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Finiteness and paradoxical decompositions in C*-dynamical systems

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Abstract. We discuss the interplay between K-theoretical dynamics and the structure theory of certain C*-algebras arising from crossed products. In the presence of sufficiently many projections we associate to each noncommutative C*-system (A, G, α) a type semigroup $S(A, G, \alpha)$ which reflects much of the spirit of the underlying action. We characterize purely infinite as well as stably finite crossed products in terms of finiteness and infiniteness in the type semigroup. We explore the dichotomy between stable finiteness and pure infiniteness in certain classes of reduced crossed products by means of paradoxical decompositions.

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1. Introduction

Dynamical systems and the theory of operator algebras are inextricably related [6, 14, 21]. Topological dynamics has long played a significant role in the study and classification of amenable C*-algebras by providing a wealth of examples that fall under the umbrella of Elliott's classification program as well as examples that lack certain regularity properties [10, 12, 30, 31]. The crossed product construction permits the exploitation of symmetry through the acting group and is generous enough to produce a variety of C*-algebraic phenomena. One would like to uncover information about the the crossed product algebra by unpacking the dynamics and, conversely, describe the nature of the system by looking at the operator algebra's structure and invariants.

Of particular interest in this paper is the deep theme common to groups, dynamical systems and operator algebras; that of finiteness, infiniteness, and proper infiniteness, the latter expressed in terms of paradoxical decompositions. The remarkable alternative theorem of Tarski establishes, for discrete groups, the dichotomy between amenability and paradoxical decomposability. This carries over into the realm of operator algebras. Indeed, if a discrete group Γ acts on itself by left-translation, the Roe algebra $C(\beta\Gamma) \rtimes_\lambda \Gamma$ is properly infinite if and only if Γ is Γ-paradoxical and this happens if and only if Γ is non-amenable [28]. This is mirrored in the von Neumann algebra setting as well; all projections in a II_1 factor are finite and the ordering of Murray–von Neumann subequivalence is determined by a unique faithful normal tracial state. Alternatively type III factors admit no traces since all non-zero projections therein are properly infinite. As for unital, simple, separable and nuclear algebras, the C*-enthusiast of old hoped that the trace/traceless divide determined a similar dichotomy between stable finiteness and pure infiniteness (the C*-algebraic analog of type III). This hope was laid to rest with Rørdam's example of a unital, simple, separable, nuclear C*-algebra containing both an infinite and a non-zero finite projection [26]. The conjecture for such a dichotomy remains open for those algebras whose projections have dense linear span. Theorem 1.3 below is a result in this direction.

Despite the failure of the above dichotomy, the classification program of Elliott in its original K-theoretic formulation has witnessed much success for stably finite algebras [11,27], as well as in the purely infinite case with the spectacular complete classification results of Kirchberg and Phillips [16,22] modulo the UCT. One motivation for studying purely infinite algebras stems from the fact that Kirchberg algebras (unital, simple, separable, nuclear, and purely infinite) are classified by their K- or KK-theory. The C*-literature has produced examples of purely infinite C*-algebras arising from dynamical systems [4,18,19,28]. In many cases the underlying algebra is abelian with spectrum the Cantor set. For example, Archbold, Spielberg, and Kumjian (independently) proved that there is an action of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor set so that the corresponding crossed product C*-algebra is isomorphic to $\mathcal{O}_2$ [29]. Laca and Spielberg [19] construct purely infinite and simple crossed products that emerge from strong boundary actions. Jolissaint and Robertson [13] generalized the idea of strong boundary action to noncommutative systems with the concept of an n-filling action. They showed that $A \rtimes_\lambda \Gamma$ is simple and purely infinite provided that the action is properly outer and n-filling and every corner pAp of A is infinite dimensional. We will give a K-theoretic proof of their result in the case that the algebra A has a well behaved K_0-group.

The transition from classical topological dynamics to noncommutative C*-dynamics presents several challenges and subtleties. One way to approach these issues is to interpret dynamical conditions K-theoretically via the induced actions on $K_0(A)$ and on the Cuntz semigroup $W(A)$ and use tools from the classification literature as well as developed techniques of Cuntz comparison to uncover pertinent algebraic information. Such an approach is seen in Brown's work [8] as well as that of the author in [23,24]. We continue this approach in the present paper by carrying over ideas of paradoxical actions on spaces to the noncommutative C*-setting. The notion of a group acting paradoxically on a set and the construction of the type semigroup goes back to the work of Tarski (see Wagon's book [32] for a good treatment). Rørdam and Sierakowski [28] looked at the type semigroup $S(X, \Gamma)$ built from an action of a discrete group on the Cantor set and tied pure infiniteness of the resulting reduced crossed product to the absence of traces on this semigroup. In effect, they prove that if a countable, discrete, and exact group Γ acts continuously and freely on the Cantor set X, and the preordered semigroup $S(X,\Gamma)$ is almost unperforated, then the following are equivalent: (i) The reduced crossed product $C(X) \rtimes_\lambda \Gamma$ is purely infinite, (ii) $C(X) \rtimes_\lambda \Gamma$ is traceless, (iii) $S(X,\Gamma)$ is purely infinite (that is $2x \leq x$ for every $x \in S(X,\Gamma)$), and (iv) $S(X,\Gamma)$ is traceless. Inspired by their work, we construct a type semigroup $S(A,\Gamma,\alpha)$ for noncommutative systems (A,Γ,α) and establish a more general result. This is Theorem 5.6 below which, in particular, implies the following.

Theorem 1.1. *Let A be a unital, separable, and exact C*-algebra with stable rank one and real rank zero. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action with $S(A,\Gamma,\alpha)$ almost unperforated. Then the following are equivalent:*

(1) *The semigroup $S(A,\Gamma,\alpha)$ is purely infinite.*

(2) *The C*-algebra $A \rtimes_\lambda \Gamma$ is purely infinite.*

(3) *The C*-algebra $A \rtimes_\lambda \Gamma$ is traceless.*

(4) *The semigroup $S(A,\Gamma,\alpha)$ admits no non-trivial state.*

As a suitable quotient of $K_0(A)^+$, this type semigroup $S(A,\Gamma,\alpha)$ is purely infinite if and only if every positive element of $K_0(A)^+$ is paradoxical under the induced action with covering multiplicity at least two. Taking covering multiplicities into account, Kerr and Nowak [15] consider completely non-paradoxical actions of a discrete group on the Cantor set. We do the same here for noncommutative systems using ordered K-theory and establish Theorem 4.9, of which the following is a special case.

Theorem 1.2. *Let A be a unital, separable and exact C*-algebra with stable rank one and real rank zero . Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal action. Then the following are equivalent:*

(1) *$A \rtimes_\lambda \Gamma$ admits a faithful tracial state.*

(2) *$A \rtimes_\lambda \Gamma$ is stably finite.*

(3) *α is completely non-paradoxical.*

Moreover, if A is AF and Γ is a free group, then (1) through (3) are all equivalent to $A \rtimes_\lambda \Gamma$ being MF in the sense of Blackadar and Kirchberg [7].

Combining these two results we obtain the desired dichotomy, albeit for a certain class of crossed products.

Theorem 1.3. *Let A be a unital, separable, and exact C*-algebra with stable rank one and real rank zero. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action with $S(A,\Gamma,\alpha)$ almost unperforated. Then the reduced crossed product $A \rtimes_\lambda \Gamma$ is simple and is either stably finite or purely infinite.*

We round off the introduction with a brief description of the contents of this article. We begin by reviewing the necessary concepts, definitions, and results that will be assumed throughout. In Section 3 we give meaning to paradoxical and completely non-paradoxical actions and construct infinite crossed products. In Section 4 we introduce the noncommutative type semigroup $S(A, \Gamma, \alpha)$ which "sees" any paradoxical phenomena and use it to characterize stably finite crossed products (Theorem 4.9). Section 5 looks at the purely infinite case. When the underlying algebra A has a well behaved K_0 group and the action is minimal and properly outer, we characterize purely infinite crossed products (Theorem 5.6) and obtain a dichotomy between the stably finite and the purely infinite (Theorem 5.7).

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2. Preliminaries

We make the blanket assumption that all C*-algebras A are separable and unital, with unit denoted by 1_A, and all groups Γ are discrete.

Most results in this article have K-theory as a main ingredient; the reader may consult [5] for a suitable treatment thereof, as well as [2] for the necessary results concerning the Cuntz semigroup. We briefly outline the storyline of $K_0(A)$ and $W(A)$ here.

If A is a C*-algebra, $M_{m,n}(A)$ will denote the linear space of all $m \times n$ matrices with entries from A. The square $n \times n$ matrices $M_n(A)$ is a C*-algebra with positive cone $M_n(A)^+$. If $a \in M_n(A)^+$ and $b \in M_m(A)^+$, write $a \oplus b$ for the matrix

$$\operatorname{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{n+m}(A)^+.$$

Set $M_\infty(A)^+ = \bigsqcup_{n \geq 1} M_n(A)^+$; the set-theoretic direct limit of the $M_n(A)^+$ with connecting maps $M_n(A) \to M_{n+1}(A)$ given by $a \mapsto a \oplus 0$. Write $\mathcal{P}(A)$ for the set of projections in A and set $\mathcal{P}_\infty(A) = \bigsqcup_{n \geq 1} \mathcal{P}(M_n(A))$. Elements a and b in $M_\infty(A)^+$ are said to be *Pedersen equivalent*, written $a \sim b$, if there is a matrix $v \in M_{m,n}(A)$ with $v^*v = a$ and $vv^* = b$. We say that a is *Cuntz subequivalent* to (or *Cuntz smaller* than) b, written $a \precsim b$, if there is a sequence $(v_k)_{k \geq 1} \subset M_{m,n}(A)$ with $\|v_k^* b v_k - a\| \to 0$ as $k \to \infty$. If $a \precsim b$ and $b \precsim a$ we say that a and b are *Cuntz equivalent* and write $a \approx b$. It is routine to check that $\sim$ and $\approx$ are equivalence relations on $M_\infty(A)^+$ and that $a \sim b$ implies $a \approx b$. Canonically set

$$V(A) = \mathcal{P}_\infty(A)/\sim \quad \text{and} \quad W(A) = M_\infty(A)^+/\approx .$$

We write $[p]$ for the equivalence class of $p \in \mathcal{P}_\infty(A)$ and $\langle a\rangle$ for the class of $a \in M_\infty(A)^+$. $W(A)$ has the structure of a preordered abelian monoid with addition given by $\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$ and preorder $\langle a\rangle \leq \langle b\rangle$ if $a \precsim b$. The monoid $W(A)$ embeds into $\mathrm{Cu}(A) := (A \otimes \mathcal{K})_+/\approx$, the *Cuntz semigroup* of A. For this work, the monoid $W(A)$ will be suitable for our purposes and we will refer to it as the Cuntz semigroup as in [2]. With addition and ordering identical to that of $W(A)$, $V(A)$ is also a preordered abelian monoid. However, there is a cardinal difference between the orderings on $V(A)$ and $W(A)$; the ordering on $W(A)$ extends the algebraic ordering ($x, y, z \in W(A)$ with $x + y = z$ implies $x \leq z$) but only in rare cases agrees with it. With $V(A)$, the ordering agrees with the algebraic one. Indeed, one verifies that for projections $p, q \in \mathcal{P}_\infty(A)$, $p \precsim q$ if and only if there is a subprojection $r \leq q$ with $p \sim r$ if and only if $p \oplus p' \sim q$ for some $p' \in \mathcal{P}_\infty(A)$. Thus $[p] \leq [q]$ implies that $[p] + [p'] = [q]$. As a brief reminder, for a unital algebra A, $K_0(A) = \mathcal{G}(V(A))$ is the Grothendieck enveloping group of $V(A)$ and $[p]_0 = \gamma([p])$ where $\gamma : V(A) \to K_0(A)$ is the canonical Grothendieck map.

A projection p in A is infinite if $p \sim q$ for some subprojection $q \lneq p$. It was shown in [17] that p infinite if and only if $p \oplus b \precsim p$ for some non-zero $b \in M_\infty(A)^+$. A unital C*-algebra A is said to be infinite if 1_A is infinite. Otherwise, A is called finite. If $M_n(A)$ is finite for every $n \in \mathbb{N}$ then A is called stably finite. Recall that a unital, stably finite C*-algebra A yields an ordered abelian group $K_0(A)$ with positive cone $K_0(A)^+ := \gamma(V(A))$ and with a distinguished order unit $[1_A]_0$. Occasionally we shall require our algebras to have *cancellation*, which simply means that γ is injective. It is routine to check that algebras with stable rank one are stably finite and have cancellation. Moreover, when A is stably finite with cancellation the map $K_0(A)^+ \to W(A)$, $[p]_0 \mapsto \langle p\rangle$ is well-defined and injective. Recall that a semigroup K has the Riesz refinement property if, whenever $\sum_{j=1}^n x_j = \sum_{i=1}^m y_i$, for members $x_1, \ldots, x_n, y_1, \ldots, y_m \in K$, there exist $\{z_{ij}\}_{i,j} \subset K$ satisfying $\sum_i z_{ij} = x_j$ and $\sum_j z_{ij} = y_i$ for each i and j. S. Zhang showed that if A is a stably finite algebra with $\mathrm{RR}(A) = 0$ then $K_0(A)^+$ has the Riesz refinement property [35].

A *transformation group* is a pair (X, Γ) where Γ is a group, and X is a locally compact Hausdorff space endowed with a continuous action $\Gamma \curvearrowright X$. By a *C*-dynamical system* we mean a triple (A, Γ, α), where A is a C*-algebra, Γ is a group, and $\alpha : \Gamma \to \mathrm{Aut}(A)$ is a group homomorphism. In the case where A is a commutative algebra, say $A = C(X)$ for some compact Hausdorff space X, C*-systems $(C(X), \Gamma, \alpha)$ are in one-to-one correspondence with transformation groups (X, Γ) via the formula $\alpha_s(f)(x) = f(s^{-1}.x)$ where $s \in \Gamma$, $f \in C(X)$, $x \in X$.

A C*-dynamical system induces a natural action at the K-theoretical level, and the order theoretical dynamics will reflect information about the nature of the action and will at times help describe the structure of the crossed product. If (G, G^+, u)

and (H, H^+, v) are ordered abelian groups each with their distinguished order units, a morphism in this category is a group homomorphism $\beta : G \to H$ which is positive and order unit preserving, i.e. $\beta(G^+) \subset H^+$, and $\beta(u) = v$ respectively. We also write

$$\mathrm{Aut}(G, G^+, u) := \{\tau \in \mathrm{Aut}(G) : \tau(G^+) = G^+, \tau(u) = u\}$$

for the group of order automorphisms of (G, G^+, u). When the context is understood we might abbreviate $\mathrm{Aut}(G, G^+, u)$ to $\mathrm{Aut}(G)$. So for every action $\alpha : \Gamma \to \mathrm{Aut}(A)$, there is an induced action $\hat{\alpha} : \Gamma \to \mathrm{Aut}(K_0(A))$ where $\hat{\alpha}(s) = \hat{\alpha}_s : K_0(A) \to K_0(A)$ is the induced automorphism given by $\hat{\alpha}_s([p]_0) = [\alpha_s(p)]_0$ for a projection p in $\mathcal{P}_\infty(A)$. In the same manner a C*-system (A, Γ, α) induces an action $\hat{\alpha} : \Gamma \to \mathrm{Aut}(W(A))$ via $\hat{\alpha}_s(\langle a \rangle) = \langle \alpha_s(a) \rangle$, where $s \in \Gamma$, and $a \in M_\infty(A)^+$. Here $\mathrm{Aut}(W(A))$ will denote the set of monoid isomorphisms of $W(A)$ which respect the ordering.

Given a C*-dynamical system (A, Γ, α), we write $A \rtimes_{\lambda,\alpha} \Gamma$ for the reduced crossed product algebra (at times we will omit the α). We briefly recall the construction and refer the reader to [9,33] and [21] for more details. First consider the algebraic crossed product $A \rtimes_{\mathrm{alg},\alpha} \Gamma$ which is the complex linear space of all finitely supported functions $C_c(\Gamma, A) = \{\sum_{s \in F} a_s u_s : F \subset \Gamma, a_s \in A\}$, equipped with a twisted multiplication and involution: for $s, t \in \Gamma, a, b \in A$

$$(au_s)(bu_t) = a\alpha_s(b)u_{st},$$
$$(au_s)^* = \alpha_{s^{-1}}(a^*)u_{s^{-1}}.$$

If $A \subset \mathbb{B}(\mathcal{H})$ is faithfully represented (the choice of representation is immaterial), the $*$-algebra $A \rtimes_{\mathrm{alg},\alpha} \Gamma$ can then be faithfully represented as operators on $\mathcal{H} \otimes \ell^2(\Gamma)$ via $au_s(\xi \otimes \delta_t) = \alpha_{st}^{-1}(a)\xi \otimes \delta_{st}$ for $\xi \in \mathcal{H}$ and $s, t \in \Gamma$. Completing with respect to the operator norm on $\mathbb{B}(\mathcal{H} \otimes \ell^2(\Gamma))$ gives the reduced crossed product $A \rtimes_{\lambda,\alpha} \Gamma$. We will at times make use of the conditional expectation $\mathbb{E} : A \rtimes_{\lambda,\alpha} \Gamma \to A$, which is a unital, contractive, completely positive map satisfying $\mathbb{E}(\sum_{s \in \Gamma} a_s u_s) = a_e$, and $\mathbb{E}^2 = \mathbb{E}$.

3. Paradoxical decompositions

In this section we study K-theoretic conditions, in the form of paradoxical phenomena, that characterize finite and infinite crossed products.

We first construct infinite algebras arising from crossed products by generalizing the notion of a local boundary action to the noncommutative setting. A continuous action $\Gamma \curvearrowright X$ of a discrete group on a locally compact space is called a *local boundary action* if for every non-empty open set $U \subset X$ there is an open set $V \subset U$ and $t \in \Gamma$ with $t.\overline{V} \subsetneq V$. Laca and Spielberg showed in [19] that such actions yield infinite projections in the reduced crossed product $C_0(X) \rtimes_\lambda \Gamma$. Sierakowski remarked that the condition $t.\overline{V} \subsetneq V$ for *some* non-empty open set V and group element $t \in \Gamma$ is equivalent to the existence of open sets $U_1, U_2 \subset X$ and elements $t_1, t_2 \in \Gamma$ such that $U_1 \cup U_2 = X$, $t_1.U_1 \cap t_2.U_2 = \emptyset$, and $t_1.U_1 \cup t_2.U_2 \neq X$. He generalized this by defining *paradoxical* actions. A transformation group (X, Γ) is n-paradoxical if there exist open subsets $U_1, \ldots, U_n \subset X$ and elements $t_1, \ldots, t_n \in \Gamma$ such that

$$\bigcup_{j=1}^{n} U_j = X, \qquad \bigsqcup_{j=1}^{n} t_j.U_j \subsetneq X.$$

He then showed that the algebra $C(X) \rtimes_\lambda \Gamma$ is infinite provided that X is compact and the action $\Gamma \curvearrowright X$ is n-paradoxical for some n. We do the same here in the noncommutative setting.

Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a C*-dynamical system where Γ is a discrete group. We look at the induced actions $\hat{\alpha} : \Gamma \curvearrowright K_0(A)^+$ and $\hat{\alpha} : \Gamma \curvearrowright W(A)$ given by $t.x = \hat{\alpha}_t(x)$ for $t \in \Gamma$ and $x \in K_0(A)^+$ or $W(A)$.

For what follows we introduce a convention: for $x, y \in W(A)$ we shall write $x < y$ to mean $x + z \leq y$ for some non-zero $z \in W(A)$.

Proposition 3.1. *Let A be a unital C*-algebra and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action which is W-paradoxical in the sense that there exist $x_1, \ldots, x_n \in W(A)$ and group elements $t_1, \ldots, t_n \in \Gamma$ with $\sum_{j=1}^{n} x_j \geq \langle 1_A \rangle$ and $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) < \langle 1_A \rangle$. Then $A \rtimes_\lambda \Gamma$ is infinite.*

Proof. Again let $\iota : A \to A \rtimes_\lambda \Gamma$ denote the canonical embedding and for $t \in \Gamma$ write u_t for the canonical unitary in $A \rtimes_\lambda \Gamma$ that implements the action $\alpha_t : A \to A$, so that $\iota(\alpha_t(a)) = u_t \iota(a) u_t^* \approx \iota(a)$ for every $a \in A$ and $t \in \Gamma$. If $a \in M_n(A)^+$ then by amplification we have $\iota^{(n)}(\alpha_t^{(n)}(a)) = (u_t \otimes 1_A)\iota^{(n)}(a)(u_t \otimes 1_n)^* \approx \iota^{(n)}(a)$ for every $t \in \Gamma$. For economy we will omit denoting the amplification when the context is understood.

For each $j = 1, \ldots, n$ set $x_j = \langle a_j \rangle$ for $a_j \in M_\infty(A)^+$. Then we have

$$\langle 1_A \rangle \leq \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \langle a_j \rangle = \langle a_1 \oplus \cdots \oplus a_n \rangle,$$

which implies $1_A \precsim \oplus_{j=1}^{n} a_j$ in $M_\infty(A)^+$. Applying ι we get $1_{A \rtimes_\lambda \Gamma} \precsim \oplus_{j=1}^{n} \iota(a_j) \approx \oplus_{j=1}^{n} \iota(\alpha_{t_j}(a_j))$ in $M_\infty(A \rtimes_\lambda \Gamma)^+$.

By our convention there is some non-zero $b \in M_\infty(A)^+$ for which

$$\langle \alpha_{t_1}(a_1) \oplus \cdots \oplus \alpha_{t_n}(a_n) \oplus b \rangle = \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) + \langle b \rangle \leq \langle 1_A \rangle.$$

Thus $\alpha_{t_1}(a_1) \oplus \cdots \oplus \alpha_{t_n}(a_n) \oplus b \precsim 1_A$ and $\iota(\alpha_{t_1}(a_1)) \oplus \cdots \oplus \iota(\alpha_{t_n}(a_n)) \oplus \iota(b) \precsim 1_{A \rtimes_\lambda \Gamma}$. Together we get

$$1_{A \rtimes_\lambda \Gamma} \oplus \iota(b) \precsim \iota(\alpha_{t_1}(a_1)) \oplus \cdots \oplus \iota(\alpha_{t_n}(a_n)) \oplus \iota(b) \precsim 1_{A \rtimes_\lambda \Gamma}.$$

Since $1_{A\rtimes_\lambda\Gamma} \oplus \iota(b) \precsim 1_{A\rtimes_\lambda\Gamma}$ and $\iota(b) \neq 0$, Lemma 3.1 in [17] implies that $A \rtimes_\lambda \Gamma$ is infinite as claimed. □

We make the brief remark that an action $\Gamma \curvearrowright A$ is W-paradoxical in the above sense with $n = 2$ if and only if there is a non-zero a in A^+ and $t \in \Gamma$ with $\hat{\alpha}_t(x) < x$, where $x = \langle a \rangle$.

Corollary 3.2. *Let A be a stably finite C*-algebra with cancellation and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a K_0-paradoxical action in the sense that there exist $x_1, \ldots, x_n \in K_0(A)^+$ and group elements $t_1, \ldots, t_n \in \Gamma$ with*

$$\sum_{j=1}^{n} x_j \geq [1_A]_0 \quad \text{and} \quad \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) < [1_A]_0.$$

Then $A \rtimes_\lambda \Gamma$ is infinite.

Proof. Given that A is stably finite and has cancellation we know that there is a well-defined, order-preserving, injective monoid homomorphism

$$K_0(A)^+ \to W(A), \quad [p]_0 \mapsto \langle p \rangle.$$

Also, if $x < y$ in $K_0(A)^+$, then $x + z = y$ for a non-zero z. The proof follows from these facts and Proposition 3.1. □

Perhaps what has been called *paradoxical* is misleading because, in a sense, paradoxicality implies the idea of duplication of sets. Gleaning from the ideas explored in [15], we define a notion of paradoxical decomposition with covering multiplicity in the noncommutative setting.

Definition 3.3. Let A be a C*-algebra, Γ a discrete group and $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action with its induced action $\hat{\alpha}$. Let $0 \neq x \in K_0(A)^+$ and $k > l > 0$ be positive integers. We say x is (Γ, k, l)*-paradoxical* if there are $x_1, \ldots, x_n$ in $K_0(A)^+$ and $t_1, \ldots, t_n$ in Γ such that

$$\sum_{j=1}^{n} x_j \geq kx \quad \text{and} \quad \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx.$$

If an element $x \in K_0(A)^+$ fails to be (Γ, k, l)-paradoxical for all integers $k > l > 0$ we call x *completely non-paradoxical*. The action α will be called *completely non-paradoxical* if every member of $K_0(A)^+$ is completely non-paradoxical.

The notion of a quasidiagonal action was first introduced in [15] and further studied in [23] from a K-theoretic viewpoint. The author of [23] observed that MF (or equivalently QD) actions of discrete groups Γ on AF algebras admit, in a local sense, Γ-invariant traces on $K_0(A)$, so it should come to no surprise that these actions do not allow paradoxical decompositions at the K-theoretic level. The next proposition illustrates this principle and provides us with our first class of examples of completely non-paradoxical actions.

Proposition 3.4. *If $\alpha : \Gamma \to \mathrm{Aut}(A)$ is an MF action of a discrete group Γ on a unital AF algebra, then α is completely non-paradoxical.*

Proof. Suppose $0 \neq x \in K_0(A)^+$ is (Γ, k, l)-paradoxical for some positive integers $k > l > 0$, so that there are $x_1, \ldots, x_n$ in $K_0(A)^+$ and $t_1, \ldots, t_n$ in Γ such that

$$y := \sum_{j=1}^{n} x_j \geq kx \quad \text{and} \quad z := \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx.$$

Consider the finite sets $F = \{t_1, \ldots, t_n\} \subset \Gamma$ and $S = \{y - kx, lx - z, x_1, \ldots, x_n, x\} \subset K_0(A)^+$. Since α is quasidiagonal, Proposition 4.8 of [23] guarantees existence of a subgroup $H \leq K_0(A)$ which contains all the F-iterates of S, and a group homomorphism $\beta : H \to \mathbb{Z}$ with $\beta(\hat{\alpha}_t(g)) = \beta(g)$ for each $t \in F$ and $g \in S$. Also, $\beta(g) > 0$ for $0 < g \in S$. Clearly y, z, kx, lx all belong to the subgroup H, and since $\beta(y - kx) \geq 0$, we have $k\beta(x) = \beta(kx) \leq \beta(y)$. Similarly, $\beta(z) \leq l\beta(x)$. Now using the Γ-invariance of β,

$$\begin{aligned} k\beta(x) \leq \beta(y) &= \beta\Big(\sum_{j=1}^{n} x_j\Big) = \sum_{j=1}^{n} \beta(x_j) = \sum_{j=1}^{n} \beta(\hat{\alpha}_{t_j}(x_j)) = \beta\Big(\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j)\Big) \\ &= \beta(z) \leq l\beta(x). \end{aligned}$$

This is absurd since $\beta(x) > 0$ and $l < k$. Thus no such non-zero x exists. □

It was shown by Kerr and Nowak [15] that quasidiagonal actions by groups whose reduced group algebras are MF give rise to MF crossed products, which are always stably finite. Indeed, it is the finiteness of the crossed product that is an obstruction to a positive element being paradoxical.

Proposition 3.5. *Consider a C*-dynamical system (A, Γ, α) with stably finite reduced crossed product $A \rtimes_\lambda \Gamma$. Then the induced $\hat{\alpha} : \Gamma \curvearrowright K_0(A)^+$ is completely non-paradoxical.*

Proof. Suppose on the contrary that $0 \neq [p]_0 := x \in K_0(A)^+$ is (Γ, k, l) paradoxical for some integers $k > l > 0$ where $p \in \mathcal{P}_m(A)$. We then have elements $x_1, \ldots, x_n$ in $K_0(A)^+$ and $t_1, \ldots, t_n \in \Gamma$ with

$$\sum_{j=1}^{n} x_j \geq kx \quad \text{and} \quad \sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx.$$

If $\iota : A \hookrightarrow A \rtimes_\lambda \Gamma$, $\iota : a \mapsto au_e$, denotes the canonical embedding, then the induced map $\hat{\iota} : K_0(A)^+ \to K_0(A \rtimes_\lambda \Gamma)^+$ is additive and order preserving. Note that for a projection q in $\mathcal{P}_\infty(A)$ and $s \in \Gamma$ we have

$$\begin{aligned} \hat{\iota}([q]_{K_0(A)}) &= [\iota(q)]_{K_0(A \rtimes_\lambda \Gamma)} = [u_s q u_s^*]_{K_0(A \rtimes_\lambda \Gamma)} = [\alpha_s(q)]_{K_0(A \rtimes_\lambda \Gamma)} \\ &= \hat{\iota}[\alpha_s(q)]_{K_0(A)} = \hat{\iota}\hat{\alpha}_s([q]_{K_0(A)}), \end{aligned}$$

so that $\hat{\iota} = \hat{\iota}\hat{\alpha}_s$ agree as maps $K_0(A)^+ \to K_0(A \rtimes_\lambda \Gamma)^+$. We now get

$$k\hat{\iota}(x) = \hat{\iota}(kx) \le \hat{\iota}\Big(\sum_{j=1}^n x_j\Big) = \sum_{j=1}^n \hat{\iota}(x_j) = \sum_{j=1}^n \hat{\iota}\hat{\alpha}_{t_j}(x_j) = \hat{\iota}\Big(\sum_{j=1}^n \hat{\alpha}_{t_j}(x_j)\Big)$$
$$\le \hat{\iota}(lx) = l\hat{\iota}(x).$$

The fact that $A \rtimes_\lambda \Gamma$ is stably finite implies that $\hat{\iota}(x) = 0$, which means that $\iota(p) = 0$, so $p = 0$, a contradiction. □

4. A noncommutative type semigroup

We wish to establish a converse to Proposition 3.5. For this we shall need more machinery. Analogous to the type semigroup of a general group action (see [32]), we associate to each suitable C*-system (A, Γ, α) a preordered abelian monoid $S(A, \Gamma, \alpha)$ which correctly reflects the above notion of paradoxicality in $K_0(A)$, and then resort to a Hahn–Banach-type extension result (Theorem 4.6 below) in the spirit of Tarski's theorem tying the existence of states on $S(A, \Gamma, \alpha)$ to non-paradoxicality. We embark on the details.

Let us first recall the notion of equidecomposability for group actions and the construction of the type semigroup. Suppose a group G acts on an arbitrary set Y, and let S be a G-invariant subalgebra of the power set $\mathcal{P}(Y)$. Subsets $E, F \in S$ are said to be *G-equidecomposable relative to S*, and we write $E \sim_{G,S} F$, if there are $E_1, \dots, E_n \in S$, and $g_1, \dots, g_n \in G$ such that:

$$E = \bigsqcup_{j=1}^n E_j \quad \text{and} \quad F = \bigsqcup_{j=1}^n g_j.E_j.$$

The notation $\sqcup$ is used to emphasize the fact that the partitioning sets are disjoint. Reflexivity and symmetry of the relation $\sim_{G,S}$ are straightforward, and transitivity follows from taking refined partitions. We will abbreviate $\sim_{G,S}$ by $\sim_G$ when the context is clear. Note that equidecomposability behaves well with respect to disjoint unions. Indeed, if $E, F, H, K \in S$ with $E \cap H = \emptyset$, $F \cap K = \emptyset$, $E \sim_G F$, and $H \sim_G K$, then it is routine to verify that $(E \sqcup H) \sim_G (F \sqcup K)$. This observation is key when defining addition in the type semigroup below.

Now suppose a group Γ acts on a set X, and let $\mathcal{C} \subset \mathcal{P}(X)$ be a Γ-invariant subalgebra of subsets. Orthogonality is built in as we amplify the action as follows. Set $X^* = X \times \mathbb{N}_0$, and $\Gamma^* = \Gamma \times \mathrm{Perm}(\mathbb{N}_0)$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. There is a natural action $\Gamma^* \curvearrowright X^*$ given by

$$(t, \sigma).(x, n) = (t.x, \sigma(n)).$$

For a set $E \subset X^*$, and $j \in \mathbb{N}_0$, the *j th level* of E is the set $E_j = \{x \in X : (x, j) \in E\}$. We say that E is *bounded* if only finitely many levels E_j are non-empty. The collection

$$S(X, \mathcal{C}) = \{E \subset X^* : E \text{ is bounded and } E_j \in \mathcal{C}, \forall j \in \mathbb{N}_0\}$$

is clearly a Γ^*-invariant subalgebra of subsets of X^*. Consider this collection $S(X, \mathcal{C})$ equipped with the equivalence relation $\sim_{\Gamma^*}$ of Γ^*-equidecomposability relative to $S(X, \mathcal{C})$. Taking a quotient we obtain

$$S(X, \Gamma, \mathcal{C}) := S(X, \mathcal{C}) \big/ \sim_{\Gamma^*},$$

and we write $[E]$ for the equivalence class of $E \in S(X, \mathcal{C})$. With enough room to enforce class representatives to be disjoint, it is readily verified that addition of classes given by

$$\left[\bigcup_{j=1}^{n} E_j \times \{j\}\right] + \left[\bigcup_{i=1}^{m} F_i \times \{i\}\right] = \left[\bigcup_{j=1}^{n} E_j \times \{j\} \cup \bigcup_{i=1}^{m} F_i \times \{n+i\}\right]$$

is well defined. Endowed with the algebraic ordering, $S(X, \Gamma, \mathcal{C})$ has the structure of a preordered abelian monoid with neutral element $[\emptyset]$. If we take $\mathcal{C} = \mathcal{P}(X)$, then $S(X, \Gamma, \mathcal{C})$ is customarily referred to as the type semigroup of the action [32].

We aim to construct a similar monoid for noncommutative C*-systems (A, Γ, α), at least in the presence of sufficiently many projections. The philosophy is that elements of the positive cone $K_0(A)^+$ would represent our "subsets" as it were, and the idea of refined partitions is reflected by suitable refinement properties displayed in the additive structure of $K_0(A)^+$. If we are to translate the notion of equidecomposability to the K_0-setting, we shall require that A be an algebra for which the monoid $K_0(A)^+$ has the the Riesz refinement property. This discussion thus motivates the following definition.

Definition 4.1. Let A be a C^*-algebra, Γ a discrete group, and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ an action. We define a relation on $K_0(A)^+$ as follows:

$$x \sim_\alpha y \quad (x, y \in K_0(A)^+)$$

$$\Longleftrightarrow$$

$$\exists \{u_j\}_{j=1}^k \subset K_0(A)^+, \{t_j\}_{j=1}^k \subset \Gamma, \quad \text{such that} \sum_{j=1}^k u_j = x \text{ and } \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) = y.$$

Lemma 4.2. *If A is a stably finite C*-algebra such that $K_0(A)^+$ has the Riesz refinement property, then $\sim_\alpha$ as defined above is an equivalence relation.*

Proof. Let $x, y \in K_0(A)^+$. Clearly $x \sim_\alpha x$, as we may simply take $u_1 = x$ and $t_1 = e$. If $x \sim_\alpha y$, via the decomposition $x = \sum_{j=1}^k u_j$ and $y = \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j)$, set $v_j = \hat{\alpha}_{t_j}(u_j)$ and $s_j = t_j^{-1}$ for $j = 1, \dots k$. It clearly follows that

$$\sum_{j=1}^k v_j = y \quad \text{and} \quad \sum_{j=1}^k \hat{\alpha}_{s_j}(v_j) = \sum_{j=1}^k \hat{\alpha}_{t_j^{-1}}(\hat{\alpha}_{t_j}(u_j)) = \sum_{j=1}^k u_j = x$$

whence $y \sim_\alpha x$. Transitivity is a little harder, and here is where the fact that $K_0(A)^+$ has the Riesz refinement property will surface. To that end, suppose $x \sim_\alpha y \sim_\alpha z$ via

$$x = \sum_{j=1}^k u_j, \quad y = \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) \quad \text{and} \quad y = \sum_{j=1}^l v_j, \quad z = \sum_{j=1}^l \hat{\alpha}_{s_j}(v_j).$$

Since $\sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) = \sum_{j=1}^l v_j$ and $K_0(A)^+$ has the Riesz refinement property, there are elements $\{w_{ij} : 1 \le j \le l, 1 \le i \le k\} \subset K_0(A)^+$ such that

$$\sum_{j=1}^l w_{ij} = \hat{\alpha}_{t_i}(u_i) \quad \text{and} \quad \sum_{i=1}^k w_{ij} = v_j.$$

We then compute

$$\sum_{i,j} \hat{\alpha}_{s_j t_i}(\hat{\alpha}_{t_i^{-1}}(w_{ij})) = \sum_{i,j} \hat{\alpha}_{s_j}(w_{ij}) = \sum_j \hat{\alpha}_{s_j}\left(\sum_i w_{ij}\right) = \sum_j \hat{\alpha}_{s_j}(v_j) = z,$$

while

$$\sum_{i,j} \hat{\alpha}_{t_i^{-1}}(w_{ij}) = \sum_i \hat{\alpha}_{t_i^{-1}}\left(\sum_j w_{ij}\right) = \sum_i \hat{\alpha}_{t_i^{-1}}(\hat{\alpha}_{t_i}(u_i)) = \sum_i u_i = x.$$

which gives the desired decomposition for $x \sim_\alpha z$. □

We can now make the following definition.

Definition 4.3. Let A be a C*-algebra such that $K_0(A)^+$ has the Riesz refinement property. Let $\Gamma \to \mathrm{Aut}(A)$ be an action. We set $S(A, \Gamma, \alpha) := K_0(A)^+ / \sim_\alpha$, and write $[x]_\alpha$ for the equivalence class with representative $x \in K_0(A)^+$.

We can define addition of classes simply by $[x]_\alpha + [y]_\alpha := [x+y]_\alpha$ for x, y in $K_0(A)^+$. It is routine to check that this operation is well defined; indeed if $x' \sim_\alpha x$ and $y' \sim_\alpha y$ via $x = \sum_{j=1}^k u_j$, $y = \sum_{j=1}^m v_j$ and $x' = \sum_{j=1}^k \hat{\alpha}_{t_j}(u_j)$, $y' = \sum_{j=1}^m \hat{\alpha}_{s_j}(v_j)$, then

$$[x']_\alpha + [y']_\alpha = [x'+y']_\alpha = \Big[\sum_{j=1}^k \hat{\alpha}_{t_j}(u_j) + \sum_{j=1}^m \hat{\alpha}_{s_j}(v_j)\Big]_\alpha$$
$$= \Big[\sum_{j=1}^k u_j + \sum_{j=1}^m v_j\Big]_\alpha = [x+y]_\alpha = [x]_\alpha + [y]_\alpha.$$

We make a few elementary observations concerning $S(A, \Gamma, \alpha)$ when A is stably finite. Firstly, $S(A, \Gamma, \alpha)$ is not just a semigroup but an abelian monoid as $[0]_\alpha$ is clearly the neutral additive element. Impose the algebraic ordering on $S(A, \Gamma, \alpha)$, that is, set $[x]_\alpha \le [y]_\alpha$ if there is a $z \in K_0(A)^+$ with $[x]_\alpha + [z]_\alpha = [y]_\alpha$. This gives $S(A, \Gamma, \alpha)$ the structure of an abelian preordered monoid. Notice at once that if $x, y \in K_0(A)^+$ with $x \le y$ (in the ordering of $K_0(A)$) then $[x]_\alpha \le [y]_\alpha$ in $S(A, \Gamma, \alpha)$. To see this, $x \le y$ implies $y - x := z \in K_0(A)^+$, so $[y]_\alpha = [x+z]_\alpha = [x]_\alpha + [z]_\alpha$ which gives $[x]_\alpha \le [y]_\alpha$. Next, we observe that if $[x]_\alpha = [0]_\alpha$, for some x in $K_0(A)^+$, then in fact $x = 0$. Indeed, say $x = \sum_i u_i$, and $\sum_i \hat{\alpha}_{t_i}(u_i) = 0$ for some elements $t_i \in \Gamma$ and $u_i \in K_0(A)^+$, then for each i, $\hat{\alpha}_{t_i}(u_i) = 0$ and so $u_i = 0$ which gives $x = 0$. Here we used the important fact that for stably finite algebras A, $K_0(A)^+ \cap (-K_0(A)^+) = (0)$. All together, there is an order preserving, faithful, monoid homomorphism

$$\rho : K_0(A)^+ \to S(A, \Gamma, \alpha) \quad \text{given by } \rho(g) = [g]_\alpha.$$

This next result shows that we have in fact constructed a noncommutative analogue of the type semigroup studied in [32].

Proposition 4.4. *Let X be the Cantor set, Γ a discrete group, and $\Gamma \curvearrowright X$ a continuous action with corresponding action $\alpha : \Gamma \to \mathrm{Aut}(C(X))$. Write $\mathscr{C}$ for the Γ-invariant algebra of all clopen subsets of X. Then the type semigroup $S(X, \Gamma, \mathscr{C})$ is isomorphic to $S(C(X), \Gamma, \alpha)$ constructed above.*

Proof. Let $f \in K_0(C(X))^+ = C(X; \mathbb{Z})^+$, then we can write $f = \sum_{j=1}^n \mathbb{1}_{E_j}$ where the E_j are clopen subsets of X. Note that such a representation is not unique.

Claim. Suppose $f = \sum_{j=1}^n \mathbb{1}_{E_j} = \sum_{j=1}^m \mathbb{1}_{F_j}$. Then

$$\bigsqcup_{j=1}^n E_j \times \{j\} =: E \sim_{\Gamma^*} F := \bigsqcup_{j=1}^m F_j \times \{j\},$$

so that $[E] = [F]$ in the type semigroup $S(X, \Gamma, \mathscr{C})$.

It is clear that $\cup_{j=1}^{n} E_j = \cup_{j=1}^{m} F_j$. By choosing a common clopen refinement, we may assume that there are disjoint clopen sets $H_1, \dots, H_r$, where $r \geq n, m$, such that each E_j and each F_j is a union of distinct H_i. For each $i = 1, \dots, r$ set the multiplicities of the H_i as

$$n_i := \big|\{j : H_i \subset E_j\}\big| = \big|\{j : H_i \subset F_j\}\big|.$$

In this case we have $f = \sum_{i=1}^{r} n_i \mathbb{1}_{H_i}$. Now set $H = \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{n_i} H_i \times \{j\}$, and for each pair (i, j) set

$$\Delta_{i,j} = \begin{cases} H_i, & \text{if } H_i \subset E_j, \\ \emptyset, & \text{if } H_i \cap E_j = \emptyset. \end{cases}$$

With a j fixed we run through all the H_i and get $\bigsqcup_{i=1}^{r} \Delta_{i,j} \times \{j\} = E_j \times \{j\}$. Then

$$\begin{aligned} E = \bigsqcup_{j=1}^{n} E_j \times \{j\} &= \bigsqcup_{j=1}^{n} \bigsqcup_{i=1}^{r} \Delta_{i,j} \times \{j\} \\ &= \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{n} \Delta_{i,j} \times \{j\} \sim_{\Gamma^*} \bigsqcup_{i=1}^{r} \bigsqcup_{j=1}^{n_i} H_i \times \{j\} = H. \end{aligned}$$

By a similar argument $F \sim_{\Gamma^*} H$, and transitivity gives $E \sim_{\Gamma^*} F$ and the claim is thus proved.

We now define a map $\psi : K_0(C(X))^+ \to S(X, \Gamma, \mathcal{C})$ by

$$\psi(f) = \Big[\bigsqcup_{j=1}^{n} E_j \times \{j\}\Big]$$

where f has representation $f = \sum_{j=1}^{n} \mathbb{1}_{E_j}$ with $E_j \subset X$ clopen. Thanks to the claim, this map is well defined as any representation of f will do. Also, it is routine to check that ψ is additive and onto. Moreover, ψ is invariant under the equivalence $\sim_\alpha$. To see this, suppose $f, g \in K_0(C(X))^+$ and $f \sim_\alpha g$. By definition and by writing members of $K_0(C(X))^+$ as sums of indicator functions on clopen sets we can find clopen sets $E_1, \dots, E_n \in \mathcal{C}$ and group elements $t_1, \dots, t_n \in \Gamma$ with

$$f = \sum_{j=1}^{n} \mathbb{1}_{E_j} \quad \text{and} \quad g = \sum_{j=1}^{n} \mathbb{1}_{t_j.E_j}.$$

Since $\bigsqcup_{j=1}^{n} E_j \times \{j\} \sim_{\Gamma^*} \bigsqcup_{j=1}^{n} t_j.E_j \times \{j\}$ we get that $\psi(f) = \psi(g)$. The map ψ thus descends to a surjective monoid homomorphism $\overline{\psi} : S(C(X), \Gamma, \alpha) \to S(X, \Gamma, \mathcal{C})$ with $\overline{\psi}([f]_\alpha) = \psi(f)$. To establish injectivity we construct a left inverse $\varphi : S(X, \Gamma, \mathcal{C}) \to S(C(X), \Gamma, \alpha)$ as follows. Set

$$\varphi\Big(\Big[\bigsqcup_{j=1}^{n} E_j \times \{j\}\Big]\Big) = \Big[\sum_{j=1}^{n} \mathbb{1}_{E_j}\Big]_\alpha.$$

To show that φ is well defined, suppose $E = \bigsqcup_{j=1}^{n} E_j \times \{j\} \sim_{\Gamma^*} F = \bigsqcup_{j=1}^{m} F_j \times \{j\}$, then there exist $l \in \mathbb{N}$, $C_k \in \mathcal{C}$, $t_k \in \Gamma$ and natural numbers n_k, m_k for $k = 1, \ldots, l$, such that

$$E = \bigsqcup_{k=1}^{l} C_k \times \{n_k\}, \qquad F = \bigsqcup_{k=1}^{l} t_k.C_k \times \{m_k\}.$$

For each fixed j, we see that $\bigsqcup_{\{k:\, n_k=j\}} C_k = E_j$, so $\sum_{\{k:\, n_k=j\}} \mathbb{1}_{C_k} = \mathbb{1}_{E_j}$. Therefore

$$\sum_{j=1}^{n} \mathbb{1}_{E_j} = \sum_{j=1}^{n} \sum_{\{k:\, n_k=j\}} \mathbb{1}_{C_k} = \sum_{k=1}^{l} \mathbb{1}_{C_k} \sim_\alpha \sum_{k=1}^{l} \mathbb{1}_{t_k.C_k} = \sum_{j=1}^{n} \mathbb{1}_{F_j},$$

where the last equality follows from same reasoning. It follows that $\varphi([E]) = \varphi([F])$. Also φ is clearly additive and onto. For an element $[f]_\alpha \in S(C(X), \Gamma, \alpha)$, where f has representation $f = \sum_{j=1}^{n} \mathbb{1}_{E_j}$, we see that

$$\varphi \circ \overline{\psi}([f]_\alpha) = \varphi \circ \psi(f) = \varphi\Big(\Big[\bigsqcup_{j=1}^{n} E_j \times \{j\}\Big]\Big) = \Big[\sum_{j=1}^{n} \mathbb{1}_{E_j}\Big]_\alpha = [f]_\alpha.$$

We conclude that $\overline{\psi}$ is a monoid isomorphism. Since both monoids are preordered with the algebraic ordering $\overline{\psi}$ is actually an isomorphism of preordered monoids. □

Next we look at how (Γ, k, l)-paradoxically is reflected in our monoid $S(A, \Gamma, \alpha)$.

Lemma 4.5. *Let A be a stably finite C*-algebra such that $K_0(A)^+$ has Riesz refinement, and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action. Then an element $0 \neq x \in K_0(A)^+$ is (Γ, k, l)-paradoxical if and only if $k[x] \leq l[x]$ in $S(A, \Gamma, \alpha)$.*

Proof. Suppose $0 \neq x \in K_0(A)^+$ is (Γ, k, l)-paradoxical. Then $kx \leq \sum_{j=1}^{n} x_j$ and $\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \leq lx$ for some x_j in $K_0(A)^+$ and t_j in Γ. Then from our above remarks:

$$k[x]_\alpha = [kx]_\alpha \leq \Big[\sum_{j=1}^{n} x_j\Big]_\alpha = \Big[\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j)\Big]_\alpha \leq [lx]_\alpha = l[x]_\alpha.$$

Now assume $k[x]_\alpha \leq l[x]_\alpha$ for integers $k > l > 0$. As the ordering is algebraic, there is a z in $K_0(A)^+$ with $k[x]_\alpha + [z]_\alpha = l[x]_\alpha$. We then have

$$[kx + z]_\alpha = [kx]_\alpha + [z]_\alpha = k[x]_\alpha + [z]_\alpha = l[x]_\alpha = [lx]_\alpha.$$

By definition there are elements $x_1, \ldots, x_n$ in $K_0(A)^+$ and $t_1, \ldots, t_n \in \Gamma$ with

$$kx \leq kx + z = \sum_{j=1}^{l} x_j \quad \text{and} \quad \sum_{j=1}^{l} \hat{\alpha}_{t_j}(x_j) = lx,$$

which witnesses the (Γ, k, l)-paradoxicality of x. The proof is complete. □

Before going any further let us recall some terminology. Let $(W, \leq)$ be a preordered abelian monoid. For positive integers $k > l > 0$, we say that an element $\theta \in W$ is (k,l)-*paradoxical* provided that $k\theta \leq l\theta$. If θ fails to be paradoxical for all pairs of integers $k > l > 0$, call θ *completely non-paradoxical*. Note that θ is completely non-paradoxical if and only if $(n+1)\theta \not\leq n\theta$ for all $n \in \mathbb{N}$. The above lemma basically states that in its setting, an element $x \in K_0(A)^+$ is completely non-paradoxical with respect to the action $\hat{\alpha}$ exactly when $[x]_\alpha$ is completely non-paradoxical in the preordered abelian monoid $S(A, \Gamma, \alpha)$. An element θ in W is said to *properly infinite* if $2\theta \leq \theta$, that is, if it is $(2,1)$-paradoxical, or equivalently it is $(k,1)$-paradoxical for any $k \geq 2$. If every member of W is properly infinite then W is said to be *purely infinite*. A *state* on W is a map $\nu : W \to [0, \infty]$ which is additive, respects the preordering $\leq$, and satisfies $\nu(0) = 0$. If a state β assumes a value other than 0 or ∞, β it said to be *non-trivial*. The monoid W is said to be *almost unperforated* if, whenever $\theta, \eta \in W$, and $n, m \in \mathbb{N}$ are such that $n\theta \leq m\eta$ and $n > m$, then $\theta \leq \eta$.

The following result is a main ingredient in the proof of Tarski's theorem. It is a Hahn–Banach type extension result and is essential in establishing a converse to Proposition 3.4. A proof can be found in [32].

Theorem 4.6. *Let* $(W, +)$ *be an abelian monoid equipped with the algebraic ordering, and let* θ *be an element of* W*. Then the following are equivalent:*

(1) $(n+1)\theta \not\leq n\theta$ *for all* $n \in \mathbb{N}$*, that is* θ *is completely non-paradoxical.*

(2) *There is a non-trivial state* $\nu : W \to [0, \infty]$ *with* $\nu(\theta) = 1$*.*

We mean to apply Theorem 4.6 to our preordered monoid $S(A, \Gamma, \alpha)$. Note that such a ν, which arises in the landscape of complete non-paradoxicality will not in general be finite on all of $S(A, \Gamma, \alpha)$. One needs the right condition on the action α, or more precisely, $\hat{\alpha}$, to guarantee finiteness everywhere. Suppose we considered $\theta = [u]_\alpha$ as in Theorem 4.6, where $u = [1]_0$ is the order unit in $K_0(A)$. If we compose the state ν with the the above $\rho : K_0(A)^+ \to S(A, \Gamma, \alpha)$, this would give us, in a sense, an invariant 'state' at the K-theoretic level, but perhaps not finitely valued everywhere, but with a finite value at $[1]_0$. To ensure finiteness at every $x \in K_0(A)^+$ we would require that finitely many Γ-iterates of x lie above $[1]_0$. This is exactly the notion of K-theoretic minimality explored in [24]. The following result is contained therein and is sufficient for our purposes. For the sake of completeness we include a proof. Recall that a C*-dynamical system (A, Γ, α) is minimal if A admits no non-trivial Γ-invariant ideals.

Proposition 4.7. *Let* (A, Γ, α) *be a C*-dynamical system with* A *stably finite. If* α *is minimal then the induced action* $\hat{\alpha} : \Gamma \curvearrowright K_0(A)$ *is* K_0*-minimal, that is, for every non-zero* $g \in K_0(A)^+$ *there are group elements* $t_1, \ldots, t_n$ *in* Γ *such that*

$$\sum_{j=1}^{n} \hat{\alpha}_{t_j}(g) \geq [1]_0.$$

Proof. Let $g = [p]_0$ be non-zero in $K_0(A)^+$ with $p \in \mathcal{P}_n(A)$. Since the algebraic ideal generated by $\{\alpha_s^{(n)}(p) : s \in \Gamma\}$ is all of $M_n(A)$, there are elements $t_1, \dots, t_m \in \Gamma$, and $x_1, \dots, x_m, y_1, \dots, y_m$ in $M_n(A)$ such that

$$\sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(p) y_j^* = \frac{1}{2} \mathbf{1}_{M_n(A)}.$$

Now set $z_j := x_j + y_j$ and observe that

$$\begin{aligned}\sum_{j=1}^{m} z_j \alpha_{t_j}^{(n)}(p) z_j^* &= \sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(p) y_j^* + \sum_{j=1}^{m} y_j \alpha_{t_j}^{(n)}(p) x_j^* \\ &\qquad + \sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(p) x_j^* + \sum_{j=1}^{m} y_j \alpha_{t_j}^{(n)}(p) y_j^* \\ &\geq \sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(p) y_j^* + \Big(\sum_{j=1}^{m} x_j \alpha_{t_j}^{(n)}(p) y_j^* \Big)^* \\ &= \mathbf{1}_{M_n(A)} \geq 1_A \oplus 0_{m-1},\end{aligned}$$

the first inequality following from the fact that the last two sums on the first line are positive. A simple Cuntz comparison now gives

$$\begin{aligned}1_A &\approx 1_A \oplus 0_{m-1} \\ &\precsim \sum_{j=1}^{m} z_j \alpha_{t_j}^{(n)}(p) z_j^* = (z_1, \dots, z_m)(\alpha_{t_1}^{(n)}(p) \oplus \cdots \oplus \alpha_{t_m}^{(n)}(p))(z_1, \dots, z_m)^* \\ &\precsim \alpha_{t_1}^{(n)}(p) \oplus \cdots \oplus \alpha_{t_m}^{(n)}(p).\end{aligned}$$

Therefore, in the ordering on $K_0(A)$ we get

$$[1]_0 \leq [\alpha_{t_1}^{(n)}(p) \oplus \cdots \oplus \alpha_{t_m}^{(n)}(p)]_0 = \sum_{j=1}^{m} [\alpha_{t_j}^{(n)}(p)]_0 = \sum_{j=1}^{m} \hat{\alpha}_{t_j}([p]_0) = \sum_{j=1}^{m} \hat{\alpha}_{t_j}(g),$$

which gives the K_0-minimality of the action. □

Proposition 4.8. *Let A be a stably finite unital C*-algebra for which $K_0(A)^+$ has Riesz refinement (*$\mathrm{sr}(A) = 1$ *and* $\mathrm{RR}(A) = 0$ *for example). Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action on A. Consider the following properties.*

(1) *For every $0 \neq g \in K_0(A)^+$, there is a faithful Γ-invariant positive group homomorphism $\beta : K_0(A) \to \mathbb{R}$ with $\beta(g) = 1$, (Γ-invariant in the sense that $\beta \circ \hat{\alpha} = \beta$ on $K_0(A)$).*

(2) *There is a faithful* Γ*-invariant state* β *on* $(K_0(A), K_0(A)^+, [1]_0)$.

(3) α *is completely non-paradoxical.*

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If the action* α *is minimal, then* (3) $\Rightarrow$ (1) *whence all the conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): Simply take $g = [1]_0$.

(2) $\Rightarrow$ (3): Assume that $x \in K_0(A)^+$ is (Γ, k, l)-paradoxical for some integers $k > l > 0$ with paradoxical decomposition $\sum_j^n x_j \geq kx$ and $\sum_j^n \hat{\alpha}_{t_j}(x_j) \leq lx$ for certain $x_j \in K_0(A)^+$ and $t_j \in \Gamma$. Apply the $\hat{\alpha}$-invariant state β and get

$$\begin{aligned} k\beta(x) = \beta(kx) \leq \beta\Bigg(\sum_j^n x_j\Bigg) = \sum_j^n \beta(x_j) = \sum_j^n \beta(\hat{\alpha}_{t_j}(x_j)) = \beta\Bigg(\sum_j^n \hat{\alpha}_{t_j}(x_j)\Bigg) \\ \leq \beta(lx) = l\beta(x). \end{aligned}$$

Now since β is faithful, we may divide by $\beta(x) > 0$ and get $k \leq l$ which is absurd.

Assuming the action α is minimal we prove (3) $\Rightarrow$ (1). Fix a non-zero $g \in K_0(A)^+$. Since the action is completely non-paradoxical, it follows from Lemma 4.5 that for every positive integer n, $(n+1)[g]_\alpha \not\leq n[g]_\alpha$. Theorem 4.6 then states that $S(A, \Gamma, \alpha)$ admits a non-trivial state $\nu : S(A, \Gamma, \alpha) \to [0, \infty]$ with $\nu([g]_\alpha) = 1$.

Claim. ν is finite.

To see this, employ K_0-minimality of the action (Proposition 4.7) to obtain group elements $t_1, \ldots, t_n$ such that $\sum_{j=1}^n \hat{\alpha}_{t_j}(g) \geq [1]_0$. Now for an arbitrary $[x]_\alpha$ in $S(A, \Gamma, \alpha)$ with x belonging to $K_0(A)^+$, there is a positive integer m with $x \leq m[1]_0 \leq m \sum_{j=1}^n \hat{\alpha}_{t_j}(g)$. Therefore

$$[x]_\alpha \leq \Bigg[m \sum_{j=1}^n \hat{\alpha}_{t_j}(g)\Bigg]_\alpha = m\Bigg[\sum_{j=1}^n \hat{\alpha}_{t_j}(g)\Bigg]_\alpha = m[ng]_\alpha = mn[g]_\alpha.$$

Applying ν yields $\nu([x]_\alpha) \leq \nu(mn[g]_\alpha) = mn\nu([g]_\alpha) = mn$. The claim is therefore proved.

We now compose ν with our above $\rho : K_0(A)^+ \to S(A, \Gamma, \alpha)$ to yield $\beta' : K_0(A)^+ \to ([0, \infty), +)$, a finite order preserving monoid homomorphism given by $\beta'(x) = \nu([x]_\alpha)$. Note how β' is invariant under the action $\hat{\alpha} : \Gamma \curvearrowright K_0(A)^+$. Indeed, for t in Γ, and x in $K_0(A)^+$,

$$\beta'(\hat{\alpha}_t(x)) = \nu([\hat{\alpha}_t(x)]_\alpha) = \nu([x]_\alpha) = \beta'(x).$$

By the universality of the Grothendieck enveloping group construction, there is a unique extension of β' to a group homomorphism on all of $K_0(A)$, which we will denote as β, given simply by $\beta(x-y) = \beta'(x) - \beta'(y)$ for x, y in $K_0(A)^+$. Clearly β is still Γ-invariant. The final product is a bona fide Γ-invariant positive group homomorphism $\beta : K_0(A) \to \mathbb{R}$, with $\beta(g) = 1$. We now show that β is faithful, which will complete this direction. Assume $0 \neq x \in K_0(A)^+$. Minimality ensures the existence of group elements $t_1, \dots t_n$ with $\sum_{j=1}^n \hat{\alpha}_{t_j}(x) \geq [1]_0$. Now we find a positive integer m for which $m[1]_0 \geq g$, so that $m\big(\sum_{j=1}^n \hat{\alpha}_{t_j}(x)\big) \geq g$. Applying β gives

$$\begin{aligned} 1 = \beta(g) &\leq \beta\left(m\left(\sum_{j=1}^n \hat{\alpha}_{t_j}(x)\right)\right) \\ &= m\left(\sum_{j=1}^n \beta(\hat{\alpha}_{t_j}(x))\right) = m\left(\sum_{j=1}^n \beta(x)\right) = mn\beta(x) \end{aligned}$$

thus $\beta(x) \neq 0$ and β is indeed faithful. □

We now are ready to establish the long desired converse.

Theorem 4.9. *Let A be a stably finite unital C*-algebra for which $K_0(A)^+$ has Riesz refinement (*$\mathrm{sr}(A) = 1$ *and* $\mathrm{RR}(A) = 0$ *for example). Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal action on A. Consider the following properties.*

(1) *There is an Γ-invariant faithful tracial state $\tau : A \to \mathbb{C}$.*

(2) *$A \rtimes_\lambda \Gamma$ admits a faithful tracial state.*

(3) *$A \rtimes_\lambda \Gamma$ is stably finite.*

(4) *α is completely non-paradoxical.*

(5) *There is a faithful Γ-invariant state β on $(K_0(A), K_0(A)^+, [1]_0)$.*

Then we have the following implications:

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5).$$

If A is exact and projections are total in A (e.g. $RR(A) = 0$) then (5) $\Leftrightarrow$ (1). *Furthermore, if A is an AF-algebra and Γ is a free group, then* (1) *through* (5) *are all equivalent to $A \rtimes_\lambda \Gamma$ being MF in the sense of Blackadar and Kirchberg [7].*

Proof. It is well known that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). Also, (3) $\Rightarrow$ (4) is Proposition 3.5 and (4) $\Rightarrow$ (5) is Proposition 4.8.

(5) $\Rightarrow$ (1): Since A exact, such a β arises from a tracial state $\tau : A \to \mathbb{C}$, via $\tau(p) = \beta([p])$ for any projection $p \in A$ ([27]). We need only to show the Γ-invariance of τ. For any $s \in \Gamma$ and projection p in A,

$$\tau(\alpha_s(p)) = \beta([\alpha_s(p)]) = \beta \circ \hat{\alpha}_s([p]) = \beta([p]) = \tau(p).$$

Using linearity, continuity, and the fact that the projections are total in A, it follows that $\tau(\alpha_s(a)) = \tau(a)$ for every $a \in A$ and $s \in \Gamma$ which yields the invariance.

Now we let $\Gamma = \mathbb{F}_r$ be the free group on r generators, and let A be an AF-algebra. In [23] the author shows that $A \rtimes_\lambda \mathbb{F}_r$ is MF if and only if it is stably finite. □

Recall that a continuous affine action of an amenable group Γ on a compact convex subset K of a locally convex space admits a fixed point.

Corollary 4.10. *Let A be a simple, unital, AF algebra and Γ a discrete amenable group. Then any action $\alpha : \Gamma \to \mathrm{Aut}(A)$ is completely non-paradoxical.*

Proof. Let $T(A)$ denote the set of all tracial states on A. The group Γ acts on $T(A)$ as $t.\tau(a) = \tau(\alpha_{t^{-1}}(a))$ for $t \in \Gamma$ and $a \in A$. Since Γ is amenable, $T(A)$ has a fixed point. Now apply Theorem 4.9. □

5. Purely infinite crossed products

In this section we want to characterize purely infinite crossed products by the "properly infinite" nature of the corresponding type semigroup constructed above. This characterization is much in line with the work of Rørdam and Sierakowski in [28], except that we will generalize their ideas by addressing the noncommutative setting. As a brief reminder, a projection $p \in A$ is properly infinite if there are two subprojections $q, r \leq p$ with $qr = 0$ and $q \sim p \sim r$. A unital C*-algebra A is properly infinite if 1_A is properly infinite. A simple algebra A is called purely infinite if every hereditary C*-subalgebra of A contains a properly infinite projection. In the simple case, S. Zhang showed that A is purely infinite if and only in $\mathrm{RR}(A) = 0$ and every projection in A is properly infinite [34]. It was a longstanding open question whether there existed a unital, separable, nuclear and simple C*-algebra which was neither stably finite or purely infinite. M. Rørdam settled the issue in [26] by exhibiting a unital, simple, nuclear, and separable C*-algebra D containing a finite projection p and an infinite projection q. It follows that $A = qDq$ is unital, separable, nuclear, simple, and properly infinite, but not purely infinite. It is natural to ask if there is a smaller class of algebras for which a stably finite/purely infinite dichotomy exists. Theorem 5.7 below is a result in this direction.

We first mention a few examples of purely infinite algebras arising as crossed products.

A continuous action $\Gamma \curvearrowright X$ of a discrete group on a compact Hausdorff space is called a *strong boundary action* if X has at least three points and for every pair U, V of non-empty open subsets of X there exists $t \in \Gamma$ with $t.U^c \subset V$. Laca and Spielberg showed in [19] that if $\Gamma \curvearrowright X$ is a strong boundary action and the induced action $\Gamma \curvearrowright C(X)$ is properly outer then $C(X) \rtimes_\lambda \Gamma$ is purely infinite and simple.

Jolissaint and Robertson [13] made a generalization valid in the noncommutative setting. They called an action $\alpha : \Gamma \to \mathrm{Aut}(A)$ *n-filling* if, for all $a_1, \dots, a_n \in A^+$, with $\|a_j\| = 1$, $1 \le j \le n$, and for all $\varepsilon > 0$, there exist $t_1, \dots, t_n \in \Gamma$ such that $\sum_{j=1}^n \alpha_{t_j}(a_j) \ge (1-\varepsilon)1_A$. They showed that $A \rtimes_\lambda \Gamma$ is purely infinite and simple provided that the action is properly outer and n-filling and every corner pAp of A is infinite dimensional. Using the tools from K-theoretic dynamics that we develop in this section, we will provide a simpler proof of their result in the real rank zero case.

First we give a necessary condition for pure infiniteness in the case where the underlying algebra is finite.

Proposition 5.1. *Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action of a discrete group on a finite C*-algebra A with induced action $\hat{\alpha}$ on $W(A)$. Suppose $A \rtimes_\lambda \Gamma$ is purely infinite. Then for every non-zero $x \in W(A)$ there is an $s \neq e$ in Γ and a non-zero $y \in W(A)$ with $y \le x$ and $\hat{\alpha}_s(y) \le x$.*

Proof. Given $x \in W(A)$, by compressing we can find a non-zero $a \in A^+$ with $\langle a \rangle \le x$. Since the crossed product is purely infinite, every non-zero positive element is properly infinite. This means that $a \oplus a \precsim a$ relative to $A \rtimes_\lambda \Gamma$. There is, therefore, a sequence $(z_n)_n \ge 1 \subset M_{1,2}(A \rtimes_\lambda \Gamma)$ with $\|z_n^* a z_n - a \oplus a\| < 1/n$. By approximating each matrix entry we can further assume that the z_n belong to $M_{1,2}(C_c(\Gamma, A))$. Writing each $z_n = (v_n, w_n)$ we get

$$\left\| \begin{pmatrix} v_n^* a v_n - a & v_n^* a w_n \\ w_n^* a v_n & w_n^* a w_n - a \end{pmatrix} \right\| = \left\| \begin{pmatrix} v_n^* a v_n & v_n^* a w_n \\ w_n^* a v_n & w_n^* a w_n \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\|$$
$$= \left\| \begin{pmatrix} v_n^* \\ w_n^* \end{pmatrix} a \begin{pmatrix} v_n & w_n \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\| < 1/n.$$

By compressing by projections or conjugating by unitaries it follows that

$$v_n^* a v_n \longrightarrow a, \quad v_n^* a w_n \longrightarrow 0, \quad w_n^* a v_n \longrightarrow 0, \quad w_n^* a w_n \longrightarrow a,$$

If both sequences $(v_n)_n$ and $(w_n)_n$ admit subsequences that belong to A, the element a would be properly infinite in A which cannot be the case since A is finite. So without loss of generality we may assume that for every n, v_n is not in A. Thus if we write

$$v_n = \sum_{s \in F_n} a_{n,s} u_s, \quad \text{where } F_n \subset \Gamma \text{ is a finite set and } a_{n,s} \in A \setminus \{0\},$$

we can assume that $\{e\} \neq F_n$ for each n. Now we apply the conditional expectation

$\mathbb{E} : A \rtimes_\lambda \Gamma \to A$ and get

$$\begin{aligned}\mathbb{E}(v_n^* a v_n) &= \mathbb{E}\Big(\sum_{s,t\in F_n} u_{n,s}^* a_{n,s}^* a a_{n,t} u_t\Big)\\ &= \sum_{s,t\in F_n} \mathbb{E}(u_s^* a_{n,s}^* a a_{n,t} u_s u_s^* u_t)\\ &= \sum_{s,t\in F_n} \mathbb{E}(\alpha_s^{-1}(a_{n,s}^* a a_{n,t}) u_{s^{-1}t}) = \sum_{t\in F_n} \alpha_t^{-1}(a_{n,t}^* a a_{n,t}).\end{aligned}$$

Now since $\mathbb{E}$ is contractive and idempotent we have

$$\Big\| a - \sum_{t\in F_n} \alpha_t^{-1}(a_{n,t}^* a a_{n,t}) \Big\| = \|\mathbb{E}(a - v_n^* a v_n)\| \leq \|a - v_n^* a v_n\| < 1/n.$$

Since $a \neq 0$ we know that there is a k with $\sum_{t\in F_k} \alpha_t^{-1}(a_{k,t}^* a a_{k,t}) \neq 0$. For economy we will suppress the k and set $F = F_k$, and $a_t = a_{k,t}$ for $t \in F$. By Lemma 2.5 in [17] we know that there is a $b \in A$ with

$$(a - 1/k)_+ = b^* \Big(\sum_{t\in F} \alpha_t^{-1}(a_t^* a a_t)\Big) b = \sum_{t\in F} (\alpha_t^{-1}(a_t) b)^* \alpha_t^{-1}(a) \alpha_t^{-1}(a_t) b.$$

Pick an $s \in F$ with $s \neq e$ so that

$$a \geq (a - 1/k)_+ \geq (\alpha_s^{-1}(a_s) b)^* \alpha_s^{-1}(a) \alpha_s^{-1}(a_s) b \neq 0,$$

and set $c = (\alpha_s^{-1}(a_s) b)^* \alpha_s^{-1}(a) \alpha_s^{-1}(a_s) b$. Then $0 \neq c \precsim a$ and

$$\alpha_s(c) = (a_s \alpha_s(x))^* a a_s \alpha_s(x) \precsim a.$$

The proof is complete with $y = \langle c \rangle$. □

In the commutative case the following is immediate.

Corollary 5.2. *Let (X, Γ) be a transformation group with Γ discrete. If $C_0(X) \rtimes_\lambda \Gamma$ is purely infinite then for every non-empty open $U \subset X$ there is an $s \neq e$ in Γ with $s.U \cap U \neq \emptyset$.*

Proof. Let $\emptyset \neq U \subset X$ be open and let $f : X \to [0, 1]$ be any non-zero continuous function whose open support $\mathrm{supp}(f) \subset U$. By Propostion 5.1 there is an $s \in \Gamma$, $s \neq e$ and a non-zero $y \in W(C_0(X))$ with $y \leq \langle f \rangle$ and $\hat{\alpha}_s(y) \leq \langle f \rangle$. We may assume $y = \langle g \rangle$ for some non-zero $g \in C_0(X)^+$. Set $V = \mathrm{supp}(g)$. Then we have $\emptyset \neq V \subset U$ and $s.V \subset U$ which implies $s.U \cap U \neq \emptyset$. □

We now concentrate on characterizing a class of purely infinite crossed products by means of the paradoxical nature of the type semigroup. Recall that an automorphism α in $\mathrm{Aut}(A)$ is said to be *properly outer* if and only if for every invariant ideal $I \subset A$ and inner automorphism β in $\mathrm{Inn}(I)$ we have $\|\alpha|_I - \beta\| = 2$. An action $\alpha : \Gamma \to \mathrm{Aut}(A)$ is said to be *properly outer* if for every $e \neq t \in \Gamma$, α_t is properly outer. The authors of [20] prove that $A \rtimes_\lambda \Gamma$ is simple provided that α is minimal and properly outer. In this setting we show that if A admits enough projections, pure infiniteness of the reduced crossed product is witnessed by the projections in A (see Theorem 5.4 below). The following lemma contains ideas from Lemma 3.2 of [28] and Lemma 7.1 of [20].

Lemma 5.3. *Let (A, Γ, α) be a C*-dynamical system with A separable and Γ countable and discrete. Assume that α is properly outer. Then for every non-zero $b \in (A \rtimes_\lambda \Gamma)^+$ there is a non-zero $a \in A^+$ with $a \precsim b$.*

Proof. We know that $\mathbb{E}(b) \neq 0$ since b is non-zero and $\mathbb{E}$ is faithful. Set $b_1 = b/\|\mathbb{E}(b)\|$ so that $\|\mathbb{E}(b_1)\| = 1$. Let $0 < \varepsilon < 1/16$. Find a $\delta > 0$ with $\frac{\delta(1+\|b_1\|)}{1-\delta} < \varepsilon$. Next find a non-zero positive $c \in C_c(\Gamma, A)^+$ with $\|c - b_1\| < \delta$. Write $c = \sum_{s\in F} c_s u_s$ where F is a finite subset of Γ. Note that $\mathbb{E}(c) = c_e \neq 0$, and also $\big|1 - \|c_e\|\big| \leq \delta$. Setting $d = c/\|c_e\|$ we estimate

$$\begin{aligned}\|b_1 - d\| = \frac{1}{\|c_e\|}\big\|\|c_e\|b_1 - c\big\| &= \frac{1}{\|c_e\|}\big\|\|c_e\|b_1 - b_1 + b_1 - c\big\| \\ &\leq \frac{1}{\|c_e\|}\big(|\|c_e\| - 1|\|b_1\| + \|b_1 - c\|\big) \\ &\leq \frac{1}{1-\delta}(\delta\|b_1\| + \delta) = \frac{\delta}{1-\delta}(1 + \|b_1\|) < \varepsilon.\end{aligned}$$

Now let $\eta > 0$ be so small that $|F|\eta < 1/8$. Since A is separable and α is properly outer, we apply Lemma 7.1 of [20] and obtain an element $x \in A^+$ with $\|x\| = 1$ satisfying

$$\|x\mathbb{E}(d)x\| = \|xd_e x\| > \|d_e\| - \eta = 1 - \eta, \quad \|xd_s\alpha_s(x)\| < \eta, \quad \forall s \in F \setminus \{e\}.$$

Then we have

$$\begin{aligned}\|x\mathbb{E}(d)x - xdx\| &\leq \Big\|\sum_{s\in F\setminus\{e\}} xd_s u_s x\Big\| \leq \sum_{s\in F\setminus\{e\}} \|xd_s u_s x\| \\ &= \sum_{s\in F\setminus\{e\}} \|xd_s u_s x u_s^*\| = \sum_{s\in F\setminus\{e\}} \|xd_s\alpha_s(x)\| \leq |F|\eta < 1/8.\end{aligned}$$

A straightforward use of the triangle inequality now gives

$$\|x\mathbb{E}(b_1)x - xb_1x\| \leq 2\varepsilon + 1/8 < 1/4, \quad \|x\mathbb{E}(b_1)x\| \geq 3/4.$$

Let $a := (x\mathbb{E}(b_1)x - 1/2)_+$. Then $a \in A$ and $a \neq 0$ since $\|x\mathbb{E}(b_1)x\| > 1/2$. Also by Proposition 2.2 of [25] we know $a \precsim xb_1x \precsim b_1 \precsim b$. □

Theorem 4.1 in [28] concentrates on the commutative case. We, however, make the observation that the same proof holds true for noncommutative algebras. Recall that a C*-algebra A has property (SP) if every non-zero hereditary subalgebra admits a non-zero projection.

Theorem 5.4. *Let (A, Γ, α) be a C*-dynamical system with A separable with property (SP) and Γ countable and discrete. Assume that α is minimal and properly outer. Then $A \rtimes_\lambda \Gamma$ is simple and the following are equivalent:*

(1) *$A \rtimes_\lambda \Gamma$ is purely infinite.*

(2) *Every non-zero projection p in A is properly infinite in $A \rtimes_\lambda \Gamma$.*

Proof. Simplicity of the reduced crossed product $A \rtimes_\lambda \Gamma$ is Theorem 7.2 in [20].

(1) $\Rightarrow$ (2): Every non-zero projection in any purely infinite algebra is properly infinite.

(2) $\Rightarrow$ (1): Since the crossed product is simple, it suffices to show that every non-zero hereditary subalgebra admits an infinite projection. To this end, let $B \subset A \rtimes_\lambda \Gamma$ be a hereditary C*-subalgebra and let $0 \neq b \in B$. By Lemma 5.3 there is a non-zero a in A with $a \precsim b$. Since A has property (SP), the hereditary subalgebra of A generated by a, $H_a = \overline{aAa}$, contains a non-zero projection $q \in H_a$. By our assumption q is properly infinite relative to $A \rtimes_\lambda \Gamma$, and $q \precsim a \precsim b$. Since q is a projection, there is a $z \in A \rtimes_\lambda \Gamma$ with $q = z^*bz$. Now consider $v := b^{1/2}z$. Then $q = v^*v \sim vv^* = b^{1/2}zz^*b^{1/2} \in B$. Thus $p := vv^*$ is the desired properly infinite projection in B. □

We now embark on studying to what extent paradoxical systems (A, Γ, α) characterize purely infinite reduced crossed product algebras $A \rtimes_\lambda \Gamma$.

Proposition 5.5. *Let (A, Γ, α) be a C*-system for which A has cancellation and $K_0(A)^+$ has the Riesz refinement property. Let $0 \neq r \in \mathcal{P}(A)$ and set $g = [r]_0 \in K_0(A)^+$. The following properties are equivalent:*

(1) *There exist $x, y \in C_c(\Gamma, A)$ that satisfy $x^*x = r = y^*y$, $xx^* \perp yy^*$, $xx^* \leq r$, $yy^* \leq r$, and whose coefficients are partial isometries.*

(2) *g is $(k, 1)$-paradoxical for some $k \geq 2$.*

(3) *$\theta = [g]_\alpha$ is properly infinite in $S(A, \Gamma, \alpha)$.*

Proof. (1) $\Rightarrow$ (2): Write $x = \sum_{s \in F} u_s v_s$ and $y = \sum_{s \in L} u_s w_s$ where $F, L \subset \Gamma$ are finite subsets, and $v_s, w_s \in A$ are partial isometries. For each s in F set $p_s := v_s^* v_s$

and $p'_s := v_s v_s^*$. Similarly for every $s \in L$ set $q_s := w_s^* w_s$ and $q'_s := w_s w_s^*$. Observe that for $s, t \in F$ we have

$$\begin{aligned}\mathbb{E}(v_s^* u_s^* u_t v_t) &= \mathbb{E}(v_s^* u_{s^{-1}t} v_t (u_{s^{-1}t})^* u_{s^{-1}t}) \\ &= \mathbb{E}(v_s^* \alpha_{s^{-1}t}(v_t) u_{s^{-1}t}) = \delta_{s,t} v_s^* v_s,\end{aligned}$$

where $\mathbb{E} : A \rtimes_\lambda \Gamma \to A$ is the conditional expectation. Consequently, applying $\mathbb{E}$ to the equality $r = x^* x$ gives

$$r = \mathbb{E}(r) = \mathbb{E}\bigg(\sum_{s,t \in F} v_s^* u_s^* u_t v_t\bigg) = \sum_{s,t \in F} \mathbb{E}(v_s^* u_s^* u_t v_t) = \sum_{s \in F} v_s^* v_s = \sum_{s \in F} p_s.$$

Therefore, the projections p_s are mutually orthogonal subprojections of r that sum to r. Similarly all the q_s, for $s \in L$, are mutually orthogonal subprojections of r with $r = \sum_{s \in L} q_s$. Thus, in $K_0(A)^+$ we have

$$\sum_{s \in F} [p_s]_0 + \sum_{s \in L} [q_s]_0 = \bigg[\sum_{s \in F} p_s\bigg]_0 + \bigg[\sum_{s \in F} q_s\bigg]_0 = 2[r]_0.$$

Now we note that for s, t in F with $s \neq t$ we have $v_s v_t^* = v_s v_s^* v_s v_t^* v_t v_t^* = v_s p_s p_t v_t^* = 0$. Computing xx^* we get

$$xx^* = \sum_{s,t \in F} u_s v_s v_t^* u_t^* = \sum_{s \in F} u_s v_s v_s^* u_s^* = \sum_{s \in F} \alpha_s(p'_s).$$

Similarly $yy^* = \sum_{s \in L} \alpha_s(q'_s)$. From

$$\sum_{s \in F} \alpha_s(p'_s) + \sum_{s \in L} \alpha_s(q'_s) = xx^* + yy^* \leq r$$

we conclude that the projections $\alpha_s(p'_s), \alpha_s(q'_s)$ are mutually orthogonal subprojections of r whence in $K_0(A)$ we have

$$\begin{aligned}[r]_0 \geq \bigg[\sum_{s \in F} \alpha_s(p'_s) + \sum_{s \in L} \alpha_s(q'_s)\bigg]_0 &= \sum_{s \in F} [\alpha_s(p'_s)]_0 + \sum_{s \in L} [\alpha_s(q'_s)]_0 \\ &= \sum_{s \in F} \hat{\alpha}_s([p'_s]_0) + \sum_{s \in L} \hat{\alpha}_s([q'_s]_0) \\ &= \sum_{s \in F} \hat{\alpha}_s([p_s]_0) + \sum_{s \in L} \hat{\alpha}_s([q_s]_0).\end{aligned}$$

Therefore $g = [r]_0$ is (2,1)-paradoxical.

(2) $\Rightarrow$ (1): Suppose $\sum_{j=1}^n x_j \geq k[r]_0$ and $\sum_{j=1}^n \hat{\alpha}_{t_j}(x_j) \leq [r]_0$ for some $k \geq 2$, group elements $t_1, \ldots, t_n \in \Gamma$, and $x_j \in K_0(A)^+$. Since $k[r]_0 \geq 2[r]_0$ we may assume $k = 2$. For some $u \in K_0(A)^+$ we then have $\sum_{j=1}^n x_j = [r]_0 + [r]_0 + u$.

Riesz refinement implies that there are subsets $\{y_j\}_{j=1}^n$, $\{z_j\}_{j=1}^n$ and $\{u_j\}_{j=1}^n$ of $K_0(A)^+$ with

$$\sum_{j=1}^n y_j = [r]_0, \quad \sum_{j=1}^n z_j = [r]_0, \quad \sum_{j=1}^n u_j \geq 0, \quad \text{and} \quad x_j = y_j + z_j + u_j, \quad \forall j.$$

Using the fact that A has cancellation we know that there are mutually orthogonal projections $p_j \in \mathcal{P}(A)$ with $[p_j]_0 = y_j$ for $j = 1, \dots, n$. Similarly there are mutually orthogonal projections $q_j \in \mathcal{P}(A)$ with $[q_j]_0 = z_j$ for $j = 1, \dots, n$. Then,

$$\begin{aligned}\sum_j [\alpha_{t_j}(p_j)]_0 + \sum_j [\alpha_{t_j}(q_j)]_0 &= \sum_j \hat{\alpha}_{t_j}(y_j) + \sum_j \hat{\alpha}_{t_j}(z_j) \\ &\leq \sum_j \hat{\alpha}_{t_j}(y_j) + \sum_j \hat{\alpha}_{t_j}(z_j) + \sum_j \hat{\alpha}_{t_j}(u_j) \\ &= \sum_j \hat{\alpha}_{t_j}(x_j) \leq [r]_0.\end{aligned}$$

We again use the fact that A has cancellation and find mutually orthogonal subprojections $e_1, \dots, e_n, f_1, \dots, f_n \in \mathcal{P}(A)$ of r with $[e_j]_0 = [\alpha_{t_j}(p_j)]_0$ and $[f_j]_0 = [\alpha_{t_j}(q_j)]_0$ for every j. Cancellation also implies that there are partial isometries v_j and w_j in A with

$$v_j^* v_j = \alpha_{t_j}(p_j), \quad v_j v_j^* = e_j, \quad w_j^* w_j = \alpha_{t_j}(q_j), \quad w_j w_j^* = f_j.$$

Now set $a = \sum_{j=1}^n v_j u_j$ and $b = \sum_{j=1}^n w_j u_j$ where $u_j = u_{t_j}$. For $i \neq j$ we compute $v_j^* v_i = v_j^* v_j v_j^* v_i v_i^* v_i = v_j^* e_j e_i v_i = 0$, so

$$\begin{aligned}a^* a = \sum_{i,j} u_j^* v_j^* v_i u_i = \sum_j u_j^* v_j^* v_j u_j &= \sum_j u_j^* \alpha_{t_j}(p_j) u_{t_j} \\ &= \sum_j \alpha_{t_j^{-1}}(\alpha_{t_j}(p_j)) = \sum_j p_j := p.\end{aligned}$$

In order to compute aa^* we note that for $i \neq j$ we have

$$\begin{aligned}v_j u_j u_i^* v_i^* &= v_j v_j^* v_j u_j u_i^* v_i^* v_i v_i^* \\ &= v_j \alpha_{t_j}(p_j) u_j u_i^* \alpha_{t_i}(p_i) v_i^* \\ &= v_j u_j p_j u_j^* u_j u_i^* u_i p_i u_i^* v_i^* = v_j u_j p_j p_i u_i^* v_i^* = 0,\end{aligned}$$

whence

$$aa^* = \sum_{i,j} v_j u_j u_i^* v_i^* = \sum_j v_j u_j u_j^* v_j^* = \sum_j v_j v_j^* = \sum_j e_j := e.$$

Similarly $b^*b = \sum_j q_j := q$, and $bb^* = \sum_j f_j = f$.

Now define $x := av$ where v is the partial isometry in A with $v^*v = r$ and $vv^* = p$. Such a v exists because $[p]_0 = \big[\sum_j p_j\big]_0 = \sum_j [p_j]_0 = \sum_j y_j = [r]_0$ and A has cancellation. Similarly define $y := bw$ where $w \in A$ satisfies $w^*w = r$ and $ww^* = q$. We compute

$$x^*x = v^*a^*av = v^*pv = v^*vv^*v = r^2 = r,$$

and

$$y^*y = w^*b^*bw = w^*qw = w^*ww^*w = r^2 = r.$$

Moreover, since a and b are partial isometries and $e \perp f$, we have

$$\begin{aligned} xx^*yy^* &= avv^*a^*bww^*b^* \\ &= avv^*a^*aa^*bb^*bww^*b^* \\ &= avv^*a^*efbww^*b^* = 0. \end{aligned}$$

Next we observe that xx^* is a subprojection of r; indeed, since $e \leq r$,

$$\begin{aligned} rxx^* = ravv^*a^* = raa^*avv^*a^* &= reavv^*a^* \\ &= eavv^*a^* = aa^*avv^*a^* = avv^*a^* = xx^* \end{aligned}$$

Similarly yy^* is a subprojection of r.

Finally we verify that the coefficients of x and y are partial isometries. Write

$$x = av = \sum_{j=1}^{n} v_j u_j v = \sum_{j=1}^{n} v_j \alpha_{t_j}(v) u_j,$$

and compute

$$\begin{aligned} (v_j\alpha_{t_j}(v))^* v_j \alpha_{t_j}(v) &= \alpha_{t_j}(v^*) v_j^* v_j \alpha_{t_j}(v) \\ &= \alpha_{t_j}(v^*)\alpha_{t_j}(p_j)\alpha_{t_j}(v) = \alpha_{t_j}(v^* p_j v). \end{aligned}$$

Now since $p_j \leq p$ for every j, v^*p_jv is a projection: $(v^*p_jv)^2 = v^*p_jvv^*p_jv = v^*p_jpp_jv = v^*p_jv$. Therefore $\alpha_{t_j}(v^*p_jv)$ is a projection for each j and so the coefficients of x, $v_j\alpha_{t_j}(v)$, are partial isometries. An identical argument works for the coefficients of y. This completes the implication (2) $\Rightarrow$ (1).

(2) $\Leftrightarrow$ (3): By definition $[g]_\alpha$ is infinite in $S(A, \Gamma, \alpha)$ if and only if $2[g]_\alpha \leq [g]_\alpha$, and by Proposition 4.5, we know this occurs if and only if g is $(2,1)$-paradoxical. Clearly g is $(2,1)$-paradoxical if and only if g is $(k,1)$-paradoxical for some $k \geq 2$. □

$$\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \geq [1]_0 \quad \text{and} \quad \sum_{j=n+1}^{2n} \hat{\alpha}_{t_j}(x_j) \geq [1]_0.$$

The following result generalizes Theorem 5.4 of [28] to the noncommutative case.

Theorem 5.6. *Let A be a unital, separable, exact C*-algebra whose projections are total. Moreover, suppose A has cancellation and $K_0(A)^+$ has the Riesz refinement property. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action. Consider the following properties:*

(1) *The semigroup $S(A, \Gamma, \alpha)$ is purely infinite.*

(2) *Every non-zero element in $K_0(A)^+$ is $(k, 1)$-paradoxical for some $k \geq 2$.*

(3) *The C*-algebra $A \rtimes_\lambda \Gamma$ is purely infinite.*

(4) *The C*-algebra $A \rtimes_\lambda \Gamma$ admits no tracial state.*

(5) *The semigroup $S(A, \Gamma, \alpha)$ admits no non-trivial state.*

Then the following implications always hold: (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). *If the semigroup $S(A, \Gamma, \alpha)$ is almost unperforated then* (5) $\Rightarrow$ (1) *and all properties are equivalent.*

Proof. (1) $\Leftrightarrow$ (2): We have already seen that $x \in K_0(A)^+$ is $(k, 1)$-paradoxical for some $k \geq 2$ if and only if $\theta = [x]_\alpha$ is properly infinite in $S(A, \Gamma, \alpha)$.

(2) $\Rightarrow$ (3): Let r be a non-zero projection in A. By assumption $[r]_0$ is $(2, 1)$-paradoxical, so by proposition 5.5 r is properly infinite in $A \rtimes_\lambda \Gamma$. Then $A \rtimes_\lambda \Gamma$ is purely infinite by Theorem 5.4.

(3) $\Rightarrow$ (4): Purely infinite C*-algebras are always traceless.

(4) $\Rightarrow$ (5): Suppose $\nu : S(A, \Gamma, \alpha) \to [0, \infty]$ is a non-trivial state. Suppose $0 < \nu([x]_\alpha) < \infty$ where $x \in K_0(A)^+$ is non-zero. Composing with the quotient map $\rho : K_0(A)^+ \to S(A, \Gamma, \alpha)$ we get an order preserving monoid homomorphism $\beta' = \nu \circ \rho : K_0(A)^+ \to [0, \infty]$ with $0 < \beta'(x) < \infty$. As in the proof of Proposition 4.8, minimality of the action ensures that β' is finite on all of $K_0(A)^+$. Extending β' to $K_0(A)$ gives a Γ-invariant positive group homomorphism, $\beta : K_0(A) \to \mathbb{R}$. Since A is exact and projections are total, β comes from a Γ-invariant trace on A (Theorem 1.1.11 in [27]), so that $A \rtimes_\lambda \Gamma$ admits a tracial state, a contradiction.

Now we assume that $S(A, \Gamma, \alpha)$ is almost unperforated and prove (5) $\Rightarrow$ (1). Let $\theta = [x]_\alpha$ be a non-zero element in $S(A, \Gamma, \alpha)$. If θ is completely non-paradoxical then by Tarski's Theorem $S(A, \Gamma, \alpha)$ admits a non-trivial state. So, assuming (5), we must have $(k + 1)\theta \leq k\theta$ for some $k \in \mathbb{N}$. So

$$(k + 2)\theta = (k + 1)\theta + \theta \leq k\theta + \theta = (k + 1)\theta \leq k\theta.$$

Repeating this trick we get $(k + 1)2\theta \leq k\theta$. Since $S(A, \Gamma, \alpha)$ is almost unperforated we conclude $2\theta \leq \theta$ and θ is properly infinite. □

Combining Theorems 4.9 and 5.6 we obtain a dichotomy.

Theorem 5.7. *Let A be a unital, separable, exact C*-algebra whose projections are total. Moreover suppose A has cancellation and $K_0(A)^+$ has the Riesz refinement property. Let Γ be a countable discrete group and let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be a minimal and properly outer action such that $S(A, \Gamma, \alpha)$ is almost unperforated. Then the reduced crossed product $A \rtimes_\lambda \Gamma$ is a simple C*-algebra which is either stably finite or purely infinite.*

It is well known that AF-algebras satisfy the conditions listed in Theorem 5.7. The following Corollary now follows from Theorems 5.7 and 4.9.

Corollary 5.8. *Let $\alpha : \mathbb{F}_r \to \mathrm{Aut}(A)$ be a minimal and properly outer action on a AF-algebra, and suppose $S(A, \mathbb{F}_r, \alpha)$ is almost unperforated. Then $A \rtimes_\lambda \Gamma$ is purely infinite or MF in the sense of Blackadar and Kirchberg [7].*

We end our discussion with a few remarks and interesting questions.

As promised, we mention that this work provides an order K-theoretic proof of Jolissaint and Robertson's main result in [13], at least in the case where $K_0(A)$ is sufficiently well-behaved. For the sake of brevity let us assume that A is an algebra of real rank zero and stable rank one so that the conditions in Theorems 5.5 and 5.4 are satisfied. Assume $\alpha : \Gamma \to \mathrm{Aut}(A)$ is a properly outer action that is n-filling in the sense of [13]. If every projection $p \in A$ generates an infinite dimensional corner pAp, it is not difficult to see that the ordered group $(K_0(A), K_0(A)^+)$ is non-atomic, that is, for every non-zero $g \in K_0(A)^+$ there is a nonzero $h \in K_0(A)^+$ with $h < g$. In this setting, if p is a non-zero projection in A we can inductively find non-zero elements $x_1, \dots, x_{2n} \in K_0(A)^+$ with $\sum_{j=1}^{2n} x_j \leq [p]_0$. The author shows in [24] that an n-filling action is minimal and satisfies a K_0-n-filling condition, which implies that there are group elements $t_1, \dots, t_{2n}$ satisfying

$$\sum_{j=1}^{n} \hat{\alpha}_{t_j}(x_j) \geq [1]_0 \quad \text{and} \quad \sum_{j=n+1}^{2n} \hat{\alpha}_{t_j}(x_j) \geq [1]_0.$$

Combining these facts we obtain $\sum_{j=1}^{2n} x_j \leq [p]_0$ and $\sum_{j=1}^{2n} \hat{\alpha}_{t_j}(x_j) \geq 2[1]_0 \geq 2[p]_0$ and thus $[p]_0$ is $(2,1)$-paradoxical. By Proposition 5.5 the projection p is properly infinite in $A \rtimes_\lambda \Gamma$. It follows by Theorem 5.4 that $A \rtimes_\lambda \Gamma$ is simple and purely infinite.

It is unknown to the author if there are examples of minimal and properly outer actions on C*-algebras satisfying the conditions in Theorem 5.7 for which the type semigroup is *not* almost unperforated. In particular, is there a free and minimal action of the free group $\mathbb{F}_2$ on the Cantor set X for which $S(X, \mathbb{F}_2, \mathcal{C})$ is not almost unperforated? Although Ara and Exel construct actions of a finitely generated free group on the Cantor set for which the type semigroup is not almost unperforated, these actions are not minimal [1]. Moreover, almost unperforation may be too strong a condition to establish (5) $\Rightarrow$ (1) in Theorem 5.6. What is required is that every "infinite element" (in the sense that $(k+1)x \leq kx$ for some k) is properly infinite ($2x \leq x$). This is a priori a weaker condition than almost unperforation.

In the context of Definition 4.3 and the remarks that follow, we note that there is a homomorphism of monoids $\varrho : S(A, \Gamma, \alpha) \to K_0(A \rtimes_\lambda \Gamma)^+$ satisfying $\varrho\rho = \hat{\iota}$, where $\hat{\iota} : K_0(A)^+ \to K_0(A \rtimes_\lambda \Gamma)^+$ is the map induced by the inclusion $\iota : A \hookrightarrow A \rtimes_\lambda \Gamma$. Under what conditions is the map ϱ injective? An answer to this question would shed considerable light on the K_0-group of certain crossed products.

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