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Blow up for the critical gKdV equation. II: Minimal mass dynamics

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Abstract. We consider the mass critical (gKdV) equation $u_t + (u_{xx} + u^5)_x = 0$ for initial data in H^1 . We first prove the existence and uniqueness in the energy space of a minimal mass blow up solution and give a sharp description of the corresponding blow up soliton-like bubble. We then show that this solution is the universal attractor of all solutions near the ground state which have a defocusing behavior. This allows us to sharpen the description of near soliton dynamics obtained in [29].

Keywords. Generalized Korteweg–de Vries equation, blow up, minimal mass solution, uniqueness of threshold solution

1. Introduction

1.1. Setting of the problem. We continue the study of the mass critical generalized Korteweg–de Vries equation

$$\text{(gKdV)} \quad \begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t, x) \in [0, T) \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.1)$$

initiated in Part I [29]. The Cauchy problem is locally well posed in the energy space H^1 from Kenig, Ponce and Vega [15], [16], and given $u_0 \in H^1$, there exists a unique (in a certain sense) maximal solution $u(t)$ of (1.1) in $\mathcal{C}([0, T), H^1)$ and

$$T < \infty \quad \text{implies} \quad \lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = \infty. \quad (1.2)$$

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The Cauchy problem for (1.1) is also locally well-posed in L^2 , and given $u_0 \in L^2$, there exists a unique maximal solution $u(t)$ of (1.1) in $C([0, T), L^2)$ with either $T = \infty$ or

$$T < \infty \quad \text{and then} \quad \|u\|_{L_x^5 L_{(0,T)}^{10}} = \infty.$$

Moreover, H^1 solutions satisfy the conservation of mass and energy:

$$M(u(t)) = \int u^2(t) = M_0, \quad E(u(t)) = \frac{1}{2} \int u_x^2(t) - \frac{1}{6} \int u^6(t) = E_0.$$

The symmetry group of (1.1) is continuous in H^1 and given by

$$\epsilon_0 \lambda_0^{1/2} u(\lambda_0^3(t - t_0), \lambda_0(x - x_0)), \quad (\epsilon_0, \lambda_0, x_0, t_0) \in \{-1, 1\} \times \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}.$$

In particular the scaling symmetry leaves the L^2 norm invariant and hence the problem is mass or L^2 critical.

Traveling wave solutions play a distinguished role in the analysis:

$$u(t, x) = Q(x - t)$$

where Q is the ground state solitary wave

$$Q(x) = \left(\frac{3}{\cosh^2(2x)} \right)^{1/4}$$

which satisfies the sharp Gagliardo–Nirenberg inequality, [50]:

$$\forall v \in H^1, \quad \int |v|^6 \leq \int v_x^2 \left(\frac{\int v^2}{\int Q^2} \right)^2. \tag{1.3}$$

The conservation of mass and energy and the blow up criterion (1.2) ensure that H^1 initial data with subcritical mass $\|u_0\|_{L^2} < \|Q\|_{L^2}$ generate global in time solutions.

1.2. The flow near the ground state. In the series of works [23]–[26], [34], Martel and Merle obtain the first qualitative information on the flow for small supercritical mass initial data $\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha^*$, $0 < \alpha^* \ll 1$, in particular the existence of finite time blow up solutions for $E_0 < 0$ and the classification of Q as the unique global attractor of all H^1 blow up solutions.

In Part I [29], we have revisited the blow up analysis in light of recent developments related to blow up for the mass critical Schrödinger equation [35]–[40] and energy critical geometrical equations [41], [46], [47].

More precisely, define the set of initial data

$$\mathcal{A} = \left\{ u_0 = Q + \varepsilon_0 : \|\varepsilon_0\|_{H^1} < \alpha_0 \text{ and } \int_{y>0} y^{10} \varepsilon_0^2 < 1 \right\},$$

and consider the L^2 tube around the family of solitary waves,

$$\mathcal{T}_{\alpha^*} = \left\{ u \in H^1 : \inf_{\lambda_0>0, x_0 \in \mathbb{R}} \left\| u - \frac{1}{\lambda_0^{1/2}} Q\left(\frac{\cdot - x_0}{\lambda_0}\right) \right\|_{L^2} < \alpha^* \right\}.$$

In [29], we have proved the following (see [29, Theorems 1.1 and 1.2] for more details).

Theorem 1.1 (Rigidity of the flow in \mathcal{A} , [29]). *Let $0 < \alpha_0 \ll \alpha^* \ll 1$ and $u_0 \in \mathcal{A}$. Let $u \in \mathcal{C}([0, T), H^1)$ be the corresponding solution to (1.1). Then one of the following three scenarios occurs:*

(Blow up) *The solution blows up in finite time $0 < T < \infty$ in the universal regime*

$$\|u(t)\|_{H^1} = \frac{\ell(u_0) + o(1)}{T - t} \quad \text{as } t \rightarrow T, \quad \ell(u_0) > 0. \quad (1.4)$$

(Soliton) *The solution is global: $T = \infty$, and converges asymptotically to a solitary wave.*

(Exit) *The solution leaves the tube \mathcal{T}_{α^*} at some time $0 < t^* < \infty$.*

Moreover, the (Blow up) and (Exit) scenarios are stable under small perturbation of the data in \mathcal{A} .

Our aim in this paper is first to classify the *minimal mass* dynamics $\|u_0\|_{L^2} = \|Q\|_{L^2}$, and then, from this classification, to complete the description obtained in Theorem 1.1 in the (Exit) regime. Indeed, we will show that for α_0 small enough, the (Exit) case is directly connected to the understanding of minimal mass dynamics.

1.3. Minimal mass dynamics. The question of existence and possibly uniqueness of minimal blow up dynamics for dispersive and parabolic PDE's has motivated several works since the pioneering result by Merle [32] for the mass critical nonlinear Schrödinger equation:

$$\text{(NLS)} \quad i\partial_t u + \Delta u + |u|^{4/N} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N. \quad (1.5)$$

Let us recall that for (NLS), the pseudo conformal symmetry generates an *explicit* minimal mass blow up solution

$$S_{\text{NLS}}(t, x) = \frac{1}{t^{N/2}} e^{-i\frac{|x|^2}{4t} - i} Q_{\text{NLS}}\left(\frac{x}{t}\right) \quad (1.6)$$

where Q_{NLS} is the ground state solution to

$$\Delta Q_{\text{NLS}} - Q_{\text{NLS}} + Q_{\text{NLS}}^{1+4/N} = 0, \quad Q > 0, \quad Q \in H^1.$$

Merle [32] proved that S_{NLS} is the unique (up to the symmetries of the equation) minimal mass blow up element in the energy space. The proof heavily relies on the pseudo conformal symmetry. Such minimal blow up dynamics has also been exhibited for the energy critical NLS and wave problems [11], [10], using the virial algebra and a fixed point argument. For the inhomogeneous mass critical (NLS) in dimension 2,

$$i\partial_t u + \Delta u + k(x)|u|^2 u = 0,$$

while Merle [33] derived sufficient conditions on $k(x)$ to ensure the *nonexistence* of minimal elements, Raphaël and Szeftel [48] introduced a more dynamical approach to existence and uniqueness under a necessary and sufficient condition on $k(x)$. A robust energy method is implemented to completely classify the minimal mass blow up, in regimes such that the inhomogeneity k influences dramatically the bubble of concentration (1.6)—in contrast with direct perturbative methods developed in [1], [3], [4]; see also [18] for existence in the one-dimensional half wave problem.

Recall that for the mass critical (gKdV) problem (1.1), Martel and Merle [27] obtained the following *global existence* result for minimal mass solutions with decay on the right.

Theorem 1.2 (Global existence at minimal mass, [27]). *Let $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and*

$$\sup_{x_0 > 0} x_0^3 \int_{x > x_0} u_0^2(x) dx < \infty. \quad (1.7)$$

Then the corresponding solution $u(t)$ of (1.1) is global for $t > 0$.

In other words, minimal mass blow up is not compatible with the decay (1.7). This is in agreement with the analysis in [29] where the threshold dynamics for data in \mathcal{A} between the stable (Blow up) and (Exit) regimes is proved to correspond to a solitary wave behavior—and not to a minimal blow up. We refer to [40] for a further discussion of threshold dynamics.

1.4. Statement of the result. The first main result of this paper is the *existence and uniqueness* in the energy space of a minimal mass blow up element:

Theorem 1.3 (Existence and uniqueness of the minimal mass blow up element).

- (i) (Existence) *There exists a solution $S(t) \in \mathcal{C}((0, \infty), H^1)$ to (1.1) with minimal mass $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ which blows up backwards at the origin:*

$$S(t, x) - \frac{1}{t^{1/2}} Q\left(\frac{x + 1/t + \bar{c}t}{t}\right) \rightarrow 0 \quad \text{in } L^2 \text{ as } t \downarrow 0$$

with speed

$$\|S(t)\|_{H^1} \sim C^*/t \quad \text{as } t \downarrow 0 \quad (1.8)$$

for some universal constants \bar{c}, C^ . Moreover, S is smooth and well-localized to the right in space:*

$$\forall x \geq 1, \quad |S(1, x)| \leq e^{-Cx}. \quad (1.9)$$

- (ii) (Uniqueness) *Let $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2}$ and assume that the corresponding solution $u(t)$ to (1.1) blows up in finite time. Then $u \equiv S$ up to the symmetries of the flow.*

Observe that the minimal element blows up with speed (1.8), which is the same as in the (Blow up) regime obtained in Theorem 1.1. However, the case of (Blow up) in Theorem 1.1 is shown to be stable under small perturbation in \mathcal{A} , while minimal mass blow up is unstable under perturbation of the data $S(0) \rightarrow (1 - \varepsilon)S(0)$, $\varepsilon > 0$, since the corresponding solution has subcritical mass and is thus global in time. This shows that the decay assumption to the right in Theorem 1.1 is essential and that the minimal blow up solution has slow decay to the left.¹ The nature of the minimal blow up is different from that of the stable blow up. We also refer to Part III [30] for examples of exotic blow up rates for initial data with slow decay.

We now relate the (Exit) case in Theorem 1.1 to the minimal mass blow up dynamics. We claim that at the (Exit) time, the solution is L^2 close up to renormalization to the unique minimal solution $S(t)$.

Theorem 1.4 (Description of the (Exit) scenario). *Let $u(t, x)$ be a solution of (1.1) corresponding to the (Exit) scenario in Theorem 1.1 and let $t_u^* \gg 1$ be the corresponding exit time. Then there exist $\sigma^* = \sigma^*(\alpha^*)$ (independent of u) and (λ_u^*, x_u^*) such that*

$$\|(\lambda_u^*)^{1/2} u(t_u^*, \lambda_u^* x + x_u^*) - S(\sigma^*, x)\|_{L^2} \leq \delta_I(\alpha_0),$$

where $\delta_I(\alpha_0) \rightarrow 0$ as $\alpha_0 \rightarrow 0$.

Note that uniqueness in Theorem 1.3 is an essential ingredient of the proof. In view of the universality of S as attractor to all defocusing solutions, and in continuation of Theorem 1.3, it is an important open problem to understand the behavior of $S(t)$ as $t \rightarrow \infty$. For the mass critical (NLS), the explicit formula (1.6) ensures that S_{NLS} scatters as $t \rightarrow \infty$, and hence it is a connection from ∞ to 0. For (gKdV), the decay in space (1.9) of $S(t, x)$ on the left, combined with Theorem 1.2, ensures that $S(t)$ is globally defined for $t > 0$, but scattering as $t \rightarrow \infty$ is an open problem.² We conjecture that $S(t)$ actually scatters, and because scattering is a property open in L^2 [16], we obtain the corollary:

Corollary 1.5. *Assume that $S(t)$ scatters as $t \rightarrow \infty$. Then any solution in the (Exit) scenario is global for positive time and scatters as $t \rightarrow \infty$.*

Related rigidity theorems near the solitary wave were recently obtained by Nakanishi and Schlag [42], [43] for supercritical wave and Schrödinger equations using the invariant set methods of Berestycki and Cazenave [2], the Kenig–Merle concentration-compactness approach [14], the classification of minimal dynamics [10]–[12] and a further “no return” lemma in the (Exit) regime. This approach relies on the virial algebra which is not known for (gKdV).

We expect the strategy of the proof of Theorem 1.4, reducing the dynamics of defocusing solutions to the sole description of the minimal mass solution, to be quite general.

¹ Remember that it blows up backwards in time.

² By *scattering* for (gKdV), we mean that there exists a solution $v(t, x)$ to the Airy equation $\partial_t v + v_{xxx} = 0$ such that $\lim_{t \rightarrow \infty} \|S(t) - v(t)\|_{L^2} = 0$.

Notation. We introduce the generator of L^2 scaling:

$$\Lambda f = \frac{1}{2}f + yf'.$$

We use the L^2 scalar product

$$(f, g) = \int_{\mathbb{R}} f(x)g(x) dx.$$

Let the linearized operator close to Q be

$$Lf = -f'' + f - 5Q^4 f. \tag{1.10}$$

For a given generic small constant $0 < \alpha^* \ll 1$, $\delta(\alpha^*)$ denotes a generic positive small constant with

$$\delta(\alpha^*) \rightarrow 0 \quad \text{as } \alpha^* \rightarrow 0.$$

Given an interval I of \mathbb{R} , we let $\mathbf{1}_I$ denote the characteristic function of I .

1.5. Strategy of the proof. Let us give a brief insight into the strategy of the proof of Theorems 1.3 and 1.4.

Step 1. Modified blow up profiles. We construct the minimal element using a variation of the compactness argument used for the construction of nondispersive objects in [31], [22], [6], [48]. Near blow up time, this solution will admit a decomposition

$$u(t, x) = \frac{1}{\lambda^{1/2}(t)}(Q_{b(t)} + \varepsilon)(s, y) \quad \text{with} \quad \frac{ds}{dt} = \frac{1}{\lambda^3}, \quad y = \frac{x - x(t)}{\lambda(t)},$$

$$\varepsilon(t) \rightarrow 0 \text{ in } H^1 \text{ as } t \downarrow 0.$$

Here Q_b is the slow modulated deformation of the ground state constructed in [29] which formally leads to the dynamical system

$$\begin{cases} b_s + 2b^2 = 0, \\ -\lambda_s/\lambda = b, \end{cases} \quad \text{i.e.} \quad \begin{cases} \lambda(t) = \ell^* t, \\ b(t) = -\ell^* \lambda^2(t), \end{cases}$$

and hence the blow up speed (1.8).

Step 2. The formal argument. Following [31], [48], we could build the minimal element by considering the solution $u_n(t)$ to (gKdV) with data

$$u(t_n) = \frac{1}{\lambda^{1/2}(t_n)} Q_{b(t_n)} \left(\frac{x - x(t_n)}{\lambda(t_n)} \right) \quad \text{with} \quad \lambda(t_n) = \ell^* t_n, \quad b(t_n) = -\ell^* \lambda^2(t_n)$$

and show that there exists a time $t_0 > 0$ independent of n such that

$$\|u_n(t_0)\|_{H^1} \lesssim 1 \quad \text{as } t_n \downarrow 0.$$

Such an estimate is at the heart of the proof and would be a consequence of the fine monotonicity properties exhibited in [29]. Passing to the limit $t_n \rightarrow 0$ automatically produces the expected blow up element.

We will argue slightly differently and propose a scheme adapted to the proof of both Theorems 1.3 and 1.4 and which as in [40] illustrates the fact that the minimal element can be obtained as the limit of a sequence of *defocusing* solutions. Indeed, we pick a sequence of *well-prepared* initial data

$$u_n(0) = Q_{b_n(0)}, \quad b_n(0) = -1/n,$$

which by construction have subcritical mass

$$\|u_n(0)\|_{L^2} = \|Q\|_{L^2} - c/n + o(1/n).$$

Such solutions are automatically in the (Exit) regime of Theorem 1.1. Moreover, we have from [29] a *complete description of the flow* for $t \in [0, t_n^*]$: the solution admits a decomposition

$$u_n(t, x) = \frac{1}{\lambda_n^{1/2}(t)} (Q_{b_n(t)} + \varepsilon_n) \left(t, \frac{x - x_n(t)}{\lambda_n(t)} \right) \quad (1.11)$$

where to leading order the modulation equations for (b_n, λ_n) are given by

$$b_n(t)/\lambda_n^2(t) \sim b_n(0) = -1/n, \quad (\lambda_n)_t \sim -b_n(0),$$

i.e.

$$\lambda_n(t) \sim 1 - b_n(0)t, \quad b_n(t) \sim b_n(0)\lambda_n^2(t). \quad (1.12)$$

The (Exit) time t_n^* is the one at which the solution moves strictly away from the solitary wave, which in our setting is equivalent to

$$b_n(t_n^*) = -\alpha^*$$

independent of n . This in particular allows us to compute t_n^* and show using (1.12) that the solution defocuses:

$$\lambda_n^2(t_n^*) \sim b_n(t_n^*)/b_n(0) \sim n\alpha^* \quad \text{as } n \rightarrow \infty.$$

We therefore renormalize the flow at t_n^* and consider the solution to (gKdV) with data at t_n^* given by the renormalized u_n at t_n^* , explicitly

$$v_n(\tau, x) = \lambda_n^{1/2}(t_n^*) u_n(t_\tau, \lambda_n(t_n^*)x + x_n(t_n^*)), \quad t_\tau = t_n^* + \tau \lambda_n^3(t_n^*).$$

Then by a direct check, v_n admits a decomposition

$$v_n(\tau, x) = \frac{1}{\lambda_{v_n}^{1/2}(\tau)} (Q_{b_{v_n}} + \varepsilon_{v_n}) \left(\tau, \frac{x - x_{v_n}(\tau)}{\lambda_{v_n}(\tau)} \right)$$

with, from the symmetries of the flow,

$$\begin{aligned} \lambda_{v_n}(\tau) &= \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)}, & x_{v_n}(\tau) &= \frac{x_n(t_\tau) - x_n(t_n^*)}{\lambda_n(t_n^*)}, \\ b_{v_n}(\tau) &= b_n(t_\tau), & \varepsilon_{v_n}(\tau) &= \varepsilon_n(t_\tau). \end{aligned}$$

The renormalized parameters can be computed approximately using (1.12):

$$\begin{aligned}\lambda_{v_n}(\tau) &\sim \frac{\lambda_n(t_n^* + \tau\lambda_n^3(t_n^*))}{\lambda_n(t_n^*)} \sim \frac{1}{\lambda_n(t_n^*)} [1 - b_n(0)(t_n^* + \tau\lambda_n^3(t_n^*))] \\ &\sim \frac{1}{\lambda_n(t_n^*)} [\lambda_n(t_n^*) - \tau b_n(0)\lambda_n^3(t_n^*)] = 1 - \tau b_n(0)\lambda_n^2(t_n^*) \\ &= 1 - \tau b_n(t_n^*) = 1 + \tau\alpha^*.\end{aligned}$$

Letting $n \rightarrow \infty$, we therefore expect to extract a weak limit $v_n(0) \rightharpoonup v(0)$ such that the corresponding solution $v(\tau)$ to (gKdV) has minimal mass $\|v(0)\|_{L^2} = \|Q\|_{L^2}$ and blows up backwards at some finite time $\tau^* \sim -1/\alpha^*$ with blow up speed $\lambda_v(\tau) \sim \tau - \tau^*$, i.e. (1.8).

The extraction of the weak limit now requires sharp control on the remaining radiation ε_{v_n} . Here an essential use is made of the fact that the set of data $u_n(0)$ is *well-prepared* as this induces uniform bounds for $\varepsilon_{v_n}(0) = \varepsilon_{u_n}(t_n^*)$ in H^1 and allows us to use the H^1 weak continuity of the flow in the limiting process.

Step 3. Solutions in the (Exit) regime. Theorem 1.4 follows similarly by considering sequences $(u_0)_n$ of data with $\|(u_0)_n\|_{L^2} \rightarrow \|Q\|_{L^2}$ such that the corresponding solution to (gKdV) is in the (Exit) regime. We write explicitly the solution at the (Exit) time in the form (1.11), renormalize the flow and now aim at extracting a weak limit as $n \rightarrow \infty$. The architecture of the proof is similar, except that we have lost the fact that the data is well-prepared, which destroys the uniform H^1 bound on $v_n(0)$. We therefore use two new tools: a concentration-compactness argument on sequences of solutions in the critical L^2 space in the spirit of [14] using the tools developed in [17], which allows us to extract a nontrivial weak limit with suitable dynamical controls; and refined local H^1 bounds on $v_n(\tau)$ in order to ensure that the L^2 limit is in fact also in H^1 . Hence the weak limit is a minimal mass H^1 blow up element.

Step 4. Uniqueness. It remains to prove the *uniqueness* in H^1 of the minimal element. This is a delicate problem and here we adapt the direct dynamical approach developed in [48]. The first step is to show that any H^1 minimal blow up element blows up with the blow up speed (1.8). Here the proof relies first on exponential decay estimates of minimal elements proved in [25], which allow us once again to apply the monotonicity machinery developed in [29]. Once the blow up speed is known, one may integrate the flow backwards from the singularity and show that the blow up element is close in a strong sense to the $S(t)$ minimal element previously constructed. It remains to show that the difference is exactly zero. This requires revisiting the monotonicity properties for the difference of two such solutions, and showing that the previously obtained a priori bound on the solution implies that the difference is exactly zero.³ Let us emphasize that as in [48], [40], we are forced to work with a finite order approximation of the solution,⁴ and therefore this step is always delicate.

³ This equivalently means that the integration of the flow from blow up time defining the minimal blow up element is a contraction mapping in a suitable function space.

⁴ And not arbitrarily degenerate as in [3] for example.

2. Nonlinear profiles and decomposition close to the soliton

In this section we collect a number of tools which can be explicitly found in the literature and which we will use in the proof of the main results. We start by recalling the status of scattering theory and profile decomposition in the critical L^2 space for (gKdV). We then recall the nonlinear decomposition of the flow for data near the ground state, and the main monotonicity formula at the heart of the analysis in [29], which will again play a distinguished role in our analysis.

2.1. Cauchy problem and scattering from [15]. We use the notion of strong solution in the sense of Kenig, Ponce and Vega [15]. For $u_0 \in L^2$, we denote by $v(t) = W(t)u_0$ the solution of the Airy equation $v_t + v_{xxx} = 0$ with $v(0) = v_0$. The following space-time Strichartz type estimate proved in [15] is essential in the resolution of the Cauchy problem for (1.1) in L^2 and H^1 :

$$\|W(t)v_0\|_{L_x^5 L_t^{10}} \lesssim \|v_0\|_{L^2}. \quad (2.1)$$

We recall the following classical results.

Theorem 2.1 (Kenig, Ponce, Vega [15]).

- (i) (L^2 theory) *The Cauchy problem (1.1) is locally well-posed in L^2 : for all $u_0 \in L^2$, there exists a unique L^2 solution of (1.1) defined on a maximal interval of existence $[0, T)$. There is continuous dependence on the data in L^2 , and the blow up alternative holds:*

$$T < \infty \quad \text{implies} \quad \|u\|_{L_x^5 L_T^{10}} = \infty.$$

Moreover, there exists $\delta > 0$ such that $\|u_0\|_{L^2} < \delta$ implies that the solution is global with $\|u\|_{L_x^5 L_\infty^{10}} < \infty$.

- (ii) (H^1 theory) *The Cauchy problem (1.1) is locally well-posed in H^1 : for all $u_0 \in H^1$, there exists a unique H^1 solution of (1.1) defined on a maximal interval of existence $[0, T)$. There is continuous dependence on the data in H^1 , and the blow up alternative holds:*

$$T < \infty \quad \text{implies} \quad \lim_{t \uparrow T} \|\partial_x u\|_{L^2} = \infty.$$

- (iii) (Scattering and stability of scattering) *Let $u(t)$ be a global L^2 solution of (1.1). If $\|u\|_{L_x^5 L_\infty^{10}} < \infty$, then the solution $u(t)$ scatters at ∞ , i.e. there exists $v_0^+ \in L^2$ such that*

$$\lim_{t \rightarrow \infty} \|u(t) - W(t)v_0^+\|_{L^2} = 0.$$

The set $\mathcal{S} = \{u_0 \in L^2 : u(t) \text{ is global and scatters at } \infty\}$ is open in L^2 .

Point (iii) of Theorem 2.1 follows from [15] and standard arguments (see e.g. [14] for similar arguments in the case of nonlinear Schrödinger equation), and means that scattering is a stable regime without any assumption on the size of the solution.

We now recall the known results from [49] on profile decomposition in the critical space of sequences of solutions to the Airy equation which describes the lack of compactness of the Strichartz estimate (2.1). For any $x_0 \in \mathbb{R}$ and $\lambda > 0$, define an operator $g_{x_0, \lambda} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by

$$[g_{x_0, \lambda} f](x) := \lambda^{-1/2} f(\lambda^{-1}(x - x_0)).$$

Definition 2.2. For $j \neq k$, $\Gamma_n^j = (\lambda_n^j, \xi_n^j, x_n^j, t_n^j)_{n \geq 1}$ and $\Gamma_n^k = (\lambda_n^k, \xi_n^k, x_n^k, t_n^k)_{n \geq 1}$ in $(0, \infty) \times \mathbb{R}^3$ are *orthogonal* if one of the following holds:

- $\lim_{n \rightarrow \infty} \left(\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \lambda_n^j |\xi_n^j - \xi_n^k| \right) = \infty$;
- $(\lambda_n^j, \xi_n^j) = (\lambda_n^k, \xi_n^k)$ and

$$\lim_{n \rightarrow \infty} \left(\frac{|t_n^k - t_n^j|}{(\lambda_n^j)^3} + \frac{3|(t_n^k - t_n^j)\xi_n^j|}{(\lambda_n^j)^2} + \frac{|x_n^j - x_n^k + 3(t_n^j - t_n^k)(\xi_n^j)^2|}{\lambda_n^j} \right) = \infty.$$

Lemma 2.3 (Profile decomposition, [49]). *Let $\{u_n\}_{n \geq 1}$ be a sequence of real-valued functions bounded in L^2 . Then, after passing to a subsequence if necessary, there exist (complex) L^2 functions $\{\phi^j\}_{j \geq 1}$ and a family of orthogonal sequences $\Gamma_n^j = (\lambda_n^j, \xi_n^j, x_n^j, t_n^j) \in (0, \infty) \times \mathbb{R}^3$ such that for all $J \geq 1$,*

$$u_n = \sum_{1 \leq j \leq J} e^{-t_n^j \partial_x^3} (g_{x_n^j, \lambda_n^j} [\operatorname{Re}(e^{ix \xi_n^j \lambda_n^j} \phi^j)]) + w_n^J, \tag{2.2}$$

where the ξ_n^j satisfy the following property: for any $1 \leq j \leq J$, either $\xi_n^j = 0$ for all $n \geq 1$, or $\xi_n^j \lambda_n^j \rightarrow \infty$ as $n \rightarrow \infty$. Here, $w_n^J \in L^2(\mathbb{R})$ is real-valued and

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\{ \|\partial_x |^{1/6} e^{-t \partial_x^3} w_n^J\|_{L_{t,x}^6(\mathbb{R} \times \mathbb{R})} + \|e^{-t \partial_x^3} w_n^J\|_{L_x^5 L_t^{10}(\mathbb{R} \times \mathbb{R})} \right\} = 0. \tag{2.3}$$

Moreover, for any $J \geq 1$,

$$\lim_{n \rightarrow \infty} \left\{ \|u_n\|_{L^2}^2 - \sum_{1 \leq j \leq J} \|\operatorname{Re}[e^{ix \xi_n^j \lambda_n^j} \phi^j]\|_{L^2}^2 - \|w_n^J\|_{L^2}^2 \right\} = 0. \tag{2.4}$$

Using this lemma for the study of the nonlinear flow (1.1) requires a suitable perturbation theory:

Lemma 2.4 (L^2 perturbation theory, [17]). *Let I be an interval of \mathbb{R} with $0 \in I$, and let \tilde{u} be an L^2 solution of*

$$u_t + (u_{xx} + u^5)_x = e_x$$

on $I \times \mathbb{R}$ for some function e . Assume that

$$\|\tilde{u}\|_{L_t^\infty L_x^2(I \times \mathbb{R})} + \|\tilde{u}\|_{L_x^5 L_t^{10}(I \times \mathbb{R})} \leq M$$

for some $M > 0$. Let $u(0) \in L^2$ be such that

$$\begin{aligned} \|u(0) - \tilde{u}(0)\|_{L^2} &\leq M', \\ \|e^{-t\partial^3}(u(0) - \tilde{u}(0))\|_{L_x^5 L_t^{10}(I \times \mathbb{R})} + \|e\|_{L_x^1 L_t^2(I \times \mathbb{R})} &\leq \epsilon, \end{aligned}$$

for $M' > 0$ and for some small $0 < \epsilon < \epsilon_0(M, M')$. Then the solution $u(t)$ of (1.1) corresponding to $u(t_0)$ is defined on I and we have the bound

$$\|u - \tilde{u}\|_{L_x^5 L_t^{10}(I \times \mathbb{R})} + \|u - \tilde{u}\|_{L_t^\infty L_x^2(I \times \mathbb{R})} \leq C(M, M')\epsilon. \tag{2.5}$$

2.2. Approximate self-similar profiles. We recall the existence of suitable approximate self-similar solutions which give the leading order profile of solutions with data near Q . The specific structure of these profiles drives both the blow speed in the (Blow up) regime and the speed of defocusing in the (Exit) regime. Let \mathcal{Y} be the set of functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\forall k = 0, 1, 2, \dots, \exists C_k, r_k > 0, \forall y \in \mathbb{R}, \quad |f^{(k)}(y)| \leq C_k(1 + |y|)^{r_k} e^{-|y|}. \tag{2.6}$$

Let $\chi \in C^\infty(\mathbb{R})$ be such that $0 \leq \chi \leq 1$, $\chi' \geq 0$ on \mathbb{R} , $\chi \equiv 1$ on $[-1, \infty)$, $\chi \equiv 0$ on $(-\infty, -2]$. Define

$$\chi_b(y) = \chi(|b|^\gamma y), \quad \gamma = 3/4. \tag{2.7}$$

Lemma 2.5 (Approximate self-similar profiles Q_b , [29]). *There exists a unique smooth function P such that $P' \in \mathcal{Y}$ and*

$$(LP)' = \Lambda Q, \quad \lim_{y \rightarrow -\infty} P(y) = \frac{1}{2} \int Q, \quad \lim_{y \rightarrow \infty} P(y) = 0, \tag{2.8}$$

$$(P, Q) = \frac{1}{16} \left(\int Q \right)^2 > 0, \quad (P, Q') = 0. \tag{2.9}$$

Moreover, the localized approximate profile

$$Q_b(y) = Q(y) + b\chi_b(y)P(y) \tag{2.10}$$

satisfies:

(i) (Estimates on Q_b) For all $y \in \mathbb{R}$,

$$|Q_b(y)| \lesssim e^{-|y|} + |b|(\mathbf{1}_{[-2,0]}(|b|^\gamma y) + e^{-|y|/2}), \tag{2.11}$$

$$|Q_b^{(k)}(y)| \lesssim e^{-|y|} + |b|e^{-|y|/2} + |b|^{1+k\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y) \quad \text{for } k \geq 1. \tag{2.12}$$

(ii) (Equation of Q_b) Let

$$-\Psi_b = (Q_b'' - Q_b + Q_b^5)' + b\Lambda Q_b. \tag{2.13}$$

Then, for all $y \in \mathbb{R}$,

$$|\Psi_b(y)| \lesssim |b|^{1+\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y) + b^2(e^{-|y|/2} + \mathbf{1}_{[-2,0]}(|b|^\gamma y)), \tag{2.14}$$

$$|\Psi_b^{(k)}(y)| \lesssim |b|^{1+(k+1)\gamma} \mathbf{1}_{[-2,-1]}(|b|^\gamma y) + b^2 e^{-|y|/2} \quad \text{for } k \geq 1. \tag{2.15}$$

(iii) (Mass and energy properties of Q_b)

$$\left| \int Q_b^2 - \left(\int Q^2 + 2b \int PQ \right) \right| \lesssim |b|^{2-\gamma}, \tag{2.16}$$

$$\left| E(Q_b) + b \int PQ \right| \lesssim b^2. \tag{2.17}$$

2.3. Geometrical decomposition of the flow. Let $u \in \mathcal{C}([0, t_0], H^1)$ be a solution of (1.1) close in L^2 to the manifold of solitary waves, i.e. we assume that there exist $(\lambda_1(t), x_1(t)) \in \mathbb{R}_+^* \times \mathbb{R}$ and $\varepsilon_1(t)$ such that

$$\forall t \in [0, t_0], \quad u(t, x) = \frac{1}{\lambda_1^{1/2}(t)} (Q + \varepsilon_1) \left(t, \frac{x - x_1(t)}{\lambda_1(t)} \right), \tag{2.18}$$

$$\forall t \in [0, t_0], \quad \|\varepsilon_1(t)\|_{L^2} + \left(\int (\partial_y \varepsilon_1)^2 e^{-|y|/2} dy \right)^{1/2} \leq \kappa_0, \tag{2.19}$$

for some small enough universal constant $\kappa_0 > 0$. This decomposition is refined using the Q_b profiles and a standard modulation argument.

Lemma 2.6 (Decomposition and H^1 properties, [29]). *Assume (2.19).*

(i) (Decomposition) *There exist \mathcal{C}^1 functions $(\lambda, x, b) : [0, t_0] \rightarrow (0, \infty) \times \mathbb{R}^2$ such that*

$$\forall t \in [0, t_0], \quad \varepsilon(t, y) = \lambda^{1/2}(t) u(t, \lambda(t)y + x(t)) - Q_{b(t)}(y) \tag{2.20}$$

satisfies the orthogonality conditions

$$(\varepsilon(t), y\Lambda Q) = (\varepsilon(t), \Lambda Q) = (\varepsilon(t), Q) = 0, \tag{2.21}$$

and

$$\|\varepsilon(t)\|_{L^2} + |b(t)| \lesssim \delta(\kappa_0), \quad \|\varepsilon(t)\|_{H^1} \lesssim \delta(\|\varepsilon_1(t)\|_{H^1}). \tag{2.22}$$

(ii) (Equation of ε) *Let*

$$s = \int_0^t \frac{dt'}{\lambda^3(t')} \quad \text{and} \quad s_0 = s(t_0). \tag{2.23}$$

For all $s \in [0, s_0]$,

$$\begin{aligned} \varepsilon_s - (L\varepsilon)_y + b\Lambda\varepsilon &= \left(\frac{\lambda_s}{\lambda} + b \right) (\Lambda Q_b + \Lambda\varepsilon) + \left(\frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y \\ &\quad + \Phi_b + \Psi_b - (R_b(\varepsilon))_y - (R_{NL}(\varepsilon))_y, \end{aligned} \tag{2.24}$$

where Ψ_b is defined in (2.13) and

$$\Phi_b = -b_s(\chi_b + \gamma y(\chi_b)_y)P, \tag{2.25}$$

$$R_b(\varepsilon) = 5(Q_b^4 - Q^4)\varepsilon, \quad R_{NL}(\varepsilon) = (\varepsilon + Q_b)^5 - 5Q_b^4\varepsilon - Q_b^5. \tag{2.26}$$

(iii) (Estimates induced by the conservation laws) *On* $[0, s_0]$,

$$\|\varepsilon(s)\|_{L^2}^2 \lesssim |b(s)| + \left| \int u_0^2 - \int Q^2 \right|, \quad (2.27)$$

$$\left| 2\lambda^2(s)E_0 + \frac{1}{8}b(s)\|Q\|_{L^1}^2 - \|\varepsilon_y(s)\|_{L^2}^2 \right| \lesssim b^2(s) + \|\varepsilon(s)\|_{L^2}^2 + \delta(\|\varepsilon\|_{L^2})\|\varepsilon_y(s)\|_{L^2}^2. \quad (2.28)$$

(iv) (Rough modulation equations) *On* $[0, s_0]$,

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + b^2, \quad (2.29)$$

$$|b_s + 2b^2| \lesssim |b| \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + |b|^3 + \int \varepsilon^2 e^{-|y|/10}. \quad (2.30)$$

(v) (Minimal mass) *If in addition* $\|u(t)\|_{L^2} = \|Q\|_{L^2}$ *then* $E_0 = E(u_0) \geq 0$ *and on* $[0, s_0]$,

$$b(s) \leq 0, \quad E_0 \lambda^2(s) \lesssim |b(s)| + \|\varepsilon(s)\|_{H^1}^2 \lesssim E_0 \lambda^2(s). \quad (2.31)$$

The proof of Lemma 2.6 is given in [29], except (2.31) which we prove now.

Proof of (2.31). Using the decomposition (2.20), one has

$$\int u^2 = \int Q_b^2 + \int \varepsilon^2 + 2 \int \varepsilon Q_b.$$

Since $(\varepsilon, Q) = 0$ and $\chi_b(y) = \chi(|b|^{3/4}y)$,

$$|(\varepsilon, Q_b)| = |b| |(\varepsilon, P\chi_b)| \lesssim |b|^{5/8} \left(\int \varepsilon^2 \right)^{1/2} \lesssim |b|^{1/8} \int \varepsilon^2 + |b|^{9/8}. \quad (2.32)$$

Moreover, by (2.16),

$$\int Q_b^2 = \int Q^2 + 2b \int PQ + O(|b|^{5/4}).$$

Thus, we obtain in general

$$\int u^2 = \int Q^2 + 2b \int PQ + \int \varepsilon^2 + O(|b|^{1/8}) \left(|b| + \int \varepsilon^2 \right). \quad (2.33)$$

In particular, using the minimal mass assumption $\int u^2 = \int Q^2$, we obtain

$$2b(P, Q) + \int \varepsilon^2 = \delta(\kappa_0) \left(|b| + \int \varepsilon^2 \right), \quad (2.34)$$

which implies $b \leq 0$. Now we write the conservation of energy using $(\varepsilon, Q) = 0$ and (2.17):

$$\begin{aligned} 2\lambda^2 E_0 &= 2E(Q_b) - 2 \int \varepsilon(Q_b)_{yy} + \int \varepsilon_y^2 - \frac{1}{3} \int [(Q_b + \varepsilon)^6 - Q_b^6] \\ &= -2b \int PQ + O(b^2) + \int \varepsilon_y^2 - 5 \int Q^4 \varepsilon^2 - \frac{1}{3} \int \varepsilon^6 \\ &\quad - 2 \int \varepsilon[(Q_b - Q)_{yy} + (Q_b^5 - Q^5)] + 5 \int (Q^4 - Q_b^4) \varepsilon^2 \\ &\quad - \frac{1}{3} \int [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon - 15Q_b^4 \varepsilon^2 - \varepsilon^6]. \end{aligned}$$

We estimate the nonlinear terms using the Sobolev bound $\|\varepsilon\|_{L^\infty} \lesssim \|\varepsilon_y\|_{L^2}^{1/2} \|\varepsilon\|_{L^2}^{1/2}$, and thus

$$2\lambda^2 E_0 = -2b \int PQ + \int \varepsilon_y^2 - 5 \int Q^4 \varepsilon^2 + O(|b|^{1/8} + \|\varepsilon\|_{H^1}) \left(|b| + \int \varepsilon^2 \right). \tag{2.35}$$

Combining with (2.34), we obtain

$$2\lambda^2 E_0 = (L\varepsilon, \varepsilon) + \delta(\kappa_0)(|b| + \|\varepsilon\|_{H^1}^2).$$

The choice of orthogonality conditions on ε ensures (see [29, Lemma 2.1]) the coercivity of the linearized energy, i.e. $(L\varepsilon, \varepsilon) \gtrsim \|\varepsilon\|_{H^1}^2$, and thus $\|\varepsilon\|_{H^1}^2 \lesssim \lambda^2 E_0 + \delta(\kappa_0)|b|$, which combined with (2.34) implies (2.31). \square

The modulation equations can be sharpened under an additional L^1 control of the solution.

Lemma 2.7 (Refined laws for H^1 solution with decay, [29]). *Under the assumptions of Lemma 2.6, assume moreover the uniform L^1 control on the right:*

$$\forall t \in [0, t_0], \quad \int_{y>0} |\varepsilon(t)| \lesssim \delta(\kappa_0). \tag{2.36}$$

Then the quantities J_1 and J_2 below are well-defined and satisfy on $[0, t_0]$:

- (Law of λ) Let

$$\rho_1(y) = \frac{4}{(\int Q)^2} \int_{-\infty}^y \Lambda Q, \quad J_1(s) = (\varepsilon(s), \rho_1). \tag{2.37}$$

Then for some universal constant c_1 ,

$$\left| \frac{\lambda_s}{\lambda} + b + c_1 b^2 - 2 \left((J_1)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_1 \right) \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + |b| \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2}. \tag{2.38}$$

- (Law of b) *Let*

$$\rho_2 = \frac{16}{(\int Q)^2} \left(\frac{(\Delta P, Q)}{\|\Delta Q\|_{L^2}^2} \Lambda Q + P - \frac{1}{2} \int Q \right) - 8\rho_1, \quad J_2(s) = (\varepsilon(s), \rho_2). \quad (2.39)$$

Then for some universal constant c_2 ,

$$\left| b_s + 2b^2 + c_2b^3 + b \left((J_2)_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J_2 \right) \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + |b|^4. \quad (2.40)$$

- (Law of b/λ^2) *Let*

$$\rho = 4\rho_1 + \rho_2, \quad J = (\varepsilon, \rho).$$

Then $\rho \in \mathcal{Y}$ so that $|J| \lesssim (\int \varepsilon^2 e^{-|y|/10})^{1/2}$ and for $c_0 = c_2 - 2c_1$,

$$\left| \frac{d}{ds} \left(\frac{b}{\lambda^2} \right) + \frac{b}{\lambda^2} \left(J_s + \frac{1}{2} \frac{\lambda_s}{\lambda} J \right) + c_0 \frac{b^3}{\lambda^2} \right| \lesssim \frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-|y|/10} + |b|^4 \right). \quad (2.41)$$

2.4. Weak H^1 stability of the decomposition. The geometrical decomposition of Lemma 2.6 is stable under weak H^1 limits.

Lemma 2.8 (Weak H^1 stability and convergence of the parameters, [23]). *Let $u_n(0)$ be a sequence of H^1 initial data such that*

$$u_n(0) \rightharpoonup u(0) \in H^1 \quad \text{as } n \rightarrow \infty.$$

Assume that for some $T_1 > 0$, for all n , the corresponding solution u_n of (1.1) exists and satisfies (2.19) on $[0, T_1]$. Assume further that the decomposition (λ_n, x_n, b_n) of u_n given by Lemma 2.6 satisfies

$$\forall t \in [0, T_1], \quad 0 < c \leq \lambda_n(t) < C, \quad \lambda_n(0) = 1, \quad x_n(0) = 0. \quad (2.42)$$

Then the H^1 solution $u(t)$ of (1.1) corresponding to $u(0)$ exists on $[0, T_1]$, satisfies (2.19) and its decomposition satisfies

$$\forall t \in [0, T_1], \quad \varepsilon_n(t) \rightharpoonup \varepsilon(t), \quad \lambda_n(t) \rightarrow \lambda(t), \quad x_n(t) \rightarrow x(t), \quad b_n(t) \rightarrow b(t). \quad (2.43)$$

This lemma is similar to a result proved in [23, Lemma 17 and Appendix D], and therefore we omit its proof.

2.5. Main monotonicity functionals from [29]. We now recall the monotonicity formula at the heart of the analysis in [29] and on which we shall heavily rely again. We refer to [29] for a further introduction to the nature of these functionals and the associated rigidity of the flow implied by (2.57).

Let $\varphi, \psi \in C^\infty(\mathbb{R})$ be such that

$$\varphi(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1 + y & \text{for } -1/2 < y < 1/2, \\ y^2 & \text{for } y > 2, \end{cases} \quad \varphi'(y) > 0, \quad \forall y \in \mathbb{R}, \quad (2.44)$$

$$\psi(y) = \begin{cases} e^{2y} & \text{for } y < -1, \\ 1 & \text{for } y > -1/2, \end{cases} \quad \psi'(y) \geq 0, \quad \forall y \in \mathbb{R}. \quad (2.45)$$

For $B \geq 100$ to be fixed, let

$$\psi_B(y) = \psi(y/B), \quad \varphi_B = \varphi(y/B),$$

and define

$$\mathcal{N}(s) = \int \varepsilon_y^2(s, y) \psi_B(y) dy + \int \varepsilon^2(s, y) \varphi_B(y) dy. \quad (2.46)$$

Proposition 2.9 (Monotonicity formula, [29]). *There exist $\mu > 0$ and $0 < \kappa^* < \kappa_0$ such that the following holds for $B > 100$ large enough. Assume that $u(t)$ is a solution of (1.1) which satisfies (2.19) on $[0, t_0]$ and thus admits on $[0, t_0]$ a decomposition (2.20) as in Lemma 2.6. Let $s_0 = s(t_0)$, and assume the following a priori bounds, for all $s \in [0, s_0]$:*

(H1) (Smallness)

$$\|\varepsilon(s)\|_{L^2} + |b(s)| + \mathcal{N}(s) \leq \kappa^*. \quad (2.47)$$

(H2) (Comparison between b and λ)

$$\frac{|b(s)| + \mathcal{N}(s)}{\lambda^2(s)} \leq \kappa^*. \quad (2.48)$$

(H3) (L^2 weighted bound on the right)

$$\int_{y>0} y^{10} \varepsilon^2(s, x) dx \leq 10 \left(1 + \frac{1}{\lambda^{10}(s)} \right). \quad (2.49)$$

For $j \in \{1, 2\}$ let

$$\mathcal{F}_j = \int [\varepsilon_y^2 \psi_B + \varepsilon^2(1 + \mathcal{J}_j) \varphi_B - \frac{1}{3}((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \psi_B] \quad (2.50)$$

with

$$\mathcal{J}_j = (1 - J_1)^{-4j} - 1. \quad (2.51)$$

Then the following bounds hold on $[0, s_0]$:

(i) (Scaling invariant Lyapunov control)

$$\frac{d\mathcal{F}_1}{ds} + \mu \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \lesssim |b|^4. \quad (2.52)$$

(ii) (Scaling weighted H^1 Lyapunov control)

$$\frac{d}{ds} \left\{ \frac{\mathcal{F}_2}{\lambda^2} \right\} + \frac{\mu}{\lambda^2} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \lesssim \frac{|b|^4}{\lambda^2}. \tag{2.53}$$

(iii) (Pointwise bounds)

$$|J_1| + |J_2| \lesssim \mathcal{N}^{1/2}, \tag{2.54}$$

$$\mathcal{N} \lesssim \mathcal{F}_j \lesssim \mathcal{N}, \quad j = 1, 2. \tag{2.55}$$

The integration of the modulation equations of Lemma 2.7 with the dispersive bounds of Proposition 2.9 implies the control of the flow by the sole parameter b :

Lemma 2.10 (Control of the flow by b , [29]). *Under the assumptions of Proposition 2.9, the following hold:*

(i) (Control of the dynamics for b) For all $0 \leq s_1 \leq s_2 < s_0$,

$$\int_{s_1}^{s_2} b^2(s) ds \lesssim \int (\varepsilon_y^2(s_1) \psi_B + \varepsilon^2(s_1) \varphi'_B) + |b(s_2)| + |b(s_1)|, \tag{2.56}$$

$$\left| \frac{b(s_2)}{\lambda^2(s_2)} - \frac{b(s_1)}{\lambda^2(s_1)} \right| \leq \frac{C^*}{10} \left[\frac{b^2(s_1)}{\lambda^2(s_1)} + \frac{b^2(s_2)}{\lambda^2(s_2)} + \frac{1}{\lambda^2(s_1)} \int (\varepsilon_y^2(s_1) \psi_B + \varepsilon^2(s_1) \varphi'_B) \right], \tag{2.57}$$

for some universal constant $C^* > 0$.

(ii) (Control of the scaling dynamics) Let $\lambda_0(s) = \lambda(s)(1 - J_1(s))^2$. Then on $[0, s_0]$,

$$\left| \frac{(\lambda_0)_s}{\lambda_0} + b + c_1 b^2 \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + |b| \mathcal{N}^{1/2} + |b|^3. \tag{2.58}$$

(iii) (Dispersive bounds) For all $0 \leq s_1 \leq s_2 < s_0$,

$$\mathcal{N}(s_2) + \int_{s_1}^{s_2} \left[\int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_B + |b|^4(s) \right] ds \lesssim \mathcal{N}(s_1) + (|b^3(s_2)| + |b^3(s_1)|). \tag{2.59}$$

$$\frac{\mathcal{N}(s_2)}{\lambda^2(s_2)} + \int_{s_1}^{s_2} \left[\int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_B + |b|^4(s) \right] \frac{ds}{\lambda^2(s)} \lesssim \frac{\mathcal{N}(s_1)}{\lambda^2(s_1)} + \left[\frac{|b^3(s_1)|}{\lambda^2(s_1)} + \frac{|b^3(s_2)|}{\lambda^2(s_2)} \right]. \tag{2.60}$$

2.6. Localization in space and decay properties of minimal mass solutions. Minimal mass blow up solutions have been studied in some detail in [27] using tools developed in [25] and [26]. Recall that the main result of [27] is the nonexistence of minimal mass blow up solutions, assuming initial decay in space. In proving this result, several general properties of minimal mass blow up solutions were derived. In the next lemma we gather all useful information which can be deduced from [27] on general minimal mass blow up solutions. Note that at this stage, we do not know whether a minimal mass blow up solution should blow up in finite or infinite time. See Proposition 4.1 in Section 4 for refined information.

Lemma 2.11 (First properties of minimal mass blow up solutions, [27]). *Let $u(t)$ be a solution of (1.1) defined on $(T, 0]$, which blows up backwards in finite or infinite time $-\infty \leq T < 0$. Assume*

$$\int u^2(0) = \int Q^2. \tag{2.61}$$

There exists $t_1 > T$ close to T such that for all $t \in (T, t_1]$, $u(t)$ or $-u(t)$ admits a decomposition $(\lambda(t), x(t), b(t), \varepsilon(t))$ as in Lemma 2.6, with

$$\lim_{t \rightarrow T} \lambda(t) = 0, \tag{2.62}$$

and for all $t \in (T, t_1)$,

$$\int_T^t \frac{b^2(t') + \int \varepsilon^2(t', y) e^{-|y|/10} dy}{\lambda^3(t')} dt' + \mathcal{N}(t) + \|\varepsilon(t)\|_{H^1}^2 + |b(t)| \lesssim \lambda^2(t) E_0. \tag{2.63}$$

Moreover,

$$\text{for all } t, t' \in (T, t_1), \quad t < t' \text{ implies } \lambda(t') \geq \frac{1}{24} \lambda(t), \tag{2.64}$$

$$\text{for all } t \in (T, t_1) \text{ and } y > 0, \quad |\varepsilon(t, y)| \lesssim e^{-y/2000}. \tag{2.65}$$

Proof. Let $u(t) \in \mathcal{C}((T, 0], H^1)$ be a general backward minimal mass blow up solution defined on $(T, 0]$ and blowing up in finite or infinite time⁵ $-\infty \leq T < 0$:

$$\lim_{t \rightarrow T} \|u_x(t)\|_{L^2} = \infty. \tag{2.66}$$

From standard concentration-compactness arguments⁶ and using the mass and energy conservations, either $u(t)$ or $-u(t)$ satisfies (2.18) and (2.19) for t close to T , with in addition

$$\|\varepsilon_1(t)\|_{H^1} \rightarrow 0 \quad \text{as } t \rightarrow T$$

thanks to the minimal mass assumption. Therefore, possibly considering $-u(t)$ instead of $u(t)$, there exists $t_0 > T$ such that the solution $u(t)$ admits on $(T, t_0]$ a decomposition given by Lemma 2.6:

$$u(t, x) = \frac{1}{\lambda^{1/2}(t)} (Q_{b(t)} + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right) \tag{2.67}$$

with

$$\forall t \in (T, t_0], \quad |b(t)| + \|\varepsilon(t)\|_{H^1} \leq \alpha^*, \tag{2.68}$$

⁵ Note that the uniqueness statement in Theorem 1.3(ii) concerns only finite time blow up solutions. Actually, in this paper we also prove the nonexistence of minimal mass solutions blowing up at infinity (see Proposition 4.1 and Section 5). However, we do not treat the case of global minimal mass solutions blowing up only on a subsequence of time.

⁶ See for example the lecture notes [45].

where $\alpha^* > 0$ is any given small constant. With this decomposition, the (finite or infinite time) blow up assumption (2.66) is equivalent to: $\lim_{t \rightarrow T} \lambda(t) = 0$, and by (2.31),

$$b(t) < 0, \quad E_0 > 0, \quad |b(t)| + \|\varepsilon(t)\|_{H^1}^2 \lesssim \lambda^2(t) E_0. \quad (2.69)$$

Now, we recall results from [27]. First, recall that the solution $u(t, x)$ is decomposed in a different way in [27, Lemma 1]. Indeed, there exist C^1 functions $\tilde{\lambda}$ and \tilde{x} such that

$$\tilde{\varepsilon}(t, y) = \tilde{\lambda}^{1/2}(t) u(t, \tilde{\lambda}(t)y + \tilde{x}(t)) - Q(y)$$

satisfies the orthogonality conditions

$$\int \Lambda Q(y) \tilde{\varepsilon}(t, y) dy = \int y \Lambda Q(y) \tilde{\varepsilon}(t, y) dy = 0.$$

Note that one easily compares this decomposition with (2.67), in particular, combining the orthogonality conditions of ε and $\tilde{\varepsilon}$, one obtains

$$\left| 1 - \frac{\lambda(t)}{\tilde{\lambda}(t)} \right| + |b(t)| \lesssim \left(\int \tilde{\varepsilon}^2(t) e^{-|y|/10} dy \right)^{1/2}. \quad (2.70)$$

Under the general assumptions of Lemma 2.11, we now claim that for some $T < t_1 < t_0$,

$$\text{for all } t', t \in (T, t_1), \quad \text{if } t \leq t' \text{ then } \tilde{\lambda}(t) \leq 4\tilde{\lambda}(t'), \quad (2.71)$$

and

$$\text{for all } t \in (T, t_1) \text{ and } y > 0, \quad |\tilde{\varepsilon}(t, y)| \lesssim e^{-y/1000}. \quad (2.72)$$

To prove (2.71) and (2.72), we invoke the arguments of [27, Section 4]. Recall that the main result of [27], stated in Theorem 1.2 of the present paper, asserts forward global existence for minimal mass solutions under the decay assumption (1.7). Unlike Section 3, based on the decay assumption on the initial data, Section 4 of [27] does not make use of this assumption, except when asserting that blow up occurs in finite time. At this point, it is important to note that here time is reversed with respect to [27], thus the left and right in space are also reversed (recall that if $u(t, x)$ is a solution of (1.1), then $u(-t, -x)$ is also a solution of (1.1)). First, using [27, Lemma 4], one obtains uniform exponential decay on the right in space, on a special sequence of times $t_n \rightarrow T$,

$$\text{for all } y > 0, \quad |\tilde{\varepsilon}(t_n, y)| \lesssim e^{-y/1000}. \quad (2.73)$$

Then, invoking Step 2 of the proof of [27, Proposition 2, p. 401], we obtain (2.71), i.e. the almost monotonicity of $\tilde{\lambda}$ and in return, using Lemma 4 again, the decay on the right (2.72) for all time.

Note that from (2.69) and (2.72), we have

$$\int_{y>0} y^{10} \tilde{\varepsilon}^2(t, y) dy \rightarrow 0 \quad \text{as } t \rightarrow T. \quad (2.74)$$

Now, using further [27, p. 405], we claim that, for all $T < t_2 < t_1$,

$$\int_{t_2}^{t_1} \frac{\int \tilde{\varepsilon}^2 e^{-|y|/10^4} dy}{\tilde{\lambda}^3} dt \lesssim \tilde{\lambda}^2(t_1) E_0. \tag{2.75}$$

Indeed, it is proved there that for all $T < t_2 < t_1$,

$$\int_{t_2}^{t_1} \frac{\int \tilde{\varepsilon}^2 e^{-|y|/10^4} dy}{\tilde{\lambda}^3} dt \lesssim \int \varepsilon^2(t_1) + \int \varepsilon^2(t_2) + \int_{t_2}^{t_1} \frac{(\int \tilde{\varepsilon} Q)^2}{\tilde{\lambda}^3} dt, \tag{2.76}$$

and that there exists $\bar{\lambda}(t)$ (related to yet another decomposition of $u(t, x)$ which requires the decay (2.72)) with $\bar{\lambda}(t) \approx \tilde{\lambda}(t)$ such that, for a universal constant $c_0 > 0$,

$$-E_0 \lambda^2 \lesssim \int \tilde{\varepsilon} Q < 0, \quad \left| 2 \int \tilde{\varepsilon} Q + c_0 \bar{\lambda}^{-2} \bar{\lambda}_t \right| \lesssim \left(\int \tilde{\varepsilon}^2 e^{-|y|/4} \right)^{3/4}. \tag{2.77}$$

Since $\int_{t_2}^{t_1} \bar{\lambda} \bar{\lambda}_t dt \lesssim \bar{\lambda}^2(t_1) \lesssim \tilde{\lambda}^2(t_1)$, we obtain (2.75) by integration.

Passing to the limit as $t_2 \rightarrow T$ in (2.75), we obtain

$$\int_T^{t_1} \frac{\int \tilde{\varepsilon}^2 e^{-|y|/10^4} dy}{\tilde{\lambda}^3} dt \lesssim \tilde{\lambda}^2(t_1) E_0 \lesssim \lambda^2(t_1) E_0, \tag{2.78}$$

and thus, using (2.70),

$$\int_T^{t_1} \frac{\int \varepsilon^2 e^{-2|y|/10^4} dy}{\lambda^3} dt \lesssim \lambda^2(t_1) E_0. \tag{2.79}$$

By (2.30), we have $\frac{b^2}{\lambda^3} \leq -b_t + \frac{C}{\lambda^3} \int \varepsilon^2 e^{-|y|/10} dy$ and thus

$$\int_T^{t_1} \frac{b^2(t)}{\lambda^3(t)} dt \lesssim \lambda^2(t_1) E_0 + C \int_T^{t_1} \frac{\int \varepsilon^2 e^{-|y|/10} dy}{\lambda^3(t)} dt \lesssim \lambda^2(t_1) E_0.$$

Now, we claim

$$\int_{y>1} y^{10} \varepsilon^2(t, y) dy \lesssim \lambda^2(t) E_0. \tag{2.80}$$

Indeed, consider a smooth function

$$\varphi_{10}(y) = \begin{cases} 0 & \text{for } y \leq 0, \\ y^{10} & \text{for } y \geq 1, \end{cases} \quad \text{with } \varphi'_{10} \geq 0.$$

Using the computations of [29, proof of Lemma 3.7] on $\tilde{\varepsilon}$ (the computations for the decomposition $(\tilde{\varepsilon}, \tilde{\lambda}, \tilde{x})$ are actually simpler, since they correspond to the choice $b = 0$), we obtain

$$\frac{1}{\tilde{\lambda}^{10}} \frac{d}{dt} \left(\tilde{\lambda}^{10} \int \varphi_{10} \tilde{\varepsilon}^2 \right) \lesssim \frac{\int \tilde{\varepsilon}^2 e^{-|y|/10} dy}{\tilde{\lambda}^3},$$

and thus, using (2.64), (2.78),

$$\int \varphi_{10} \tilde{\varepsilon}^2(t_1, y) dy \lesssim \int \varphi_{10} \tilde{\varepsilon}^2(t_2, y) dy + \lambda^2(t_1) E_0.$$

Passing to the limit as $t_2 \rightarrow T$ and using (2.74) we obtain, for all $T < t < t_0$,

$$\int_{y>1} y^{10} \tilde{\varepsilon}^2(t, y) dy \lesssim \lambda^2(t) E_0.$$

Finally, by (2.70) and (2.69), we obtain (2.80). \square

3. Construction of a minimal element

This section is devoted to the proof of the existence of a minimal blow up element. We propose a strategy of proof slightly different from the recent approach developed for the construction of nondispersive solutions in [5]–[7], [18], [19], [22], [28], [31], [48], mainly to prepare the analysis of the (Exit) regime in Theorem 1.4; see also Section A.2 and Remark 3.2 below.

The strategy of the proof is as follows. We consider a *well-prepared* sequence of initial data (u_n) with

$$\|u_n\|_{L^2} < \|Q\|_{L^2} \quad \text{and} \quad u_n(0) \rightarrow Q \text{ in } H^1.$$

By Theorem 1.1, such solutions are in the (Exit) scenario and we denote by $t_n^* > 0$ the corresponding exit time. The estimates extracted from [29] allow for a complete dynamical description of the (Exit) regime and in particular of the defocusing structure of the solution at t_n^* . This explicit detailed knowledge allows us to renormalize the flow and extract in the limit $n \rightarrow \infty$ a solution $v \in \mathcal{C}((t^*, 0], H^1)$ which blows up at time $t^* < 0$ and has subcritical mass $\|v\|_{L^2} \leq \|Q\|_{L^2}$. But then the global well-posedness below the ground state mass implies $\|v\|_{L^2} = \|Q\|_{L^2}$ and v is an H^1 minimal mass blow up element.

Step 1. Well-prepared data. Let $u_n(0) = Q_{b_n(0)}$, where $b_n(0) = -1/n$ so that

$$u_n(0) \in \mathcal{A} \subset H^1, \quad u_n(0) \rightarrow Q \quad \text{in } H^1 \text{ as } n \rightarrow \infty.$$

By (2.16), we have $\int u_n^2(0) < \int Q^2$. In particular, from energy and mass conservation, and the Gagliardo–Nirenberg inequality (1.3), the solution $u_n(t)$ is global. We take $n > 0$ large enough and we apply Theorem 1.1. The solution u_n being global, the (Blow up) scenario is ruled out. The solution cannot converge locally to a solitary wave because of mass conservation and the strictly subcritical mass assumption, hence (Soliton) is also ruled out. Hence (Exit) holds and we define the *exit time* (related to the constant α^* of Theorem 1.1) by

$$t_n^* = \sup\{t > 0 : \forall t' \in [0, t], u_n(t') \in \mathcal{T}_{\alpha^*}\}.$$

Note that $t_n^* \rightarrow \infty$ as $n \rightarrow \infty$ from the continuous dependence of the solution of (1.1) on the initial data, and the fact that $Q(x - t)$ is a solution of (1.1).

Now, we use refined information given in the (Exit) case by [29, Proposition 4.1]. In particular, $u_n(t)$ satisfies (2.19) and has a decomposition $(\lambda_n, x_n, b_n, \varepsilon_n)$ as in Lemma 2.6 on $[0, t_n^*]$. Moreover, (H1)–(H3) are satisfied on $[0, t_n^*]$, and by definition of $u_n(0)$,

$$\lambda_n(0) = 1, \quad x_n(0) = 0, \quad b_n(0) = -1/n, \quad \varepsilon_n(0) = 0. \tag{3.1}$$

In addition, from [29, proof of Proposition 4.1, (4.41)], we also have

$$\text{for all } 0 \leq t_1 \leq t_2 \leq t_n^*, \quad \lambda_n(t_2) \geq \frac{1}{2}\lambda_n(t_1). \tag{3.2}$$

Note also that by continuity in time and the definition of t_n^* ,

$$\inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \|u_n(t_n^*) - \lambda_0^{-1/2} \mathcal{Q}(\lambda_0^{-1}(\cdot + x_0))\|_{L^2} = \alpha^*, \tag{3.3}$$

$$\alpha^* \leq \|u_n(t_n^*) - \lambda_n^{-1/2}(t_n^*) \mathcal{Q}(\lambda_n^{-1}(t_n^*)(\cdot + x_n(t_n^*)))\|_{L^2} \leq \delta(\alpha^*). \tag{3.4}$$

Step 2. Structure of the defocusing bubble. From Lemma 2.10, (3.1) and (3.4), we will deduce:

Lemma 3.1.

(i) (Estimates on $[0, t_n^*]$)

$$\forall t \in [0, t_n^*], \quad \frac{1 - \delta(\alpha^*)}{n} \leq -\frac{b_n(t)}{\lambda_n^2(t)} \leq \frac{1 + \delta(\alpha^*)}{n}, \tag{3.5}$$

$$\|\varepsilon_n(t)\|_{H^1}^2 \lesssim \lambda_n^2(t)/n \lesssim \delta(\alpha^*). \tag{3.6}$$

(ii) (Estimates at t_n^*) For all n ,

$$(\alpha^*)^2 \lesssim -b_n(t_n^*) \lesssim \delta(\alpha^*), \tag{3.7}$$

$$(\alpha^*)^2 \lesssim \int \varepsilon_n^2(t_n^*) \lesssim \delta(\alpha^*), \quad (\alpha^*)^2 \lesssim \frac{\lambda_n^2(t_n^*)}{n} \lesssim \delta(\alpha^*), \tag{3.8}$$

$$0 < c(\alpha^*) \leq \frac{t_n^*}{\lambda_n^3(t_n^*)} \leq C(\alpha^*). \tag{3.9}$$

(iii) (Control of the dynamics on $[0, t_n^*]$)

$$-(1 - \delta(\alpha^*)) \frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)} \leq (\lambda_{0n})_t(t) \leq -(1 + \delta(\alpha^*)) \frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)}. \tag{3.10}$$

Proof. Using (2.33) and (2.35) at $t = 0$, we obtain

$$\int u_n^2(0) = \int Q^2 - \frac{2}{n} \int PQ + O(n^{-9/8}), \quad 2E(u_n(0)) = \frac{2}{n} \int PQ + O(n^{-9/8}).$$

Combining the conservation of the L^2 norm, the conservation of energy and (2.33), (2.35), we obtain, at any $t \in (0, t_n^*)$,

$$|b_n| \lesssim \int \varepsilon^2 + 1/n, \tag{3.11}$$

and

$$(L\varepsilon_n(t), \varepsilon_n(t)) = 2\lambda_n^2 E(u_n(0)) + \int u_n^2(0) - \int Q^2 + O(n^{-9/8} + \|\varepsilon_n(t)\|_{H^1}^{9/4}).$$

Thus, from $\|\varepsilon_n(t)\|_{H^1}^2 \lesssim (L\varepsilon_n(t), \varepsilon_n(t))$, we obtain

$$\|\varepsilon_n(t)\|_{H^1}^2 \lesssim \lambda_n^2(t)/n. \tag{3.12}$$

Next, recall that by (3.4),

$$\alpha^* \leq \|b_n(t_n^*)\chi_{b_n(t_n^*)}P + \varepsilon_n(t_n^*)\|_{L^2} \leq \delta(\alpha^*),$$

and use (2.32) and (3.11) to obtain

$$(\alpha^*)^2 \lesssim \int \varepsilon_n^2(t_n^*) \lesssim \delta(\alpha^*), \quad (\alpha^*)^2 \lesssim -b_n(t_n^*) \lesssim \delta(\alpha^*). \tag{3.13}$$

Now, we use the dynamical information given by the rigidity property (2.57) and the initialization (3.1): for all $t \in [0, t_n^*]$,

$$(1 + \delta(\alpha^*))b_n(0) \leq b_n(t)/\lambda_n^2(t) \leq (1 - \delta(\alpha^*))b_n(0), \tag{3.14}$$

and thus by (3.13),

$$(\alpha^*)^2 \lesssim \lambda_n^2(t_n^*)/n \lesssim \delta(\alpha^*). \tag{3.15}$$

Finally, let us prove (3.10) and (3.9). By (2.58), we have

$$\left| (\lambda_{0n})_t \frac{\lambda_n}{\lambda_{0n}} + \frac{b_n}{\lambda_n^2} \right| \lesssim \frac{\int \varepsilon_n^2 e^{-|y|/10}}{\lambda_n^2} + \frac{|b_n|}{\lambda_n^2} (\mathcal{N}_n^{1/2} + |b_n|) \lesssim \frac{1}{\lambda_n^2} (\mathcal{N}_n + |b_n|^2).$$

By (2.60), (3.1), and then (3.14), we have

$$\frac{\mathcal{N}_n(t)}{\lambda_n^2(t)} \lesssim \frac{|b_n(t)|^3}{\lambda_n^2(t)} + \frac{|b_n(0)|^3}{\lambda_n^2(0)} \lesssim \delta(\alpha^*) \frac{|b_n(t_n^*)|}{\lambda_n^2(t_n^*)}.$$

Thus, again by (3.14),

$$-(1 - \delta(\alpha^*)) \frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)} \leq (\lambda_{0n})_t(t) \leq -(1 + \delta(\alpha^*)) \frac{b_n(t_n^*)}{\lambda_n^2(t_n^*)},$$

which is (3.10). We integrate on $[0, t_n^*]$ and then divide by $\lambda_{0,n}(t_n^*)$ to obtain

$$-t_n^* \frac{b_n(t_n^*)}{\lambda_n^3(t_n^*)} (1 - \delta(\alpha^*)) \leq 1 - \frac{1}{\lambda_{0,n}(t_n^*)} \leq -t_n^* \frac{b_n(t_n^*)}{\lambda_n^3(t_n^*)} (1 + \delta(\alpha^*)). \tag{3.16}$$

Hence using (3.15) and $\lambda_n/\lambda_{0,n} \lesssim \mathcal{N}_n^{1/2} \lesssim \delta(\alpha^*)$, we get

$$-\frac{1 - \delta(\alpha^*)}{b_n(t_n^*)} \leq \frac{t_n^*}{\lambda_n^3(t_n^*)} \leq -\frac{1 + \delta(\alpha^*)}{b_n(t_n^*)}, \tag{3.17}$$

which together with (3.13) implies (3.9). □

Step 3. *Renormalization and extraction of the limit.* Let

$$\forall \tau \in [-t_n^*/\lambda_n^3(t_n^*), 0], \quad t_\tau = t_n^* + \tau \lambda_n^3(t_n^*),$$

$$v_n(\tau, x) = \lambda_n^{1/2}(t_n^*) u_n(t_\tau, \lambda_n(t_n^*)x + x(t_n^*)) \tag{3.18}$$

$$= \frac{\lambda_n^{1/2}(t_n^*)}{\lambda_n^{1/2}(t_\tau)} (Q_{b_n(t_\tau)} + \varepsilon_n) \left(t_\tau, \frac{\lambda_n(t_n^*)}{\lambda_n(t_\tau)} x + \frac{x(t_n^*) - x(t_\tau)}{\lambda_n(t_\tau)} \right), \tag{3.19}$$

so that v_n is a solution of (1.1) and belongs to the tube \mathcal{T}_{α^*} for $\tau \in [-t_n^*/\lambda_n^3(t_n^*), 0]$. Moreover, its decomposition $(\lambda_{v_n}, x_{v_n}, \varepsilon_{v_n})$ satisfies, on $[-t_n^*/\lambda_n^3(t_n^*), 0]$,

$$\lambda_{v_n}(\tau) = \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)}, \quad x_{v_n}(\tau) = \frac{x_n(t_\tau) - x_n(t_n^*)}{\lambda_n(t_n^*)}, \quad b_{v_n}(\tau) = b_n(t_\tau), \quad \varepsilon_{v_n}(\tau) = \varepsilon_n(t_\tau). \tag{3.20}$$

By (3.7), (3.8) and (3.5), we have

$$\forall \tau \in [-t_n^*/\lambda_n^3(t_n^*), 0], \quad \|\varepsilon_{v_n}(\tau)\|_{H^1}^2 \lesssim \delta(\alpha^*),$$

$$\lambda_{v_n}(0) = 1, \quad x_{v_n}(0) = 0, \quad (\alpha^*)^2 \lesssim -b_{v_n}(0) \leq \delta(\alpha^*).$$

Therefore, there exists a subsequence of (v_n) , which we will still denote by (v_n) , and $v(0) \in H^1$ such that

$$v_n(0) \rightharpoonup v(0) \text{ weakly in } H^1, \quad \text{and} \quad \|v(0) - Q\|_{H^1} \lesssim \delta(\alpha^*),$$

and, by (3.8) and (3.9),

$$\tau_n^* = -t_n^*/\lambda_n^3(t_n^*) \rightarrow -\tau^*, \quad \tau^* > 0, \quad -b_n(t_n^*) \rightarrow b^* > 0.$$

Moreover, by (3.17),

$$\frac{1 - \delta(\alpha^*)}{b^*} \leq \tau^* \leq \frac{1 + \delta(\alpha^*)}{b^*}. \tag{3.21}$$

We let $v(\tau)$ be the backward H^1 solution of (1.1) with initial data $v(0)$ at $\tau = 0$.

Step 4. *Minimal mass blow up.* We claim that v is a minimal mass blow up element, $\|v\|_{L^2} = \|Q\|_{L^2}$ which blows up at finite negative time $-\tau^*$ with, for τ close enough to $-\tau^*$,

$$\frac{1 - \delta(\alpha^*)}{1 + \tau/\tau^*} \leq \|v_x(\tau)\|_{L^2} \leq \frac{1 + \delta(\alpha^*)}{1 + \tau/\tau^*}. \tag{3.22}$$

Indeed, we integrate (3.10) and obtain for $t \in [0, t_n^*]$, n large enough,

$$\frac{b^* t}{\lambda_n^2(t_n^*)} (1 - \delta(\alpha^*)) \leq \lambda_{0,n}(t) - \lambda_{0,n}(0) \leq \frac{b^* t}{\lambda_n^2(t_n^*)} (1 + \delta(\alpha^*)).$$

We conclude from (3.20) and the definition of τ_n^* that for all $\tau \in [\tau_n^*, 0]$,

$$b^*(\tau_n^* + \tau)(1 - \delta(\alpha^*)) \leq \lambda_{0,v_n}(\tau) - \lambda_{0,v_n}(\tau_n^*) \leq b^*(\tau_n^* + \tau)(1 + \delta(\alpha^*)). \tag{3.23}$$

Let $\tau_0 \in (-\tau^*, 0)$. From (3.8), (3.20) and $\varepsilon_{v_n}(\tau_n^*) = \varepsilon_n(0)$, we have

$$\lambda_{0,v_n}(\tau_n^*) = \lambda_n(0)/\lambda_n(\tau_n^*) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and $1 - \delta(\alpha^*) \leq \lambda_{0,v_n}/\lambda_{v_n} \leq 1 + \delta(\alpha^*)$. Thus, we conclude from (3.23) that for n large enough depending on τ_0 ,

$$\forall \tau \in [\tau_0, 0], \quad b^*(\tau^* + \tau)(1 - \delta(\alpha^*)) \leq \lambda_{v_n}(\tau) \leq b^*(\tau^* + \tau)(1 + \delta(\alpha^*)),$$

and $\frac{1}{2}b^*(\tau^* + \tau_0) \leq \lambda_{v_n}(\tau)$. It follows from Lemma 2.8 that $v(\tau)$ is well-defined and $\lambda_{v_n}(\tau) \rightarrow \lambda_v(\tau)$ on $[\tau_0, 0]$. In particular, v exists on $(-\tau^*, 0]$, and for all $\tau \in (-\tau^*, 0]$,

$$b^*(\tau^* + \tau)(1 - \delta(\alpha^*)) \leq \lambda_v(\tau) \leq b^*(\tau^* + \tau)(1 + \delta(\alpha^*)),$$

which together with (3.21) implies (3.22). Finally, by weak H^1 convergence we have $\int v^2(0) \leq \lim_{n \rightarrow \infty} \int v_n^2(0) = \int Q^2$, and since v blows up in finite time, $\int v^2(0) = \int Q^2$. This concludes the proof of the existence of the minimal element.

Remark 3.2. We may rewrite this proof by saying that understanding the minimal mass blow up scenario is in some sense equivalent to understanding how subcritical solutions initially near the ground state move away from the ground state and start *defocusing*, and here the sharp knowledge of the speed of defocusing is fundamental for the proof. Another approach for the construction of the minimal blow up element in the continuation of [5]–[7], [19], [22], [28], [31], [48] would be to take the initial data $Q_{b(t_n)}$ at some time $t_n \downarrow 0$ with $b(t_n) = t_n$ and to obtain uniform H^1 bounds on the corresponding forward solution $u_n(t)$ to (1.1) at a time $t_0 > 0$ independent of n using the monotonicity machinery of Proposition 2.9 and Lemma 2.10. It is not clear to us whether a direct fixed point approach as in [3], [10], [11], [21] is applicable here due to the poor localization in space of the minimal element.

4. Sharp description of minimal mass blow up

We now turn to the proof of uniqueness in H^1 of the minimal element. Let us stress that uniqueness is always a delicate problem, in particular in the absence of suitable symmetries as in [28]. As in [48], the first crucial information is to derive the blow up speed for all minimal elements, and here we shall use the a priori localization in space of minimal elements given by Lemma 2.11 which allows us to use the monotonicity tools, Proposition 2.9 and Lemma 2.10. Once the minimal mass blow up regime is sufficiently well described, we may rerun the analysis of Proposition 2.9 for the difference of two such bubbles and conclude that they are equal; this is done in Section 5.

4.1. Finite time blow up and blow up speed for minimal mass blow up solutions.

Our aim in this section is to derive sharp qualitative bounds on minimal mass blow up solutions, improving general results stated in Lemma 2.11. In particular, we prove that the blow up time is finite, $T > -\infty$, and we specify the blow up speed and the behavior

of the concentration point, which are essential preliminary pieces of information on the singularity formation. Note that the additional information below requires the sharper analysis of [29] and cannot be derived from [27]. We consider a minimal mass blow up solution $u(t)$, and in the setting of Lemma 2.11, we introduce the rescaled time

$$s(t) = - \int_t^{t_0} \frac{ds}{\lambda^3}. \tag{4.1}$$

Recall that $s(T) = -\infty$ by a standard argument (see e.g. [25]).

Proposition 4.1 (Sharp bounds). *Let $u(t)$ be a solution of (1.1) defined on $(T, 0]$, which blows up backwards in finite or infinite time $-\infty \leq T < 0$. Assume*

$$\int u^2(0) = \int Q^2.$$

(i) (Finite time blow up) *We have*

$$T > -\infty.$$

(ii) (Sharp controls near blow up time) *There exist universal constants c_λ, c_x, c_b and $\ell^* = \ell^*(u) < 0, x^* = x^*(u) \in \mathbb{R}$ such that, for t close to T ,*

$$\lambda(t) = |\ell^*|(t - T) + c_\lambda |\ell^*|^4 (t - T)^3 + O[(t - T)^4], \tag{4.2}$$

$$x(t) = - \frac{1}{(\ell^*)^2(t - T)} + x^* + c_x \ell^* (t - T) + O[(t - T)^2], \tag{4.3}$$

$$\frac{b(t)}{\lambda^2(t)} = \ell^* + c_b (\ell^*)^4 (t - T)^2 + O[(t - T)^3], \tag{4.4}$$

$$\mathcal{N}(t) \lesssim (t - T)^6. \tag{4.5}$$

(iii) (Estimates in rescaled time) *For $-s$ large,*

$$\|\varepsilon(s)\|_{L^\infty}^2 \lesssim \|\varepsilon(s)\|_{H^1}^2 \lesssim \lambda^2(s) \lesssim \frac{1}{|s|}, \tag{4.6}$$

$$\mathcal{N}(s) + \int_{-\infty}^s \int (\varepsilon_y^2 + \varepsilon^2)(s') \varphi'_B ds' \lesssim \frac{1}{|s|^3}, \tag{4.7}$$

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \frac{1}{|s|^{3/2}}, \quad |b_s| \lesssim \frac{1}{|s|^2}, \tag{4.8}$$

$$b(s) = \frac{1}{2s} + \frac{c_1^* \log |s|}{s^2} + \frac{c_2^*}{s^2} + o\left(\frac{1}{s^2}\right) \quad \text{as } s \rightarrow -\infty, \tag{4.9}$$

for some universal constants $(c_1^, c_2^*) \in \mathbb{R} \times \mathbb{R}$.*

(iv) (Global forward behavior) *The solution is globally defined for $t > T, u \in \mathcal{C}((0, \infty) \times \mathbb{R})$ and for some $C(t), \gamma(t) > 0$,*

$$\forall t > T, \forall x > 0, \quad |u(t, x)| \leq C(t)e^{-\gamma(t)x}. \tag{4.10}$$

(v) (Time decay of weighted Sobolev norms) For $|s|$ large,

$$\int \varepsilon^2(s, y)e^{\lambda(s)y} dy \lesssim \frac{1}{|s|^2}. \tag{4.11}$$

For all $9/B \leq \omega \leq 1/10$, for $|s|$ large,

$$\sum_{k=0}^3 \int (\partial_y^k \varepsilon)^2(s, y)e^{\omega y} dy + \int_{-\infty}^s \sum_{k=0}^4 \int (\partial_y^k \varepsilon)^2(s', y)e^{\omega y} dy ds' \lesssim \frac{1}{|s|}, \tag{4.12}$$

$$\|((\varepsilon_i)_{yy}^2 + (\varepsilon_i)_{yy}^2)(s)e^{\omega y}\|_{L^\infty} \lesssim \frac{1}{|s|}. \tag{4.13}$$

Remark 4.2. The constant ℓ^* in (4.2) depends on the solution, and the scaling $u(t, x) \mapsto u_{\lambda_0}(t, x) = \lambda_0^{1/2}u(\lambda_0^3 t, \lambda_0 x)$ leads to

$$\ell^*(u_{\lambda_0}) = \lambda_0^2 \ell^*(u). \tag{4.14}$$

Proof of Proposition 4.1. From Lemma 2.11, $E(u(t)) > 0$. Using the scaling invariance of the (gKdV) equation, we consider the solution

$$u_{\lambda_0}(t, x) = \lambda_0^{1/2}u(\lambda_0^3 t, \lambda_0 x),$$

where $\lambda_0 > 0$ is chosen so that $E(u_{\lambda_0}(t)) \ll \kappa^*$, κ^* being the small constant in Proposition 2.9. We work on u_{λ_0} instead of working on u , all statements being scaling invariant. Hereafter, we denote u_{λ_0} simply by $u(t)$.

Step 1. Entering the monotonicity regime. Note first that from Lemma 2.11 and $E_0 \ll \kappa^*$, (H1)–(H3) hold on $(-\infty, s_0]$ for $-s_0$ large enough. The solution is therefore in the monotonicity regime of Proposition 2.9 and Lemma 2.10 on $(-\infty, s_0]$.

Step 2. Rigidity and blow up speed. We state the key nondegeneracy relation:

$$\mathcal{N} \lesssim O(\lambda^6), \quad \left| \frac{b}{\lambda^2} - \ell^* - \frac{c_0}{2} \frac{b^2}{\lambda^2} \right| \lesssim O(\lambda^3) \tag{4.15}$$

for some constant $\ell^* > 0$.

Let $C^* > 0$ be the universal constant in (2.57). Let us first remark that there exists a sequence $s_n \rightarrow -\infty$ such that

$$\forall n \geq 1, \quad b(s_n) \leq -C^* \int (\varepsilon_y^2 + \varepsilon^2)(s_n)\varphi'_B. \tag{4.16}$$

Indeed, assume for contradiction that there exists a time $s^* \leq s_0$ such that (recall that $b < 0$)

$$\forall s < s^*, \quad |b(s)| \leq C^* \int (\varepsilon_y^2 + \varepsilon^2)(s)\varphi'_B. \tag{4.17}$$

Thus, (2.58) implies

$$\left| \frac{(\lambda_0)_s}{\lambda_0} \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + |b| \lesssim \int (\varepsilon_y^2 + \varepsilon^2)(s) \varphi'_B,$$

where $\lambda_0(s) = \lambda(s)(1 - J_1(s))^2$. Using (2.59) we obtain

$$\forall s < s_2^*, \quad \left| \log \left(\frac{\lambda_0(s)}{\lambda_0(s_2^*)} \right) \right| \lesssim \int_s^{s_2^*} \int (\varepsilon_y^2 + \varepsilon^2)(s') \varphi'_B ds' \lesssim 1;$$

but together with

$$|\lambda/\lambda_0 - 1| \lesssim |J_1| \lesssim \mathcal{N}^{1/2} \quad (4.18)$$

and (2.63), this contradicts the blow up assumption: $\lambda(s) \rightarrow 0$ as $s \rightarrow -\infty$. Since $b < 0$, this concludes the proof of (4.16).

Inserting (4.16) in (2.57) yields the rigidity: for all $n \geq 1$ and all s such that $s_n \leq s \leq s_0$,

$$2 \frac{b(s_n)}{\lambda^2(s_n)} \leq \frac{b(s)}{\lambda^2(s)} \leq \frac{b(s_n)}{2\lambda^2(s_n)}.$$

We conclude using $b < 0$ and $\lambda(s_0) = 1$ that for all $s < s_0$,

$$4b(s_0) \leq \frac{b(s)}{\lambda^2(s)} \leq \frac{b(s_0)}{4} < 0. \quad (4.19)$$

By (2.62) and (2.63) we have $\lim_{s \rightarrow -\infty} \mathcal{N}(s) = 0$. Using (2.59), we obtain, for $s_1 < s_2$,

$$\mathcal{N}(s_2) + \int_{s_1}^{s_2} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B ds \lesssim \mathcal{N}(s_1) + |b^3(s_1)| + |b^3(s_2)|.$$

Thus, passing to the limit $s_1 \rightarrow -\infty$, and using (2.69),

$$\mathcal{N}(s_2) + \int_{-\infty}^{s_2} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B ds \lesssim |b^3(s_2)| \lesssim \lambda^6(s_2). \quad (4.20)$$

From (2.41), we have

$$\left| \frac{d}{ds} \left(\frac{b}{\lambda^2} e^J \right) + c_0 \frac{b^3}{\lambda^2} \right| \leq \frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-|y|/10} + |b|^4 \right). \quad (4.21)$$

Letting $s_1 \rightarrow -\infty$ in (2.60) yields

$$\int_{-\infty}^{s_2} \frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-|y|/10} + |b|^4 \right) ds \lesssim \frac{b^3}{\lambda^2} \lesssim \lambda^4. \quad (4.22)$$

Next, by (2.30), (4.22),

$$\frac{b^3}{\lambda^2} = -\frac{1}{2} \frac{b_s b}{\lambda^2} + O \left(\frac{1}{\lambda^2} \left(\int \varepsilon^2 e^{-|y|/10} + |b|^4 \right) \right),$$

so that by integration by parts and (2.29),

$$\int_{-\infty}^s \frac{b^3}{\lambda^2} = -\frac{1}{4} \frac{b^2(s)}{\lambda^2(s)} - \frac{1}{2} \int_{-\infty}^s \frac{b^2 \lambda_s}{\lambda^3} + O(\lambda^4) = -\frac{1}{4} \frac{b^2(s)}{\lambda^2(s)} + \frac{1}{2} \int_{-\infty}^s \frac{b^3}{\lambda^2} + O(\lambda^4),$$

and thus

$$\int_{-\infty}^s \frac{b^3}{\lambda^2} = -\frac{1}{2} \frac{b^2(s)}{\lambda^2(s)} + O(\lambda^4).$$

It follows by integrating (4.21) and using (4.19) that

$$\lim_{s \rightarrow -\infty} \frac{b}{\lambda^2}(s) = \ell^* < 0, \tag{4.23}$$

and more precisely, using $|J| \lesssim \mathcal{N}^{1/2} \lesssim \lambda^3$,

$$\left| \frac{b}{\lambda^2} - \ell^* - \frac{c_0}{2} \frac{b^2}{\lambda^2} \right| \lesssim O(\lambda^4) + \frac{|b|}{\lambda^2} (1 - e^J) \lesssim O(\lambda^3). \tag{4.24}$$

Step 3. Finite time blow up. From (4.20) and (2.54), we have

$$\lambda_0 = \lambda + O(\lambda J_1) = \lambda + O(\lambda \mathcal{N}^{1/2}) = \lambda + O(\lambda^4), \tag{4.25}$$

and then by (2.58), (4.24),

$$\begin{aligned} -\lambda^3 \frac{(\lambda_0)_t}{\lambda_0} &= -\frac{(\lambda_0)_s}{\lambda_0} = b + c_1 b^2 + O(\mathcal{N} + |b| \mathcal{N}^{1/2} + |b|^3) \\ &= \ell^* \lambda^2 + c(\ell^*)^2 \lambda^4 + O(\lambda^5); \end{aligned}$$

here and below c denotes various universal constants. Hence

$$-(\lambda_0)_t = \ell^* + c(\ell^*)^2 \lambda_0^2 + O(\lambda_0^3). \tag{4.26}$$

For t close to T , we obtain $(\lambda_0)_t > \ell^*/2 > 0$, and thus λ_0 vanishes backwards at some finite time $T > -\infty$; in particular, the solution blows up in finite time. Moreover, integrating (4.26) on $(T, t]$ for $t > T$ close to T , using (4.25), yields

$$\lambda(t) = \lambda_0(t) + O[(t - T)^4] = |\ell^*|(t - T) + c|\ell^*|^4(t - T)^3 + O[(t - T)^4].$$

Together with (4.20), (4.15), this concludes the proof of (4.4), (4.2), (4.5).

We now integrate the modulation equation (2.29) for the blow up point:

$$x_t = \frac{1}{\lambda^2} \frac{x_s}{\lambda} = \frac{1}{\lambda^2} [1 + O(b^2 + \mathcal{N}^{1/2})] = \frac{1}{\lambda^2} [1 + O(\lambda^3)],$$

and thus, using (4.2), we obtain

$$x_t(t) = \frac{1}{(\ell^*)^2 (T - t)^2} - 2c\ell^* + O(t - T),$$

which implies (4.3) by integration in time.

Step 4. *Sharp estimates in rescaled time.* From (4.2),

$$s(t) = - \int_t^{t_0} \frac{dt'}{|\ell^*|^3(t' - T)^3(1 + O((T - t')^2))} = - \frac{1}{2|\ell^*|^3(t - T)^2}(1 + O(t - T)).$$

From Step 3 and (2.29)–(2.30), we thus get, in terms of the variable s ,

$$\lambda(s) = \frac{1 + o(1)}{\sqrt{2|\ell^*s|}}, \quad \mathcal{N}(s) + \int_{-\infty}^s \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B ds' \lesssim \frac{1}{|s|^3}, \quad b(s) = \frac{1 + o(1)}{2s}, \tag{4.27}$$

$$\left| \frac{\lambda_s}{\lambda}(s) \right| \lesssim |b(s)| + \mathcal{N}^{1/2}(s) \lesssim \frac{1}{s}, \quad \left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \frac{1}{|s|^{3/2}}, \tag{4.28}$$

and $|b_s| \lesssim 1/|s|^2$, so that (4.6) and (4.8) are proved.

Now, we prove (4.9). We rewrite the sharp modulation equation (2.40) for b as

$$|(b(1 + J_2))_s + 2b^2 + c_2b^3| \lesssim \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B + \frac{1}{s^4} + \frac{1}{s^2}\mathcal{N}^{1/2} \lesssim \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B + \frac{1}{|s|^{7/2}}.$$

Let

$$\tilde{b} = b(1 + J_2) = b + O(1/|s|^{5/2}). \tag{4.29}$$

Then equivalently

$$|\tilde{b}_s + 2\tilde{b}^2 + c_2\tilde{b}^3| \lesssim \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B + \frac{1}{|s|^{7/2}}.$$

If $c_2 \leq 1$, let $b_0 = -1$, otherwise let $b_0 = -1/c_2$. In order to integrate this differential inequality, we let

$$F(b) = \int_{b_0}^b \frac{d\beta}{2\beta^2 + c_2\beta^3} = -\frac{1}{2b} - \frac{c_2 \log |b|}{4} + c_0 + O(b) \quad \text{as } b \rightarrow 0, \tag{4.30}$$

for some universal constant $c_0 \in \mathbb{R}$. Then

$$\frac{d}{ds} F(\tilde{b}) = -1 + O\left(s^2 \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B + \frac{1}{|s|^{3/2}}\right). \tag{4.31}$$

By (2.60) with $s_1 \rightarrow -\infty$, we have

$$s^2\mathcal{N}(s) + \int_{-\infty}^s (s')^2 \int (\varepsilon_y^2 + \varepsilon^2)(s')\varphi'_B ds' \lesssim \frac{1}{|s|}.$$

Therefore, integrating (4.31) on $[s, s_0]$ and using (4.30) yields

$$F(\tilde{b}(s)) = -\frac{1}{2\tilde{b}(s)} - \frac{c_2 \log |\tilde{b}(s)|}{4} + c'_0 + O\left(\frac{1}{s}\right) = -s + O\left(\frac{1}{\sqrt{|s|}}\right),$$

which is easily inverted to give

$$\tilde{b}(s) = \frac{1}{2s} + \frac{c_1^* \log |s|}{s^2} + \frac{c_2^*}{s^2} + O\left(\frac{1}{|s|^{5/2}}\right)$$

for some universal constants c_1^*, c_2^* . The estimate (4.29) now implies (4.9).

Step 5. Global existence for $t > t_0$. Recall that $|\varepsilon(s_0, y)| \lesssim e^{-y/20}$ for all $y > 0$. Thus, $u(t)$ has exponential decay in space on the right ($x > 0$), in particular, $\int_{x>0} x^{10} u^2(t_0) < \infty$. From this fact and since $u(t)$ has critical mass, we conclude from Theorem 1.2 that u is globally defined for $t > t_0$. Since Q has exponential decay at ∞ , the exponential decay (2.65) of ε translates into exponential decay (4.10) of u . Finally, it is proved in [13] that a solution of the (gKdV) equation with such exponential decay on the right is smooth, i.e. $u \in C^\infty((0, \infty) \times \mathbb{R})$.

The proofs of (4.11) and (4.12)–(4.13) are given in Appendix A. □

4.2. Sharp description of $S(t)$. We conclude from Proposition 4.1 that the minimal element constructed in Section 3 satisfies the following sharp bounds which conclude the proof of statements (i) and (iii) of Theorem 1.3.

Corollary 4.3. *There exists a solution $S \in C((0, \infty), H^1) \cap C^\infty((0, \infty) \times \mathbb{R})$ to (1.1) with critical mass $\|S(t)\|_{L^2} = \|Q\|_{L^2}$ such that*

$$\|\partial_x S(t)\|_{L^2} \sim \|\partial_x Q\|_{L^2}/t \quad \text{as } t \downarrow 0, \tag{4.32}$$

$$S(t, x) - \frac{1}{t^{1/2}} Q\left(\frac{x + 1/t + \bar{c}t}{t}\right) \rightarrow 0 \quad \text{in } L^2 \quad \text{as } t \downarrow 0, \tag{4.33}$$

$$\forall x > 0, \quad |S(1, x)| \lesssim e^{-\gamma x}, \tag{4.34}$$

for some universal constants $(\bar{c}, \gamma) \in \mathbb{R} \times \mathbb{R}_+^*$. Moreover,

$$\frac{d}{dt} \left(\inf_{\lambda_1 > 0, x_1 \in \mathbb{R}} \|S(t) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2}^2 \right) = 4t(P, Q) + O(t^2). \tag{4.35}$$

Proof. Let $v(t, x)$ be the minimal mass blow up solution constructed in Section 3 with finite backward blow up time $T < 0$. Let $\ell^* = \ell^*(v)$ and $x^* = x^*(v)$ be the constants corresponding to v in Proposition 4.1. From the invariances of the equation and Remark 4.2, $S(t)$ defined by

$$S(t, x) = (\ell^*)^{-1/4} v((\ell^*)^{-3/2}t + T, (\ell^*)^{-1/2}x + x^*)$$

satisfies equation (1.1), and the estimates of Proposition 4.1 with $\ell^*(S) = 1, x^*(S) = 0$ and S blows up backwards in time at the origin. In particular, there exist $\varepsilon(t), b(t), \lambda(t)$ and $x(t)$ such that

$$S(t, x) = \frac{1}{\lambda^{1/2}(t)} (Q_{b(t)} + \varepsilon) \left(t, \frac{x - x(t)}{\lambda(t)} \right), \tag{4.36}$$

$$b(t) = -t^2 + O(t^4), \quad \lambda(t) = t + O(t^3), \quad x(t) = -1/t + \bar{c}t + O(t^2), \tag{4.37}$$

$$\|\varepsilon(t)\|_{L^2} \lesssim t, \quad \int (e^{-|y|/10} + \mathbf{1}_{y>0}(y)) (\varepsilon_y^2 + \varepsilon^2)(t, y) dy \lesssim t^6. \tag{4.38}$$

We now prove (4.33). Since

$$\left\| \frac{1}{\lambda^{1/2}(t)} \varepsilon \left(t, \frac{\cdot - x(t)}{\lambda(t)} \right) \right\|_{L^2} = \|\varepsilon(t)\|_{L^2} \lesssim t,$$

we are reduced to estimating

$$\begin{aligned} & \left\| \frac{1}{\lambda^{1/2}(t)} \mathcal{Q}_{b(t)} \left(\frac{\cdot - x(t)}{\lambda(t)} \right) - \frac{1}{t^{1/2}} \mathcal{Q} \left(\frac{\cdot + 1/t + \bar{c}t}{t} \right) \right\|_{L^2} \\ &= \left\| \mathcal{Q}_{b(t)} - \frac{\lambda^{1/2}(t)}{t^{1/2}} \mathcal{Q} \left(\frac{\lambda(t)}{t} x + \frac{1}{t} \left(\frac{1}{t} + \bar{c}t + x(t) \right) \right) \right\|_{L^2} \\ &\lesssim |b(t)|^{5/8} + \left| 1 - \frac{\lambda(t)}{t} \right| + \left| \frac{1}{t} \left(\frac{1}{t} + \bar{c}t + x(t) \right) \right| \lesssim t, \end{aligned}$$

and (4.33) is proved.

Let (see (4.36))

$$A(t) = \inf_{\lambda_0 > 0, x_0 \in \mathbb{R}} \|S(t) - \mathcal{Q}_{\lambda_0}(\cdot - x_0)\|_{L^2}^2 = \inf_{\lambda_1 > 0, x_1 \in \mathbb{R}} \|\mathcal{Q}_{b(t)} + \varepsilon(t) - \mathcal{Q}_{\lambda_1}(\cdot - x_1)\|_{L^2}^2.$$

Let $\lambda_1(t)$ and $x_1(t)$ realize the infimum in the definition of $A(t)$. (The existence, uniqueness and regularity of $\lambda_1(t)$ and $x_1(t)$ follow by standard arguments.)

Note that by extremality of λ_1 and x_1 ,

$$\int (\mathcal{Q}_{b+\varepsilon} - \mathcal{Q}_{\lambda_1}(\cdot - x_1)) \frac{\partial \mathcal{Q}_{\lambda_1}}{\partial \lambda_1}(\cdot - x_1) = 0, \quad \int (\mathcal{Q}_{b+\varepsilon} - \mathcal{Q}_{\lambda_1}(\cdot - x_1)) \frac{\partial \mathcal{Q}_{\lambda_1}}{\partial x_1}(\cdot - x_1) = 0, \tag{4.39}$$

and from $\|S(t)\|_{L^2} = \|\mathcal{Q}_b + \varepsilon\|_{L^2} = \|\mathcal{Q}\|_{L^2}$,

$$\int (\mathcal{Q}_b + \varepsilon) \frac{\partial}{\partial t} (\mathcal{Q}_b + \varepsilon) = 0,$$

so that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} A(t) \\ &= \int (\mathcal{Q}_b + \varepsilon - \mathcal{Q}_{\lambda_1}(\cdot - x_1)) \left(\frac{\partial}{\partial t} (\mathcal{Q}_b + \varepsilon) - \lambda_1' \frac{\partial \mathcal{Q}_{\lambda_1}}{\partial \lambda_1}(\cdot - x_1) + x_1' \frac{\partial \mathcal{Q}_{\lambda_1}}{\partial x_1}(\cdot - x_1) \right) \\ &= - \int \mathcal{Q}_{\lambda_1}(\cdot - x_1) \frac{\partial}{\partial t} (\mathcal{Q}_b + \varepsilon) = - \int \mathcal{Q}_{\lambda_1}(\cdot - x_1) \left(b_t \frac{\partial \mathcal{Q}_b}{\partial b} + \varepsilon_t \right) \\ &= -b_t \int \mathcal{Q}_{\lambda_1}(\cdot - x_1) \frac{\partial \mathcal{Q}_b}{\partial b} - \int \varepsilon_t \left(\mathcal{Q}_{\lambda_1}(\cdot - x_1) - \mathcal{Q} - (\lambda_1 - 1)\Lambda \mathcal{Q} \right), \end{aligned}$$

where we have used $\int \varepsilon \mathcal{Q} = \int \varepsilon \Lambda \mathcal{Q} = 0$.

To estimate this term, we now claim that from (4.39) and (4.37), (4.38),

$$|\lambda_1 - 1| \lesssim \left| \int (\mathcal{Q}_b + \varepsilon - \mathcal{Q}) \Lambda \mathcal{Q} \right| \lesssim t^2, \quad |x_1| \lesssim \left| \int (\mathcal{Q}_b + \varepsilon - \mathcal{Q}) \mathcal{Q}' \right| \lesssim t^3$$

(the extra smallness of $|x_1|$ is due to $(P, \mathcal{Q}') = 0$). Using $b_t \sim -2t$ and the equation of ε_t (after integration by parts, and using (4.37) and (4.38)), we obtain

$$\frac{1}{2} \frac{d}{dt} A(t) = 2t \int P \mathcal{Q} + O(t^2). \quad \square$$

5. Uniqueness

In this section we prove the uniqueness statement, i.e. part (ii) of Theorem 1.3. The strategy is to rerun the monotonicity machinery of Proposition 2.9 for the difference of two solutions. The reintegration of the Lyapunov functional backwards from blow up time *using the sharp a priori bounds of Proposition 4.1* will show that this difference is zero. The proof is delicate because as in [48], we only have a finite order expansion of the approximate solution and of the error. Therefore, reintegrating the difference of the modulation equations requires sharp dispersive controls on the difference of two solutions to close the estimates.

5.1. Reduction of the proof. We consider the minimal mass blow up solution constructed in Corollary 4.3. Let $u_2(t)$ be another minimal mass solution $S(t, x) = u_1(t, x)$ of (1.1) which blows up in finite time. From Proposition 4.1, $u_2(t)$ is defined on a maximal time interval of the form $(-\infty, T)$ or (T, ∞) for a finite time T . By time translation invariance, we may assume that $u_2(t)$ is defined on $(0, \infty)$ and blows backwards as $t \downarrow 0$. Let $t_0 > 0$ be small such that u_1 and u_2 admit the decomposition of Lemma 2.6 on $(0, t_0]$ (see also Lemma 2.11)

$$\varepsilon_i(s, y) = \lambda_i^{1/2}(s)u_i(t_i(s), \lambda_i(s)y + x_i(s)) - Q_{b_i(s)}(y),$$

where $t_i(s)$ satisfies $dt_i/ds = \lambda_i^3$, $t_i(-1) = t_0$. Applying Proposition 4.1 to u_2 shows that estimates (4.2)–(4.12) hold for $u_2(t)$, for some $\ell^*(u_2)$ and $x^*(u_2)$.

Using scaling and translation invariance (see Remark 4.2), we assume further that the limits as defined in Proposition 4.1 are equal:

$$\ell^*(u_2) = \ell^*(S) = 1, \quad x^*(u_2) = x^*(S) = 0.$$

The uniqueness statement reduces to proving that

$$u_1 \equiv u_2. \tag{5.1}$$

Note that for $i = 1, 2$, ε_i satisfies, on $(-\infty, -1] \times \mathbb{R}$,

$$(\varepsilon_i)_s - (L\varepsilon_i)_y + b_i \Lambda \varepsilon_i = \Gamma_i (\Lambda Q_{b_i} + \Lambda \varepsilon_i) + X_i (Q_{b_i} + \varepsilon_i)_y + \Psi_i - (R_i(\varepsilon_i))_y$$

with

$$\begin{aligned} \Gamma_i &= (\lambda_i)_s / \lambda_i + b_i, & X_i &= (x_i)_s / \lambda_i - 1, \\ \Psi_i &= \Psi_{b_i} - (b_i)_s (\chi_{b_i} + \gamma y (\chi_{b_i})_y) P, & \Psi_b &\text{ being defined in (2.13),} \\ R_i(\varepsilon_i) &= 5(Q_{b_i}^4 - Q^4)\varepsilon_i + (\varepsilon_i + Q_{b_i})^5 - 5Q_{b_i}^4 \varepsilon_i - Q_{b_i}^5. \end{aligned}$$

We form the difference

$$\varepsilon(s, y) = \varepsilon_2(s, y) - \varepsilon_1(s, y),$$

which satisfies the orthogonality conditions (2.21) and the equation

$$\varepsilon_s - (L\varepsilon)_y = \Gamma \Lambda Q_{b_2} + X(Q_{b_2})_y + \frac{(\lambda_2)_s}{\lambda_2} \Lambda \varepsilon + E + F_y \tag{5.2}$$

with

$$\begin{aligned} \Gamma &= \Gamma_2 - \Gamma_1, \quad b = b_2 - b_1, \quad X = X_2 - X_1, \\ E &= (\Gamma - b)\Lambda\varepsilon_1 + \Gamma_1\Lambda(Q_{b_2} - Q_{b_1}) + (\Psi_2 - \Psi_1), \end{aligned} \tag{5.3}$$

$$F = X_1(Q_{b_2} - Q_{b_1}) + X_2\varepsilon + X\varepsilon_1 - R_2(\varepsilon_2) + R_1(\varepsilon_1). \tag{5.4}$$

For B as in Proposition 2.9, we consider $100 < \bar{B} < \frac{1}{50}B$, large enough (in the next lemma, we need \bar{B} large, so we take a possibly larger universal B in Proposition 2.9). We define the norms

$$\bar{\mathcal{N}}(s) = \int \varepsilon_y^2(s, y)\psi_{\bar{B}}(y) dy + \int \varepsilon^2(s, y)\varphi_{\bar{B}}(y) dy, \quad \bar{\mathcal{N}}_{\text{loc}}(s) = \int \varepsilon^2(s, y)\varphi'_{\bar{B}}(y) dy.$$

The key to the proof of uniqueness is the following proposition which revisits Proposition 2.9 for ε :

Proposition 5.1 (Bounds on the difference). *For $|s|$ large, we have:*

(i) (Refined control of b) *Let $J_2 = (\varepsilon, \rho_2)$ with ρ_2 given by (2.39). Then*

$$|J_2| \lesssim \bar{\mathcal{N}}^{1/2}, \quad \left| \frac{d}{ds} \left\{ s^2 \left(b + \frac{J_2}{2s} \right) \right\} \right| \lesssim s^2 \int \varepsilon^2 e^{-|y|/10} + |s|^{1/2}|b| + |s|^{1/2}\bar{\mathcal{N}}^{1/2}. \tag{5.5}$$

(ii) (Refined bounds) *Let*

$$\begin{aligned} \mathcal{F}(s) &= \int [\psi_{\bar{B}}(\varepsilon_y^2 - 5Q^4\varepsilon^2 - \varepsilon^6/3) + \varphi_{\bar{B}}\varepsilon^2](s, y) dy \\ &\quad + \frac{1}{\sqrt{|s|}} \int e^{\lambda_2(s)y} \varepsilon^2(s, y) dy. \end{aligned} \tag{5.6}$$

Then

$$\bar{\mathcal{N}} + \frac{1}{\sqrt{|s|}} \int e^{\lambda_2 y} \varepsilon^2 \lesssim \mathcal{F} \lesssim \bar{\mathcal{N}} + \frac{1}{\sqrt{|s|}} \int e^{\lambda_2 y} \varepsilon^2. \tag{5.7}$$

Moreover, there exists $\mu > 0$ such that, for $|s|$ large,

$$\frac{d}{ds}(s^2\mathcal{F}) + \mu s^2 \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_{\bar{B}} \lesssim |s|^{11/10}b^2. \tag{5.8}$$

Remark 5.2. The first term in the definition of \mathcal{F} in (5.6) corresponds to a refined combination of virial estimates and monotonicity properties, which was used in [29] (see also Proposition 2.9 of the present paper). Unfortunately, the scaling term of the equation of ε , i.e. the term $((\lambda_2)_s/\lambda_2)\Lambda\varepsilon$, produces bad a priori lower order terms which prevent us from closing the estimates as in [29]. To control these terms we have to add to the definition of \mathcal{F} the second term $|s|^{-1/2} \int e^{\lambda_2 y} \varepsilon^2$, which is a lower order corrective term. Note that this term is scaling invariant and thus it does not produce such bad terms.

The next two sections are devoted to the proof of Proposition 5.1.

5.2. Proof of (i). We start with the control of the modulation parameters and the proof of the improved bound (5.5).

Step 1. Modulation equations. We start by computing the modulation equations and state the following bounds:

$$|\Gamma| + |X| \lesssim \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|}, \tag{5.9}$$

$$\left| \Gamma - \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{1}{|s|} \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|}, \tag{5.10}$$

$$|b_s| \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{1}{|s|} \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|}. \tag{5.11}$$

Indeed, we compute the modulation parameters Γ, X using (5.2) and the orthogonality conditions (2.21). We argue as for the proof of (2.29), (2.30) (see [29]), taking the scalar product of the equation of ε by $\Lambda Q, y\Lambda Q$ and then by Q . We obtain

$$\begin{aligned} |\Gamma| + |X| &\lesssim \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} \\ &\quad + |b| \left(|b_1| + |b| + |\Gamma_1| + |X_1| + \left(\int \varepsilon_1^2 e^{-|y|/10} \right)^{1/2} \right) + |b_s|, \\ |b_s| &\lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2}}{|s|} \\ &\quad + |b| \left(|b_1| + |b| + |\Gamma_1| + |X_1| + \left(\int \varepsilon_1^2 e^{-|y|/10} \right)^{1/2} \right) + |b_1| (|\Gamma| + |X|). \end{aligned}$$

Next, using estimates (4.6)–(4.9) for ε_1 , we find (5.9) and (5.11).

Note that estimate (5.9) can be improved to

$$\begin{aligned} \left| \Gamma - \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| &\lesssim \int \varepsilon^2 e^{-|y|/10} + \left(|b| + \frac{1}{|s|} \right) \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|} \\ &\lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{1}{|s|} \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|}, \end{aligned}$$

which is (5.10).

Step 2. Proof of (i). The estimate $|J_2| \lesssim \bar{\mathcal{N}}^{1/2}$ follows from the properties of ρ_2 :

$$|\rho_2| \lesssim \mathbf{1}_{y>0} + e^{-|y|/10} \mathbf{1}_{y<0}.$$

We now turn to the proof of the refined equation of b . We claim that

$$|b_s + 4b_2b + b_2(J_2)_s| \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{\bar{\mathcal{N}}^{1/2}}{|s|^{3/2}} + \frac{|b|}{|s|^{3/2}}, \tag{5.12}$$

which follows by combining the following two estimates:

$$|b_s + 4b_2b - b_2(\varepsilon, L(\rho'_2))| \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|^{3/2}} + \frac{|b|}{|s|^{3/2}}, \tag{5.13}$$

$$|(J_2)_s + (\varepsilon, L(\rho'_2))| \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{\bar{\mathcal{N}}^{1/2}}{|s|} + \frac{|b|}{|s|}. \tag{5.14}$$

Assume (5.13), (5.14). From (5.14),

$$|(J_2)_s| \lesssim \int \varepsilon^2 e^{-|y|/10} + \bar{\mathcal{N}}^{1/2} + \frac{|b|}{|s|},$$

and expanding b_2 in (5.12) according to (4.9) yields (5.5).

Proof of (5.13). Taking the scalar product of the equation of ε by Q , we obtain (using $LQ' = 0$)

$$0 = \frac{d}{ds}(\varepsilon, Q) = \Gamma(\Lambda Q_{b_2}, Q) + X((Q_{b_2})_y, Q) + \frac{(\lambda_2)_s}{\lambda_2}(\Lambda \varepsilon, Q) + (E, Q) - (F - \varepsilon^5, Q'). \tag{5.15}$$

Using the definition of Q_b in (2.10), we have $|(\Lambda Q_{b_2}, Q) - (\Lambda P, Q)b_2| \lesssim |s|^{-10}$, and thus using (5.10), we obtain

$$\left| \Gamma(\Lambda Q_{b_2}, Q) - b_2(\Lambda P, Q) \frac{(\varepsilon, L(\Lambda Q)')}{\|\Lambda Q\|_{L^2}^2} \right| \lesssim \frac{\int \varepsilon^2 e^{-|y|/10}}{|s|} + \frac{|b|}{|s|^2} + \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{s^2}.$$

Similarly, since $|((Q_{b_2})_y, Q)| \lesssim |s|^{-10}$, using (5.9), we get

$$|X((Q_{b_2})_y, Q)| \leq \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2} + |b|}{|s|^{10}}.$$

Now, we compute (E, Q) . By the expression of Ψ_b (see [29, (2.17)]), and the formula $((10P^2Q^3)' + \Lambda P, Q) = \frac{1}{8}\|Q\|_{L^1}^2$, we have

$$(\Psi_{b_2} - \Psi_{b_1}, Q) = -\frac{1}{4}b_2b\|Q\|_{L^1}^2 + O(b/s^2).$$

Next, using the expression of Φ_b and $(P, Q) = \frac{1}{16}\|Q\|_{L^1}^2$,

$$(\Phi_{b_2} - \Phi_{b_1}, Q) = -\frac{1}{16}b_s\|Q\|_{L^1}^2 + O(b/s^{10}).$$

Thus,

$$(\Psi_2 - \Psi_1, Q) = -\frac{1}{16}\|Q\|_{L^1}^2(b_s + 4b_2b) + O(b/s^2). \tag{5.16}$$

Since $(\Lambda \varepsilon_1, Q) = -(\varepsilon_1, \Lambda Q) = 0$ and (using (4.8) on ε_1)

$$|\Gamma_1(\Lambda(Q_{b_2} - Q_{b_1}), Q)| \lesssim |s|^{-3/2}|b|,$$

we obtain

$$(E, Q) = -\frac{1}{16} \|Q\|_{L^1}^2 (b_s + 4b_2b) + O(b/|s|^{3/2}).$$

Now, we compute $(F - \varepsilon^5, Q')$. First, since $|X_1| + |X_2| + \int |\varepsilon_1| e^{-|y|/10} \lesssim |s|^{-3/2}$, we have

$$|X_1(Q_{b_2} - Q_{b_1}, Q')| + |X_2(\varepsilon, Q')| + |X(\varepsilon_1, Q')| \lesssim \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|^{3/2}} + \frac{|b|}{|s|^{3/2}}.$$

Second, we estimate $(R_2(\varepsilon_2) - R_1(\varepsilon_1), Q')$. From the expression of $R_i(\varepsilon_i)$, we observe that

$$\begin{aligned} |R_2(\varepsilon_2) - R_1(\varepsilon_1) - 20b_2PQ^3\varepsilon| &\leq |R_2(\varepsilon_2) - R_2(\varepsilon_1) - 20b_2PQ^3\varepsilon| + |R_2(\varepsilon_1) - R_1(\varepsilon_1)| \\ &\lesssim |\varepsilon|(|s|^{-2} + |\varepsilon_1| + |\varepsilon_2|) + |b| |\varepsilon_1|, \end{aligned}$$

and so by (4.7),

$$|(R_2(\varepsilon_2) - R_1(\varepsilon_1), Q') - 20b_2(\varepsilon, PQ^3Q')| \lesssim \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|^{3/2}} + \frac{|b|}{|s|^{3/2}}.$$

We have thus obtained, for this term,

$$|((F - \varepsilon^5)_y, Q) + 20b_2(\varepsilon, PQ^3Q')| \lesssim \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|^{3/2}} + \frac{|b|}{|s|^{3/2}}.$$

Inserting the above computations into (5.15), we obtain

$$\begin{aligned} \left| b_s + 4b_2b - \frac{16}{\|Q\|_{L^1}^2} b_2 \left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q') \right] \right| \\ \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|^{3/2}} + \frac{|b|}{|s|^{3/2}}. \end{aligned}$$

Using the formula⁷ from [29, proof of Lemma 2.7],

$$(\varepsilon, L(\rho_2')) = \frac{16}{\|Q\|_{L^1}^2} \left[\frac{(\Lambda P, Q)}{\|\Lambda Q\|_{L^2}^2}(\varepsilon, L(\Lambda Q)') + 20(\varepsilon, PQ^3Q') \right],$$

we obtain (5.13).

Proof of (5.14). To complete the proof of (5.12), we take the scalar product of the equation of ε by ρ_2 . We obtain first

$$\begin{aligned} \frac{d}{ds} J_2 + \frac{(\lambda_2)_s}{\lambda_2}(\varepsilon, \Lambda\rho_2) &= -(\varepsilon, (L\rho_2)') + \Gamma(\Lambda Q_{b_2}, \rho_2) + X((Q_{b_2})_y, \rho_2) \\ &\quad + (E, \rho_2) + (-F + \varepsilon^5, \rho_2'). \end{aligned}$$

⁷ Which of course motivates the definition of ρ_2 in (2.39).

Note that

$$\left| \frac{(\lambda_2)_s}{\lambda_2}(\varepsilon, \Lambda\rho_2) \right| \leq \left| \frac{(\lambda_2)_s}{\lambda_2} \right| (|J_2| + |(\varepsilon, y\rho'_2)|) \lesssim \frac{\bar{N}^{1/2}}{|s|}.$$

Using the orthogonality $(\Lambda Q, \rho_2) = 0$ (see [29]), and (5.9), we have

$$|\Gamma(\Lambda Q_{b_2}, \rho_2)| = |\Gamma b_2(\Lambda P \chi_{b_2}, \rho_2)| \lesssim \left[\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|} \right] \frac{1}{|s|},$$

and similarly, using $(Q, \rho'_2) = 0$,

$$\begin{aligned} |X((Q_{b_2})_y, \rho_2)| + |(E, \rho_2)| + |(-F + \varepsilon^5, \rho'_2)| \\ \lesssim \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2} + |b| + \bar{N}^{1/2}}{|s|} + \int \varepsilon^2 e^{-|y|/10}, \end{aligned}$$

and (5.14) is proved.

5.3. Proof of (ii). The functional \mathcal{F} in (5.6) is defined similarly to Proposition 2.9 for a parameter \bar{B} large enough but smaller than $B/10$ where B is used in Proposition 2.9.

Step 1. Coercivity of \mathcal{F} . The upper and lower bounds (5.7) on \mathcal{F} follow from the coercivity of the linearized energy $\int(\varepsilon_y^2 + \varepsilon^2 - 5Q^4\varepsilon^2)$ under the orthogonality conditions (2.21) together with standard localization arguments. We refer to [29, proof of Proposition 3.1(iii)], for example, for more details.

Step 2. Proof of (5.8). We now turn to the proof of the monotonicity (5.8). We decompose $\mathcal{F} = \bar{\mathcal{F}}_1 + |s|^{-1/2}\bar{\mathcal{F}}_2$ with

$$\bar{\mathcal{F}}_1 = \int [\psi_{\bar{B}}(\varepsilon_y^2 - 5Q^4\varepsilon^2) + \varphi_{\bar{B}}\varepsilon^2], \quad \bar{\mathcal{F}}_2 = \int e^{\lambda_2 y} \varepsilon^2,$$

and state the following monotonicity formulas for $|s|$ large enough:

$$\frac{d\bar{\mathcal{F}}_1}{ds} + \mu_1 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\bar{B}} \lesssim \frac{b^2}{|s|} + \frac{1}{|s|} \int_{y<0} |y| e^{-|y|/\bar{B}} \varepsilon^2, \tag{5.17}$$

$$\frac{d\bar{\mathcal{F}}_2}{ds} + \frac{\mu_2}{\sqrt{|s|}} \int e^{\lambda_2 y} (\varepsilon_y^2 + \varepsilon^2) \lesssim \int \varepsilon^2 e^{-|y|/10} + \frac{|b|^2}{|s|^{2/5}}. \tag{5.18}$$

Assume (5.17), (5.18); then for $|s|$ large,

$$\begin{aligned} \frac{d}{ds}(s^2\bar{\mathcal{F}}_1 + |s|^{3/2}\bar{\mathcal{F}}_2) &= 2s\bar{\mathcal{F}}_1 - \frac{3}{2}|s|^{1/2}\bar{\mathcal{F}}_2 + s^2\frac{d}{ds}\bar{\mathcal{F}}_1 + |s|^{3/2}\frac{d}{ds}\bar{\mathcal{F}}_2 \\ &\leq -\mu_1 s^2 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\bar{B}} - \mu_2 |s| \int e^{\lambda_2 y} \varepsilon^2 + C|s|^{11/10} b^2 + C|s| \int_{y<0} |y| e^{-|y|/\bar{B}} \varepsilon^2 \\ &\quad + C|s|^{3/2} \int \varepsilon^2 e^{-|y|/10} \\ &\leq -\frac{\mu_1}{2} s^2 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\bar{B}} - \mu_2 |s| \int e^{\lambda_2 y} \varepsilon^2 + C|s|^{11/10} b^2 + C_1 |s| \int_{y<0} |y| e^{-|y|/\bar{B}} \varepsilon^2. \end{aligned}$$

Then, for $0 < \kappa < \mu_1/(4\bar{B}C_1)$ and $|s|$ large enough (depending on \bar{B}),

$$\begin{aligned} C_1|s| \int_{y<0} |y|e^{-|y|/\bar{B}} \varepsilon^2 &= C_1|s| \int_{\kappa s < y < 0} |y|e^{-|y|/\bar{B}} \varepsilon^2 + C_1|s| \int_{y < \kappa s} |y|e^{-|y|/\bar{B}} \varepsilon^2 \\ &\leq C_1\bar{B}\kappa s^2 \bar{\mathcal{N}}_{\text{loc}} + C_1|s| \sup_{y < \kappa s} (|y|e^{-|y|/(2\bar{B})}) \int e^{-|y|/(2\bar{B})} \varepsilon^2 \\ &\leq \frac{\mu_1}{4} s^2 \bar{\mathcal{N}}_{\text{loc}} + \frac{\mu_2}{2} |s| \int e^{\lambda_2 y} \varepsilon^2, \end{aligned}$$

and thus

$$\frac{d}{ds} (s^2 \bar{\mathcal{F}}_1 + |s|^{3/2} \bar{\mathcal{F}}_2) + \frac{\mu_1}{4} s^2 \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + \frac{\mu_2}{2} |s| \int e^{\lambda_2 y} \varepsilon^2 \lesssim |s|^{11/10} b^2,$$

which implies (5.8).

Step 3. *Proof of (5.17).* We compute the time derivative of $\bar{\mathcal{F}}_1$ using (5.2):

$$\begin{aligned} \frac{1}{2} \frac{d\bar{\mathcal{F}}_1}{ds} &= \int \partial_s \varepsilon [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \\ &= \int \left[(L\varepsilon)_y + \Gamma \Lambda Q_{b_2} + X(Q_{b_2})_y + \frac{(\lambda_2)_s}{\lambda_2} \Lambda \varepsilon + E + F_y \right] \\ &\quad \times [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon]. \end{aligned}$$

We now estimate all these terms as in [29, proof of Proposition 3.1(i)].

- First, we claim that for \bar{B} large enough, for some $\mu_1 > 0$,

$$\int (L\varepsilon)_y [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \leq -\mu_1 \int \varphi'_B (\varepsilon_y^2 + \varepsilon^2). \tag{5.19}$$

The proof is mainly based on local virial estimates for ε and explicit computations similar to the ones for the term $f_{1,1}^{(i)}$ in [29, proof of Proposition 3.1]. Here, computations are easier than in [29]. We sketch these computations and estimates for the sake of completeness.

By explicit computations (mainly integrations by parts, see [29] for details), one gets

$$\begin{aligned} &2 \int (L\varepsilon)_y [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \\ &= - \int [3\psi'_B \varepsilon_{yy}^2 + (3\varphi'_B + \psi'_B - \psi_B'') \varepsilon_y^2 + (\varphi'_B - \varphi_B'') \varepsilon^2] \\ &\quad + \int 5Q^4 \varepsilon^2 (\varphi'_B - \psi'_B) + \int 20Q^3 Q' \varepsilon^2 (\psi_B - \varphi_B) + 10 \int \psi'_B \varepsilon_y \{4Q' Q^3 \varepsilon + Q^4 \varepsilon_y\} \\ &\quad - \int \psi'_B \{ (5Q^4 \varepsilon)^2 - 10Q^4 \varepsilon (-\varepsilon_{yy} + \varepsilon) \} \\ &= I^< + I^{\sim} + I^> \end{aligned}$$

where $I^<, \sim, >$ respectively correspond to integration on $y < -\bar{B}/2$, $|y| \leq \bar{B}/2$, $y > \bar{B}/2$.

In the region $y > \bar{B}/2$, we have $\psi'_{\bar{B}}(y) = 0$, and thus

$$I^> = - \int_{y>\bar{B}/2} [3\varphi'_{\bar{B}}\varepsilon_y^2 + (\varphi'_{\bar{B}} - \varphi'''_{\bar{B}})\varepsilon^2] + \int_{y>\bar{B}/2} 5Q^4\varepsilon^2\varphi'_{\bar{B}} + \int_{y>\bar{B}/2} 20Q^3Q'\varepsilon^2(1 - \varphi_{\bar{B}}).$$

Using $\varphi'''_{\bar{B}} \lesssim \bar{B}^{-2}\varphi'_{\bar{B}}$ and the exponential decay of Q we obtain, for \bar{B} large enough,

$$I^> \leq - \int_{y>\bar{B}/2} \varphi'_{\bar{B}}(\varepsilon_y^2 + \varepsilon^2) + \frac{C}{\bar{B}^2} \int_{y>\bar{B}/2} \varphi'_{\bar{B}}\varepsilon^2 \leq -\frac{1}{2} \int_{y>\bar{B}/2} \varphi'_{\bar{B}}(\varepsilon_y^2 + \varepsilon^2).$$

In the region $|y| < \bar{B}/2$, we have $\varphi_{\bar{B}}(y) = 1 + y/\bar{B}$, $\psi_{\bar{B}}(y) = 1$ and $\psi'''_{\bar{B}} = \psi'_{\bar{B}} = 0$. Thus,

$$I^{\sim} = -\frac{1}{\bar{B}} \int_{|y|<\bar{B}/2} [3\varepsilon_y^2 + \varepsilon^2 - 5Q^4\varepsilon^2 + 20yQ^3Q'\varepsilon^2] + \frac{1}{\bar{B}} \frac{5}{3} \int_{|y|<\bar{B}/2} \varepsilon^6.$$

From [29, Lemma 3.4] (local virial estimate), for some $\mu > 0$ and for \bar{B} large,

$$\int_{|y|<\bar{B}/2} [3\varepsilon_y^2 + \varepsilon^2 + 15Q^4\varepsilon^2 - 20yQ^3Q'\varepsilon^2] \geq \mu \int_{|y|<\bar{B}/2} (\varepsilon_y^2 + \varepsilon^2) - \frac{1}{\bar{B}} \int \varepsilon^2 e^{-|y|/2},$$

and since $\|\varepsilon\|_{L^\infty}^4 \lesssim |s|^{-2}$, we have $\int_{|y|<\bar{B}/2} \varepsilon^6 \lesssim |s|^{-2} \int_{|y|<\bar{B}/2} \varepsilon^2$, so that for \bar{B} large and $|s|$ large,

$$I^{\sim} \leq -\frac{\mu}{2} \frac{1}{\bar{B}} \int_{|y|<\bar{B}/2} (\varepsilon_y^2 + \varepsilon^2) + \frac{1}{\bar{B}^2} \int \varepsilon^2 e^{-|y|/2}.$$

In the region $y < -\bar{B}/2$, we use $\psi'''_{\bar{B}} \lesssim \bar{B}^{-2}\psi'_{\bar{B}}$, $\psi'_{\bar{B}} \lesssim \varphi'_{\bar{B}}$, $\varphi'''_{\bar{B}} \lesssim \bar{B}^{-2}\varphi'_{\bar{B}}$, and the exponential decay of Q to obtain as before, for \bar{B} large enough,

$$I^< \leq -\frac{1}{2} \int_{y<-\bar{B}/2} \varphi'_{\bar{B}}(\varepsilon_y^2 + \varepsilon^2).$$

Gathering the estimates for $I^>$, I^{\sim} and $I^<$, we get (5.19).

- Next, arguing as for estimating $f_{1,2}^{(i)}$ and $f_{1,3}^{(i)}$ in [29, proof of Proposition 3.1], we find

$$\left| \Gamma \int \Lambda Q_{b_2} [-\psi'_{\bar{B}}\varepsilon_y + \psi_{\bar{B}}(L\varepsilon) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| \leq \frac{\mu_1}{100} \bar{N}_{\text{loc}} + C \frac{b^2}{s^2}, \tag{5.20}$$

$$\left| X \int (Q_{b_2})_y [-\psi'_{\bar{B}}\varepsilon_y + \psi_{\bar{B}}(L\varepsilon) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| \leq \frac{\mu_1}{100} \bar{N}_{\text{loc}} + C \frac{b^2}{s^2}. \tag{5.21}$$

Indeed, using the algebraic facts

$$(\Lambda Q, L\varepsilon) = -2(Q, \varepsilon) = 0, \quad (\varepsilon, y\Lambda Q) = (\varepsilon, \Lambda Q) = (\varepsilon, yQ') = 0, \quad LQ' = 0,$$

the exponential decay of Q , $|b_2| \leq 1/\sqrt{|s|}$, and integrating by parts to remove all derivative from ε , we obtain

$$\begin{aligned} \left| \int \Lambda Q_{b_2}[-\psi'_{\bar{B}}\varepsilon_y + \psi_{\bar{B}}(L\varepsilon) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| &\lesssim \left(\frac{1}{\bar{B}} + \frac{\sqrt{\bar{B}}}{\sqrt{|s|}} \right) \bar{\mathcal{N}}_{\text{loc}}^{1/2}, \\ \left| \int (Q_{b_2})_y[-\psi'_{\bar{B}}\varepsilon_y + \psi_{\bar{B}}(L\varepsilon) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| &\lesssim \left(\frac{1}{\bar{B}} + \frac{\sqrt{\bar{B}}}{\sqrt{|s|}} \right) \bar{\mathcal{N}}_{\text{loc}}^{1/2}. \end{aligned}$$

Thus, using (5.9), the estimates (5.20) and (5.21) follow for \bar{B} large and $|s|$ large.

At this point, \bar{B} is fixed and thus B in Proposition 2.9 is also fixed. B and \bar{B} are universal constants.

- The next term is similar to $f_3^{(i,j)}$ in [29]. Using the properties of $\psi_{\bar{B}}$ and $\varphi_{\bar{B}}$ and (2.29) we have

$$\begin{aligned} \frac{(\lambda_2)_s}{\lambda_2} \int \Lambda \varepsilon[-(\psi_{\bar{B}}\varepsilon_y)_y + \varphi_{\bar{B}}\varepsilon - \psi_{\bar{B}}(5Q^4\varepsilon + \varepsilon^5)] \\ = -\frac{1}{2} \frac{(\lambda_2)_s}{\lambda_2} \int y\varphi'_{\bar{B}}\varepsilon^2 + O\left(\frac{1}{|s|} \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_{\bar{B}}\right). \end{aligned}$$

From (4.5), (4.8) and (4.9), we have, for s large,

$$\frac{1}{s} \leq -\frac{(\lambda_2)_s}{\lambda_2} \leq \frac{1}{4s} < 0,$$

and thus

$$-\frac{1}{2} \frac{(\lambda_2)_s}{\lambda_2} \int y\varphi'_{\bar{B}}\varepsilon^2 \lesssim \frac{1}{|s|} \int_{y<0} |y|e^{-|y|/B}\varepsilon^2.$$

Eventually, we have proved that for s large.

$$\begin{aligned} \frac{(\lambda_2)_s}{\lambda_2} \int \Lambda \varepsilon[-(\psi_{\bar{B}}\varepsilon_y)_y + \varphi_{\bar{B}}\varepsilon - \psi_{\bar{B}}(5Q^4\varepsilon + \varepsilon^5)] \\ \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_{\bar{B}} + \frac{C}{|s|} \int_{y<0} |y|e^{-|y|/B}\varepsilon^2. \end{aligned}$$

We now estimate terms coming from E and F . For this, we will need higher order Sobolev estimates on ε_1 and ε_2 coming from Proposition 4.1. Since $\bar{B} < B/50$, from (4.12) and (4.13) we have, for all $\frac{9}{50}\frac{1}{\bar{B}} < \omega < \frac{1}{10}$ and $i = 1, 2$,

$$\sum_{k=0}^3 \int (\partial_y^k \varepsilon_i)^2(s, y)e^{\omega y} dy + \int_{-\infty}^s \sum_{k=0}^4 \int (\partial_y^k \varepsilon_i)^2(s', y)e^{\omega y} dy ds' \lesssim \frac{1}{|s|}, \tag{5.22}$$

$$\|((\varepsilon_i)_{yy}^2 + (\varepsilon_i)_y^2)(s)e^{\omega y}\|_{L^\infty} \lesssim \frac{1}{|s|}. \tag{5.23}$$

- *Estimate for E.* In view of the expression of E in (5.3), the first term to estimate is

$$\begin{aligned} & \left| (\Gamma - b) \int \Lambda \varepsilon_1 [-\psi'_B \varepsilon_y + \psi_{\bar{B}}(L\varepsilon) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| \\ & \lesssim \left(|b| + \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} \right) \left(\int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \right)^{1/2} \\ & \quad \times \left(\int ((\varepsilon_1)_{yy}^2 + (\varepsilon_1)_y^2 + \varepsilon_1^2) (1 + |y|^3) \varphi_{\bar{B}} \right)^{1/2} \\ & \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C b^2 \int ((\varepsilon_1)_{yy}^2 + (\varepsilon_1)_y^2 + \varepsilon_1^2) e^{2/\bar{B}y} dy \\ & \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C \frac{b^2}{|s|}, \end{aligned}$$

using (5.9), integration by parts, and then (5.22). For the next term, we need to estimate $\Lambda(Q_{b_2} - Q_{b_1})$. From Lemma 2.5, we have

$$Q_{b_2} - Q_{b_1} = bP\chi_{b_2} + b_1P(\chi_{b_2} - \chi_{b_1}),$$

and using (4.9),

$$|\chi_{b_2} - \chi_{b_1}| = \left| \int_{b_1}^{b_2} \frac{\partial \chi_b}{\partial b} db \right| \lesssim \sup |y\chi'| \frac{|b_1 - b_2|}{|b_1|}.$$

Thus, $|Q_{b_2} - Q_{b_1}| \lesssim |b| |P|$. Arguing similarly, we obtain

$$|(Q_{b_2} - Q_{b_1})_y| + |(\Lambda Q_{b_2} - \Lambda Q_{b_1})_y| + |\Lambda(Q_{b_2} - Q_{b_1})| \lesssim |b| (\mathbf{1}_{y < 0} + e^{-|y|/10}). \tag{5.24}$$

Using (5.24) and the estimates for Γ_1 from Proposition 4.1 yields

$$\begin{aligned} \left| \int \Gamma_1 \Lambda(Q_{b_2} - Q_{b_1}) [-\psi'_B \varepsilon_y + \psi_{\bar{B}}(L\varepsilon - \varepsilon^5) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| & \lesssim \frac{|b|}{|s|^{3/2}} \bar{\mathcal{N}}_{\text{loc}}^{1/2} \\ & \leq \frac{\mu_1}{100} \bar{\mathcal{N}}_{\text{loc}} + C \frac{b^2}{s^2}. \end{aligned}$$

Next, from the definition of Ψ_{b_i} and Φ_{b_i} (see [29, Lemma 2.5 and (2.17)]), (5.11), (2.30), (4.7), we have

$$\begin{aligned} |\Psi_{b_2} - \Psi_{b_1}| & \lesssim \frac{|b|}{\sqrt{|s|}} (\mathbf{1}_{y < 0} + e^{-|y|/10}), \\ |\Phi_{b_2} - \Phi_{b_1}| & \lesssim \left(|b_s| + \frac{|(b_1)_s| |b|}{|b_1|} \right) (\mathbf{1}_{y < 0} + e^{-|y|/10}) \\ & \lesssim \left(\int \varepsilon^2 e^{-|y|/10} + \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|} + \frac{|b|}{|s|} \right) (\mathbf{1}_{y < 0} + e^{-|y|/10}), \end{aligned}$$

and similar estimates for the derivatives of these terms. In particular, we obtain

$$\begin{aligned}
 & |\Psi_2 - \Psi_1| + |(\Psi_2 - \Psi_1)_y| + |(\Psi_2 - \Psi_1)_{yy}| \\
 & \lesssim \left(\int \varepsilon^2 e^{-|y|/10} + \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|} + \frac{|b|}{\sqrt{|s|}} \right) (\mathbf{1}_{y < 0} + e^{-|y|/10}) \\
 & \lesssim \left(\overline{\mathcal{N}}_{\text{loc}} + \frac{\overline{\mathcal{N}}_{\text{loc}}^{1/2}}{|s|} + \frac{|b|}{\sqrt{|s|}} \right) (\mathbf{1}_{y < 0} + e^{-|y|/10}). \tag{5.25}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & \left| \int (\Psi_2 - \Psi_1) [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \right| \\
 & \lesssim \left(\overline{\mathcal{N}}_{\text{loc}} + \frac{\overline{\mathcal{N}}_{\text{loc}}^{1/2}}{|s|} + \frac{|b|}{\sqrt{|s|}} \right) \overline{\mathcal{N}}_{\text{loc}}^{1/2} \leq \frac{\mu_1}{100} \overline{\mathcal{N}}_{\text{loc}} + C \frac{b^2}{|s|}.
 \end{aligned}$$

In conclusion, for E we have obtained

$$\left| \int E [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \right| \leq \frac{3\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C \frac{b^2}{|s|}.$$

• *Estimate for F .* Similarly, we easily get the following three estimates:

$$\begin{aligned}
 & \left| X_1 \int (Q_{b_2} - Q_{b_1})_y [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \right| \\
 & \lesssim \frac{|b|}{|s|^{3/2}} \overline{\mathcal{N}}_{\text{loc}}^{1/2} \lesssim \frac{\mu_1}{100} \overline{\mathcal{N}}_{\text{loc}} + C \frac{b^2}{s^2}, \\
 & \left| X_2 \int \varepsilon_y [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \right| \\
 & \lesssim \frac{1}{|s|^{3/2}} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B, \\
 & \left| X \int (\varepsilon_1)_y [-\psi'_B \varepsilon_y + \psi_B(L\varepsilon) + (\varphi_B - \psi_B)\varepsilon] \right| \\
 & \lesssim \left(\frac{|b|}{|s|} + \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} \right) \left(\int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B \right)^{1/2} \\
 & \quad \times \left(\int ((\varepsilon_1)_{yy}^2 + (\varepsilon_1)_y^2 + \varepsilon_1^2) e^{2/\overline{B}y} dy \right)^{1/2} \\
 & \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_B + C \frac{b^2}{|s|}.
 \end{aligned}$$

The remaining nonlinear term for F is estimated using the Sobolev bound (5.22). We decompose the nonlinear term as follows:

$$R_2(\varepsilon_2) - R_1(\varepsilon_1) = F_1 + F_2$$

where

$$F_1 = R_2(\varepsilon_1) - R_1(\varepsilon_1), \quad F_2 = R_2(\varepsilon_2) - R_2(\varepsilon_1).$$

Using the expression for R_i and $\|\varepsilon_1\|_{L^\infty} \lesssim |s|^{-1/2}$, we obtain

$$|(F_1)_y| + |(F_1)_{yy}| \lesssim |b|(|\varepsilon_1| + |(\varepsilon_1)_y| + |(\varepsilon_1)_y^2| + |(\varepsilon_1)_{yy}|),$$

and then by (5.23),

$$(|(F_1)_y| + |(F_1)_{yy}|)e^{\frac{y}{4B}} \lesssim |b|(|\varepsilon_1| + |(\varepsilon_1)_y| + |(\varepsilon_1)_{yy}|)e^{\frac{y}{4B}} + \frac{|b|}{\sqrt{|s|}}|(\varepsilon_1)_y|.$$

Thus, integrating by parts, and using the Cauchy–Schwarz inequality and (5.22), we get

$$\begin{aligned} & \left| \int (F_1)_y [-\psi'_B \varepsilon_y + \psi_{\bar{B}}(L\varepsilon) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| \\ & \lesssim \int (|(F_1)_y| + |(F_1)_{yy}|)e^{\frac{y}{2B}}(|\varepsilon| + |\varepsilon_y|)\sqrt{\varphi'_B} \\ & \lesssim |b| \int [(|\varepsilon_1| + |(\varepsilon_1)_y| + |(\varepsilon_1)_{yy}|)e^{\frac{y}{4B}} + |(\varepsilon_1)_y|]e^{\frac{y}{4B}}(|\varepsilon| + |\varepsilon_y|)\sqrt{\varphi'_B} \\ & \lesssim \frac{|b|}{\sqrt{|s|}} \left(\int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B \right)^{1/2} \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B + C \frac{b^2}{s}. \end{aligned}$$

We decompose F_2 as follows:

$$\begin{aligned} F_2 &= 5(Q_{b_2}^4 - Q^4)\varepsilon + 10Q_{b_2}^3(\varepsilon_2^2 - \varepsilon_1^2) + 10Q_{b_2}^2(\varepsilon_2^3 - \varepsilon_1^3) + 5Q_{b_2}(\varepsilon_2^4 - \varepsilon_1^4) \\ &\quad + \varepsilon_2^5 - \varepsilon_1^5 \\ &= \varepsilon[5(Q_{b_2}^4 - Q^4) + 10Q_{b_2}^3(\varepsilon_2 + \varepsilon_1) + 10Q_{b_2}^2(\varepsilon_2^2 + \varepsilon_1\varepsilon_2 + \varepsilon_1^2) \\ &\quad + 5Q_{b_2}(\varepsilon_2^3 + \varepsilon_2^2\varepsilon_1 + \varepsilon_2\varepsilon_1^2 + \varepsilon_1^3) + (\varepsilon_2^4 + \varepsilon_2^3\varepsilon_1 + \varepsilon_2^2\varepsilon_1^2 + \varepsilon_2\varepsilon_1^3 + \varepsilon_1^4)]. \end{aligned}$$

Therefore, by suitable integration by parts, we have

$$\begin{aligned} & \left| \int (F_2)_y [-\psi'_B \varepsilon_y + \psi_{\bar{B}}(L\varepsilon - \varepsilon^5) + (\varphi_{\bar{B}} - \psi_{\bar{B}})\varepsilon] \right| \\ & \lesssim |b_2| \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B + \int (\varepsilon_y^2 + \varepsilon^2)\varphi_{\bar{B}}(|\varepsilon_1| + |\varepsilon_2|) \\ & \quad + \int (\varepsilon_y^2 + \varepsilon^2)\psi_{\bar{B}} \sum_{i=1,2} (|\varepsilon_i| + |(\varepsilon_i)_y| + |(\varepsilon_i)_y^2| + |(\varepsilon_i)_{yy}|). \end{aligned}$$

These terms are next treated as follows:

$$C|b_2| \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2)\varphi'_B,$$

and using (5.23), $\|\varepsilon_i\|_{L^\infty} \lesssim 1/\sqrt{|s|}$ and the notation $y_+ = \max(0, y)$,

$$\begin{aligned} \int (\varepsilon_y^2 + \varepsilon^2) \varphi_{\overline{B}}(|\varepsilon_1| + |\varepsilon_2|) &\lesssim (\|\varepsilon_1(1 + |y_+|)\|_{L^\infty} + \|\varepsilon_2(1 + |y_+|)\|_{L^\infty}) \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\overline{B}} \\ &\leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\overline{B}}. \end{aligned}$$

Finally, using $\psi_{\overline{B}} \lesssim \varphi'_{\overline{B}} e^{y/\overline{B}}$ and (5.23), we get

$$\begin{aligned} C \int (\varepsilon_y^2 + \varepsilon^2) \psi_{\overline{B}} \sum_{i=1,2} (|\varepsilon_i| + |(\varepsilon_i)_y| + |(\varepsilon_i)_y^2| + |(\varepsilon_i)_{yy}|) \\ \lesssim \|(|\varepsilon_i| + |(\varepsilon_i)_y| + |(\varepsilon_i)_y^2| + |(\varepsilon_i)_{yy}|) e^{y/\overline{B}}\|_{L^\infty} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\overline{B}} \leq \frac{\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\overline{B}}. \end{aligned}$$

Combining the above estimates yields the bound

$$\left| \int F_y [-\psi'_{\overline{B}} \varepsilon_y + \psi_{\overline{B}} (L\varepsilon - \varepsilon^5) + (\varphi_{\overline{B}} - \psi_{\overline{B}}) \varepsilon] \right| \leq \frac{7\mu_1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \varphi'_{\overline{B}} + C \frac{b^2}{|s|}.$$

Step 4. *Proof of (5.18).* We compute the time derivative of $\overline{\mathcal{F}}_2$ using (5.2):

$$\begin{aligned} \frac{1}{2} \frac{d\overline{\mathcal{F}}_2}{ds} &= \frac{(\lambda_2)_s}{2} \int y e^{\lambda_2 y} \varepsilon^2 + \int e^{\lambda_2 y} (\partial_s \varepsilon) \varepsilon \\ &= \frac{(\lambda_2)_s}{2} \int y e^{\lambda_2 y} \varepsilon^2 \\ &\quad + \int \left[(L\varepsilon)_y + \Gamma \Lambda Q_{b_2} + X(Q_{b_2})_y + \frac{(\lambda_2)_s}{\lambda_2} \Lambda \varepsilon + E + F_y \right] e^{\lambda_2 y} \varepsilon. \end{aligned}$$

Since $\int \Lambda \varepsilon e^{\lambda_2 y} \varepsilon = -\frac{1}{2} \lambda_2 \int \varepsilon^2 y e^{\lambda_2 y}$, the scaling term cancels. By usual integrations by parts, we get

$$\begin{aligned} \frac{1}{2} \frac{d\overline{\mathcal{F}}_2}{ds} &= -\frac{3}{2} \lambda_2 \int \varepsilon_y^2 e^{\lambda_2 y} - \frac{1}{2} \lambda_2 (1 - \lambda_2^2) \int \varepsilon^2 e^{\lambda_2 y} \\ &\quad + \int (-10Q^3 Q_y + \frac{5}{2} \lambda_2 Q^4) \varepsilon^2 e^{\lambda_2 y} + \int [\Gamma \Lambda Q_{b_2} + X(Q_{b_2})_y + E + F_y] e^{\lambda_2 y} \varepsilon. \end{aligned}$$

Using $\lambda_2 \sim 1/\sqrt{2|s|}$ and the decay properties of Q we get, for $|s|$ large,

$$\begin{aligned} \frac{1}{2} \frac{d\overline{\mathcal{F}}_2}{ds} &\leq -\frac{1}{4\sqrt{|s|}} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} + \int \varepsilon^2 e^{-|y|/10} \\ &\quad + \left| \int [\Gamma \Lambda Q_{b_2} + X(Q_{b_2})_y + E + F_y] e^{\lambda_2 y} \varepsilon \right|. \end{aligned}$$

Now, we estimate the remaining terms. First, from (5.9) and the definition of Q_b ,

$$\begin{aligned} & \left| \Gamma \int \Lambda Q_{b_2} e^{\lambda_2 y} \varepsilon \right| + \left| X \int (Q_{b_2})_y e^{\lambda_2 y} \varepsilon \right| \\ & \leq C \left(\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|} \right) \left(\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b_2|}{\sqrt{\lambda_2}} \left(\int \varepsilon^2 e^{\lambda_2 y} \right)^{1/2} \right) \\ & \leq \frac{\lambda_2}{100} \int \varepsilon^2 e^{\lambda_2 y} + C \int \varepsilon^2 e^{-|y|/10} + C \frac{b^2}{s^2}. \end{aligned}$$

Second, we estimate the terms coming from E . Since

$$\int \Lambda \varepsilon_1 \varepsilon e^{\lambda_2 y} = - \int \varepsilon_1 \Lambda \varepsilon e^{\lambda_2 y} - \lambda_2 \int y \varepsilon_1 \varepsilon e^{\lambda_2 y},$$

we get

$$\begin{aligned} & \left| (\Gamma - b) \int \Lambda \varepsilon_1 e^{\lambda_2 y} \varepsilon \right| \\ & \lesssim \left(\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + |b| \right) \left(\int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} \right)^{1/2} \left(\int (1 + y^2) \varepsilon_1^2 e^{\lambda_2 y} \right)^{1/2}. \end{aligned}$$

We estimate the term involving ε_1 using Proposition 4.1: for some $0 < \omega < \omega' < 1/10$, using $\lambda_2 \geq \frac{19}{20} \lambda_1$ for $|s|$ large, we get

$$\begin{aligned} \int y^2 \varepsilon_1^2 e^{\lambda_2 y} & \leq \int_{y < 0} y^2 \varepsilon_1^2 e^{\lambda_2 y} + \int_{y > 0} y^2 \varepsilon_1^2 e^{\omega y} \\ & \lesssim \sup_{y < 0} [y^2 e^{\lambda_1 y/20}] \int_{y < 0} \varepsilon_1^2 e^{9\lambda_1 y/10} + \int_{y > 0} \varepsilon_1^2 e^{\omega' y} \\ & \lesssim \lambda_1^{-2} \left(\int_{y < 0} \varepsilon_1^2 \right)^{1/10} \left(\int_{y < 0} \varepsilon_1^2 e^{\lambda_1 y} \right)^{9/10} + \int_{y > 0} \varepsilon_1^2 e^{\omega' y} \lesssim |s|^{-9/10}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \left| (\Gamma - b) \int \Lambda \varepsilon_1 e^{\lambda_2 y} \varepsilon \right| & \lesssim |s|^{-9/20} \left(\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + |b| \right) \left(\int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} \right)^{1/2} \\ & \leq C \int \varepsilon^2 e^{-|y|/10} + C \frac{|b|^2}{|s|^{2/5}} + \frac{\lambda_2}{100} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y}. \end{aligned}$$

For the second term coming from E , we use (5.24) and $|\Gamma_1| \lesssim 1/|s|$, so that

$$\begin{aligned} \left| \Gamma_1 \int \Lambda (Q_{b_2} - Q_{b_1}) \varepsilon e^{\lambda_2 y} \right| & \lesssim \frac{|b|}{|s|} \left(\int_{y < 0} |\varepsilon| e^{\lambda_2 y} + \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} \right) \\ & \lesssim \frac{|b|}{|s|} \left(|s|^{1/4} \left(\int_{y < 0} \varepsilon^2 e^{\lambda_2 y} \right)^{1/2} + \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} \right) \\ & \leq \frac{\lambda_2}{100} \int \varepsilon^2 e^{\lambda_2 y} + C \int \varepsilon^2 e^{-|y|/10} + C \frac{|b|^2}{|s|}. \end{aligned}$$

For the last term in E , we use (5.25) and argue similarly:

$$\begin{aligned} & \left| \int (\Psi_2 - \Psi_1) \varepsilon e^{\lambda_2 y} \right| \\ & \lesssim \left(\int \varepsilon^2 e^{-|y|/10} + \frac{(\int \varepsilon^2 e^{-|y|/10})^{1/2}}{|s|} + \frac{|b|}{\sqrt{|s|}} \right) \left(\int_{y < 0} |\varepsilon| e^{\lambda_2 y} + \left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} \right) \\ & \leq \frac{\lambda_2}{100} \int \varepsilon^2 e^{\lambda_2 y} + C \int \varepsilon^2 e^{-|y|/10} + C \frac{|b|^2}{|s|}. \end{aligned}$$

Now, we estimate the terms coming from F . Arguing as before, since $|X_1| \lesssim |s|^{-3/2}$ and using (5.24), we obtain

$$\left| X_1 \int (Q_{b_2} - Q_{b_1})_y \varepsilon e^{\lambda_2 y} \right| \lesssim \frac{\lambda_2}{100} \int \varepsilon^2 e^{\lambda_2 y} + \int \varepsilon^2 e^{-|y|/10} + \frac{|b|^2}{|s|}.$$

Next, since $|X_2| \lesssim |s|^{-3/2}$, $\lambda_2 \lesssim |s|^{-1/2}$, by integration by parts

$$\left| X_2 \int \varepsilon_y \varepsilon e^{\lambda_2 y} \right| \lesssim \frac{1}{s^2} \int \varepsilon^2 e^{\lambda_2 y} \leq \frac{\lambda_2}{100} \int \varepsilon^2 e^{\lambda_2 y}.$$

Then, by integration by parts, (5.9) and (4.11),

$$\begin{aligned} \left| X \int (\partial_y \varepsilon_1) \varepsilon e^{\lambda_2 y} \right| & \lesssim \left(\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + \frac{|b|}{|s|} \right) \left(\int \varepsilon_1^2 e^{\lambda_2 y} \right)^{1/2} \left(\int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} \right)^{1/2} \\ & \lesssim \frac{\lambda_2}{100} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} + \int \varepsilon^2 e^{-|y|/10} + \frac{|b|^2}{|s|}. \end{aligned}$$

Finally, we decompose $R_2(\varepsilon_2) - R_1(\varepsilon_1)$ as follows, using $\|\varepsilon_i\|_{L^\infty} \lesssim |s|^{-1/2}$ and $|b_i| \sim |s|^{-1}$:

$$\begin{aligned} |R_2(\varepsilon_2) - R_1(\varepsilon_1)| & \leq |R_2(\varepsilon_2) - R_2(\varepsilon_1)| + |R_2(\varepsilon_1) - R_1(\varepsilon_1)| \\ & \lesssim |\varepsilon| (|b_2| + e^{-|y|/10}) + |b| |\varepsilon_1| \lesssim |s|^{-1} |\varepsilon| + |\varepsilon| e^{-|y|/10} + |b| |\varepsilon_1|, \end{aligned}$$

and thus, using also (4.11), we estimate the last term coming from F as follows:

$$\begin{aligned} & \left| \int (R_2(\varepsilon_2) - R_1(\varepsilon_1)) (\varepsilon_y + \lambda_2 \varepsilon) e^{\lambda_2 y} \right| \\ & \lesssim \frac{1}{|s|} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} + \int \varepsilon^2 e^{-|y|/10} + |b| \left(\int \varepsilon_1^2 e^{\lambda_2 y} \right)^{1/2} \left(\int \varepsilon^2 e^{\lambda_2 y} \right)^{1/2} \\ & \lesssim \frac{\lambda_2}{100} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda_2 y} + \int \varepsilon^2 e^{-|y|/10} + \frac{|b|^2}{|s|}. \end{aligned}$$

Combining the above bounds yields (5.18).

5.4. Conclusion. We are now in a position to conclude the proof of uniqueness (5.1).

First, recall the following estimates from Proposition 4.1 as $s \rightarrow -\infty$:

$$\bar{\mathcal{N}} \lesssim 1/|s|^3, \quad |J_2| \lesssim \bar{\mathcal{N}}^{1/2} \lesssim 1/|s|^{3/2}, \quad b = o(1/|s|^2). \tag{5.26}$$

Moreover, from (4.11), (4.12) we estimate

$$\begin{aligned} \int e^{\lambda_2 y} \varepsilon^2 &\lesssim \int e^{\lambda_2 y} \varepsilon_1^2 + \int e^{\lambda_2 y} \varepsilon_2^2 \lesssim \int e^{\lambda_2 y} \varepsilon_1^2 + \frac{1}{s^2} \lesssim \int_{y<0} e^{-\lambda_2 |y|} \varepsilon_1^2 + \int_{y>0} e^{\lambda_2 y} \varepsilon_1^2 + \frac{1}{s^2} \\ &\lesssim \int_{y<0} e^{-9/10 \lambda_1 |y|} \varepsilon_1^2 + \int_{y>0} e^{y/10} \varepsilon_1^2 + \frac{1}{s^2} \lesssim \left(\int e^{-\lambda_1 |y|} \varepsilon_1^2 \right)^{9/10} + \frac{1}{s^2} \lesssim \frac{1}{s^{9/5}}. \end{aligned}$$

This yields in particular from (5.7) the bound

$$\mathcal{F} \lesssim 1/s^{2+3/10}. \tag{5.27}$$

Recall from (5.5) and (5.8) (using $\int \varepsilon^2 e^{-|y|/10} \lesssim \bar{\mathcal{N}}_{\text{loc}}$) that

$$\begin{aligned} \frac{d}{ds} \left\{ s^2 \left(b + \frac{J_2}{2s} \right) \right\} &\lesssim s^2 \bar{\mathcal{N}}_{\text{loc}} + |s|^{1/2} |b| + |s|^{1/2} \bar{\mathcal{N}}^{1/2}, \\ \frac{d}{ds} (s^2 \mathcal{F}) + \mu s^2 \bar{\mathcal{N}}_{\text{loc}} &\lesssim |s|^{11/10} b^2. \end{aligned} \tag{5.28}$$

Using also (5.26), it follows that for $K_0 > 0$ large enough,

$$\frac{d}{ds} \left\{ s^2 \left(b + \frac{J_2}{2s} + K_0 \mathcal{F} \right) \right\} \lesssim |s|^{1/2} |b| + |s|^{1/2} \bar{\mathcal{N}}^{1/2} + s^{11/10} b^2 \lesssim |s|^{1/2} |b| + |s|^{1/2} \bar{\mathcal{N}}^{1/2}. \tag{5.29}$$

We now deduce from (5.26), (5.27) the a priori bound

$$s^2 (|b| + |J_2|/|s| + \mathcal{F}) \rightarrow 0 \quad \text{as } s \rightarrow -\infty.$$

We then integrate (5.28) on $(-\infty, s]$:

$$\begin{aligned} s^2 \mathcal{F}(s) &\lesssim \int_{-\infty}^s |s'|^{11/10} b^2(s') ds' \lesssim \int_{-\infty}^s (s')^4 b^2(s') \frac{ds'}{|s'|^{2+9/10}} \\ &\lesssim \frac{1}{|s|^{19/10}} \left(\sup_{s' \in (-\infty, s]} (s')^2 |b(s')| \right)^2, \end{aligned}$$

so that

$$|s| \mathcal{F}^{1/2}(s) \lesssim \frac{1}{|s|^{19/20}} \sup_{s' \in (-\infty, s]} (s')^2 |b(s')|. \tag{5.30}$$

Next, by integration of (5.29) on $(-\infty, s]$ and (5.7) and (5.26),

$$\begin{aligned}
 s^2|b(s)| &\lesssim |s| |J_2| + s^2 \mathcal{F} + \int_{-\infty}^s (|s'|^{1/2}|b| + |s'|^{1/2}\overline{\mathcal{N}}^{1/2}) ds' \\
 &\lesssim |s|\overline{\mathcal{N}}^{1/2} + s^2 \mathcal{F} + \int_{-\infty}^s |b| |s'|^2 \frac{ds'}{|s'|^{3/2}} + \int_{-\infty}^s |s'|^{7/4}\overline{\mathcal{N}}^{1/2} \frac{ds'}{|s'|^{5/4}} \\
 &\lesssim \frac{1}{|s|^{1/4}} \sup_{s' \in (-\infty, s]} \{(s')^2|b(s')| + |s'|^{7/4}\mathcal{F}^{1/2}(s')\}. \tag{5.31}
 \end{aligned}$$

Putting together (5.30) and (5.31), we get

$$s^2|b(s)| + |s|^{7/4}\mathcal{F}^{1/2}(s) \lesssim \frac{1}{|s|^{1/5}} \sup_{s' \in (-\infty, s]} \{(s')^2|b(s')| + (s')^{7/4}\mathcal{F}^{1/2}(s')\}. \tag{5.32}$$

This gives immediately for $|s|$ large, by (5.7), $|b(s)| + \overline{\mathcal{N}}(s) =$ and thus $\varepsilon(s, y) \equiv 0$. Therefore, for some $t > 0$, $u_2(t)$ is a rescaling and translation of $S(t)$, and thus it is so for all time by uniqueness of the Cauchy problem in H^1 . This concludes the proof of (5.1) and Theorem 1.3.

6. Description of the (Exit) scenario

This section is devoted to the proof of Theorem 1.4. The argument relies first on an extension of the compactness argument of Section 3 and second on the uniqueness up to symmetries of the minimal mass blow up solution.

6.1. Reduction of the proof. Theorem 1.4 is a direct consequence of the following proposition which describes the defocusing bubble in the (Exit) regime at the exit time.

Proposition 6.1 (Compactness of sequences of solutions at the (Exit) time). *There exists a small universal constant $\alpha^* > 0$ such that the following holds. Let $(u_n(0))$ be a sequence in H^1 satisfying:*

- (1) $u_n(0) \in \mathcal{A}$;
- (2) $\|u_n(0) - Q\|_{H^1} \leq 1/n$;
- (3) *the solution $u_n \in \mathcal{C}([0, T_n], H^1)$ of (1.1) corresponding to $(u_n(0))_{n \geq 1}$ satisfies the (Exit) scenario, i.e. for all n large enough,*

$$t_n^* = \sup\{t > 0 : \forall t' \in [0, t], u_n(t') \in \mathcal{T}_{\alpha^*}\} < T_n. \tag{6.1}$$

Then there exists $\sigma^* = \sigma^*(\alpha^*)$ (independent of the sequence u_n) such that

$$\lambda_n^{1/2}(t_n^*)u_n(t_n^*, \lambda_n(t_n^*) \cdot + x_n(t_n^*)) \rightarrow \lambda_S^{1/2}(\sigma^*)S(\sigma^*, \lambda_S(\sigma^*) \cdot + x_S(\sigma^*)) \text{ in } L^2 \tag{6.2}$$

as $n \rightarrow \infty$.

6.2. Proof of Proposition 6.1. The strategy of the proof is similar to the proof of existence of Theorem 1.3 in Section 3. However, the initial data in Section 3 are *well-prepared* and in particular generate H^1 bounded sequences after renormalization (see (3.6)). Here the H^1 bound is lost, and one needs to invoke a concentration-compactness argument in the critical L^2 space for sequences of solutions to (1.1) and uniform local estimates to recover a nontrivial weak limit.

Step 1. Renormalization. Let $C^* > 0$ be the universal constant in (2.57) of Lemma 2.10. Let t_n^* be the exit time (6.1), and consider the decomposition of $u_n(t)$ on $[0, t_n^*]$ given by Lemma 2.6. It follows from [29, proof of Theorem 1.2, Section 4.3] that there exists a time $0 \leq t_{1,n}^* < t_n^*$ such that

$$b_n(t_{1,n}^*) \leq -C^* \int ((\partial_y \varepsilon_n)^2(t_{1,n}^*) \psi_B + \varepsilon_n^2(t_{1,n}^*) \varphi'_B), \tag{6.3}$$

$$\lambda_n(t_{1,n}^*) \approx 1, \quad |b_n(t_{1,n}^*)| + \mathcal{N}_n(t_{1,n}^*) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{6.4}$$

where \mathcal{N}_n denotes the quantity \mathcal{N} defined in (2.46) for u_n . This time corresponds to when the (Exit) regime is decided (b_n is negative and becomes predominant in the sense (6.3)), and it is proved in [29] that such a time $t_{1,n}^*$ can be chosen so that the solution has moved only $\delta(\|u_n(0) - Q\|_{H^1})$ away from the initial data (see [29, (4.37)]), which implies (6.4) in the present situation (since $\|u_n(0) - Q\| \rightarrow 0$ as $n \rightarrow \infty$).

Recall also from [29] that $u_n(t)$ satisfies (H1)–(H3) on $[0, t_n^*]$.

Define

$$\forall \tau \in [\tau_n^*, 0], \quad t_\tau = t_n^* + \tau \lambda_n^3(t_n^*), \quad \tau_n^* = -\frac{t_n^* - t_{1,n}^*}{\lambda_n^3(t_n^*)},$$

and consider on $[\tau_n^*, 0]$ the renormalized solution $v_n(\tau)$ at the exit time t_n^* ,

$$\begin{aligned} v_n(\tau, x) &= \lambda_n^{1/2}(t_n^*) u_n(t_\tau, \lambda_n(t_n^*)x + x(t_n^*)) \\ &= \frac{\lambda_n^{1/2}(t_n^*)}{\lambda_n^{1/2}(t_\tau)} (Q_{b_n(t_\tau)} + \varepsilon_n) \left(t_\tau, \frac{\lambda_n(t_n^*)}{\lambda_n(t_\tau)} x + \frac{x_n(t_n^*) - x_n(t_\tau)}{\lambda_n(t_\tau)} \right). \end{aligned} \tag{6.5}$$

Then v_n is solution of (1.1) and belongs to the L^2 tube \mathcal{T}_{α^*} for $\tau \in [\tau_n^*, 0]$. Moreover, its decomposition $(\lambda_{v_n}, x_{v_n}, \varepsilon_{v_n})$ satisfies on $[\tau_n^*, 0]$:

$$\lambda_{v_n}(\tau) = \frac{\lambda_n(t_\tau)}{\lambda_n(t_n^*)}, \quad x_{v_n}(\tau) = \frac{x_n(t_\tau) - x_n(t_n^*)}{\lambda_n(t_n^*)}, \quad b_{v_n}(\tau) = b_n(t_\tau), \quad \varepsilon_{v_n}(\tau) = \varepsilon_n(t_\tau). \tag{6.6}$$

Step 2. Preliminary estimates on the renormalized sequence. We claim:

Lemma 6.2. *There exist b^*, τ^* such that, possibly after extracting a subsequence,*

$$b_n(t_n^*) \rightarrow -b^*, \quad (\alpha^*)^2 \lesssim b^* \leq \delta(\alpha^*), \tag{6.7}$$

$$\tau_n^* = -\frac{t_n^*}{\lambda_n^3(t_n^*)} \rightarrow -\tau^*, \quad \tau^* b^* \approx 1. \tag{6.8}$$

Moreover, for all n large and $\tau \in [\tau_n^*, 0]$,

$$\lambda_{v_n}(0) = 1, \quad x_{v_n}(0) = 0, \tag{6.9}$$

$$|b_{v_n}(\tau)| + \mathcal{N}_{v_n}(\tau) + \|\varepsilon_{v_n}(\tau)\|_{L^2} \lesssim \delta(\alpha^*), \tag{6.10}$$

$$\int_{\tau_{1,n}^*}^{\tau_n^*} \left[\int ((\partial_y \varepsilon_{v_n})^2 + \varepsilon_{v_n}^2) \frac{\varphi_B}{1 + y_+^2} + |b_{v_n}|^4 \right] \frac{d\tau}{\lambda_{v_n}^5(\tau)} \lesssim \delta(\alpha^*), \tag{6.11}$$

$$|b_{v_n}(\tau_n^*)|/\lambda_{v_n}^2(\tau_n^*) \lesssim \delta(\alpha^*), \quad \lambda_{v_n}(\tau_n^*) = 1/\lambda_n(\tau_n^*) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{6.12}$$

$$(\lambda_{0,v_n})_\tau(\tau) \approx b^*, \tag{6.13}$$

$$x_{v_n}(\tau_n^*) \lesssim -1/(b^* \lambda_{v_n}(\tau_n^*)) \rightarrow -\infty \quad \text{as } n \rightarrow \infty. \tag{6.14}$$

(Recall that λ_0 is defined in Lemma 2.10. Here, λ_{0,v_n} denotes this quantity for v_n . Similarly, \mathcal{N}_{v_n} denotes the quantity \mathcal{N} for v_n . As usual $y_+ = \max(0, y)$.)

Proof of Lemma 6.2. Arguing as in the proof of Lemma 3.1, using conservation of mass and energy of $u_n(t)$, we first obtain

$$(\alpha^*)^2 \lesssim -b_n(t_n^*) \lesssim \delta(\alpha^*). \tag{6.15}$$

Next, using (2.57) on $[t_{1,n}^*, t_n^*]$, and (6.3), (6.4), one obtains

$$-b_n(t)/\lambda_n^2(t) \approx -b_n(t_{1,n}^*), \tag{6.16}$$

and thus, by (6.15),

$$(\alpha^*)^2/|b_n(t_{1,n}^*)| \lesssim \lambda_n^2(t_n^*) \lesssim \delta(\alpha^*)/|b_n(t_{1,n}^*)|, \tag{6.17}$$

which implies $|b_{v_n}(\tau_n^*)|/\lambda_{v_n}^2(\tau_n^*) \lesssim \delta(\alpha^*)$. Next, by definition of t_n^* , $\|\varepsilon_{v_n}(\tau)\|_{L^2} = \|\varepsilon_n(t_\tau)\|_{L^2} \leq \delta(\alpha^*)$. By (2.59) and (6.4), for n large,

$$\begin{aligned} \mathcal{N}_n(t) + \int_{t_{1,n}^*}^{t_n^*} \left[\int ((\partial_y \varepsilon_n)^2 + \varepsilon_n^2) \varphi_B' \right] \frac{dt}{\lambda_n^3} \\ \lesssim \mathcal{N}_n(t_{1,n}^*) + |b_n(t)|^3 + |b_n(t_{1,n}^*)|^3 \leq \delta(\alpha^*). \end{aligned} \tag{6.18}$$

Now, we use [29, Lemma 4.3] to obtain a slightly different estimate. From [29, (4.12)] with $i = 1$, using the definition of $\varphi_{i,B}$ in [29, p. 84], and then using (6.3), we obtain

$$\begin{aligned} \int_{t_{1,n}^*}^{t_n^*} \left[\int ((\partial_y \varepsilon_n)^2 + \varepsilon_n^2) \frac{\varphi_B}{1 + y_+^2} + |b_n|^4 \right] \frac{dt}{\lambda_n^5} \\ \lesssim \frac{\int ((\partial_y \varepsilon_n)^2(t_{1,n}^*) \psi_B + \varepsilon_n^2(t_{1,n}^*) \varphi_B')}{\lambda_n^2(t_{1,n}^*)} + \frac{|b_n(t_n^*)|^3}{\lambda_n^2(t_n^*)} + \frac{|b_n(t_{1,n}^*)|^3}{\lambda_n^2(t_{1,n}^*)} \lesssim \frac{|b_n(t_{1,n}^*)|}{\lambda_n^2(t_{1,n}^*)} \\ \lesssim \delta(\alpha^*). \end{aligned} \tag{6.19}$$

Moreover, by (2.58) and (6.16),

$$(\lambda_{0n})_t(t) \approx |b_n(t_n^*)|/\lambda_n^2(t_n^*). \tag{6.20}$$

By definition of v_n , we obtain (6.9)–(6.13) from the above estimates.

Now, we prove (6.14). Integrating the estimate of $(\lambda_{0,v_n})_\tau$, we obtain

$$\lambda_{0,v_n}(\tau) - \lambda_{0,v_n}(\tau_n^*) \approx b^*(\tau - \tau_n^*).$$

Finally, since $(x_{v_n})_\tau \approx 1/\lambda_{0,v_n}^2$, we obtain by integration on $[\tau_n^*, 0]$ and $x_{v_n}(0) = 0$, for n large,

$$-x_{v_n}(\tau_n^*) = x_{v_n}(0) - x_{v_n}(\tau_n^*) \approx \frac{1}{b^*} \frac{1}{\lambda_{v_n}(\tau_n^*)},$$

and (6.14) is proved. □

Step 3. Monotonicity estimates. We now state the following bound on v_n which will allow us to recover H^1 bounds in the limit:

$$\int_{x > -\lambda_n^2(t_n^*)} (\partial_x v_n)^2(0, x) dx \lesssim 1. \tag{6.21}$$

In fact, we prove the following estimate on $\varepsilon_{v_n}(0)$, which together with Lemma 6.2 and $\lambda_{v_n}(0) = 1$ implies (6.21):

$$\int_{y > -2\lambda_n^2(t_n^*)} (\partial_y \varepsilon_{v_n})^2(0, y) dy \lesssim \delta(\alpha^*). \tag{6.22}$$

Proof of (6.22). For $\tau \in [\tau_n^*, 0]$, let $s = -\int_\tau^0 d\tau/\lambda_{v_n}^3(\tau)$ be the rescaled time for v_n , and $s_n^* = -\int_{\tau_n^*}^0 d\tau/\lambda_{v_n}^3(\tau)$. We perform monotonicity estimates on ε_{v_n} to complement the ones obtained in (2.59). We define $\phi \in C^\infty(\mathbb{R})$ such that

$$\phi(y) = \begin{cases} e^y & \text{for } y < -1, \\ 1 - \frac{1}{10}e^{-y} & \text{for } y > -1/2, \end{cases} \quad \phi'(y) > 0, \quad \forall y \in \mathbb{R}, \tag{6.23}$$

and we consider ψ defined as in (2.45). Let

$$\phi_B(s, y) = \phi\left(\frac{y + \frac{1}{2}(s - s_n^*)}{B}\right), \quad \tilde{\phi}_B(s, y) = \psi\left(\frac{y + \frac{1}{2}(s - s_n^*)}{B}\right),$$

and

$$\begin{aligned} \mathcal{F}_B(s) = \frac{1}{\lambda_{v_n}^2(s)} \int \left[(\partial_y \varepsilon_{v_n})^2 \tilde{\phi}_B + \frac{\lambda_{v_n}^2(s)}{\lambda_{0,v_n}^2(s)} \varepsilon_{v_n}^2 \phi_B \right. \\ \left. - \frac{1}{3}((\varepsilon_{v_n} + Q_b)^6 - Q_b^6 - 6\varepsilon_{v_n} Q_b^5) \tilde{\phi}_B \right] (s, y) dy. \end{aligned}$$

We state the following estimates, proved in Appendix B.

Lemma 6.3. *For B large enough,*

$$\begin{aligned} \frac{d}{ds} \mathcal{F}_B &\leq -\frac{1}{4} \frac{1}{\lambda_{v_n}^2} \int ((\partial_y \varepsilon_{v_n})^2 + \varepsilon_{v_n}^2) \phi'_B \\ &\quad + \frac{C}{\lambda_{v_n}^2} \int ((\partial_y \varepsilon_{v_n})^2 + \varepsilon_{v_n}^2) \frac{\varphi_B}{1 + y_+^2} + \frac{C}{\lambda_{v_n}^2} |b_{v_n}|^4, \end{aligned} \tag{6.24}$$

$$\mathcal{F}_B \approx \frac{1}{\lambda_{v_n}^2} \int ((\partial_y \varepsilon_{v_n})^2 \tilde{\phi}_B + \varepsilon_{v_n}^2 \phi_B). \tag{6.25}$$

We now integrate (6.24) on $[\tau_n^*, 0]$ and then use (6.19) and Lemma 6.2:

$$\begin{aligned} \mathcal{F}_B(0) &\leq \mathcal{F}_B(\tau_n^*) + C \int_{\tau_n^*}^0 \left(\int ((\partial_y \varepsilon_{v_n})^2 + \varepsilon_{v_n}^2) \frac{\varphi_B}{1 + y_+^2} + |b_{v_n}|^4 \right) \frac{d\tau}{\lambda_{v_n}^5(\tau)} \\ &\lesssim \mathcal{N}_{v_n}(\tau_n^*) + \frac{|b_{v_n}(\tau_n^*)|^3}{\lambda_{v_n}^2(\tau_n^*)} + |b_{v_n}(0)|^3 \lesssim \delta(\alpha^*). \end{aligned}$$

And thus, by (6.25), $\lambda_{v_n}(0) = 1$ and the definition of $\phi_B, \tilde{\phi}_B$, since

$$s_n^* = - \int_{\tau_n^*}^0 \frac{d\tau}{\lambda_{v_n}^3(\tau)} \approx - \int_{\tau_n^*}^0 \frac{d\tau}{(\lambda_{v_n}(\tau_n^*) + b^*(\tau - \tau_n^*))^3} \approx - \frac{1}{b^*} \frac{1}{\lambda_{v_n}^2(\tau_n^*)},$$

we finally obtain

$$\int_{y > -2\lambda_{v_n}^{-2}(\tau_n^*)} (\partial_y \varepsilon_{v_n})^2 \leq \int_{y > \frac{1}{2}s_n^*} (\partial_y \varepsilon_{v_n})^2 \lesssim \delta(\alpha^*). \tag{6.26}$$

Step 4. Extraction of the limit. Since $\|v_n(0) - Q\|_{L^2} \leq \delta(\alpha^*)$, there exists $v(0) \in L^2$ and a subsequence still denoted $(v_n(0))$ such that $v_n(0) \rightharpoonup v(0)$ weakly in L^2 as $n \rightarrow \infty$. Moreover, by the properties of weak convergence,

$$\|v(0)\|_{L^2} \leq \|Q\|_{L^2}, \quad \|v(0) - Q\|_{L^2} \leq \delta(\alpha^*), \tag{6.27}$$

and since $\lambda_n(t_n^*) \rightarrow \infty$, it follows from (6.21) that $v(0) \in H^1$.

Let $\delta_0 > 0$ be small enough, $\delta_0 < \delta$ where δ is defined in Theorem 2.1(i). We consider α^* small enough, but universal, such that

$$\|v_n(0) - v(0)\|_{L^2} \leq \delta_0/2. \tag{6.28}$$

In order to exhibit a nontrivial weak limit, we decompose the sequence $(v_n(0) - v(0))$ into profiles according to Lemma 2.3: there exist

$$U_n^j(0) = e^{-i t_n^j \partial_x^3} (g_n^j [\text{Re}(e^{ix \xi_n^j \lambda_n^j} \phi^j)])$$

and $w_n^J(0) \in L^2$ such that (up to a subsequence)

$$v_n(0) - v(0) = \sum_{j=1}^J U_n^j(0) + w_n^J(0), \quad \lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|e^{-t\partial_x^3} w_n^J(0)\|_{L_x^5 L_t^{10}(\mathbb{R} \times \mathbb{R})} = 0, \tag{6.29}$$

$$\|v_n(0) - v(0)\|_{L^2}^2 - \sum_{j=1}^J \|U_n^j(0)\|_{L^2}^2 - \|w_n^J(0)\|_{L^2}^2 = o_n(1). \tag{6.30}$$

Moreover, by the weak convergence $v_n(0) \rightharpoonup v(0)$, we have

$$\int v(0)(v_n(0) - v(0)) = o_n(1),$$

and thus

$$\|v_n(0)\|_{L^2}^2 - \|v(0)\|_{L^2}^2 - \sum_{j=1}^J \|U_n^j(0)\|_{L^2}^2 - \|w_n^J(0)\|_{L^2}^2 = o_n(1). \tag{6.31}$$

In particular, by (6.29) and (6.31), $v(0)$ is interpreted as the first profile U^0 of the decomposition of $v_n(0)$ with $g_n^0 = g_{1,0}$, $t_n^0 = 0$ and $\lambda_n^0 = 1$. By (6.28) and (6.30), for n large,

$$\sum_{j=1}^J \|U_n^j(0)\|_{L^2}^2 + \|w_n^J(0)\|_{L^2}^2 \lesssim \delta(\alpha^*) \leq \delta_0^2/2.$$

Define $U_n^j(\tau)$ and $w_n^J(\tau)$ to be the (global) solutions of the nonlinear equation (1.1) corresponding to the initial data $U_n^j(0)$ and $w_n^J(0)$. Let $\tau_0 < 0$ be such that $v(\tau)$ exists on $[\tau_0, 0]$. We claim that, for n large, v_n exists on $[\tau_0, 0]$ and

$$\lim_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{\tau \in [\tau_0, 0]} \|v_n(\tau) - v(\tau) - \sum_{j=1}^J U_n^j(\tau) - w_n^J(\tau)\|_{L^2} = 0. \tag{6.32}$$

Indeed, (6.32) is a by now standard corollary of the perturbation Lemma 2.4 (see e.g. [8, Proposition 2.8]). In particular, there exist $n_0 > 1$ and $J_0 \geq 1$ such that for $n > n_0$,

$$\|v_n(\tau) - v(\tau) - \sum_{j=1}^{J_0} U_n^j(\tau) - w_n^{J_0}(\tau)\|_{L^2} \leq \delta_0,$$

and thus, for all $\tau \in [\tau_0, 0]$,

$$\|v_n(\tau) - v(\tau)\|_{L^2} \leq \left\| \sum_{j=1}^{J_0} U_n^j(\tau) \right\|_{L^2} + \|w_n^{J_0}(\tau)\|_{L^2} + \delta_0 \lesssim \delta_0 < \frac{1}{10} \|Q\|_{L^2}, \tag{6.33}$$

choosing now δ_0 small but universal. In particular, let A be such that, for all $\tau \in [\tau_0, 0]$,

$$\int_{|x|>A} v^2(\tau, x) dx < \frac{1}{100} \int Q^2.$$

Then from (6.33),

$$\int_{|x|>A} v_n^2(\tau, x) dx \leq \frac{1}{25} \int Q^2.$$

Now, recall from (6.14) that $x_{v_n}(\tau_n^*) \rightarrow -\infty$ as $n \rightarrow \infty$, and in particular, for n large enough, $-x_{v_n}(\tau_n^*) \ll A$; thus,

$$\int_{x<-A} v_n^2(\tau_n^*, x) dx \geq \frac{3}{4} \int Q^2.$$

We conclude that necessarily $\tau_0 > \tau_n^*$ for n large enough, and thus

$$\tau_0 \geq -\tau^* = \lim_n \tau_n^*.$$

It follows that $v(\tau)$ blows up at a finite time $\tau_{\max}(v) \geq -\tau^* = \lim_n \tau_n^*$. Since $\|v(0)\|_{L^2} \leq \|Q\|_{L^2}$ and $v(0) \in H^1$, we have $\|v\|_{L^2} = \|Q\|_{L^2}$. In particular, by weak convergence, and $\lim_{n \rightarrow \infty} \|v_n(0)\|_{L^2} = \|v(0)\|_{L^2}$, we obtain $\lim_{n \rightarrow \infty} \|v_n(0) - v(0)\|_{L^2} = 0$.

From the uniqueness statement in Theorem 1.3, there exist $\lambda^* > 0$, $x^* \in \mathbb{R}$ and $\sigma^* > 0$ such that

$$v(0, x) = (\lambda^*)^{1/2} S(\sigma^*, \lambda^* x + x^*).$$

Moreover, denoting by (b_S, λ_S, x_S) the parameters of the decomposition of S , we observe that

$$\lambda_v(0) = 1 = (\lambda^*)^{-1} \lambda_S(\sigma^*), \quad x_v(0) = 0 = x_S(\sigma^*) - x^*,$$

and thus

$$v(0, x) = \lambda_S^{1/2}(\sigma^*) S(\sigma^*, \lambda_S(\sigma^*)x + x_S(\sigma^*)).$$

In particular, by scaling,

$$v(\tau, x) = \lambda_S^{1/2}(\sigma^*) S(\sigma^* + \lambda_S^3(\sigma^*)\tau, \lambda_S(\sigma^*)x + x_S(\sigma^*)).$$

Since v blows up at $\tau_{\max}(v)$, and S blows up at time 0 (by convention), we have

$$\sigma^* = -\lambda_S^3(\sigma^*)\tau_{\max}(v) < \lambda_S^3(\sigma^*)\tau_*$$

From the definition of t_n^* and then strong L^2 convergence, we have

$$\begin{aligned} \alpha^* &= \inf_{\lambda_1, x_1} \|u_n(t_n^*) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2} = \inf_{\lambda_1, x_1} \|v_n(0) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2} \\ &= \inf_{\lambda_1, x_1} \|S(\sigma^*) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2}. \end{aligned} \tag{6.34}$$

Moreover, recall that by definition of t_n^* and τ_n^* , for all $\tau \in [\tau_n^*, 0]$ we have

$$\inf_{\lambda_1, x_1} \|v_n(\tau) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2} \leq \alpha^*,$$

and so for all $t \in (0, \sigma^*]$,

$$\inf_{\lambda_1, x_1} \|S(t) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2} \leq \alpha^*. \tag{6.35}$$

From (4.35), we fix $t_S > 0$ such that the distance of $S(t)$ to the family of solitons is increasing on $(0, t_S)$. Take $\alpha^* > 0$ small enough so that

$$\alpha^* \leq \frac{1}{2} \inf_{\lambda_1, x_1} \|S(t_S) - Q_{\lambda_1}(\cdot - x_1)\|_{L^2}.$$

By (6.35), it is clear that $\sigma^* \in (0, t_S)$. Moreover, σ^* is uniquely defined by (6.34) (small for α^* small) and thus does not depend on the subsequence but only on α^* . In particular, the whole sequence converges to the same limit, and the proposition is proved.

Appendix A. End of the proof of Proposition 4.1

In this appendix, we finish the proof of Proposition 4.1 by proving (4.11)–(4.13). For the reader’s convenience, we recall the main estimates already proved on ε and the parameters b, λ, x : for $|s|$ large,

$$\|\varepsilon(s)\|_{L^\infty} \lesssim \|\varepsilon(s)\|_{H^1} \lesssim \frac{1}{\sqrt{|s|}}, \quad \frac{c_1(u_0)}{\sqrt{|s|}} \leq \lambda(s) \leq \frac{c_2(u_0)}{\sqrt{|s|}}, \quad b(s) \sim \frac{1}{2s}, \quad (\text{A.1})$$

$$\mathcal{N}(s) + \int_{-\infty}^s \int (\varepsilon_y^2 + \varepsilon^2)(s') \varphi'_B ds' \lesssim \frac{1}{|s|^3}, \quad (\text{A.2})$$

$$\left| \frac{\lambda_s}{\lambda} + b \right| + \left| \frac{x_s}{\lambda} - 1 \right| \lesssim \frac{1}{|s|^{3/2}}, \quad |b_s| \lesssim \frac{1}{|s|^2}. \quad (\text{A.3})$$

A.1. Proof of (4.11). Since $u(t)$ is a minimal mass blowing up solution and $\lambda(s)$ is increasing for $|s|$ large, from Lemma 2.11 and then using the properties of Q_b (see Lemma 2.5) we obtain, for $|s|$ large,

$$\forall y > 0, \quad |\varepsilon(s, y)| \lesssim e^{-y/20}. \quad (\text{A.4})$$

Thus, by (A.1),

$$\lim_{s \rightarrow -\infty} \int \varepsilon^2(s, y) e^{\lambda(s)y} dy = 0. \quad (\text{A.5})$$

Now, to prove (4.11), we compute the time derivative of $\int \varepsilon^2 e^{\lambda y}$. Using the equation of ε (see (2.24)), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{ds} \int \varepsilon^2 e^{\lambda y} &= \frac{\lambda_s}{2} \int y e^{\lambda y} \varepsilon^2 + \int \varepsilon_s \varepsilon e^{\lambda y} \\ &= \frac{\lambda_s}{2} \int y e^{\lambda y} \varepsilon^2 + \int (L\varepsilon)_y \varepsilon e^{\lambda y} + \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_b \varepsilon e^{\lambda y} + \frac{\lambda_s}{\lambda} \int \varepsilon \Lambda \varepsilon e^{\lambda y} \\ &\quad + \left(\frac{x_s}{\lambda} - 1 \right) \int (Q_b + \varepsilon)_y \varepsilon e^{\lambda y} + \int \Phi_b \varepsilon e^{\lambda y} + \int \Psi_b \varepsilon e^{\lambda y} \\ &\quad - \int (R_b(\varepsilon))_y \varepsilon e^{\lambda y} - \int (R_{NL}(\varepsilon))_y \varepsilon e^{\lambda y}. \end{aligned}$$

Since $\int \varepsilon \Lambda \varepsilon e^{\lambda y} = -\frac{1}{2} \lambda \int \varepsilon^2 y e^{\lambda y}$, the scaling terms cancel (this is because the quantity is scaling invariant). Next, using (A.3), we have

$$\begin{aligned} \int (L\varepsilon)_y \varepsilon e^{\lambda y} &= -\frac{3}{2} \lambda \int \varepsilon_y^2 e^{\lambda y} - \frac{1}{2} \lambda (1 - \lambda^2) \int \varepsilon^2 e^{\lambda y} + \int (-10Q^3 Q' - \frac{5}{2} \lambda Q^4) \varepsilon^2 e^{\lambda y} \\ &\leq -\frac{\lambda}{4} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda y} + C \mathcal{N}_{\text{loc}}, \\ \left| \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_b \varepsilon e^{\lambda y} \right| + \left| \left(\frac{x_s}{\lambda} - 1 \right) \int (Q_b)_y \varepsilon e^{\lambda y} \right| \\ &\lesssim \frac{1}{|s|^{3/2}} \left[\left(\int \varepsilon^2 e^{-|y|/10} \right)^{1/2} + |b| \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} \left(\int_{y<0} e^{\lambda y} \right)^{1/2} \right] \\ &\lesssim \frac{1}{|s|^{3/2}} \left[\frac{1}{|s|^{3/2}} + \frac{1}{\sqrt{|s|}} \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} \right] \lesssim \frac{\lambda}{100} \int \varepsilon^2 e^{\lambda y} + \frac{1}{|s|^3}, \\ \left| \left(\frac{x_s}{\lambda} - 1 \right) \int \varepsilon_y \varepsilon e^{\lambda y} \right| &= \frac{\lambda}{2} \left| \left(\frac{x_s}{\lambda} - 1 \right) \int \varepsilon^2 e^{\lambda y} \right| \leq \frac{\lambda}{100} \int \varepsilon^2 e^{\lambda y}, \\ \left| \int \Phi_b \varepsilon e^{\lambda y} \right| &\lesssim |b_s| \int |P| |\varepsilon| e^{\lambda y} \lesssim \frac{1}{|s|^2} \left(\int P^2 e^{\lambda y} \right)^{1/2} \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} \\ &\lesssim \frac{1}{|s|^{7/4}} \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} \leq \frac{\lambda}{100} \int \varepsilon^2 e^{\lambda y} + \frac{C}{|s|^3}. \end{aligned}$$

Using (2.14) (recall $\gamma = 3/4$), we get

$$\begin{aligned} \left| \int \Psi_b \varepsilon e^{\lambda y} \right| &\lesssim |b|^{7/4} \int_{-2|b|^{-3/4} < y < -|b|^{-3/4}} |\varepsilon| e^{\lambda y} + |b|^2 \int_{y<0} |\varepsilon| e^{\lambda y} + |b|^2 \int_{y>0} |\varepsilon| e^{-y/4} \\ &\lesssim \frac{1}{|s|^{7/4}} \left(\int_{y<-|b|^{-3/4}} e^{\lambda y} \right)^{1/2} \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} + \frac{1}{|s|^2} \left(\int_{y<0} e^{\lambda y} \right)^{1/2} \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} + \frac{1}{|s|^3} \\ &\lesssim \left(\frac{1}{|s|^{3/2}} e^{-\lambda|b|^{-3/4}} + \frac{1}{|s|^{7/4}} \right) \left(\int \varepsilon^2 e^{\lambda y} \right)^{1/2} + \frac{1}{|s|^3} \leq \frac{\lambda}{100} \int \varepsilon^2 e^{\lambda y} + \frac{C}{|s|^3}. \end{aligned}$$

Next, since $|R_b(\varepsilon)| = 5|Q_b^4 - Q^4| |\varepsilon| \lesssim |b| |\varepsilon|$, we have

$$\begin{aligned} \left| \int (R_b(\varepsilon))_y \varepsilon e^{\lambda y} \right| &\leq \left| \int R_b(\varepsilon) (\lambda |\varepsilon| + |\varepsilon_y|) e^{\lambda y} \right| \\ &\lesssim |b| \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda y} \leq \frac{\lambda}{100} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda y}. \end{aligned}$$

Finally, since $|R_{\text{NL}}(\varepsilon)| \leq \|\varepsilon\|_{L^\infty} (\|\varepsilon\|_{L^\infty}^3 + |b| + e^{-y/10}) |\varepsilon|$, we get

$$\begin{aligned} \left| \int (R_{\text{NL}}(\varepsilon))_y \varepsilon e^{\lambda y} \right| &\leq \left| \int R_{\text{NL}}(\varepsilon) (\lambda |\varepsilon| + |\varepsilon_y|) e^{\lambda y} \right| \\ &\lesssim \frac{1}{|s|^{3/2}} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda y} \leq \frac{\lambda}{100} \int (\varepsilon_y^2 + \varepsilon^2) e^{\lambda y}. \end{aligned}$$

Combining the above bounds gives

$$\frac{d}{ds} \int \varepsilon^2 e^{\lambda y} \lesssim \frac{1}{|s|^3},$$

and integration on $(-\infty, s]$ using (A.5) yields (4.11).

A.2. Proofs of (4.12)–(4.13). Note first that by standard arguments,

$$\|((\partial_y^2 \varepsilon)^2 + (\partial_y \varepsilon)^2)(s) e^{\omega y}\|_{L^\infty} \lesssim \int ((\partial_y^3 \varepsilon)^2 + (\partial_y^2 \varepsilon)^2 + (\partial_y \varepsilon)^2 + \varepsilon^2)(s, y) e^{\omega y} dy,$$

and so it is sufficient to prove (4.12).

The proof is similar to [22, Section 3.4] and involves some computations originally introduced in [13]. To prove (4.12), we need only rough bounds on ε and it is therefore simpler to decompose

$$\varepsilon + Q_b = \tilde{\varepsilon} + Q,$$

which satisfies

$$\partial_s \tilde{\varepsilon} + \partial_y (\partial_y^2 \tilde{\varepsilon} - \tilde{\varepsilon} + F(\tilde{\varepsilon})) = \frac{\lambda_s}{\lambda} (\Lambda Q + \Lambda \tilde{\varepsilon}) + \left(\frac{x_s}{\lambda} - 1\right) (\partial_y Q + \partial_y \tilde{\varepsilon}) \tag{A.6}$$

with

$$F(\tilde{\varepsilon}) = (Q + \tilde{\varepsilon})^5 - Q^5.$$

From (A.1), (A.2) and $Q_b - Q = bP\chi_b$ (see Lemma 2.5) we have the following estimates on $\tilde{\varepsilon}$:

$$\|\tilde{\varepsilon}\|_{L^\infty} \lesssim \frac{1}{\sqrt{|s|}}, \quad \int \tilde{\varepsilon}^2 e^{-|y|/10} \leq \frac{1}{|s|^2}. \tag{A.7}$$

From (A.2),

$$\int \varepsilon^2(s) \varphi_B + \int_{-\infty}^s \int (\varepsilon_y^2 + \varepsilon^2)(s') \varphi_B ds' \lesssim \frac{1}{|s|^3},$$

and thus, since $|b(s)| \lesssim 1/|s|$, for $|s|$ large we have

$$\int \tilde{\varepsilon}^2(s) \varphi_B + \int_{-\infty}^s \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2)(s') \varphi_B ds' \lesssim \frac{1}{|s|}. \tag{A.8}$$

Moreover, since $u(t)$ is a minimal mass blowing up solution and $\lambda(s)$ is increasing for $|s|$ large, from Lemma 2.11 and then using the properties of Q_b (see Lemma 2.5) we obtain, for $|s|$ large,

$$\forall y > 0, |\varepsilon(s, y)| \lesssim e^{-y/20} \quad \text{and so} \quad \forall y > 0, |\tilde{\varepsilon}(s, y)| \lesssim e^{-y/20}. \tag{A.9}$$

In particular, it follows that for all $1/B \leq \omega < 1/10$,

$$\lim_{s \rightarrow -\infty} \int \tilde{\varepsilon}^2(s, y) e^{\omega y} dy = 0. \tag{A.10}$$

Step 1. For all $1/B < \omega < 1/10$, for $|s|$ large,

$$\int \tilde{\varepsilon}^2(s, y)e^{\omega y} dy + \int_{-\infty}^s \int (\tilde{\varepsilon}_y^2(s', y) + \tilde{\varepsilon}^2(s', y))e^{\omega y} dy ds' \lesssim \frac{1}{|s|}. \tag{A.11}$$

Define

$$H_0(s) = \frac{1}{2} \int \tilde{\varepsilon}^2(s, y)e^{\omega y} dy.$$

Then

$$\begin{aligned} \frac{d}{ds} H_0 &= \int \tilde{\varepsilon}_s \tilde{\varepsilon} e^{\omega y} \\ &= - \int (-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))(\tilde{\varepsilon} e^{\omega y})_y + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon}) \tilde{\varepsilon} e^{\omega y} + \left(\frac{x_s}{\lambda} - 1\right) \int (Q' + \tilde{\varepsilon}_y) \tilde{\varepsilon} e^{\omega y} \\ &= -\frac{3}{2} \omega \int \tilde{\varepsilon}_y^2 e^{\omega y} - \frac{1}{2} \omega (1 - \omega^2) \int \tilde{\varepsilon}^2 e^{\omega y} + \int F(\tilde{\varepsilon})(\tilde{\varepsilon} e^{\omega y})_y \\ &\quad + \frac{\lambda_s}{\lambda} \int \Lambda Q \tilde{\varepsilon} e^{\omega y} - \frac{\omega}{2} \frac{\lambda_s}{\lambda} \int \tilde{\varepsilon}^2 y e^{\omega y} + \left(\frac{x_s}{\lambda} - 1\right) \int Q' \tilde{\varepsilon} e^{\omega y} - \frac{\omega}{2} \left(\frac{x_s}{\lambda} - 1\right) \int \tilde{\varepsilon}^2 e^{\omega y}. \end{aligned}$$

First, by the decay properties of Q and since $\|\tilde{\varepsilon}\|_{L^\infty}^2 \lesssim 1/|s|$ (by (A.7)), for $|s|$ large we have

$$\begin{aligned} \left| \int F(\tilde{\varepsilon})(\tilde{\varepsilon} e^{\omega y})_y \right| &\lesssim \int (|\tilde{\varepsilon}| Q^4 + |\tilde{\varepsilon}|^5)(|\tilde{\varepsilon}_y| + |\tilde{\varepsilon}|) e^{\omega y} \\ &\lesssim \frac{\omega}{100} \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y} + \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) \varphi_B. \end{aligned}$$

Second, from $|\lambda_s/\lambda| + |x_s/\lambda - 1| \lesssim 1/|s|$, the decay properties of Q and (A.7),

$$\left| \frac{\lambda_s}{\lambda} \int \Lambda Q \tilde{\varepsilon} e^{\omega y} \right| + \left| \left(\frac{x_s}{\lambda} - 1\right) \int Q' \tilde{\varepsilon} e^{\omega y} \right| \lesssim \frac{1}{|s|} \left(\int \tilde{\varepsilon}^2 e^{-|y|/10} \right)^{1/2} \lesssim \frac{1}{|s|^2}. \tag{A.12}$$

Finally, for $1/B < \omega'' < \omega < \omega' \leq 1/10$, and then using (A.10), we obtain

$$\begin{aligned} \left| \frac{\lambda_s}{\lambda} \int \tilde{\varepsilon}^2 y e^{\omega y} \right| &\lesssim \frac{1}{|s|} \left(\int \tilde{\varepsilon}^2 y^2 e^{\omega y} \right)^{1/2} \left(\int \tilde{\varepsilon}^2 e^{\omega y} \right)^{1/2} \\ &\lesssim \frac{1}{|s|} \left(\int \tilde{\varepsilon}^2 (e^{\omega'' y} + e^{\omega' y}) \right)^{1/2} \left(\int \tilde{\varepsilon}^2 e^{\omega y} \right)^{1/2} \lesssim \frac{\omega}{100} \int \tilde{\varepsilon}^2 e^{\omega y} + \frac{1}{s^2}. \end{aligned}$$

In conclusion,

$$\frac{d}{ds} H_0 \leq -\frac{\omega}{4} \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y} dy + C \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) \varphi_B + \frac{C}{|s|^2}.$$

Integrating on $(-\infty, s]$, using (A.8) and $\lim_{s \rightarrow -\infty} H_0(s) = 0$ by (A.10), we get (A.11).

In particular, for some sequence $s_n \rightarrow -\infty$,

$$\lim_{n \rightarrow \infty} \int \tilde{\varepsilon}_y^2(s_n, y) e^{\omega y} dy = 0. \tag{A.13}$$

Step 2. For all $1/B < \omega < 1/10$, for $|s|$ large,

$$\int \tilde{\varepsilon}_y^2(s, y)e^{\omega y} dy + \int_{-\infty}^s \int \tilde{\varepsilon}_{yy}^2(s', y)e^{\omega y} dy ds' \lesssim \frac{1}{|s|}. \tag{A.14}$$

Define

$$H_1(s) = \int \left[\frac{1}{2}(\tilde{\varepsilon}_y^2(s, y) + \tilde{\varepsilon}^2(s, y))e^{\omega y} - \left(\frac{(Q + \tilde{\varepsilon})^6}{6} - \tilde{\varepsilon}Q^5 - \frac{Q^6}{6} \right) e^{\omega y} \right] dy.$$

Then

$$\begin{aligned} \frac{d}{ds}H_1 &= \int \tilde{\varepsilon}_s(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} - \omega \int \tilde{\varepsilon}_s \tilde{\varepsilon}_y e^{\omega y} \\ &= -\frac{\omega}{2} \int (-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))^2 e^{\omega y} - \omega \int (-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))_y \tilde{\varepsilon}_y e^{\omega y} \\ &\quad + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon})(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} + \omega \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon}) \tilde{\varepsilon}_y e^{\omega y} \\ &\quad + \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y)(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} + \omega \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y) \tilde{\varepsilon}_y e^{\omega y} \\ &\leq -\omega \int \tilde{\varepsilon}_{yy}^2 e^{\omega y} - \omega(1 - \frac{1}{2}\omega^2) \int \tilde{\varepsilon}_y^2 e^{\omega y} - \omega \int F(\tilde{\varepsilon})(\tilde{\varepsilon}_y e^{\omega y})_y \\ &\quad + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon})(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} + \omega \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon}) \tilde{\varepsilon}_y e^{\omega y} \\ &\quad + \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y)(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} + \omega \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y) \tilde{\varepsilon}_y e^{\omega y}. \end{aligned}$$

First, as in Step 1, for $|s|$ large,

$$\begin{aligned} \left| \int F(\tilde{\varepsilon})(\tilde{\varepsilon}_y e^{\omega y})_y \right| &\lesssim \int (|\tilde{\varepsilon}|Q^4 + |\tilde{\varepsilon}|^5)(|\tilde{\varepsilon}_{yy}| + |\tilde{\varepsilon}_y|)e^{\omega y} \\ &\lesssim \frac{\omega}{100} \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2)e^{\omega y} + \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2)\varphi_B. \end{aligned}$$

Second, the following estimates are proved as in Step 1, (A.12), after possible integrations by parts:

$$\begin{aligned} &\left| \frac{\lambda_s}{\lambda} \int \Lambda Q(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} \right| + \left| \frac{\lambda_s}{\lambda} \int \Lambda Q \tilde{\varepsilon}_y e^{\omega y} \right| \\ &\quad + \left| \left(\frac{x_s}{\lambda} - 1 \right) \int Q'(-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))e^{\omega y} \right| + \left| \left(\frac{x_s}{\lambda} - 1 \right) \int Q' \tilde{\varepsilon}_y e^{\omega y} \right| \lesssim \frac{1}{s^2}. \end{aligned}$$

For example, by the decay properties of Q and (A.8),

$$\left| \frac{\lambda_s}{\lambda} \int \Lambda Q \tilde{\varepsilon}_{yy} e^{\omega y} \right| \lesssim \frac{1}{|s|} \int |(\Lambda Q e^{\omega y})_{yy}| |\tilde{\varepsilon}| \lesssim \frac{1}{s^2}.$$

Finally, we observe that

$$\int (\Lambda \tilde{\varepsilon}) \tilde{\varepsilon}_{yy} e^{\omega y} = \int (-\tilde{\varepsilon}_y^2 e^{\omega y} + \frac{1}{2} \omega^2 \tilde{\varepsilon}^2 e^{\omega y} + \tilde{\varepsilon}_y^2 (y e^{\omega y})_y),$$

and thus for some $1/B < \omega'' < \omega < \omega' < 1/10$,

$$\left| \frac{\lambda_s}{\lambda} \int (\Lambda \tilde{\varepsilon}) \tilde{\varepsilon}_{yy} e^{\omega y} \right| \lesssim \frac{1}{|s|} \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) (e^{\omega' y} + e^{\omega'' y}).$$

All the remaining terms are easier and are treated similarly to Step 1.

Combining the above bounds yields

$$\begin{aligned} \frac{d}{ds} H_1 &\lesssim - \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y} dy + \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) \varphi_B \\ &\quad + \int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) (e^{\omega' y} + e^{\omega'' y}) + \frac{1}{|s|^2}. \end{aligned}$$

Note that $\lim_{n \rightarrow \infty} H_1(s_n) = 0$ by (A.10) and (A.13). Integrating on $[s_n, s]$, and then letting $n \rightarrow \infty$, using (A.8) and (A.11) for ω' and ω'' , we find (A.14). In particular, there exists a subsequence still denoted (s_n) such that

$$\lim_{n \rightarrow \infty} \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2)(s_n, y) e^{\omega y} dy = 0. \tag{A.15}$$

Step 3. For all $3/B < \omega < 1/10$, for $|s|$ large,

$$\int \tilde{\varepsilon}_{yy}^2(s, y) e^{\omega y} dy + \int_{-\infty}^s \int \tilde{\varepsilon}_{yyy}^2(s', y) e^{\omega y} dy ds' \lesssim \frac{1}{|s|}. \tag{A.16}$$

Define

$$H_2(s) = \frac{1}{2} \int \tilde{\varepsilon}_{yy}^2 e^{\omega y} - \frac{25}{6} \int \tilde{\varepsilon}_y^2 \tilde{\varepsilon}^4 e^{\omega y}.$$

Then

$$\frac{d}{ds} H_2 = \int \tilde{\varepsilon}_{yys} \tilde{\varepsilon}_{yy} e^{\omega y} - \frac{25}{3} \int (\tilde{\varepsilon}_{ys} \tilde{\varepsilon}_y \tilde{\varepsilon}^4 + 2 \tilde{\varepsilon}_s \tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3) e^{\omega y} = H_{2,1} + H_{2,2}.$$

First,

$$\begin{aligned} H_{2,1} &= \int (-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))_{yyy} \varepsilon_{yy} e^{\omega y} \\ &\quad + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon})_{yy} \tilde{\varepsilon}_{yy} e^{\omega y} + \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y)_{yy} \tilde{\varepsilon}_{yy} e^{\omega y} \\ &= -\frac{3}{2} \omega \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y} - \frac{\omega}{2} (1 - \omega^2) \int \tilde{\varepsilon}_{yy}^2 e^{\omega y} - 50 \int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}_y \tilde{\varepsilon}^3 e^{\omega y} \\ &\quad + \int (F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5)_{yy} (\varepsilon_{yy} e^{\omega y})_y \\ &\quad + \frac{5}{2} \omega \int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}^4 e^{\omega y} + 30 \int \tilde{\varepsilon}_y^5 \tilde{\varepsilon} e^{\omega y} + 15 \omega \int \tilde{\varepsilon}_y^4 \tilde{\varepsilon}^2 e^{\omega y} \\ &\quad + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon})_{yy} \tilde{\varepsilon}_{yy} e^{\omega y} + \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y)_{yy} \tilde{\varepsilon}_{yy} e^{\omega y}. \end{aligned}$$

Second,

$$\begin{aligned}
 H_{2,2} &= \frac{25}{3} \int \tilde{\varepsilon}_s ((\tilde{\varepsilon}_y \tilde{\varepsilon}^4 e^{\omega y})_y - 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 e^{\omega y}) \\
 &= \frac{25}{3} \int (-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))_y (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y} \\
 &\quad + \frac{25}{3} \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon}) (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y} \\
 &\quad + \frac{25}{3} \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y) (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y} \\
 &= 50 \int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}_y \tilde{\varepsilon}^3 e^{\omega y} + \frac{25}{3} \omega \int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}^4 e^{\omega y} - \frac{25}{2} \int \tilde{\varepsilon}_y^5 \tilde{\varepsilon} e^{\omega y} - \frac{175}{12} \omega \int \tilde{\varepsilon}_y^4 \tilde{\varepsilon}^2 e^{\omega y} \\
 &\quad - \frac{25}{9} \omega^3 \int \tilde{\varepsilon}_y^3 \tilde{\varepsilon}^3 e^{\omega y} - \frac{25}{3} \int (F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5)_y (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y} \\
 &\quad + \frac{25}{3} \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon}) (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y} \\
 &\quad + \frac{25}{3} \left(\frac{x_s}{\lambda} - 1 \right) \int (Q' + \tilde{\varepsilon}_y) (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y}.
 \end{aligned}$$

The main observation when looking at the above expressions of $H_{2,1}$ and $H_{2,2}$ is that the higher order nonlinear term $\int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}_y \tilde{\varepsilon}^3 e^{\omega y}$ cancels in the expression of $\frac{d}{ds} H_2$. All other terms are now controlled as follows.

First, by (A.7),

$$\left| \int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}^4 e^{\omega y} \right| \lesssim \frac{1}{|s|^2} \int \tilde{\varepsilon}_{yy}^2 e^{\omega y}.$$

Second, by the Hölder inequality, (1.3), and then (A.11), (A.14), for $3/100 < \omega < 1/10$,

$$\begin{aligned}
 \left| \int \tilde{\varepsilon}_y^5 \tilde{\varepsilon} e^{\omega y} \right| &\lesssim \left(\int \tilde{\varepsilon}_y^6 e^{\omega y} \right)^{5/6} \left(\int \tilde{\varepsilon}^6 e^{\omega y} \right)^{1/6} \\
 &\lesssim \left(\int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y/3} \right)^{5/6} \left(\int (\tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y/3} \right)^{13/6} \\
 &\lesssim \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y/3} + \frac{1}{|s|^{13}}.
 \end{aligned}$$

Similar estimates are proved for $|\int \tilde{\varepsilon}_y^4 \tilde{\varepsilon}^2 e^{\omega y}|$ and $|\int \tilde{\varepsilon}_y^3 \tilde{\varepsilon}^3 e^{\omega y}|$. Next, for terms containing $F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5$, we argue as follows. A first observation is (using (A.1))

$$|(F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5)_y| \lesssim (|\tilde{\varepsilon}_y| + |\tilde{\varepsilon}|) Q, \quad |(F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5)_{yy}| \lesssim (|\tilde{\varepsilon}_{yy}| + |\tilde{\varepsilon}_y|^2 + |\tilde{\varepsilon}_y| + |\tilde{\varepsilon}|) Q.$$

Thus,

$$\begin{aligned} \left| \int (F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5)_{yy} (\tilde{\varepsilon}_{yy} e^{\omega y})_y \right| &\leq C \int (|\tilde{\varepsilon}_{yy}| + |\tilde{\varepsilon}_y|^2 + |\tilde{\varepsilon}_y| + |\tilde{\varepsilon}|)(|\tilde{\varepsilon}_{yyy}| + |\tilde{\varepsilon}_{yy}|) Q e^{\omega y} \\ &\leq \frac{1}{100} \int (\tilde{\varepsilon}_{yyy}^2 + \tilde{\varepsilon}_{yy}^2) e^{\omega y} + C \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^4 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) Q \\ &\leq \frac{1}{100} \int (\tilde{\varepsilon}_{yyy}^2 + \tilde{\varepsilon}_{yy}^2) e^{\omega y} + C \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y} + \frac{C}{s^2}. \end{aligned}$$

The term $|\int (F(\tilde{\varepsilon}) - \tilde{\varepsilon}^5)_y (\tilde{\varepsilon}_{yy} \tilde{\varepsilon}^4 + 2\tilde{\varepsilon}_y^2 \tilde{\varepsilon}^3 + \omega \tilde{\varepsilon}_y \tilde{\varepsilon}^4) e^{\omega y}|$ is treated similarly and easier.

Finally, terms containing λ_s/λ and $x_s/\lambda - 1$ are treated as in Steps 1 and 2. For example, let us consider the term $(\lambda_s/\lambda) \int (\Lambda Q + \Lambda \tilde{\varepsilon})_{yy} \tilde{\varepsilon}_{yy} e^{\omega y}$. We first have

$$\left| \frac{\lambda_s}{\lambda} \int (\Lambda Q)_{yy} \tilde{\varepsilon}_{yy} e^{\omega y} \right| = \left| \frac{\lambda_s}{\lambda} \int ((\Lambda Q)_{yy} e^{\omega y})_{yy} \tilde{\varepsilon} \right| \lesssim \frac{1}{s^2}.$$

Since

$$\int (\Lambda \tilde{\varepsilon})_{yy} \tilde{\varepsilon}_{yy} e^{\omega y} = \frac{3}{2} \int \tilde{\varepsilon}_{yy}^2 e^{\omega y} - \omega \int \tilde{\varepsilon}_{yy}^2 y e^{\omega y},$$

we get, for some $1/100 < \omega'' < \omega < \omega' < 1/10$,

$$\left| \frac{\lambda_s}{\lambda} \int (\Lambda \tilde{\varepsilon})_{yy} \tilde{\varepsilon}_{yy} e^{\omega y} \right| \lesssim \frac{1}{s^2} \int \tilde{\varepsilon}_{yy}^2 (e^{\omega' y} + e^{\omega'' y}).$$

Gathering all the previous estimates, we obtain

$$\begin{aligned} \frac{d}{ds} H_2 &\lesssim - \int (\tilde{\varepsilon}_{yyy}^3 + \tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) e^{\omega y} dy \\ &\quad + \int (\tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) (e^{\omega y/3} + e^{\omega'' y}) + \frac{1}{|s|^2}. \end{aligned}$$

Integrating on $[s_n, s]$ and letting $n \rightarrow \infty$ using (A.8), (A.15) and (A.14), we get (A.16).

For some sequence $s'_n \rightarrow -\infty$, this implies

$$\lim_{n \rightarrow \infty} \int (\tilde{\varepsilon}_{yyy}^2 + \tilde{\varepsilon}_{yy}^2 + \varepsilon_y^2 + \tilde{\varepsilon}^2)(s'_n, y) e^{\omega y} dy = 0. \tag{A.17}$$

Note also that by standard arguments, (A.16) implies directly that

$$\|\varepsilon_y^2 e^{\omega y}\|_{L^\infty} \lesssim 1/|s|. \tag{A.18}$$

Step 4. For all $9/B < \omega < 1/10$, for $|s|$ large,

$$\int \tilde{\varepsilon}_{yyy}^2(s, y) e^{\omega y} dy + \int_{-\infty}^s \int (\partial_y^4 \tilde{\varepsilon})^2(s', y) e^{\omega y} dy ds' \lesssim \frac{1}{|s|}. \tag{A.19}$$

Define

$$H_3(s) = \frac{1}{2} \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y}.$$

Then

$$\begin{aligned} \frac{d}{ds} H_3 &= \int \tilde{\varepsilon}_{yyyy} \tilde{\varepsilon}_{yyy} e^{\omega y} = \int (-\tilde{\varepsilon}_{yy} + \tilde{\varepsilon} - F(\tilde{\varepsilon}))_{yyyy} \varepsilon_{yyy} e^{\omega y} \\ &\quad + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon})_{yyy} \tilde{\varepsilon}_{yyy} e^{\omega y} + \left(\frac{x_s}{\lambda} - 1\right) \int (Q' + \tilde{\varepsilon}_y)_{yyy} \tilde{\varepsilon}_{yyy} e^{\omega y} \\ &= -\frac{3}{2} \omega \int \tilde{\varepsilon}_{yyyy}^2 e^{\omega y} - \frac{\omega}{2} (1 - \omega^2) \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y} + \int (F(\tilde{\varepsilon}))_{yyy} (\varepsilon_{yyy} e^{\omega y})_y \\ &\quad + \frac{\lambda_s}{\lambda} \int (\Lambda Q + \Lambda \tilde{\varepsilon})_{yyy} \tilde{\varepsilon}_{yyy} e^{\omega y} + \left(\frac{x_s}{\lambda} - 1\right) \int (Q' + \tilde{\varepsilon}_y)_{yyy} \tilde{\varepsilon}_{yyy} e^{\omega y}. \end{aligned}$$

The last two terms $(\lambda_s/\lambda) \int (\Lambda Q + \Lambda \tilde{\varepsilon})_{yyy} \tilde{\varepsilon}_{yyy} e^{\omega y}$ and $(x_s/\lambda - 1) \int (Q' + \tilde{\varepsilon}_y)_{yyy} \tilde{\varepsilon}_{yyy} e^{\omega y}$ are treated exactly as in the previous steps, and thus we omit the estimates.

We focus on the nonlinear term $\int (F(\tilde{\varepsilon}))_{yyy} (\varepsilon_{yyy} e^{\omega y})_y$. Expanding $F(\tilde{\varepsilon}) = 5Q^4 \tilde{\varepsilon} + 10Q^3 \tilde{\varepsilon}^2 + 10Q^2 \tilde{\varepsilon}^3 + 5Q \tilde{\varepsilon}^4 + \tilde{\varepsilon}^5$ and integrating by parts, we obtain many different terms. We check the worst ones; the others can be handled similarly. See also [22, Section 3.4] for similar arguments.

First, we remark that the following term, which is only quadratic in $\tilde{\varepsilon}$, is easily controlled:

$$\left| \int \tilde{\varepsilon}_{yyy}^2 (Q^4)' e^{\omega y} \right| \lesssim \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y}.$$

Second, we treat some terms coming from $\tilde{\varepsilon}^5$:

$$\begin{aligned} \left| \int \tilde{\varepsilon}_{yyy}^2 \tilde{\varepsilon}_y \tilde{\varepsilon}^3 e^{\omega y} \right| &\lesssim \|\tilde{\varepsilon}\|_{L^\infty}^3 \|\tilde{\varepsilon}_y e^{\omega y/2}\|_{L^\infty} \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y/2} \lesssim \frac{1}{|s|^2} \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y/2}; \\ \left| \int \tilde{\varepsilon}_{yy}^3 \tilde{\varepsilon}_y \tilde{\varepsilon}^2 e^{\omega y} \right| &\lesssim \|\tilde{\varepsilon}\|_{L^\infty}^2 \|\tilde{\varepsilon}_y e^{\omega y/4}\|_{L^\infty} \left| \int \tilde{\varepsilon}_{yy}^3 e^{3\omega y/4} \right| \\ &\lesssim \frac{1}{|s|^{3/2}} \left(\int (\tilde{\varepsilon}_{yyy}^2 + \tilde{\varepsilon}_{yy}^2) e^{\omega y/2} \right)^{1/4} \left(\int \tilde{\varepsilon}_{yy}^2 e^{\omega y/2} \right)^{5/4} \\ &\lesssim \frac{1}{|s|^{11/3}} + \int \tilde{\varepsilon}_{yyy}^2 e^{\omega y/2}, \\ \left| \int \tilde{\varepsilon}_{yy}^2 \tilde{\varepsilon}_y^3 \tilde{\varepsilon} e^{\omega y} \right| &\lesssim \|\tilde{\varepsilon}\|_{L^\infty} \|\tilde{\varepsilon}_y e^{\omega y/5}\|_{L^\infty}^3 \left| \int \tilde{\varepsilon}_{yy}^2 e^{2\omega y/5} \right| \lesssim \frac{1}{|s|^3}. \end{aligned}$$

Thus, we get

$$\frac{d}{ds} H_3 \lesssim - \int (\partial_y^4 \varepsilon)^2 e^{\omega y} + \frac{1}{|s|^2} + \int (\tilde{\varepsilon}_{yyy}^2 + \tilde{\varepsilon}_{yy}^2 + \tilde{\varepsilon}_y^2 + \tilde{\varepsilon}^2) (e^{2\omega y/5} + e^{\omega y}).$$

Integrating on $[s'_n, s]$ and letting $n \rightarrow \infty$, using (A.17) and (A.16), we obtain (A.19) for $15/(2B) < \omega < 1/10$.

Appendix B. Proof of Lemma 6.3

For simplicity of notation, we denote ε_{v_n} , λ_{v_n} , b_{v_n} and x_{v_n} simply by ε , λ , b and x .

Step 1. Algebraic computations. We follow closely the computations in [29, proof of Proposition 3.1]. First,

$$\begin{aligned} \frac{d}{ds} \mathcal{F}_B &= \frac{1}{\lambda^2} \left(\frac{1}{2} \int \left[\varepsilon_y^2 \tilde{\phi}'_B + \frac{\lambda^2}{\lambda_0^2} \varepsilon^2 \phi'_B - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}'_B \right] \right. \\ &\quad \left. + 2 \int \tilde{\phi}_B (\varepsilon_y)_s \varepsilon_y + 2\varepsilon_s \left[\frac{\lambda^2}{\lambda_0^2} \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \right] \right. \\ &\quad \left. - 2 \int \tilde{\phi}_B (Q_b)_s ((\varepsilon + Q_b)^5 - Q_b^5 - 5\varepsilon Q_b^4) \right) \\ &\quad - 2 \frac{\lambda_s}{\lambda^3} \int \left[\varepsilon_y^2 \tilde{\phi}_B - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}_B \right] - 2 \frac{(\lambda_0)_s}{\lambda_0^3} \int \varepsilon^2 \phi_B \\ &= \frac{1}{\lambda^2} (f_1 + f_2 + f_3 + f_4), \end{aligned}$$

where

$$\begin{aligned} f_1 &= \frac{1}{2} \int \left[\varepsilon_y^2 \tilde{\phi}'_B + \varepsilon^2 \phi'_B - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}'_B \right] \\ &\quad + 2 \int \left(\varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon \right) \left(-(\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \right), \\ f_2 &= 2 \left(1 - \frac{\lambda^2}{\lambda_0^2} \right) \int \varepsilon_s \varepsilon \phi_B - 2 \frac{(\lambda_0)_s}{\lambda_0} \int \varepsilon^2 \phi_B \\ f_3 &= 2 \frac{\lambda_s}{\lambda} \int \Lambda \varepsilon \left(-(\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \right) \\ &\quad - 2 \frac{\lambda_s}{\lambda} \int \left[\varepsilon_y^2 \tilde{\phi}_B - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}_B \right] \\ f_4 &= -2 \int \tilde{\phi}_B (Q_b)_s ((\varepsilon + Q_b)^5 - Q_b^5 - 5\varepsilon Q_b^4). \end{aligned}$$

We use the equation of ε in the following form:

$$\begin{aligned} \varepsilon_s - \frac{\lambda_s}{\lambda} \Lambda \varepsilon &= (-\varepsilon_{yy} + \varepsilon - (\varepsilon + Q_b)^5 + Q_b^5)_y \\ &\quad + \left(\frac{\lambda_s}{\lambda} + b \right) \Lambda Q_b + \left(\frac{x_s}{\lambda} - 1 \right) (Q_b + \varepsilon)_y + \Phi_b + \Psi_b, \quad (\text{B.1}) \end{aligned}$$

where $\Phi_b = -b_s (\chi_b + \gamma \gamma (\chi_b)_y) P$ and $-\Psi_b = (Q_b'' - Q_b + Q_b^5)' + b \Lambda Q_b$.

Step 2. *Control of f_1 .* We have

$$\begin{aligned}
 f_1 &= \frac{1}{2} \int [\varepsilon_y^2 \tilde{\phi}'_B + \varepsilon^2 \phi'_B - \frac{1}{3}((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}'_B] \\
 &\quad + 2 \int (-\varepsilon_{yy} + \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5))_y (-\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B [(\varepsilon + Q_b)^5 - Q_b^5] \\
 &\quad + 2 \left(\frac{\lambda_s}{\lambda} + b \right) \int \Lambda Q_b (-\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
 &\quad + 2 \left(\frac{x_s}{\lambda} - 1 \right) \int (Q_b + \varepsilon)_y (-\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
 &\quad + 2 \int \Phi_b (-\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
 &\quad + 2 \int \Psi_b (-\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B ((\varepsilon + Q_b)^5 - Q_b^5) \\
 &= f_{1,1} + f_{1,2} + f_{1,3} + f_{1,4} + f_{1,5}.
 \end{aligned}$$

As in [29], after some computations we obtain

$$\begin{aligned}
 f_{1,1} &= - \int [3\tilde{\phi}'_B \varepsilon_{yy}^2 + (3\phi'_B + \frac{1}{2}\tilde{\phi}'_B - \tilde{\phi}'''_B) \varepsilon_y^2 + (\frac{1}{2}\phi'_B - \phi'''_B) \varepsilon^2] \\
 &\quad - \frac{1}{6} \int ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}'_B \\
 &\quad - 2 \int \left[\frac{(\varepsilon + Q_b)^6}{6} - \frac{Q_b^6}{6} - Q_b^5 \varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5) \varepsilon \right] (\phi'_B - \tilde{\phi}'_B) \\
 &\quad + 2 \int [(\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon] (Q_b)_y (\tilde{\phi}_B - \phi_B) \\
 &\quad + 10 \int \tilde{\phi}'_B \varepsilon_y \{ (Q_b)_y [(Q_b + \varepsilon)^4 - Q_b^4] + (Q_b + \varepsilon)^4 \varepsilon_y \} \\
 &\quad + \int \tilde{\phi}'_B [-2\varepsilon_{yy} + 2\varepsilon - ((\varepsilon + Q_b)^5 - Q_b^5)] [(\varepsilon + Q_b)^5 - Q_b^5].
 \end{aligned}$$

Using the following estimates (see [29] for more details):

$$\tilde{\phi}'''_B \lesssim \frac{1}{B^2} \phi'_B, \quad \phi'''_B \lesssim \frac{1}{B^2} \phi'_B \quad \text{for all } y \in \mathbb{R}, \tag{B.2}$$

$$|Q_b(y)| + |(Q_b)_y(y)| \lesssim e^{-|y|} + |b| \quad \text{for all } y \in \mathbb{R}, \tag{B.3}$$

$$\int \varepsilon^6 \phi'_B \lesssim \delta(\alpha^*) \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B, \tag{B.4}$$

$$\int \varepsilon_y^2 \varepsilon^4 \tilde{\phi}'_B \leq \delta(\alpha^*) \left(\int \varepsilon_{yy}^2 \tilde{\phi}'_B + \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B \right), \tag{B.5}$$

and the bound on the L^2 norm of ε (see Lemma 6.2), we obtain, for B large and α^* small,

$$\begin{aligned} f_{1,1} &\leq - \int \tilde{\phi}'_B \varepsilon_{yy}^2 - \frac{1}{4} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C \int (\varepsilon_y^2 + \varepsilon^2) e^{-|y|/10} + C|b|^4 \\ &\leq - \int \tilde{\phi}'_B \varepsilon_{yy}^2 - \frac{1}{4} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C \int (\varepsilon_y^2 + \varepsilon^2) \frac{\varphi_B}{1+y_+^2} + C|b|^4. \end{aligned}$$

Next,

$$\begin{aligned} f_{1,2} &= 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda Q(L\varepsilon) - 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \varepsilon(1 - \phi_B) \Lambda Q \\ &\quad + 2b\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda(\chi_b P) (-\tilde{\phi}_B \varepsilon_y)_y + \varepsilon \phi_B - \tilde{\phi}_B [(Q_b + \varepsilon)^5 - Q_b^5] \\ &\quad + 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda Q (-\tilde{\phi}_B)_y \varepsilon_y - (1 - \tilde{\phi}_B) \varepsilon_{yy} + (1 - \tilde{\phi}_B) [(Q_b + \varepsilon)^5 - Q_b^5] \\ &\quad + 2\left(\frac{\lambda_s}{\lambda} + b\right) \int \Lambda Q [(Q_b + \varepsilon)^5 - Q_b^5 - 5Q_b^4 \varepsilon]. \end{aligned}$$

The main term $\int \Lambda Q(L\varepsilon)$ is zero by the orthogonality conditions on ε , and the other terms are controlled as in [29] using (2.29), (B.2), (B.3) and (B.5), to obtain

$$|f_{1,2}| \leq \frac{1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C \int (\varepsilon_y^2 + \varepsilon^2) \frac{\varphi_B}{1+y_+^2} + C|b|^4.$$

The next term is

$$\begin{aligned} f_{1,3} &= 2\left(\frac{x_s}{\lambda} - 1\right) \int \frac{1}{6} \tilde{\phi}'_B [(Q_b + \varepsilon)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \\ &\quad + 2\left(\frac{x_s}{\lambda} - 1\right) \int (b\chi_b P + \varepsilon)_y [-\tilde{\phi}'_B \varepsilon_y - \tilde{\phi}_B \varepsilon_{yy} + \varepsilon \phi_B] \\ &\quad + 2\left(\frac{x_s}{\lambda} - 1\right) \int Q' [L\varepsilon - \tilde{\phi}'_B \varepsilon_y + (1 - \tilde{\phi}_B) \varepsilon_{yy} - \varepsilon(1 - \phi_B)] \\ &\quad + 10\left(\frac{x_s}{\lambda} - 1\right) \int \varepsilon \tilde{\phi}_B (Q_b^4 (Q_b)_y - Q_b^4 Q_y). \end{aligned}$$

Using $LQ' = 0$ and arguing similarly, we obtain

$$|f_{1,3}| \leq \frac{1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi'_B + C \int (\varepsilon_y^2 + \varepsilon^2) \frac{\varphi_B}{1+y_+^2} + C|b|^4.$$

Step 3. Control of f_2 . First, by (6.13), we have

$$-\frac{(\lambda_0)_s}{\lambda_0} \int \varepsilon^2 \phi_B < 0.$$

Next, by the definition of λ_0 in Lemma 2.10, we have

$$\left| 1 - \frac{\lambda^2}{\lambda_0^2} \right| \lesssim \mathcal{N}^{1/2} \lesssim \delta(\alpha^*),$$

and thus, proceeding for $\int \varepsilon_s \varepsilon \phi_B$ as in the previous step, we find

$$\left| 1 - \frac{\lambda^2}{\lambda_0^2} \right| \left| \int \varepsilon_s \varepsilon \phi_B \right| \leq \frac{1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C \int (\varepsilon_y^2 + \varepsilon^2) \frac{\varphi_B}{1 + y_+^2} + C|b|^4.$$

Step 4. *Control of f_3 .* From computations in [29],

$$\begin{aligned} f_3 &= \frac{\lambda_s}{\lambda} \int [2\tilde{\phi}_B - y\tilde{\phi}'_B] \varepsilon_y^2 - \frac{\lambda_s}{\lambda} \int y\tilde{\phi}'_B \varepsilon^2 \\ &\quad - \frac{1}{3} \frac{\lambda_s}{\lambda} \int [2\tilde{\phi}_B - y\tilde{\phi}'_B] [(\varepsilon + Q_b)^6 - Q_b^6 - 6Q_b^5 \varepsilon] \\ &\quad + 2 \frac{\lambda_s}{\lambda} \int \tilde{\phi}_B \Lambda Q_b ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon) \\ &\quad - 2 \frac{\lambda_s}{\lambda} \int [\varepsilon_y^2 \tilde{\phi}_B - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6\varepsilon Q_b^5) \tilde{\phi}_B]. \end{aligned}$$

After simplification of the last line with terms in the first and second lines, we obtain

$$\begin{aligned} f_3 &= - \frac{\lambda_s}{\lambda} \int y\tilde{\phi}'_B [\varepsilon_y^2 + \varepsilon^2 - \frac{1}{3} ((\varepsilon + Q_b)^6 - Q_b^6 - 6Q_b^5 \varepsilon)] \\ &\quad + 2 \frac{\lambda_s}{\lambda} \int \tilde{\phi}_B \Lambda Q_b ((\varepsilon + Q_b)^5 - Q_b^5 - 5Q_b^4 \varepsilon). \end{aligned}$$

For this term we observe, from the definition of ϕ_B and $\tilde{\phi}_B$,

$$\int |y| \tilde{\phi}'_B (\varepsilon_y^2 + \varepsilon^2 + |\varepsilon|^6) \lesssim \int (\varepsilon_y^2 + \varepsilon^2) \phi_B'$$

and $|\lambda_s/\lambda| \lesssim \delta(\alpha^*)$. The other terms in the expression of f_3 are treated as before, so that

$$|f_3| \leq \frac{1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C \int (\varepsilon_y^2 + \varepsilon^2) \frac{\varphi_B}{1 + y_+^2} + C|b|^4.$$

Step 5. *Control of f_4 .* Arguing exactly as in [29] (using (2.30)), we obtain

$$|f_4| \leq \frac{1}{100} \int (\varepsilon_y^2 + \varepsilon^2) \phi_B' + C \int (\varepsilon_y^2 + \varepsilon^2) \frac{\varphi_B}{1 + y_+^2} + C|b|^4.$$

Gathering these estimates, we get (6.24).

Step 6. *Proof of (6.25).* This is a standard fact by localization arguments (see e.g. [25, Appendix A]).

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