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A Gromov–Winkelmann type theorem for flexible varieties

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Abstract. An affine variety X of dimension ≥ 2 is called *flexible* if its *special automorphism group* $\mathrm{SAut}(X)$ acts transitively on the smooth locus X_{reg}. Recall that $\mathrm{SAut}(X)$ is the subgroup of the automorphism group $\mathrm{Aut}(X)$ generated by all one-parameter unipotent subgroups [2]. Given a normal, flexible, affine variety X and a closed subvariety Y in X of codimension at least 2, we show that the pointwise stabilizer subgroup of Y in the group $\mathrm{SAut}(X)$ acts infinitely transitively on the complement $X \setminus Y$, that is, m-transitively for any $m \geq 1$. More generally we prove such a result for any quasi-affine variety X and codimension ≥ 2 subset Y of X.

In the particular case of $X = \mathbb{A}^n$, $n \geq 2$, this yields a theorem of Gromov and Winkelmann [8], [18].

Keywords. Affine varieties, group actions, one-parameter subgroups, transitivity

1. Introduction

Throughout the paper X will be an algebraic variety of dimension ≥ 2 over an algebraically closed field $\Bbbk$ of characteristic 0. The *special automorphism group* $\mathrm{SAut}(X)$ of X is the subgroup of the full automorphism group $\mathrm{Aut}(X)$ generated by all one-parameter unipotent subgroups of $\mathrm{Aut}(X)$.[1] Let $\mathcal{U}(X)$ denote the set of all those subgroups. A quasi-affine variety X is called *flexible* if the tangent space $T_x X$ at any smooth point $x \in X_{\mathrm{reg}}$ is spanned by the tangent vectors at x to the orbits $U.x$, where U runs over $\mathcal{U}(X)$. Thus a normal quasi-affine variety X is flexible if and only if X_{reg} is.

If X is affine then this amounts to the notion of flexibility as introduced in [2, 1]. For such varieties flexibility is equivalent to transitivity, and even to infinite transitivity of the group $\mathrm{SAut}(X)$ acting on X_{reg} (see [1, Theorem 0.1]). (We say that a group action is *infinitely transitive* if it is m-transitive for any $m \geq 1$.) These characterizations of

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[1] That is, by subgroups isomorphic to $\mathbb{G}_{\mathrm{a}}$. By abuse of language we do not distinguish between one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ and effective $\mathbb{G}_{\mathrm{a}}$-actions on X.

flexibility can be extended to any quasi-affine variety (see Remarks 2.7 and Theorem 2.12, and [3, Theorem 2]).

It is worth mentioning that the class of flexible varieties is rather wide and contains objects with sophisticated topology. It includes in particular

- the orbits of dimension at least 2 of the group $\mathrm{SAut}(X)$ on a quasi-affine variety X;
- homogeneous spaces of semisimple groups (and even homogeneous spaces of extensions of semisimple groups by unipotent radicals);
- nondegenerate toric varieties (i.e. toric varieties without nonconstant invertible regular functions);
- cones over flag varieties and anticanonical cones over del Pezzo surfaces of degree at least 4;
- normal hypersurfaces of the form $uv = p(\bar{x})$ in $\mathbb{C}^{n+2}_{u,v,\bar{x}}$,
- homogeneous affine Gizatullin surfaces, etc. (see [2], [1], [12], [14]).

There are general constructions (suspensions, tensor bundles etc., see [2, 1]) that allow one to associate with a given flexible variety a rich collection of other such varieties. Furthermore, if a normal variety of dimension ≥ 2 can be covered by several copies of $\mathbb{A}^n$ then its universal torsor is a flexible quasi-affine variety [3].

In its simplest form the main result of this paper is the following theorem (see Sect. 2 for generalizations and refinements).

Theorem 1.1. *Let X be a smooth quasi-affine variety of dimension ≥ 2 and $Y \subseteq X$ a closed subscheme of codimension ≥ 2. If X is flexible then so is $X \setminus Y$.*

That is, if $\mathrm{SAut}(X)$ acts transitively on X then $\mathrm{SAut}(X\setminus Y)$ acts transitively on $X\setminus Y$. More generally, our main result (Theorem 2.6) shows that the pointwise stabilizer $\mathrm{SAut}_Y(X)$ acts transitively on $X \setminus Y$. This answers in the affirmative a question posed in [1, 4.22(2)]. Partial results in this direction were obtained in [1, Theorem 2.5 and Proposition 4.19] (see also Proposition 2.12 below). We note that Theorem 1.1 does not hold for subsets Y of X of codimension 1, in general [1, Proposition 4.13]. In this sense the result above is optimal.

For an affine space $X = \mathbb{A}^n$, $n \geq 2$, the flexibility of $X \setminus Y$ was first observed by M. Gromov [8, §2.1.5, p. 72, Exercise (b′)] (cf. also [9, 4.6(b) and 5.3(c)]). The transitivity of $\mathrm{SAut}_Y(X)$ in $X \setminus Y$ was proven in this particular case by J. Winkelmann [18, §2, Proposition 1].

To get an idea of the proof of Theorem 1.1, let us recall the argument used by both Gromov and Winkelmann. Say, for flexibility, given a point $x \in \mathbb{A}^n \setminus Y$ it suffices to find a $\mathbb{G}_a$-action moving x in a prescribed generic direction $v \in T_x(\mathbb{A}^n) \cong \mathbb{A}^n$ and fixing Y pointwise. Indeed, then the orbit $G.x$ of $G := \mathrm{SAut}(\mathbb{A}^n \setminus Y)$ of every point $x \in \mathbb{A}^n \setminus Y$ is open. Since any two nonempty open subsets meet, the result follows.

Let us consider the linear projection $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ parallel to v. By the genericity of v the closure $Y' := \overline{\pi(Y)}$ is a proper subvariety of $\mathbb{A}^{n-1}$ not containing $x' := \pi(x)$. Hence there is a regular function f on $\mathbb{A}^{n-1}$ such that $f(x') = 1$ and $f|_{Y'} \equiv 0$. Assuming without loss of generality that the coordinate form of π is $(x_1, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$

consider the locally nilpotent vector field $f \cdot \partial/\partial x_1$. Its phase flow preserves Y and moves x in direction v, as required.

A similar argument is crucial in the analytic version of this theorem for the complement to a closed tame analytic subset of $\mathbb{C}^n$ of codimension at least 2 [6, Propositions 4.11.6 and 4.11.7]. It is also implicitly present in the theorem of the second author and Kutzschebauch [11, Theorem 4] where some deeper properties of the group $\mathrm{SAut}_Y(\mathbb{A}^n)$ are described.

In the case of arbitrary algebraic manifolds as treated here, the problem is considerably more complicated. Though for a general flexible variety X it is natural to replace a generic linear projection π as before by the quotient morphism $\varrho : X \to X/\!/U$ with respect to the action of a generic $U \in \mathcal{U}(X)$, it is not obvious anymore that for a given $x \in X \setminus Y$ the group U can be chosen so that the closure of $\varrho(Y)$ does not contain $\varrho(x)$, i.e. one cannot guarantee the existence of a regular function on $X/\!/U$ that separates $\varrho(x)$ and $\varrho(Y)$. This turns out to be the essential obstacle to carrying out the original Gromov–Winkelmann argument, which we overcome as follows.

By a result in [1] the pointwise stabilizer $\mathrm{SAut}_Y(X)$ of Y in $\mathrm{SAut}(X)$ has an open orbit, say O, in X. As an important ingredient of the proof we show that for any flexible variety X one can find a subgroup H of $\mathrm{SAut}(X)$ acting with an open orbit on X, which is generated by two locally nilpotent derivations δ_0, δ_1 along with all their *replicas* $f_0\delta_0$, $f_1\delta_1$, where $f_0 \in \ker\delta_0$ and $f_1 \in \ker\delta_1$ (see Proposition 2.15). We now consider a completion $\bar{X}$ of X compatible with partial quotients by the two $\mathbb{G}_a$-subgroups $U^0 = \exp(\Bbbk\delta_0)$ and $U^1 = \exp(\Bbbk\delta_1)$. These quotients define on $\bar{X}$ two $\mathbb{P}^1$-fibrations $\bar{\varrho}_0, \bar{\varrho}_1$ with privileged components D_0 and D_1 of the boundary of X, for which the restrictions $\bar{\varrho}_0|_{D_0}$ and $\bar{\varrho}_1|_{D_1}$ are birational. Acting with a suitable replica of U^0 one can move the part $\partial Y \cap D_1$ of the boundary to a fixed proper subset of D_1, and symmetrically for U^1 and $\partial Y \cap D_0$ (see Proposition 4.11). Up to a controllable (and so negligible) proper subset of $D_0 \cup D_1$, this property is preserved when we iterate subsequently actions by suitable replicas of U^0 and U^1 (see Proposition 5.11). Using the transitivity property of $\mathrm{SAut}(X)$ and the subgroup H we can move a given codimension ≥ 2 subset Y as in Theorem 1.1 and, simultaneously, a given point $x \in X \setminus Y$ to a generic fiber, say F, of the $\mathbb{P}^1$-fibration $\bar{\varrho}_0$ so that F does not meet $\partial Y \cap D_0$. Using the Transversality Theorem from [1] we can achieve that F does not meet Y, hence in total F and $\bar{Y}$ are disjoint. This enables us to find a U^0-invariant function $f \in \mathcal{O}_X(X)$ which vanishes on Y but not at x. The corresponding replica U^0_f of U^0 fixes Y and moves x along F. Since the fiber F is generic, it meets the open orbit O of $\mathrm{SAut}_Y(X)$, hence so does $U^0_f.x$. Thus x belongs to O, and so $O = X \setminus Y$, as stated.

In order to prove Propositions 4.11 and 5.11 mentioned above, we develop in Sections 3 and 4 a machinery which allows us to reduce the proof to the model case of a standard birational transformation of a ruled surface induced by a $\mathbb{G}_a$-action. This reduction is the longest part of the proof.

The paper is organized as follows. In Section 2 we recall some useful facts from [1] and formulate, after introducing necessary definitions, a stronger version of Theorem 1.1 (see Theorem 2.6). In the second part of Sect. 2 we give a proof of Proposition 2.15 mentioned above. In Sections 3 and 4 we prepare the setup for the proof of our Main

Theorem 2.6. The proof is then contained in Section 5. To get a first impression of the line of argument, it is advisable after reading Section 2 to go directly to Section 5, addressing results in Sections 3 and 4 when necessary.

2. Main theorem

2.1. Basic notions and the main result

We let $\mathbb{A}^n = \mathbb{A}^n_{\Bbbk}$ and $\mathbb{G}_a = \mathbb{G}_a(\Bbbk)$. In what follows, X denotes a quasi-affine variety over $\Bbbk$. Thus X can be embedded into an affine variety $X' = \operatorname{Spec} B$ as an open subset. We let $A = \mathcal{O}_X(X)$ so that B is a finitely generated $\Bbbk$-subalgebra of A. The embedding $X \hookrightarrow X'$ factors as $X \to \operatorname{Spec} A \to \operatorname{Spec} B$. Furthermore $X \hookrightarrow \operatorname{Spec} A$ is an open embedding. We note that A is not in general a finitely generated algebra over $\Bbbk$.

Lemma 2.1. *With the notation as above the following hold.*

(a) *Every action of an algebraic group on X extends in a canonical way to* $\operatorname{Spec} A$.
(b) *Every subgroup $U \in \mathcal{U}(X)$ with infinitesimal generator δ yields a locally nilpotent $\Bbbk$-derivation on A.*

Proof. (a) is standard, and (b) is a consequence of (a). □

Let us recall some notions and useful facts from [1]. Given a subgroup $U \in \mathcal{U}(X)$ we let δ denote an infinitesimal generator of U; the latter is uniquely determined up to a nonzero constant factor. Thus δ is a locally nilpotent derivation of the algebra $A = \mathcal{O}_X(X)$ such that $U = \exp(\Bbbk\delta)$. Geometrically δ can be viewed as a complete vector field on X with phase flow $u_t = \exp(t\delta)$, $t \in \Bbbk$. The tangent vector at $x \in X$ given by this vector field is denoted δ_x.

Lemma 2.2. *Let Y be a closed (not necessarily reduced) subscheme of the quasi-affine variety X with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$, and consider the ideal of global sections $I = \mathcal{I}(X) \subseteq A = \mathcal{O}_X(X)$. Given $U \in \mathcal{U}(X)$ with an infinitesimal generator δ the following hold.*

(a) *$\delta(A) \subseteq I$ if and only if $u|Y = \mathrm{id}_Y$ for any $u \in U$.*
(b) *$\delta(I) \subseteq I$ if and only if $u.Y \subseteq Y$ for any $u \in U$.*[2]

Notation 2.3. (a) Let as before X be a quasi-affine variety and $A = \mathcal{O}_X(X)$ be its ring of regular functions. If $\mathfrak{a} \subseteq A$ is the ideal of the complement $\operatorname{Spec}(A) \setminus X$, then the set of nonzero locally nilpotent derivations δ of A with $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ is denoted by $\operatorname{LND}(X)$. In view of Lemmas 2.1 and 2.2(b) any element $\delta \in \operatorname{LND}(X)$ gives rise to a one-parameter subgroup $U = \exp(\Bbbk\delta)$ in $\mathcal{U}(X)$ and vice versa.

(b) In order to deal with quasi-affine varieties we choose a $\Bbbk$-subalgebra Λ of A such that the induced map $X \to \operatorname{Spec} \Lambda$ is an open embedding. Letting $\mathfrak{b}$ be the ideal of the

[2] In the terminology of [7, p. 10] this means that I is an integral ideal.

complement $\operatorname{Spec}(\Lambda) \setminus X$ we let $\mathrm{LND}_\Lambda(X)$ denote the set of all locally nilpotent derivations δ on Λ with $\delta(\mathfrak{b}) \subseteq \mathfrak{b}$. Every such derivation induces as before a one-parameter subgroup $U \in \mathcal{U}(X)$ and consequently extends to an element in $\mathrm{LND}(X)$. Thus $\mathrm{LND}_\Lambda(X)$ can be considered as a subset of $\mathrm{LND}(X)$.

(c) Given a collection $\mathcal{N} \subseteq \mathrm{LND}_\Lambda(X)$ of nonzero locally nilpotent derivations we let $G = G_{\mathcal{N}} = \langle \mathcal{N} \rangle$ be the subgroup of $\mathrm{SAut}(X)$ generated by the one-parameter unipotent subgroups $U = \exp(\Bbbk\delta)$, $\delta \in \mathcal{N}$.

Remarks 2.4. 1. We emphasize that the subring Λ of A is not supposed to be finitely generated over $\Bbbk$ so that the choice $\Lambda = A$ is also possible. In other words, we consider X as an open subset of an affine $\Bbbk$-scheme $\operatorname{Spec}\Lambda$, which is not necessarily an algebraic variety, in contrast with [1] (see also Remarks 2.7 below).

2. We observe as well that the G-action on X as in 2.3(c) extends to a G-action on the affine scheme $\operatorname{Spec}\Lambda$.

Let us recall some notation and standard facts.

2.5. (1) Given a group $G = G_{\mathcal{N}}$ as before, the set of all one-parameter unipotent subgroups of G will be denoted by $\mathcal{U}(G)$, and the set of all nonzero locally nilpotent derivations on Λ generating one-parameter subgroups of G by $\mathrm{LND}_\Lambda(G)$ or simply $\mathrm{LND}(G)$.

(2) A Λ-*replica* of a subgroup $U = \exp(\Bbbk\delta) \in \mathcal{U}(G)$ is a subgroup $U_f = \exp(\Bbbk f\delta) \in \mathcal{U}(G)$, where $f \in \Lambda$ is in the kernel of δ ([1]). Note that $f\delta$ is again a locally nilpotent derivation on Λ.

(3) We say that $\mathcal{N}$ is Λ-*saturated* if $\mathcal{N}$ is closed under conjugation by elements in G and taking Λ-replicas, i.e.,

$$f\delta \in \mathcal{N} \quad \forall\delta \in \mathcal{N} \text{ and } \forall f \in \ker_\Lambda \delta.$$

Hereafter Λ will be fixed, hence in most cases we omit the symbol Λ and say simply 'replica' or 'saturated'.

(4) A point $x \in X$ is called G-*flexible* if $T_x X = \operatorname{Span}(\mathcal{N}(x))$, where $\mathcal{N}(x)$ denotes the set of tangent vectors δ_x with $\delta \in \mathcal{N}$. We say that X is G-*flexible* if X_{reg} consists of G-flexible points.

(5) Given a (not necessarily reduced) closed subscheme Y in X we let $G_{\mathcal{N},Y}$ denote the subgroup of G generated by all replicas $f\delta$ in $\mathcal{N}$ vanishing on Y in the ideal-theoretic sense (see Lemma 2.2(a)). Therefore $G_{\mathcal{N},Y} \subseteq G_Y$, where $G_Y = \{g \in G \mid g|Y = \mathrm{id}_Y\}$ stands for the 'pointwise' stabilizer of Y in G in the scheme-theoretic sense.

The following is our main result.

Main Theorem 2.6. *Let X be a quasi-affine variety of dimension ≥ 2 and $X \hookrightarrow \operatorname{Spec}\Lambda$ be an open embedding into an affine $\Bbbk$-scheme (see 2.3(b)). Let $G = \langle \mathcal{N} \rangle$ be a subgroup of the group $\mathrm{SAut}(X)$ generated by a Λ-saturated set $\mathcal{N}$ of locally nilpotent derivations as in 2.5. Suppose that X is G-flexible. If Y is a closed (possibly nonreduced[3]) subscheme of X of codimension ≥ 2, then the complement $X \setminus Y$ is $G_{\mathcal{N},Y}$-flexible.*

[3] The authors are grateful to M. Gizatullin for the suggestion to take also into account nonreduced subschemes Y of X.

In the case of a smooth variety X, applying Theorem 2.6 to the group $G = \mathrm{SAut}(X)$ we get Theorem 1.1 from the Introduction.

Remarks 2.7. 1. Since $G \subseteq \mathrm{Aut}(\mathrm{Spec}\, \Lambda)$ the variety X satisfies the requirements of Theorem 2.6 whenever so does its (G-stable) regular locus X_{reg}. Therefore it suffices to prove Theorem 2.6 under the assumption that X is smooth. This explains the necessity to fix a subring $\Lambda \subseteq A$ as in 2.3(b). Indeed, A can be properly contained in $A' = \mathcal{O}_{X_{\mathrm{reg}}}(X_{\mathrm{reg}})$. If instead of fixing Λ we always consider LND's and their replicas with respect to the ring $A = \mathcal{O}_X(X)$, then an A'-replica need not be an A-replica, and so the notion of saturated set of derivations could change when passing from X to X_{reg}.

2. The viewpoint of [1] is slightly different, as it deals with open subsets X of affine algebraic varieties $Z = \mathrm{Spec}\, B$, and with subgroups G of $\mathrm{SAut}(Z)$ stabilizing X. It might happen in principle that although $\mathrm{Aut}(X)$ acts transitively on X_{reg}, there is no subgroup G of $\mathrm{Aut}(Z)$ acting transitively on X_{reg}, whatever the choice of an embedding of X into an affine variety Z (cf. Question 2.11 below). Thus *a priori* our viewpoint here is more general.

3. Working with quasi-affine varieties has yet another advantage: given a subgroup $G \subseteq \mathrm{SAut}(X)$, in the subsequent proofs we may at any step replace X by an open orbit of G. This considerably simplifies our notation.

It is worth noting that if X as in Theorem 2.6 is normal and affine then $\mathrm{SAut}(X \setminus Y)$ is in a natural way a subgroup of $\mathrm{SAut}(X)$. More generally, this holds for algebraically generated groups (see [1]), by which we mean groups which are generated by algebraic subgroups acting algebraically on $X \setminus Y$. This is a consequence of the following lemma.

Lemma 2.8. *Let X be a normal affine variety over $\mathbb{k}$ and $Y \subseteq X$ a closed subset of codimension ≥ 2. Let G be an algebraically generated group acting on $X \setminus Y$. Then this action extends to an action on X stabilizing Y.*

Proof. By our assumptions, $A := \mathcal{O}_X(X \setminus Y) = \mathcal{O}_X(X)$. Hence the action of an algebraic group on $X \setminus Y$ induces an action of G on A, and thus extends to X. Since G is algebraically generated, the result follows. □

Remark 2.9. (1) If in the lemma above the variety X is only quasi-affine then X embeds into the affine scheme $Z := \mathrm{Spec}\, A$, where $A := \mathcal{O}_X(X)$. Note that the algebra A is not in general finitely generated over $\mathbb{k}$, so $Z := \mathrm{Spec}\, A$ is only an affine scheme and not necessarily an algebraic variety. Given a closed subset $Y \subseteq X$ of codimension ≥ 2 and an action of an algebraically generated group G on $X \setminus Y$, this action extends now with the same argument as before to an action on Z.

(2) It would be interesting to know whether there is an algebraic variety $Z' = \mathrm{Spec}\, A'$ containing X as an open subset such that the action of G extends to Z'. If G is an algebraic group, this is a consequence of Lemma 2.10 below.

Lemma 2.10. *With the notation as in Remark 2.9, given a finite-dimensional $\mathbb{k}$-subspace $E \subseteq A$ there is a finitely generated G-stable $\mathbb{k}$-subalgebra $A' \subseteq A$ containing E such that X embeds as an open subset in the affine variety* $\mathrm{Spec}\, A'$.

Proof. Since X is quasi-affine, there is a finitely generated $\Bbbk$-subalgebra C of $A = \mathcal{O}_X(X)$ such that X embeds as an open subset in $\operatorname{Spec} C$. We may suppose that E contains a finite set of generators of C. By the lemma of Cartier [15, Chapt. I, §1], E is contained in a G-stable subspace E' of A of finite dimension. Obviously the $\Bbbk$-subalgebra A' of A generated by E' has the desired properties. □

Lemma 2.10 shows, in particular, that any locally nilpotent derivation δ of A stabilizes some finitely generated subalgebra A' of A such that X embeds as an open subset in the affine variety $\operatorname{Spec} A'$. We do not know whether the latter remains true for any finite collection of locally nilpotent derivations (cf. Remark 2.9(2)). More precisely:

Question 2.11. *Suppose that $\mathcal{N} \subseteq \operatorname{LND}(X)$ is finite. Does there exist a finitely generated $\mathcal{N}$-stable $\Bbbk$-subalgebra A' of $A = \mathcal{O}_X(X)$ such that X embeds into* $\operatorname{Spec} A'$ *as an open subset?*

2.2. *Transitivity versus flexibility on quasi-affine varieties*

Let $X = \operatorname{Spec} A$ be an affine variety. By the main result in [1] the flexibility of X is equivalent to the transitivity of $\operatorname{SAut}(X)$ on X_{reg}, which in turn is equivalent to infinite transitivity. We will need this and related facts in the more general setting of quasi-affine varieties.

We will state the necessary results in the generality that we need below. The proofs in [1] can be carried over to our more general quasi-affine setup without any difficulty. Let us start with the main result of [1] (see 1.11 and 2.2 there).

Theorem 2.12. *Let X be a smooth quasi-affine variety of dimension ≥ 2, and let $G = \langle \mathcal{N} \rangle$ be a subgroup of* $\operatorname{SAut}(X)$ *generated by a Λ-saturated set $\mathcal{N} \subseteq \operatorname{LND}_\Lambda(X)$ as in Notation 2.3 and 2.5. Then the following are equivalent:*

(i) *X is $G_{\mathcal{N}}$-flexible.*
(ii) *$G_{\mathcal{N}}$ acts transitively on X.*
(iii) *$G_{\mathcal{N}}$ acts infinitely transitively on X.*

In the proof of Theorem 2.6 we use the following auxiliary results. They are established in [1, 2.5, 4.19, and 4.2] in the case of affine schemes X and reduced subvarieties Y of X. The proofs given there immediately carry over to our more general situation.

Proposition 2.13. *Let X and $G_{\mathcal{N}}$ be as in Theorem 2.12, and let Y be a closed subscheme of X. If X is $G_{\mathcal{N}}$-flexible[4] then:*

(1) *The group $G_{\mathcal{N},Y}$ acts on $X \setminus Y$ with a dense open orbit, say O_Y, which consists of all $G_{\mathcal{N},Y}$-flexible points of $X \setminus Y$. Consequently, the $G_{\mathcal{N},Y}$-action on O_Y is infinitely transitive.*
(2) *If Y is finite then $O_Y = X \setminus Y$.*

[4] Equivalently, if $G_{\mathcal{N}}$ acts transitively on X.

(3) *If $x \in X$ then the image of the tangent representation $G_{\mathcal{N},x} \to \mathbf{GL}(T_x X)$ given by the differential coincides with the special linear group $\mathbf{SL}(T_x X)$.*

Finally, we need the following interpolation result (see [1, Theorem 4.14 and Remark 4.16]).

Proposition 2.14. *Let X and $G_{\mathcal{N}}$ be as in Theorem 2.12. If G acts transitively on X then for any finite subset $Z \subseteq X$ there exists an automorphism $g \in G$ with $g(x) = x$ for all $x \in Z$ and with prescribed tangent map $d_x g \in \mathbf{SL}(T_x X)$ at all $x \in Z$.*[5]

2.3. *Generation of subgroups by LND's*

Let as before X be a quasi-affine algebraic variety of dimension $n \geq 2$ equipped with an open embedding into an affine $\Bbbk$-scheme Spec Λ, where $\Lambda \subseteq \mathcal{O}_X(X)$. Given a set $\mathcal{N} \subseteq \mathrm{LND}_\Lambda(X)$ of locally nilpotent derivations we enrich it by adding all the Λ-replicas of derivations in $\mathcal{N}$. Letting $\tilde{\mathcal{N}}$ be this enlarged set, we consider the subgroup $\langle\langle \mathcal{N} \rangle\rangle := \langle \tilde{\mathcal{N}} \rangle$ of the group $\mathrm{Aut}(X)$ generated by $\tilde{\mathcal{N}}$.

In this section we prove the following result.

Proposition 2.15. *Let $G = \langle \mathcal{N} \rangle \subseteq \mathrm{SAut}(X)$ be the subgroup generated by a Λ-saturated set $\mathcal{N}$ of locally nilpotent derivations. Suppose that G acts transitively on X. Then for any locally nilpotent derivation $\delta_0 \in \mathcal{N}$ one can find another one $\delta_1 \in \mathcal{N}$ such that the subgroup*

$$H = \langle\langle \delta_0, \delta_1 \rangle\rangle \tag{2.1}$$

generated by δ_0, δ_1 and all their replicas acts with an open orbit on X.

To deduce this result let us recall a few facts. Let U be a one-parameter unipotent subgroup with an infinitesimal generator $\delta \in \mathrm{LND}_\Lambda(X)$ (see Notation 2.3). By assumption, X is contained as an open subset in Spec Λ, and by Lemma 2.10 even in Spec Λ' for some δ-stable finitely generated subalgebra Λ' of Λ. By the Rosenlicht Theorem (see [16, Theorem 2.3]) one can find a finite set of U-invariant functions $f_1, \ldots, f_m \in \Lambda'^U$ which separate general U-orbits. Let B be the integral closure of the finitely generated $\Bbbk$-algebra $\Bbbk[f_1, \ldots, f_m]$. It is a standard result that B is again finitely generated (see e.g. [5, Theorem 4.14]).

Definition 2.16. The normal affine variety $Q_U = \mathrm{Spec}\, B$ will be called a *partial quotient* of X by U. In general it depends on the choice of the functions $f_1, \ldots, f_m$.[6] The inclusion $B \hookrightarrow \mathcal{O}_X(X)$ defines a dominant morphism $\varrho_U : X \to Q_U$ such that the general fibers of ϱ_U are general orbits of U.

[5] In fact, this proposition holds more generally for any finite collection of m-jets provided these jets fix the corresponding points and preserve local volume forms on X at these points (see [1, Remark 4.16]).

[6] Alternatively, one could use the *Winkelmann quotient* [19]. This quasi-affine quotient is canonically defined, but has the disadvantage of being nonaffine in general.

Proof of Proposition 2.15. Let as before $\varrho_0 : X \to Q_0$ be a partial quotient of X by U^0, where $\dim Q_0 = n-1$. Since $n \geq 2$ there exists $\sigma \in \mathcal{N}$ such that $\ker\sigma \neq \ker\delta_0$, and so U^0 and $U = \exp(\mathbb{C}\sigma)$ have different general orbits. We can choose $x \in X$ such that the tangent vector $\delta_{0,x}$ of δ_0 at x is nonzero, hence $\dim U^0.x = 1$. If we choose x in an appropriate way, there are points $x_1, \dots, x_{n-1}$ on the orbit $U^0.x$ such that the vectors $v_i' = \sigma_{x_i} \in T_{x_i}X$ are all nonzero. Letting $q = \varrho_0(x) \in Q_0$ we fix for each $i = 1, \dots, n-1$ a tangent vector $v_i \in T_{x_i}X$ in such a way that the vectors $d\varrho_0(v_i) \in T_q Q_0$, $i = 1, \dots, n-1$, generate the tangent space $T_q Q_0$ to Q_0 at q. For every $i = 1, \dots, n-1$ we can choose a 1-jet of a local automorphism at x_i that fixes x_i and sends v_i' to v_i. This amounts to choosing $\alpha_i \in \mathbf{SL}(T_{x_i}X)$ such that $\alpha_i(v_i') = v_i$. According to Proposition 2.14 one can interpolate these jets by an automorphism, say $\alpha \in G$, such that $\alpha(x_i) = x_i$ and $d\alpha(v_i') = v_i$ for $i = 1, \dots, n-1$. Replacing U by

$$U^1 = \alpha \circ U \circ \alpha^{-1} = \exp(\mathbb{C}\delta_1) \in \mathcal{U}(G),$$

we obtain a one-parameter unipotent subgroup with tangent vector v_i at x_i, $i = 1, \dots, n-1$. We claim that the locally nilpotent derivation δ_1 satisfies our requirement. Indeed, $\delta_1 \in \mathcal{N}$ since $\mathcal{N}$ is saturated, and so in particular closed under conjugation in G. Consider the conjugate one-parameter subgroups

$$U_i^1 = \alpha_i^{-1} \circ U^1 \circ \alpha_i = \exp(\mathbb{C}\sigma_i) \in \mathcal{U}(H), \qquad i = 1, \dots, n-1,$$

where $\alpha_i \in U^0$ is an element which maps x to x_i. Here H is as in (2.1) and σ_i is a conjugate of δ_1 under the action of H for $i = 1, \dots, n-1$. For any i in this range the vector $u_i = d\alpha_i(v_i)$ is tangent to the orbit $U_i^1.x$ at $x \in X$. Furthermore, the vectors $d\varrho_0(u_i) = d\varrho_0(v_i) \in T_q Q_0$, $i = 1, \dots, n-1$, still generate $T_q Q_0$. Hence the vectors

$$u_0 = \delta_{0,x},\ u_1 = \sigma_{1,x},\ \dots,\ u_{n-1} = \sigma_{n-1,x} \in T_x X$$

span $T_x X$ as well. Consequently, x is an H-flexible point, and so the H-orbit $H.x$ is open and dense in X (see [1, Corollary 1.11(a)]). □

3. m-blowups, tangency, and m-contractions

This section is technical; we use its results and notions (see especially Definitions 3.5 and 3.8 and Proposition 3.15) in the proof of Proposition 4.11 in the next section.

3.1. In the following we deal with rational maps $g : X \dashrightarrow Y$ which fit into a diagram

$$\begin{tikzcd} & \hat{X} \arrow[dl, "h"'] \arrow[dr, "g'"] & \\ X \arrow[rr, dashed, "g"] & & Y \end{tikzcd}$$

where h is a sequence of blowups and g' is a proper morphism. This somewhat restricted class of rational maps is suitable for our purposes. Given subsets $A \subseteq X$ and $B \subseteq Y$ we let

$$g(A) = g'(h^{-1}(A)) \quad \text{and} \quad g^{-1}(B) = h(g'^{-1}(B))$$

denote the total image and preimage, respectively.[7] Since any two resolutions of the indeterminacy set are dominated by a third one, the total image and the total preimage are well defined.

3.1. m-blowups and tangency

In the next definition we introduce a setup which is used repeatedly in this and the next section.

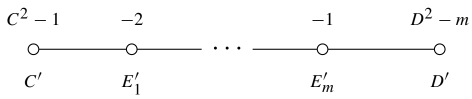
Definition 3.2. Let X be an algebraic variety and C, D be divisors in X, which are Cartier near $C \cap D$. The *m-blowup* $\sigma_m : X_m \to X$ *of D along C* is defined recursively as follows. With $X_0 = X$ we let X_1 be the blowup of X along the subscheme $C \cap D$. If X_{m-1} is already defined for some $m \geq 2$, then we let $X_m \to X_{m-1}$ be the blowup along $D^{(m-1)} \cap E_{m-1}$, where $D^{(m-1)}$ is the proper transform of D in X_{m-1} and E_{m-1} the exceptional set of the previous blowup $X_{m-1} \to X_{m-2}$.

In the following we call the proper transforms

$$E'_1, \ldots, E'_m \subseteq X' = X_m$$

of the exceptional sets E_i of $X_i \to X_{i-1}$ the *exceptional sets of the m-blowup* of D along C. The proper transforms of C and D will always be denoted C', D', respectively.

Example 3.3. Suppose that S is a complete smooth surface and $C \cap D = \{p\}$, where the intersection is transversal. Then the dual graph of $C' \cup E'_1 \cup \cdots \cup E'_m \cup D'$ is a linear chain:



(3.1)

Let us next consider the effect of an m-blowup as in Definition 3.2 on the boundary of a closed subset of X.

Proposition 3.4. *We keep the notation and assumptions as in Definition 3.2. Given a closed subset $Y \subseteq X$ we let Y' denote its proper transform in X', and $\partial Y'$ its boundary $\partial Y' = Y' \cap \sigma_m^{-1}(C \cup D)$. Then with $P = \overline{Y \cap D \setminus C}$, for $m \gg 0$,*

$$\partial Y' \setminus C' \subseteq E'_1 \cup \cdots \cup E'_{m-1} \cup \sigma_m^{-1}(P).$$

[7] These notions should be treated with caution, because they are not compatible with composition of rational maps.

Proof. The assertion is local around points in $\overline{C\cap D\setminus P}$. Thus we may assume that $P=\emptyset$, $X=\operatorname{Spec} A$ is affine, and that $D=V(x)$, $C=V(y)$ with some functions $x, y\in A$. The subset

$$U'=X'\setminus\bigcup_{i=0}^{m-1}E'_i$$

of X' is affine with coordinate ring $A'=A[u]$, where $u=x/y^m$ (cf. Lemma 3.10 below for the special case of surfaces). Furthermore

$$U'\cap E'_m=\{y=0\}\quad\text{and}\quad U'\cap D'=\{u=0\}.\tag{3.2}$$

If $I\subseteq A$ is the ideal of Y then $B=A/I$ is the affine coordinate ring of Y. Since $\overline{Y\cap D\setminus C}=\emptyset$, the set $Y\cap D$ is contained in $C\cap D$ and so the localization $(B/xB)_y$ is zero. Hence there exists a natural number m such that $y^{m-1}\in xB$. In other words, we can find $a\in A$ such that

$$y^{m-1}-a\cdot x\in I.\tag{3.3}$$

In the blowup ring A' the ideal I' of Y' is given by

$$I'=\{g\in A'\mid\exists k\in\mathbb{N}: y^k g\in IA'\}.$$

Since $u=x/y^m$, condition (3.3) can be rewritten in the form

$$y^{m-1}\cdot(1-yau)\in IA'.$$

Hence $1-yau\in I'$. This shows that in the affine coordinate ring $B'=A'/I'$ of $U'\cap Y'$ the residue classes of y and u are units. In view of (3.2) this implies that

$$U'\cap Y'\cap E'_m=\emptyset\quad\text{and}\quad U'\cap Y'\cap D'=\emptyset,$$

which immediately yields the required result. □

Definition 3.5. We say that a closed subset Y of X is *at most m-tangent* to D along C if the conclusion of Proposition 3.4 holds with this particular value of m. The subset $N=C\cap\overline{Y\cap D\setminus C}$ of $C\cap D$ will be called the *defect set*.

We note that if Y is at most m-tangent to C along D then it is also at most m'-tangent to C along D for all $m'\geq m$. The following observation is important.

Lemma 3.6. *If* $\operatorname{codim}_X Y\geq 1$ *and* $Y\setminus D$ *is dense in* Y *then the defect set* N *is nowhere dense in* $C\cap D$.

Proof. If $\operatorname{codim}_X Y\geq 1$ then the set $Y\cap D$ has codimension ≥ 1 in D. Hence its closure cannot contain any component of $C\cap D$. □

Remark 3.7. In the setup of Proposition 3.4, suppose that $(Y_s)_{s\in S}$ is a family of proper closed subsets of X. Then there is a natural m such that Y_s is at most m-tangent to D along C for any $s\in S$.

This follows easily from the fact that the construction of Proposition 3.4 can be done al least generically in the given family and that it is then compatible with restriction to the general fiber. More precisely, one can find an open dense subset $U \subseteq S$ such that all fibers Y_s are at most m-tangent to D along C with m independent of $s \in U$, and with a defect set $N_s = C \cap \overline{Y_s \cap D \setminus C}$. Restricting the family to $S' = S \setminus U$ and applying induction on $\dim S$, we may assume that Y_s is at most m-tangent to D along C for any $s \in S'$. Hence the assertion follows.

3.2. *m-contractions*

Definition 3.8. Let C, D be divisors on an the algebraic variety X which are Cartier near $C \cap D$. Consider a birational map $g : X \dashrightarrow X$ and a resolution of the indeterminacy set of g which factors through the m-blowup $\sigma_m : X' = X_m \to X$ of D along C (see Definition 3.2):

$$\begin{tikzcd} & \hat{X} \arrow[dl, "h_m"'] \arrow[d, "h"] \arrow[dr, "g'"] & \\ X' = X_m \arrow[r, "\sigma_m"] & X \arrow[r, dashed, "g"] & X \end{tikzcd}$$

g is called an *m-contraction for C along D* if:

- g is biregular at the points of $X \setminus C$;
- with $g_m = g \circ \sigma_m$, the total image[8] $g_m(C' + E'_1 + \cdots + E'_{m-1})$ is a subset of D, where $E'_1, \ldots, E'_m$ are as in Definition 3.2.

Clearly, an m-contraction for C along D is also an m'-contraction for C along D for any $m' \le m$. The following example is important and serves as a model case.

Notation 3.9. Let $\Gamma = (\Gamma, o)$ be a germ of a smooth affine curve with a uniformizing parameter u such that $u(o) = 0$, and let $d(u)$ denote a nowhere vanishing function on Γ. We consider homogeneous coordinates $(\zeta_1 : \zeta_2)$ on $\mathbb{P}^1$ and an affine coordinate $v = \zeta_1/\zeta_2$ on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{(1:0)\}$. The product $S := \Gamma \times \mathbb{P}^1$ is a $\mathbb{P}^1$-fibered surface over Γ. Its fiber, say C, over $o \in \Gamma$ and the section $D = \Gamma \times \{(0:1)\} \subseteq \Gamma \times \mathbb{A}^1$ can be described in coordinates by

$$C = \{u = 0\} \quad \text{and} \quad D = \{v = 0\}.$$

Let us study the rational map $g_m : S \dashrightarrow S$, where $m \in \mathbb{N}$, given in affine coordinates by

$$g_m(u, v) = \left(u, \frac{u^m v}{d(u)v + u^m}\right). \tag{3.4}$$

Its indeterminacy set consists of the intersection point $C \cap D = \{u = v = 0\}$, which will be denoted by $\bar{0}$.

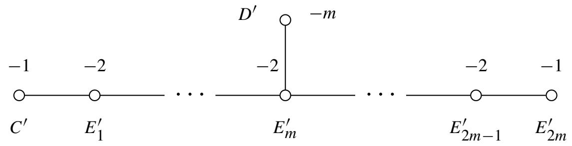
---

[8] See 3.1.

Lemma 3.10. *Let*

$$\begin{tikzcd} & S' \arrow[dl, "\sigma_m"'] \arrow[dr, "g'_m"] & \\ S \arrow[rr, dashed, "g_m"] & & S \end{tikzcd}$$

be the minimal resolution of indeterminacies of g_m, where σ_m is a sequence of blowups and g'_m is a morphism. Then the total transform of $C + D$ on S' under σ_m has weighted dual graph



(3.5)

where C' and D' are the proper transforms of C and D, respectively. The map σ_m contracts the components $E'_1, \ldots, E'_{2m}$ to the origin $\bar{0} \in S$, while g'_m contracts the curves $C', E'_1, \ldots, E'_{2m-1}$ to $\bar{0} \in S$. Furthermore $g'_m(D') = D$ and $g'_m(E'_{2m}) = C$.

Proof. Letting $v_0 = v$ we define a sequence of coordinate charts (u, v_i) on S', $i = 0, \ldots, 2m$, so that the $2m$ blowdowns over the origin with exceptional curves $E'_1, \ldots, E'_{2m}$ that constitute the map

$$\sigma : (u, v_{2m}) \mapsto (u, v_{2m-1}) \mapsto \cdots \mapsto (u, v_1) \mapsto (u, v)$$

can be described by the formulae

$$v_1 = v/u, \quad v_2 = v_1/u = v/u^2, \; \ldots, \; v_m = v_{m-1}/u = v/u^m, \tag{3.6}$$

and

$$v_{m+1} = (1+d(u)v_m)/u, \quad v_{m+2} = v_{m+1}/u, \; \ldots, \; v_{2m} = v_{2m-1}/u = (1+d(u)v_m)/u^m. \tag{3.7}$$

The map g_m can be written in these coordinate charts as

$$\begin{aligned}(u, v) \mapsto \left(u, \frac{u^m v}{d(u)v + u^m}\right) &= \left(u, \frac{u^m v_1}{d(u)v_1 + u^{m-1}}\right) = \cdots = \left(u, \frac{u^m v_m}{1 + d(u)v_m}\right) \\ &= \left(u, \frac{d(u)u^m v_{m+1} - u^{m-1}}{v_{m+1}}\right) = \cdots = \left(u, \frac{d(u)u^m v_{2m} - 1}{v_{2m}}\right).\end{aligned}$$

Hence the curve E'_i given in the chart (u, v_i) by the equation $u = 0$ is contracted under g'_m for every $i = 0, \ldots, 2m - 1$, while the curve E'_{2m} given by the same equation in the chart (u, v_{2m}) maps birationally onto the curve C in S. Now the assertion follows. □

An immediate consequence is the following corollary.

Corollary 3.11. *The birational map g_m in (3.4) is an m-contraction of C along D.*

Note that g_m is not an $(m+1)$-contraction of C along D. This example can be generalized to higher dimensions as follows.

Notation 3.12. Instead of a curve Γ in 3.9 we now consider a smooth affine algebraic variety Q and a smooth divisor $T \subseteq Q$ given by the equation $\{u = 0\}$, where $u \in \mathcal{O}_Q(Q)$. The product $X = Q \times \mathbb{P}^1$ is $\mathbb{P}^1$-fibered over Q and contains the divisors

$$C = T \times \mathbb{P}^1 \quad \text{and} \quad D = Q \times \{(0:1)\} \subseteq Q \times \mathbb{A}^1,$$

where we equip $\mathbb{P}^1$ with homogeneous coordinates $(\zeta_1 : \zeta_2)$. As before, $v = \zeta_1/\zeta_2$ stands for an affine coordinate on $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{(1:0)\}$. Thus we have

$$C = \{u = 0\} \quad \text{and} \quad D = \{v = 0\}.$$

Lemma 3.13. *Given a nowhere vanishing function $d(q)$ on Q and $m \in \mathbb{N}$, the rational map*

$$g_m : X \dashrightarrow X, \quad \text{where} \quad g_m(q, v) = \left(q, \frac{u(q)^m v}{d(q)v + u(q)^m}\right), \tag{3.8}$$

is an m-contraction of C along D.

Proof. A resolution

$$\begin{tikzcd} & X' \arrow[dl, "\sigma_m"'] \arrow[dr, "g'_m"] & \\ X \arrow[rr, dashed, "g_m"] & & X \end{tikzcd}$$

of the indeterminacy points of g_m can be obtained (with obvious changes) by the same sequence of blowups as in the proof of Lemma 3.10. Letting $v_0 = v$ we define a sequence of coordinate charts $(q, v_i) \in U_i = Q \times \mathbb{A}^1$ on X', $i = 0, \ldots, 2m$, so that the $2m$ blowdowns over $C \cap D$ with exceptional divisors $E'_1, \ldots, E'_{2m}$ that constitute the map

$$\sigma : (q, v_{2m}) \mapsto (q, v_{2m-1}) \mapsto \cdots \mapsto (q, v_1) \mapsto (q, v)$$

can be described by the formulae in (3.6) and (3.7), where u is now the function $u(q)$. With the same calculation as before, the map g_m can be written in these coordinate charts as

$$(q, v) \mapsto \left(q, \frac{d(q)u(q)^m v_{2m} - 1}{v_{2m}}\right).$$

As in the proof of 3.10, the exceptional set E'_i is given in the chart U_i by the equation $u = 0$, and it is contracted under g'_m to the subset $C \cap D$ for every $i = 0, \ldots, 2m-1$. Finally, the exceptional set E'_{2m} given by $\{u = 0\}$ in the chart U_{2m} maps under g'_m isomorphically onto the divisor C in X. Since the divisors $C', E'_1, \ldots, E'_{m-1}$ in X' are contracted under g'_m to $C \cap D$, the result follows. □

Next we show that m-contractions are compatible with certain blowups.

Proposition 3.14. *Let X be an algebraic variety and C, D be connected divisors on X, which are Cartier near $C \cap D$. Let $g : X \dashrightarrow X$ be an m-contraction of C along D and $p : Z \to X$ be a modification, which is an isomorphism over $D \cup (X \setminus C)$. Then the rational map $f : Z \dashrightarrow Z$ induced by g is an m-contraction of $C_Z = p^{-1}(C)$ along $D_Z = p^{-1}(D) \cong D$.*

Proof. Let $X_m \to X$ and $Z_m \to Z$ be the m-blowups of X and Z, respectively. Since p is an isomorphism at the points near D, the exceptional sets $E'_1, \ldots, E'_m$ of $X_m \to X$ can be identified in a natural way with the exceptional sets, say $E'_{1,Z}, \ldots, E'_{m,Z}$, of $Z_m \to Z$. Consider the composed rational maps

$$Z'_m \overset{f_m}{\dashrightarrow} Z \quad \text{and} \quad X'_m \overset{g_m}{\dashrightarrow} X$$

and a diagram

$$\begin{tikzcd} & \hat{Z} \arrow[dl, "h_m"'] \arrow[dr, "f'_m"] & \\ Z'_m \arrow[rr, dashed, "f_m"] \arrow[d, "p'"'] & & Z \arrow[d, "p"] \\ X'_m \arrow[rr, dashed, "g_m"] & & X \end{tikzcd}$$

where $\hat{Z}$ is a resolution of the indeterminacy locus of f_m and then also of g_m. By our assumption the set

$$(p' \circ h_m)^{-1}(C' \cup E'_1 \cup \cdots \cup E'_{m-1}) = h_m^{-1}(C'_Z \cup E'_{1,Z} \cup \cdots \cup E'_{m-1,Z})$$

is contracted under $p \circ f'_m$ to a subset of D. Since p is an isomorphism near D, the latter set is already contracted under f'_m to a subset of D. This proves the assertion. □

Let us now study the effect of an m-contraction of C along D on the boundary of a closed subset Y of X.

Proposition 3.15. *Let X be an algebraic variety and C, D divisors on X, which are Cartier near $C \cap D$. Assume that $g : X \dashrightarrow X$ is an m-contraction of C along D and that $Y \subseteq X$ is a closed subset which is at most m-tangent to C along D with defect set $N = C \cap \overline{Y \cap D \setminus C}$. Then the proper image $\hat{Y}$ of Y under g satisfies*

$$\partial \hat{Y} \subseteq D \cup g(N),$$

where $g(N)$ is the total image of N, and $\partial \hat{Y}$ denotes the intersection of $\hat{Y}$ with $D \cup C$.

Proof. Let $\sigma_m : X' = X_m \to X$ be the m-blowup of C along D with exceptional sets $E'_1, \ldots, E'_m$ and consider the composition $g_m = g \circ \sigma_m : X' \dashrightarrow X$. We can find a resolution of the indeterminacy locus of g_m

$$\begin{tikzcd} & \hat{X} \arrow[dl, "h_m"'] \arrow[dr, "g'_m"] & \\ X' \arrow[rr, dashed, "g_m"] & & X \end{tikzcd}$$

Since Y is at most m-tangent to C along D, the boundary $\partial Y'$ of the proper transform Y' of Y in X' satisfies

$$\partial Y' \subseteq E'_1 \cup \cdots \cup E'_{m-1} \cup \sigma_m^{-1}(P),$$

where $P = \overline{Y \cap D \setminus C}$ (see Proposition 3.4). By condition (2) in Definition 3.8,

$$h_m^{-1}(C' \cup E'_1 \cup \cdots \cup E'_{m-1})$$

is contracted under g'_m to a subset of D. Hence

$$g'_m(h_m^{-1}(\partial Y')) \subseteq D \cup g'_m(h_m^{-1}(\sigma_m^{-1}(P))) = D \cup g(P).$$

Since g'_m is proper, the set on the right is easily seen to contain $\partial \hat{Y}$, as stated. □

4. Replicas as m-contractions

Notation 4.1. (a) Let X be a smooth quasi-affine algebraic variety and $G_{\mathcal{N}}$ a group of automorphisms on X generated by a set of Λ-saturated locally nilpotent derivations $\mathcal{N} \subseteq \mathrm{LND}_\Lambda(X)$ (see Notation 2.3 and 2.5). Suppose that $G_{\mathcal{N}}$ acts transitively on X.

(b) We choose two locally nilpotent derivations $\delta, \delta_0 \in \mathrm{LND}_\Lambda(X)$ such that

$$\ker \delta \neq \ker \delta_0.$$

Let U, U^0 denote the associated one-parameter subgroups and choose partial quotients

$$\varrho : X \to Q \quad \text{and} \quad \varrho_0 : X \to Q_0$$

as introduced in 2.16.

(c) We can embed Q and Q_0 into normal projective varieties $\bar{Q}$ and $\bar{Q}_0$, respectively. Let $\bar{X}$ be a smooth projective completion of X. After blowing up $\bar{X}$ in the boundary $\partial X = \bar{X} \setminus X$, if necessary, we may extend ϱ and ϱ_0 to morphisms

$$\begin{tikzcd} \bar{X} \arrow[r, "\bar{\varrho}_0"] \arrow[d, "\bar{\varrho}"] & \bar{Q}_0 \\ \bar{Q} & \end{tikzcd}$$

The general fiber of ϱ is an orbit of U isomorphic to $\mathbb{A}^1$. Clearly $\bar{\varrho}^{-1}(q) \cong \mathbb{P}^1$ for a general point $q \in Q$. Hence there is a unique divisor $D \subseteq \bar{X} \setminus X$ which maps birationally onto $\bar{Q}$. Similarly there is a unique divisor D_0 in $\bar{X} \setminus X$ mapping birationally onto $\bar{Q}_0$. Thus both D and D_0 are contained in the boundary $\partial X = \bar{X} \setminus X$.

The following observations will be important.

Lemma 4.2. (a) *Let $\varphi \in \ker\delta \setminus \ker\delta_0$ be a regular function on X. Then φ is a rational function on $\bar{X}$ with poles at general points of D_0.*
(b) *$\bar{\varrho}(D_0) \subseteq \bar{Q} \setminus Q$ and $\bar{\varrho}_0(D) \subseteq \bar{Q}_0 \setminus Q_0$. In particular, $D \neq D_0$.*

Proof. (a) Since $D_0 \to \bar{Q}_0$ is dominant, an orbit closure $\overline{H_0.x}$ of a general point $x \in X$ meets D_0 at a general point $\bar{x} \in D_0$. Let us consider φ as a rational map $\bar{X} \dashrightarrow \mathbb{P}^1$. Since the indeterminacy set of φ on $\bar{X}$ is of codimension at least 2, φ is regular on the orbit closure $\overline{H_0.x} \cong \mathbb{P}^1$ for a general $x \in X$. Since $\varphi \notin \ker\delta_0$, this map is not constant on general orbits of H_0. In particular it restricts to a dominant morphism $\varphi : \overline{H_0.x} \to \mathbb{P}^1$ such that $\varphi(\bar{x}) = \infty$.

(b) It is sufficient to prove the first part. If $\bar{\varrho}(D_0) \cap Q \neq \emptyset$ then a function $\varphi \in \mathcal{O}(Q) \setminus \ker\delta_0$ would be holomorphic at a general point of D_0, contradicting (1). □

Lemma 4.3. *After blowing up the boundaries $\partial X = \bar{X} \setminus X$ and $\partial Q = \bar{Q} \setminus Q$ suitably we can achieve that*

(a) *$T = \bar{\varrho}(D_0)$ is a divisor in $\bar{Q}$, and*
(b) *$\bar{X}$, D and D_0 are smooth.*

Proof. (a) By Lemma 4.2(b), T sits in the boundary of $\bar{Q}$. According to a theorem of Zariski (see [20] and [13, Chapter VI, Theorem 1.3][9]), there is a blowup $\bar{Q}' \to \bar{Q}$ with a center in $\bar{\varrho}(D_0)$ such that the proper transform of D_0 in $\bar{X}_{\bar{Q}'}$ maps onto a divisor in $\bar{Q}'$. Thus replacing $\bar{Q}$ by $\bar{Q}'$ we can achieve that T is a divisor.

Since X is smooth and does not meet $D \cup D_0$, by a suitable blowup of the boundary $\bar{X} \setminus X$ we can achieve that (b) holds. □

Lemma 4.4. *There is a closed subset B_0 of $\bar{Q}$ with $\operatorname{codim}_{\bar{Q}} B_0 \geq 2$ such that*

(a) *$\operatorname{Sing}\bar{Q} \cup \operatorname{Sing} T \subseteq B_0$,*
(b) *$D \to \bar{Q}$ is an isomorphism at the points of $D \setminus \bar{\varrho}^{-1}(B_0)$, and*
(c) *$\bar{X} \to \bar{Q}$ is flat at the points over $\bar{Q} \setminus B_0$.*

Proof. (a) can be satisfied as $\bar{Q}$ is normal and T is reduced. Since $D \to \bar{Q}$ is a birational map, also (b) can be achieved.

(c) By the theorem on generic flatness [5, Theorem 14.4] there is a proper closed subset E in $\bar{Q}$ such that $\bar{\varrho}$ is flat at the points over $\bar{Q} \setminus E$. Applying the theorem on generic flatness again shows that $\bar{\varrho}|E : \bar{\varrho}^{-1}(E) \to E$ is flat over a subset $E \setminus B'$ of E, where B' is a nowhere dense closed subset of E. Using [5, Corollary 6.9] it follows that f is flat over $\bar{Q} \setminus B''$, where

$$B'' = B' \cup \{s \in E \mid E \text{ is not a Cartier divisor in } \bar{Q} \text{ at } x\}.$$

Since $\bar{Q}$ is normal, this set has codimension ≥ 2 in $\bar{Q}$. After adding B'' to B_0, also (c) is satisfied. □

[9] We are grateful to Y. Prokhorov for pointing out to us this reference.

The following facts should be well known; for lack of a reference we provide a brief argument.

Lemma 4.5. *Let $p : S \to \Gamma$ be a $\mathbb{P}^1$-fibration of a smooth surface S over a smooth affine curve Γ admitting a smooth section $D \subseteq S$ so that $D \cong \Gamma$. Then for any point $t \in \Gamma$ the fiber $F = p^{-1}(t)$ is a tree of rational curves. Furthermore:*

(a) *If $\{x\} = F \cap D$ then $h^0(F, \mathcal{O}_F(x)) = 2$ and $H^i(F, \mathcal{O}_F(x)) = 0$ for $i \geq 1$.*
(b) *The sheaf $\mathcal{O}_F(x)$ is generated by its global sections.*
(c) *If $s_0, s_1 \in H^0(F, \mathcal{O}_F(x))$ is a basis, then the map $(s_0 : s_1) : F \to \mathbb{P}^1$ is an isomorphism near x.*

Proof. Blowing down successively (-1)-curves in the fibers of p not meeting D we obtain a locally trivial $\mathbb{P}^1$-bundle $\mathcal{V} \to \Gamma$. The curve D can also be considered as a section of $\mathcal{V} \to \Gamma$ and so we have an isomorphism $\mathcal{V} \cong \mathrm{Proj}_\Gamma(p_*(\mathcal{O}_\mathcal{V}(D)))$. If $S = \mathcal{V}$ then assertions (a)–(c) are trivial. By blowing up subsequently points in the fibers we prove these assertions also for $p : S \to \Gamma$. □

In what follows we may assume that conditions (a), (b) in Lemma 4.3 are satisfied.

Lemma 4.6. *Let $\bar{X}_q = \bar{\varrho}^{-1}(q)$ and $D_q = D \cap \bar{X}_q$. Then there is a closed subset B of codimension ≥ 2 in $\bar{Q}$ such that for $q \in \bar{Q} \setminus B$ the following assertions hold:*

$(\mathrm{a})_q$ *$h^0(\bar{X}_q, \mathcal{O}_{\bar{X}_q}(D_q)) = 2$ and $H^i(\bar{X}_q, \mathcal{O}_{\bar{X}_q}(D_q)) = 0$ for $i \geq 1$.*
$(\mathrm{b})_q$ *The sheaf $\mathcal{O}_{\bar{X}_q}(D_q))$ is generated by its global sections.*
$(\mathrm{c})_q$ *If $s_0, s_1 \in H^0(\bar{X}_q, \mathcal{O}_{\bar{X}_q}(D_q))$ is a basis, then the map $(s_0 : s_1) : \bar{X}_q \to \mathbb{P}^1$ is an isomorphism near D_q.*
$(\mathrm{d})_q$ *The map $\bar{\varrho}_*(\mathcal{O}_{\bar{X}}(D))_q \to H^0(\bar{X}_q, \mathcal{O}_{\bar{X}_q}(D_q))$ is surjective, and $\bar{\varrho}_*(\mathcal{O}_{\bar{X}}(D))_q$ is free of rank 2.*

Proof. Let $B_0 \subseteq \bar{Q}$ be a set as in Lemma 4.4. We choose a proper closed subset P of $\bar{Q}$ such that any fiber over $\bar{Q} \setminus P$ is isomorphic to $\mathbb{P}^1$. For any $q \in \bar{Q} \setminus P$ assertions $(\mathrm{a})_q$–$(\mathrm{d})_q$ follow easily.

Let a curve Γ in $\bar{Q}$ be the intersection of $n-1$ general ample divisors in $\bar{Q}$. Since $\bar{Q}$ is normal and codim $B_0 \geq 2$, Γ meets neither Sing $\bar{Q}$ nor B. By Bertini's theorem both Γ and the surface $S = \bar{\varrho}^{-1}(\Gamma)$ are smooth. The restriction $\bar{\varrho}|S : S \to \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration. This $\mathbb{P}^1$-fibration admits a section, namely $D \cap S$. The intersection $D \cap S$ is smooth in view of Bertini's theorem and Lemma 4.3(b). The fiber of $S \to \Gamma$ over $q \in \Gamma \subseteq \bar{Q}$ coincides with $\bar{X}_q$. By Lemma 4.5, $\bar{X}_q$ is a tree of rational curves satisfying $(\mathrm{a})_q$–$(\mathrm{c})_q$. Since Γ meets every component, say P_i, of P of codimension 1 and does not meet B_0, for some $q_i \in P_i \setminus B_0$ conditions $(\mathrm{a})_{q_i}$–$(\mathrm{c})_{q_i}$ are satisfied. By semicontinuity (see [10, III, 12.8]) we obtain

$$h^j(\bar{X}_p, \mathcal{O}_{\bar{X}_p}(D_p)) \leq h^j(\bar{X}_q, \mathcal{O}_{\bar{X}_q}(D_q)) \leq h^j(\bar{X}_{q_i}, \mathcal{O}_{\bar{X}_{q_i}}(D_{q_i})), \quad j \geq 0,$$

where $q \in P_i$ is a point near q_i and $p \in \bar{Q} \setminus P$ is a point near q. Since the outer terms are equal, condition $(\mathrm{a})_q$ holds for q in some open dense subset P_i^o of P_i.

By Grauert's criterion (see [10, III, 12.9]) now also $(d)_q$ is satisfied. Since $(b)_q$ and $(c)_q$ are open conditions on P_i^o, which are satisfied for some $q \in P_i^o$, they are satisfied generically on P_i. Now the lemma follows. □

Corollary 4.7. *There is a proper closed subset $B \subseteq \bar{Q}$ containing $\operatorname{Sing} T$ and $\operatorname{Sing} \bar{Q}$ with $\operatorname{codim}_T(T \cap B) \geq 1$ such that if we let*

$$X^o = \bar{X} \setminus \bar{\varrho}^{-1}(B), \quad Q^o = \bar{Q} \setminus B, \quad T^o = T \setminus B \quad \text{and} \quad C = \bar{\varrho}^{-1}(T),$$

then there is a birational morphism

$$\varphi : X^o \to \mathcal{X} = Q^o \times \mathbb{P}^1, \tag{4.1}$$

compatible with the projection to Q^o, which restricts to a biregular morphism

$$X^o \setminus C \to \mathcal{X} \setminus \mathcal{C} = (Q^o \setminus T) \times \mathbb{P}^1, \tag{4.2}$$

where $\mathcal{C} = T^o \times \mathbb{P}^1$. Furthermore φ is biregular in a neighborhood of $D^o = D \cap X^o$.

Proof. Let $B \subseteq \bar{Q}$ be the subset constructed in Lemma 4.6. Enlarging it in a suitable way we may assume that it contains $\operatorname{Sing} T \cup \operatorname{Sing} \bar{Q}$. According to Lemma 4.6(c) the sheaf $\mathcal{E} = \bar{\varrho}_*(\mathcal{O}_{X^o}(D))$ is locally free of rank 2 on Q^o. Thus enlarging B we may suppose that $\bar{\varrho}_*(\mathcal{O}_{X^o}(D))$ is free. Choose two sections s_0, s_1 which form a basis of this bundle. They provide a morphism

$$\varphi = (\bar{\varrho}, (s_0 : s_1)) : X^o \to Q^o \times \mathbb{P}^1.$$

Restricting to a fiber over $q \in Q^o$, in view of Lemma 4.6$(c)_q$, yields an isomorphism near D_q. Hence φ is an isomorphism near D^o. Enlarging B further we may also assume that all fibers in $Q^o \setminus T$ are isomorphic to $\mathbb{P}^1$. This implies that the restricted morphism (4.2) is an isomorphism. □

Notation 4.8. Consider the restriction of the locally nilpotent vector field δ to $X^o \cap X$. The associated action of $U = \exp(\Bbbk\delta)$ has no fixed points in this set and extends to an action on $X^o \setminus C$, where as before $C = \bar{\varrho}^{-1}(T)$. The fibers of $X^o \setminus C \to Q^o \setminus T$ are preserved under U.

Under the isomorphism $X^o \setminus C \simeq \mathcal{X} = (Q^o \setminus T) \times \mathbb{P}^1$ the second factor can be equipped with a homogeneous coordinate system $(\zeta_1 : \zeta_2)$ such that the image, say $\mathcal{D}$, of $D^o = D \cap X^o$ in X^o is defined by the equation $\zeta_1 = 0$. We treat $v = \zeta_1/\zeta_2$ as a coordinate in the neighborhood $\mathcal{X} \setminus \{\zeta_2 = 0\}$ of $\mathcal{D}$ in $\mathcal{X}$.

We fix a function $f \in \Bbbk[Q]$ whose pullback on X belongs to $\ker \delta \setminus \ker \delta_0$. This pullback induces rational functions on X^o and on $\mathcal{X}$, denoted by the same symbol f. By Lemma 4.2(a), f has poles along $D_0 \cap X^o$.

By our choice of B in Corollary 4.7, T^o is a submanifold of Q^o. Thus locally the ideal of T^o is generated by some function, say u, on Q^o. On Q^o the function f is of the form a/u^s. Here $s \geq 1$ is the pole order of f along T^o, so a is a rational function on Q^o, which is nonzero at the general point of T^o.

Later on we will replace f by a sufficiently large power f^k. Thus we can achieve that the pole order s is arbitrarily large.

Recall that U_f stands for the replica of U associated with the locally nilpotent vector field $f\delta$. We note that U_f is well defined on the set

$$X^o \setminus C \cong (Q^o \setminus T^o) \times \mathbb{P}^1$$

(cf. Corollary 4.7). Its element at time $\tau \in \Bbbk$ will be denoted by $h_{f,\tau}$. Considered as an automorphism of $(Q^o \setminus T) \times \mathbb{P}^1$ it preserves the first factor but not the second one. The action of $h_{f,\tau}$ on v is described by the following lemma.

Lemma 4.9. *There exist a regular function $d = d(f)$ on Q^o, which does not vanish at general points of T, and an integer l such that the automorphism of $(Q^o \setminus T) \times \mathbb{P}^1$ defined by $h_{f,\tau}$ is given in the coordinates (q, v) by the formula*

$$h_{f,\tau} : (q, v) \mapsto \left(q, \frac{u(q)^m v}{u(q)^m + \tau d(q) v}\right),$$

where $m = s - l$. In particular $\mathcal{D} \cap \mathcal{C} = \{u = v = 0\}$ is the set of indeterminacy points of $h_{f,\tau}$.

Proof. In homogeneous coordinates $(\zeta_1 : \zeta_2)$ the action of $U = \exp(\Bbbk\delta)$ on $(Q^o \setminus T) \times \mathbb{P}^1$ is of the form $(\zeta_1 : \zeta_2) \mapsto (\zeta_1, \zeta_2 + \tau c \zeta_1)$ where c is a nonvanishing function on $Q^o \setminus T$. That is, $c = c_0 u^l$ where c_0 is a nonvanishing function on Q^o and $l \in \mathbb{Z}$. Hence $h_{f,\tau}$ is of the form $(\zeta_1 : \zeta_2) \mapsto \left(\zeta_1 : \zeta_2 + \frac{\tau d}{u^{s-l}} \zeta_1\right)$, where d does not vanish at general points of T^o. Note that $m > 0$ since $f\delta$ has a pole along D_0. Passing to the affine coordinate $v = \zeta_1/\zeta_2$ yields the desired conclusion. □

Letting s be the pole order of f along T we consider the set

$$P_f = \{q \in T \mid \text{locally } f = a/u^s \text{ with } a(q) = 0 \text{ or } a \notin \mathcal{O}_{\bar{Q},q}\}, \tag{4.3}$$

where u is as before (i.e. $u = 0$ is a local equation of T near q) and a is a rational function. This set is a proper closed subset of T. The next proposition is the main result of this section.

Proposition 4.10. *Given m and a function $f \in \Bbbk[Q] \cap \ker \delta \setminus \ker \delta_0$ there exists a positive integer k_0 such that any transformation*

$$h \in U_{f^k}, \quad h \neq \mathrm{id}, \ k \geq k_0,$$

is an m-contraction of C along D over the points of $Q^o \setminus P_f$.

Proof. Let s, l be as in Notation 4.8 and Lemma 4.9. If we choose k_0 in such a way that $m' = k_0 s - l \geq m$ then by Lemma 3.13 the map $h = h_{f^k,\tau}$ is indeed an m-contraction for any $\tau \neq 0$. □

Let now $Y \subseteq X$ be a closed subset. Consider the partial boundary

$$\partial_0 Y = \bar{Y} \cap D_0.$$

For $U \in \mathcal{U}(X)$ we let $U^* = U \setminus \{\mathrm{id}\}$. With this notation the following result holds.

Proposition 4.11. *Let the notation and conventions be as in Notation 4.1 and assume that* (a) *and* (b) *in Lemma 4.3 are satisfied. Let $(Y_{\alpha,\beta})_{(\alpha,\beta)\in A\times B}$ be a flat family of proper closed subsets of X. Suppose that there is a flat family $(E_\alpha)_{\alpha\in A}$ of proper closed subsets of D such that*

$$\partial Y_{\alpha,\beta} \cap D \subseteq E_\alpha \quad \textit{for all } (\alpha, \beta) \in A \times B.$$

Given an invariant function $f \in \ker \delta \setminus \ker \delta_0$, there is a dense open subset A^o of A and a flat family $(E'_\alpha)_{\alpha\in A^o}$ of proper closed subsets of D_0 satisfying

$$\partial_0 h.Y_{(\alpha,\beta)} \subseteq E'_\alpha \qquad \forall(\alpha, \beta) \in A^o \times B,\ \forall h \in U^*_{fk},\ \forall k \geq k_0.$$

Proof. According to Proposition 3.4 and Remark 3.7 the closure $\bar{Y}_{\alpha\beta}$ of $Y_{\alpha\beta}$ in $\bar{X}$ is at most m-tangent to D along C for $m \gg 0$ and for all $(\alpha, \beta) \in A \times B$ simultaneously. Let $N_{\alpha\beta} = C \cap \overline{D \cap Y_{\alpha\beta} \setminus C}$ denote the defect set. By Proposition 4.10 for $k \gg 0$ any map $h \in U^*_{fk}$ is an m-contraction of C along D over the points of $Q^o \setminus P_f$. By Proposition 3.15 the image $h.Y_{\alpha\beta}$ satisfies

$$\overline{h.Y_{\alpha\beta}} \cap (D^o \cup C^o) \subseteq D \cup h(N_{\alpha\beta}) \cup \bar{\varrho}^{-1}(P_f) \tag{4.4}$$

where $h(N_{\alpha\beta})$ stands for the total transform of $N_{\alpha\beta}$ under h. By our assumption the defect set $N_{\alpha\beta}$ is contained in $N_\alpha = C \cap \overline{E_\alpha \setminus C}$. Since our birational transformation h is compatible with the fibration $\bar{\varrho}$, the total image $h(N_{\alpha\beta})$ is contained in $\bar{\varrho}^{-1}(\bar{\varrho}(N_\alpha))$. Taking in (4.4) the intersection with D_0 gives

$$\partial_0(h.Y_{\alpha\beta}) \subseteq E'_\alpha = \big(D \cup \bar{\varrho}^{-1}(B \cup \bar{\varrho}(N_\alpha) \cup P_f)\big) \cap D_0,$$

where $B = \bar{Q} \setminus \bar{Q}^o$ is as in Corollary 4.7. Using the theorem on generic flatness it is easily seen that over an open dense subset A^o of A the sets E'_α form a flat family of closed subsets of D_0. This yields the assertion. ◻

5. Proof of the main theorem

5.1. *Algebraic families of automorphisms*

Following Ramanujam [17] let us introduce the following notion.

Definition 5.1. Given irreducible algebraic varieties X and A and a map $\varphi : A \to \mathrm{Aut}(X)$ we say that (A, φ) is an *algebraic family of automorphisms on X* if the induced map $A \times X \to X$, $(\alpha, x) \mapsto \varphi(\alpha).x$, is a morphism.

By abuse of notation, we will not distinguish A and its image $\varphi(A)$, and we identify $\alpha \in A$ with its image $\varphi(\alpha)$ in $\mathrm{Aut}(X)$. As in the case of group action, given a point $x \in X$ the set $A.x$ will be called the *A-orbit* of x, and the set $A_x = \{\alpha \in A \mid \alpha(x) = x\}$ the *stabilizer* of x in A. The stabilizer admits a natural linear representation $d_x : A_x \to \mathbf{GL}(T_xX)$, $\alpha \mapsto d\alpha|T_xX$, called the *tangent representation*.

The following result allows one to work with finite-dimensional algebraic families instead of infinite-dimensional groups of automorphisms.

Lemma 5.2. *Let X be a smooth quasi-affine variety and $G = G_{\mathcal{N}}$ a group of automorphisms generated by a saturated set of locally nilpotent derivations such that G acts transitively on X. Then there exists an algebraic family of automorphisms $A \subseteq G$ such that for any $x \in X$ we have*

(a) $A.x = X$ *and*
(b) $d_x(A_x) = \mathbf{SL}(T_x X)$.

Proof. According to [1, Proposition 1.5] there exist one-parameter unipotent subgroups $H_1, \dots, H_s$ of G such that with $H = H_1 \cdot \ldots \cdot H_s \subseteq G$ we have $H.x = G.x$ for any $x \in X$. In particular, (a) holds with the algebraic family $A = H$.

By [1, Theorem 4.2 and its proof], for a fixed point $x \in X$ the group $\mathbf{SL}(T_x X)$ is equal to the image $d_x(H') \subseteq \mathbf{GL}(T_x X)$ for an algebraic family $H' = H'_1 \cdot \ldots \cdot H'_r$, where $H'_1, \dots, H'_r$ are suitable one-parameter subgroups of $G_{\mathcal{N},x}$. Taking the product $A = HH'H^{-1}$, where H is as in (a) and $H^{-1} = H_s \cdot \ldots \cdot H_1$, we thus achieve that both (a) and (b) are satisfied at every point $x \in X$. ◻

Notation 5.3. (a) As before, we let X be a smooth quasi-affine variety and $G = G_{\mathcal{N}}$ a group of automorphisms generated by a saturated set of locally nilpotent derivations as in Notation 4.1(a). We suppose that G acts transitively on X. According to Theorem 2.15 there are derivations $\delta_0, \delta_1 \in \mathcal{N}$ such that the group

$$H = \langle\langle \delta_0, \delta_1 \rangle\rangle \subseteq G$$

generated by δ_0, δ_1 and their replicas acts with an open orbit on X.[10] These locally nilpotent vector fields generate one-parameter unipotent subgroups $U^0, U^1 \in \mathcal{U}(G)$. Any function $f \in \ker\delta_0 \setminus \ker\delta_1$ yields a replica U^0_f, and similarly $g \in \ker\delta_1 \setminus \ker\delta_0$ yields a replica U^1_g.

(b) To any sequence of invariant functions

$$\mathcal{F} = \{f_1, \dots, f_s, g_1, \dots, g_s\}, \quad \text{where} \quad f_i \in \ker\delta_1 \setminus \ker\delta_0 \quad \text{and} \quad g_i \in \ker\delta_0 \setminus \ker\delta_1, \tag{5.1}$$

we associate an algebraic family of automorphisms $\mathbb{A}^{2s} \to \mathrm{Aut}(X)$ defined by the product

$$U^{\mathcal{F}} = U^1_{f_s} \cdot U^0_{g_s} \cdot \ldots \cdot U^1_{f_1} \cdot U^0_{g_1} \subseteq H. \tag{5.2}$$

More generally, given a tuple $\kappa = (k_i, l_i)_{i=1,\dots,s} \in \mathbb{N}^{2s}$, the product

$$U_\kappa = U^{\mathcal{F}}_\kappa = U^1_{f_s^{k_s}} \cdot U^0_{g_s^{l_s}} \cdot \ldots \cdot U^1_{f_1^{k_1}} \cdot U^0_{g_1^{l_1}} \subseteq H \tag{5.3}$$

is also an algebraic family of automorphisms.

Corollary 5.4. *There is a finite collection $\mathcal{F}$ of invariant functions as in (5.1) such that for any sequence $\kappa = (k_i, l_i)_{i=1,\dots,s} \in \mathbb{N}^{2s}$ the algebraic family of automorphisms U_κ as in (5.3) has a dense open orbit in X. This orbit $O(U_\kappa)$ coincides with $O(H)$ and so does not depend on the choice of $\kappa \in \mathbb{N}^{2s}$.*

[10] In contrast to Notation 4.1(a), in this section the roles of δ_0 and δ_1 will be symmetric so that it is convenient to replace the former δ by δ_1.

Proof. According to [1, Proposition 1.5] there is a sequence $\mathcal{F}$ as in (5.1) such that

$$H.x = U^{\mathcal{F}}.x \quad \forall x \in X.$$

In particular, for $x \in O(H)$ the orbit $U^{\mathcal{F}}.x = O(H)$ is open in X. It is easily seen that for any $\kappa \in \mathbb{N}^{2s}$ we have $O(U_\kappa) = O(U^{\mathcal{F}}) = O(H)$. Indeed, $O(H)$ consists of all the $U^{\mathcal{F}}$-flexible points in X. Now the assertions follow. □

5.2. Proof of the main theorem

Notation 5.5. We keep the notation and assumptions of 5.3(a).

(a) Let $\varrho_0 : X \to Q_0$ and $\varrho_1 : X \to Q_1$ be partial quotients with respect to the unipotent subgroups U^0 and U^1, respectively. Let us choose open embeddings $X \hookrightarrow \bar{X}$, $Q_0 \hookrightarrow \bar{Q}_0$, and $Q_1 \hookrightarrow \bar{Q}_1$ into normal projective varieties (see Notation 4.1). We may assume that the following conditions are satisfied:

(1) ϱ_0 and ϱ_1 extend to morphisms $\bar{\varrho}_0 : \bar{X} \to \bar{Q}_0$ and $\bar{\varrho}_1 : \bar{X} \to \bar{Q}_1$. Let D_0 and D_1 as in 4.1 be the unique horizontal divisors that map birationally onto $\bar{Q}_0$ and $\bar{Q}_1$, respectively.
(2) $\bar{X}$, D_0, and D_1 are smooth (see Lemma 4.3(b)).
(3) $T_0 = \bar{\varrho}_1(D_0)$ and $T_1 = \bar{\varrho}_0(D_1)$ are divisors in $\bar{Q}_1$ and $\bar{Q}_0$, respectively (see Lemma 4.3(a)).

(b) Given a closed subscheme $Y \subseteq X$ of codimension ≥ 2, we call

$$\partial_0 Y = \bar{Y} \cap D_0 \quad \text{and} \quad \partial_1 Y = \bar{Y} \cap D_1$$

the *partial boundaries*. Furthermore O_Y will denote the open orbit of $G_{\mathcal{N},Y}$ in $X \setminus Y$.

5.6. In the course of the proof of our Main Theorem 2.6 we move the given pair (Y, x) to another one (Y_α, x_α) by means of an automorphism $\alpha \in G_{\mathcal{N}}$, where $Y_\alpha = \alpha.Y$ and $x_\alpha = \alpha.x$. In this way we can make our pair adopt the position with respect to the $\mathbb{P}^1$-fibration $\bar{\varrho}_0 : \bar{X} \to \bar{Q}_0$ so that conditions (i)–(iii) below hold.

(i) $U^0.x_\alpha \cap O_{Y_\alpha} \neq \emptyset$;
(ii) $U^0.x_\alpha \cap Y_\alpha = \emptyset$;
(iii) $\partial_0(U^0.x_\alpha) \notin \partial_0(Y_\alpha)$.

The following lemma allows one to deduce Theorem 2.6 provided that (i)–(iii) hold for any $x \in X \setminus Y$ with some $\alpha \in G$ depending on x.

Lemma 5.7. *If for a point $x \in X \setminus Y$ and for some $\alpha \in G$ conditions* (i)–(iii) *in* 5.6 *are fulfilled then $x \in O_Y$. If these conditions are fulfilled for any $x \in X \setminus Y$ with some $\alpha \in G$ depending on x, then the conclusion of Theorem* 2.6 *holds.*

Proof. Since $O_{Y\alpha} = \alpha.O_Y$ we have

$$x \in O_Y \;\Leftrightarrow\; x_\alpha \in O_{Y_\alpha}.$$

Replacing (Y, x) by (Y_α, x_α) we will assume that (i)–(iii) hold for the pair (Y, x) and $\alpha = \mathrm{id}$. We need to show that then $x \in O_Y$. Conditions (ii) and (iii) yield

$$\varrho_0(x) \in \varrho_0(O_Y) \setminus \overline{\varrho_0(Y)}.$$

Therefore there exists a regular function $h \in \mathcal{O}(Q_0)$ such that $h(\varrho_0(x)) = 1$ and h vanishes on $\varrho_0(Y)$. Replacing h by a suitable power of h we may suppose that the δ_0-invariant function $f = h \circ \varrho_0$ on X vanishes on Y. Thus the replica $U_f^0 = \exp(\Bbbk f \delta_0)$ of U^0 fixes Y pointwise, i.e. $U_f^0 \in \mathcal{U}(G_{\mathcal{N},Y})$. By (i) one can find $u \in U_f^0$ such that $u.x \in O_Y$. Hence also $x \in O_Y$, as stated. □

Thus to prove Theorem 2.6 it is enough to show that (i)–(iii) hold for every $x \in X \setminus Y$ with a suitable $\alpha \in G$ depending on x.

Lemma 5.8. *Given $x \in X \setminus Y$ and an algebraic family of automorphisms $\varphi : A \to \mathrm{Aut}(X)$, the following hold:*

(a) *The set of all $\alpha \in A$ satisfying* (i) *is open in A.*
(b) *The set of all $\alpha \in A$ satisfying* (ii) *is constructible in A.*

Proof. (a) The subset $B \subseteq A$ where (i) does not hold is the set of $\alpha \in A$ satisfying

$$U^0.x_\alpha \subseteq Y_\alpha \quad \text{or equivalently} \quad \alpha^{-1}U^0\alpha.x \subseteq Y.$$

Thus $B = \bigcap_{u \in U^0} B_u$, where $B_u = \{\alpha \in A \mid \alpha^{-1}u\alpha.x \in Y\}$ is the preimage of Y under the morphism $A \to X$, $\alpha \mapsto \alpha^{-1}u\alpha.x$. Hence B is closed in A. This proves (a).

(b) Similarly, the subset $C \subseteq A$ where (ii) does not hold is the set of $\alpha \in A$ with $\alpha^{-1}U^0\alpha \cap Y \neq \emptyset$. Consider the set

$$C' = \{(\alpha, u) \in A \times U^0 \mid \alpha^{-1}u\alpha.x \in Y\}.$$

This set is closed in $A \times U^0$ since it is the preimage of Y under the morphism $A \times U^0 \to X$, $(\alpha, u) \mapsto \alpha^{-1}u\alpha.x$. Since C is the image of C' under the projection to A, (b) follows. □

The next proposition allows one to verify conditions (i) and (ii).

Proposition 5.9. *Let as before $x \in X \setminus Y$.*

(a) *If A is an algebraic family of automorphisms of X with $d_x(A_x) \supseteq \mathbf{SL}(T_xX)$, then the set of all $\alpha \in A$ satisfying* (i) *is a dense open subset of A.*
(b) *There exists an algebraic family $A^* \subseteq G_x$ transitive in $X^* = X \setminus \{x\}$ such that for any subgroup $U^0 \in \mathcal{U}(X)$ condition* (ii) *holds for a general $\alpha \in A^*$.*
(c) *Given an algebraic family $B \subseteq \mathrm{Aut}(X)$ let $\tilde{A} = B \cdot A^* \subseteq \mathrm{Aut}(X)$, where $A^* \subseteq G_x$ is as in* (b). *Then* (ii) *holds for a general $\tilde{\alpha} \in \tilde{A}$.*

Proof. (a) By Lemma 5.8 it suffices to find $\alpha \in A$ satisfying (i), or equivalently such that $\alpha^{-1}U^0\alpha.x \cap O_Y \neq \emptyset$. By our assumptions in (a), for any nonzero vector $v \in T_xX$ there is an element $\alpha \in A_x$ such that v is tangent to the orbit through x of the one-parameter group $\alpha^{-1}U^0\alpha \subseteq \mathrm{Aut}(X)$. These orbits form an algebraic family of smooth rational curves in X through the point x that dominates X and so meets the open orbit O_Y, as required.

(b) By the Transversality Theorem [1, 1.16] there exists an algebraic family $A^* \subseteq G_x$ transitive in X^* such that for any two subvarieties $Y, Z \subseteq X$ there is a dense open subset $A_0 \subseteq A^*$ with the property that for any $\alpha \in A_0$ the varieties $\alpha.Y$ and Z are transversal. Applying this to $Z = U^0.x$ we find that the varieties $U^0.x$ and $\alpha.Y$ are disjoint, because under our assumptions

$$\dim U^0.x + \dim Y < \dim X.$$

Since $x_\alpha = x$, (b) follows.

To deduce (c) we note that the set, say C, of points $\tilde{\alpha} \in \tilde{A}$ where (ii) fails is the set of $\tilde{\alpha} = (\beta, \alpha)$ with $\alpha^{-1}\beta^{-1}U^0\beta\alpha.x \cap Y \neq \emptyset$. Consider, similarly to the proof of Lemma 5.8(b), the closed subset of $B \times A^* \times U^0$ given by

$$C' = \{(\beta, \alpha, u) \in B \times A^* \times U^0 \mid \alpha^{-1}\beta^{-1}u\beta\alpha.x \in Y\},$$

where A^* satisfies the conclusion of (b). According to (b), for any $\beta \in B$ the set

$$C'_\beta = C' \cap (\{\beta\} \times A^* \times U^0)$$

maps under the projection to A^* to a nowhere dense subset. Hence also the image C of C' under the projection to $\tilde{A} = B \times A^*$ will be nowhere dense. Thus its complement contains an open dense subset, proving (c). □

Notation 5.10. Given a one-parameter group $U \in \mathcal{U}(X)$ we let as before $U^* = U \setminus \{\mathrm{id}\}$. Given a collection $\mathcal{F}$ of invariant functions

$$f_1, \ldots, f_s \in \ker \delta_1 \setminus \ker \delta_0 \quad \text{and} \quad g_1, \ldots, g_s \in \ker \delta_0 \setminus \ker \delta_1$$

and $U_\kappa = U^1_{f_s^{k_s}} \cdot U^0_{g_s^{l_s}} \cdot \ldots \cdot U^1_{f_1^{k_1}} \cdot U^0_{g_1^{l_1}}$ as in (5.2), we let

$$U^*_\kappa = U^{1*}_{f_s^{k_s}} \cdot U^{0*}_{g_s^{l_s}} \cdot \ldots \cdot U^{1*}_{f_1^{k_1}} \cdot U^{0*}_{g_1^{l_1}}.$$

Using Proposition 4.11 we can deduce the following result.

Proposition 5.11. *Let $(Y_\alpha)_{\alpha \in A}$ be a flat family of proper closed subsets of X. Assume that the partial boundaries $\partial_i Y_\alpha$ (see Notation 5.5) are contained in $E_{\alpha,i}$, where the $(E_{\alpha,i})_{\alpha \in A}$, $i = 0, 1$, form flat families of proper closed subsets of D_i. Then one can find an open dense subset A^o of A, flat families $(E^o_{\alpha,i})_{\alpha \in A^o}$ of proper closed subsets of D_i $(i = 0, 1)$, and a sequence $\kappa = (k_1, l_1, \ldots, k_s, l_s) \in \mathbb{N}^{2s}$ such that for any $h \in U^*_\kappa$ we have*

$$\partial_i(h.Y_\alpha) \subseteq E^o_{\alpha,i}, \quad i = 0, 1, \ \forall \alpha \in A^o.$$

Proof. The proof proceeds by induction on s. For $s = 0$ the assertion clearly holds with $A^o = A$ and $E_{\alpha,i} = \partial_i Y_\alpha$, $i = 0, 1$. Assume that it holds at step $s - 1$, i.e. we can find $\kappa' = (k_j, l_j)_{j=1,\dots,s-1} \in \mathbb{N}^{2s-2}$, a dense open subset $A' \subseteq A$ and flat families $(E_{\alpha,i})_{\alpha\in A'}$ of proper closed subsets of D_i such that for $\alpha \in A'$,

$$\partial_i(h.Y_\alpha) \subseteq E_{\alpha,i}, \quad i = 0, 1,\ \forall h \in U^*_{\kappa'}.$$

The varieties $(h.Y_\alpha)_{(h,\alpha)\in U^*_{\kappa'}\times A'}$ form a flat algebraic family. By Proposition 4.11 one can find an open dense subset $A'' \subseteq A'$ and flat families $(E'_{\alpha,i})_{\alpha\in A''}$, $i = 0, 1$, of proper closed subsets of D_i such that

$$\partial_i(h'h.Y_\alpha) \subseteq E'_{\alpha,i} \quad (i = 0, 1) \quad \forall l_s \gg 0,\ \forall\alpha \in A'',\ \forall(h', h) \in U^{0*}_{g_s^{l_s}} \times U^*_{\kappa'}.$$

Fixing a sufficiently large l_s and applying the same argument again, one can find an open dense subset $A^o \subseteq A''$ and flat families $(E^o_{\alpha,i})_{\alpha\in A^o}$, $i = 0, 1$, of proper closed subsets of D_i such that

$$\partial_i(h''h'h.Y) \subseteq E^o_{\alpha,i} \quad (i = 0, 1) \quad \forall k_1 \gg 0,\ \forall\alpha \in A^o,\ \forall(h'', h', h) \in U^{1*}_{f_s^{k_s}} \times U^{0*}_{g_s^{l_s}} \times U^*_{\kappa'}.$$

This concludes the induction. □

Using Proposition 5.11 and Corollary 5.4 we can now deduce Theorem 2.6.

Proof of Theorem 2.6.. Fix $x \in X \setminus Y$. We show that for a suitable choice of an algebraic family A of automorphisms conditions (i)–(iii) are satisfied for the pair (Y_α, x_α) if $\alpha \in A$ is generic. Then our theorem follows by applying Lemma 5.6.

Step 1. Consider an algebraic family $A \subseteq G$ satisfying conditions (a) and (b) of Lemma 5.2. By Proposition 5.9(a) condition (i) holds when α varies in a dense open subset of A. Replacing the original pair (Y, x) by a suitable new one $(Y_\alpha, x_\alpha) = (\alpha.Y, \alpha.x)$ we may suppose that (Y, x) satisfies (i).

Step 2. In the following we construct an algebraic family B of automorphisms such that for a generic choice of $\beta \in B$ the translates (Y_β, x_β) satisfy (ii), (iii). Since by Proposition 5.9(a) condition (i) is open, the pair (Y_β, x_β) also satisfies (i).

Let A^* be a family of automorphisms as in Proposition 5.9(b). The translates $Y_\alpha = \alpha.Y$, $\alpha \in A^*$, form a flat family of proper closed subsets of X. Using the theorem of generic flatness it is easily seen that over an open dense subset $A' \subseteq A^*$ also the partial boundaries $E_{\alpha,i} = \partial_i Y_\alpha$, $\alpha \in A'$, form flat families of proper closed subsets of D_i, $i = 0, 1$. Let now $\mathcal{F}$, U_κ, and U^*_κ be as in Notation 5.10. By Proposition 5.11 we can find $\kappa = (k_1, l_1, \dots, k_s, l_s) \in \mathbb{N}^{2s}$, a dense open subset $A^o \subseteq A'$, and families $(E^o_{\alpha,i})_{\alpha\in A^o}$, $i = 0, 1$, of proper closed subsets of D_i such that $\partial_i(h.Y_\alpha) \subseteq E^o_{\alpha,i}$ for $i = 0, 1$, $\alpha \in A^o$ and all $h \in U^*_\kappa$.

We claim that for a generic choice of $(h, \alpha) \in B = U^*_\kappa \times A^*$ conditions (ii) and (iii) are satisfied for $h.Y_\alpha$. To check (ii) we note that $h.Y_\alpha = h\alpha.Y$. Thus applying Proposition 5.9(c) to the family $B = U^*_\kappa \times A^*$ we see that condition (ii) is indeed satisfied for a generic choice of (h, α).

It remains to show that (iii) is satisfied for a generic choice of (h, α). Condition (iii) is equivalent to $\bar{\varrho}_0(h.x_\alpha) \notin \partial_0(h.Y_\alpha)$. By construction $\partial_0(h.Y_\alpha) \subseteq E_{\alpha,0} \subseteq D_0$ for any $h \in U_\kappa^*$, while for a fixed $\alpha \in A^o$ the points $h.x_\alpha$, $h \in U_\kappa^*$, fill in a dense subset of X, and so their images $\bar{\varrho}_0(h.x)$ fill in a dense subset of $Q_0 \subseteq \bar{Q}_0 \setminus \bar{\varrho}_0(D_0)$. Thus (iii) holds for a generic choice of $(h, \alpha) \in U_\kappa^* \times A^o$. This concludes the proof of Main Theorem 2.6. $\square$

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