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The Bernstein–Sato b-function of the Space of Cyclic Pairs

by

Robin WALTERS

Abstract

We compute the Bernstein–Sato polynomial of f, a function which given a pair (M, v) in $X = M_n(\mathbb{C}) \times \mathbb{C}^n$ tests whether v is a cyclic vector for M. The proof includes a description of shift operators corresponding to the Calogero–Moser operator L_k in the rational case.

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§1. Introduction

Let f be an algebraic function on a variety X over $\mathbb{C}$. Let D_X be the ring of algebraic differential operators on X. The *Bernstein–Sato b-function* of f is defined to be the minimal degree monic function $b(s)$ in $\mathbb{C}[s]$ such that

$$Df^{s+1} = b(s)f^s \tag{1.1}$$

for some operator D in $D_X[s] = \mathbb{C}[s] \otimes D_X$. We call D the *Bernstein operator* and (1.1) the *Bernstein equation*. A minimal $b(s)$ must exist since the set of all $b(s)$ satisfying (1.1) form an ideal in $\mathbb{C}[s]$. Existence of non-zero solutions to (1.1) was proved by Bernstein in 1971 [1]. The rationality of the roots of $b(s)$ was proved by Kashiwara in 1976 [9]. The b-function is interesting, in part, because it is an invariant of the singularities of the divisor given by f.

In [11], Opdam proves a conjecture of Yano and Sekiguchi [14] by computing the b-function corresponding to $I = \prod_{\alpha \in R^+} \alpha^2$, a W-invariant function on $\mathfrak{h}$. (The notation is defined in Section 5.) In type A_n, the function I is the square of the

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R. Walters: Department of Mathematics, University of Chicago,
5734 S. University Avenue, Chicago, IL 60637, USA;
e-mail: robin@math.uchicago.edu

Vandermonde determinant. Opdam proves the result by realizing the Bernstein operator as a shift operator related to the Calogero–Moser operator.

For this paper, let $X = M_n(\mathbb{C}) \times \mathbb{C}^n$. For $(M, v) \in X$, we say v is *cyclic* for M or that (M, v) is a *cyclic pair* if the set $\{v, Mv, M^2v, \ldots\}$ is a spanning set of $\mathbb{C}^n$. Let $C(M, v)$ denote the square matrix

$$[v\ Mv\ M^2v\ \ldots\ M^{n-1}v].$$

We define $f(M, v) = \det(C(M, v))$, a polynomial on X. Then (M, v) is cyclic if and only if $f(M, v) \neq 0$.

The function f maps to I via radial reduction. Since radial reduction maps the standard Laplacian operator Δ to the Calogero–Moser operator on $\mathfrak{h}$ [5], we can think of X and f as describing a broader, yet simpler, precursor situation to the one studied in [11]. More generally, the function f and the space X are relevant to the study of mirabolic D-modules and rational Cherednik algebras [2–4].

The main result of the paper is the computation of the b-function of f.

Theorem 1.1. *The b-function of f is*

$$\tilde{b}(s) = \prod_{0 \leq c < d \leq n} (s + 1 + c/d). \tag{1.2}$$

Since f is a semi-invariant, the calculation has some similarity to the prehomogeneous case originally considered by Sato in [12] and [13]. In that case, G acts on a vector space with open dense orbit. However, our space is not prehomogeneous, so some additional work is required.

The proof will proceed in three parts. First we will define a differential operator S in $D_X[s]$, and show that one has an equation

$$Sf^{s+1} = b'(s)f^s \tag{1.3}$$

for an unknown function $b'(s)$ in $\mathbb{C}[s]$. Secondly, we show $\tilde{b}(s)$, our specific candidate function, is the monic associate of $b'(s)$. Thirdly, we will show that $\tilde{b}(s)$ has, in fact, the *minimal* degree, completing the proof.

Section 2 contains some results about f and cyclic vectors. Then we continue to the proof of our main theorem. This is the content of Sections 3 through 6. In the Appendix (Section 7), we will give a proof characterizing the structure of the space of shift operators of the rational Calogero–Moser operator, an analog of a similar result for the trigonometric case in [11].

§2. Cyclic vectors and semi-invariants

The following result is well-known.

Proposition 2.1. *The matrix M has a cyclic vector if and only if each Jordan block B_i has a distinct eigenvector λ_i, that is, if each eigenspace is one-dimensional.*

We conjecture that f is irreducible. The above result allows us to prove something weaker but still sufficient for our purposes. Let $\mathfrak{g} = M_n(\mathbb{C})$.

Proposition 2.2. *The function f has no non-constant, proper factor $h \in \mathbb{C}[\mathfrak{g}]$.*

Proof. Such a proper factor would correspond to a set of codimension 1 in $\mathfrak{g}$ containing matrices M with no cyclic vector. By the assumption that at least two Jordan blocks share an eigenvalue, we have $k-1$ choices of λ_i. Assuming $\lambda_1 = \lambda_2 = \lambda$ and reordering the basis so that the true eigenvectors of B_1 and B_2 are the first two vectors followed by all the generalized eigenvectors, we see that $B_1 \oplus B_2$ is

$$\left(\begin{array}{c|c} \lambda \operatorname{Id}_2 & A \\ \hline 0 & B \end{array}\right).$$

The submatrix B is upper triangular with λ on its diagonal, and the entries of A and of B above the diagonal are 1 or 0. Then this matrix is stabilized by

$$\left(\begin{array}{c|c} \mathrm{GL}(2) & 0 \\ \hline 0 & \mathbb{C}^* \operatorname{Id}_{n-2} \end{array}\right).$$

Thus the dimension of this partition is less than or equal to

$$n^2 + (k-1) - (k-2) - 1 - 4 = n^4 - 4.$$

So we are done. □

Given a space Y with a G-action, we denote the G-invariant operators on Y as D_Y^G. Denote $p^g(y) = p(g \cdot y)$. The function p is said to be a *G-semi-invariant* corresponding to a character χ if

$$p^g = \chi(g)p \quad \text{for all } g \in G.$$

We denote the semi-invariant functions corresponding to χ as $\mathbb{C}[Y]^\chi$ and differential operators corresponding to χ as D_Y^χ.

In our case, we have $G = \mathrm{GL}_n(\mathbb{C})$ acting on $X = \mathfrak{gl}_n(\mathbb{C}) \times \mathbb{C}^n$ via conjugation on the first factor and multiplication on the second factor. The one-dimensional representations (or characters) are just powers of the determinant, $\chi = \det^r$. Note that $f \in \mathbb{C}[X]^{\det}$. The space of diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$ is isomorphic to $\mathbb{C}^n$. We denote it by $\mathfrak{h}$. Let $Z = \mathfrak{h} \times V \subset X$.

The following result is originally due to Weyl, although we give a different proof.

Proposition 2.3. *Let $p \in \mathbb{C}(s)[X]^\chi$ where $\chi = \det^r$. Then*

$$p = f^r h,$$

where $h \in \mathbb{C}(s)[\mathfrak{g}]^G$.

Proof. Consider $q = pf^{-r} \in \mathbb{C}(s)(X)^G$.

Now define the matrix

$$d = \operatorname{diag}(1, \dots, 1, \lambda, 1, \dots, 1) \in G$$

containing λ in its i^{th} entry. This acts on Z by fixing $\mathfrak{h}$ and scaling v^i, the i^{th} coordinate of V. However, since $q|_Z$ is invariant, it must be homogeneous of degree 0 in v^i, that is, it is independent of v^i. By invariance, q is independent of v^i on $G \cdot Z$, which is dense in X, and thus $q \in \mathbb{C}(s)(\mathfrak{g})^G$.

It is not hard to see that

$$\mathbb{C}(s)(\mathfrak{g})^G = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{C}(s)[\mathfrak{g}]^G \right\}$$

since a general element of $\mathbb{C}(s)(\mathfrak{g})^G$ is a ratio of semi-invariants, but $\mathbb{C}(s)[\mathfrak{g}]$ has no non-invariant semi-invariants.

So we know

$$p = f^r \frac{a}{b}$$

where $a, b \in \mathbb{C}(s)[\mathfrak{g}]^G$ and are, we can assume, relatively prime. This equation implies that b divides f^r. However, by Proposition 2.2, f^r has no factor in $\mathbb{C}[\mathfrak{g}]$ and thus none in $\mathbb{C}(s)[\mathfrak{g}]$ either. □

§3. The operator S

When we use the term *order* in reference to a differential operator in $D_X[s]$, we refer to the traditional filtration in which $\partial_{x_i} = \partial/\partial x_i$ has degree 1 and functions in $\mathbb{C}[s][X]$ have degree 0. We will also refer to a $\mathbb{Z}$-grading on $D_X[s]$ in which $|\partial_{x_i}| = -1$ and $|x_i| = 1$. The ring $\mathbb{C}[s]$ lives in the grade 0. Note that the grading is well-defined on $D_X[s]$ since it respects the defining relation $\partial_{x_i} x_i - x_i \partial_{x_i} = 1$. When we refer to this grading, we will use term *degree*.

We define a differential operator S which is given by taking the function f and replacing all variables with the corresponding partial derivatives. That is,

$$S = \det([\partial_v \ \partial_M \partial_v \ \partial_M^2 \partial_v \ \dots \ \partial_M^{n-1} \partial_v])$$

where $[\partial_v]_i = \partial_{v_i}$ and $[\partial_M]^i_j = \partial_{m^i_j}$. Then S is an order $n(n+1)/2$ differential operator and $S \in D_X^{\det^{-1}}$. The definition of S here is analogous to that of the Bernstein operator in Sato's prehomogeneous case.

We can now prove the following:

Proposition 3.1. *There exists a function $b' \in \mathbb{C}[s]$ such that $Sf^{n+1} = b'(s)f^n$. Further,* $\deg(b'(s)) \leq n(n+1)/2$.

Proof. Since S has weight $\det^{-1}$ and f^{s+1} has weight $\det^{s+1}$, the operator $S \circ \hat{f}^{s+1}$ has weight $\det^s$. Applying this operator to 1, we get a polynomial $p \in \mathbb{C}[s][X]^{\det^s}$. Therefore by Proposition 2.3, there exists $q \in \mathbb{C}[s][X]^G$ such that

$$Sf^{s+1} = f^s q.$$

Taking the degree of both sides we see

$$-\frac{n(n+1)}{2} + (s+1)\frac{n(n+1)}{2} = s\frac{n(n+1)}{2} + \deg(q).$$

Thus $\deg(q) = 0$ and so $q \in \mathbb{C}[s]$ as desired. Denote q as $b'(s)$. Moreover since S is an $n(n+1)/2$-order operator, we see $\deg(b') \leq n(n+1)/2$. □

§4. Localization

In this and the next section, we will show that $b'(s)$ equals

$$\tilde{b}(s)\alpha_n = \prod_{0 \leq c < d \leq n} (d(s+1) + c), \tag{4.1}$$

where $\alpha_n = \prod_{d=1}^n d^d \in \mathbb{C}$. We factor this as:

$$b_1(s) = n!(s+1)^n,$$
$$b_2(s) = \prod_{1 \leq c < d \leq n} (d(s+1) + c),$$

so that $b_1 b_2 = \tilde{b}\alpha_n$.

Proposition 4.1. *The polynomial $b_1(s)$ divides the b-function of f.*

Proof. Consider the following local coordinates for X:

$$\mathbb{C}^{n(n-1)} \times \mathbb{C}^n \times \mathbb{C}^n \to X,$$
$$\varphi : (\{t^i_j\}_{i \neq j}, \{a_i\}_{i=1}^n, \{v_i\}_{i=1}^n) \mapsto (TAT^{-1}, Tv),$$

where

$$T = \begin{pmatrix} 1 & t_2^1 & t_3^1 & \dots & t_n^1 \\ t_1^2 & 1 & t_3^2 & \dots & t_n^2 \\ t_1^3 & t_2^3 & 1 & \dots & t_n^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1^n & t_2^n & t_3^n & \dots & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}.$$

Let $p \in X$ be defined by $t_j^i = 0, a_i = i$ and $v_i = 0$. It is straightforward to compute that $\det(D\varphi)|_p \neq 0$, and thus φ gives local coordinates at p.

In these local coordinates we have

$$f = \det(T)\det(C(A, v)).$$

Note $C(A, v)_j^i = v_i a_i^{j-1}$ is the Vandermonde matrix with rows multiplied by v_i. So

$$\det(C(A, v)) = v_1 \cdots v_n \prod_{1 \le i < j \le n} (a_j - a_i).$$

Let $\overline{1} = (1, \dots, 1)$. Since at p, $a_i = i$, the quantity

$$\det(C(A, \overline{1})) = \prod_{1 \le i < j \le n} (a_j - a_i) \neq 0.$$

Similarly, $T = I_n$ at p, so $\det(T) = 1$. Thus in a small open neighborhood U of p, the function $\det(T)\det(C(A, \overline{1}))$ is invertible.

We will show that the b-function for f on U is $b_1(s)$, which will finish the proof since the global b-function is the least common multiple of the local b-functions [6, Proposition 2.1].

Let

$$D = \det(C(A, \overline{1}))^{-1} \det(T)^{-1} \partial_{v_1} \dots \partial_{v_n}.$$

Then

$$Df^{s+1} = b_1(s) f^s.$$

Since v_i are not invertible near p, the Bernstein operator D must include the factor $\partial_{v_1} \dots \partial_{v_n}$. Thus b_1 is minimal. □

§5. Radial parts reduction

For b_2, we relate our problem to the situation of [11]. To do this we require the radial parts map from [5, Appendix].

We introduce more notation. Denote by R and R^+ the sets of roots and of positive roots respectively for type A_{n-1}. Let $W = S_n$ be the Weyl group corresponding to $\mathfrak{gl}_n$. Consistent with the notation in [5], we define X^{reg} to consist of pairs (M, v) where $M \in \mathfrak{g}^{\mathrm{rs}}$. Lastly define $\mathfrak{h}^{\mathrm{reg}} \subset \mathfrak{h}$ to be those points which avoid the root hyperplanes.

The radial parts map is more clearly described in [5]. We give an overview. To derive it, we start with the map

$$\rho : X^{\mathrm{reg}}/G \to \mathfrak{h}^{\mathrm{reg}}/W, \qquad (M, v) \mapsto \text{eigenvalues of } M,$$

which induces the map

$$\rho^* : \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]^W \to \mathbb{C}[X^{\mathrm{reg}}]^G.$$

For $k \in \mathbb{C}$, we define the radial parts map $\mathrm{Rad}_k : D_X^G \to D_{\mathfrak{h}^{\mathrm{reg}}}^W$ as follows. Let $D \in D_X^G$ and $g \in \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]^W$. Then

$$\mathrm{Rad}_k(D)(g) = f^{-k} D(f^k \rho^*(g))|_{\mathfrak{h}}.$$

Let $\delta = f|_{\mathfrak{h}^{\mathrm{reg}}}$ be the Vandermonde determinant and L_k be the *Calogero–Moser* operator

$$L_k = \Delta_{\mathfrak{h}} - \sum_{\alpha \in R^+} k(k+1) \frac{(\alpha, \alpha)}{\alpha^2}.$$

Note that since δ is the product of the positive roots, α^{-1} as well as δ^{-1} are in $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$. Let $\Delta_{\mathfrak{g}}$ be the standard Laplacian on $\mathfrak{g}$. Since $X = \mathfrak{g} \times V$, we can view $\Delta_{\mathfrak{g}}$ as a differential operator on X. By [5, Appendix],

$$\mathrm{Rad}_k(\Delta_{\mathfrak{g}}) = \delta^{-k-1} L_k \delta^{k+1}. \tag{5.1}$$

Define $P^+ = \sum_{\alpha \in R^+} \alpha^{-1} \partial_{X_\alpha}$. Then we can simplify (5.1) using the equations $\delta^{-1} \Delta_{\mathfrak{h}} \delta = \Delta_{\mathfrak{h}} + 2P^+$ and $\delta^{-1} P^+ \delta = P^+ + 2 \sum_{\alpha \in R^+} (\alpha, \alpha) \alpha^{-2}$ to get

$$\mathrm{Rad}_k(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}} + 2(k+1) P^+, \tag{5.2}$$

an operator which we will call $L_{\mathfrak{h}}(k+1)$.

We say that $D \in D_{\mathfrak{h}^{\mathrm{reg}}}^W \otimes \mathbb{C}[k]$ is a *shift operator* if

$$D L_{\mathfrak{h}}(k) = L_{\mathfrak{h}}(k+r) D. \tag{5.3}$$

Denote the set of shift operators with shift r as $\mathbb{S}_{\mathfrak{h}}(r, k)$, or just $\mathbb{S}_{\mathfrak{h}}(r)$ if k is clear from context. Note that $\mathbb{S}_{\mathfrak{h}}(0)$ is just the centralizer of $L_{\mathfrak{h}}(k)$.

The following is a rational case analog to the trigonometric case given in [11, Theorem 3.1].

Proposition 5.1. *The set $\mathbb{S}_{\mathfrak{h}}(r, k)$ is a free, rank-one $\mathbb{S}_{\mathfrak{h}}(0, k)$-module.*

We defer the proof until Section 7. We denote the single generator of $\mathbb{S}_{\mathfrak{h}}(r, k)$ over $\mathbb{S}_{\mathfrak{h}}(0, k)$ by $g(r, k)$. We denote the operator $p \mapsto f \cdot p$ by $\hat{f}$.

Proposition 5.2. *The operator* $\mathrm{Rad}_k(S\hat{f})$ *belongs to* $\mathbb{S}_{\mathfrak{h}}(-1, k+2)$.

Proof. As noted before, $S\hat{f}$ is a semi-invariant with character $\det \cdot \det^{-1} = 1$, so $S\hat{f} \in D_X^G$, and thus we can take its radial part $\mathrm{Rad}(S\hat{f}) \in D^W_{\mathfrak{h}^{\mathrm{reg}}}$.

The set $\mathbb{S}_{\mathfrak{h}}(-1, k+2)$ is a subset of $D^W_{\mathfrak{h}^{\mathrm{reg}}} \otimes \mathbb{C}[k]$. So we need to show that $k \mapsto \mathrm{Rad}_k(S\hat{f})$ is a polynomial map $\mathbb{C} \to D^W_{\mathfrak{h}^{\mathrm{reg}}}$. Given $p \in \mathbb{C}[X]$, we denote the corresponding differential operator as ∂_p or $\partial(p)$. Let $p = x_1 \cdots x_m$. Denote $(k+1)_r = (k+1)\dots(k-r+2)$, the falling Pochhammer symbol. Then from the formula

$$f^{-k}\partial_p f^{k+1} g = \sum_{q\cdot p_1 \cdots p_r = p} (k+1)_r f^{1-r}\Big(\prod_{i=1}^{r} \partial_{p_i} f\Big)\partial_q g,$$

we can see $\mathrm{Rad}_k(S\hat{f})$ is a polynomial in k, since k appears only in the polynomial coefficients $(k+1)_r$.

Now we need to show that $\mathrm{Rad}_k(S\hat{f})$ satisfies (5.3). Since S and $\Delta_{\mathfrak{g}}$ are constant coefficient operators, they commute, which gives the equation

$$S\hat{f}\widehat{f^{-1}}\Delta_{\mathfrak{g}}\hat{f} = \Delta_{\mathfrak{g}} S\hat{f}.$$

Applying Rad_k, which is a homomorphism, we get

$$\mathrm{Rad}_k(S\hat{f})\,\mathrm{Rad}_k(\widehat{f^{-1}}\Delta_{\mathfrak{g}}\hat{f}) = \mathrm{Rad}_k(\Delta_{\mathfrak{g}})\,\mathrm{Rad}_k(S\hat{f}).$$

Thus we obtain

$$\mathrm{Rad}_k(S\hat{f}) L_{\mathfrak{h}}(k+2) = L_{\mathfrak{h}}(k+1)\,\mathrm{Rad}_k(S\hat{f}),$$

as required. ☐

The operator $\mathrm{Rad}_k(S\hat{f})$ helps us compute $b'(k)$, since

$$\mathrm{Rad}_k(S\hat{f})(1) = (f^{-k} S f^{k+1})|_{\mathfrak{h}} = b'(k).$$

We now introduce the relation to the trigonometric case studied in [11]. Let H be the complex torus with $\mathrm{Lie}(H) = \mathfrak{h}$. For $h \in H$, denote $h^\alpha = \exp(\alpha)(h)$. We then set

$$H^{\mathrm{reg}} = \{h \in H \mid h^\alpha \neq 0 \text{ for all } \alpha \in R^+\}.$$

Define the trigonometric Calogero–Moser operator with parameter $k \in \mathbb{C}$ on H^{reg} :

$$L_H(k) = \Delta_{\mathfrak{h}} - \sum_{\alpha \in R^+} k \frac{1 + h^\alpha}{1 - h^\alpha} \partial(X_\alpha).$$

We also define

$$\rho(k) = k\frac{1}{2}\sum_{\alpha\in R^+} \alpha.$$

Then we define the related operator $\tilde{L}_H(k) = L_H(k) + (\rho(k), \rho(k))$. Let $\mathbb{S}_H(r,k)$ be the space of shift operators with respect to this trigonometric operator,

$$\mathbb{S}_H(r,k) = \{D \in D_H^W \mid D\tilde{L}_H(k) = \tilde{L}_H(k+r)D\}.$$

The relationship to the rational case comes from the map $\epsilon : D_H \to D_{\mathfrak{h}}$ which takes the lowest homogeneous part of the operator with respect to the grading defined by the *degree* of the differential operator. This map is more carefully defined in [11, Section 3]. The lowest homogeneous degree or lhd of $\tilde{L}_H(k)$ is -2, and so

$$\epsilon(\tilde{L}_H(k)) = \Delta_{\mathfrak{h}} + 2k \sum_{\alpha \in R^+} \frac{1}{\alpha} \partial_{X_\alpha},$$

which is $L_{\mathfrak{h}}(k)$ from above. Note that ϵ satisfies

$$\epsilon(D_1 D_2) = \epsilon(D_1)\epsilon(D_2) \qquad \text{for all } D_1, D_2, \tag{5.4}$$

$$\epsilon(D_1 + D_2) = \epsilon(D_1) + \epsilon(D_2) \qquad \text{for all } D_1, D_2 \text{ with } \mathrm{lhd}(D_1) = \mathrm{lhd}(D_2). \tag{5.5}$$

Thus if $D \in \mathbb{S}_H(r,k)$, we get

$$\epsilon(D) L_{\mathfrak{h}}(k) = L_{\mathfrak{h}}(k+r)\epsilon(D).$$

So $\epsilon(D) \in \mathbb{S}_{\mathfrak{h}}(r,k)$.

From [11, Theorem 3.1] we know that $\mathbb{S}_H(r,k)$ is a rank-one $\mathbb{S}_H(0,k)$-module, similar to the case of $\mathbb{S}_{\mathfrak{h}}$. We denote the generator by $G(r,k)$. Moreover, we prove the following proposition in Section 7.

Proposition 5.3. *The map ϵ sends $G(-1,k)$ to $g(-1,k)$.*

Since $\mathrm{Rad}_k(S\hat{f}) \in \mathbb{S}_{\mathfrak{h}}(-1,k+2)$, we know there exists some $D_0 \in \mathbb{S}_{\mathfrak{h}}(0,k+2)$ such that $D_0 \cdot \epsilon(G(-1,k+2)) = \mathrm{Rad}_k(S\hat{f})$. By [11, Theorem 3.3], we know $G(-1,k+2)$ has lowest homogeneous degree 0. Thus it can be written

$$G(-1,k+2) = \sum_{i \in \mathcal{I}} p_i \partial(q_i)$$

for some p_i and q_i, homogeneous polynomials with $\deg(p_i) \geq \deg(q_i)$. Thus

$$\epsilon(G(-1,k+2)) = \sum_i p_i \partial(q_i)$$

where the sum is over $\{i \in \mathcal{I} \mid \deg(p_i) = \deg(q_i)\}$. Given a differential operator P, we define the *constant term* of P, denoted $\mathrm{CT}(P)$, to be the scalar part of the operator, i.e., the summand with order zero and degree zero. If we write $P = \sum_i p_i \partial(q_i)$, then $\mathrm{CT}(P) = p_k \partial(q_k)$ where $\deg(p_k) = \deg(q_k) = 0$. We see that

$$\epsilon(G(-1,k+2)) \cdot 1 = \mathrm{CT}(\epsilon(G(-1,k+2))) = \mathrm{CT}(G(-1,k+2)).$$

Let $r_0 = \mathrm{CT}(D_0) \in \mathbb{C}[k]$. Then

$$\begin{aligned}\mathrm{CT}(\mathrm{Rad}_k(Sf)) &= \mathrm{Rad}_k(Sf)\cdot 1 = D_0\epsilon(G(-1,k+2)))\cdot 1\\ &= D_0\cdot \mathrm{CT}(G(-1,k+2)) = r_0\,\mathrm{CT}(G(-1,k+2)).\end{aligned}$$

We deduce that $\mathrm{CT}(G(-1,k+2))$ divides $\mathrm{CT}(\mathrm{Rad}_k(Sf)) = b'(k)$.

Let Γ be the Gamma function. For a reduced root system define

$$\tilde{c}(\lambda,k) = \prod_{\alpha\in R^+} \frac{\Gamma(-(\lambda,\alpha^\vee))}{\Gamma(-(\lambda,\alpha^\vee)+k)}.$$

In the special case of type A_n we have

$$\frac{1}{\tilde{c}(-\rho(k),k)} = \prod_{d=2}^{n} \frac{\Gamma(dk)}{\Gamma(k)}. \tag{5.6}$$

By [11, Theorem 3.1, Corollary 3.4, Corollary 5.2], we know that

$$\mathrm{CT}(G(-1,k+2)) = \frac{\tilde{c}(-\rho(k+1),k+1)}{\tilde{c}(-\rho(k+2),k+2)}. \tag{5.7}$$

So substituting (5.6) into (5.7) and canceling, we arrive at

$$\mathrm{CT}(G(-1,k+2)) = n!\prod_{d=2}^{n}\prod_{j=1}^{d-1}(d(k+1)+j).$$

In summary, we have proved the following.

Proposition 5.4. *The polynomial* $b_2(s) = \prod_{d=2}^{n}\prod_{j=1}^{d-1}(d(s+1)+j)$ *divides* $b'(s)$ *in* $\mathbb{C}[s]$.

Thus we have shown b_1 and b_2 divide b'. Since b_1 and b_2 are coprime and $\deg(\tilde{b}) \le n(n+1)/2$, it follows that $b' = \tilde{b}\alpha_n = b_1 b_2$.

§6. Minimality of $\tilde{b}$

To complete the proof of Theorem 1.1, we must show that $\tilde{b}$ has minimal degree. In our case, this will follow directly from Proposition 6.1. Set $m = n(n+1)/2 = \deg(f)$.

The proof that Sato's b-function in the prehomogeneous case gives the Bernstein polynomial (see, e.g., [8]) is general enough to cover our situation. The argument given in [8] proves the following result.

Proposition 6.1. *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous of degree d. If the operator $D \in \mathbb{C}[s, \partial_{x_1}, \ldots, \partial_{x_n}]$ satisfies (1.1) for some polynomial $b_D(s)$ and is not divisible by any non-scalar factor in $\mathbb{C}[s]$, then b_D is the Bernstein polynomial for f.*

Proof. The proof is identical to the proof of [8, Theorem 6.3.1]. We give a sketch of it for the reader's convenience.

For any polynomial $p(s) = \prod_\mu (s - \mu)$, we define

$$\gamma_p(s) = \prod_\mu \Gamma(s - \mu).$$

Let $b(s)$ be a polynomial satisfying (1.1) with respect to $f \in \mathbb{C}[X]$. Define the zeta function corresponding to f as

$$Z_f(s) = \int_X |f(x)|^s e^{-2\pi|x|^2} dx.$$

By [8, Theorem 5.3.2], $Z_f(s)/\gamma_b(s)$ is holomorphic on $\mathbb{C}$. Each root of b corresponds to infinitely many zeros of $Z_f(s)/\gamma_b(s)$. Thus if $Z_f(s)/\gamma_b(s)$ is nowhere-vanishing, b must be minimal. We will now show this for b_D.

The integration by parts argument from [8, Theorem 6.3.1], applied with D instead of $f(\partial)$, shows that

$$Z_f(s) = (2\pi)^{-ds} \gamma_{b_D}(s) \prod_\mu \frac{1}{\Gamma(-\mu)}$$

for $\mathrm{Re}(s) > 0$. Thus $Z_f(s)/\gamma_{b_D}(s)$ is a nowhere-vanishing holomorphic function. □

Remark 1. Let $D \in \mathbb{C}[s, \partial_{x_1}, \ldots, \partial_{x_n}]$ be an operator satisfying (1.1) with respect to the Bernstein polynomial b. Let $\tilde{D}$ be the degree $-m$ part of D. Then taking the degree ms part of (1.1) we get $\tilde{D} f^{s+1} = b(s) f^s$. Since G acts locally finitely on D_X, we can write $\tilde{D} = \sum_i \tilde{D}_i$ where $g \cdot \tilde{D}_i = \det(g)^i \tilde{D}_i$. Then since $gf = \det(g) f$, this means $\tilde{D}_i f^{s+1} = 0$ unless $i = -1$. Thus we have $\tilde{D}_{-1} f^{s+1} = b(s) f^s$. Then by Proposition 2.3, since $\deg(f) = \deg(S)$ we have

$$\tilde{D}_{-1} = cS$$

where $c \in \mathbb{C}$. So we have, in fact, shown that any Bernstein operator independent of $x \in X$ realizing $b(s)$ is our differential operator S up to a constant factor.

§7. Appendix: Proof of Propositions 5.1 and 5.3

In this section, we will prove Propositions 5.1 and 5.3.

Given $\mathbf{j} \in \mathbb{Z}^{R^+}$, an index on R^+, we define a partial order by $\mathbf{j} \leq 0$ if $\mathbf{j}_\alpha \leq 0$ for all α. We represent the basis of $\mathbb{Z}^{R^+}$ by e_α where $\alpha \in R^+$. Also denote $\overline{\alpha}^{\mathbf{j}} = \prod_{\alpha \in R^+} \alpha^{\mathbf{j}_\alpha}$. If $p \in \mathbb{C}[k] \otimes \mathbb{C}[\mathfrak{h}]$, then as above, we denote the corresponding differential operator in $\mathbb{C}[k] \otimes D_{\mathfrak{h}}$ as ∂_p or $\partial(p)$.

Let $h_1, \ldots, h_n$ be an orthonormal basis of $\mathfrak{h}^*$. As a shorthand we write $\partial_i = \partial(h_i)$. So $\Delta_{\mathfrak{h}} = \sum_{i=1}^n \partial_i^2$.

The following lemma is a straightforward calculation.

Lemma 7.1. *Let* $\sum \mathbf{r} = \sum_{i=1}^n \mathbf{r}_i$ *and* $p^{(\mathbf{r})} = \prod_{i=1}^n \partial_{t_i}^{\mathbf{r}_i} p(t)$ *and*

$$\mathrm{AV}(\mathbf{r}) = \frac{(\sum \mathbf{r})!}{\prod_{i=1}^n (\mathbf{r}_i!)} (-1)^{\sum \mathbf{r}}.$$

Then

$$\partial(p)(\alpha^{-1} e^{t\lambda}) = \sum_{\mathbf{r}=0}^{\infty} \mathrm{AV}(\mathbf{r}) \alpha^{-1-\sum \mathbf{r}} \prod_{i=1}^{n} (\partial_i \alpha)^{r_i} p^{(\mathbf{r})}. \tag{7.1}$$ □

The key lemma we need for proving Proposition 5.1 reads as follows.

Lemma 7.2. *The map*

$$p_{\mathbf{N}} : \mathbb{S}_{\mathfrak{h}}(-1, k) \to \mathbb{C}[k] \otimes \mathbb{C}[\mathfrak{h}], \qquad \sum_{\mathbf{j} \leq \mathbf{N}} \overline{\alpha}^{\mathbf{j}} \partial(p_{\mathbf{j}}) \mapsto p_{\mathbf{N}},$$

is injective.

Proof. First note that if $D \in \mathbb{S}_{\mathfrak{h}}(r, k)$ and $S \in \mathbb{S}_{\mathfrak{h}}(0, k)$, then commuting S and $L_{\mathfrak{h}}(k)$ yields

$$(DS) L_{\mathfrak{h}}(k) = L_{\mathfrak{h}}(k + r)(DS).$$

So $\mathbb{S}_{\mathfrak{h}}(r, k)$ is an $\mathbb{S}_{\mathfrak{h}}(0, k)$-module.

We will solve for an arbitrary element D of the set $\mathbb{S}_{\mathfrak{h}}(r, k)$. Since D is in $\mathbb{C}[k] \otimes D^W_{\mathfrak{h}^{\mathrm{reg}}}$, we can write

$$D = \sum_{\mathbf{j} \leq \mathbf{N}} \overline{\alpha}^{\mathbf{j}} \partial(p_{\mathbf{j}}).$$

Let $e^{t\lambda} = e^{t_1 \lambda_1 + \cdots + t_n \lambda_n}$. If $\alpha = \sum_i c_i \lambda_i$, then denote $t_\alpha = \sum_i c_i t_i$. Then applying (5.3) to $e^{t\lambda}$, we solve for D (i.e. solve for the p_i) in

$$D L_{\mathfrak{h}}(k) e^{t\lambda} = L_{\mathfrak{h}}(k + r) D e^{t\lambda}.$$

Using Lemma 7.1, we expand this into

$$\begin{aligned}0 &= DL_{\mathfrak{h}}(k)e^{t\lambda} - L_{\mathfrak{h}}(k+r)De^{t\lambda}\\ &= e^{t\lambda}\sum_{\mathbf{j}\le\mathbf{N}}\overline{\alpha}^{\mathbf{j}}\Big[2k\sum_{\alpha\in R^+} t_\alpha\sum_{\mathbf{r}=0}^{\infty}\mathrm{AV}(\mathbf{r})\alpha^{-1-\sum\mathbf{r}}\prod_i(\partial_i\overline{\alpha})^{\mathbf{r}_i}p^{(\mathbf{r})}(t)\\ &\qquad -p_{\mathbf{j}}(t)\Big\{\sum_{\alpha\in R^+}(\mathbf{j}_\alpha^2-\mathbf{j}_\alpha)\alpha^{-2}\Big(\sum_{l=1}^{n}(\partial_l\alpha)^2\Big)+2\mathbf{j}_\alpha\alpha^{-1}t_\alpha\\ &\qquad +2\mathbf{j}_\alpha\sum_{\alpha\ne\beta}\mathbf{j}_\beta\alpha^{-1}\beta^{-1}\Big(\sum_{l=1}^{n}(\partial_l\alpha)(\partial_l\beta)\Big)\\ &\qquad +2(k+r)\Big(\sum_{\alpha\in R^+}\alpha^{-2}\mathbf{j}_\alpha 2+\alpha^{-1}t_\alpha+2\sum_{\alpha\ne\beta}\beta^{-1}\alpha^{-1}\mathbf{j}_\alpha(\partial_\beta\alpha)\Big)\Big\}\Big].\end{aligned}$$

Hence extracting the coefficient of $\overline{\alpha}^{\mathbf{N}-e_\beta}e^{t\lambda}$, we get

$$0 = p_{\mathbf{N}}t_\beta 2(k-\mathbf{N}_\beta-(k+r))$$

and so assuming that $p_{\mathbf{N}}\ne 0$ we find that $\mathbf{N}_\beta = -r$ for all β. That is, $\mathbf{N} = (-r,\dots,-r)$.

Denote $\nabla\alpha = (\partial_1\alpha,\dots,\partial_m\alpha)$. In general, if we look at the coefficient of $\overline{\alpha}^{\mathbf{M}}$ we see that the following expression is 0:

$$\begin{aligned}&2k\sum_{\alpha\in R^+}\sum_{\mathbf{r}=0}^{\infty}p^{(\mathbf{r})}_{\mathbf{M}+e_\alpha(1+\sum\mathbf{r})}t_\alpha\,\mathrm{AV}(\mathbf{r})\prod_{i=1}^{n}(\partial_i\alpha)^{\mathbf{r}_i}\\ &\quad-\sum_{\alpha\in R^+}p_{\mathbf{M}+2e_\alpha}(\mathbf{M}_\alpha+2)(\mathbf{M}_\alpha+1)\nabla\alpha\cdot\nabla\alpha-\sum_{\alpha\in R^+}p_{\mathbf{M}+e_\alpha}2(\mathbf{M}_\alpha+1)t_\alpha\\ &-\sum_{\alpha\ne\beta}p_{\mathbf{M}+e_\alpha+e_\beta}2(\mathbf{M}_\alpha+1)(\mathbf{M}_\beta+1)\nabla\alpha\cdot\nabla\beta-\sum_{\alpha\in R^+}p_{\mathbf{M}+2e_\alpha}4(k+r)(\mathbf{M}_\alpha+2)\\ &\quad-\sum_{\alpha\in R^+}p_{\mathbf{M}+e_\alpha}2(k+r)t_\alpha-\sum_{\alpha\in R^+}p_{\mathbf{M}+e_\alpha+e_\beta}4(k+r)(\mathbf{M}_\alpha+1)(\alpha,\beta).\end{aligned}\tag{7.2}$$

This equation involves only $p_{\mathbf{j}}$ with $\mathbf{j}\ge\mathbf{M}$. Define $\deg(\mathbf{j}) = \sum_{\alpha\in R^+}\mathbf{j}_\alpha$ and $|\mathbf{j}| = |\deg(\mathbf{j})|$. Then note that even though some of the $p_{\mathbf{j}}$ are differentiated, none of the ones with $|\mathbf{M}-\mathbf{j}| = 1$ are. We can use (7.2) to solve for one of the $p_{\mathbf{j}}$ in terms of $p_{\mathbf{j}}$ of greater degree. If we move through the $\overline{\alpha}^{\mathbf{M}}$ with decreasing degree, we can thus show that all $p_{\mathbf{j}}$ are ultimately determined by $p_{\mathbf{N}}$ alone. It is lengthy although not difficult to give a precise algorithm for how one should progress downward through the indices.

Thus we know that $p_{\mathbf{N}}$ determines D. □

Proof of Proposition 5.3. We first note that since $L_{\mathfrak{h}}(k)$ is homogeneous of degree -2, the generator of $\mathbb{S}_{\mathfrak{h}}(r,k)$ must be homogeneous. If $D \in \mathbb{S}_{\mathfrak{h}}(r,k)$ is not homogeneous, then $DL_{\mathfrak{h}}(k) = L_{\mathfrak{h}}(k+r)D$ implies $\epsilon(DL_{\mathfrak{h}}(k)) = \epsilon(L_{\mathfrak{h}}(k+r)D)$, and since $L_{\mathfrak{h}}(k)$ is already homogeneous, $\epsilon(D)L_{\mathfrak{h}}(k) = L_{\mathfrak{h}}(k+r)\epsilon(D)$. So $\epsilon(D)$ is in $\mathbb{S}_{\mathfrak{h}}(r,k)$ but is not a multiple of D.

Now since the degree of differential operators is additive, $g(-1,k)$ must have the minimal degree. We know from [10, Corollary 3.12] that $\deg(G(-1,k)) = n(n+1)/2$, and it has lowest homogeneous degree 0 [10, Theorem 4.4]. Thus, since $\mathbf{N} = (1,\dots,1)$, we can only have $p_{\mathbf{j}} \neq 0$ for $\mathbf{j}$ whith $\deg(p_{\mathbf{j}}) = \sum \mathbf{j} \geq 0$.

Now we know $D \in D_{\mathfrak{h}}^W$, so $w \cdot p_{\mathbf{j}}(t) = p_{w.\mathbf{j}}(t)$ where W acts on the indices $\mathbf{j}$ by permutation. Thus $p_{\mathbf{N}}$ is W-invariant.

Let $\alpha \in R^+$. By (7.2) with $\mathbf{M} = (1,\dots,1) - 2e_\alpha$, we see that

$$0 = 2kp_{\mathbf{N}-e_\alpha}t_\alpha + \sum_{i=1}^{n} 2k(\partial_i p_{\mathbf{N}})t_\alpha \operatorname{AV}(e_i)(\partial_i\alpha) - 4(1-k)p_{\mathbf{N}} - t_\alpha p_{\mathbf{N}-e_\alpha}2(k-1)$$

since $\mathbf{M}_\alpha = -1$. Then we see that t_α divides $p_{\mathbf{N}}$ unless

$$0 = 2kp_{\mathbf{N}-e_\alpha}t_\alpha + \sum_{i=1}^{n} 2k(\partial_i p_{\mathbf{N}})t_\alpha \operatorname{AV}(e_i)(\partial_i\alpha) - t_\alpha p_{\mathbf{N}-e_\alpha}2(k-1).$$

This requires $p_{\mathbf{N}}$ to be 0, which makes $g(-1,k)$ zero, contradicting the fact that it is a generator.

Thus for all α, t_α divides $p_{\mathbf{N}}$. Then the positive roots give $n(n+1)/2$ independent divisors of $p_{\mathbf{N}}$ and so $\deg(g(-1,k)) \geq n(n+1)/2$. Thus the $p_{\mathbf{N}}$ term of $g(-1,k)$ is a scalar multiple of the $p_{\mathbf{N}}$ term of $\epsilon(G(-1,k))$. So the operators are scalar multiples of each other and thus $\epsilon(G(-1,k))$ is a generator. ☐

Proof of Proposition 5.1. By Lemma 7.2, D is determined by $p_{\mathbf{N}}(D)$. Given $D \in \mathbb{S}_{\mathfrak{h}}(-1,k)$ and $S \in \mathbb{S}_{\mathfrak{h}}(0,k)$, the lead coefficients are multiplicative,

$$p_{\mathbf{N}}(SD) = p_{\mathbf{N}}(S)p_{\mathbf{N}}(D).$$

Hence Lemma 7.2 reduces the proof to showing that $p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(-1,k))$ is a rank-one $p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(0,k))$-module.

By [7, Theorem 1.7] we know that

$$\mathbb{C}[t_1,\dots,t_n]^W \subset p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(0,k)).$$

By Proposition 5.3, we know that $\prod_{\alpha\in R^+} t_\alpha$ divides $p_{\mathbf{N}}(D)$ for all $D \in \mathbb{S}_{\mathfrak{h}}(-1,k)$. Now since D is W-invariant and $\mathbf{N}$ is fixed by W, we know that the lead term

$$\Big(\prod_{\alpha\in R^+} \alpha\Big)\partial(p_{\mathbf{N}})$$

is W-invariant. Since w acts on $\prod_{\alpha\in R^+}$ by $\mathrm{sgn}(w)$, this means w acts on $p_{\mathbf{N}}$ by $\mathrm{sgn}(w)$ and is thus equal to

$$q(t)\Big(\prod_{\alpha\in R^+}\alpha\Big)$$

where $q(t)\in\mathbb{C}[t_1,\dots,t_n]^W\subset p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(0,k))$. Since $p_{\mathbf{N}}(\epsilon(G(-1,k)))=c(\prod_{\alpha\in R^+}\alpha)$, we know $\prod_{\alpha\in R^+}\alpha\in p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(-1,k))$. Thus

$$p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(-1,k))=\mathbb{C}[t_1,\dots,t_n]^W\Big(\prod_{\alpha\in R^+}\alpha\Big)$$

is clearly a rank-one $p_{\mathbf{N}}(\mathbb{S}_{\mathfrak{h}}(0,k))$-module. □

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