phi_i(\sigma^m) \subset \sigma^m$ for every $\sigma^m \in \mathcal{S}^m$,
(c) $\mathrm{Supp}\,\Phi_i \subset T^{i^*} + Q^m_{2\mu\eta}$,

(d) $\Phi_i(T^r + Q^m_{\tau\mu\eta}) \supset T^r + Q^m_{\tau\mu\eta}$ for every $r \in \{i^*, \ldots, m-1\}$,

(e) $\Phi_i(T^{i^*} + Q^m_{\tau\mu\eta}) \supset T^{i^*} + Q^m_{\mu_i\eta}$,

(f) for every $0 < \beta < i + 1$, every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,

$$(\mu\eta)^{j-1}|D^j\Phi_i(x)| \leq C(\mathrm{jac}\,\Phi_i(x))^{j/\beta},$$

for some constant $C > 0$ depending on β, j, and m,

(g) for every $0 < \beta < i + 1$, every $j \in \mathbb{N}_*$ and every $x \in \Phi_i^{-1}(T^{i^*} + Q^m_{\mu_i\eta})$,

$$(\mu\eta)^{j-1}|D^j\Phi_i(x)| \leq C'\tau^{j(\frac{i+1}{\beta}-1)}(\mathrm{jac}\,\Phi_i(x))^{j/\beta},$$

for some constant $C' > 0$ depending on β, j, and m.

The map Φ_ℓ will satisfy the conclusion of the proposition.

Let $i \in \{\ell + 1, \ldots, m\}$ and let Θ_i be the map obtained from Proposition 8.3 with parameters $\underline{\ell} = i$, $\underline{\mu} = \mu_{i-1}$, $\mu = \nu_i$, $\overline{\mu} = \mu_i$ and $\tau\mu/\nu_i$. Given $\sigma^i \in \mathcal{S}^i$, we may identify σ^i with $Q^i_\eta \times \{0^{m-i}\}$ and $T^{(i-1)^*} \cap (\sigma^i + Q^m_{2\mu\eta})$ with $\{0^i\} \times Q^{m-i}_{2\mu\eta}$. The map Θ_i induces by isometry a map which we shall denote by Θ_{σ^i}.

Let $\Psi_i : \mathbb{R}^m \to \mathbb{R}^m$ be defined for every $x \in \mathbb{R}^m$ by

$$\Psi_i(x) := \begin{cases} \Theta_{\sigma^i}(x) & \text{if } x \in \sigma^i + Q^m_{(1-\nu_i)\eta} \text{ for some } \sigma^i \in \mathcal{S}^i, \\ x & \text{otherwise.} \end{cases}$$

We explain why Ψ_i is well-defined. Since Θ_{σ^i} coincides with the identity map on $\partial\sigma^i + Q^m_{(1-\nu_i)\eta}$, it follows that for every $\sigma_1^i, \sigma_2^i \in \mathcal{S}^i$, if $x \in (\sigma_1^i + Q^m_{(1-\nu_i)\eta}) \cap (\sigma_2^i + Q^m_{(1-\nu_i)\eta})$ and $\sigma_1^i \neq \sigma_2^i$, then

$$\Theta_{\sigma_1^i}(x) = x = \Theta_{\sigma_2^i}(x).$$

One also verifies that Ψ_i is smooth.

Assuming that Φ_i has been defined satisfying properties (a)–(g), we let

$$\Phi_{i-1} = \Psi_i \circ \Phi_i.$$

We check that Φ_{i-1} satisfies all required properties. Up to an exchange of coordinates, for every $\sigma^i \in \mathcal{S}^i$, we may assume that $\sigma^i = Q^i_\eta \times \{0^{m-i}\}$ and Θ_{σ^i} can be written as $\Theta_{\sigma^i}(x) = (\lambda(x)x', x'')$, with $\lambda(x) \geq 1$. Hence, for every $0 < s \leq 1$ and every $r \in \{0, \ldots, m-1\}$,

$$\Psi_i(T^r + Q^m_{s\eta}) \supset T^r + Q^m_{s\eta}. \tag{8.1}$$

Moreover, in the new coordinates, the set

$$(\sigma^i \times Q^{m-i}_\eta) \cap \left((T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \setminus (T^{i^*} + Q^m_{\mu_i\eta})\right) \tag{8.2}$$

becomes

$$Q^i_{\tau\mu\eta} \times Q^{m-i}_{(1-\mu_i)\eta}. \tag{8.3}$$

In view of Proposition 8.3(i)&(iii),

$$\Theta_{\sigma^i}(Q^i_{\tau\mu\eta} \times Q^{m-i}_{(1-\mu_i)\eta}) \supset Q^i_{\mu_{i-1}\eta} \times Q^{m-i}_{(1-\mu_i)\eta}.$$

Since this property holds for every $\sigma^i \in \mathcal{S}^i$,

$$\Psi_i\big((T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \setminus (T^{i^*} + Q^m_{\mu_i\eta})\big) \supset (T^{(i-1)^*} + Q^m_{\mu_{i-1}\eta}) \setminus (T^{i^*} + Q^m_{\mu_i\eta}). \quad (8.4)$$

Proof of (d). Let $r \in \{(i-1)^*, \ldots, m-1\}$. By induction hypothesis and by (8.1) with $s = \tau\mu$,

$$\Phi_{i-1}(T^r + Q^m_{\tau\mu\eta}) \supset \Psi_i(T^r + Q^m_{\tau\mu\eta}) \supset T^r + Q^m_{\tau\mu\eta}. \qquad \square$$

Proof of (e). By induction hypothesis (properties (d) and (e)),

$$\Phi_i(T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \supset (T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \cup (T^{i^*} + Q^m_{\mu_i\eta}).$$

Thus,

$$\Phi_{i-1}(T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \supset \Psi_i(T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \cup \Psi_i(T^{i^*} + Q^m_{\mu_i\eta}).$$

By inclusion (8.4) and by inclusion (8.1) with $r = i^*$ and $s = \mu_i$,

$$\Phi_{i-1}(T^{(i-1)^*} + Q^m_{\tau\mu\eta}) \supset \big((T^{(i-1)^*} + Q^m_{\mu_{i-1}\eta}) \setminus (T^{i^*} + Q^m_{\mu_i\eta})\big) \cup (T^{i^*} + Q^m_{\mu_i\eta})$$
$$= T^{(i-1)^*} + Q^m_{\mu_{i-1}\eta}. \qquad \square$$

Proof of (g). Let $j \in \mathbb{N}_*$ and $0 < \beta < i$. By the chain rule for higher order derivatives, for every $x \in \mathbb{R}^m$ we have

$$|D^j\Phi_{i-1}(x)| \le C_1 \sum_{r=1}^j \sum_{\substack{1 \le t_1 \le \cdots \le t_r \\ t_1+\cdots+t_r=j}} |D^r\Psi_i(\Phi_i(x))| |D^{t_1}\Phi_i(x)| \cdots |D^{t_r}\Phi_i(x)|.$$

Let $x \in \Phi_{i-1}^{-1}(T^{(i-1)^*} + Q^m_{\mu_{i-1}\eta})$. By induction hypothesis (property (f)), for every $r \in \{1, \ldots, j\}$, if $1 \le t_1 \le \cdots \le t_r$ and $t_1 + \cdots + t_r = j$, then

$$|D^{t_1}\Phi_i(x)| \cdots |D^{t_r}\Phi_i(x)| \le C_2 \frac{(\mathrm{jac}\,\Phi_i(x))^{j/\beta}}{(\mu\eta)^{j-r}}.$$

If in addition $x \in \Phi_{i-1}^{-1}((T^{(i-1)^*} + Q^m_{\mu_{i-1}\eta}) \setminus (T^{i^*} + Q^m_{\mu_i\eta}))$, then $\Phi_i(x) \in \Psi_i^{-1}((T^{(i-1)^*} + Q^m_{\mu_{i-1}\eta}) \setminus (T^{i^*} + Q^m_{\mu_i\eta}))$. By the correspondence between the sets given by (8.2) and (8.3), by inclusion (8.4), and by Proposition 8.3(v), for every $0 < \alpha < i$ we have

$$|D^r\Psi_i(\Phi_i(x))| \le C_3 \tau^{r(i/\alpha-1)} \frac{(\mathrm{jac}\,\Psi_i(\Phi_i(x)))^{r/\alpha}}{(\mu\eta)^{r-1}}.$$

Take $\alpha = \beta r/j$. Since $r \le j$ and $\tau \le 1$, we get

$$|D^r\Psi_i(\Phi_i(x))| \le C_3 \tau^{r(\frac{ij}{\beta r}-1)} \frac{(\mathrm{jac}\,\Psi_i(\Phi_i(x)))^{j/\beta}}{(\mu\eta)^{r-1}} \le C_3 \tau^{j(i/\beta-1)} \frac{(\mathrm{jac}\,\Psi_i(\Phi_i(x)))^{j/\beta}}{(\mu\eta)^{r-1}}.$$

Thus, for every $x \in \Phi_{i-1}^{-1}((T^{(i-1)^*} + Q_{\mu_{i-1}\eta}^m) \setminus (T^{i^*} + Q_{\mu_i\eta}^m))$,

$$|D^j\Phi_{i-1}(x)| \le C_4\tau^{j(i/\beta-1)} \frac{(\text{jac } \Psi_i(\Phi_i(x)))^{j/\beta}}{(\mu\eta)^{r-1}} \frac{(\text{jac } \Phi_i(x))^{j/\beta}}{(\mu\eta)^{j-r}}$$

$$= C_4\tau^{j(i/\beta-1)} \frac{(\text{jac } \Phi_{i-1}(x))^{j/\beta}}{(\mu\eta)^{j-1}}.$$

On the other hand, if $x \in \Phi_{i-1}^{-1}(T^{i^*} + Q_{\mu_i\eta}^m)$, then $\Phi_i(x) \in \Psi_i^{-1}(T^{i^*} + Q_{\mu_i\eta}^m)$. By inclusion (8.1) with $r = i^*$ and $s = \mu_i$, $\Phi_i(x) \in T^{i^*} + Q_{\mu_i\eta}^m$. By induction hypothesis (property (g)), we deduce that for every $r \in \{1, \dots, j\}$, if $1 \le t_1 \le \cdots \le t_r$ and $t_1 + \cdots + t_r = j$, then

$$|D^{t_1}\Phi_i(x)| \cdots |D^{t_r}\Phi_i(x)| \le C_5\tau^{j(i/\beta-1)} \frac{(\text{jac } \Phi_i(x))^{j/\beta}}{(\mu\eta)^{j-r}}.$$

By Proposition 8.3(iv),

$$|D^r\Psi_i(\Phi_i(x))| \le C_6 \frac{(\text{jac } \Psi_i(\Phi_i(x)))^{j/\beta}}{(\mu\eta)^{r-1}}.$$

We deduce as above that

$$|D^j\Phi_{i-1}(x)| \le C_7\tau^{j(i/\beta-1)} \frac{(\text{jac } \Phi_{i-1}(x))^{j/\beta}}{(\mu\eta)^{j-1}}. \qquad \square$$

The other properties can be checked as in the proof of Proposition 4.1.

By downward induction, we conclude that properties (a)–(g) hold for every i in $\{\ell, \dots, m-1\}$. In particular, we deduce properties (i)–(v) of Proposition 8.3. $\qquad \square$

We need a couple of lemmas in order to prove Proposition 8.3:

Lemma 8.4. *Let $\eta > 0$, $0 < \underline{\mu} < \mu < \overline{\mu} < 1$ and $0 < \kappa < \underline{\mu}/\mu$. There exists a smooth function $\lambda : \mathbb{R}^m \to [1, \infty)$ such that if $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is defined for $x = (x', x'')$ in $\mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by $\Phi(x) = (\lambda(x)x', x'')$, then*

(i) Φ *is a diffeomorphism,*
(ii) $\text{Supp } \Phi \subset Q_{\mu\eta}^\ell \times Q_{(1-\mu)\eta}^{m-\ell}$,
(iii) $\Phi(Q_{\kappa\mu\eta}^\ell \times Q_{(1-\overline{\mu})\eta}^{m-\ell}) \supset Q_{\underline{\mu}\eta}^\ell \times Q_{(1-\overline{\mu})\eta}^{m-\ell}$,
(iv) *for every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,*

$$(\mu\eta)^{j-1}|D^j\Phi(x)| \le C,$$

for some constant $C > 0$ depending on j, m, $\mu/\underline{\mu}$, $\overline{\mu}/\mu$ and κ,

(v) *for every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,*

$$C' \le \text{jac } \Phi(x) \le C'',$$

for some constants $C', C'' > 0$ depending on m, $\mu/\underline{\mu}$, $\overline{\mu}/\mu$ and κ.

Proof. By scaling, we may assume that $\mu\eta = 1$. Let $\psi : \mathbb{R} \to [0, 1]$ be a smooth function such that

- ψ is nonincreasing on $\mathbb{R}_+$ and nondecreasing on $\mathbb{R}_-$,
- $\psi(t) = 1$ for $|t| \leq \underline{\mu}/\mu$,
- $\psi(t) = 0$ for $|t| \geq 1$.

Let $\theta : \mathbb{R} \to [0, 1]$ be a smooth function such that

- $\theta(t) = 1$ for $|t| \leq (1 - \overline{\mu})/\mu$,
- $\theta(t) = 0$ for $|t| \geq (1 - \mu)/\mu$.

Since $(1 - \mu)/\mu - (1 - \overline{\mu})/\mu = \overline{\mu}/\mu - 1$, we may require that for every $j \in \mathbb{N}_*$ and every $t \geq 0$, $|D^j\theta(t)| \leq C$, for some constant $C > 0$ depending only on j and $\overline{\mu}/\mu$.

Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be the function defined for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ by

$$\varphi(x) = \prod_{i=1}^{\ell} \psi(x_i) \prod_{i=\ell+1}^{m} \theta(x_i).$$

Let $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ be the function defined for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by

$$\Psi(x) = \big((1 - \alpha\varphi(x))x', x''\big),$$

where $\alpha \in \mathbb{R}$. In particular, for every $x = (x', x'') \in Q^\ell_{\underline{\mu}/\mu} \times Q^{m-\ell}_{(1-\overline{\mu})/\mu}$,

$$\Psi(x) = ((1 - \alpha)x', x'').$$

Taking $\alpha = 1 - \kappa\mu/\underline{\mu}$, we deduce that Ψ is a bijection between $Q^\ell_{\underline{\mu}/\mu} \times Q^{m-\ell}_{(1-\overline{\mu})/\mu}$ and $Q^\ell_\kappa \times Q^{m-\ell}_{(1-\overline{\mu})/\mu}$. As in Lemma 4.4 we can prove that $\Phi = \Psi^{-1}$ satisfies the required properties. $\qquad\square$

Lemma 8.5. *Let $\ell \in \{1, \ldots, m\}$, $\eta > 0$, $0 < \underline{\mu} < \mu < \overline{\mu} < 1$ and $0 < \tau < \underline{\mu}/\mu$. There exists a smooth function $\lambda : \mathbb{R}^m \to [1, \infty)$ such that if $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ is defined for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$ by $\Phi(x) = (\lambda(x)x', x'')$, then*

  (i) *Φ is injective,*
 (ii) *$\mathrm{Supp}\,\Phi \subset Q^\ell_{\mu\eta} \times Q^{m-\ell}_{(1-\mu)\eta}$,*
(iii) *$\Phi(B^\ell_{\tau\mu\eta} \times Q^{m-\ell}_{(1-\overline{\mu})\eta}) \supset B^\ell_{\underline{\mu}\eta} \times Q^{m-\ell}_{(1-\overline{\mu})\eta}$,*
 (iv) *for every $0 < \beta < \ell$, every $j \in \mathbb{N}_*$ and every $x \in \mathbb{R}^m$,*

$$(\mu\eta)^{j-1}|D^j\Phi(x)| \leq C(\mathrm{jac}\,\Phi(x))^{j/\beta},$$

 for some constant $C > 0$ depending on β, j, m, $\mu/\underline{\mu}$ and $\overline{\mu}/\mu$,
  (v) *for every $\beta > 0$, every $j \in \mathbb{N}_*$ and every $x \in B^\ell_{\tau\mu\eta} \times Q^{m-\ell}_{(1-\overline{\mu})\eta}$,*

$$(\mu\eta)^{j-1}|D^j\Phi(x)| \leq C'\tau^{j(\ell/\beta-1)}\big(\mathrm{jac}\,\Phi(x)\big)^{j/\beta},$$

 for some constant $C' > 0$ depending on β, j, m, $\mu/\underline{\mu}$ and $\overline{\mu}/\mu$.

Proof. By scaling, we may assume that $\mu\eta = 1$. Given $\varepsilon > 0$ and $b > 0$, let $\varphi : (0, \infty) \to [1, \infty)$ be a smooth function such that

- $\varphi(s) = \dfrac{\underline{\mu}/\mu}{s}\sqrt{1+\varepsilon}\left(1 + \dfrac{b}{\ln 1/s}\right)$ for $0 < s \le \tau\sqrt{1+\varepsilon}$,
- $\varphi(s) = 1$ for $s \ge 1$,
- the function $(0, \infty) \ni s \mapsto s\varphi(s)$ is increasing.

Note that such a function φ exists if we take $\varepsilon > 0$ such that

$$(\underline{\mu}/\mu)\sqrt{1+\varepsilon} < 1$$

and thus $\tau\sqrt{1+\varepsilon} < 1$ and if we take $b > 0$ such that

$$(\underline{\mu}/\mu)\sqrt{1+\varepsilon}\left(1 + \dfrac{b}{\ln\dfrac{1}{(\underline{\mu}/\mu)\sqrt{1+\varepsilon}}}\right) < 1.$$

Let $\theta : \mathbb{R}^{m-\ell} \to [0, 1]$ be a smooth function such that

- $\theta(y) = 0$ for $y \in Q^{m-\ell}_{(1-\overline{\mu})/\mu}$,
- $\theta(y) = 1$ for $y \in \mathbb{R}^{m-\ell} \setminus Q^{m-\ell}_{(1-\mu)/\mu}$.

We now introduce, for $x = (x', x'') \in \mathbb{R}^\ell \times \mathbb{R}^{m-\ell}$,

$$\zeta(x) = \sqrt{|x'|^2 + \theta(x'')^2 + \varepsilon\tau^2}.$$

Let $\lambda : \mathbb{R}^m \to \mathbb{R}$ be the function defined for $x \in \mathbb{R}^m$ by

$$\lambda(x) = \varphi(\zeta(x)).$$

As in the proof of Lemma 4.5, one may check that the map Φ defined in the statement satisfies all the required properties:

Proof of (iii). Let $x \in B^\ell_{\underline{\mu}/\mu} \times Q^{m-\ell}_{(1-\overline{\mu})/\mu}$. For every $s \ge 0$,

$$\Phi(sx', x'') = \left(s\varphi\left(\sqrt{s^2|x'|^2 + \varepsilon\tau^2}\right)x', x''\right).$$

Consider the function $h : [0, \infty) \to \mathbb{R}$ defined by

$$h(s) = s\varphi\left(\sqrt{s^2 + \varepsilon\tau^2}\right).$$

Then, assuming that $x' \ne 0$,

$$\Phi(sx', x'') = \left(h(s|x'|)\dfrac{x'}{|x'|}, x''\right).$$

We have $h(0) = 0$ and $h(\tau) > \underline{\mu}/\mu \ge |x'|$. By the intermediate value theorem, there exists $t \in (0, \tau)$ such that $h(t) = |x'|$. Thus, $tx'/|x'| \in B^\ell_\tau$ and $\Phi(tx'/|x'|, x'') = x$. $\square$

Proof of (v). Proceeding as in the proof of Lemma 4.5, for every $x \in B_1^\ell \times \mathbb{R}^{m-\ell}$ one gets

$$|D^j \Phi(x)| \leq C_1 / \zeta(x)^j.$$

Since $\zeta(x) \geq \tau \sqrt{\varepsilon}$, we deduce that

$$|D^j \Phi(x)| \leq C_1 / (\tau \sqrt{\varepsilon})^j \leq C_2 / \tau^j.$$

On the other hand,

$$\text{jac } \Phi(x) = \varphi(\zeta(x))^{\ell-1} \left(\varphi(\zeta(x)) \left(1 - \frac{|x'|^2}{\zeta(x)^2} \right) + \left(\varphi^{(1)}(\zeta(x)) \zeta(x) + \varphi(\zeta(x)) \right) \frac{|x'|^2}{\zeta(x)^2} \right).$$

Since $\varphi^{(1)}(s)s + \varphi(s) \geq 0$ for every $s > 0$, we have

$$\text{jac } \Phi(x) \geq \varphi(\zeta(x))^\ell \left(1 - \frac{|x'|^2}{\zeta(x)^2} \right) \geq \frac{C_3}{\zeta(x)^\ell} \left(1 - \frac{|x'|^2}{\zeta(x)^2} \right).$$

If $x \in B_\tau^\ell \times Q_{(1-\overline{\mu})/\mu}^{m-\ell}$, then $\zeta(x) \leq \tau \sqrt{1+\varepsilon}$ and $\zeta(x)^2 \geq (1+\varepsilon)|x'|^2$. Thus,

$$\text{jac } \Phi(x) \geq \frac{C_3}{(\tau \sqrt{1+\varepsilon})^\ell} \frac{\varepsilon}{1+\varepsilon} = \frac{C_4}{\tau^\ell}.$$

Combining the estimates of $|D^j \Phi|$ and jac Φ, we have the conclusion. □

In order to establish the remaining properties stated in Lemma 8.5, we only need to repeat the proof of Lemma 4.5 with obvious modifications. □

Proof of Proposition 8.3. Define Φ to be the composition of the map Φ_1 given by Lemma 8.4 with $\kappa = \mu/(\mu\sqrt{\ell})$ with the map Φ_2 given by Lemma 8.5; more precisely, $\Phi = \Phi_1 \circ \Phi_2$. The properties of Φ can be established as in the case of thickening. □

9. Proof of Theorem 1

Let $\mathcal{K}^m$ be a cubication of Q_1^m of radius $\eta > 0$ and let $\mathcal{T}^{\ell^*}$ be the dual skeleton with respect to $\mathcal{K}^\ell$ for some $\ell \in \{0, \dots, m-1\}$.

Claim. *Let* $v \in C^\infty(K^m \setminus T^{\ell^*}; N^n) \cap W^{k,p}(K^m; N^n)$. *If* $\pi_\ell(N^n) = \{0\}$ *and* $\ell^* < m - kp$, *then there exists a family of smooth maps* $v_{\tau_\mu, \mu}^{\text{sh}} : K^m \to N^n$ *such that*

$$\lim_{\mu \to 0} \|v_{\tau_\mu, \mu}^{\text{sh}} - v\|_{W^{k,p}(K^m)} = 0.$$

This claim is a removable singularity property of topological nature for $W^{k,p}$ maps. Theorem 1 follows from Theorem 3 and this claim. Indeed, by Theorem 3 the class of maps v in the claim is dense in $W^{k,p}(K^m; N^n)$ when $\ell = \lfloor kp \rfloor$. Since the maps $v_{\tau_\mu, \mu}^{\text{sh}}$ are smooth and converge to v in $W^{k,p}$, we deduce that smooth maps are dense in $W^{k,p}(K^m; N^n)$.

Proof of the Claim. Assuming that $\pi_\ell(N^n) = \{0\}$, we can modify v in a neighborhood of T^{ℓ^*} in order to obtain a smooth map $v_\mu^{\text{ex}} : K^m \to N^n$. More precisely, for every $0 < \mu < 1$, by Propositions 7.1 and 7.3, there exists $v_\mu^{\text{ex}} \in C^\infty(K^m; N^n)$ such that $v_\mu^{\text{ex}} = v$ in $K^m \setminus (T^{\ell^*} + Q_{\mu\eta}^m)$.

Although v and v_μ^{ex} coincide on a large set, $\|v_\mu^{\text{ex}}\|_{W^{k,p}(K^m)}$ can be much larger than $\|v\|_{W^{k,p}(K^m)}$ since the extension is of topological nature and does not take into account the values of v in a neighborhood of T^{ℓ^*}. In order to get a better extension of v, we have to shrink $T^{\ell^*} + Q_{\mu\eta}^m$ into a smaller neighborhood of T^{ℓ^*}.

Assume that $\mu < 1/2$ and take $0 < \tau < 1/2$. Let $\Phi_{\tau,\mu}^{\text{sh}} : \mathbb{R}^m \to \mathbb{R}^m$ be the smooth diffeomorphism given by Proposition 8.1. Define

$$v_{\tau,\mu}^{\text{sh}} = v_\mu^{\text{ex}} \circ \Phi_{\tau,\mu}^{\text{sh}}.$$

In particular $v_{\tau,\mu}^{\text{sh}} \in C^\infty(K^m; N^n)$.

Since $v_{\tau,\mu}^{\text{sh}} = v$ in the complement of $T^{\ell^*} + Q_{2\mu\eta}^m$, for every $j \in \mathbb{N}_*$,

$$\|D^j v_{\tau,\mu}^{\text{sh}} - D^j v\|_{L^p(K^m)} = \|D^j v_{\tau,\mu}^{\text{sh}} - D^j v\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))}$$

$$\leq \|D^j v_{\tau,\mu}^{\text{sh}}\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))} + \|D^j v\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))}.$$

If $\ell^* < m - kp$, or equivalently $\ell + 1 > kp$, then by Corollary 8.2 we have, for every $j \in \{1, \ldots, k\}$,

$$(\mu\eta)^j \|D^j v_{\tau,\mu}^{\text{sh}}\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))} \leq C_1 \sum_{i=1}^{j} (\mu\eta)^i \|D^i v_\mu^{\text{ex}}\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m) \setminus (T^{\ell^*} + Q_{\mu\eta}^m))}$$

$$+ C_1 \tau^{(\ell+1-kp)/p} \sum_{i=1}^{j} (\mu\eta)^i \|D^i v_\mu^{\text{ex}}\|_{L^p(K^m \cap (T^{\ell^*} + Q_{\mu\eta}^m))}.$$

Since $v_\mu^{\text{ex}} = v$ in the complement of $T^{\ell^*} + Q_{\mu\eta}^m$, we deduce that

$$(\mu\eta)^j \|D^j v_{\tau,\mu}^{\text{sh}} - D^j v\|_{L^p(K^m)} \leq C_2 \sum_{i=1}^{j} (\mu\eta)^i \|D^i v\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))}$$

$$+ C_1 \tau^{(\ell+1-kp)/p} \sum_{i=1}^{j} (\mu\eta)^i \|D^i v_\mu^{\text{ex}}\|_{L^p(K^m \cap (T^{\ell^*} + Q_{\mu\eta}^m))}.$$

We show that

$$\lim_{\mu \to 0} \sum_{i=1}^{j} (\mu\eta)^{i-j} \|D^i v\|_{L^p(K^m \cap (T^{\ell^*} + Q_{2\mu\eta}^m))} = 0. \tag{9.1}$$

Since N^n is a compact subset of $\mathbb{R}^\nu$, v is bounded. By the Gagliardo–Nirenberg interpolation inequality, for every $i \in \{1, \ldots, k-1\}$, $D^i v \in L^{kp/i}(K^m)$. By Hölder's inequality, for every $i \in \{1, \ldots, k\}$ we then have

$$(\mu\eta)^{i-j}\|D^i v\|_{L^p(K^m \cap (T^{\ell^*}+Q_{2\mu\eta}^m))}$$

$$\leq (\mu\eta)^{i-j}\left|K^m \cap (T^{\ell^*}+Q_{2\mu\eta}^m)\right|^{\frac{k-i}{kp}}\|D^i v\|_{L^{kp/i}(K^m \cap (T^{\ell^*}+Q_{2\mu\eta}^m))}$$

$$= \eta^{i-j}\mu^{k-j+(\ell+1-kp)\frac{k-i}{kp}}\left(\frac{\left|K^m \cap (T^{\ell^*}+Q_{2\mu\eta}^m)\right|}{\mu^{\ell+1}}\right)^{\frac{k-i}{kp}}\|D^i v\|_{L^{kp/i}(K^m \cap (T^{\ell^*}+Q_{2\mu\eta}^m))}.$$

Since $|K^m \cap (T^{\ell^*}+Q_{2\mu\eta}^m)| \leq C_3\mu^{\ell+1}$, the limit follows.

For every $0 < \mu < 1/2$, take $0 < \tau_\mu < 1/2$ such that

$$\lim_{\mu\to 0}\tau_\mu^{(\ell+1-kp)/p}\sum_{i=1}^{j}(\mu\eta)^{i-j}\|D^i v_\mu^{\text{ex}}\|_{L^p(K^m \cap (T^{\ell^*}+Q_{\mu\eta}^m))} = 0. \qquad (9.2)$$

From (9.1) and (9.2), we deduce that for every $j \in \{1, \ldots, k\}$,

$$\lim_{\mu\to 0}\|D^j v_{\tau_\mu,\mu}^{\text{sh}} - D^j v\|_{L^p(K^m)} = 0.$$

Since $v_{\tau_\mu,\mu}^{\text{sh}}$ converges in measure to v as μ tends to 0, we then have

$$\lim_{\mu\to 0}\|v_{\tau_\mu,\mu}^{\text{sh}} - v\|_{W^{k,p}(K^m)} = 0.$$

This establishes the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

10. Concluding remarks

10.1. Other domains

The proof of Theorem 1 can be adapted to more general domains $\Omega \subset \mathbb{R}^m$. In order to apply the extension argument at the beginning of the proof of Theorem 3, it suffices that Ω be starshaped.

Concerning Theorem 1, the crucial tool is the extension property of Proposition 7.4. This can be enforced by assuming that for every $\ell \in \{0, \ldots, \lfloor kp \rfloor - 1\}$,

$$\pi_\ell(\Omega) = \{0\}.$$

This contains in particular the case where Ω is starshaped. Another option is to require that for some CW-complex structure, $\overline{\Omega}$ has the $\lfloor kp \rfloor - 1$ extension property with respect to N^n. More precisely, for every $u \in C^0(\overline{\Omega}^{\lfloor kp \rfloor}; N^n)$, the restriction $u|_{\overline{\Omega}^{\lfloor kp \rfloor - 1}}$ of u to the skeleton of $\overline{\Omega}$ of dimension $\lfloor kp \rfloor - 1$ has a continuous extension to $\overline{\Omega}$. It can be shown that this property does not depend on the CW-complex structure of $\overline{\Omega}$ (see remark following [28, Definition 2.3]).

10.2. Complete manifolds

The proofs of Theorems 1 and 3 still apply for complete manifolds N^n that are embedded in $\mathbb{R}^\nu$ and for which there exists a projection Π defined on a uniform neighborhood of size ι around N^n. The compactness of N^n ensures the Gagliardo–Nirenberg interpolation inequality that for every $i \in \{1, \ldots, k-1\}$, $D^i u \in L^{kp/i}(Q_1^m)$. This inequality still holds if the assumption $u \in L^\infty$ is replaced by $u \in W^{1,kp}$. In this case, one proves that if $\pi_{\lfloor kp \rfloor}(N^n) = \{0\}$, then for every $u \in W^{k,p}(Q^m; N^n) \cap W^{1,kp}(Q^m; N^n)$ there exists a family of maps $u_\eta \in C^\infty(Q^m; N^n)$ such that for every $i \in \{1, \ldots, k\}$,

$$\lim_{\eta \to 0} \|D^i u_\eta - D^i u\|_{L^{kp/i}(Q^m)} = 0$$

and u_η converges to u in measure as η tends to 0. Hence,

$$\lim_{\eta \to 0} \|u_\eta - u\|_{W^{k,p}(Q^m) \cap W^{1,kp}(Q^m)} = 0.$$

10.3. $\lfloor kp \rfloor$ connected manifolds

Under the additional assumption that for every $\ell \in \{0, \ldots, \lfloor kp \rfloor\}$,

$$\pi_\ell(N^n) = \{0\},$$

it is possible to give a simpler proof of $H^{k,p}(Q^m; N^n) = W^{k,p}(Q^m; N^n)$ without relying on the density of maps in $R_{m-\lfloor kp \rfloor-1}(Q^m; N^n)$. This approach is inspired by previous works of Escobedo [17] and Hajłasz [26]; see [7] for details.

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