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The Brauer category and invariant theory

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Abstract. A category of Brauer diagrams, analogous to Turaev's tangle category, is introduced, a presentation of the category is given, and full tensor functors are constructed from this category to the category of tensor representations of the orthogonal group $\mathrm{O}(V)$ or the symplectic group $\mathrm{Sp}(V)$ over any field of characteristic zero. The first and second fundamental theorems of invariant theory for these classical groups are generalised to the category-theoretic setting. The major outcome is that we obtain presentations for the endomorphism algebras of the module $V^{\otimes r}$, which are new in the classical symplectic case and in the orthogonal and symplectic quantum case, while in the orthogonal classical case, the proof we give here is more natural than in our earlier work. These presentations are obtained by appending to the standard presentation of the Brauer algebra of degree r one additional relation. This relation stipulates the vanishing of a single element of the Brauer algebra which is quasi-idempotent, and which we describe explicitly both in terms of diagrams and algebraically. In the symplectic case, if $\dim V = 2n$, the element is precisely the central idempotent in the Brauer subalgebra of degree $n + 1$, which corresponds to its trivial representation. Since this is the Brauer algebra of highest degree which is semisimple, our generator is an exact analogue for the Brauer algebra of the Jones idempotent of the Temperley–Lieb algebra. In the orthogonal case the additional relation is also a quasi-idempotent in the integral Brauer algebra. Both integral and quantum analogues of these results are given, the latter involving the BMW algebras.

Keywords. Brauer category, invariant theory, second fundamental theorem, quantum group

Contents

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1. Introduction

The basic problems of classical invariant theory are to describe generators and relations for invariants of group actions. Depending on the context, the problem may be formulated in different ways. A linear formulation describes a spanning set of the vector space of invariant linear functionals on a module, and all the linear relations among the elements of this set. A commutative algebraic formulation describes the invariants of the coordinate ring of an appropriate module, where one seeks a presentation of the algebra of invariant functions as a commutative algebra. A third formulation, which is more frequently encountered in representation theory, is in terms of the non-commutative endomorphism algebra of a module of tensors. In this context, the problem is to give a presentation of the subalgebra of invariants as a generally non-commutative associative algebra.

The celebrated first and second fundamental theorems [W] of classical invariant theory solve the problem for classical group actions on tensor modules in the linear formulation, and on coordinate algebras of multi-copies of the natural modules in the commutative-algebraic formulation. These two formulations are in fact equivalent. In the endomorphism algebra formulation, the first fundamental theorem (FFT) [GW] describes the endomorphism algebra as the homomorphic image of some known algebra, viz. the group algebra of the symmetric group in the case of the general linear group following Schur, and the Brauer algebra [Br] with appropriate parameters in the case of the orthogonal or symplectic group by work of Brauer.

However, for over three quarters of a century after Brauer's work, there has been no second fundamental theorem (SFT) in the endomorphism algebra formulation, except in the case of type A (GL_n). For the orthogonal or symplectic group, an SFT cannot be simply deduced in any useful form from the SFTs in the other two formulations. This is because one is seeking generators of an ideal of a non-commutative associative algebra, and a knowledge of a linear spanning set is not very useful. It is the purpose of this paper to address the second fundamental theorems for these classical groups in its third formulation (as described above).

In [LZ2, LZ3], the orthogonal group $O(V)$ over $\mathbb{C}$ with $\dim V = 3$ was investigated together with its quantum analogue at generic q. We obtained a single idempotent E in the Brauer algebra of degree $r \geq 4$, which generates a two-sided ideal that is equal to the kernel of the algebra homomorphism from the Brauer algebra to the endomorphism algebra $\mathrm{End}_{O(V)}(V^{\otimes r})$ (the kernel is trivial if $r < 4$). Thus $\mathrm{End}_{O(V)}(V^{\otimes r})$ can be presented in terms of the standard generators and relations of the Brauer algebra with the single additional relation $E = 0$.

Remarkably, the situation turned out to be the same for all the orthogonal groups [LZ4] over any field K of characteristic zero. It was shown in [HX] that this should also be the case for the symplectic group, even though the corresponding element E was not explicitly constructed there.

In the present paper we develop a categorical approach to the invariant theory of the orthogonal and symplectic groups, obtaining a unified treatment of the fundamental theorems in both the linear and endomorphism algebra formulations. This yields an explicit formula for the generator E of the kernel of the algebra homomorphism from the Brauer algebra to the endomorphism algebra $\mathrm{End}_{Sp(V)}(V^{\otimes r})$ in the case of the symplectic group, and provides new and more conceptual proofs of the main results of [LZ4] for the orthogonal group.

We then use integral forms of our main results, as well as the cellular structure of both the Brauer algebras and the BMW algebras, to generalise these results to the quantum group case in §8.

The methods used in the papers [LZ2, LZ3, LZ4] and [HX] are quite different. In [LZ2, LZ3], an analysis of the radical of the Brauer algebra is performed, which makes an extensive use of the theory of cellular algebras [GL96, GL03, GL04]. The paper [HX] relied on results on the detailed structure and representations [DHW, HW, RS, X] of the Brauer algebra and BMW algebra [BW]. In particular, it made essential use of a series of earlier papers of Hu and collaborators. In contrast, invariant theory featured much more prominently in [LZ4].

Our approach here is inspired by works on quantum invariants of links [J, T1, R, RT, ZGB], as well as works such as [PM], which have discussed relationships between certain diagram categories and representations of algebras, particularly those arising in questions of statistical mechanics. Recall that a key algebraic result in quantum topology is that the category of tangles is a strict monoidal category with braiding [FY1, FY2, T1] (also see [RT, T2]) in the sense of Joyal and Street [JS]. The set of objects of this category is $\mathbb{N} = \{0, 1, 2, \dots\}$, and the vector spaces of morphisms have bases consisting of non-isotopic tangle diagrams. We define a similar, but much simpler category $\mathcal{B}(\delta)$, the *category of Brauer diagrams* with parameter $\delta \in K$. The space of morphisms of $\mathcal{B}(\delta)$ is spanned by *Brauer diagrams* (see Definition 2.1), which include the usual Brauer diagrams [Br] as a special case, as endomorphisms of an object of the category. Similar categories, such as the partition category, are discussed in [PM, §§4, 5, 6].

Let G be either the orthogonal group $O(V)$ or the symplectic group $Sp(V)$, and denote by $\mathcal{T}_G(V)$ the full subcategory of the category of finite-dimensional G-representations with objects $V^{\otimes r}$ ($r \in \mathbb{N}$). There exists an additive functor $F : \mathcal{B}(\epsilon m) \to \mathcal{T}_G(V)$, which is given by Theorem 3.4. Here $\epsilon m = \epsilon(G) \dim V$ with $\epsilon(G) = 1$ for $O(V)$ and -1

for $\mathrm{Sp}(V)$. The functor F is shown to be full in Theorem 4.8(1). This significantly generalises the FFTs for the orthogonal and symplectic groups. Both the linear and endomorphism algebra versions of FFT are now special cases of Theorem 4.8(1), and their equivalence becomes entirely transparent.

For each pair of objects r, s in the category $\mathcal{B}(\epsilon m)$, the functor F induces a linear map $F_r^s : \mathrm{Hom}_{\mathcal{B}(\epsilon m)}(r, s) \to \mathrm{Hom}_{\mathcal{T}_G(V)}(V^{\otimes r}, V^{\otimes s})$. A simple description of the subspace $\mathrm{Ker}\, F_r^s$ is obtained in Theorem 4.8(2), which contains the linear version of SFT as a special case.

When $s = r$, the domain of F_r^r is the Brauer algebra of degree r, the range is the endomorphism algebra $\mathrm{End}_{\mathcal{T}_G(V)}(V^{\otimes r})$, and the map is an algebra homomorphism. In this case, we want to understand the algebraic structure of the kernel of the map F_r^r.

We explicitly construct an element in the Brauer algebra which generates $\mathrm{Ker}\, F_r^r$ as a two-sided ideal ($\mathrm{Ker}\, F_r^r \neq 0$ only when $r > d$, see Theorem 4.6). The result for the symplectic group is given in Theorem 5.9, and that for the orthogonal group in Theorem 6.10. This leads to a presentation of $\mathrm{End}_{\mathcal{T}_G(V)}(V^{\otimes r})$ upon imposing the condition that this element vanishes.

The significance of the Brauer categorical approach may be described as follows. As linear space, $\mathrm{Hom}_G(V^{\otimes r}, V^{\otimes s})$ depends only on $r + s$, which must be even for the Hom space to be non-zero. But when $s = r = (r + s)/2$, this space additionally has the structure of an associative algebra, and the calculus of diagrams in the category makes it possible to economically transform linear relations into algebraic ones, involving composition.

In the case of $\mathrm{O}(V)$, the generating element we obtain is shown to be equal to that obtained in [LZ4]. In the symplectic case, the element anticipated by [HX] is a scalar multiple of the one obtained here (Remark 5.10). However, our approach yields an explicit formula for the element, both in terms of generators and relations, and in terms of diagrams; moreover we show that the element is (a multiple of) the central idempotent corresponding to the trivial representation of the Brauer algebra on $n + 1$ strings, if $r = 2n$. We note that $B_r(-2n)$ is semisimple if and only if $r \leq n + 1$ (see §7 below). Thus our generating element is an exact analogue of Jones' 'augmentation' idempotent [J, GL98].

Notwithstanding the fact that convenient formulae for our generating elements involve rational numbers with large denominators, the elements are actually sums of diagrams with coefficients ± 1. This permits reduction modulo primes, and an approach to the case of positive characteristic (§7).

2. The category of Brauer diagrams

We begin with a discussion on Brauer diagrams, which could be thought of as a highly simplified version of the tangle diagrams of [FY1, FY2, T1] (also see [RT, T2]). Tangles in this paper are neither oriented nor framed. In fact we shall find it easier to work with the (equivalent) category of Brauer diagrams, with no reference to tangles.

2.1. The category of Brauer diagrams

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 2.1. For any pair $k, \ell \in \mathbb{N}$, a (k, ℓ) *Brauer diagram*, or *Brauer diagram from k to ℓ*, is a partitioning of the set $\{1, \ldots, k + \ell\}$ as a disjoint union of pairs.

This is thought of as a diagram where $k + \ell$ points (the nodes, or vertices) are placed on two parallel horizontal lines, k on the lower line and ℓ on the upper, with arcs drawn to join points which are paired. We shall speak of the lower and upper nodes or vertices of a diagram. The pairs will be known as *arcs*. If $k = \ell = 0$, there is by convention just one $(0, 0)$ Brauer diagram. Figure 1 below is a $(6, 4)$ Brauer diagram.

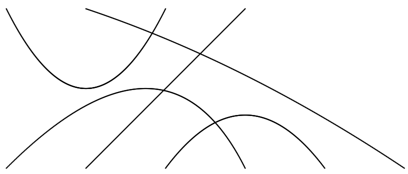


Fig. 1. A $(6, 4)$ Brauer diagram.

Remark 2.2. Such a diagram may be thought of as the image of a tangle diagram (i.e. ambient isotopy class of (k, ℓ) tangles) under projection to a plane. It is straightforward that if overcrossings and undercrossings are identified in a tangle projection, the only invariants of a tangle are the number of free loops and the set of pairs of boundary points, each of which is the boundary of a connected component of the tangle. Hence the identification with Brauer diagrams. We shall therefore not use tangles explicitly.

There are two operations on Brauer diagrams: *composition* defined using concatenation of diagrams and *tensor product* defined using juxtaposition (see below).

Definition 2.3. Let K be a commutative ring with identity, and fix $\delta \in K$. Denote by $B_k^\ell(\delta)$ the free K-module with a basis consisting of (k, ℓ) Brauer diagrams. Note that $B_k^\ell(\delta) \neq 0$ if and only if $k + \ell$ is even, since the free K-module with basis the empty set is zero. By convention there is one diagram in $B_0^0(\delta)$, viz. the empty diagram. Thus $B_0^0(\delta) = K$.

There are two K-bilinear operations on diagrams.

$$\begin{aligned} &\text{composition} \quad \circ : B_\ell^p(\delta) \times B_k^\ell(\delta) \to B_k^p(\delta), \quad \text{and} \\ &\text{tensor product} \quad \otimes : B_p^q(\delta) \times B_k^\ell(\delta) \to B_{k+p}^{q+\ell}(\delta) \end{aligned} \tag{2.1}$$

These operations are defined as follows.

- The composite $D_1 \circ D_2$ of the Brauer diagrams $D_1 \in B_\ell^p(\delta)$ and $D_2 \in B_k^\ell(\delta)$ is defined as follows. First, the concatenation $D_1 \# D_2$ is obtained by placing D_1 above D_2, and identifying the ℓ lower nodes of D_1 with the corresponding upper nodes of D_2. Then $D_1 \# D_2$ is the union of a Brauer (k, p) diagram D with a certain number, $f(D_1, D_2)$ say, of free loops. The composite $D_1 \circ D_2$ is the element $\delta^{f(D_1, D_2)} D \in B_k^p(\delta)$.

- The tensor product $D \otimes D'$ of any two Brauer diagrams $D \in B_p^q(\delta)$ and $D' \in B_k^l(\delta)$ is the $(p+k, q+l)$ diagram obtained by juxtaposition, that is, placing D' on the right of D without overlapping.

Both operations are clearly associative.

Definition 2.4. The *category of Brauer diagrams*, denoted by $\mathcal{B}(\delta)$, is the following pre-additive small category equipped with a bi-functor $\otimes$ (which will be called the tensor product):

- the set of objects is $\mathbb{N} = \{0, 1, 2, \dots\}$, and for any pair of objects k, l, $\mathrm{Hom}_{\mathcal{B}(\delta)}(k, l)$ is the K-module $B_k^l(\delta)$; the composition of morphisms is given by the composition of Brauer diagrams defined by (2.1);
- the tensor product $k \otimes l$ of objects k, l is $k+l$ in $\mathbb{N}$, and the tensor product of morphisms is given by the tensor product of Brauer diagrams of (2.1).

It follows from the associativity of composition of Brauer diagrams that $\mathcal{B}(\delta)$ is indeed a pre-additive category.

Remark 2.5. The operations in $\mathcal{B}(\delta)$ mirror the operations in the tangle category considered in [FY1, FY2, T1, RT, T2], and the *category of Brauer diagrams* is a quotient category of the category of tangles in the sense of [M, §II.8].

2.2. *Involutions*

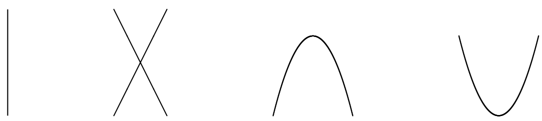
The category $\mathcal{B}(\delta)$ has a *duality functor* $^* : \mathcal{B}(\delta) \to \mathcal{B}(\delta)^{\mathrm{op}}$, which takes each object to itself, and takes each diagram to its reflection in a horizontal line. More formally, for any (k, ℓ) diagram D, D^* is the (ℓ, k) diagram with precisely the same pairs identified as D. Further, there is an involution $^\sharp : \mathcal{B}(\delta) \to \mathcal{B}(\delta)$ which also takes objects to themselves, but takes a diagram D to its reflection in a vertical line. Formally, if the upper nodes of the diagram D are labelled $1, \dots, \ell$ and the lower nodes are labelled $1', 2', \dots, k'$, we apply the permutation $i \mapsto \ell + 1 - i$, $j' \mapsto k + 1 - j'$ to the nodes to get the arcs of $D^\sharp$. We shall meet the contravariant functor $D \mapsto *D := D^{*\circ\sharp}$ later.

It is easily checked that $(D_1 \circ D_2)^* = D_2^* \circ D_1^*$, $(D_1 \otimes D_2)^* = D_1^* \otimes D_2^*$, and that $(D_1 \circ D_2)^\sharp = D_1^\sharp \circ D_2^\sharp$ and $(D_1 \otimes D_2)^\sharp = D_2^\sharp \otimes D_1^\sharp$.

2.3. *Generators and relations*

Generators and relations for tangle diagrams were described in [FY1, FY2, T1, RT, T2]. The next theorem is the corresponding result for Brauer diagrams.

Theorem 2.6. (1) *The four Brauer diagrams*



generate all Brauer diagrams by composition and tensor product (*i.e., juxtaposition*).

We shall refer to these generators as the elementary Brauer diagrams, and denote them by I, X, A and U respectively. Note that these diagrams are all fixed by $^\sharp$, *and that* * *fixes I and X, while $A^* = U$ and $U^* = A$.*

(2) *A complete set of relations among these four generators is given by the following, and their transforms under* * *and* $^\sharp$. *This means that any equation relating two words in these four generators can be deduced from the given relations.*

$$I \circ I = I, \ (I \otimes I) \circ X = X, \ A \circ (I \otimes I) = A, \ (I \otimes I) \circ U = U, \tag{2.2}$$

$$X \circ X = I, \tag{2.3}$$

$$(X \otimes I) \circ (I \otimes X) \circ (X \otimes I) = (I \otimes X) \circ (X \otimes I) \circ (I \otimes X), \tag{2.4}$$

$$A \circ X = A, \tag{2.5}$$

$$A \circ U = \delta, \tag{2.6}$$

$$(A \otimes I) \circ (I \otimes X) = (I \otimes A) \circ (X \otimes I), \tag{2.7}$$

$$(A \otimes I) \circ (I \otimes U) = I. \tag{2.8}$$

The relations (2.3)–(2.8) are depicted diagrammatically in Figures 2–4.

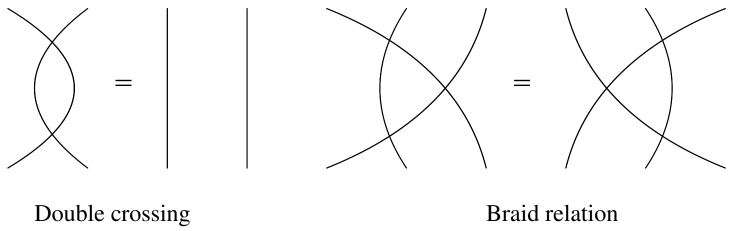


Fig. 2. Relations (2.3) and (2.4).

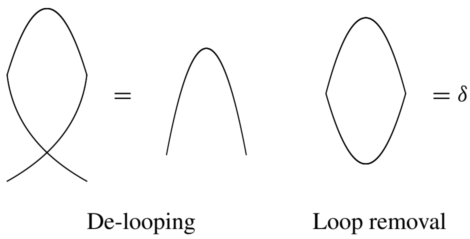


Fig. 3. Relations (2.5) and (2.6).

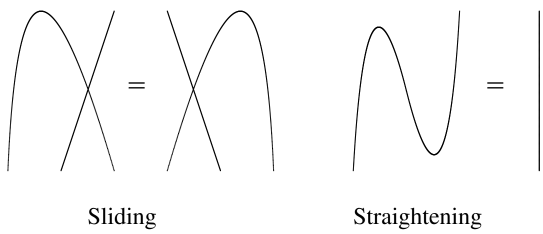


Fig. 4. Relations (2.7) and (2.8).

In principle, the proof of Theorem 2.6 may be deduced from of [T2, §I.4]. In the Appendix, we provide an independent proof of the completeness of the above relations, which is purely algebraic.

2.4. *Some useful diagrams*

We shall find the following diagrams useful in later sections of this work. Let $A_q = A \circ (I \otimes A \otimes I) \dots (I^{\otimes(q-1)} \otimes A \otimes I^{\otimes(q-1)})$, $U_q = (I^{\otimes(q-1)} \otimes U \otimes I^{\otimes(q-1)}) \circ \dots \circ (I \otimes U \otimes I) \circ U$ and $I_q = I^{\otimes q}$. These are depicted as diagrams in Figure 5.

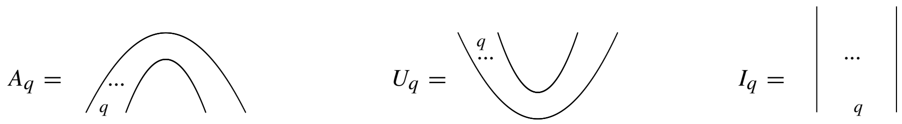


Fig. 5. A_q, U_q and I_q.

We shall also need $X_{s,t}$, the $(s+t, s+t)$ Brauer diagram shown in Figure 6.

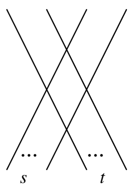


Fig. 6. $X_{s,t}$.

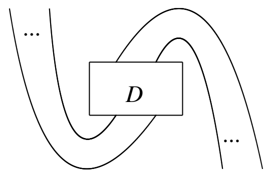


Fig. 7. $*D$.

The following result is easy.

Lemma 2.7. (1) *For any Brauer diagrams* $D_1 \in B_k^r(\delta)$ *and* $D_2 \in B_r^q(\delta)$*, we have* $I_r \circ D_1 = D_1$ *and* $D_2 \circ I_r = D_2$*. That is,* $I_r = \mathrm{id}_r$ *for any object* r *of* $\mathcal{B}(\delta)$.
(2) *The following relation holds:*

$$(I_q \otimes A_q) \circ (U_q \otimes I_q) = (A_q \otimes I_q) \circ (I_q \otimes U_q) = I_q.$$

Corollary 2.8. *For all* p, q *and* r*, define the linear maps*

$$\mathbb{U}_p^q = (- \otimes I_q) \circ (I_p \otimes U_q) : B_{p+q}^r(\delta) \to B_p^{r+q}(\delta),$$
$$\mathbb{A}_q^r = (I_r \otimes A_q) \circ (- \otimes I_q) : B_p^{r+q}(\delta) \to B_{p+q}^r(\delta).$$

Then $\mathbb{U}_p^q = R^q$ *and* $\mathbb{A}_q^r = L^q$ *(see Definition A.5). These are mutually inverse.*

This is clear since by Lemma A.6, L and R are mutually inverse.

Let $* : B_p^q(\delta) \to B_q^p(\delta)$ be the linear map defined for any $D \in B_p^q(\delta)$ by $*D = (I_p \otimes A_q) \circ (I_p \otimes D \otimes I_q) \circ (U_p \otimes I_q)$. Pictorially, $*D$ is obtained from D as in Figure 7.

Lemma 2.9. *The map* $*$ *coincides with the anti-involution* $D \mapsto *D := D^{*\circ\sharp}$ *discussed in* §2.2.

This is easily seen in terms of diagrams.

2.5. *The Brauer algebra*

For any object r in $\mathcal{B}(\delta)$, the set $B_r^r(\delta)$ of morphisms from r to itself forms a unital associative K-algebra under composition of Brauer diagrams. This is the Brauer algebra of degree r with parameter δ, which we will denote by $B_r(\delta)$. The first two results of the following lemma are well known.

Lemma 2.10. (1) *For $i = 1, \dots, r-1$, let s_i and e_i respectively be the (r, r) Brauer diagrams shown in Figure 8 below. Then $B_r(\delta)$ has the following presentation as a K-algebra with anti-involution $*$. The generators are $\{s_i, e_i \mid i = 1, \dots, r-1\}$, and relations*

$$s_i s_j = s_j s_i,\ s_i e_j = e_j s_i,\ e_i e_j = e_j e_i \quad \textit{if } |i-j| \geq 2,$$
$$s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$
$$s_i e_i = e_i s_i = e_i,$$
$$e_i^2 = \delta e_i,$$
$$e_i e_{i\pm 1} e_i = e_i,$$
$$s_i e_{i+1} e_i = s_{i+1} e_i,$$

with the last five relations being valid for all applicable i.

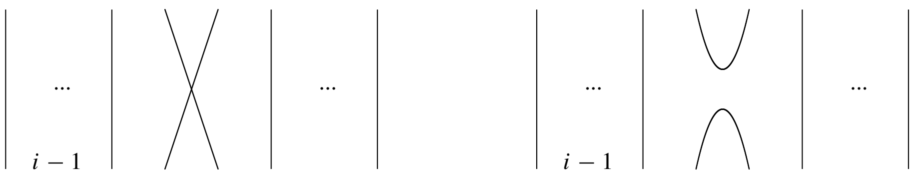


Fig. 8. s_i (left) and e_i (right).

(2) *The elements $s_1, \dots, s_{r-1}$ generate a subalgebra of $B_r(\delta)$, isomorphic to the group algebra $K\mathrm{Sym}_r$ of the symmetric group Sym_r.*
(3) *The map $*$ of Lemma 2.9 restricts to an anti-involution of the Brauer algebra.*

Parts (1) and (2) follow from Theorem 2.6, noting that any regular expression (see (A.1) in the Appendix for the definition of this term) for a diagram in $B_r(\delta)$ contains an equal number of factors of type A and U. The stated relations are precise analogues of the relations in Theorem 2.6(2). Part (3) is easy to prove. However, we note that $*s_i = s_{r+1-i}$ and $*e_i = e_{r+1-i}$. This is different from the standard cellular anti-involution * of the Brauer algebra.

We remark that multiplying the last relation above by e_i on the left and using two of the earlier relations, we obtain

$$e_i s_{i+1} e_i = e_i,$$

a relation which we shall often use, together with its transform under $*$: $e_i s_{i-1} e_i = e_i$.

Now we prove some technical lemmas for later use.

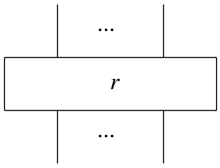
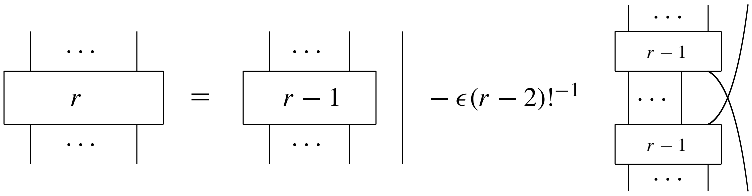


Fig. 9. $\Sigma_\epsilon(r)$.

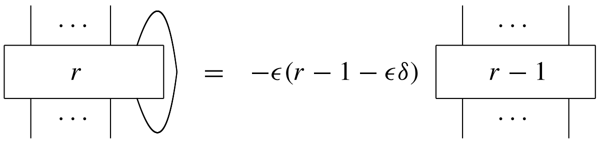
Lemma 2.11. *Let $\Sigma_\epsilon(r) = \sum_{\sigma\in \mathrm{Sym}_r}(-\epsilon)^{|\sigma|}\sigma \in B_r(\delta)$, where $\epsilon = \pm 1$ and $|\sigma|$ is the length of σ. Represent $\Sigma_\epsilon(r)$ pictorially by Figure 9.*

Then the following relations hold for all r.

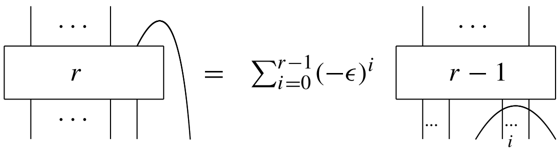
(1)



(2)



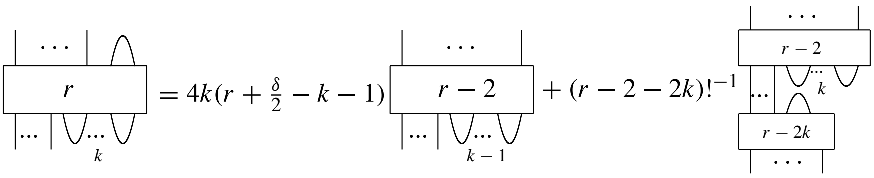
(3)



Proof. Part (1) generalises [LZ4, Lemma 5.1(i)] and is a simple consequence of the double coset decomposition of Sym_r into $\mathrm{Sym}_{r-1} \amalg \mathrm{Sym}_{r-1}s_{r-1}\mathrm{Sym}_{r-1}$. Part (2) immediately follows from (1). Statement (3) can be obtained from (1) by induction on r. □

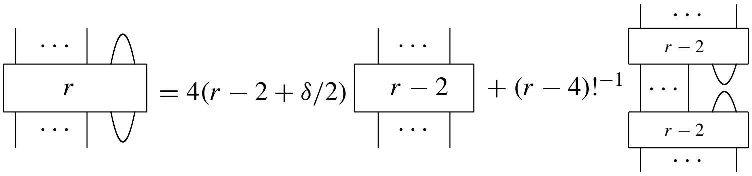
Remark 2.12. Symmetry considerations easily show that the second diagram on the right hand side of Lemma 2.11(1) is a $(r-2)!$-multiple of a $\mathbb{Z}$-linear combination of Brauer diagrams; thus the second term is still defined over $\mathbb{Z}$ despite having the coefficient $\frac{1}{(r-2)!}$. The same remark applies to similar terms appearing in Lemma 2.13 and its proof.

Lemma 2.13. *Set $\epsilon = -1$. Then for all $k \geq 0$,*



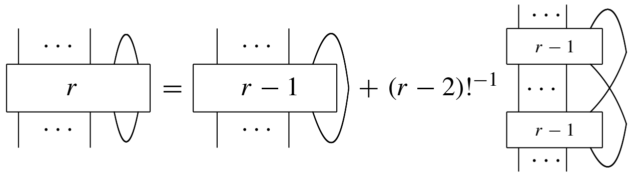
(2.9)

Proof. For $k = 0$, the formula is an identity. The important case is $k = 1$, where the formula becomes



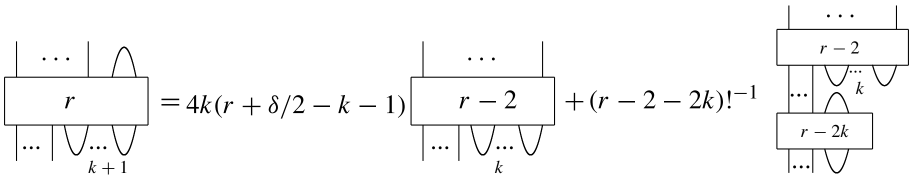
(2.10)

To prove it, we first obtain from Lemma 2.11(1) with $\epsilon = -1$ the following relation.



Using Lemma 2.11(2) for the first diagram on the right hand side, and applying Lemma 2.11(3) and the corresponding relation under the anti-involution $*$ to the second diagram, we obtain (2.10).

The general case can be proven by induction on k. From (2.9) at k, we obtain



Using (2.10) in the second term on the right hand side, we arrive at the $k+1$ case of (2.9). This completes the proof. □

3. A covariant functor

Let K be a field. Let $V = K^m$ be an m-dimensional vector space with a non-degenerate bilinear form $(-,-)$, which is either symmetric or skew symmetric. When the form is skew symmetric, non-degeneracy requires $m = 2n$ to be even. Let G denote the isometry group of the form, so that $G = \{g \in \mathrm{GL}(V) \mid (gv, gw) = (v, w), \forall v, w \in V\}$. Then G is the orthogonal group $\mathrm{O}(V)$ if the form is symmetric, and the symplectic group $\mathrm{Sp}(V)$ if the form is skew symmetric.

Given a basis $\{b_1, \ldots, b_m\}$ for V, let $\{\bar{b}_1, \ldots, \bar{b}_m\}$ be the dual basis of V, identified with V^* via the map $v(\in V) \mapsto \phi_v\ (\in V^*)$ where $\phi_v(x) := (v, x)$; thus $(\bar{b}_i, b_j) = \delta_{ij}$.

For any positive integer t, the space $V^{\otimes t}$ is a G-module in the usual way: $g(v_1 \otimes \cdots \otimes v_t) = gv_1 \otimes \cdots \otimes gv_t$. Moreover the form on V induces a non-degenerate bilinear form $[-,-]$ on $V^{\otimes t}$, given by $[v_1 \otimes \cdots \otimes v_t, w_1 \otimes \cdots \otimes w_t] := \prod_{i=1}^{t}(v_i, w_i)$, which permits the identification of $V^{\otimes t}$ with its dual space $V^{\otimes t *} = \mathrm{Hom}_K(V^{\otimes t}, K)$.

Define $c_0 \in V \otimes V$ by $c_0 = \sum_{i=1}^{m} b_i \otimes \bar{b}_i$ in $V \otimes V$. Then c_0 is canonical in that it is independent of the basis, and is invariant under G. We shall consider various G-equivariant maps $\beta : V^{\otimes s} \to V^{\otimes t}$ for $s, t \in \mathbb{Z}_{\geq 0}$. Among these we have the following.

$$\begin{aligned} P : V \otimes V \to V \otimes V, &\quad v \otimes w \mapsto w \otimes v, \\ \check{C} : K \to V \otimes V, &\quad 1 \mapsto c_0, \\ \hat{C} : V \otimes V \to K, &\quad v \otimes w \mapsto (v, w). \end{aligned} \tag{3.1}$$

They have the following properties.

Lemma 3.1. *Let $\epsilon = \epsilon(G)$ be 1 (resp. -1) if $G = \mathrm{O}(V)$ (resp. $\mathrm{Sp}(V)$). Denote the identity map on V by* id.

(1) *The element c_0 belongs to $(V \otimes V)^G$ and satisfies $P(c_0) = \epsilon c_0$.*
(2) *The maps P, $\check{C}$ and $\hat{C}$ are all G-equivariant, and*

$$P^2 = \mathrm{id}^{\otimes 2}, \quad (P \otimes \mathrm{id})(\mathrm{id} \otimes P)(P \otimes \mathrm{id}) = (\mathrm{id} \otimes P)(P \otimes \mathrm{id})(\mathrm{id} \otimes P), \tag{3.2}$$

$$P\check{C} = \epsilon\check{C}, \quad \hat{C}P = \epsilon\hat{C}, \tag{3.3}$$

$$\hat{C}\check{C} = \epsilon \dim V, \quad (\hat{C} \otimes \mathrm{id})(\mathrm{id} \otimes \check{C}) = \mathrm{id} = (\mathrm{id} \otimes \hat{C})(\check{C} \otimes \mathrm{id}), \tag{3.4}$$

$$(\hat{C} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes P) = (\mathrm{id} \otimes \hat{C}) \circ (P \otimes \mathrm{id}), \tag{3.5}$$

$$(P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \check{C}) = (\mathrm{id} \otimes P) \circ (\check{C} \otimes \mathrm{id}). \tag{3.6}$$

Proof. Equation (3.2) reflects standard properties of permutations, and the relations (3.3) are evident. We prove the other relations. Consider for example $\hat{C}\check{C} = \hat{C}(\sum_i b_i \otimes \bar{b}_i) = \sum_i (b_i, \bar{b}_i)$. The far right hand side is $\sum_i \epsilon = \epsilon \dim V$. This proves the first relation of (3.4). The proofs of the remaining relations are similar, and therefore omitted. □

Definition 3.2. We denote by $\mathcal{T}_G(V)$ the full subcategory of G-modules with objects $V^{\otimes r}$ $(r = 0, 1, \ldots)$, where $V^{\otimes 0} = K$ by convention. The usual tensor product of G-modules and of G-equivariant maps is a bi-functor $\mathcal{T}_G(V) \times \mathcal{T}_G(V) \to \mathcal{T}_G(V)$, which will be called the tensor product of the category. We call $\mathcal{T}_G(V)$ the *category of tensor representations of G*.

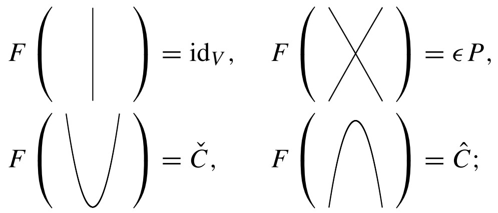
Note that $\mathrm{Hom}_G(V^{\otimes r}, V^{\otimes t}) = 0$ unless $r + t$ is even. The zero module is not an object of $\mathcal{T}_G(V)$, thus the category is only pre-additive but not additive.

Remark 3.3. The category $\mathcal{T}_G(V)$ is also a strict monoidal category with a symmetric braiding in the sense of [JS], where the braiding is given by the permutation maps $V^{\otimes r} \otimes V^{\otimes t} \to V^{\otimes t} \otimes V^{\otimes r}$, $v \otimes w \mapsto w \otimes v$.

We have the following result.

Theorem 3.4. *There is a unique additive covariant functor $F : \mathcal{B}(\epsilon m) \to \mathcal{T}_G(V)$ of pre-additive categories with the following properties:*

(i) *F sends the object r to $V^{\otimes r}$ and morphism $D : k \to \ell$ to $F(D) : V^{\otimes k} \to V^{\otimes l}$ where $F(D)$ is defined on the generators of Brauer diagrams by*



(3.7)

(ii) *F respects tensor products, so that for any objects r, r' and morphisms D, D' in $\mathcal{B}(\epsilon m)$,*

$$F(r \otimes r') = V^{\otimes r} \otimes V^{\otimes r'} = F(r) \otimes F(r'), \qquad F(D \otimes D') = F(D) \otimes F(D').$$

Proof. We want to show that the functor F is uniquely defined, and gives rise to an additive covariant functor from $\mathcal{B}(\epsilon m)$ to $\mathcal{T}_G(V)$.

By Lemma 3.1, the linear maps in (3.7) are all G-module maps, and by Theorem 2.6(1), the above requirements define F on all objects of $\mathcal{B}(\epsilon m)$; it is clear that F respects tensor products of objects. As a covariant functor, F preserves composition of Brauer diagrams, and by (ii) F respects tensor products of morphisms. It remains only to show that F is well defined.

To prove this, we need to show that the images of the generators satisfy the relations in Theorem 2.6(2). This is precisely the content of equations (3.4)–(3.6) in Lemma 3.1(2).

Hence for any morphism D in $\mathcal{B}(\epsilon m)$, $F(D)$ is indeed a well defined morphism in $\mathcal{T}_G(V)$. □

Remark 3.5. The functor F is a tensor functor between braided strict monoidal categories.

Lemma 3.6. *Let $H_s^t = \mathrm{Hom}_G(V^{\otimes s}, V^{\otimes t})$ for all $s, t \in \mathbb{N}$.*

(1) *The K-linear maps*

$$F\mathbb{U}_p^q := (- \otimes \mathrm{id}_V^{\otimes q})(\mathrm{id}_V^{\otimes p} \otimes F(U_q)) : H_{p+q}^r \to H_p^{r+q},$$
$$F\mathbb{A}_q^r := (\mathrm{id}_V^{\otimes r} \otimes F(A_q))(- \otimes \mathrm{id}_V^{\otimes q}) : H_p^{r+q} \to H_{p+q}^r$$

are well defined and are mutually inverse isomorphisms.

(2) *For each pair k, ℓ of objects in $\mathcal{B}(\epsilon m)$, the functor F induces a linear map*

$$F_k^\ell : B_k^\ell(\epsilon m) \to H_k^\ell = \mathrm{Hom}_G(V^{\otimes k}, V^{\otimes \ell}), \qquad D \mapsto F(D), \tag{3.8}$$

and the following diagrams are commutative:

$$\begin{array}{ccc} B_p^{r+q}(\epsilon m) & \xrightarrow{\mathbb{A}_q^r} & B_{p+q}^r(\epsilon m) \\ \downarrow{\scriptstyle F_p^{r+q}} & & \downarrow{\scriptstyle F_{p+q}^r} \\ H_p^{r+q} & \xrightarrow{F\mathbb{A}_q^r} & H_{p+q}^r \end{array} \qquad \begin{array}{ccc} B_{p+q}^r(\epsilon m) & \xrightarrow{\mathbb{U}_p^q} & B_p^{r+q}(\epsilon m) \\ \downarrow{\scriptstyle F_{p+q}^r} & & \downarrow{\scriptstyle F_p^{r+q}} \\ H_{p+q}^r & \xrightarrow{F\mathbb{U}_p^q} & H_p^{r+q} \end{array}$$

Proof. Part (1) follows by applying the functor F to Corollary 2.8, using Theorem 3.4.

Now for any $D \in B_p^{r+q}(\epsilon m)$, $\mathbb{A}_q^r(D) = (I_r \otimes A_q) \circ (D \otimes I_q)$. Since F preserves both composition and tensor product of Brauer diagrams,

$$F(\mathbb{A}_q^r(D)) = (\mathrm{id}_V^{\otimes r} \otimes F(A_q))(F(D) \otimes \mathrm{id}_V^{\otimes q}) = F\mathbb{A}_q^r(F(D)).$$

This proves the commutativity of the first diagram in part (2). The commutativity of the other diagram is proved in the same way. □

We shall require the next lemma, which is surely well known. Nevertheless, we supply a proof by adapting some computations in [ZGB] to the present context.

Lemma 3.7. *For any endomorphism $L \in \mathrm{End}_K(V^{\otimes r})$, define the* Jones trace $J(L)$ *by*

$$J(L) = F(A_r) \circ (L \otimes \mathrm{id}_V^{\otimes r}) \circ F(U_r) \in \mathrm{End}_K(K) \simeq K, \tag{3.9}$$

where A_r and U_r are the capping and cupping operations defined above. Then

$$\mathrm{Tr}(L, V^{\otimes r}) = \epsilon^r J(L).$$

In particular, if $L = F(D)$, then for $D \in B_r^r(\epsilon m)$, we have

$$\mathrm{Tr}(F(D), V^{\otimes r}) = \epsilon^r F(A_r \circ (D \otimes I_r) \circ U_r) =: J(D).$$

The map $J : B_r^r(\epsilon m) \to K$ is referred to as the Jones trace on the Brauer algebra.

Proof. Let $L \in \mathrm{End}_K(V^{\otimes r})$. Since (3.9) is linear in L, it suffices to prove the conclusion for $L = L_1 \otimes \cdots \otimes L_r$, where $L_i \in \mathrm{End}_K(V)$ for each i. Now observe that if we write $\Gamma : \mathrm{End}_K(V^{\otimes i}) \to \mathrm{End}_K(V^{\otimes(i-1)})$ for the map defined by

$$\Gamma(M) = (\mathrm{id}_V^{\otimes(i-1)} \otimes F(A)) \circ (M \otimes \mathrm{id}_V) \circ (\mathrm{id}_V^{\otimes(i-1)} \otimes F(U)),$$

then $J(L) = \Gamma^r(L)$. We therefore compute $\Gamma(L)$. We have

$$\begin{aligned}
&\Gamma(L)(v_1 \otimes \cdots \otimes v_{r-1})\\
&\quad = (\mathrm{id}_V^{\otimes(r-1)} \otimes F(A)) \circ (L \otimes \mathrm{id}_V) \circ (\mathrm{id}_V^{\otimes(r-1)} \otimes \check{C})(v_1 \otimes \cdots \otimes v_{r-1} \otimes 1)\\
&\quad = (\mathrm{id}_V^{\otimes(r-1)} \otimes F(A)) \circ (L \otimes \mathrm{id}_V)(v_1 \otimes \cdots \otimes v_{r-1} \otimes c_0)\\
&\quad = (\mathrm{id}_V^{\otimes(r-1)} \otimes F(A))\Big(L_1 v_1 \otimes \cdots \otimes L_{r-1} v_{r-1} \otimes \sum_i L_r b_i \otimes \bar{b}_i\Big)\\
&\quad = (\mathrm{id}_V^{\otimes(r-1)} \otimes \hat{C})\Big(L_1 v_1 \otimes \cdots \otimes L_{r-1} v_{r-1} \otimes \sum_i L_r b_i \otimes \bar{b}_i\Big)\\
&\quad = \sum_i (L_r b_i, \bar{b}_i)(L_1 v_1 \otimes \cdots \otimes L_{r-1} v_{r-1})\\
&\quad = \epsilon\, \mathrm{Tr}(L_r, V)(L_1 v_1 \otimes \cdots \otimes L_{r-1} v_{r-1}).
\end{aligned}$$

It follows that $\Gamma(L_1 \otimes \cdots \otimes L_r) = \epsilon\, \mathrm{Tr}(L_r, V) L_1 \otimes \cdots \otimes L_{r-1}$, and hence by induction $J(L) = \Gamma^r(L) = \epsilon^r\, \mathrm{Tr}(L, V^{\otimes r})$. The result follows. □

4. Theory of invariants of the orthogonal and symplectic groups

Henceforth we assume that K is a field of characteristic zero.

4.1. The fundamental theorems of invariant theory

Let G be either the orthogonal group $\mathrm{O}(V)$ or the symplectic group $\mathrm{Sp}(V)$. For any $t \in \mathbb{N}$, the space $V^{\otimes t}$ is a G-module, and hence so is its dual space $V^{\otimes t*} = \mathrm{Hom}_K(V^{\otimes t}, K)$. The space of invariants $(V^{\otimes t*})^G = \mathrm{Hom}_G(V^{\otimes t}, K)$ consists of linear functions on $V^{\otimes t}$ which are constant on G-orbits. One formulation of the first fundamental theorem of classical invariant theory for the orthogonal and symplectic groups [W, GW] is as follows.

Theorem 4.1. *The space $(V^{\otimes t*})^G$ is zero if t is odd. If $t = 2r$ is even, any element of $(V^{\otimes t*})^G$ is a linear combination of maps of the form γ_α ($\alpha \in \mathrm{Sym}_{2r}$), where*

$$\gamma_\alpha : v_1 \otimes \cdots \otimes v_{2r} \mapsto \prod_{i=1}^{r} (v_{\alpha(2i-1)}, v_{\alpha(2i)}). \tag{4.1}$$

Now Sym_{2r} evidently acts transitively on the set of γ_α through its action on $V^{\otimes r}$ by place permutations: for $\pi \in \mathrm{Sym}_{2r}$, $\pi.\gamma_\alpha := \gamma_\alpha \circ \pi^{-1} = \gamma_{\pi\alpha}$. Moreover the centraliser H in Sym_{2r} of the involution $(12)(34)\ldots(2r-1, 2r)$, which is isomorphic to $\mathrm{Sym}_r \ltimes (\mathbb{Z}/2\mathbb{Z})^r$, clearly takes γ_1 to $\pm\gamma_1$. Hence if $\mathcal{T}_r := \mathrm{Sym}_{2r}/(\mathrm{Sym}_r \ltimes (\mathbb{Z}/2\mathbb{Z})^r)$ is a left transversal of $\mathrm{Sym}_r \ltimes (\mathbb{Z}/2\mathbb{Z})^r$ in Sym_{2r}, it follows that each function γ_α is equal to $\pm\gamma_\beta$, with $\beta \in \mathcal{T}_r$, and hence we obtain

Corollary 4.2. *With notation as in Theorem 4.1, and writing $\mathcal{T}_r$ for the transversal above, $(V^{\otimes t*})^G$ is spanned by $\{\gamma_\alpha \mid \alpha \in \mathcal{T}_r\}$.*

Remark 4.3. Note that since $\mathcal{T}_r := \mathrm{Sym}_{2r}/(\mathrm{Sym}_r \ltimes (\mathbb{Z}/2\mathbb{Z})^r)$ is evidently identified with the set of all pairings of the elements of $\{1, \ldots, 2r\}$, $\mathcal{T}_r$ is in bijection with the diagrams in $B_r^r(\epsilon m)$.

For any subset $S \subseteq [1, t]$, let $\mathrm{Sym}(S)$ be the symmetric group of S, regarded as the subgroup of Sym_t which fixes all elements in $[1, t] \setminus S$. The next lemma provides some linear relations among the γ_α.

Lemma 4.4. *Let S be any subset of $[1, t]$ with $|S| = m + 1$. Then for any $\gamma = \gamma_\alpha$ as in (4.1), we have $\sum_{\pi \in \mathrm{Sym}(S)} (-1)^{|\pi|} \pi\gamma = 0$. In particular, for $\alpha \in \mathcal{T}_r$,*

$$\sum_{\pi \in \mathrm{Sym}(S)} (-1)^{|\pi|} \gamma_{\alpha\pi} = 0. \tag{4.2}$$

Proof. For any $S \subset [1, t]$ of cardinality $m + 1$ and $\gamma \in (V^{\otimes t*})^G$, we have

$$\begin{aligned}\sum_{\pi \in \mathrm{Sym}(S)} (-1)^{|\pi|} \pi\gamma(v_1 \otimes \cdots \otimes v_t) &= \sum_{\pi \in \mathrm{Sym}(S)} (-1)^{|\pi|} \gamma(\pi^{-1}(v_1 \otimes \cdots \otimes v_t)) \\ &= \gamma\Big(\sum_{\pi \in \mathrm{Sym}(S)} (-1)^{|\pi|} \pi^{-1}(v_1 \otimes \cdots \otimes v_t)\Big) = 0, \end{aligned} \tag{4.3}$$

since $\mathrm{Sym}(S)$ acts on $m+1$ positions, and therefore the alternating sum has a factor which is an element of $\Lambda^{m+1}(V)$, which is zero since $m = \dim(V)$. □

Remark 4.5. (i) When the form is symmetric, the inner sum in the second line of (4.3) may be zero for the trivial reason that an involution in S might fix γ. Thus some of the relations above are trivial in the orthogonal case.

(ii) Although $\alpha\pi$ may not be in $\mathcal{T}_r$ above, it is always the case that $\gamma_{\alpha\pi} = \pm\gamma_\beta$ for some $\beta \in \mathcal{T}$. Thus the lemma does provide linear relations among the γ_α for $\alpha \in \mathcal{T}_r$.

The second fundamental theorem for the orthogonal and symplectic groups [W] may be stated as follows [GW].

Theorem 4.6. *Write $m = \dim(V)$ and let $d = m$ if $G = \mathrm{O}(V)$, and $d = m/2$ if $G = \mathrm{Sp}(V)$. If $r \le d$, the linear functions $\{\gamma_\alpha \mid \alpha \in \mathcal{T}_r\}$ of Corollary 4.2 form a basis of the space of G-invariants on $V^{\otimes 2r}$. If $r > d$, any linear relation among the functionals γ_α is a linear consequence of the relations in Lemma 4.4.*

4.2. *Categorical generalisations of the fundamental theorems*

We now return to the category $\mathcal{B}(\epsilon m)$ of Brauer diagrams with parameter ϵm (where $\epsilon = \epsilon(G)$) and the covariant functor $F : \mathcal{B}(\epsilon m) \to \mathcal{T}_G(V)$. Recall that the group algebra $K\mathrm{Sym}_r$ is embedded in the Brauer algebra $B_r(\epsilon m)$ of degree r. In particular, $\Sigma_\epsilon(r)$ belongs to $B_r^r(\epsilon m)$. Let $\phi_r = F(\Sigma_\epsilon(r)) \in \mathrm{End}_G(V^{\otimes r}) = H_r^r$. Then for any r vectors v_i in V,

$$\phi_r(v_1 \otimes \cdots \otimes v_r) = \sum_{\sigma \in \mathrm{Sym}_r} (-1)^{|\sigma|} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}.$$

In particular, if $r = m+1$, then $\phi_r = 0$ as an element in H_{m+1}^{m+1}.

Definition 4.7. Denote by $\langle \Sigma_\epsilon(m+1) \rangle$ the subspace of $\bigoplus_{k,\ell} B_k^\ell(\epsilon m)$ spanned by the morphisms in $\mathcal{B}(\epsilon m)$ obtained from $\Sigma_\epsilon(m+1)$ by composition and tensor product. Set $\langle \Sigma_\epsilon(m+1) \rangle_k^\ell = \langle \Sigma_\epsilon(m+1) \rangle \cap B_k^\ell(\epsilon m)$.

The first and second fundamental theorems of classical invariant theory for the orthogonal and symplectic groups can be respectively interpreted as parts (1) and (2) of the following theorem.

Theorem 4.8. *Assume that K has characteristic 0 and write $d = m$ if $G = \mathrm{O}(V)$, and $d = m/2$ if $G = \mathrm{Sp}(V)$, where $m = \dim(V)$.*

(1) *The functor $F : \mathcal{B}(\epsilon m) \to \mathcal{T}_G(V)$ is full. That is, F is surjective on* Hom *spaces.*
(2) *The map F_k^ℓ is injective if $k + \ell \le 2d$, and* $\mathrm{Ker}\, F_k^\ell = \langle \Sigma_\epsilon(m+1) \rangle_k^\ell$ *if $k + \ell > 2d$.*

Proof. It follows from Lemma 3.6 that we have a canonical isomorphism $B_k^\ell \simeq B_{k+\ell}^0$, and the study of F_k^ℓ is equivalent to that of $F_{k+\ell}^0$. Hence without loss of generality, we may assume that $\ell = 0$. When $\ell = 0$, the theorem is true trivially when k is odd. Thus we only need to consider the case $\ell = 0$ and $k = 2r$.

(1) By Corollary 4.2, every element of H_{2r}^0 is a linear combination of functionals γ_α for $\alpha \in \mathcal{T}_r$. As remarked in Remark 4.3, the elements of $\mathcal{T}_r$ are in canonical bijection with pairings of the set $[1, 2r]$, i.e. the partitioning of $[1, 2r]$ into a disjoint union of pairs. Let D be the diagram corresponding to $\alpha \in \mathcal{T}_r$. Then $F(D) = \gamma_\alpha$. Thus F_{2r}^0 is surjective, and so is also F_k^ℓ for all k and ℓ. This proves part (1) of the theorem.

(2) Note that every $(2r, 0)$ Brauer diagram is mapped by F to a γ_α of the form (4.1). Thus if $r \le d$, then $\operatorname{Ker} F_{2r}^0 = 0$ by Theorem 4.6, the second fundamental theorem.

Now consider the case $r > d$. By Theorem 4.6 it suffices to show that every relation of the form (4.2) arises by applying F_{2r}^0 to an element of $\langle \Sigma_\epsilon(m+1)\rangle_{2r}^0$. Fix $\alpha \in \mathcal{T}_r$, and let $D \in B_{2r}^0$ be the diagram such that $F(D) = \gamma_\alpha$. This is the diagram corresponding to $\alpha \in \mathcal{T}_r$ by Remark 4.3.

Write $\gamma_\alpha(S) = \sum_{\pi \in \mathrm{Sym}(S)} (-1)^{|\pi|}\gamma_{\alpha\pi}$ for the left side of (4.2).

If $\sigma \in \mathrm{Sym}_{2r}$ satisfies $\{\sigma([1, m+1])\} = S$, then $\mathrm{Sym}(S) = \sigma \mathrm{Sym}([1, m+1])\sigma^{-1}$. Now regard Sym_{2r} as embedded in $B_{2r}^{2r}(\epsilon m)$, and define the element

$$D_S := \sum_{\pi \in \mathrm{Sym}([1,m+1])} (-\epsilon)^{|\pi|} D \circ \sigma \circ \pi \circ \sigma^{-1}$$

in $B_{2r}^0(\epsilon m)$. Then $\gamma_\alpha(S) = F(D_S)$, and since we have

$$D_S = D \circ \sigma \circ \Sigma_\epsilon(m+1) \circ \sigma^{-1} \in \langle \Sigma_\epsilon(m+1)\rangle_{2r}^0,$$

it follows that $\gamma_\alpha(S) \in F(\langle \Sigma_\epsilon(m+1)\rangle_{2r}^0)$.

By Theorem 4.6, all relations among invariant functionals on $V^{\otimes 2r}$ are linear consequences of the relations $\gamma_\alpha(S) = 0$. Using the bijection between diagrams and $\mathcal{T}_r$, it follows that $\operatorname{Ker} F_{2r}^0$ is spanned by elements of the form D_S.

Conversely, it is evident that $\langle \Sigma_\epsilon(m+1)\rangle_{2r}^0 \subset \operatorname{Ker} F_{2r}^0$ since $\phi_{m+1} = F(\Sigma_\epsilon(m+1)) = 0$. This proves part (2) for $r > d$, completing the proof of the theorem. □

Remark 4.9. Theorem 4.8(1) with $k = 2r$, $\ell = 0$ yields the linear version of FFT, while the endomorphism algebra formulation arises from the case $k = \ell = r$. The equivalence of the two versions is an obvious consequence of Lemma 3.6.

Corollary 4.10. *If $k + \ell \le 2d$, then $\langle \Sigma_\epsilon(m+1)\rangle_k^\ell = 0$.*

Proof. Since $\phi_{m+1} = 0$, $\langle \Sigma_\epsilon(m+1)\rangle_k^\ell$ is contained in $\operatorname{Ker} F_k^\ell$. But $\operatorname{Ker} F_k^\ell = 0$ for $k + \ell \le 2d$, and the lemma follows. □

5. Structure of the endomorphism algebra: the symplectic case

Recall from Section 2.5 that $B_r^r(\epsilon m)$ is the Brauer algebra of degree r. Thus $\operatorname{Ker} F_r^r$ is a two-sided ideal of $B_r^r(\epsilon m)$, and $B_r^r(\epsilon m)/\operatorname{Ker} F_r^r$ is canonically isomorphic to the endomorphism algebra $\mathrm{End}_G(V^{\otimes r})$ by Theorem 4.8(2). In order to understand the algebraic structure of $\mathrm{End}_G(V^{\otimes r})$, we need to understand that of $\operatorname{Ker} F_r^r$, and this is what we shall do in this section and the next.

Here we take $G = \mathrm{Sp}(V)$ with $\dim V = 2n$ and $\epsilon = -1$. Denote $\Sigma_{-1}(r)$ by $\Sigma(r)$.

5.1. Generators of the kernel

For any $s < r$, there is a natural embedding $B_s^s(-2n) \hookrightarrow B_r^r(-2n)$, $b \mapsto b \otimes I_{r-s}$, of the Brauer algebra of degree s in that of degree r as associative algebras. Thus we may regard $B_s^s(-2n)$ as the subalgebra of $B_r^r(-2n)$ consisting of all elements of the form $b \otimes I_{r-s}$.

Let $D(p, q)$ denote the element of the Brauer algebra $B_k^k(-2n)$ of degree $k = 2n + 1 - p + q$ shown in Figure 10.

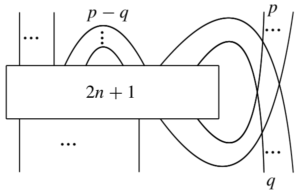
$D(p,q) =$ 

Fig. 10

Proposition 5.1. *Assume that $r > n$. As a two-sided ideal of the Brauer algebra $B_r^r(-2n)$,* Ker F_r^r *is generated by $D(p, q)$ and $*D(p, q)$ with $p + q \leq r$ and $p \leq n$.*

Proof. Let A be a single $(2r, 0)$ Brauer diagram with $r > n$. Then $F(A)$ is some functional γ on $V^{\otimes 2r}$ defined by (4.1). For any $\pi \in \text{Sym}_{2r} \subset B_{2r}^{2r}(-2n)$, $A \circ \pi$ is defined. Note that A has only one row of vertices at the bottom, which will be labelled $1, \ldots, 2r$ from left to right. Choose a subset S of $[1, 2r]$ of cardinality $2n + 1$ as in Lemma 4.4, and consider $\text{Sym}_S \subset \text{Sym}_{2r} \subset B_{2r}^{2r}(-2n)$. Define

$$A_S = \sum_{\pi \in \text{Sym}_S} A \circ \pi. \tag{5.1}$$

Then by Theorem 4.6, or equivalently Theorem 4.8(2), Ker F_{2r}^0 is spanned by A_S for all A and S. Given A_S, we define

$$A_S^\natural = A_S \circ (I_r \otimes U_r) \in B_r^r(-2n).$$

Then Ker F_r^r is spanned by $A_S^\natural$ for all A and S by Lemma 3.6(2).

We can considerably simplify the description of Ker F_{2r}^0 and Ker F_r^r. There exist elements $\sigma = (\sigma_1, \sigma_2)$ in the parabolic subgroup $\text{Sym}_r \times \text{Sym}_r$ of Sym_{2r}, which map S to $S' = \{i+1, i+2, \ldots, i+2n+1\} \subset [1, 2r]$ for some $i \leq 2r - 2n - 1$. Let $\sigma_2^{-\tau} = *(\sigma_2^{-1})$, where $*$ is the anti-involution of $B_r^r(-2n)$. Then

$$\sigma_2^{-\tau} \circ A_S^\natural \circ \sigma_1^{-1} = (A_S \circ \sigma^{-1})^\natural, \quad A_S \circ \sigma^{-1} = \sum_{\pi \in \text{Sym}_{S'}} (A \circ \sigma^{-1}) \circ \pi. \tag{5.2}$$

By appropriately choosing σ, we can ensure that $A \circ \sigma^{-1}$ is of the form shown in Figure 11. The vertices labelled by $\bullet$ are those in S', which all appear in the middle, and the other vertices all appear at the left end and right end. Here t denotes the number of edges in $A \circ \sigma^{-1}$ with both vertices in $\{1, \ldots, i\}$, and t' that of the edges with both vertices in $\{i + 2n + 2, i + 2n + 3, \ldots, 2r\}$. Note that after such a σ is chosen, $\pi \in \text{Sym}_{S'}$ acting

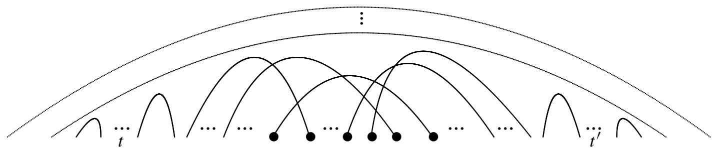


Fig. 11

on $A \circ \sigma^{-1}$ permutes only vertices labelled by •. Thus every term on the right hand side of (5.2) is of the form of Figure 11 with the same t and t'.

Now $(A_S \circ \sigma^{-1})^\natural$ can be expressed as $D_1 \otimes D_2$, where $D_1 \in B_{r_1}^{r_1}(-2n)$ for r_1 maximal, $D_2 \in B_k^k(-2n)$ with $k > n$ satisfying $r_1 + k = r$. There are several possibilities for D_2 depending on i, t and t'. Assume $i + 2n + 1 > r$. If $t = t'$, then D_2 is as shown in Figure 12.

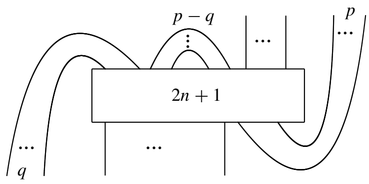


Fig. 12

If $t < t'$, then $D_2 = E \circ (I_s \otimes D_3)$ for some s, where D_3 is as shown in Figure 12, and E is the product of some e_i's composed with a permutation in $\mathrm{Sym}_{2n+1+q-p}$ (D_3 and E may not be unique). Analogously, $D_2 = (D_3 \otimes I_s) \circ E$ if $t > t'$. Assume that $i + 2n + 1 \leq r$. Then $D_2 = E \circ (I_{s_1} \otimes \Sigma(2n + 1) \otimes I_{s_2})$ for some E in $B_k^k(-2n)$, and fixed non-negative integers s_1 and s_2 satisfying $s_1 + s_2 + 2n + 1 = k$.

Therefore, Ker F_r^r is generated as a two-sided ideal of $B_r^r(-2n)$ by elements of the form of Figure 12 with $2n + 1 + q - p \leq r$. If $p > n$, we apply the anti-involution $*$ of $B_k^k(-2n)$ to the element of Figure 12 to obtain the element shown in Figure 13, which we denote by D. Recall the element $X_{s,t}$ of Figure 6, which belongs to Sym_{s+t}, where Sym_{s+t} is regarded as embedded in $B_{s+t}^{s+t}(-2n)$. Then $X_{2n+1-p,q} \circ D \circ X_{2n+1-2p+q,p}$ is of the form shown in Figure 12, but with p replaced by $2n + 1 - p \leq n$.

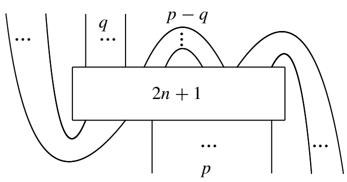


Fig. 13

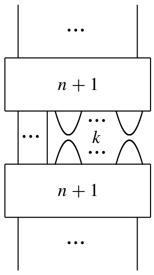
Therefore, we only need to consider Figure 12 with $p \leq n$ and its $*$ image. Post-composing $X_{2n+1-p,q}$ to Figure 12 turns the latter into the form shown in Figure 10. Since $X_{2n+1-p,q}$ is invertible in $B_r^r(-2n)$, Ker F_r^r as a two-sided ideal of $B_r^r(-2n)$ is generated by elements of $D(p, q)$ and $*D(p, q)$ with $2n + 1 + q - p \leq r$ and $p \leq n$. □

5.2. The element Φ

For each k such that $0 \leq k \leq [(n+1)/2]$, define the element $E(k) = \prod_{j=1}^{k} e_{n+2-2j}$ of $B_{n+1}^{n+1}(-2n)$, where $E(0)$ is the identity by convention. Then define

$$\Xi_k = \Sigma(n+1)E(k)\Sigma(n+1),$$

which may be represented pictorially as



Now define the following element of $B_{n+1}^{n+1}(-2n)$:

$$\Phi = \sum_{k=0}^{[(n+1)/2]} a_k \Xi_k \quad \text{with} \quad a_k = \frac{1}{(2^k k!)^2 (n+1-2k)!}. \tag{5.3}$$

Lemma 5.2. *The element* Φ *is the sum of all the Brauer diagrams in* $B_{n+1}^{n+1}(-2n)$*. In particular* Φ *is defined over the ring* $\mathbb{Z}$ *of integers.*

Proof. Note that $\Xi_k = \sum_{(\pi,\sigma)\in(\mathrm{Sym}_{n+1})^2} \pi E(k)\sigma$ is simply the sum of all the diagrams with $t = n+1-2k$ through strings, each one occurring with coefficient equal to the order of the centraliser in $(\mathrm{Sym}_{n+1})^2$ of $E(k)$. But this order is evidently a_k^{-1}. □

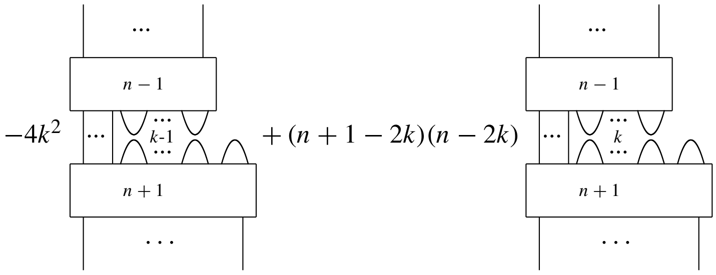
We have the following result.

Lemma 5.3. *The element* Φ *has the following properties:*

(1) $e_i\Phi = \Phi e_i = 0$ *for all* $e_i \in B_{n+1}^{n+1}(-2n)$*;*
(2) $\Phi^2 = (n+1)!\Phi$*;*
(3) $*\Phi = \Phi$*;*
(4) $\Phi \in \mathrm{Ker}\, F_{n+1}^{n+1}$.

Proof. Part (3) follows from the fact that $*\Xi_k = \Xi_k$ for all k. Part (2) immediately follows from (1).

Since $*(e_i \circ \Phi) = \Phi \circ e_{n+1-i}$, we only need to show that $e_i \circ \Phi = 0$ for all i in order to prove part (1). In view of the symmetrising property of $\Sigma(n+1)$, it suffices to show that $e_n \circ \Phi = 0$. Consider $(I_{n-1} \otimes A_1) \circ \Xi_k$, which can be shown to be equal to



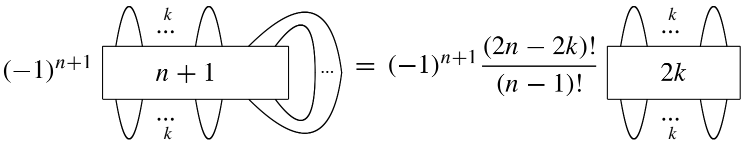
(5.4)

by using Lemma 2.13 with $\delta = -2n$. Note that each Brauer diagram summand of the first term has $n+1-2k$ through strings, while the summands in the second term have $n-1-2k$ through strings. Using (5.4) one shows by simple calculation that

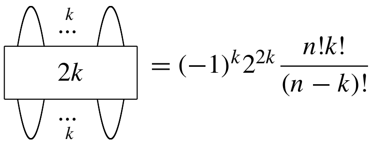
$$\sum a_k (I_{n-1} \otimes A_1) \circ \Xi_k = 0.$$

Hence $(I_{n-1} \otimes A_1) \circ \Phi = 0$, which implies statement (1).

To prove part (4), we note that the trace of $F(\Phi)/(n+1)!$ is equal to the dimension of the subspace $F(\Phi)(V^{\otimes(n+1)})$, since $F(\Phi)/(n+1)!$ is an idempotent by part (2). In order to evaluate $\mathrm{tr}(F(\Phi)/(n+1)!)$, we first consider $\mathrm{tr}(F(\Xi_k)/(n+1)!)$, which is given by



where the last step uses Lemma 2.11(2) with $\epsilon = -1$. Using (2.10), one can show that



Putting these formulae together, we arrive at

$$\begin{aligned}\mathrm{tr}\left(\frac{F(\Phi)}{(n+1)!}\right) &= \frac{n!}{(n-1)!} \sum_{k=0}^{[(n+1)/2]} a_k (-1)^k 2^{2k} \frac{k!(2n-2k)!}{(n-k)!} \\ &= \sum_{k=0}^{[(n+1)/2]} (-1)^k \binom{n}{k} \binom{2n-2k}{n-1}.\end{aligned}$$

There is a binomial coefficient identity stating that the far right hand side is equal to zero. Hence $F(\Phi)$ is the zero map on $V^{\otimes(n+1)}$. □

The corollary below follows from Lemma 6.2 and the fact that $\pi \Sigma(n+1)\pi' = \Sigma(n+1)$ for all $\pi, \pi' \in \mathrm{Sym}_{n+1}$.

Corollary 5.4. *The element $\Phi/(n+1)!$ is the central idempotent in $B_{n+1}^{n+1}(-2n)$ which corresponds to the trivial representation ρ_1 of $B_{n+1}^{n+1}(-2n)$, defined by $\rho_1(s_i) = 1$ and $\rho_1(e_i) = 0$ for all i. It generates a 1-dimensional two-sided ideal of $B_{n+1}^{n+1}(-2n)$.*

Remark 5.5. Another formula for $\Phi/(n+1)!$ was given in terms of Jucys–Murphy elements in [IMO].

5.3. The main theorem

Recall the natural embedding of the Brauer algebra of degree s in that of degree t for any $t > s$.

Definition 5.6. For each $r > n$, let $\langle\Phi\rangle_r$ be the two-sided ideal in the Brauer algebra $B_r^r(-2n)$ generated by Φ.

Remark 5.7. A priori, elements such as $(I_{r-q} \otimes A_q \otimes I_q)(z \otimes X_{q,q})(I_{r-q} \otimes U_q \otimes I_q)$ are not included in $\langle\Phi\rangle_r$ even if $z \in \langle\Phi\rangle_r$.

We have the following result.

Lemma 5.8. *The element $\Sigma(2n+1)$ belongs to $\langle\Phi\rangle_{2n+1}$.*

Proof. Consider $B_r^r(-2n)$ for $r > n$. Let $E_r^r(k) = \prod_{j=1}^k e_{r-2j+1}$, and define

$$\begin{aligned} \Upsilon(r)_k &= \Sigma(r)E_r^r(k)\Sigma(r), \qquad k \geq 1, \\ \Upsilon(r)_{\geq k} &= \text{linear span of } \langle\Phi\rangle_r \cup \{\Upsilon(r)_i \mid i \geq k\}. \end{aligned}$$

We first want to show that

$$\Sigma(r) \in \Upsilon(r)_{\geq[(r+1-n)/2]}. \tag{5.5}$$

From the formula for Φ, we obtain

$$r!(n+1)!\Sigma(r) = \Sigma(r)\Big(\Big(\Phi - \sum_{k=1}^{[(n+1)/2]} a_k \Xi_k\Big) \otimes I_{r-n-1}\Big)\Sigma(r).$$

Thus $\Sigma(r) \in \Upsilon(r)_{\geq 1}$.

Note that for any $z \in \Upsilon(r-2k)_{\geq 1}$, $\Sigma(r)(z \otimes I_{2k})E_r^r(k)\Sigma(r)$ belongs to $\Upsilon(r)_{\geq k+1}$. We can always rewrite $\Upsilon(r)_k$ as

$$\Upsilon(r)_k = \frac{1}{(r-2k)!}\Sigma(r)(\Sigma(r-2k) \otimes I_{2k})E_r^r(k)\Sigma(r).$$

If $r - 2k > n$, then $\Sigma(r-2k) \in \Upsilon(r-2k)_{\geq 1}$. This implies that $\Upsilon(r)_k \in \Upsilon(r)_{\geq k+1}$ if $r - 2k > n$. Hence $\Upsilon(r)_{\geq 1} = \Upsilon(r)_{\geq 2} = \cdots = \Upsilon(r)_{\geq[(r+1-n)/2]}$, and (5.5) is proved.

Now consider $\Sigma(2n+1)$. It follows from (5.5) that $\Sigma(2n+1)^2$ can be expressed as a linear combination of elements in $\langle\Phi\rangle_{2n+1}$ and also elements of the form

$$\Sigma(2n+1)E_{2n+1}^{2n+1}(i)\Sigma(2n+1)E_{2n+1}^{2n+1}(j)\Sigma(2n+1), \qquad i, j \geq 1 + [n/2].$$

Using the symmetrising property of $\Sigma(2n+1)$, we can write this element as $\Sigma(2n+1) \cdot (I_{2n+1-2i} \otimes U_i)\Psi_{ij}(I_{2n+1-2j} \otimes A_j)\Sigma(2n+1)$ with

$$\Psi_{ij} = (I_{2n+1-2i} \otimes A_i)\Sigma(2n+1)(I_{2n+1-2j} \otimes U_j).$$

By Corollary 4.10, $\Psi_{ij} = 0$ for all $i, j \geq 1+[n/2]$. Hence $\Sigma(2n+1)^2$ belongs to $\langle\Phi\rangle_{2n+1}$, and so does also $\Sigma(2n+1)$. □

The following is one of the main results of this paper.

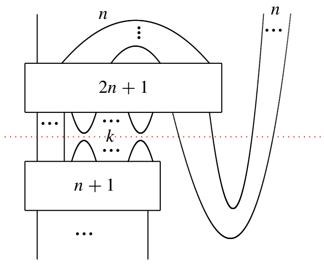
Theorem 5.9. *The algebra homomorphism $F_r^r : B_r^r(-2n) \to \mathrm{Hom}_{\mathrm{Sp}(V)}(V^{\otimes r}, V^{\otimes r})$ is injective if $r \leq n$. If $r \geq n+1$, then* Ker F_r^r *is the two-sided ideal of the Brauer algebra $B_r^r(-2n)$ which is generated by the element Φ defined by* (5.3).

Proof. Only the second statement requires proof. Thus we assume that $r \geq n+1$. Consider first the case $r = n+1$. Then there is only one $D(p,q)$ with $p = n$ and $q = 0$ (see Figure 10). Using $\Sigma(n+1) = \Phi - \sum_{k=1}^{[(n+1)/2]} a_k \Xi_k$, we have

$$D(n,0) = \frac{D(n,0)\Phi}{(n+1)!} - \sum_{k=1}^{[(n+1)/2]} a_k \frac{D(n,0)\Xi_k}{(n+1)!}.$$

Note that

$$\frac{D(n,0)\Xi_k}{(n+1)!} =$$



where the dotted line indicates that the diagram is the composition of the two diagrams above and below the line. The diagram above the dotted line is the tensor product of an element in $\langle\Sigma(2n+1)\rangle^1_{n+1-2k}$ with I_n. Since $\langle\Sigma(2n+1)\rangle^1_{n+1-2k} = 0$ for all $k \geq 1$ by Corollary 4.10, we have $\frac{D(n,0)\Xi_k}{(n+1)!} = 0$. This proves $D(n,0) \in \langle\Phi\rangle_{n+1}$.

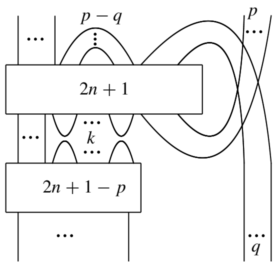
Now we use induction on r to show that the theorem holds for $r > n+1$. If $p = 0$, the diagram corresponds to $\Sigma(2n+1)$, which belongs to $\langle\Phi\rangle_{2n+1}$ by Lemma 5.8. Assume $n \geq p \geq 1$, and let $r = 2n+1-p+q$. Consider $D(p,q) \circ \Sigma(2n+1-p)$ by using the the formula

$$\Sigma(2n+1-p) = \left(\left(\Phi - \sum_{k=1}^{[(n+1)/2]} a_k \Xi_k\right) \otimes I_{n-p}\right) \frac{\Sigma(2n+1-p)}{(n+1)!}.$$

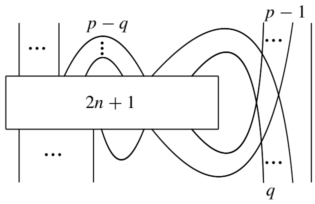
We obtain an expression for $D(p,q)$ of the form

$$D(p,q) = \sum_{k \geq 1} c_k D(p,q;k) + D^0, \tag{5.6}$$

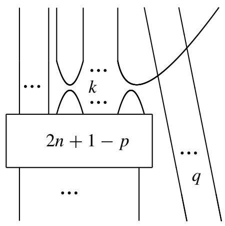
where c_k are scalars, $D^0 \in \langle\Phi\rangle_r$, and

$D(p, q; k) =$ 

The diagram $D(p, q; k)$ is the composition of

$D' \otimes | =$ 

with the following element of $B_r(-2n)$:



Note that D' belongs to Ker F_{r-1}. Thus $D' \in \langle\Phi\rangle_{r-1}$ by the induction hypothesis and it follows that $D(p, q; k) \in \langle\Phi\rangle_r$. This completes the proof. □

Remark 5.10. Any element which generates the kernel Ker $F_{n+1}^{n+1} = \langle\Phi\rangle_{n+1}$ must be a non-trivial scalar multiple of Φ. It was proved in [HX] that for all $r \geq n + 1$, Ker F_r^r is generated by a single generator belonging to Ker F_{n+1}^{n+1}. Therefore, our Φ provides an explicit formula for this generator (up to a scalar multiple). In particular, our Lemma 5.2 shows that $\lim_{q\to 1} Y_{n+1}$ is the sum of all the Brauer diagrams in $B_{n+1}(-2n)$, up to a scalar multiple, where Y_{n+1} is as in [HX].

6. Structure of the endomorphism algebra: the orthogonal case

We now study the algebraic structure of Ker F_r^r in the case of the orthogonal group. Throughout this section, we take $G = \mathrm{O}(V)$ with dim $V = m$ and $\epsilon = 1$.

6.1. Generators of the kernel

For $p = 0, 1, \ldots, m + 1$, let E_{m+1-p} denote the element of the Brauer algebra $B_{m+1}^{m+1}(m)$ of degree $m + 1$ shown in Figure 14.

$E_{m+1-p} =$

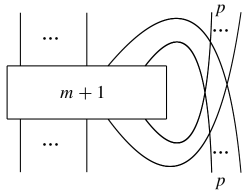


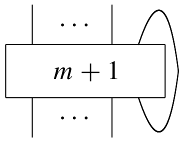
Fig. 14

Lemma 6.1. *For all* $0 \le k \le m+1$*, the elements* E_k *are linear combinations of Brauer diagrams over* $\mathbb{Z}$*.*

This is evident from the definition of these elements. They also have the following properties.

Lemma 6.2. (1) $*E_p = E_{m+1-p}$ *for all* p*.*
(2) $F_p E_p = E_p F_p = p!(m+1-p)!E_p$*.*
(3) $e_i E_p = E_p e_i = 0$ *for all* $i \le m$*.*

Proof. Both (1) and (2) follow easily from the pictorial representation of E_p given in Figure 14. If $i \neq p$, then $e_i F_p = F_p e_i = 0$. Thus (3) holds for all $i \neq p$. The $i = p$ case of (3) follows from the fact that



$= 0$

which is implied by Lemma 2.11(2) when $r = m+1$ and $\epsilon = 1$. □

The arguments used in the proof of [LZ4, Corollary 5.13] lead to

Corollary 6.3 ([LZ4]). *Let* D *be any diagram in* $B_{m+1}^{m+1}(n)$ *which has fewer than* $m+1$ *through strings. Then* $DE_i = E_i D = 0$ *for all* i*.*

Note that $E_0 = E_{m+1} = \Sigma_{+1}(m+1)$.

Problem 6.4. *Assume* $r > m$*. As a two-sided ideal of the Brauer algebra* $B_r^r(m)$*,* $\operatorname{Ker} F_r^r$ *is generated by* E_p *for all* $0 \le p \le m+1$*.*

Proof. The proof of Proposition 5.1 can easily be modified to prove the assertion above. The two required modifications are that for any $(2r, 0)$ Brauer diagram A with associated invariant functional $\gamma = F(A)$, (i) the definition (5.1) of A_S needs to be changed to

$$A_S = \sum_{\pi \in \mathrm{Sym}_S} (-1)^{|\pi|} A \circ \pi;$$

(ii) we only need to consider subsets S of $[1, 2r]$ which will not lead to the trivial vanishing of A_S discussed in Remark 4.5(i). With these modifications, the arguments following

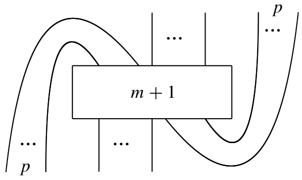


Fig. 15

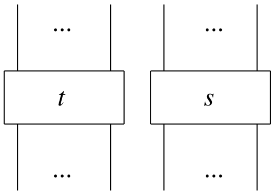
(5.1) may be repeated verbatim, leading to the conclusion that Ker F_r^r is generated as a two-sided ideal of $B_r^r(-2n)$ by elements of the form of Figure 15.

Postmultiplying the diagram in Figure 15 by the invertible element $X_{m+1-p,p}$, we obtain Figure 14 up to a sign. This completes the proof. □

Remark 6.5. Figure 15 is the $p = q$ analogue of Figure 12. In the present case, diagrams of the form Figure 12 with $p > q$ vanish identically, since $\Sigma_{+1}(m + 1)$ is the total antisymmetriser in Sym_{m+1}.

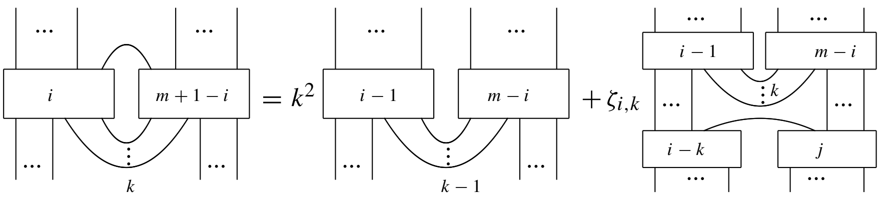
6.2. Formulae for the E_i

If k, l are integers such that $1 \le k < l$, write $A(k, l) := \Sigma_{+1}(\mathrm{Sym}_{\{k,k+1,...,l\}})$ for the total antisymmetriser in $\mathrm{Sym}_{\{k,k+1,...,l\}}$. By convention, $A(k, l) = 1$ if $k \ge l$. Represent $A(1, t)A(t + 1, t + s)$ in $B_{t+s}^{t+s}(m)$ pictorially by



The lemma below is the graphical reformulation of some of the computations in the proofs of [LZ4, Corollary 5.2 and Theorem 5.10].

Lemma 6.6. *For all $k = 0, 1, \ldots, i$,*

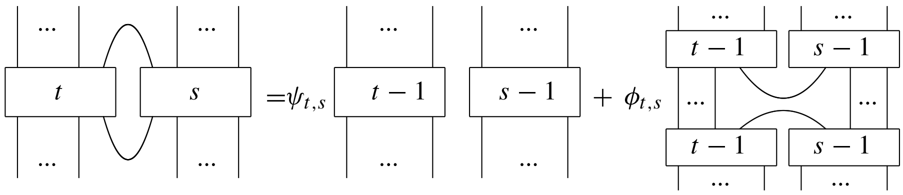


(6.1)

where $j = m + 1 - i - k$ and $\zeta_{i,k} = \frac{1}{(i-k-1)!(m-i-k)!}$.

Proof. When $k = 0$, (6.1) is an identity.

We use Lemma 2.11(1) twice to obtain



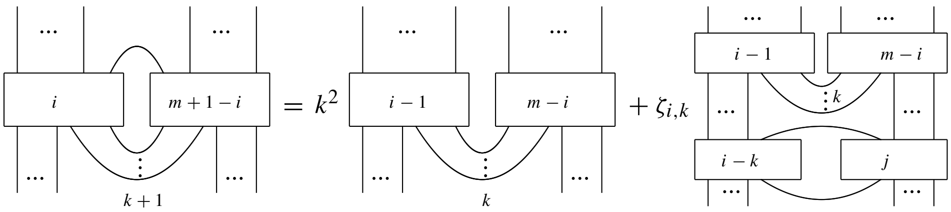
(6.2)

where

$$\psi_{t,s} = m + 2 - t - s, \qquad \phi_{t,s} = \frac{1}{(t-2)!(s-2)!}.$$

The case $k = 1$ of (6.1) can be obtained by setting $t = i$ and $s = m + 1 - i$.

Now use induction on k. Postcomposing $I_{i-k-1} \otimes U \otimes I_{m-i-k}$ to (6.1) we obtain



By using (6.2) in the bottom half of the second diagram on the right hand side, we obtain (6.1) for $k + 1$, completing the proof. □

Following [LZ4, §4.2], we introduce the elements

$$F_p := A(1, p)A(p + 1, m + 1)$$

of $B^{m+1}_{m+1}(m)$ for $p = 0, 1, \ldots, m + 1$, where F_0 is interpreted as $A(1, m + 1)$. For $j = 0, 1, \ldots, i$, define $e_i(j) = e_{i,i+1}e_{i-1,i+2}\ldots e_{i-j+1,i+j}$. Note that $e_i(0) = 1$ by convention. We have the following formulae for the E_i.

Lemma 6.7. *For $i = 0, 1, \ldots, m + 1$, let $\min_i = \min(i, m + 1 - i)$. Then*

$$E_i = \sum_{j=0}^{\min_i} (-1)^j c_i(j)\Xi_i(j) \quad \textit{with} \quad \Xi_i(j) = F_i e_i(j) F_i, \tag{6.3}$$

where $c_i(j) = \big((i - j)!(m + 1 - i - j)!(j!)^2\big)^{-1}$.

Remark 6.8. For $0 \leq i \leq [(m + 1)/2]$, the lemma states that the E_i are the elements defined in [LZ4, Definition 4.2] with the same notation.

Proof. We have $*\Xi_i(j) = \Xi_{m+1-i}(j)$. For $i \leq [m/2]$,

$$*\Big(\sum_{j=0}^{\min_i}(-1)^j c_i(j)\Xi_i(j)\Big) = \sum_{j=0}^{\min_i}(-1)^j c_{m+1-i}(j)\Xi_{m+1-i}(j),$$

since $c_i(j) = c_{m+1-i}(j)$. Therefore, equation (6.3) will hold for all i by Lemma 6.2(1), if we can show that it holds for $0 \leq i \leq [m/2]$. This will be done in two steps.

(i) We first show that for each $i \leq [m/2]$, there exist scalars $x_i(j)$ such that

$$E_i = \sum_{j=0}^{i} x_i(j)\Xi_i(j).$$

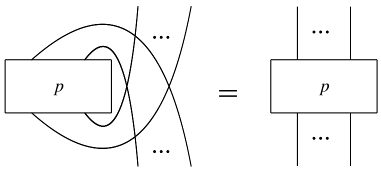
The case $i = 0$ is obvious as we have $E_0 = A(1, m+1)$. Thus we only need to consider the case with $i \geq 1$.

Let us label the vertices of E_i (see Figure 14) in the bottom row by $1, \ldots, m+1$ from left to right, and those in the top row by $1', \ldots, (m+1)'$ from left to right. Let $L = \{1, \ldots, i\}$, $R = \{i+1, \ldots, m+1\}$, $L' = \{1', \ldots, i'\}$ and $R' = \{(i+1)', \ldots, (m+1)'\}$. Since $A(1, m+1)$ has through strings only, a Brauer diagram in E_i can only have the following types of edges (an edge is represented by its pair of vertices):

$$\begin{aligned} &(a, t) \in L \times R, \qquad (a', t') \in L' \times R', \\ &(a', b) \in L' \times L, \qquad (s', t) \in R' \times R, \end{aligned}$$

and the numbers of edges in $L \times R$ and in $L' \times R'$ must be equal. Thus it follows from Lemma 6.2(2) and the antisymmetrising property of $A(1, i)$ and $A(i+1, m+1)$ that E_i is a linear combination of $\Xi_i(j)$.

(ii) To determine the scalar $x_i(0)$, we observe that the terms in $A(1, m+1)$ which do not contain s_i make up $F_i = A(1, i)A(i+1, m+1)$. Note that



Thus $x_i(0)\Xi_i(0) = A(1, i)A(i+1, m+1)$, and hence $x_i(0) = (i!(m+1-i)!)^{-1} = c_i(0)$.

Now we determine the $x_i(k)$ for all $k > 0$. By Lemma 6.2(3), $e_i E_i = 0$. Using (6.1) in this relation, we obtain

$$(k+1)^2 x_i(k+1) + (i-k)!(m+1-i-k)!\zeta_{i,k} x_i(k) = 0, \qquad 0 \leq k \leq i.$$

The recurrent relation with $x_i(0) = c_i(0)$ yields $x_i(k) = (-1)^j c_i(k)$. □

The following result is an easy consequence of Lemma 6.7. Recall the elements $X_{s,t} \in \mathrm{Sym}_{s+t}$ shown in Figure 6.

Corollary 6.9. *For all* $i = 0, 1, \ldots, m+1$, *we have* $X_{i,m+1-i} E_i X_{m+1-i,i} = E_{m+1-i}$.

Proof. It is easy to show pictorially that $X_{i,M+1-i}\Xi_i(j)X_{m+1-i,i} = \Xi_{m+1-i}(j)$ for all $j \leq i$. Since $c_i(j) = c_{m+1-i}(j)$, this proves the claim of the corollary. □

6.3. The main theorem

The following theorem is Theorem 4.3 in [LZ4], which is the main result of that paper.

Theorem 6.10 ([LZ4]). *The algebra map $F_r^r : B_r^r(m) \to \mathrm{Hom}_{\mathrm{O}(V)}(V^{\otimes r}, V^{\otimes r})$ is injective if $r \le m$. If $r > m$, the two-sided ideal $\mathrm{Ker}\, F_r^r$ of the Brauer algebra $B_r^r(m)$ is generated by the element $E = E_\ell$ with $\ell = [(m+1)/2]$.*

Proof. Only the second part of the theorem needs explanation. By Proposition 6.4 and Corollary 6.9, the elements E_i with $i = 0, 1, \dots, \ell = [(m+1)/2]$ generate $\mathrm{Ker}\, F_r^r$. Using some general properties of the symmetric group and Corollary 6.3, we showed in [LZ4, §7] that E_{i-1} is contained in the ideal generated by E_i for each $i = 1, \dots, \ell$. The theorem follows. □

7. The case of positive characteristic

The following statement is an immediate consequence of [RS, Theorem 2.3].

Lemma 7.1. *Let $n, r \in \mathbb{Z}_{>0}$. The following are equivalent for the Brauer algebras over $\mathbb{Z}$.*

(1) *The Brauer algebra $B_r(n)$ is semisimple.*
(2) *The Brauer algebra $B_r(-2n)$ is semisimple.*
(3) $r \le n + 1$.

It follows from this that $n + 1$ is the largest value of r such that $B_r(n)$ and $B_r(-2n)$ are semisimple. The idempotents we have found are thus each in the 'last' Brauer algebra which is semisimple. This is in complete analogy with the situation in the Temperley–Lieb algebra when q is a root of unity, where the radical of the Jones trace function is the idempotent corresponding to the trivial representation of the 'last' semisimple Temperley–Lieb algebra (see [GL96, Cor. 3.7, Remark 3.8]).

Note that our basic setup in this paper remains the same over the ring $\mathbb{Z}$ of integers. Since we will deal with the orthogonal and symplectic groups simultaneously in this section, we write the functor F as $F_\epsilon : \mathcal{B}(\epsilon m) \to \mathcal{T}_G(V)$, and F_k^l as $F_{\epsilon,k}^l$ for easy reference. Recall that $m = \dim V$ and $\epsilon = -1$ if $G = \mathrm{Sp}(V)$ and $\epsilon = 1$ if $G = \mathrm{O}(V)$. We also set $d = m/2$ if $\epsilon = -1$, and $d = m$ if $\epsilon = 1$.

By Lemmas 5.2 and 6.1, the element Φ defined by equation (5.3) and the elements E_k $(0 \le k \le [(m+1)/2])$ of Lemma 6.2 are linear combinations of Brauer diagrams over $\mathbb{Z}$.

Lemma 7.2. *We have $\Phi \in \mathrm{Ker}\, F_{-1,r}^r$ and $E_k \in \mathrm{Ker}\, F_{1,r}^r$ (for all k) over any field K.*

Proof. For the elements E_k, the claim immediately follows from their definition and from Theorem 7.3(2) below. It was also proved in [LZ4].

Next note that by Lemma 5.2, Φ is defined over $\mathbb{Z}$. It follows from Lemma 7.1 that $B_r(-2n)$ is semisimple over K, and $\mathrm{Ker}\, F_{-1,r}^r$ is the two-sided ideal of $B_r(-2n)$ corresponding to the one-dimensional simple module. The element Φ is a central quasi-idempotent contained in this two-sided ideal. □

The following result is a generalisation of Theorem 4.8 to fields of positive characteristic.

Theorem 7.3. *Over any field K with* $\mathrm{char}(K) \geq m+2$,

(1) *the functor $F_\epsilon : \mathcal{B}(\epsilon m) \to \mathcal{T}_G(V)$ is full;*
(2) *the map $F^\ell_{\epsilon,k}$ is injective if $k+\ell \leq 2d$, and* $\mathrm{Ker}\, F^\ell_{\epsilon,k} = \langle \Sigma_\epsilon(m+1)\rangle^\ell_k$ *if $k+\ell > 2d$.*

Proof. In the orthogonal case, this was proved in [LZ4, Theorem 9.4] as an application of [Ri, Prop. 21]. Although the symplectic case is surely in the literature, we have been unable to find it, and therefore we provide the following sketch of the argument, which may be found in [ALZ]. Note that it gives a proof of the second fundamental theorem in positive characteristic for the symplectic groups.

Let $R = \mathbb{Z}[((m+1)!)^{-1}]$. Then we may consider the symplectic Lie algebra $\mathfrak{G}_R$ over R, and the corresponding R-forms V_R and $B_R = (B_r(-m))_R$. Note that by Lemma 5.2 we may regard Φ as an element of B_R. It is shown in [ALZ] that if M is a tilting module for $\mathfrak{G}_R$ and K is a field with $\phi : R \to K$ a ring homomorphism, then $\mathrm{End}_{\mathfrak{G}_K}(M \otimes_R K) \simeq \mathrm{End}_{\mathfrak{G}_R}(M) \otimes_R K$. It also follows from [ALZ] that $V_R \otimes V_R^*$ is a tilting module. This implies (cf. [ALZ, Cor. 3.4]) that $\dim_K(B_R/\langle\Phi\rangle) \otimes_R K = \dim_{\mathbb{C}}(B_r(-m)/\langle\Phi\rangle) = \dim \mathrm{End}_{\mathfrak{G}_K}(V_K^{\otimes r})$, and the result follows. □

Scholium 7.4. *Let K be a field with* $\mathrm{char}(K) \geq m+2$. *Then the kernel of the algebra homomorphism $F^r_{\epsilon,r} : B_r(\epsilon m) \to \mathrm{End}_G(V^{\otimes r})$ as a two-sided ideal in the Brauer algebra is generated by Φ in the case of the symplectic group (i.e., $\epsilon = -1$), and by $E = E_\ell$ with $\ell = [(m+1)/2]$ in the case of the orthogonal group (i.e., $\epsilon = 1$).*

Remark 7.5. Recent results of Hu and Xiao show that Scholium 7.4 is valid for all fields K such that $\mathrm{char}(K) > 2$.

8. Quantum analogues

8.1. Background

Let U_q^+ (resp. U_q^-) be the quantised enveloping algebra in the sense of [LZ1, §6] of the Lie algebra $\mathfrak{o}_m(\mathbb{C})$ (see [LZ1, 8.1.2] for the definition) (resp. $\mathfrak{sp}_m(\mathbb{C})$), over the field $\mathcal{K} = \mathbb{C}(q)$, where in the latter case we require that $m = 2n$ is even. Write $\mathcal{A}_q$ for the subring of $\mathbb{C}(q)$ consisting of all rational functions with no pole at $q = 1$. Denote by $V_q = \mathcal{K}^m$ the quantum analogue of the natural representation of U_q. The study of the endomorphism algebras $\mathrm{End}_{\mathrm{U}_q}(V_q^{\otimes r})$ is closely analogous to the classical case we have been considering, which may be thought of as the limit as $q \to 1$ of the quantum case, in a way we shall shortly make precise.

In particular, there are homomorphisms from certain specialisations of the Birman–Murakami–Wenzl algebra $\mathrm{BMW}_r(q)$ to $\mathrm{End}_{\mathrm{U}_q}(V_q^{\otimes r})$, and the classical case is essentially the limit of the quantum case in the sense that $\lim_{q\to 1} \mathrm{BMW}_r(q) = B_r$, the Brauer algebra. Let us recall the details (see [LZ2, §4]). Let y, z be indeterminates over $\mathbb{C}$ and write $\mathcal{A} = \mathbb{C}[y^{\pm 1}, z]$. The BMW algebra $\mathrm{BMW}_r(y, z)$ over $\mathcal{A}$ is the associative $\mathcal{A}$-algebra with generators $g_1^{\pm 1}, \ldots, g_{r-1}^{\pm 1}$ and $e_1, \ldots, e_{r-1}$, subject to the following relations:

- the braid relations for the g_i:

$$\begin{aligned} g_i g_j &= g_j g_i \quad \text{if } |i-j| \geq 2, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} \quad \text{for } 1 \leq i \leq r-1; \end{aligned} \tag{8.1}$$

- the Kauffman skein relations:

$$g_i - g_i^{-1} = z(1 - e_i) \quad \text{for all } i; \tag{8.2}$$

- the de-looping relations:

$$\begin{aligned} g_i e_i &= e_i g_i = y e_i, \\ e_i g_{i-1}^{\pm 1} e_i &= y^{\mp 1} e_i, \\ e_i g_{i+1}^{\pm 1} e_i &= y^{\mp 1} e_i. \end{aligned} \tag{8.3}$$

The next four relations are easy consequences of the previous three:

$$e_i e_{i\pm 1} e_i = e_i, \tag{8.4}$$

$$(g_i - y)(g_i^2 - z g_i - 1) = 0, \tag{8.5}$$

$$z e_i^2 = (z + y^{-1} - y) e_i, \tag{8.6}$$

$$-y z e_i = g_i^2 - z g_i - 1. \tag{8.7}$$

It is easy to show that $\mathrm{BMW}_r(y, z)$ may be defined using the relations (8.1), (8.3), (8.5) and (8.7) instead of (8.1), (8.2) and (8.3), i.e. that (8.2) is a consequence of (8.5) and (8.7).

8.2. *Specialisations and integral forms*

Now in both the orthogonal and symplectic cases, V_q is the simple U_q-module corresponding to the highest weight ε_1 using the standard notation for the weights as in [Bour], and we have the following decomposition of $V_q^{\otimes 2}$:

$$V_q \otimes V_q = L_{2\varepsilon_1} \oplus L_{\varepsilon_1 + \varepsilon_2} \oplus L_0, \tag{8.8}$$

where L_λ is the simple module corresponding to the dominant weight λ, and L_0 is the trivial module. The eigenvalues of the R-matrix $\check{R}$ on the respective components are as follows (see [LZ1, (6.12)]):

$$\mathrm{U}_q(\mathfrak{o}_m) : q, -q^{-1}, q^{1-m}; \qquad \mathrm{U}_q(\mathfrak{sp}_m) : q, -q^{-1}, -q^{-1-m}.$$

Now define two $\mathbb{C}$-algebra homomorphisms $\psi^\pm : \mathcal{A} \to \mathcal{A}_q$ as follows: $\psi^+(y) = q^{1-m}$, $\psi^+(z) = q - q^{-1}$, $\psi^-(y) = -q^{-1-m}$, $\psi^-(z) = q - q^{-1}$. We then obtain two $\mathcal{A}_q$-algebras $\mathrm{BMW}_r^\pm(q) := \mathcal{A}_q \otimes_{\psi^\pm} \mathrm{BMW}_r(y, z)$, and we write $\mathrm{BMW}_r^\pm(\mathcal{K}) := \mathcal{K} \otimes_\iota \mathrm{BMW}_r^\pm(q)$, where ι is the inclusion of $\mathcal{A}_q$ into $\mathcal{K}$.

It follows from (8.6) that in these two specialisations, we have $e_i^2 = \delta^\pm(q) e_i$, where $\delta^+(q) = [m-1]_q + 1$ and $\delta^-(q) = -([m+1]_q - 1)$. Here we use the standard notation for q-numbers: for any integer t, $[t]_q = \frac{q^t - q^{-t}}{q - q^{-1}}$.

It is a consequence of [LZ1, Theorem 7.5] that we have surjective homomorphisms

$$\mathrm{BMW}_r^\pm(\mathcal{K}) \xrightarrow{\eta_q} \mathrm{End}_{\mathrm{U}_q^\pm}(V_q^{\otimes r}). \tag{8.9}$$

To relate the above statement to the classical ($q = 1$) case, it was shown in [LZ1, §8.2] that U_q and the modules $V_q^{\otimes r}$ have $\mathcal{A}_q$-forms $\mathrm{U}_q(\mathcal{A}_q)$, $V_q^{\otimes r}(\mathcal{A}_q)$ such that $\mathrm{U}_q(\mathcal{A}_q)$ acts on $V_q^{\otimes r}(\mathcal{A}_q)$, and the projections to the components in (8.8) are defined over $\mathcal{A}_q$, so that the decomposition (8.8) is compatible with the $\mathcal{A}_q$ forms. We may therefore take $\lim_{q\to 1} := \mathbb{C} \otimes_{\psi_1} -$ of all $\mathcal{A}_q$-modules in (8.9), where $\psi_1 : \mathcal{A}_q \to \mathbb{C}$ takes q to 1. It is well known that $\lim_{q\to 1} \mathrm{U}_q^\epsilon = \mathfrak{sp}_m(\mathbb{C})$ if $\epsilon = -1$, and $\mathfrak{o}_m(\mathbb{C})$ if $\epsilon = +1$, and that $\lim_{q\to 1} \mathrm{BMW}_r^\epsilon(q) = B_r(\epsilon m)$. In the proof of the next result we shall make extensive use of the cellular structure of $\mathrm{BMW}_r^\epsilon(q)$ and its relationship to the cellular structure of $B_r(\epsilon m)$, as described in [LZ2, Proposition 7.1].

We therefore recall the following facts.

Lemma 8.1 ([LZ2, Proposition 7.1]).

(1) *For each r, the algebras $\mathrm{BMW}_r^\epsilon(q)$ and $B_r(\epsilon m)$ have a cellular structure with the same cell datum (Λ, M, C).*
(2) *The structure constants of $B_r(\epsilon m)$ are obtained from those of $\mathrm{BMW}_r^\epsilon(q)$ by putting $q = 1$.*
(3) *For each $\lambda \in \Lambda$, denote the cell module of $\mathrm{BMW}_r^\epsilon(q)$ by $W_q(\lambda)$ and that of $B_r(\epsilon m)$ by $W(\lambda)$. Then $W(\lambda) = \lim_{q\to 1} W_q(\lambda) = \mathbb{C} \otimes_{\psi_1} W_q(\lambda)$; further, the Gram matrix of the canonical form on $W(\lambda)$ is obtained from that of $W_q(\lambda)$ by setting $q = 1$, as is the matrix of $\lim_{q\to 1} b \in B_r(\epsilon m)$ from that of b.*

The main result of this section is the following.

Theorem 8.2. (i) *With notation as above, suppose Φ is an idempotent in $B_r(\epsilon m)$ such that the ideal $\langle \Phi \rangle$ is equal to $\mathrm{Ker}(\eta : B_r(\epsilon m) \to \mathrm{End}_G(V^{\otimes r}))$. Suppose that $\Phi_q \in \mathrm{BMW}_r^\epsilon(q)$ is such that*

(1) $\Phi_q^2 = f(q)\Phi_q$ *where* $f(q) \in \mathcal{A}_q$.
(2) $\lim_{q\to 1} \Phi_q = c\Phi$, *where* $c \neq 0$.

Then Φ_q generates $\mathrm{Ker}(\eta_q : \mathrm{BMW}_r^\epsilon(q) \to \mathrm{End}_{\mathrm{U}_q}(V_q^{\otimes r}))$.

(ii) *In the symplectic case, $\mathrm{BMW}_{d+1}^-(\mathcal{K})$ is semisimple, and the kernel of η_q is generated by the idempotent corresponding to the trivial representation of $\mathrm{BMW}_{d+1}^-(\mathcal{K})$, where $m = 2d$.*

(iii) *In the orthogonal case, there is an idempotent in $\mathrm{BMW}_{m+1}^+(q)$ which generates $\mathrm{Ker}(\eta_q)$.*

Proof. It is clear from Lemma 8.1 that $\mathrm{rank}_{\mathcal{A}_q}\langle \Phi_q \rangle \geq \dim_{\mathbb{C}}\langle \Phi \rangle$ (this also follows from the fact that $\lim_{q\to 1} \mathrm{BMW}_r^\epsilon(q)\Phi_q\mathrm{BMW}_r^\epsilon(q) = B_r(\epsilon m)\Phi B_r(\epsilon m)$), and hence

$$\dim_{\mathcal{K}}(\mathrm{BMW}_r^\epsilon(\mathcal{K})/\langle \Phi_q \rangle) \leq \dim_{\mathbb{C}}(B_r(\epsilon m)/\langle \Phi \rangle).$$

It follows that if we knew that $\Phi_q \in \mathrm{Ker}(\eta_q)$, then

$$\begin{aligned}\dim_{\mathbb{C}}(B_r(\epsilon m)/\langle \Phi \rangle) &\geq \dim_{\mathcal{K}}(\mathrm{BMW}_r^\epsilon(\mathcal{K})/\langle \Phi_q \rangle)\\ &\geq \dim_{\mathcal{K}}(\mathrm{BMW}_r^\epsilon(\mathcal{K})/\mathrm{Ker}(\eta_q)) = \dim_{\mathbb{C}}(B_r(\epsilon m)/\langle \Phi \rangle),\end{aligned}$$

whence (i) follows. Hence we turn to the proof that $\Phi_q \in \mathrm{Ker}(\eta_q)$.

Let $M_q = V_q^{\otimes r}$ and $M = V^{\otimes r} = \lim_{q\to 1} M_q$. We wish to show that $\Phi_q M_q = 0$. Now $\lim_{q\to 1} \Phi_q M_q = c\Phi M = 0$. It follows that $\Phi_q M_q \subseteq (q-1)M_q$. We shall show that $\Phi_q M_q \subseteq (q-1)^i M_q$ for each integer i, which will show that $\Phi_q M_q = 0$.

Assume that $\Phi_q M_q \subseteq (q-1)^i M_q$; then operating by Φ_q, we obtain $\Phi_q^2 M_q = f(q)\Phi_q M_q \subseteq (q-1)^{i+1} M_q$. But $f(q)$ is not divisible by $q-1$, since $\lim_{q\to 1} \Phi_q^2 = c^2\Phi = f(1)\Phi \neq 0$. Hence $\Phi_q M_q \subseteq (q-1)^{i+1} M_q$, and it follows by induction that $\Phi_q M_q \subseteq (q-1)^i M_q$ for all i, completing the proof of (i).

(ii) We are now in the symplectic case, and by Theorem 5.9, the idempotent $\Phi \in B_{d+1}(-m)$ which corresponds to the trivial representation generates $\mathrm{Ker}(\eta)$. Since the Gram matrix $G(W(\lambda))$ of the cell module $W(\lambda)$ of $B_{d+1}(-m)$ is obtained from the Gram matrix $G(W_q(\lambda))$ of the corresponding cell module of $\mathrm{BMW}^-_{d+1}(q)$ by taking $\lim_{q\to 1}$, it follows that since the former is non-singular for each λ, so is the latter. Hence $\mathrm{BMW}^-_{d+1}(q)$ is semisimple. Hence there is a central idempotent $\tilde{\Phi}_q \in \mathrm{BMW}^-_{d+1}(\mathcal{K})$ which corresponds to the trivial representation. This is characterised by the property that $e_i\tilde{\Phi}_q = \tilde{\Phi}_q e_i = 0$ and $g_i\tilde{\Phi}_q = \tilde{\Phi}_q g_i = q\tilde{\Phi}_q$ for all i. Now there is an element $f(q) \in \mathcal{A}_q$ such that $f(q)\tilde{\Phi}_q \in \mathrm{BMW}^-_{d+1}(q)$ and $f(1) \neq 0$. Write $\Phi_q = f(q)\tilde{\Phi}_q$. Using an argument by descent similar to that used above, it is easily shown that $\lim_{q\to 1} \Phi_q \neq 0$, i.e. $\Phi_q \notin (q-1)\mathrm{BMW}^-_{d+1}(q)$.

If we write $\sigma_i \in B_r(-m)$ for the transposition $(i, i+1)$, then with a slight abuse of notation, we have $\lim_{q\to 1}(g_i) = \sigma_i$ and $\lim_{q\to 1} e_i = e_i$. Taking limits, the relations above show that $\Phi_1 := \lim_{q\to 1} \Phi_q$ is central in $B_{d+1}(-m)$ and satisfies $e_i\Phi_1 = \Phi_1 e_i = 0$ and $\sigma_i\Phi_1 = \Phi_1\sigma_i = \Phi_1$ for all i. It follows that $\Phi_1 = c\Phi$, for some non-zero scalar c, and hence by (i), that Φ_q generates $\mathrm{Ker}(\eta_q)$.

(iii) In the orthogonal case, it follows from Theorem 6.10 that $\mathrm{Ker}(\eta)$ is generated by an idempotent element $\Phi \in B_{m+1}(m)$, which may be taken to be a scalar multiple of E_ℓ. Now $B_{m+1}(m)$ is semisimple, and hence there are primitive central idempotents $I_1, \dots, I_s \in B_{m+1}(m)$ such that $I_1 + \cdots + I_s = 1$. Hence $\Phi = \Phi I_1 + \cdots + \Phi I_s$. Suppose without loss of generality that $\Phi I_j \neq 0$ if $j \leq t$, and $\Phi I_j = 0$ if $j > t$. Then the ideal generated by Φ is equal to that generated by $\Psi := I_1 + \cdots + I_t$. For clearly $\langle\Phi\rangle \subseteq \langle I_1 + \cdots + I_t\rangle$, but conversely if $\Phi I_j \neq 0$, the two-sided ideal generated by ΦI_j includes the simple ideal generated by I_j, and hence I_j itself. So $\langle I_1 + \cdots + I_t\rangle \subseteq \langle I_1, \dots, I_t\rangle \subseteq \langle\Phi\rangle$.

We shall show that there is an element $\Psi_q \in \mathrm{BMW}^+_{m+1}(q)$ with properties analogous to those of Φ_q in (i), but for Ψ. First observe that by the same agrgument as in (ii) (using Lemma 8.1) the algebra $\mathrm{BMW}^+_{m+1}(q)$, and hence $\mathrm{BMW}^+_{m+1}(\mathcal{K})$, whose cell modules have the same Gram matrices, is semisimple. It follows that there are unique primitive central idempotents $\Phi_{1,q}, \dots, \Phi_{t,q} \in \mathrm{BMW}^+_{m+1}(\mathcal{K})$ which correspond to the same cells as $\Phi_1, \dots, \Phi_t$ respectively (recall that Λ parametrises the cells of both $\mathrm{BMW}^+_{m+1}(\mathcal{K})$ and $B_{m+1}(m)$, and hence also their minimal two-sided ideals). For each i, there is an element $f_i(q) \in \mathcal{A}_q$ such that $f_i(q)\Psi_{i,q} \in \mathrm{BMW}^+_{m+1}(q)$. Using the same argument as in (ii), one may choose $f_j(q)$ so that $\lim_{q\to 1} f_j(q)\Psi_{j,q} = f_j(1)I_j \neq 0$. Then $\Psi_q := f_1(q)\dots f_t(q)(\Phi_{1,q} + \cdots + \Phi_{t,q}) \in \mathrm{BMW}^+_{m+1}(q)$, and satisfies: (i) $\Psi_q^2 = F(q)\Psi_q$, where $F(q) = f_1(q)\dots f_t(q)$ and (ii) $\lim_{q\to 1} \Psi_q = F(1)\Psi$. It now follows from (i) that Ψ_q generates $\mathrm{Ker}(\eta_q)$. □

We remark finally that Hu and Xiao [HX] have also contributed to the subject of this section.

8.3. Further comments

The invariant theory of quantum groups [D, L] in a broad sense has been extensively studied. One aspect of it is the quantum group theoretical construction [R, RT, ZGB] (see [T2] for a review) of the Jones polynomial of knots [J] and its cousins. It was in this context that the braided monoidal category structure of the category of quantum group representations rose to prominence.

In the quantum case, the appropriate replacement for the category of Brauer diagrams is the category of (non-directed) ribbon graphs [RT, T2], also known as the category of framed tangles. The Reshetikhin–Turaev functor [RT] gives rise to a full tensor functor from this category to the category of tensor representations of the symplectic quantum group, or the orthogonal quantum group defined in [LZ1]. This is the quantum analogue of Theorem 4.8(1).

The FFT of invariant theory for quantum groups is best understood in terms of endomorphism algebras (see e.g. [DPS, LZ1]). However, in order to establish a quantum analogue of FFT in the polynomial formulation, one has to go beyond commutative algebra and consider quantum group actions on noncommutative algebras. This was developed in [LZZ].

Appendix. Proof of Theorem 2.6

We first prove (1). The fact that the elementary Brauer diagrams I, X, A and U generate all Brauer diagrams under the operations of $\circ$ and $\otimes$ may be seen as follows. Fix the nodes of an arbitrary diagram D from k to ℓ, and draw all the arcs as piecewise smooth curves, in such a way that there are at most two arcs through any point, and no two crossings or turning points have the same vertical coordinate. We may now draw a set of horizontal lines (possibly after a small perturbation of the diagram) such that

(i) no line is tangent to any of the arcs,
(ii) between successive lines there is precisely one crossing or turning point.

Then the part of the diagram between successive lines may be thought of as the $\otimes$-product of the four generators, all except one being equal to I. Thus we have exhibited D as a word in the generators, of the form $D = D_1 \circ \cdots \circ D_n$, where each D_i is of the form

$$D_i = I^{\otimes r} \otimes Y \otimes I^{\otimes s}, \tag{A.1}$$

with Y being one of A, U or X. Such an expression will be called a *regular expression*, and the factors D_i *elementary diagrams*. A product of elementary diagrams in which $Y = X$ for each factor will be called a *permutation diagram*. An example of a particular regular expression is given in Figure 16.

This completes the proof of (1).

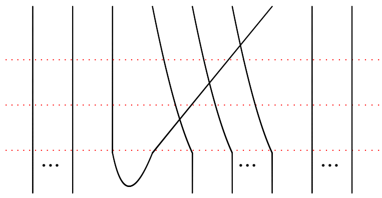


Fig. 16. Regular expression.

We now turn to the proof that the stated relations form a complete set. Observe first that any expression for a diagram D as a word in the generators provides a regular expression for D by repeated use of the relation (2.2) and its dual. Accordingly we say that two regular expressions $\mathfrak{D}, \mathfrak{D}'$ are *equivalent*, and write $\mathfrak{D} \sim \mathfrak{D}'$, if one can be obtained from the other by a sequence of applications of the relations in part (2) of the theorem. This is clearly an equivalence relation on regular expressions.

However, a word in the generators does not in general yield a Brauer diagram, but rather a diagram multiplied by δ^k for some non-negative integer k, where k is the number of deleted loops. For any Brauer diagram D and any $N \in \mathbb{Z}_+$, the above argument shows that we can always represent $\delta^N D$ as a word in the generators, and hence also as a regular expression. We therefore need to work with morphisms of the form $\delta^N D$, where D is a diagram. We refer to such a morphism as a *scaled Brauer diagram*, or simply a *scaled diagram*. Every Brauer diagram is clearly a scaled diagram.

The discussion above shows that to prove the theorem, it will suffice to show that

$$\text{any two regular expressions for a scaled diagram are equivalent.} \tag{A.2}$$

We shall extend the notion of equivalence to any expression of the form $D_1 \circ \cdots \circ D_n$, where the D_i are diagrams.

Definition A.1. The two compositions $D_1 \circ \cdots \circ D_n$ and $D'_1 \circ \cdots \circ D'_m$ are said to be *equivalent* if one can be obtained from the other using only the relations in Theorem 2.6(2), and the properties of $\circ$ and $\otimes$.

To prove (A.2) we require some analysis of regular expressions and equivalence. We shall return to the proof after carrying this out.

Definition A.2. (1) The *valency* of a scaled diagram $D \in B_k^l$ is the pair (k, l).
(2) If $D = I^{\otimes r} \otimes Y \otimes I^{\otimes s}$ is elementary, the *abscissa* $a(D)$ of D is $r + 1$, while the *type* $t(D) = Y$ $(= A, U$ or $X)$.
(3) The *length* of a regular expression $E_1 \circ \cdots \circ E_n$ is n.

We shall repeatedly apply the following elementary observation, which we refer to as the 'commutation principle'.

Remark A.3. (1) Let E_1, E_2 be elementary diagrams such that $E_1 \circ E_2$ makes sense. If $|a(E_1) - a(E_2)| > 1$ then $E_1 \circ E_2 \sim E'_1 \circ E'_2$, where $t(E'_1) = t(E_2)$ and $t(E'_2) = t(E_1)$.

(2) If D, D' are scaled diagrams of valency (k, l) and (k', l') respectively, then $D \otimes D' = (I^{\otimes l} \otimes D') \circ (D \otimes I^{\otimes k'}) = (D \otimes I^{\otimes l'}) \circ (I^{\otimes k} \otimes D')$.

Part (2) of the remark states the obvious relations among diagrams depicted in Figure 17.

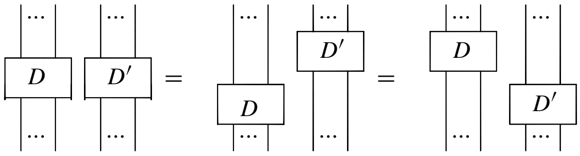


Fig. 17. Commutativity.

This follows from the fact that $(A \otimes B) \circ (A' \otimes B') \sim (A \circ A') \otimes (B \circ B')$ for A, A', B, B' of appropriate valency, and the relation (2.2).

The next two results will be used in the reduction of the proof of Theorem 2.6(2) to a single case.

Lemma A.4. *Let P, Q be permutation diagrams of valency (l, l) and (k, k) respectively and let $D \in B_k^l$ be a scaled diagram. If any two regular expressions for $P \circ D \circ Q$ are equivalent, then so are any two regular expressions for D.*

Proof. Let $\mathfrak{D}, \mathfrak{D}'$ be two regular expressions for D, and suppose for the moment that P is an elementary permutation diagram. Then $P \circ \mathfrak{D}$ and $P \circ \mathfrak{D}'$ are regular expressions for $P \circ D$, and hence are equivalent by hypothesis. Now $P \circ P \circ \mathfrak{D}$ is a regular expression, and it is evident that $P \circ P \circ \mathfrak{D}$ is equivalent to $P \circ P \circ \mathfrak{D}'$. But from (2.3), $P \circ P \circ \mathfrak{D} \sim \mathfrak{D}$ and $P \circ P \circ \mathfrak{D}' \sim \mathfrak{D}'$, whence $\mathfrak{D}$ and $\mathfrak{D}'$ are equivalent. This proves the lemma for elementary P and $Q = \mathrm{id}$.

Applying the above statement repeatedly, we see that for any permutation diagram P, if any two regular expressions for $P \circ D$ are equivalent, the same is true for D. A similar argument applies to prove the corresponding statement for $D \circ Q$, for any permutation diagram Q. □

It follows that in proving (A.2), we may pre- and postmultiply D by arbitrary permutation diagrams, and replace D by the resulting scaled diagram.

For the second reduction, we require the following definitions.

Definition A.5. (1) Define $R : B_k^l \to B_{k-1}^{l+1}$ (for $k \geq 1$) (the *raising operator*) by $R(D) = (D \otimes I) \circ (I^{\otimes(k-1)} \otimes U)$, and (the *lowering operator*) $L : B_k^l \to B_{k+1}^{l-1}$ by $L(D) = (I^{\otimes(l-1)} \otimes A) \circ (D \otimes I)$.

(2) If $\mathfrak{D} = D_1 \circ \cdots \circ D_n$ is a regular expression for the scaled diagram $D \in B_k^l$, define the regular expression $R(\mathfrak{D})$ for $R(D)$ by $R(\mathfrak{D}) = (D_1 \otimes I) \circ \cdots \circ (D_n \otimes I) \circ (I^{\otimes k-1} \otimes U)$, and similarly define the regular expression $L(\mathfrak{D})$ for $L(D)$. Note that if E is elementary, then so is $E \otimes I$, so that the above definition makes sense.

Lemma A.6. (1) *For any regular expression $\mathfrak{D}$ for a scaled diagram $D \in B_k^l$, we have $R \circ L(\mathfrak{D}) \sim \mathfrak{D}$ and $L \circ R(\mathfrak{D}) \sim \mathfrak{D}$.*

(2) *Suppose D is a scaled diagram of valence (k, l) with $k \geq 1$. The regular expressions $\mathfrak{D}, \mathfrak{D}'$ for D are equivalent if and only if $L(\mathfrak{D})$ and $L(\mathfrak{D}')$ (or $R(\mathfrak{D})$ and $R(\mathfrak{D}')$) are equivalent.*

Proof. To prove (1), let $\mathfrak{D} = E_1 \circ \cdots \circ E_n$ be a regular expression for $D \in B_k^l$. Then

$$\begin{aligned}
R \circ L(\mathfrak{D}) &= R\big((I^{\otimes(l-1)} \otimes A) \circ (E_1 \otimes I) \cdots \circ (E_n \otimes I)\big) \\
&= (I^{\otimes(l-1)} \otimes A \otimes I) \circ (E_1 \otimes I \otimes I) \cdots \circ (E_n \otimes I) \circ (I^{\otimes k} \otimes U) \\
&\sim (I^{\otimes(l-1)} \otimes A \otimes I) \circ (I^{\otimes l} \otimes U) \circ E_1 \circ \cdots \circ E_n \quad \text{by several applications of A.3} \\
&\sim I^{\otimes l} \circ E_1 \circ \cdots \circ E_n \quad \text{by (2.8)} \\
&\sim E_1 \circ \cdots \circ E_n \quad \text{by (2.2)} \\
&= \mathfrak{D}.
\end{aligned}$$

Thus $R \circ L(\mathfrak{D}) \sim \mathfrak{D}$, and the proof that $L \circ R(\mathfrak{D}) \sim \mathfrak{D}$ is similar.

Now to prove (2), suppose first that $\mathfrak{D}, \mathfrak{D}'$ are equivalent regular expressions for D. Then the same sequence of moves using the relations in Theorem 2.6(2) which convert $\mathfrak{D}$ into $\mathfrak{D}'$ may be applied to $L(\mathfrak{D})$ to convert it into $L(\mathfrak{D}')$. This shows that if $\mathfrak{D}, \mathfrak{D}'$ are equivalent regular expressions for D, then $L(\mathfrak{D}), L(\mathfrak{D}')$ are equivalent regular expressions for $L(D)$. A similar argument proves the corresponding statement for $R(D)$.

To prove the converse, suppose that any two regular expressions for $R(D)$ are equivalent, and that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are two regular expressions for D. Then $R(\mathfrak{D}_1)$ and $R(\mathfrak{D}_2)$ are two regular expressions for $R(D)$, and hence by hypothesis are equivalent. Hence by the above, $L \circ R(\mathfrak{D}_1)$ and $L \circ R(\mathfrak{D}_2)$ are two equivalent regular expressions for $L \circ R(D)$, which is equal to D by (1). But by (1), $L \circ R(\mathfrak{D}_1) \sim \mathfrak{D}_1$ and $L \circ R(\mathfrak{D}_2) \sim \mathfrak{D}_2$, whence $\mathfrak{D}_1 \sim \mathfrak{D}_2$. □

The following lemma is the key computation involving the relations in Theorem 2.6(2).

Lemma A.7. *Let $\mathfrak{T}_s := E_s \circ E_{s-1} \circ \cdots \circ E_0$ be a regular expression, where $t(E_0) = U$, $a(E_0) = a$, $t(E_i) = X$ and $a(E_i) = a + i$ for $i \geq 1$. The diagram $\mathfrak{T}_s$ is shown in Figure 16. Let E be an elementary diagram of type A or X which does not 'commute with' $E_s \circ E_{s-1} \circ \cdots \circ E_0$, i.e. such that $a - 1 \leq a(E) \leq a + s + 1$. Then*

(1) *If $t(E) = A$, then $E \circ \mathfrak{T}_s$ is equivalent to a shorter regular expression unless $s = 0$ and $a(E) = a(E_0)$. In the latter case, $E \circ \mathfrak{T}_s$ is the identity multiplied by δ.*
(2) *Suppose $t(E) = X$. Then*
   (i) *If $a + 1 \leq a(E) \leq a + s - 1$, then $E \circ \mathfrak{T}_s \sim \mathfrak{T}_s \circ E'$ for an elementary diagram E' of type X. (Thus E may be 'moved through' $E \circ \mathfrak{T}_s$.)*
   (ii) *If $a(E) = a$ or $a + s$, then $E \circ \mathfrak{T}_s$ is equivalent to a shorter regular expression.*
   (iii) *If $a(E) = a - 1$ or $a + s + 1$ then $E \circ \mathfrak{T}_s \sim \mathfrak{T}_{s+1}$.*
(3) *Let $\mathfrak{T}_s$ be as above and let E be elementary of type A or X. Then $E \circ \mathfrak{T}_s$ is equivalent to a shorter regular expression (possibly multiplied by δ) or to $\mathfrak{T}_s \circ E'$ for some elementary E', or to $\mathfrak{T}_{s+1}$.*

Proof. Consider first the case where $t(E) = A$.

If $s = 0$ and $a(E) = a(E_0)$, the claim follows from the loop removal relation (2.6).

If $a(E) = a + s + 1$, then (2.7) yields $E \circ E_s \sim E' \circ E'_s$, where $t(E') = t(E) = A$, $t(E'_s) = t(E_s) = X$, $a(E') = a + s$ and $a(E'_s) = a + s + 1$. It now follows by repeated application of Remark A.3 about commutation that $E \circ \mathfrak{T}_s \sim E'' \circ \mathfrak{T}_{s-1} \circ E'''$, where $t(E'') = A$ and $a(E'') = a + s$. Repeating this argument s times, we see that $E \circ \mathfrak{T}_s$ is equivalent to a regular expression of length $s+1$ which includes $F \circ E_0$ as a subexpression, where $t(F) = A$ and $a(F) = a + 1$. By (2.8), we see that $F \circ E \sim I^{\otimes k}$ for some k, and hence $E \circ \mathfrak{T}_s$ is equivalent to a regular expression of length $s - 1$.

If $a(E) = a + s$, then by (2.5), $E \circ E_s \sim E$, and we have again shortened $E \circ \mathfrak{T}_s$.

If $a \le a(E) \le a + s - 1$, then by commutation, $E \circ \mathfrak{T}_s$ is equivalent to a regular expression with a subexpression of the form $E \circ E_i \circ E_{i-1}$, where $t(E_i) = X$ and $a(E) = a(E_i) - 1$. Applying (2.8), this is equivalent to an expression $E' \circ E'_i \circ E_{i-1}$, where $a(E'_i) = a(E_{i-1})$, and $t(E'_i) = X$. Using either (2.3) (if $i > 1$) or the * of (2.5), we again reduce the length to show that $E \circ \mathfrak{T}_s$ is equivalent to a shorter regular expression.

Finally, if $a(E) = a - 1$, we use commutation to show that $E \circ \mathfrak{T}_s$ is equivalent to a regular expression of length $s + 1$ with a subexpression of the form $E' \circ E_0$, where $t(E') = A$ and $a(E') = a - 1 = a(E_0) - 1$. Applying (2.8), we see that $E' \circ E_0 \sim I^{\otimes k}$ for some k, and this completes the proof of (1).

Now consider the case where $t(E) = X$.

If $a + 1 \le a(E) \le a + s - 1$, then after applying the commutation rule, we see that $E \circ \mathfrak{T}_s$ is equivalent to a regular expression of length $s + 1$ which has a subexpression of the form $E \circ E_{a(E)+1} \circ E_{a(E)}$. But the braid relation (2.4) implies that this is equivalent to $E' \circ E_{a(E)} \circ E_{a(E)+1}$, where $E' = E_{a(E)+1}$. Again using commutation, we may now move the last factor below E_0 (since $a(E)+1 \ge a+2$). It follows that $E \circ \mathfrak{T}_s \sim \mathfrak{T}_s \circ E'$, where $t(E') = X$. This proves (i).

If $a(E) = a + s + 1$ then evidently $E \circ \mathfrak{T}_s = \mathfrak{T}_{s+1}$. If $a(E) = a + s$, the relation $X \circ X = I \otimes I$ (2.3) shows that $E \circ E_s \sim I^{\otimes r}$ for some r, and hence $E \circ \mathfrak{T}_s$ is equivalent to a shorter regular expression. If $a(E) = a - 1$, then we may use commutation to see that $E \circ \mathfrak{T}_s \sim E_s \circ \cdots \circ E_1 \circ E \circ E_0$. Using (2.7) we see that this is equivalent to $E_s \circ \cdots \circ E_1 \circ E_1 \circ E'_0$, where $t(E'_0) = U$. Applying (2.3), we see that $E \circ \mathfrak{T}_s$ is equivalent to a shorter regular expression. Finally, if $a(E) = a$, we again use commutation to see that $E \circ \mathfrak{T}_s$ is equivalent to $E_s \circ E_{s-1} \circ \cdots \circ E \circ E_1 \circ E_0$. Again applying (2.7), we obtain a factor $E \circ E$, and applying (2.3), we again shorten the regular expression $E \circ \mathfrak{T}_s$. This completes the proof of (2).

The statement (3) is a summary of the previous two statements. □

Completion of the proof of Theorem 2.6(2). It remains to prove (A.2). It follows from Lemmas A.6 and A.4 that to complete the proof of the theorem, it suffices to prove (A.2) for any scaled diagram which can be obtained from D by raising or lowering, or multiplication by a permutation diagram. It follows that we may take D to be the scaled diagram $D = \delta^N U^{\otimes r}$ ($N \in \mathbb{Z}_+$). Hence we shall be done if we prove the following result:

$$\text{Any two regular expressions for } D = \delta^N U^{\otimes r} \text{ are equivalent.} \tag{A.3}$$

We shall prove (A.3) by induction on r, starting with $r = 0$. For convenience, we adopt the following local convention:

- scaled diagrams will be simply called 'diagrams';
- a regular expression $\mathfrak{D}$ is said to be δ-*equivalent* to another regular expression $\mathfrak{D}'$ if it can be changed to $\delta^k \mathfrak{D}'$ for some $k \in \mathbb{Z}_+$ by the relations in Theorem 2.6(2).

Let $r = 0$ and suppose $\mathfrak{D} := D_1 \circ \cdots \circ D_n$ is a regular expression for the empty scaled diagram δ^N in B_0^0. We need to show that $\mathfrak{D}$ is δ-equivalent to the empty regular expression; we do this by showing that every non-empty regular expression for the empty scaled diagram is δ-equivalent to one of shorter length.

Now by valency considerations, we must have $D_1 = A$ and $D_n = U$. Let i be the least integer such that $t(D_i) = U$; then for all $j < i$, $t(D_j) = A$ or X. Applying Lemma A.7 repeatedly, we see that since at least one of the D_j for $j < i$ is of type A, $\mathfrak{D}$ is δ-equivalent to a shorter regular expression. This proves the result for $r = 0$.

Now take $r > 0$ and let $\mathfrak{D} = D_1 \circ \cdots \circ D_n$ be a regular expression for D. Then since at least r of the D_i must have type U, we have $n \geq r$. Moreover if $n = r$, which can happen only if $N = 0$, then the D_i are all of type U, and have odd abscissa, and any such regular expression represents D. Any two such regular expressions (which will be called *minimal*) are equivalent by the commutation rule (see Remark A.3).

It therefore suffices to show that if $n > r$, then $\mathfrak{D}$ is δ-equivalent to a shorter regular expression.

Clearly we have $t(D_n) = U$; if $t(D_1) = U$ then $\mathfrak{D}' := D_2 \circ \cdots \circ D_n$ is a regular expression for $U^{\otimes(r-1)}$, and we conclude by induction on r that $\mathfrak{D}'$ is δ-equivalent to a shorter regular expression. Thus we are finished. Let $p = p(\mathfrak{D})$ be the least index such that D_p is of type U. We have seen that if $p = 1$ then we are finished by induction. It will therefore suffice to show that $\mathfrak{D}$ is either equivalent to a regular expression $\mathfrak{D}'$ with $p(\mathfrak{D}') < p(\mathfrak{D})$, or is δ-equivalent to a shorter regular expression $\mathfrak{D}'$.

Thus we take $p > 1$; then $t(D_p) = U$, and $t(D_i) = A$ or X for $i < p$. We now apply Lemma A.7 to conclude that either we may commute one of the D_i $(i < p)$ past D_p, or $D_1 \circ \cdots \circ D_p \sim \mathfrak{T}_{p-1}$ or at least one of the D_i $(i < p)$ is of type A. In the first case, we obtain a regular expression with small p-value; in the second case, in the diagram $D_1 \circ \cdots \circ D_n$ if the nodes are numbered $1, \ldots, 2r$ from left to right, node $a(D_p)$ would be joined to node $a(D_p) + p$. Hence $p = 1$, which has been excluded.

In the third case, suppose i is the largest index such that $1 \leq i \leq p - 1$ and D_i is of type A. Then either some D_j $(i \leq j \leq p - 1)$ can be commuted past D_p by application of Remark A.3, or else we are in the situation of Lemma A.7(1). In the former case, we have reduced p; in the latter, by the same remark $D_i \circ \cdots \circ D_p$ is δ-equivalent to a shorter regular expression.

We have now shown that either $\mathfrak{D}$ is δ-equivalent to a shorter regular expression, or equivalent to a regular expression which has the same length as $\mathfrak{D}$ but a smaller p value.

This completes the proof of (A.3), and hence of Theorem 2.6. □

Remark A.8. We note that to prove part (2) of the theorem, we could have proceeded by regarding $\mathcal{B}(\delta)$ as a quotient category of the category of (unoriented) tangles (see

Remark 2.5) and deduce the relations among the generators of Brauer diagrams from a complete set of relations among the generators of tangles given in [T1, §3.2] (suppressing information about orientation). This way we obtain all relations except the one which enforces the removal of free loops and multiplication by powers of δ, i.e., (2.6).

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