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On the topology of polynomials with bounded integer coefficients

Received August 27, 2012 and in revised form January 28, 2015

Abstract. For a real number $q > 1$ and a positive integer m , let

$$Y_m(q) := \left\{ \sum_{i=0}^n \epsilon_i q^i : \epsilon_i \in \{0, \pm 1, \dots, \pm m\}, n = 0, 1, \dots \right\}.$$

In this paper, we show that $Y_m(q)$ is dense in \mathbb{R} if and only if $q < m + 1$ and q is not a Pisot number. This completes several previous results and answers an open question raised by Erdős, Joó and Komornik [8].

Keywords. Pisot numbers, iteration function systems

1. Introduction

For a real number $q > 1$ and a positive integer m , let

$$Y_m(q) := \left\{ \sum_{i=0}^n \epsilon_i q^i : \epsilon_i \in \{0, \pm 1, \dots, \pm m\}, n = 0, 1, \dots \right\}.$$

In this paper, we consider the following old question regarding the topological structure of $Y_m(q)$:

Question 1.1. For which pairs (q, m) is the set $Y_m(q)$ dense in \mathbb{R} ?

It is well known that $Y_m(q)$ is not dense in \mathbb{R} in the following two cases: when q is a Pisot number (Garsia [12]) or $q \geq m + 1$ (Erdős and Komornik [9]). Recall that a *Pisot number* is an algebraic integer > 1 all of whose conjugates have modulus < 1 (cf. [22]). For the reader's convenience, we include a brief proof. First assume that q is a Pisot number. Denote by q_1, \dots, q_d the algebraic conjugates of q . Then $\rho := \max_{1 \leq j \leq d} |q_j| < 1$. Let $P(x) = \sum_{i=0}^n \epsilon_i x^i$ be a polynomial with coefficients in $\{0, \pm 1, \dots, \pm m\}$. Suppose that

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Mathematics Subject Classification (2010): Primary 11J17; Secondary 11K16, 28A78

$P(q) \neq 0$. Then $P(q_j) \neq 0$ for $1 \leq j \leq d$. Hence $P(q) \prod_{j=1}^d P(q_j)$ is a non-zero integer. Therefore

$$|P(q)| \geq \prod_{j=1}^d \frac{1}{|P(q_j)|} \geq \left(\frac{1}{\sum_{i=0}^n m \rho^i} \right)^d > m^{-d} (1 - \rho)^d.$$

It follows that $Y_m(q)$ is not dense in \mathbb{R} since 0 is an isolated point of $Y_m(q)$. The same argument also shows that 0 is an isolated point of $Y_{2m}(q) = Y_m(q) - Y_m(q)$, therefore $Y_m(q)$ is uniformly discrete in \mathbb{R} . Next assume that $q \geq m + 1$. Then for any $n \in \mathbb{N}$,

$$q^n - \sum_{i=0}^{n-1} m q^i = \frac{q^n(q-1-m) + m}{q-1} \geq \frac{(q-1-m) + m}{q-1} = 1.$$

It follows that $|P(q)| \geq 1$ for any polynomial P with degree ≥ 1 and coefficients in $\{0, \pm 1, \dots, \pm m\}$. Hence $Y_m(q) \cap (-1, 1) = \{0\}$, so $Y_m(q)$ is not dense in \mathbb{R} .

In this paper, by proving the reverse direction we obtain the following theorem, which provides a complete answer to Question 1.1.

Theorem 1.2. $Y_m(q)$ is dense in \mathbb{R} if and only if $q < m + 1$ and q is not a Pisot number.

We remark that Question 1.1 is closely related to a project proposed by Erdős, Joó and Komornik in the late 90's. For $q > 1$ and $m \in \mathbb{N}$, let

$$X_m(q) = \left\{ \sum_{i=0}^n \epsilon_i q^i : \epsilon_i \in \{0, 1, \dots, m\}, n = 0, 1, \dots \right\}.$$

Since $X_m(q)$ is discrete, we may arrange the points of $X_m(q)$ into an increasing sequence:

$$0 = x_0(q, m) < x_1(q, m) < x_2(q, m) < \dots.$$

Denote

$$\ell_m(q) = \liminf_{n \rightarrow \infty} (x_{n+1}(q, m) - x_n(q, m)), \quad L_m(q) = \limsup_{n \rightarrow \infty} (x_{n+1}(q, m) - x_n(q, m)).$$

Motivated by the study of expansions in non-integer bases, Erdős, Joó and Komornik [7, 8, 9] proposed to characterize all the pairs (q, m) so that $\ell_m(q)$ and $L_m(q)$ vanish. By definition, $\ell_m(q) = 0$ is equivalent to 0 being an accumulation point of $Y_m(q)$. However, it was proved by Drobot [5] (see also [6]) that $Y_m(q)$ is dense in \mathbb{R} if and only if 0 is an accumulation point of $Y_m(q)$. Hence $\ell_m(q) = 0$ if and only if $Y_m(q)$ is dense in \mathbb{R} . In [8], Erdős, Joó and Komornik asked whether $\ell_1(q) = 0$ for any non-Pisot number $q \in (1, 2)$. This question was also formulated in [23, 1]. As a direct corollary of Theorem 1.2 and Drobot's result, we can provide an affirmative answer to this question.

Corollary 1.3. $\ell_m(q) = 0$ if and only if $q < m + 1$ and q is not a Pisot number.

In the literature there are some partial results on Question 1.1 and the project of Erdős et al. It was shown in [5, 6] that if $q \in (1, m + 1)$ does not satisfy an algebraic equation with coefficients $0, \pm 1, \dots, \pm m$, then $\ell_m(q) = 0$. In [3] Bugeaud showed that if q is not a Pisot number, then there exists an integer m such that $\ell_m(q) = 0$. The approach of Bugeaud did not provide any estimate of m . A substantial progress was made later by Erdős and Komornik [9], who proved that $\ell_m(q) = 0$ if q is not a Pisot number and $m \geq \lceil q - q^{-1} \rceil + \lceil q - 1 \rceil$, where $\lceil x \rceil$ denotes the smallest integer $\geq x$. Recently Akiyama and Komornik [1] showed that $\ell_1(q) = 0$ if $q \in (1, \sqrt{2}]$ is not a Pisot number smaller than the golden ratio $(1 + \sqrt{5})/2$. Sidorov and Solomyak [23] proved that if $q \in (1, m + 1)$ and q is not a Perron number, then $\ell_m(q) = 0$. Recall that an algebraic integer $q > 1$ is called a *Perron number* if each of its conjugates is less than q in modulus.

As for the value of $L_m(q)$, Erdős and Komornik [9] proved that $L_m(q) > 0$ if q is a Pisot number or $q \geq (m + \sqrt{m^2 + 4})/2$. It remains an open problem whether $L_m(q) = 0$ for all other pairs (q, m) with $q > 1$ and $m \in \mathbb{N}$. Komornik [14] conjectured that this is true in the case when $m = 1$, i.e., $L_1(q) = 0$ for any non-Pisot number smaller than the golden ratio. Some partial results were obtained by Erdős–Komornik and Akiyama–Komornik: $L_m(q) = 0$ if q is non-Pisot and $m \geq \lceil q - q^{-1} \rceil + 2\lceil q - 1 \rceil$ (see [9]); furthermore, $L_1(q) = 0$ if $1 < q \leq \sqrt[3]{2} \approx 1.2599$ (see [9, 1]). Here the second part was only proved in [9] for all $1 < q \leq \sqrt[4]{2} \approx 1.1892$ with the possible exception of the square root of the second Pisot number.

By directly applying Corollary 1.3 and [1, Lemma 2.5] (which says that $\ell_m(q^2) = 0$ implies¹ $L_m(q) = 0$), we have the following theorem which improves the results in [9, 1].

Theorem 1.4. *If $1 < q < \sqrt{m+1}$ and q^2 is not a Pisot number, then $L_m(q) = 0$. In particular, if $q \in (1, \sqrt{2})$ and q^2 is not a Pisot number, then $L_1(q) = 0$.*

Let us mention some other important results related to Question 1.1. In [9], Erdős and Komornik showed that if $q > 1$ is not a Pisot number and $m \geq q - q^{-1}$, then $Y_m(q)$ has a finite accumulation point. Very recently, Akiyama and Komornik [1] characterized all pairs (q, m) such that $Y_m(q)$ has a finite accumulation point, completing the previous results of Erdős and Komornik [9] and Zaimi [25] on this topic.

Theorem 1.5 (Akiyama and Komornik [1]). *$Y_m(q)$ has a finite accumulation point in \mathbb{R} if and only if $q < m + 1$ and q is not a Pisot number.*

In this paper, we shall prove the following result.

Theorem 1.6. *Assume that $1 < q \leq m + 1$. Then $Y_m(q)$ has no finite accumulation points in \mathbb{R} if and only if 0 is not an accumulation point of $Y_m(q)$.*

Theorem 1.6 was conjectured at the end of [1], where the authors observed that, combined with Theorem 1.5, this would imply that 0 is an accumulation point of $Y_m(q)$ (equivalently, $Y_m(q)$ is dense in \mathbb{R}) if and only if $q < m + 1$ and q is not a Pisot number. Hence Theorem 1.2 follows from Theorems 1.6 and 1.5.

¹ This implication was first proved in [8, Theorem 5] in the case $m = 1$. It extends to $m > 1$ directly.

We remark that the separation property of $Y_m(q)$ was also considered by Lau [15] in his study of Bernoulli convolutions (see [19] for a survey about Bernoulli convolutions). Following Lau [15], we call $q \in (1, 2)$ an *F-number* if

$$Y_1(q) \cap \left[-\frac{1}{q-1}, \frac{1}{q-1} \right] \text{ is a finite set.}$$

Clearly, each Pisot number in $(1, 2)$ is an F-number. Lau [15] raised the question of whether or not there exists an F-number which is non-Pisot. As a corollary of Theorem 1.2 (it also follows from Theorem 1.5 together with Remark 1.10 and Lemma 2.1), we have the following answer to Lau's question.

Corollary 1.7. *Every F-number is a Pisot number.*

As a closely related topic, for $q \in (1, 2)$, the topological structure of the set

$$A(q) = \left\{ \sum_{i=0}^n \epsilon_i q^i : \epsilon_i \in \{-1, 1\}, n = 0, 1, \dots \right\}$$

has been studied [20, 2, 24, 1]. It was proved that if $1 < q \leq \sqrt{2}$ is not a Pisot number, then $A(q)$ is dense in \mathbb{R} [1]; moreover, for almost all $q \in (\sqrt{2}, 2)$, $A(q)$ is dense in \mathbb{R} [20]. Meanwhile, there exist non-Pisot numbers $q \in (\sqrt{2}, 2)$ such that $A(q)$ is discrete [2]. It is an interesting question to characterize all $q \in (\sqrt{2}, 2)$ such that $A(q)$ is dense in \mathbb{R} .

The proof of Theorem 1.6 is based on our study of separation properties of homogeneous iterated function systems (IFS) on \mathbb{R} . Let m be a positive integer and $\Phi = \{\phi_i\}_{i=0}^m$ a family of contractive maps on \mathbb{R} of the form

$$\phi_i(x) = \rho x + b_i, \quad i = 0, 1, \dots, m,$$

where

$$0 < \rho < 1 \quad \text{and} \quad 0 = b_0 < \dots < b_m = 1 - \rho. \quad (1.1)$$

Then Φ is called a *homogeneous iterated function system* on \mathbb{R} . According to Hutchinson [13], there is a unique compact set $K := K_\Phi \subset \mathbb{R}$ such that

$$K = \bigcup_{i=0}^m \phi_i(K).$$

We call K the *attractor* of Φ . It is easy to check that

$$K = \left\{ \sum_{n=0}^{\infty} b_{i_n} \rho^n : i_n \in \{0, 1, \dots, m\} \text{ for } n \geq 0 \right\}.$$

The condition (1.1) implies that the convex hull of K is the unit interval $[0, 1]$.

For any finite word $I = i_1 \dots i_n \in \{0, 1, \dots, m\}^n$, write $\phi_I = \phi_{i_1} \circ \dots \circ \phi_{i_n}$. Clearly, $\phi_I(0) = b_{i_1} + \rho b_{i_2} + \dots + \rho^{n-1} b_{i_n}$.

Definition 1.8. Say that Φ satisfies the *weak separation condition* if there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and any $I, J \in \{0, 1, \dots, m\}^n$,

$$\text{either } \rho^{-n}|\phi_I(0) - \phi_J(0)| = 0 \quad \text{or} \quad \rho^{-n}|\phi_I(0) - \phi_J(0)| \geq c.$$

Definition 1.9. Say that Φ satisfies the *finite type condition* if there exists a finite set $\Gamma \subset [0, 1)$ such that for any $n \in \mathbb{N}$ and any $I, J \in \{0, 1, \dots, m\}^n$,

$$\text{either } \rho^{-n}|\phi_I(0) - \phi_J(0)| \geq 1 \quad \text{or} \quad \rho^{-n}|\phi_I(0) - \phi_J(0)| \in \Gamma.$$

Remark 1.10. By definition, for $1 < q < 2$, q is an F-number if and only if the IFS $\{q^{-1}x, q^{-1}x + (1 - q^{-1})\}$ satisfies the finite type condition.

The concept of weak separation condition was first introduced by Lau and Ngai [16] for more general IFSs. One is referred to [26, 4] for some equivalent definitions. The above definition of finite type condition was adopted from [10], and is slightly stronger than the one introduced by Ngai and Wang [17]. For a homogeneous IFS on \mathbb{R} , it is unknown whether the open set condition (cf. [13]) always implies the finite type condition.

It is easy to see that in our setting, the finite type condition implies the weak separation condition (this is also true in the general settings of [16, 17]; see [18] for a proof). However, it is not clear whether the weak separation condition also implies the finite type condition in our setting. The following theorem gives this implication under an additional assumption on Φ .

Theorem 1.11. Let $\Phi = \{\phi_i(x) = \rho x + b_i\}_{i=0}^m$ be an IFS satisfying (1.1). Assume in addition that

$$b_{i+1} - b_i \leq \rho \quad \text{for all } 0 \leq i \leq m - 1. \tag{1.2}$$

Suppose Φ satisfies the weak separation condition. Then Φ also satisfies the finite type condition.

We remark that (1.2) is equivalent to $[0, 1] = \bigcup_{i=0}^m \phi_i([0, 1])$, i.e., $K_\Phi = [0, 1]$.

Now for a given pair (q, m) with $1 < q \leq m + 1$, consider a special IFS $\Phi = \{\rho x + b_i\}_{i=0}^m$ with $\rho = q^{-1}$ and $b_i = i(1 - q^{-1})/m$ for $0 \leq i \leq m$. Then Φ satisfies the assumptions in Theorem 1.11. However Φ satisfies the weak separation condition if and only if 0 is not an accumulation point of $Y_m(q)$; whilst Φ satisfies the finite type condition if and only if $Y_m(q)$ has no finite accumulation points in \mathbb{R} (see Lemma 2.1). Hence according to Theorem 1.11, the condition that 0 is not an accumulation point of $Y_m(q)$ implies that $Y_m(q)$ has no finite accumulation points in \mathbb{R} ; from which Theorem 1.6 follows. As a corollary of this and Theorem 1.2, we have

Corollary 1.12. For a given pair (q, m) with $1 < q < m + 1$, let Φ denote the IFS $\{\phi_i(x) = q^{-1}x + i(1 - q^{-1})/m\}_{i=0}^m$ on \mathbb{R} . Then Φ satisfies the weak separation condition (resp. the finite type condition) if and only if q is a Pisot number.

The paper is organized as follows. In Section 2, we prove Theorem 1.11. In Section 3, we give some final remarks and questions.

2. Separation properties of IFSs and the proof of Theorem 1.11

Before giving the proof of Theorem 1.11, we present two lemmas.

Lemma 2.1. *Let $\Phi = \{\phi_i(x) = \rho x + b_i\}_{i=0}^m$ be an IFS on \mathbb{R} with*

$$0 < \rho < 1, \quad 0 = b_0 < \dots < b_m = 1 - \rho.$$

Denote

$$Y = \left\{ \sum_{i=1}^n \epsilon_i \rho^{-i} : \epsilon_i \in \{b_s - b_t : 0 \leq s, t \leq m\}, n = 1, 2, \dots \right\}.$$

Then Φ satisfies the weak separation condition if and only if 0 is not an accumulation point of Y ; whilst Φ satisfies the finite type condition if and only if Y has no finite accumulation points in \mathbb{R} .

Proof. For $n \geq 1$, $I = i_1 \dots i_n$, $J = j_1 \dots j_n \in \{0, 1, \dots, m\}^n$, we have

$$\rho^{-n}(\phi_I(0) - \phi_J(0)) = \sum_{s=1}^n (b_{i_s} - b_{j_s}) \rho^{-(n+1-s)} = \sum_{s=1}^n (b_{i_{n+1-s}} - b_{j_{n+1-s}}) \rho^{-s}. \quad (2.1)$$

Hence by Definition 1.8, Φ satisfies the weak separation condition if and only if 0 is not an accumulation point of Y . In the following we show that Y has no finite accumulation points if and only if Φ satisfies the finite type condition.

By (2.1) and Definition 1.9, we see that Φ satisfies the finite type condition if and only if $Y \cap [-1, 1]$ contains only finitely many points. It is straightforward to see that Y having no finite accumulation points implies $Y \cap [-1, 1]$ contains only finitely many points. Hence to finish the proof, we only need to show that the finiteness of $Y \cap [-1, 1]$ implies that Y has no finite accumulation points.

From now on, we assume that $Y \cap [-1, 1]$ contains only finitely many points. Set $A = Y \cap [-1, 1]$ and $B = \{b_i - b_j : 0 \leq i, j \leq m\}$. Since A and B are finite sets, we can pick $u > 1$ such that $(1, u) \cap \rho^{-1}(A + B) = \emptyset$, where

$$\rho^{-1}(A + B) := \{\rho^{-1}(x + \epsilon) : x \in A, \epsilon \in B\}.$$

Since $0 \in A$, we have $(1, u) \cap \rho^{-1}B = \emptyset$. We first claim that $Y \cap (1, u) = \emptyset$. To see this, for any $y \in Y$, let $\deg(y)$ denote the smallest $n \in \mathbb{N}$ such that $y = \sum_{i=1}^n \epsilon_i \rho^{-i}$ for some $\epsilon_1, \dots, \epsilon_n \in B$. Assume on the contrary that $Y \cap (1, u) \neq \emptyset$. Define

$$N = \min\{\deg(y) : y \in Y \cap (1, u)\}.$$

Then $N \in \mathbb{N}$. Pick $z \in Y \cap (1, u)$ so that $\deg(z) = N$. Since $(1, u) \cap \rho^{-1}B = \emptyset$, we have $z \notin \rho^{-1}B$ and thus $N = \deg(z) \geq 2$. Then there exist $\epsilon_1, \dots, \epsilon_N \in B$ such that

$$z = \sum_{i=1}^N \epsilon_i \rho^{-i}.$$

Denote $w = \sum_{i=1}^{N-1} \epsilon_{i+1} \rho^{-i}$. Then $w \in Y$ and $z = \rho^{-1}w + \rho^{-1}\epsilon_1$. Notice that $w \notin A$ (and hence $|w| > 1$); for otherwise we have $z \in \rho^{-1}(A + B)$, contradicting $(1, u) \cap \rho^{-1}(A + B) = \emptyset$ and $z \in (1, u)$. On the other hand, we must have $|w| < z$; if not,

$$|\rho^{-1}\epsilon_1| = |\rho^{-1}w - z| \geq \rho^{-1}|w| - z \geq (\rho^{-1} - 1)z > \rho^{-1} - 1 = \rho^{-1} \max B,$$

leading to a contradiction. Therefore, we have $1 < |w| < z < u$, and thus $|w| \in Y \cap (1, u)$. However, $\deg(|w|) \leq N - 1 < \deg(z)$, contradicting the minimality of $\deg(z)$. Therefore, we must have $Y \cap (1, u) = \emptyset$.

Since $Y = -Y$, we also have $Y \cap (-u, -1) = \emptyset$. Thus $Y \cap (-u, u)$ contains only finitely many points. Finally, we show that Y has no finite accumulation points. Assume on the contrary that Y has a finite accumulation point, say v . Since $Y \cap (-u, u)$ contains only finitely many points, we must have $|v| \geq u$. Note that for any $n \in \mathbb{N}$,

$$Y = \rho^{-n}Y + D_n, \tag{2.2}$$

where $D_n := \{\sum_{i=1}^n \epsilon_i \rho^{-i} : \epsilon_i \in B \text{ for all } i\}$. Take a large n such that $\rho^n|v| + 1 < u$. By (2.2), Y has a finite accumulation point w (it is possible that $w \notin Y$), and $z \in D_n$ such that $v = \rho^{-n}w + z$. Then

$$|w| = |\rho^n(v - z)| \leq \rho^n|v| + \rho^n \sum_{i=1}^n (1 - \rho)\rho^{-i} < \rho^n|v| + 1 < u.$$

This contradicts the fact that Y has no accumulation points in $(-u, u)$. □

Lemma 2.2. Let $\Phi = \{\phi_i(x) = \rho x + b_i\}_{i=0}^m$ be an IFS satisfying

$$0 < \rho < 1, \quad 0 = b_0 < \dots < b_m = 1 - \rho, \\ b_{i+1} - b_i \leq \rho \quad \text{for all } 0 \leq i \leq m - 1.$$

Then:

- (1) For any $n \in \mathbb{N}$, we have $[0, 1] = \bigcup_{I \in \{0, 1, \dots, m\}^n} \phi_I([0, 1])$.
- (2) For $n, k \in \mathbb{N}$ and $J \in \{0, 1, \dots, m\}^n$, if $[c, d]$ is a subinterval of $\phi_J([0, 1])$ of length $\geq \rho^{n+k}$, then there exists $J' \in \{0, 1, \dots, m\}^k$ such that $\phi_{JJ'}(0) \in [c, d]$.

Proof. It is direct to check that $[0, 1] = \bigcup_{i=0}^m \phi_i([0, 1])$. Iterating this relation n times yields (1).

To see (2), note that $\phi_J^{-1}([c, d])$ is a subinterval of $[0, 1]$ of length $\geq \rho^k$. By (1), there exists $J' \in \{0, 1, \dots, m\}^k$ such that $\phi_{J'}(0) \in \phi_J^{-1}([c, d])$. Hence $\phi_{JJ'}(0) \in [c, d]$. □

Proof of Theorem 1.11. We divide the proof into smaller steps.

Step 1. Let $0 < \delta < 1$. We claim that there is a finite set $\Gamma_\delta \subset [0, 1 - \delta]$ such that for each $n \in \mathbb{N}$ and $I, J \in \{0, 1, \dots, m\}^n$,

$$\text{either } \rho^{-n}|\phi_I(0) - \phi_J(0)| > 1 - \delta \quad \text{or} \quad \rho^{-n}|\phi_I(0) - \phi_J(0)| \in \Gamma_\delta. \tag{2.3}$$

To prove the above claim, we use an idea in [11]. Since Φ satisfies the weak separation condition, according to the pigeon-hole principle we have

$$\sup_{x \in [0,1], k \in \mathbb{N}} \#\{\phi_I(0) : \phi_I(0) \in [x, x + \rho^k], I \in \{0, 1, \dots, m\}^k\} =: \ell < \infty, \quad (2.4)$$

where $\#X$ denotes the cardinality of X . Indeed, we have $\ell \leq 1/c + 1$, where c is the constant in Definition 1.8.

Pick $x \in [0, 1]$ and $k \in \mathbb{N}$ so that the supremum in (2.4) is attained at (x, k) . Clearly, the supremum is then also attained at $(\phi_I(x), n+k)$ for any $n \in \mathbb{N}$ and $I \in \{0, 1, \dots, m\}^n$. Pick a large integer k' so that $\rho^{k'} + \rho^{k'+k} < 1$ and let

$$x_0 = \phi_{0^{k'}}(x), \quad k_0 = k' + k.$$

Then $[x_0, x_0 + \rho^{k_0}] \subset [0, 1]$ and the supremum in (2.4) is attained at (x_0, k_0) . Choose $W_1, \dots, W_\ell \in \{0, 1, \dots, m\}^{k_0}$ such that $\phi_{W_1}(0), \dots, \phi_{W_\ell}(0)$ are different points in $[x_0, x_0 + \rho^{k_0}]$.

Fix $0 < \delta < 1$. Pick $k_1 \in \mathbb{N}$ so that

$$\rho^\delta \leq \rho^{k_1} < \delta. \quad (2.5)$$

Now suppose that $I, J \in \{0, 1, \dots, m\}^n$ for some $n \in \mathbb{N}$ are such that

$$|\phi_I(0) - \phi_J(0)| \leq (1 - \delta)\rho^n.$$

Without loss of generality, assume that $\phi_I(0) \leq \phi_J(0)$. Denote $\Delta = [\phi_J(0), \phi_I(0) + \rho^n]$. Clearly $\Delta \subset \phi_I([0, 1]) \cap \phi_J([0, 1])$, and $|\Delta| \geq \delta\rho^n$, where $|\Delta|$ denotes the length of Δ . Since $\phi_I(0) + \rho^n = \phi_I(1)$, we see that $\phi_I^{-1}(\Delta) = [u, 1]$ for some $u \in (0, 1)$ with $1 - u \geq \delta > \rho^{k_1}$. Set $I' = m \dots m$ (k_1 letters). Since $\phi_m(1) = 1$, we have $\phi_{I'}(1) = 1$. Observe that $\phi_{I'}([0, 1])$ has length ρ^{k_1} , therefore $\phi_{I'}([0, 1]) = [1 - \rho^{k_1}, 1] \subset [u, 1]$, and thus $\phi_{I'}([0, 1]) \subset \phi_I([u, 1]) = \Delta$; in particular,

$$\phi_{I'}([x_0, x_0 + \rho^{k_0}]) \subset \Delta \subset \phi_J([0, 1]).$$

Note that $\phi_{I'}([x_0, x_0 + \rho^{k_0}])$ is a subinterval of $\phi_J([0, 1])$ with length $\rho^{n+k_0+k_1}$. By Lemma 2.2(2), there exists $J' \in \{0, 1, \dots, m\}^{k_0+k_1}$ such that $\phi_{J'}(0) \in \phi_{I'}([x_0, x_0 + \rho^{k_0}])$. Let $x_1 = \phi_{I'}(x_0)$. Then $\phi_{I'}([x_0, x_0 + \rho^{k_0}]) = [x_1, x_1 + \rho^{n+k_0+k_1}]$. Recall that $\phi_{W_1}(0), \dots, \phi_{W_\ell}(0)$ are different points in $[x_0, x_0 + \rho^{k_0}]$, hence $\phi_{I'W_1}(0), \dots, \phi_{I'W_\ell}(0)$ are ℓ distinct points in $[x_1, x_1 + \rho^{n+k_0+k_1}]$. Since $\phi_{J'}(0) \in [x_1, x_1 + \rho^{n+k_0+k_1}]$, by the maximality of ℓ (cf. (2.4)) we must have

$$\phi_{J'}(0) \in \{\phi_{I'W_j}(0) : 1 \leq j \leq \ell\}.$$

That is,

$$\phi_J(0) + \rho^n \phi_{J'}(0) \in \{\phi_I(0) + \rho^n \phi_{I'W_j}(0) : 1 \leq j \leq \ell\}.$$

It follows that

$$\begin{aligned} \rho^{-n}(\phi_J(0) - \phi_I(0)) &\in \{\phi_{I'W_j}(0) - \phi_{J'}(0) : 1 \leq j \leq \ell\} \\ &\subset \{\phi_{\tilde{I}}(0) - \phi_{\tilde{J}}(0) : \tilde{I}, \tilde{J} \in \{0, 1, \dots, m\}^{k_0+k_1}\}. \end{aligned}$$

Hence we can finish the proof of the claim in Step 1 by setting

$$\Gamma_\delta = \{\phi_{\tilde{I}}(0) - \phi_{\tilde{J}}(0) : \tilde{I}, \tilde{J} \in \{0, 1, \dots, m\}^{k_0+k_1}\} \cap [0, 1 - \delta]. \quad (2.6)$$

Step 2. Denote $\gamma = \min\{b_1, b_m - b_{m-1}\}$ and $B = \{b_i - b_j : 0 \leq i, j \leq m\}$. By (1.1) and (1.2), $0 < \gamma \leq \rho < 1$. Let Γ_γ be as in Step 1 (with $\delta = \gamma$). Set

$$\eta := \max(\rho^{-1}(\pm\Gamma_\gamma + B) \cap [0, 1]).$$

Clearly $0 \leq \eta < 1$. We claim that for any $n \in \mathbb{N}$ and $I, J \in \{0, 1, \dots, m\}^n$,

$$\text{either } \rho^{-n}|\phi_I(0) - \phi_J(0)| \geq 1 \quad \text{or} \quad \rho^{-n}|\phi_I(0) - \phi_J(0)| \leq \eta. \quad (2.7)$$

Assume the claim is not true. Then we can find $n \in \mathbb{N}$ and $I, J \in \{0, 1, \dots, m\}^n$ such that

$$\eta < \rho^{-n}(\phi_J(0) - \phi_I(0)) < 1. \quad (2.8)$$

Assume further that the above n is the smallest possible.

First we show that $n \geq 2$. Indeed, otherwise $n = 1$ and by (2.8), $0 < \rho^{-1}(\phi_J(0) - \phi_I(0)) < 1$, and hence $\rho^{-1}(\phi_J(0) - \phi_I(0)) \in \rho^{-1}B \cap [0, 1)$; by the definition of η and the fact $0 \in \Gamma_\gamma$, we have $\rho^{-1}(\phi_J(0) - \phi_I(0)) \leq \eta$, contrary to (2.8).

Since $n \geq 2$, we can write

$$I = I'i, \quad J = J'j,$$

where $I', J' \in \{0, 1, \dots, m\}^{n-1}$ and $i, j \in \{0, 1, \dots, m\}$. Then we have

$$\phi_I(0) = \phi_{I'}(0) + \rho^{n-1}b_i, \quad \phi_J(0) = \phi_{J'}(0) + \rho^{n-1}b_j.$$

Therefore,

$$\phi_{J'}(0) - \phi_{I'}(0) = \phi_J(0) - \phi_I(0) + \rho^{n-1}(b_i - b_j). \quad (2.9)$$

By (2.9) and (2.8), we have

$$|\phi_{J'}(0) - \phi_{I'}(0)| < \rho^n + \rho^{n-1}(1 - \rho) = \rho^{n-1}. \quad (2.10)$$

In the following we show further that

$$|\phi_{J'}(0) - \phi_{I'}(0)| \leq (1 - \gamma)\rho^{n-1}. \quad (2.11)$$

By (2.9) and the fact that $\phi_J(0) > \phi_I(0)$, we have

$$\phi_{J'}(0) - \phi_{I'}(0) > \rho^{n-1}(b_i - b_j) \geq -\rho^{n-1}(1 - \rho) \geq -\rho^{n-1}(1 - \gamma). \quad (2.12)$$

To get an upper bound for $\phi_{J'}(0) - \phi_{I'}(0)$, we consider the following two scenarios separately:

- (i) $(i, j) = (m, 0)$;
- (ii) $(i, j) \neq (m, 0)$.

First assume that (i) occurs. Then by (2.9),

$$\phi_{J'}(0) - \phi_{I'}(0) = \phi_J(0) - \phi_I(0) + \rho^{n-1}(1 - \rho),$$

from which and (2.8) we obtain

$$\begin{aligned} \frac{\phi_{J'}(0) - \phi_{I'}(0)}{\rho^{n-1}} &= \frac{\phi_J(0) - \phi_I(0)}{\rho^{n-1}} + (1 - \rho) \\ &= \frac{\phi_J(0) - \phi_I(0)}{\rho^n} + (1 - \rho) \left(1 - \frac{\phi_J(0) - \phi_I(0)}{\rho^n} \right) \\ &> \frac{\phi_J(0) - \phi_I(0)}{\rho^n} > \eta. \end{aligned}$$

This together with (2.10) yields $1 > \rho^{-(n-1)}(\phi_{J'}(0) - \phi_{I'}(0)) > \eta$, contradicting the minimality of n . Hence (i) cannot happen, and (ii) must occur. Since $(i, j) \neq (m, 0)$, we have

$$b_j - b_i \geq \min\{b_1 - b_m, b_0 - b_{m-1}\} = \min\{b_1 - (1 - \rho), -b_{m-1}\}.$$

This together with (2.9) yields

$$\begin{aligned} \phi_{J'}(0) - \phi_{I'}(0) &\leq \rho^n - \rho^{n-1} \cdot \min\{b_1 - (1 - \rho), -b_{m-1}\} \\ &= \rho^{n-1} \cdot \max\{1 - b_1, 1 - (b_m - b_{m-1})\} \\ &= \rho^{n-1}(1 - \gamma). \end{aligned} \tag{2.13}$$

Now (2.11) follows from (2.12) and (2.13).

According to (2.11) and the claim in Step 1, we have $\rho^{-(n-1)}|\phi_{J'}(0) - \phi_{I'}(0)| \in \Gamma_\gamma$. Then by (2.9),

$$\begin{aligned} \rho^{-n}(\phi_J(0) - \phi_I(0)) &= \rho^{-n}(\phi_{J'}(0) - \phi_{I'}(0)) + \rho^{-1}(b_j - b_i) \\ &\in \rho^{-1}(\pm\Gamma_\gamma + B). \end{aligned}$$

This together with (2.8) yields $\rho^{-n}(\phi_J(0) - \phi_I(0)) \in \rho^{-1}(\pm\Gamma_\gamma + B) \cap [0, 1)$. By the definition of η , we have $\rho^{-n}(\phi_J(0) - \phi_I(0)) \leq \eta$, which contradicts (2.8). This proves (2.7).

Step 3. Let $\eta \in [0, 1)$ be as in Step 2. Combining (2.7) with the claim in Step 1, for any $n \in \mathbb{N}$ and $I, J \in \{0, 1, \dots, m\}^n$ we have

$$\text{either } \rho^{-n}|\phi_I(0) - \phi_J(0)| \geq 1 \quad \text{or} \quad \rho^{-n}|\phi_I(0) - \phi_J(0)| \in \Gamma_{1-\eta},$$

where $\Gamma_1 := \{0\}$. Hence Φ satisfies the finite type condition. This finishes the proof of Theorem 1.11. \square

3. Final remarks and open questions

3.1

It is worth mentioning a connection between the topological property of $Y_m(q)$ and the following famous unsolved question. Suppose $q > 1$ is such that $\|\lambda q^n\| \rightarrow 0$ as $n \rightarrow \infty$ for some real number $\lambda > 0$; can we assert that q is a Pisot number? Here $\|x\|$ denotes the absolute value of the difference between x and the nearest integer. This was answered positively by Pisot [21] (see also [22]) if in addition one of the following conditions is satisfied: (i) $\|\lambda q^n\|$ tends to 0 rapidly enough so that $\sum_{n=1}^{\infty} \|\lambda q^n\|^2 < \infty$, or (ii) q is an algebraic number.

We remark that Theorem 1.2 (also Bugeaud’s result in [3]) implies the following weaker result:

$$\sum_{n=1}^{\infty} \|\lambda q^n\| < \infty \Rightarrow q \text{ is a Pisot number.} \tag{3.1}$$

To see this, assume that $\sum_{n=1}^{\infty} \|\lambda q^n\| < \infty$. Pick a positive integer $m > q - 1$. Take a large integer N so that $\sum_{n \geq N} \|\lambda q^n\| < 1/(3m)$. Then $\|y\| < 1/3$ for any real number y in the set $F = \{\sum_{i=N}^{n+N} \epsilon_i \lambda q^i : \epsilon_i \in \{0, \pm 1, \dots, \pm m\}, n = 0, 1, \dots\}$. Hence F is not dense in \mathbb{R} . Note that $Y_m(q) = F/(\lambda q^N)$. So $Y_m(q)$ is not dense in \mathbb{R} . Therefore by Theorem 1.2, q is a Pisot number.

As pointed out by an anonymous referee, using Theorem 1.2, the implication (3.1) also follows from the following inequality:

$$\ell_1(q) \geq (\lambda q^N)^{-1} \left(1 - \sum_{n=N}^{\infty} \|\lambda q^n\|\right) \quad \text{if} \quad \sum_{n=N}^{\infty} \|\lambda q^n\| < \frac{1}{q+1}.$$

This inequality is only formulated in [8, Theorem 1] in the case when $\lambda = 1$, but it extends to $\lambda > 0$ with the identical proof.

3.2

We remark that the proof of Theorem 1.11 implies the following result, which is of interest in its own right.

Proposition 3.1. *Under the assumptions of Theorem 1.11, there exists $k \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ and $I, J \in \{0, 1, \dots, m\}^n$, if $\rho^{-n} |\phi_I(0) - \phi_J(0)| < 1$, then there exist $I', J' \in \{0, 1, \dots, m\}^k$ such that $\phi_{I'}(0) = \phi_{J'}(0)$.*

As a corollary, we have

Corollary 3.2. *Assume that $m \in \mathbb{N}$ and q is a Pisot number in $(1, m + 1]$. Then there exists $k \in \mathbb{N}$ such that if $|\sum_{i=0}^{n-1} \epsilon_i q^i| < m/(q - 1)$ for some $n \in \mathbb{N}$ and $\epsilon_0, \dots, \epsilon_{n-1} \in \{0, \pm 1, \dots, \pm m\}$, then there exist $\epsilon_n, \dots, \epsilon_{n+k-1} \in \{0, \pm 1, \dots, \pm m\}$ such that*

$$\sum_{i=0}^{n+k-1} \epsilon_i q^i = 0.$$

Similar to Pisot numbers, there is a certain separation property for Salem numbers. Recall that $q > 1$ is called a *Salem number* if it is an algebraic integer whose algebraic conjugates all have modulus no greater than 1, and at least one of them is on the unit circle. It follows from Lemma 1.51 in Garsia [12] that if q is a Salem number and $m \in \mathbb{N}$, then there exist $c > 0$ and $k \in \mathbb{N}$ (c, k depend on q and m) such that

$$Y_m^n(q) \cap (-cn^{-k}, cn^{-k}) = \{0\}, \quad \forall n \in \mathbb{N}, \quad (3.2)$$

where $Y_m^n(q) := \{\sum_{i=0}^{n-1} \epsilon_i q^i : \epsilon_i \in \{0, \pm 1, \dots, \pm m\}\}$. We end the paper by posing the following questions.

- For $m \in \mathbb{N}$ and a non-Pisot number $q \in (1, m + 1)$, does the property (3.2) imply that q is a Salem number?
- Does Theorem 1.11 still hold without the assumption (1.2)?

Acknowledgments. The research was supported by RGC grants in the Hong Kong Special Administrative Region, China (projects CUHK401112, CUHK401013). The author thanks Shigeki Akiyama for sending him a copy of [1]. He is grateful to Nikita Sidorov for pointing out the implication (3.1) in Section 3.1, and to Shigeki Akiyama, Vilmos Komornik and Nikita Sidorov for pointing out the fact that Theorem 1.2 implies Theorem 1.4. He also thanks the anonymous referees and Toufik Zaimi for a very careful reading of the original manuscript and many valuable comments to improve the paper.

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