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Convexity at infinity and Brownian motion on manifolds with unbounded negative curvature

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Harmonic functions on complete simply connected Riemannian manifolds with negative sectional curvatures have been extensively studied in the last two decades, and several basic questions in this field ([Dyn, p.19], [GW2, p.3], [GW1], [Yau]) are by now essentially solved, at least if the sectional curvatures are also assumed to be bounded from below : for such a manifold M, and with respect to its compactification $\widehat{M} = M \cup S_\infty(M)$ with the sphere at infinity $S_\infty(M)$ (for definitions see [EO]), the Dirichlet problem, the behavior at infinity of positive harmonic functions and, in probabilistic terms, the asymptotic behavior of the Brownian motion on M are all well understood ([Pra], [Kif1], [Cho], [And], [Sul], [AS], [Anc1], [Kif2], [Anc2]). In particular, the following property (P_M) holds: the Brownian path on M converges a.s. to some exit point $\xi \in S_\infty(M)$ ([Pra], [Sul]), and moreover, for each $\zeta \in S_\infty(M)$, the distribution of the exit point ξ converges to the Dirac measure δ_ζ when the starting point tends to ζ ([Sul]). In analytic terms, (P_M) means that the Dirichlet problem on M is solvable for each given continuous boundary datum $f \in C(S_\infty(M); \mathbb{R})$ ([And]; see also [AS], [Anc1]). By a theorem of Choi ([Cho]), (P_M) may also be deduced from the following convexity property (C_M): each point $\zeta \in S_\infty(M)$ has a fundamental system of neighborhoods V in $\widehat{M}$ such that $V \cap M$ is convex ([And]). In fact, [Cho] shows that it is sufficient to check the weaker property (C'_M): for each pair (ζ, ζ') of distinct points on $S_\infty(M)$, there is a neighborhood V of ζ in $\widehat{M}$ such that ζ' is not in the closure (in $\widehat{M}$) of the convex hull of $M \cap V$ (see the Appendix in Section 6 below). Note however, that in contrast to (C'_M), property (C_M) guarantee that the harmonic measure does not charge points (see Corollary 6.3).

In this paper, we consider the case of complete simply connected Riemannian manifolds whose sectional curvatures are negative (say bounded from above by -1) but not bounded from below, and we show that property (P_M) does not then hold in general (See [HM] for property (P_M) under a growth condition on the curvature).

Theorem A. *There is a complete simply connected Riemannian manifold M of dimension* 3, *with sectional curvatures ≤ -1, and a point $\zeta_0 \in S_\infty(M)$ such that*

i) *the Brownian motion B_s on M has a.s. an infinite lifetime,*

ii) *with probability* 1, *B_s exits from M at ζ_0.*

Clearly, for such an M, there is no non-trivial bounded harmonic function f which may be extended continuously on $\widehat{M} = M \cup S_\infty(M)$ (or even such that $\lim_{m \to \zeta_0} f(m)$ exists). A variant of the method gives also the following.

Theorem B. *There exists a complete simply connected Riemannian manifold M of dimension* 3, *with sectional curvatures ≤ -1 and such that*

i) *the Brownian motion B_s on M has a.s. an infinite lifetime,*

ii) *with probability* 1, *every point on the sphere at infinity $S_\infty(M)$ is a cluster point of B_s (when $s \to \infty$).*

It will be clear that we may as well construct examples such that the set of cluster points of the Brownian motion in M is a.s. a fixed continuum $K \subset S_\infty(M)$, the pair $(S_\infty(M), K)$ being prescribed up to topological equivalence. Also, both theorems extend to higher dimensions. See final remarks in Section 5.C.

In our framework (no lower bound assumption on the curvature), the basic tools available in the "bounded geometry" case collapse with the notable exception of Choi's Theorem. For example, if $\dim(M) = 2$, (C_M) evidently holds (each geodesic in M divides M in two convex regions) so that (P_M) is still true; for M rotationaly symmetric and $\dim(M) \geq 3$, (C_M) holds and thus also (P_M) ([Cho]). Here, from Choi's results (see Corollary 6.3 below), we have the following purely geometric consequence of Theorem A.

Corollary C. *There exists a complete simply connected Riemannian manifold M of dimension 3, with sectional curvatures ≤ -1, such that for some point $\zeta_0 \in S_\infty(M)$ and every neighborhood V of ζ_0 in $\widehat{M}$ the closed convex hull of $V \cap M$ is M.*

(Since a convex open subset of M is equal to the interior of its closure -the usual proof of the similar well-known statement in $\mathbb{R}^N$ is easily adapted-, one may in fact remove the word "closed" in the statement). Theorem B shows that we can also construct M with the above property for all $\zeta_0 \in S_\infty(M)$; we have stated Corollary C because, for sake of simplicity, we shall first give a direct and nonprobabilistic proof of this corollary.

AN EXAMPLE. It is interesting to observe that one may easily construct examples (of dimension ≥ 3) with the following property: there is a point $\zeta_0 \in S_\infty(M)$ such that the Brownian motion B_s on M has a (strictly) positive probability to converge to ζ_0 in $\widehat{M}$ when $s \to S$ (S being the lifetime of the Brownian particle). To see this, fix a complete simply connected Riemannian manifold (N, g) of dimension 2 whose sectional curvatures are ≤ -1 and whose Brownian motion has a.s. a finite lifetime; let $M = N \times \mathbb{R} = \{(\xi, u) : \xi \in N, u \in \mathbb{R}\}$ equipped with the metric

$$ds^2 = du^2 + e^{2u} g(d\xi, d\xi).$$

Then:

i) M is complete and its sectional curvatures are ≤ -1 (simple computations; M is in fact a special case of the warped products in [BO]),

ii) each region $\{u < a\}$ is a horoball in M at $\zeta_0 = \lim_{u \to -\infty}(\xi_0, u) \in S_\infty(M)$, ($\zeta_0$ is independent of $\xi_0 \in N$),

iii) from the standard description of the Brownian motion on M in terms of a Brownian motion with a drift on $\mathbb{R}$ and an independent

Brownian motion on N with a change of time, there is a strictly positive probability that u_s, the u-component of B_s, satisfies $|u_s| \leq 1$ for $s \leq 1$ and that the N-component explodes also before time $s = 1$ (so that $\lim_{s \to S} B_s = \zeta_0$ in $\widehat{M}$).

It follows that the distance of any fixed given point in M to the closed convex hull C_V of $V \cap M$, with V neighborhood of ζ_0 in $\widehat{M}$ is bounded (independently of V) (see Corollary 6.3); thus C_V contains some fixed point $z_0 \in M$ independent of V. However, as was pointed out to me by W. S. Kendall, (P_M) holds. We may for example observe that the sectional curvatures of M are bounded (from below) near each boundary point $\zeta \in S_\infty(M) \setminus \{\zeta_0\}$, so that by the constructions in [And, Section 2], there is for each $\zeta \in S_\infty(M) \setminus \{\zeta_0\}$ arbitrarily small neighborhoods V in $\widehat{M}$ such that $M \setminus V$ (or $M \cap V$) is convex. Hence, (C'_M) holds but not (C_M) and the harmonic measure has a non-trivial discrete part, though its support is the whole boundary. However, it seems difficult to deduce from this construction an example proving Theorem A (even if we drop there condition i)).

PROBLEMS WHICH REMAIN OPEN. We should also point out that our counter-examples leave open several related natural questions: does a Cartan-Hadamard manifold with sectional curvatures ≤ -1 always supports a non-trivial bounded (respectively, positive) harmonic function? Does the Martin boundary of such a manifold M always have dimension $n-1$ ($n = \dim(M)$)? ([Dyn, p.19]), or dimension $\geq n-1$? We do not know (even when the sectional curvatures are bounded from below) if positive (respectively, bounded) harmonic functions may fail to separate points in M!

PLAN OF THIS PAPER. Our proof of Theorem A will follow from a careful study of a class of metrics which is in some sense the simplest for which property (C'_M) is not clear. We show that a nice "convexity" property (connected to Choi's criterion) is related to the integrability along some rays of a positive function derived from the metric (Proposition 2.1); on the other hand it is shown that the curvature assumptions imply that this function in general tends to zero along the rays (Proposition 2.2), but in general not rapidly enough so as to ensure the integrability (Section 4). Using also some technical gluing lemmas (Section 3), we construct an example proving Corollary C. This then leads to examples proving Theorem A or B. Though the material in Section 2 is not really necessary for the proof of these theorems, it should bring some light on the problem and its difficulties, and could be useful for other questions. It would be interesting to know if Proposition 2.3 below could be in some way extended to all Cartan-Hadamard manifold of sectional curvatures ≤ -1.

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1. A class of Riemannian metrics.

1.1. In the sequel, we consider the manifold $M = \mathbb{R}^3 = \{(x,y,t) : x,y,t \in \mathbb{R}\}$ equipped with a Riemannian metric $\gamma = \gamma(g,h)$ in the form

$$ds_\gamma{}^2 = dt^2 + g(x,t)^2\, dx^2 + h(x,t)^2\, dy^2\,,$$

g and h being two smooth positive functions on $\mathbb{R}^2$ which are nondecreasing with respect to t and such that $\inf\{g(x,t) : \ t_0 < t < t_1\} > 0$ for all $t_0, t_1 \in \mathbb{R}$. It is easily checked that (M,γ) is then a complete Riemannian manifold.

It is also easy to compute the sectional curvatures of M on using the natural moving frame given by

$$e_1 = \frac{1}{g(x,t)}\,(1,0,0)\,,\quad e_2 = \frac{1}{h(x,t)}\,(0,1,0)\quad \text{and}\quad e_3 = (0,0,1)\,,$$

the corresponding Cartan forms being $\alpha_1 = g(x,t)\,dx$, $\alpha_2 = h(x,t)\,dy$, $\alpha_3 = dt$.

Since

$$d\alpha_1 = -\frac{g'_t}{g}\,\alpha_1 \wedge \alpha_3\,,\qquad d\alpha_2 = -\frac{h'_t}{h}\,\alpha_2 \wedge \alpha_3 + \frac{h'_x}{gh}\,\alpha_1 \wedge \alpha_2\,,$$

and $d\alpha_3 = 0$, the connexion matrix $\Omega = \{\omega_{ij}\}_{1\leq i,j\leq 3}$ is

$$\Omega = \begin{pmatrix} 0 & -\dfrac{h'_x}{hg}\,\alpha_2 & \dfrac{g'_t}{g}\,\alpha_1 \\ \dfrac{h'_x}{hg}\,\alpha_2 & 0 & \dfrac{h'_t}{h}\,\alpha_2 \\ -\dfrac{g'_t}{g}\,\alpha_1 & -\dfrac{h'_t}{h}\,\alpha_2 & 0 \end{pmatrix}$$

(Ω is skew symmetric and $d\alpha = -\Omega \wedge \alpha$); thus, the curvature matrix $\mathcal{K} = \Omega \wedge \Omega + d\Omega$ is the skew 3×3 matrix $\mathcal{K} = \{K_{ij}\}$ with

$$K_{12} = \Big(-\frac{g'_t h'_t}{gh} - \frac{h''_{xx}}{g^2 h} + \frac{h'_x g'_x}{g^3 h} \Big) \alpha_1 \wedge \alpha_2 + \Big(\frac{h''_{xt}}{gh} - \frac{h'_x g'_t}{g^2 h} \Big) \alpha_2 \wedge \alpha_3 \, ,$$

$$K_{13} = -\frac{g''_{tt}}{g} \alpha_1 \wedge \alpha_3 \, ,$$

and

$$K_{23} = -\frac{h''_{tt}}{h} \alpha_2 \wedge \alpha_3 + \Big(\frac{h''_{tx}}{gh} - \frac{g'_t h'_x}{g^2 h} \Big) \alpha_1 \wedge \alpha_2 \, .$$

In other words, the curvature tensor R is given by the formula

$$\begin{aligned} \langle R(u,v)u, v \rangle = {} & A\,(u_1 v_2 - v_1 u_2)^2 + B\,(u_1 v_3 - u_3 v_1)^2 \\ & + C\,(u_2 v_3 - u_3 v_2)^2 \\ & + 2\,D\,(u_1 v_2 - u_2 v_1)(u_2 v_3 - u_3 v_2) \, , \end{aligned}$$

where the u_i, v_j denote the coordinates of the vectors u and v with respect to the moving frame, and where

$$A = \Big(\frac{g'_t h'_t}{gh} + \frac{h''_{xx}}{g^2 h} - \frac{h'_x g'_x}{g^3 h} \Big) , \qquad B = \frac{g''_{tt}}{g} \, ,$$

$$C = \frac{h''_{tt}}{h} \, , \qquad D = -\Big(\frac{h''_{xt}}{gh} - \frac{h'_x g'_t}{g^2 h} \Big) \, .$$

Since $|u \wedge v|^2 = \sum_{i<j} (u_i v_j - u_j v_i)^2$, it is then clear that γ has all its sectional curvatures $\leq -\alpha^2$, $\alpha \geq 0$, if and only if the following four inequalities hold for all $(x,t) \in \mathbb{R}^2$

$$\frac{g''_{tt}}{g} \geq \alpha^2 \, , \tag{1}$$

$$\frac{h''_{tt}}{h} \geq \alpha^2 \, , \tag{2}$$

$$\Big(\frac{g'_t h'_t}{gh} + \frac{h''_{xx}}{g^2 h} - \frac{h'_x g'_x}{g^3 h} \Big) \geq \alpha^2 \, , \tag{3}$$

$$\Big(\frac{h''_{xt}}{gh} - \frac{h'_x g'_t}{g^2 h} \Big)^2 \leq \Big(\frac{h''_{tt}}{h} - \alpha^2 \Big) \Big(\frac{g'_t h'_t}{gh} + \frac{h''_{xx}}{g^2 h} - \frac{h'_x g'_x}{g^3 h} - \alpha^2 \Big) \, . \tag{4}$$

Note that this set of inequalities expresses that the quadratic forms

$$q(X,Y,Z) = (A-\alpha^2)\,X^2 + (B-\alpha^2)\,Y^2 \\ + (C-\alpha^2)\,Z^2 + 2\,D\,X\,Z$$

are nonnegative (for all $(x,t) \in \mathbb{R}^2$). In the sequel, we shall say that a metric $\gamma(g,h)$ is of class $\gamma(-\alpha^2)$, $(\alpha \geq 0)$, if the above inequalities (1) to (4) hold throughout $\mathbb{R}^2$.

We note also for later use that for $g > 0$ nondecreasing with respect to t, (1) implies that

$$g(x,t) \geq g(x,s)\,\cosh(\alpha(t-s)) + \alpha^{-1}\,g'_t(x,s)\,\sinh(\alpha(t-s))$$

for $t \geq s$, $x \in \mathbb{R}$.

1.2. Assuming now that g and h define a $\gamma(-\alpha^2)$ metric $(\alpha > 0)$, it is easy to describe $S_\infty(M)$, the sphere at infinity of M (See [EO] for definitions and basic facts concerning the compactification $\widehat{M} = M \bigcup S_\infty(M)$ with the sphere at infinity, and for the basic relations of these objects with the geodesics in M). Clearly, the curves $\tau_{(x,y)}$: $t \mapsto (x,y,t)$ $(x,y \in \mathbb{R}$ fixed) are the unit-speed geodesics emanating from a point on $S_\infty(M)$; this point shall be denoted ∞_M. Denoting also $\zeta_{(x,y)}$ the end point (for $t \to +\infty$) of $\tau_{x,y}$ on $S_\infty(M)$, we have $S_\infty(M) = \{\zeta_{x,y} : x,y \in \mathbb{R}\} \cup \{\infty_M\}$, the mapping $(x,y) \mapsto \zeta_{(x,y)}$ being a homeomorphism from $\mathbb{R}^2$ on $S_\infty(M) \backslash \{\infty_M\}$. It is also easily seen that a basis $\{V_n\}_{n\geq 1}$ of neighborhoods of ∞_M can be obtained by setting $V_n = \widehat{M} \setminus \overline{W_n}$ and $W_n = \{(x,y,t) \in M : |x| \leq n,\ |y| \leq n,\ t \geq -n\}$, (here $\overline{W_n} = W_n \cup \{\zeta_{(x,y)} : |x| \leq n,\ |y| \leq n, \}$).

1.3. Finally, let us write the expression of the Laplace-Beltrami operator Δ induced by the metric $\gamma(g,h)$ with respect to the coordinates x, y, t; on using the standard formula (or direct computations), we have

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{1}{g^2}\,\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\,\frac{\partial^2}{\partial y^2} + \left(\frac{g'_t}{g} + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \left(\frac{h'_x}{hg^2} - \frac{g'_x}{g^3}\right)\frac{\partial}{\partial x}\,.$$

In order to obtain our examples, we shall try below to construct metrics of class $\gamma(-\alpha^2)$ $(\alpha > 0)$ with g independent of x and such that the x-drift term $h'_x/(hg^2)$ is "large" compared to the t-drift term $g'_t/g+$

h'_t/h, and thus force the Brownian motion on M to move away from each geodesic $\tau_{(x,y)}$; in the next section, we study the ratio $h'_x/(h'_t g^2)$ in relation with the curvature assumptions and the Choi property C'_M.

2. Filling by convex hulls.

In this section, we assume that g and h define a $\gamma(-\alpha^2)$ metric ($\alpha > 0$), and that $g = g(t)$ is a function of t (*i.e.* g is independent of x). Of course, g is convex and increasing. Recall that M denotes the resulting manifold.

We denote by $M(\theta_0, a, b)$ the set $\{(x, y, t);\ t > \theta_0,\ a < x \leq b,\ y \in \mathbb{R}\}$ for given reals θ_0, a, b with $a < b$; let $N(\theta_0, a, b)$ denotes the projection of $M(\theta_0, a, b)$ on the (x, t)-plane, and let $F(\theta_0, b)$ be the "vertical" half plane $F(\theta_0, b) = \{(b, y, t) :\ t > \theta_0,\ y \in \mathbb{R}\}$.

Proposition 2.1. *Let t_0, a, b be three reals such that*

i) *$h(x, t)$ is nondecreasing in x (for each fixed t) on $M(\theta_0, a, b')$ for some $b' > b$ (*) and*

ii) $\int_{\theta_0}^{\infty} \delta(t)\, dt = +\infty$, *where*

$$\delta(t) = \inf\left\{ \frac{h'_x(x,t)}{h'_t(x,t)\, g^2(t)} \ :\ a \leq x \leq b \right\}.$$

Then, the closed convex hull C of $F(\theta_0, b)$ in $M(\theta_0, a, b)$ (with respect to $\gamma(g, h)$) is $M(\theta_0, a, b)$.

We shall first establish the following lemma.

Lemma 2.2. *Let $\sigma : u \mapsto (x(u), y(u), t(u))$ be a unit speed geodesic in M with $\sigma(0) = m_0 = (x_0, y_0, t_0)$, and $\sigma'(0) = e_2 = h(x_0, y_0, t_0)^{-1}(0, 1, 0)$. The map $\tau : u \mapsto (x(u), t(u))$ is even, induces a global diffeomorphism of $]0, \infty)$ onto $\tau(]0, \infty))$, and the curve $\tau([0, +\infty))$ admits at $\tau(0)$ a tangent directed by the vector $V(x_0, t_0) = (h'_x(x_0, t_0), g(t_0)^2 h'_t(x_0, t_0))$. Also, the tangent to $\tau[0, +\infty)$ at $\tau(u)$ is (as a line) a continuous function of u, x_0, y_0, t_0.*

(*) The proposition is in fact also valid if we allow $b' = b$.

PROOF. Since the metric γ is invariant under the map $(x, y, t) \mapsto (x, -y, t)$, the points $\sigma(u)$ and $\sigma(-u)$ are symmetric with respect to the plane $y = y_0$ so that $\tau(u) = \tau(-u)$. Writing the Euler equations for the functional

$$F = \dot{t}^2 + g(t)^2 \dot{x}^2 + h(x,t)^2 \dot{y}^2 ,$$

we have the following three differential equations

$$t'' = g\, g'\, x'^2 + h\, h'_t\, y'^2 ,$$
$$g^2\, x'' + 2\, g\, g'\, x'\, t' = h\, h'_x\, y'^2 ,$$
$$h^2\, y' = C ,$$

where C is a positive constant. Let $p_0 = (x_0, t_0)$. Since $t'(0) = x'(0) = 0$ and $y'(0) = (h(p_0))^{-1}$, it follows that $t''(0) = h(p_0)^{-1}\, h'_t(p_0)$ and $x''(0) = g(t_0)^{-2}\, h(p_0)^{-1}\, h'_x(p_0)$; also $h'_t(x_0, t_0) > 0$ ($t \mapsto h(x_0, t)$ is strictly convex and nondecreasing). Thus the third (and main) assertion of the lemma follows from the Taylor formula.

The first Euler equation above shows that t is a convex function of u, so that $t(u)$ is strictly increasing on $[0, \infty)$ and $t'(u) > 0$ for $u > 0$. Hence, τ is regular for $u > 0$ and injective on $[0, \infty[$. If we let $W(u) = (x'(u)/u, t'(u)/u) = \int_0^1 (x''(u\theta), t''(u\theta))\, d\theta$ for $u > 0$ and $W(0) = g(t_0)^{-2}\, h(p_0)^{-1}\, V(x_0, t_0)$, then $W(u)$ supports the tangent to τ at $\tau(u)$ and $W(u)$ depends continuously on $u \in [0, \infty[$, x_0, y_0, t_0 by the standard continuity theorems for solutions of differential equations.

REMARK. If $m_0 \in M(\theta_0, a, b')$, $x_0 < b'$, then $x(u)$ is increasing on any interval $J = [0, T[$ such that $\sigma(J) \subset M(\theta_0, a, b')$ (the second Euler equation above shows that $x'(u) > 0$ when $0 < u < T$).

PROOF OF PROPOSITION 2.1. C is invariant under translation in the y-variable so that $C = \{(x, y, t) : (x, t) \in \Phi\}$, Φ being a closed subset of $N(\theta_0, a, b)$.

We claim that the vector field $-V$ (V is defined in the previous lemma) is an inward vector field for Φ, which means that

$$\lim_{\substack{t>0 \\ t\to 0}} t^{-1} d(m - t\, V(m), \Phi) = 0$$

for each $m \in \Phi$.

To see this, fix $m_0 = (x_0, y_0, t_0) \in C$ and consider the (unit speed and oriented) geodesic arc σ_ε connecting $p(\varepsilon) = (x_0, y_0 - \varepsilon, t_0)$ to $p'(\varepsilon) =$

$(x_0, y_0+\varepsilon, t_0)$ in M, and the projection τ_ε of σ_ε on the (x,t)-plane. Observe that the middle point $q(\epsilon)$ of σ_ε is in the plane $y = y_0$, with a tangent parallel to the y-axis; choose the parametrization of σ_ε such that $\sigma_\varepsilon(0) = q(\varepsilon)$, and let $\eta_\varepsilon > 0$ be the value of u for which $\tau_\varepsilon(u) = \tau_\varepsilon(-u) = (x_0, t_0)$. By the remark after Lemma 2.2, it is clear that for $\varepsilon > 0$ and small, one has $\sigma_\varepsilon \subset M(\theta_0, a, b)$ so that $\sigma_\varepsilon \subset C$ and $\tau_\varepsilon \subset \Phi$; also τ_ε is smooth at $m_1 = (x_0, t_0)$ and admits a tangent there, directed by $-\tau'_\varepsilon(\eta_\varepsilon)$.

It follows from the lemma above that when $\varepsilon \to 0$, the limit position of this tangent is the half-line emanating from m_1 and directed by $-V(m_1)$. This proves the claim, since a limit of Φ-inward vectors at fixed $m_1 \in \Phi$ is again a Φ-inward vector at m_1.

V being smooth (Lipschitz would be enough) and inward for Φ, it follows from a well-known theorem ([Bre]) that every curve $\gamma : [0, T[\to N(\theta_0, a, b)$ with $\gamma'(s) = -V(\gamma(s))$ for $s \in [0, T)$ and such that $\gamma(0) \in \Phi$, has all its image in Φ. Thus, to finish the proof it suffices now to observe that each maximal V-integral curve $\beta : J \to N(\theta_0, a, b)$ in $N(\theta_0, a, b)$ hits the line $x = b$. In fact, one has (letting $t(u)$ to denote the t-component of $\beta(u)$)

$$b - a \geq \int_\beta dx \geq \int_J \delta(t(u))\, dt(u) = \int_\beta \delta(t)\, dt$$

(both coordinates being increasing functions on J), so that our assumption on the function $\delta(t)$ implies that we must have $\sup\{t(u) : u \in J\} < +\infty$ and hence a hit with the line $x = b$ at the end point of β.

We shall see later that the assumptions of Proposition 2.1 may really occur. In this connection, it is interesting to note the following effect of the negative curvature for our class of metrics.

Proposition 2.3. *For every choice of functions $g(t)$ and $h(x,t)$ defining a $\gamma(-\alpha^2)$ metric ($\alpha > 0$) on M, and for every $a > 0$, we have $\lim_{t\to+\infty} \delta_1(t) = 0$, where*

$$\delta_1(t) = \sup\left\{\left|\frac{h'_x(x,t)}{h'_t(x,t)\, g^2(t)}\right| : \ -a \leq x \leq a\right\}.$$

PROOF. The proof is based on the consideration of the level curves of h in $\mathbb{R}^2$. Let $x = \varphi(t)$, $t \in I$, be a maximal solution of $h'_x(\varphi(t), t)\, \varphi'(t) +$

$h'_t(\varphi(t),t) = 0$ with $h'_x(\varphi(t),t) \neq 0$ on I, say $h'_x(\varphi(t),t) < 0$ on I (*i.e.* φ is an increasing function). One immediately checks that

$$\varphi'' = \frac{1}{|h'_x|}\left(h''_{tt} + 2\,h''_{tx}\,\varphi' + h''_{xx}\,\varphi'^2\right)$$

on I. Because $\gamma(g,h)$ is of class $\gamma(0)$ (see the end of Section 1.1), we have (for all $(x,t) \in \mathbb{R}^2$)

$$h''_{tt}\,u^2 + 2\,\partial^2_{tx}\left(\frac{h}{g}\right)u\,v + \left(g^{-2}\,h''_{xx} + \frac{h'_t\,g'}{g}\right)v^2 \geq 0\,.$$

for all $u, v \in \mathbb{R}$.

On using $u = 1$ and $v = g\,\varphi'$ in this inequality, we derive from the expression of φ''

$$\varphi'' \geq -2\frac{g'}{g}\,\varphi' - g'\,g\,(\varphi')^3 \qquad \text{on } I\,,$$

where h has been eliminated. To "solve" this differential inequality, we first solve the differential equation

$$(E) \qquad \Psi'' = -2\frac{g'}{g}\,\Psi' - g'\,g\,(\Psi')^3\,.$$

If we let $\Psi'(t) = \lambda(t)\,g(t)^{-2}$, with $\lambda > 0$, (E) is equivalent to

$$\lambda' = -\frac{g'}{g^3}\lambda^3\,,$$

so that

$$\frac{1}{\lambda^2} = -\frac{1}{g^2} + C\,,$$

where C is some positive constant.

Whence the maximal solutions of (E)

$$\Psi(t) = \pm\int_t^{\infty} \frac{1}{g(s)\,\sqrt{Cg(s)^2 - 1}}\,ds + C'\,,$$

$t \in (t_0, +\infty)$, with $C = 1/g(t_0)^2$.

For each $t_1 \in I$, there exists a unique maximal E solution Ψ as above with $\Psi(t_1) = \varphi(t_1) = x_1$ and $\Psi'(t_1) = \varphi'(t_1)$, the corresponding t_0 and C being given by

$$C = \frac{1}{g(t_1)^2} + \frac{1}{\varphi'(t_1)^2 g(t_1)^4} \qquad \text{and} \qquad C\,g(t_0)^2 = 1$$

(see Figure 1).

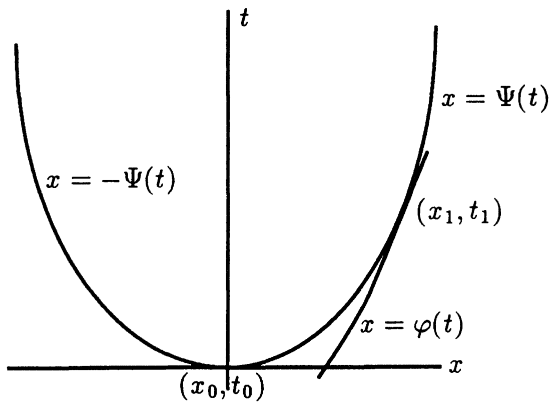

Figure 1

By the standard comparison theorem for first order differential equations, we see that when $t_0 < t \le t_1$ and $t \in I$,

$$\varphi'(t) \le \Psi'(t)\,.$$

In particular, $(t_0, t_1] \subset I$ and for all $t \in [t_0, t_1]$, $\Psi(t) \le \varphi(t) \le \Psi(t_1)$.

We now make two observations. Firstly, we have the following lower bound of

$$\mu_\varphi(t_1) = \varphi'(t_1)\, g(t_1)^2 = \frac{g(t_1)^2 h_t'(x_1, t_1)}{|h_x'(x_1, t_1)|}$$

in terms of t_0; from the above expression of C and $C\, g(t_0)^2 = 1$, we get

$$(1) \qquad g(t_0)^2 = \frac{1}{C} = \frac{1}{\dfrac{1}{g(t_1)^2} + \dfrac{1}{g(t_1)^4 \varphi'(t_1)^2}} = \frac{\mu_\varphi^2}{1 + \dfrac{\mu_\varphi^2}{g(t_1)^2}} \le \mu_\varphi^2\,.$$

Also, since

$$\Psi(\infty) - \Psi(t) = \int_t^\infty \frac{g(t_0)}{\sqrt{g(s)^2 - g(t_0)^2}}\,\frac{ds}{g(s)}\,,$$

and

$$g(s) \geq g(t_0)\cosh(\alpha(s-t_0)) + \alpha^{-1} g'(t_0)\sinh(\alpha(s-t_0))$$
$$\geq g(t_0)\cosh(\alpha(s-t_0)),$$

for $s \geq t_0$, we have, letting $t_0' = \max\{t_0+1, 0\}$, $x_0' = \varphi(t_0')$ (and assuming $t_0' \leq t_1$),

$$x_1 - x_0' \leq \Psi(\infty) - \Psi(t_0') \leq \frac{C_1}{g(t_0')} \tag{2}$$

for some absolute constant C_1 depending only on α.

Suppose now that there is a sequence of points (x_j, t_j) with $t_j \to \infty$, $|x_j| \leq c$ for some fixed $c > 0$, and such that

$$\mu_j = g(t_j)^2 \frac{h_t'(x_j,t_j)}{|h_x'(x_j,t_j)|}$$

is bounded. Then, for the coresponding (E) curves through (x_j, t_j) and with the above notations we have (we omit the index j): i) t_0 is bounded from above by (1), so that t_0' is bounded, and ii) x_0' also stays bounded because of (2). Since, the level curve of h through (x_j, t_j) must hit $\{(x,t) : t = t_0', |x - x_0'| \leq |\Psi(\infty) - \Psi(t_0')|\}$ (Ψ depends on j), it is seen that this line meets a compact subset of $\mathbb{R}^2$ (independent of j), so that $h(x_j, t_j)$ stays bounded. But this is in contradiction with the exponential growth of h with respect to t in each strip $\{|x| \leq A\}$ $(A > 0)$, and Proposition 2.3 is proved.

3. Two extension properties for γ-metrics.

In this section and the next, we let $g(t) = e^t$. If α is a (strictly) positive number, and if $h(x,t)$ is a smooth non-negative function on some region A of $\mathbb{R}^2$, we shall say that h is of class $\mathcal{H}(\alpha)$ on A if the following three inequalities hold on A:

$$h_{tt}'' - \alpha h \geq 0,$$
$$h_t' + \frac{h_{xx}''}{g^2} - \alpha h \geq 0,$$

and

$$\left|\partial_{tx}^2\left(\frac{h}{g}\right)\right|^2 \leq (h_{tt}'' - \alpha h)\left(h_t' + \frac{h_{xx}''}{g^2} - \alpha h\right),$$

and if $h(x,t)$ is nondecreasing with respect to t. If $A = \mathbb{R}^2$, $\alpha \leq 1$ and $h > 0$, this just means that g and h define a $\gamma(-\alpha)$ metric on $\mathbb{R}^3$.

We shall need the following two elementary observations.

3.1. If h_1 and h_2 are of type $\mathcal{H}(\alpha)$ on $A \subset \mathbb{R}^2$ and if $\lambda, \mu \in \mathbb{R}_+$, then $h = \lambda h_1 + \mu h_2$ is of type $\mathcal{H}(\alpha)$ on A. Observe that the $\mathcal{H}(\alpha)$ condition is equivalent to the non-negativity of a quadratic form on $\mathbb{R}^2$ whose coefficients depends linearly on h.

3.2. Let h be of type $\mathcal{H}(\alpha)$ on a region A such that $\inf\{t : (x,t) \in A\} > -\infty$ and let $\alpha' < \alpha$. If h admits a > 0 lower bound on A and if h_1 is any smooth bounded real function on A with bounded first and second order derivatives on A, then $h_1 + Ch$ is of type $\mathcal{H}(\alpha')$ on A provided that the constant C is chosen large enough.

In the following proposition, we denote by a_0 an absolute positive constant whose value is fixed before the statement of Lemma 3.4.

Proposition 3.3. *Let $h(x,t)$ be a smooth (strictly) positive function of type $\mathcal{H}(\alpha)$ (for some $\alpha \in (0,1)$) on the region $\omega(\varepsilon) = \{(x,t) : t_0 < t < t_1 + \varepsilon\} \cup \{(x,t) : |x| < x_0 + \varepsilon,\ t_1 \leq t < t_2 + \varepsilon\}$, where $t_1 = t_0 + 1$, $t_2 \geq t_1 + 2$, $\varepsilon > 0$, $x_0 > a_0 e^{-t_1}$, and assume that $h(x,t) = e^t$ for $t_0 < t < t_1 + \varepsilon$.*

Then, we may find a smooth positive function h_1 on $A = \{(x,t) : t_0 < t < \infty\}$ such that

i) *h_1 is of type $\mathcal{H}(\alpha)$ on A,*

ii) *$h_1(x,t) = e^t$ if $t \leq t_1$, or if t is large enough, and*

iii) *$h_1 = h$ for $|x| \leq x_0 - a_0 e^{-t_1}$ and $t_0 < t < t_2$.*

The proof will show that we may also require $h_1(x,t)$ to be for $|x|$ large a function of t only.

Before proving this proposition, we fix some notations. Let $\theta = f_0 * \varphi$ be a standard regularization of $f_0(x) = \inf\{x, 1\}$, with $\varphi \geq 0$ smooth, even, and such that $\operatorname{supp}(\varphi) = [-1/2, 1/2]$; observe that θ is concave, $\theta(x) = x$ for $x \leq 1/2$ and $\theta(x) = 1$ if and only if $x \geq 3/2$. We then choose and fix a positive number a, sufficiently large so that $-\theta''\theta + \theta'^2 \leq a^2$, and let $\Phi(x) = -\log(\theta(x/a))$ for $x > 0$. Clearly Φ is ≥ 0, convex non-increasing and smooth on $(0, +\infty)$, $\operatorname{supp}(\Phi) = (0, 3a/2]$

and $\Phi(x) = -\log(x/a)$ on $(0, a/2]$. We let $a_0 = 3a/2$ and observe the following.

Lemma 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth convex function with* $\operatorname{supp}(f) = [0, \infty)$, *and let $h(x,t) = g(t)f(t - \Phi(x))$ ($= 0$ if $x \leq 0$). Then h is smooth on $\mathbb{R}^2$, nondecreasing in t, $\{h > 0\} = \{(x,t) : x > 0,\ t > \Phi(x)\}$ and $h(x,t) = g(t)f(t)$ for $x > a_0$. Moreover h is of type $\mathcal{H}(1)$ on $\mathbb{R}^2$.*

PROOF. We have (with obvious notations and for $x > 0$)

$$\left|\partial^2_{tx}\left(\frac{h}{g}\right)\right|^2 = |f''\Phi'|^2,$$
$$h''_{tt} - h = e^t\,(2f' + f''),$$
$$h'_t\frac{g'}{g} + \frac{h''_{xx}}{g^2} - h = e^t\left(f'e^{-2t}(e^{2t} - \Phi'') + f''\,\Phi'^2 e^{-2t}\right).$$

Clearly, h is of type $\mathcal{H}(1)$ if $\Phi''(x) \leq e^{2t}$ for $t \geq \Phi(x)$, $0 < x < a_0$, or equivalently if $\Phi''(x) \leq \exp(2\,\Phi(x))$ for $0 < x \leq a_0$. But with our previous choices we have

$$\Phi''(x) = a^{-2}\,\theta\left(\frac{x}{a}\right)^{-2}\left(-\theta''\left(\frac{x}{a}\right)\theta\left(\frac{x}{a}\right) + \theta'\left(\frac{x}{a}\right)^2\right) \quad \text{and} \quad e^{2\Phi} = \theta^{-2},$$

so that the lemma follows from the choice of a.

REMARKS.

3.5. If $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and if we let $k(x,t) = h(e^{t_0}(x - x_0), t - t_0)$, h being the function in the previous lemma, we obtain a function k of type $\mathcal{H}(1)$ on $\mathbb{R}^2$ with $\{k > 0\} = \{(x,t) : t > t_0,\ x > x_0,\ t > t_0 + \Phi(e^{t_0}(x - x_0))\}$.

3.6. Let Ψ be the function : $\Psi(x) = \Phi(x)$ for $0 < x \leq a_1$ and $\Psi(x) = \Phi(2a_1 - x)$ for $a_1 \leq x < 2a_1$, where $a_1 \geq 3a_0/2$. Then, with f as above, the function $h_1(x,t) = g(t)f(t - \Psi(x))$ ($= 0$ if $x \notin (0, 2a_1)$) is of type $\mathcal{H}(1)$ and $\{h_1 > 0\} = \{(x,t) : 0 < x < 2a_1,\ t > \Psi(x)\}$.

PROOF OF PROPOSITION 3.3. We break the construction into three steps.

a) We first construct h_1 on the region $t_0 < t < t_2$. After multiplying h by a standard cut off function, we may assume that h is a smooth ≥ 0 function on $\mathbb{R}^2$ whose derivatives of order ≤ 2 are bounded, whose support is contained in $\omega'(\varepsilon) = \omega(\varepsilon) \cup \{(x,t) : t \leq t_0\}$, and which is of type $\mathcal{H}(\alpha)$ and > 0 on $\omega(2\varepsilon/3)$. Moreover $h = e^t$ if $t < t_1 + \varepsilon/2$.

Using the Remark 3.5 above, we construct a smooth and non-negative function $k_0(x,t)$ of type $\mathcal{H}(1)$ on $\mathbb{R}^2$ with $\{ k_0 > 0 \} = \{(x,t) : x > x_0 - a_0 e^{-t_1},\ t - t_1 > \Phi(e^{t_1}(x - x_0) + a_0)\}$. If we let $k_1 = h + Ck_0$ where C is a large positive constant, then, by 3.2, k_1 is of type $\mathcal{H}(\alpha)$ on the set $B = \{(x,t) : x > x_0 + \varepsilon/2,\ t > t_1 + \varepsilon/2\}$.

On the other hand, for all positive value of C, $h + Ck_0$ is of type $\mathcal{H}(\alpha)$ on $\omega(\varepsilon/2)$. Thus, we obtain a function k_1 on $\mathbb{R}^2$ of type $\mathcal{H}(\alpha)$ on $\omega(\varepsilon/2) \cup B$.

On applying the similar procedure to k_1 for the region $B' = \{ x < -x_0 - \varepsilon/2,\ t > t_1 + \varepsilon/2\}$, we obtain a smooth positive function k_2 on $\mathbb{R}^2$ which is of type $\mathcal{H}(\alpha)$ on the region $\{t_0 < t < t_2 + \varepsilon/2\}$ and which agrees with h for $|x| \leq x_0 - a_0 e^{-t_1}$, $t_0 < t < t_2$ and for $t_0 < t \leq t_1$. Note that for $|x| \geq x_0 + \varepsilon$ and $t > t_0$, $k_2(x,t)$ is a function of t only, which is increasing and such that $\partial_t^2 k_2(t,x) \geq k_2(t,x)$.

b) On multiplying k_2 by a cut-off function we may assume that k_2 has its support contained in $\{t \leq t_2 + \varepsilon/2\}$, is (strictly) positive of type $\mathcal{H}(\alpha)$ on $\{t_0 < t \leq t_2 + \varepsilon/4\}$, and that k_2 is for $|x| \geq x_0 + \varepsilon$ a function of t only. Let $k(t)$ be a smooth positive function on $\mathbb{R}$ with support $[t_2, \infty)$ and such that $k'' \geq k$, $k' \geq k$; we choose k in the form $k(t) = e^t\, \Psi(t)$ with Ψ convex, smooth, and $\operatorname{supp}(\Psi) = [t_2, \infty)$. Let $k_3(t,x) = k_2(t,x) + Ck(t)$. As before, if C is a large positive constant, then k_3 is of type $\mathcal{H}(\alpha)$ on $\{t_0 < t < +\infty\}$. Also, for $t > t_2 + \varepsilon$, $k_3 = Ce^t\Psi(t)$ is of type $\mathcal{H}(1)$ and is a function of t only.

c) To finish the proof, we observe that we can easily modify k_3 for $t > t_2' = t_2 + 2\varepsilon$ into a new function h_1 of type $\mathcal{H}(\alpha)$ on $\{t > t_2'\}$ in such a way that $h_1(t) = e^t$ for large t: if we let $\beta = \sqrt{\alpha}$, $u(t) = Ce^{(1-\beta)t}\Psi(t)$, u is convex increasing on $[t_2', \infty)$, and since $v(t) = e^{(1-\beta)t}$ is also convex and such that $\lim_{t\to\infty} t^{-1} v(t) = +\infty$, there exists a smooth convex function $w(t)$ on $[t_2', \infty)$ such that $w(t) = u(t)$ for $t_2' \leq t \leq t_2' + 1$, and $w(t) = e^{(1-\beta)t}$ for t large enough. So that $h_1(t) = e^{\beta t} w(t)$ agrees with k_3 on $[t_2', t_2' + 1]$, with e^t for large t, and verifies $\partial_t^2 h_1 \geq \beta^2 h_1$ for $t > t_2'$; thus, h_1 is of type $\mathcal{H}(\alpha)$. (We have used the following simple fact: if u and v are two smooth convex functions on $[0, \infty)$ and if $\lim_{t\to\infty} t^{-1} v(t) = +\infty$, there is a smooth convex function φ on $[0, +\infty[$ such that $\varphi(t) = u(t)$ if $0 \leq t \leq 1$ and $\varphi(t) = v(t)$ for large t).

We shall also use the following two variants of Proposition 3.3:

3.7. Replace in the statement $\omega(\varepsilon)$ by the region $\omega'(\varepsilon) = \{(x,t) : t_0 < t < t_1 + \varepsilon\} \cup \{(x,t) : x < x_0 + \varepsilon, \ t_1 \leq t < t_2 + \varepsilon\}$, where $x_0 \in \mathbb{R}$, $t_1 = t_0 + 1$, $t_2 \geq t_1 + 2$, and $\varepsilon > 0$. If h and its derivatives of order ≤ 2 are bounded (for large $|x|$), then the conclusions of Proposition 3.3 still hold, iii) being replaced by

iii)' $h_1 = h$ for $x \leq x_0 - a_0 e^{-t_1}$ and $t_0 < t < t_2$.

Moreover, if the given function $h(t,x)$ is increasing with respect to x, h_1 may also be chosen increasing with respect to x.

3.8. One may more generally replace the region $\omega(\varepsilon)$ by a region $\omega'(\varepsilon) = \{(x,t) : t_0 < t < t_1 + \varepsilon\} \cup \{(x,t) : t_1 < t < t_2 + \varepsilon, \ x \in B\}$ where B is the union of a finite number of intervals. Then, assuming that h and its derivatives of order ≤ 2 are bounded, a simple adaptation of the proof above (using 3.4 and 3.6) shows that the conclusions of Proposition 3.3 hold, iii) being replaced by

iii)' $h_1 = h$ for $d(x, B^c) \geq 2a_0 e^{-t_1}$, $t_0 \leq t \leq t_2$.

If B is the union of two intervals $I =]-\infty, -a]$ and $J = -I$ where $a > 0$, and if h is even, increasing with respect to $x \in I$, then we may choose a function h_1, even with respect to x, and decreasing with respect to $x \in \mathbb{R}_+$.

We shall need another "pasting" lemma which says that given $t_0 \in \mathbb{R}$, $\alpha \in (0,1)$ and a function of type $\mathcal{H}(1)$ in the form $h(x,t) = e^t\, b(x)$, $x \in J$, where b is smooth convex and ≥ 1 on the open interval J (with $\|b'\|_\infty < +\infty$), we may construct on the region $\{(x,t) : x \in J\}$ a function h_1 of type $\mathcal{H}(\alpha)$ equal to e^t when $t \leq t_0$ and equal to h for t large enough. To state this lemma, we fix a smooth non-negative and non-increasing function φ on $\mathbb{R}$ such that $\varphi(t) = 0$ for $t \geq 1 - 1/16$, $\varphi(t) = 1$ for $t \leq 3/4$; we also assume as we may that $\varphi(t)$ is convex for $t \geq 7/8$. Let $\Psi(t) = \varphi(1-t)$.

Lemma 3.9. *Let b be a smooth convex function on the open interval $J \subset \mathbb{R}$ such that $b \geq 1$ and $\|b'\|_\infty < +\infty$, and let t_0, ε, α be real numbers with $\varepsilon > 0$, $0 < \alpha < 1$. If $h(t,x) = e^t\big(\varphi(\varepsilon(t-t_0)) + \Psi(\varepsilon(t-t_0))\, b(x)\big)$, for $t \in \mathbb{R}$, $x \in J$, and $h(t,x) = e^t$ for all x and $t \leq t_0 + 1/(16\varepsilon)$, then for ε small enough (depending only on α and*

$e^{-t_0}\|b'\|_\infty$), *h is of type* $\mathcal{H}(\alpha)$ *on the region* $\{(x,t):\ x \in J\} \cup \{(x,t):$ $t < t_0 + 1/(16\varepsilon)\}$.

PROOF. We have

$$\begin{aligned}\frac{h''_{tt}}{h} - \alpha &= 1 - \alpha + 2\varepsilon\,\frac{\varphi' + b\Psi'}{\varphi + b\Psi} + \varepsilon^2\,\frac{\varphi'' + b\Psi''}{\varphi + b\Psi}\\ &\geq 1 - \alpha - 2\varepsilon\,\|(\varphi')^-\|_\infty - \varepsilon^2\left\|\left(\frac{\varphi''}{\varphi}\right)^-\right\|_\infty\end{aligned}$$

and

$$\begin{aligned}\frac{g'}{g}\,\frac{h'_t}{h} + \frac{h''_{xx}}{g^2 h} - \alpha &= 1 - \alpha + \varepsilon\,\frac{\varphi' + \Psi' b}{\varphi + \Psi\, b} + \frac{1}{g^2}\,\frac{\Psi\, b''}{\varphi + \Psi\, b}\\ &\geq 1 - \alpha - \varepsilon\|(\varphi')^-\|_\infty\end{aligned}$$

so that for ε small enough both quantities are greater than $(1-\alpha)/2$.

Observe next that if $t \geq t_0 + \varepsilon^{-1}/4$, or if $t \leq t_0 + \varepsilon^{-1}/16$, the mixed curvature term $h^{-1}\partial_t(h'_x/g)$ vanishes. For $t_0 + \varepsilon^{-1}/16 \leq t \leq t_0 + \varepsilon^{-1}/4$, and with obvious notations, we have

$$h^{-1}\left|\partial_t\left(\frac{h'_x}{g}\right)\right| = \varepsilon\left|\frac{\Psi'\, b'}{g(1 + \Psi\, b)}\right| \leq \varepsilon\, e^{-t_0}\,\|\Psi'\|_\infty\,\|b'\|_\infty$$

and the lemma follows.

4. Proof of Corollary C.

We first exhibit a simple way of producing on a region of the type $U(t_1, t_2; J) = \{(x,y,t):\ t_1 < t < t_2,\ x \in J\}$ a $\gamma(-1)$ metric such that the integral $\int_{t_1}^{t_2} \delta_K(t)\,dt$ is very large, where we have let

$$\delta_K(t) = \inf\left\{\frac{h'_x(x,t)}{g(t)^2\, h'_t(x,t)}\ :\ x \in K\right\}.$$

Here, $J \subset \mathbb{R}$ denotes an interval with a finite upper bound, K is a compact subinterval, and g will simply be $g(t) = e^t$ as in the previous section, so that h should be positive and of type $\mathcal{H}(1)$ on $U(t_1, t_2; J)$.

We fix a smooth positive and convex function β on J such that $\beta'(x) > 0$, $\beta(x) \geq 1$ and

$$2\,\beta'(x)^2\Big(1+\frac{1}{2\beta(x)}\Big) \leq \beta(x)\,\beta''(x)\,, \tag{4.1}$$

for all $x \in J$. For example, we may take $\beta(x) = a + e^x$ with $a \geq 1$ sufficiently large depending on the upper bound of J.

Let $\Phi(t)$ be any smooth increasing function of t such that $\Phi'' + 2\Phi' \geq 0$ and $\Phi(t) \geq 1$ on (t_1, t_2). We then define

$$h(x,t) = e^t \exp(\Phi(t)\,\beta(x))\,, \qquad \text{for } \; x \in J\;,\; t_1 < t < t_2\;, \tag{4.2}$$

and may easily check the following lemma.

Lemma 4.1. *The functions g and h define a $\gamma(-1)$ metric on the set $U(t_1, t_2; J)$.*

PROOF. Clearly, h is positive, increasing with respect to the t variable,

$$\begin{aligned}\frac{h''_{tt}(x,t)}{h(x,t)} - 1 &= 2\,\Phi'(t)\,\beta(x) + \beta(x)\,\Phi''(t) + \beta(x)^2\,\Phi'(t)^2\\ &= \beta(x)\,(\Phi''(t) + 2\,\Phi'(t)) + \beta(x)^2\,\Phi'(t)^2\,,\end{aligned}$$

$$\begin{aligned}\frac{h'_t(x,t)}{h(x,t)} + \frac{h''_{xx}(x,t)}{g^2(t)\,h(x,t)} - 1 = {}& \Phi'(t)\,\beta(x)\\ &+ \frac{1}{g(t)^2}\left(\Phi(t)^2\,\beta'(x)^2 + \Phi(t)\,\beta''(x)\right),\end{aligned}$$

and

$$\Big|\frac{1}{h}\,\partial_t\big(\frac{h'_x}{g}\big)\Big|^2 = \frac{1}{g(t)^2}\,\Phi'(t)^2\,\beta'(x)^2\,\big(1+\Phi(t)\,\beta(x)\big)^2\,.$$

Thus, it suffices to check that

$$\Phi'(t)^2\,\beta'(x)^2\,(1+\Phi(t)\,\beta(x))^2 \leq \beta(x)^2\,\Phi'(t)^2\big(\Phi(t)^2\,\beta'(x)^2 + \Phi(t)\,\beta''(x)\big)\,,$$

which is the same as

$$\Phi'(t)^2\,\beta'(x)^2\,(1+2\Phi(t)\,\beta(x)) \leq \Phi(t)\,\Phi'(t)^2\,\beta(x)^2\,\beta''(x)$$

or

$$2\,\beta'(x)^2\Big(1+\frac{1}{2\Phi(t)\beta(x)}\Big)\le\beta(x)\,\beta''(x)\,,$$

which follows from (4.1) since $\Phi(t)\ge 1$.

Lemma 4.2. *Let A be a positive number and $K\subset J$ be a compact interval. For each given t_1, we may choose $t_2>t_1+2$ and a function Φ as above such that*

i) $\displaystyle\int_{t_1}^{t_2}\delta_K(t)\,dt>A,$ *and*

ii) $\Phi(t)=1$ *for* $t_1<t<t_1+1/2$.

Here

$$\delta_K(t)=\inf\Big\{\frac{h'_x(x,t)}{g(t)^2\,h'_t(x,t)}\ :\ x\in K\Big\}\,.$$

PROOF. The inequality $\Phi''+2\,\Phi'\ge 0$ means that $\Phi'(t)\,e^{2t}$ is a nondecreasing function; thus, choosing

$$\Phi(t)=1+\int_{t_1}^{t}\varphi(s)\,e^{-2s}\,ds$$

with φ smooth, increasing on $(t_1,+\infty)$ and $\varphi(t)=0$ on $(t_1,t_1+1/2)$ guarantee property ii) above and the required differential inequality for Φ. Also, $\Phi(t)\ge 1$.

On the other hand, we have, for $t\ge t_1$, $x\in K$,

$$\frac{h'_x(x,t)}{g(t)^2\,h'_t(x,t)}=e^{-2t}\,\frac{\Phi(t)\beta'(x)}{1+\Phi'(t)\beta(x)}\ge c\,e^{-2t}\,\frac{\Phi(t)}{1+\Phi'(t)}\,,$$

where c is some positive constant (depending on β only). Now, assuming that $\varphi(t)$ is a (large) constant φ_0 on the interval $(t_1+3/4,T)$, we have the following lower bound (we let $t'_1=t_1+1$ and assume $T>t'_1+1$)

$$\int_{t'_1}^{T}\delta_K(t)\,dt\ge\frac{c}{4}\int_{t'_1}^{T}e^{-2t}\,\frac{\varphi_0\,e^{-2t'_1}}{1+e^{-2t}\varphi_0}\,dt\,.$$

So that, if moreover φ_0 is so large that $\varphi_0\,e^{-2T}\ge 1$,

$$\int_{t'_1}^{T}\delta_K(t)\,dt\ge\frac{c}{8}\,e^{-2t'_1}\,(T-t'_1)\,.$$

Finally, it is seen that if we choose $t_2 = T$ so large that

$$\frac{c}{8}\, e^{-2t_1'}\,(T - t_1') \geq A$$

and then construct φ such that $\varphi(t) = \varphi_0$ on $(t_1 + 3/4, T)$, with φ_0 larger than e^{2T}, we obtain a number t_2 and a function Φ with all the required properties.

It is now easy to construct a function h on $\mathbb{R}^2$ which produces an example establishing Corollary C. Using propositions 3.3 (see Remark 3.7), 3.9 and the above lemmas, one constructs by induction a smooth positive function h on $\mathbb{R}^2$ and an increasing sequence $\{t_j\}_{j\geq 0}$ of reals such that (recall that $g(t) = e^t$ for all t)

i) $t_0 > 0$, $t_{j+1} - t_j \geq 1$,

ii) h is increasing with respect to each variable,

iii) $h(x,t) = g(t)$ for $t \leq t_0$ or $t_{4j+3} < t < t_{4(j+1)}$,

iv) g and h define a $\gamma(-1/4)$ metric on $\mathbb{R}^3$, and

v) $\displaystyle\int_{t_{4j+1}}^{t_{4j+2}} \delta_j(t)\, dt \geq 1$, where

$$\delta_j(t) = \inf\left\{\frac{h_x'(x,t)}{g(t)^2 h_t'(x,t)} \; : \; |x| \leq j\right\}, \qquad t_{4j+1} \leq t \leq t_{4j+2}\,.$$

Now, for $M = \mathbb{R}^3$ equipped with the corresponding γ metric, Proposition 2.1 shows that every neighborhood V of the point $\infty_M \in S_\infty(M)$ in the compactification $\widehat{M}$ (see Section 1.3) is such that the closed convex hull of $V \cap M$ in M is M itself; on the other hand, the sectional curvatures of M are all $\leq -1/4$.

5. The Brownian motion's behavior.

A. Let $g(t) = e^t$ and $h(x,t)$ define a $\gamma(-1/4)$ metric on $\mathbb{R}^3$ (*i.e.* h is smooth, positive and of type $\mathcal{H}(1/4)$ on $\mathbb{R}^2$) and, as before, let M to denote the corresponding Riemannian manifold. We assume once for all that on each region $\{(x,t) : \; t < a\}$, $a \in \mathbb{R}$, the partial derivatives of h of order ≤ 2 are bounded and that $h(x,t) = e^t$ for $t \leq 0$; thus, the sectional curvatures of $\gamma_{(g,h)}$ are bounded on each region $\{t < a\}$. Clearly, this is verified in the above construction in Section 4.

Let $\{\Omega,\ \mathcal{F},\ \{P_x\}_{x\in M},\ \{B_s\}_{s\geq 0}\}$ be the Brownian motion on M which we see as a continuous Markov process on $[0,+\infty]$ with value in the Alexandroff compactification $M \cup \{c(M)\}$ of M, $c(M)$ being the cemetery point (this is possible because M is transient). Denote by S the lifetime of B_s ($B_s = c(M)$ if and only if $s \geq S$), by $X(s)$, $Y(s)$ and $T(s)$ the components in $\mathbb{R}^3$ of B_s for $s < S$. We start with the following observations.

Lemma 5.1. *We have*

1) *Almost surely,* $\lim_{s\to S} T_s(\omega) = +\infty$,

2) *Almost surely,* $\lim_{s\to S} Y_s(\omega)$ *exists and is finite,*

3) *If h is nondecreasing with respect to x, then* $\lim_{s\to S} X_s(\omega)$ *exists in* $(-\infty,+\infty]$ *almost surely.*

PROOF. The lemma follows from several application of the following basic (and standard) fact: if u is a continuous ≥ 0 supersolution on M (with respect to the Laplace-Beltrami operator Δ_M), then $\{u(B_s)\}_{s\geq 0}$ (with the usual convention that $u(c(M)) = 0$) is a nonnegative right-continuous supermartingale with respect to each probability measure P_x, so that almost surely, $s \mapsto u(B_s)$ admits a left-hand limit at every $s_0 \in (0,+\infty]$ ([DM, p.75 and p.79]). In particular, as $s \to S-0$, $u(B_s)$ has a finite limit almost surely. This being true for any Riemannian manifold, it is also seen that if v is a positive continuous Δ_M-superharmonic function on a region $\Omega \subset M$ such that $\sup\{s : B_s \in M\setminus\Omega\} < S$ a.s. (Ω is *absorbing*), then $\lim_{s\to S} v(B_s)$ exists and is finite a.s. Recall also from Section 1.3 that

$$\Delta_M = \frac{\partial^2}{\partial t^2} + \frac{1}{g^2}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \left(1 + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{hg^2}\frac{\partial}{\partial x}.$$

It is clear that $u(x,y,t) = e^{-t}$ is Δ_M-superharmonic on M; it follows, by the above remark, that $T(s)$ admits a limit in $(-\infty,+\infty]$ a.s. On the other hand, since the sectional curvatures of M are bounded in each region $\{t < a\}$ the Brownian motion is a.s. bounded before leaving any such region, and the first assertion follows.

We then prove the third assertion in the lemma. From the form of Δ_M, and because h is nondecreasing in x, we see that $u(x,t)$ is superharmonic on M (or a region of M) if $u(x,t)$ is smooth nonincreasing in x and in t and is superharmonic with respect to to

$$L = \frac{\partial^2}{\partial t^2} + \frac{1}{g(t)^2}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\,.$$

L is the Laplace-Beltrami operator on N, the (x,t)-plane equipped with the hyperbolic metric $dt^2 + e^{2t}\,dx^2$. Thus, for each real x_0, if we let

$$u(x,t) = \begin{cases} 1\,, & \text{if } x \le x_0\,,\\ 1 - \dfrac{2}{\pi}\operatorname{Arctg}\left((x-x_0)e^t\right), & \text{otherwise}\,,\end{cases}$$

(u is the harmonic measure in N of the region $\{x \le x_0\}$), it is easily seen that u as a function on M is Δ_M-superharmonic. Since $u(B_s)$ converges a.s. when $s \to S$, and x_0 is chosen in a dense sequence of reals, and since $\lim_{s\to S} T(s) = +\infty$, it is clear that X_s converge a.s. in $[-\infty,+\infty]$ when $s \to S$. The value $-\infty$ is excluded by the supermartingale inequality

$$u(x) \ge E_x(\lim_{s\to S-0} u(B_s)) \ge P_x(\lim_{s\to S} X_s \; < x_0)\,.$$

The second claim of the lemma may be proved similarly. Observe that $h(x,t) \ge e^{t/2}$ when $t \ge 0$, since $h''_{tt} \ge h/4$, and $h(x,t) = e^t$ for $t \le 0$, so that $h(x,t) \ge \cosh(t/2) + 2\sinh(t/2)$ for $t \ge 0$; thus, each positive function $u(y,t)$ which is nonincreasing in t, convex in y and superharmonic on the (y,t)-plane with respect to

$$L = \frac{\partial^2}{\partial t^2} + e^{-t}\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial t}$$

is also Δ_M-superharmonic on the absorbing region $\Omega = \{t > 0\} \subset M$. It follows easily that for each real y_0, the function

$$u(x,y,t) = \begin{cases} 1\,, & \text{if } y \le y_0 \quad (\text{resp. } y \ge y_0)\,,\\ 1 - \dfrac{2}{\pi}\operatorname{Arctg}\left(\dfrac{1}{2}\,|y-y_0|\,e^{\frac{t}{2}}\right), & \text{otherwise}\,,\end{cases}$$

is superharmonic on Ω. The claim follows then as above. (One could also use the convexity in M of the sets $C = \{y \le y_0\}$ and the corresponding superharmonic functions given by Proposition 6.1).

Lemma 5.2. *Let t_1, β and J be fixed as in Section 4, and let $K = [a,b] \subset J$; let η be some positive number and let τ to denote the first exit time of B_s out of $U = \{(x,y,t) : \ a < x < b, \ t_1'' = t_1 + 2 < t < t_2\}$. We may choose, in the statement of Lemma 4.2, Φ and $t_2 \geq t_1 + 10$ such that, if $h(x,t) = e^t \exp(\Phi(t)\,\beta(x))$ on U, then $P_m(T_\tau = t_2$ or $T_\tau = t_1'') \leq \eta$ for all $m = (x,y,t)$ such that $a < x < b$ and $t = (t_1'' + t_2)/2$.*

PROOF. It will suffice to use once again a supermartingale argument. We let $\sigma(m) = \sigma(x,y,t) = \exp(\varepsilon\, t - x)$ where ε will be chosen small (depending only on t_1, β and K) so that σ will be superharmonic on U if the constant φ_0 of the construction in Lemma 4.2 is taken large enough (depending on t_2). In fact, we have in U (with the notations of 4.2)

$$\sigma^{-1} \Delta_M(\sigma) = \varepsilon^2 + \varepsilon\,(2 + \beta(x)\,\varphi_0\, e^{-2t}) + e^{-2t} - e^{-2t}\,\Phi(t)\,\beta'(x)$$

and

$$\sigma^{-1} \Delta_M(\sigma) \leq \varepsilon^2 + 2\,\varepsilon + c_1\,\varepsilon\,\varphi_0\, e^{-2t} + e^{-2t} - c_2\, e^{-2t}\,\varphi_0\, e^{-2t_1}\,,$$

where the c_j are positive and depend only on β and K. We fix $\varepsilon > 0$ and small enough so that $c_1\,\varepsilon - c_2\, e^{-2t_1}/2 \leq 0$, and then may choose t_2 and φ_0 (in that order) so large that

$$\exp(b-a)\, \exp(-\frac{1}{2}\,\varepsilon\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,, \qquad \exp(-\frac{1}{2}\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,,$$

and $\Delta_M(s) \leq 0$ on U. Then, from the supermatingale inequality $E_m(\sigma(B_\tau)) \leq \sigma(m)$, we have

$$P_m(T_\tau = t_2) \leq \exp(b-a)\, \exp(-\frac{1}{2}\,\varepsilon\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,,$$

if $m = (x,y,t)$, $t = (t_1'' + t_2)/2$, and $a < x < b$. On the other hand, by the superharmonicity of e^{-t}, we also have

$$P_m(t_\tau = t_1'') \leq \exp(-\frac{1}{2}\,(t_2 - t_1'')) \leq \frac{\eta}{2}\,.$$

We also need the following obvious lemma.

Lemma 5.3. *Assume that there is a sequence* $\{t_j\}$ *with* $\lim_{j\to\infty} t_j = +\infty$ *and such that* $h(t) = e^t$ *when* $t_j < t < t_j + 1$. *Then* $S = +\infty$ *almost surely.*

PROOF. Let $t'_j = t_j + 1/2$ and let τ_j to denote the first exit time from $\{t_j < t < t_j + 1\}$. Then, for $m = (x, y, t'_j) \in M$, $P_m(\tau_j \geq 1) = c$ where c is positive and independent of j, x, and y. The result then follows from the Borel-Cantelli Lemma (we may assume $t_j + 1 < t_{j+1}$).

With the above three lemmas, we may now derive Theorem A.

Proposition 5.4. *The construction performed in Section* 4 *can be achieved so that*

i) $S = +\infty$ *a.s.,* *and*

ii) $\lim_{s\to\infty} X_s(\omega) = +\infty$ *a.s.*

PROOF. It suffices to achieve the construction above (with a function h nondecreasing in x) in such a way that for a sequence of "boxes" $U_j = \{(x, y, t) : \ |x| < j, \ t_{4j+1} < t < t_{4j+2}\}$ ($\{t_j\}$ being a rapidly increasing sequence of reals, $t_0 > 0$) we have $P_m(|X_{\tau_j}| = j) \geq 1 - 2^{-j}$ when $m = (x, y, t)$ with $|x| < j$, $t = (t_{4j+1} + t_{4j+2})/2$, τ_j denoting the exit time from U_j, and $h(t) = e^t$ for $t_{4j+3} \leq t \leq t_{4j+4}$. The t_j may be chosen by induction, using Lemma 5.2 , Proposition 3.3 (and 3.7) above. By Lemma 5.1, the Markov property and the first Borel-Cantelli Lemma, $\lim_{s\to S} X_s = +\infty$ a.s. We may choose the gaps $t_{4j+4} - t_{4j+3}$ as large as we wish, whence i) by Lemma 5.3.

B. Let us now indicate the changes that should be made in order to construct an example proving Theorem B. We first notice that we may adapt the above construction in such a way that the Brownian motion converges a.s. to the end point (for $s \to +\infty$) on $S_\infty(M)$ of the geodesic $s \to (0, 0, s)$; however, the metric cannot be chosen among γ metrics.

First step. We construct a Riemannian manifold $\widetilde{M}$ in the following way: start with a $\gamma(-1/4)$ metric related to $g(t) = e^t$ and a function $h(x, t)$ such that

i) h is increasing with x for $x \leq 0$ and decreasing in x when $x \geq 0$,

ii) there is an increasing sequence $\{t_j\}_{j\geq 0}$ of positive numbers, with say $t_{j+1} > t_j + 10$, such that

$$ds^2 = dt^2 + e^{2t}\,dx^2 + e^{2t}\,dy^2$$

(the standard hyperbolic metric) for $t < t_0$ or $t_{4j+3} - 1 \leq t \leq t_{4j+4} + 1$.

$\widetilde{M}$ is $\mathbb{R}^3$ equipped with the metric d obtained from the metric $\gamma(g, h)$ by rotating the regions $\{t_{8j} \leq t \leq t_{8j+4}\}$ by $\pi/2$ around the $t-$axis, while the others regions $\{t < t_0\}$, $\{t_{8j+4} < t < t_{8j+8}\}$ are kept fixed. It is clear that $\widetilde{M}$ is a (smooth) Riemannian manifold with sectional curvatures $\leq -1/4$ and for which the description of the sphere at infinity in Paragraph 1.2 is still valid. Also, the Brownian motion $\{B_s\}_{s\geq 0}$ on $\widetilde{M}$ satisfies the following properties (X_s, Y_s, T_s denote the coordinates of B_s, S is the lifetime of B_s).

Lemma 5.5. *Almost surely,* $\lim_{s\to S} T_s(\omega) = +\infty$ *and both limits* $\lim_{s\to S} Y_s(\omega)$, $\lim_{s\to S} X_s(\omega)$ *exist and are finite.*

PROOF. The first point follows of course exactly as in Lemma 5.1. To prove the second claim, we first note that in N, the (x, t) plane equipped with the hyperbolic metric $ds^2 = dt^2 + e^{2t}\,dx^2$, the harmonic measure of $|x| = \pi/(2a)$ in the region $|x| < \pi/(2a)$ is explicitly given by

$$u_a(x, t) = 1 - \frac{2}{\pi}\,\text{Arctg}\left(\frac{\cos(ax)}{\sinh(ae^{-t})}\right).$$

Let $u_a(x, t) = 1$ when $|x| \geq \pi/(2a)$. u_a is convex with respect to to x on $[-\pi/(2a), \pi/(2a)]$, and is decreasing with respect to to t. It follows in particular that the function $f_a(x, t) = u(x/2, t/2)$ is superharmonic with respect to to

$$L = \frac{\partial^2}{\partial t^2} + e^{-t}\,\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}.$$

Recall that

$$\Delta_N = \frac{\partial^2}{\partial t^2} + e^{-2t}\,\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}.$$

From the construction of the metric of $\widetilde{M}$ and similar observations as in the proof of Lemma 5.1 (using in particular the monotonicity properties of h and u_a with respect to to x), it is then easily checked that the functions $(x, y, t) \mapsto f_a(x, t)$ are superharmonic on the region $\{t > 0\}$ of $\widetilde{M}$, and similarly for $(x, y, t) \mapsto f_a(y, t)$. The second assertion of the lemma follows from this and the supermartingale argument.

Now, by Lemma 5.2 and the extension lemmas of Section 3, for each given t_{4j} we may choose t_{4j+1}, t_{4j+2}, t_{4j+3} and the function $h(x, t)$ on $\{t_{4j} < t \leq t_{4j+3}\}$ in such a way that

a) $h = e^t$ if $t_{4j} \leq t \leq t_{4j} + 1$, or if $t_{4j+3} - 1 \leq t \leq t_{4j+3}$,

b) $h(x, t)$ is an even function of x which is decreasing on $\mathbb{R}_+$,

c) on $U_j = \{(t, x) : t_{4j+1} < t < t_{4j+2},\ x < -2a_0 e^{-t_{4j}}\}$, h is as in Lemma 4.2 (with $J = (-\infty, -a_0 e^{-t_{4j}}])$,

d) if we start B_s (the Brownian motion on $(M, \gamma(g, h))$) from (t_0', x_0, y_0) , $t_0' = (t_{4j+1} + t_{4j+2})/2$, $-j \leq x_0 \leq -2a_0 e^{-t_{4j}}$, the probability to hit $\{x = -j\}$ or $\{x = -a_0\, e^{-t_{4j}}\}$ before $\{t = t_{4j+1}\}$ or $\{t = t_{4j+2}\}$ is larger than $1 - 2^{-j}$ (a_0 was defined in Section 3).

Thus, we may construct h with the above properties. By Lemma 5.5, it follows that, for the Brownian motion B_s on the corresponding Riemannian manifold $\widetilde{M}$, $\lim_{s \to S} X_s = \lim_{s \to S} Y_s = 0$ a.s. Since we may again choose the gaps $t_{4j+4} - t_{4j+3}$ very large, we may also realise $S = +\infty$ a.s.

Second step. We now consider $M = \{(x, y, t) : x, y, t \in \mathbb{R}\}$ equipped with a metric for which there is a sequence of regions $V_j = \{\rho_j \leq t \leq \rho_{j+1}\}$ $(j \geq 1)$ with $\{\rho_j\}$ rapidly increasing,

$$ds^2 = dt^2 + e^{2t}\, dx^2 + e^{2t}\, dy^2 \qquad \text{on } V_{2j} \ ,$$

the metric on V_{2j+1} being obtained by some translation $x \mapsto x + a_j$, $y \mapsto y + b_j$ from a metric of the type considered in the first step. Again, M has sectional curvatures $\leq -1/4$ and the description of $S_\infty(M)$ in Paragraph 1.2 holds. Choose and fix a dense sequence (a_j, b_j) in $\mathbb{R}^2$. By the first step, it is clear (and easy to prove) that one may successively choose the strips and the metric on these so that the Brownian motion $\{B_s\}$ on M starting from $m_0 = (0, 0, 0)$ hits the set $\{|x_j - a_j| + |y_j - b_j| \leq 4e^{-\rho_j},\ t = \rho_{j+1}\}$ with a probability $\geq 1 - 2^{-j}$; it is also clear that the lifetime of B_s is $+\infty$ a.s. The desired construction is then obtained and Theorem B is proved.

C. FINAL REMARKS.

1) We first sketch a more accurate variant of the construction. Let $\{a_j\}_{j\geq 0}$ be a sequence of real numbers and let M to denote the manifold $M = \{(x, y, t) : x, y, t \in \mathbb{R}\}$ equipped with a metric of the following type, for some rapidly increasing sequence $\{\theta_j\}_{j\geq 0}$ of positive numbers

$$ds^2 = dt^2 + e^{2t}\,dx^2 + e^{2t}\,dy^2\,,$$

when $t \leq \theta_0$,

$$ds^2 = dt^2 + e^{2t}\,dx^2 + h_{2j}(x - a_{2j}, t)^2\,dy^2\,,$$

when $\theta_{2j} \leq t \leq \theta_{2j+1}$, and

$$ds^2 = dt^2 + h_{2j+1}(y - a_{2j+1}, t)^2\,dx^2 + e^{2t}\,dy^2\,,$$

when $\theta_{2j+1} \leq t \leq \theta_{2j+2}$. Here $h_j(x,t)$ is a smooth positive and even function of type $\mathcal{H}(1/4)$ on $\mathbb{R}^2$ such that $h_j(x,t) = e^t$ when $t \leq \theta_j + \ell_j$ or $t \geq \theta_{j+1} - \ell_j$, ℓ_j being (much) smaller than $\theta_{j+1} - \theta_j$. We also require that $D^2 h_j$ is bounded on $\{t \leq \theta_{j+1}\}$.

Then, M is complete, its sectional curvatures are $\leq -1/4$, and again the Brownian motion $B_s = (X_s, Y_s, T_s)$ on M is such that $S = +\infty$ a.s., $\lim_{s\to\infty} T_s = \infty$ a.s. (S being the lifetime of Brownian motion). Moreover, for each given sequence $\{\varepsilon_j\}_{j\geq 0}$ of positive reals, we may (using a variation of the methods above) choose by induction the $(\ell_j, \theta_{j+1}, h_j)$ so that for each $m = (x, y, \theta_j + \ell_j)$, with $|x| + |y| \leq \varepsilon_j^{-1}$, and if τ_j denotes the first hitting time of B_s with $\{t = \theta_j\}$ or $\{t = \theta_{j+1} + \ell_{j+1}\}$, we have $P_m\{T_{\tau_j} = \theta_j\} \leq \varepsilon_j$,

$$P_m\{\,|X_{\tau_{2j}} - a_{2j}| + \sup_{s\leq\tau_{2j}} (d(X_s, [X_0, a_{2j}]) + |Y_s - Y_0|) \geq \varepsilon_j\} \leq 2^{-j}\,,$$

$$P_m\{|Y_{\tau_{2j+1}} - a_{2j+1}| + \sup_{s\leq\tau_{2j+1}} (d(Y_s, [Y_0, a_{2j+1}]) + |X_s - X_0|) \geq \varepsilon_j\} \leq 2^{-j}\,,$$

and also

$$P_{m_0}\{|X_{\tau_j'}| + |Y_{\tau_j'}| \geq \varepsilon_{j+1}^{-1}\} \leq 2^{-j}\,,$$

where $m_0 = (0,0,0)$ and τ_j' is the first hitting time with $\{t = \theta_{j+1} + \ell_{j+1}\}$. Choosing the ε_j sufficiently small, it is then seen that the set of cluster values (for $s \to +\infty$) of $b_s = (X_s, Y_s)$ is a.s. equal to the set of cluster values C_Γ of the polygonal ray $\Gamma = \bigcup_{j\geq 1}[A_j, A_{j+1}]$, where $A_{2j} = (a_{2j}, a_{2j-1})$, $A_{2j+1} = (a_{2j}, a_{2j+1})$. Since any continuum in $\mathbb{R}^2$ may be realized as a set C_Γ, this explains the remark after the statement of Theorem B.

2) Fix an integer $m \geq 1$, let M be as in Section 5.A and let $\widetilde{M} = \{(x, y, z_1, \ldots, z_m, t) : \ x, y, t, \ z_j \in \mathbb{R}\}$ with metric

$$ds^2 = dt^2 + g(t)^2\, dx^2 + h(x,t)^2\, dy^2 + e^{2t} \sum_j dz_j^2 .$$

Then, simple direct computations show that $\widetilde{M}$ is a C.H. manifold with sectional curvatures $\leq -1/4$. Since

$$\Delta_{\widetilde{M}} = \frac{\partial^2}{\partial t^2} + \frac{1}{g^2}\frac{\partial^2}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2}{\partial y^2} + \frac{1}{g^2}\sum_{j=1}^{m}\frac{\partial^2}{\partial {z_j}^2} + \left(m + 1 + \frac{h'_t}{h}\right)\frac{\partial}{\partial t} + \frac{h'_x}{hg^2}\frac{\partial}{\partial x},$$

it is not difficult, using the argument in Section 5.A, to choose h such that Brownian motion on $\widetilde{M}$ has infinite lifetime and satisfies $\lim_{s\to\infty} X_s = +\infty$ ($X_s = x-$component of Brownian motion); thus Theorem A extends to all dimensions ≥ 3. It is also clearly seen how one may adapt the constructions above in Section 5.B and extend similarly Theorem B.

6. Appendix.

The following statement is essentially in [Cho]. That the smoothness assumptions in [Cho] may be dropped is already observed in [And] Theorem 1.4.

Proposition 6.1. *Let M be a complete simply connected Riemannian manifold whose sectional curvatures are ≤ -1, and let C be a closed convex subset of M, $C \neq \varnothing$. Set $u(m) = 1 - \tanh(d(m,C)/2)$, $m \in M$. Then, u is a superharmonic function on M.*

PROOF. If C is smooth, Theorem 4.3 in [Cho] says that u is superharmonic on $M \setminus C$. To settle the general case, we argue as follows. Let $m_0 \in M \setminus C$, let $m_1 = p_C(m_0)$ be the unique point in C with $d(m_0, m_1) = d(m_0, C)$, and let $C_0 = B(m_1, 1) \cap C$. By Lemma 6.2 below, C_0 is the limit of a decreasing sequence of smooth compact convex sets C_n. On the other hand, the projection map p_C is continuous, so that for m in some neighborhood V of m_0, $d(m, C) = d(m, C_0) = \lim_{n\to\infty} d(m, C_n)$. Thus, $u = \sup_{n\geq 1} u_n$ on V, with $u_n = 1 - \tanh(d(m, C_n)/2)$, and each u_n is superharmonic on V. Hence, u is continuous on M, superharmonic and ≤ 1 on $M \setminus C$, and equal to 1 on C. It is then clear that u is superharmonic on M.

Lemma 6.2. *Let M be a Cartan-Hadamard manifold and let K be a compact convex set in M. Then K is the intersection of a decreasing sequence $\{K_n\}_{n\geq 1}$ of smooth compact convex subsets of M.*

PROOF. Note that $F : m \mapsto d(m, K)$ is convex ([BO]) and that there is a smooth bounded function h on $U = \{m : d(m, K) < 2\}$ such that $\text{Hess}_m(h) > cI$ for $m \in U$ and some $c > 0$ (*e.g.* $h(m) = |d(m, m_0)|^2$ with $m_0 \in M$). Approximating $F + \varepsilon h$ by smooth functions, it is seen that $F = \lim_{n\to\infty} F_n$ uniformly on $U' = \{m : d(m, K) < 1\}$, F_n being smooth and convex on $\bar{U}'$. For given $\varepsilon \in (0, 1)$ and large n, $K(n, \varepsilon) = \{F_n \leq \varepsilon + \max_K F_n\}$ is a compact neighborhood of K contained in $\{F < 2\varepsilon\}$, and $K(n, \varepsilon)$ is convex and smooth.

It follows from Proposition 6.1 (and the method of barriers) that property (C'_M) in the introduction (for a complete, simply connected, and negatively curved Riemannian manifold M) implies (P_M) ([Cho], [And]). From the probabilistic point of view, we have also the following simple corollary (by the usual supermatingale argument).

Corollary 6.3. *Let M, C and u be as in Proposition 6.1. Then the probability for the Brownian motion on M starting from $m_0 \in M$ to hit C is at most $u(m_0)$.*

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