Regularization of Hyperfunctions

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Abstract. We show that there are no continuous regularization procedures for the extension of hyperfunctions. We also show that there is a continuous projection operator from the space of hyperfunctions with support in a given compact set onto the subspace of hyperfunctions with support on a given *closed* subset if and only if the subset is a countable intersection of sets that are closed and open.

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1. Introduction

One of the key characteristics of Sato's hyperfunctions [21, 22] is the fact that they form a flabby sheaf [14, 19]. This means that if U is an open set and S is a locally closed set in $\mathbb{R}^n$ with $U \subset S$, then any hyperfunction f defined on U admits an extension $\widetilde{f}$, defined on S. Alternatively, the canonical projection

$$\pi : \mathfrak{B}(S) \longrightarrow \mathfrak{B}(U) , \tag{1}$$

where $\mathfrak{B}(\mathcal{X})$ denotes the space of hyperfunctions defined on $\mathcal{X}$, is surjective. We say that $\widetilde{f}$ is a regularization of f.

The regularization of generalized functions is a very important subject, not only from the mathematical point of view, but also from the mathematical physics perspective. Indeed, the problem of renormalization in quantum field theory is nothing but a problem of regularization of generalized functions [2, 23]. Actually, the normalized coupling constants are not determined by the theory, but must be fixed by experiment, and correspond exactly to undetermined constants in regularized generalized functions.

There are several methods of regularization of generalized functions [9, 11] such as principal values, analytic continuation, Hadamard finite parts, or methods based on distributional continuity. For hyperfunctions one uses function

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theoretical tools, such as the Mittag-Leffer theorem [14, 19]. Moreover, in some cases there is a controversy as to which is the best method of regularization [9, 12]. Whether any of the regularization methods can be established as "correct" is of physical interest, because it addresses the philosophical question of whether the "infinities" in quantum field theory represent genuine physical ambiguities, or merely arise from a poor formulation of the technical mathematical problems that appear in the theory [23].

It was proved recently [7] that for distributions it is not possible to define a *continuous* regularization procedure that can be applied to all distributions that admit regularizations. Naturally, distributions are not a flabby sheaf and thus not all distributions admit regularizations, but this result applies to the space of extendable distributions as introduced by Orton [20]; extendable distributions were characterized in [6], and they correspond to the distributions obtained by da Silva's construction [4].

In the present article we show that the corresponding result also holds for hyperfunctions. In other words, there is no continuous extension operator

$$R : \mathfrak{B}(a, b] \longrightarrow \mathfrak{B}[a, b], \tag{2}$$

such that

$$\pi R g = g, \qquad g \in \mathfrak{B}(a, b], \tag{3}$$

where $\pi : \mathfrak{B}[a, b] \longrightarrow \mathfrak{B}(a, b]$ is the restriction. The method of proof is very different from that of [7], however, since the space of distributions concentrated at one point is a $(\mathcal{DF})$ space, an inductive limit of finite dimensional spaces, in fact, while the space of hyperfunctions concentrated at a point is an $(\mathcal{F})$ space.

A related question is the possibility of restriction of generalized functions to a *closed* set. This is a fundamental question in the study of integral equations in spaces of distributions [10, 20]; recently it was a key component in the study of multiplicative regularizers [8], and it is expected to be play an important role in the study of hypersingular integral equations [3, 15, 18]. In this article we prove a necessary and sufficient condition for the existence of continuous restriction operators for spaces of hyperfunctions on closed sets. Indeed, we demonstrate that if H and K are compact sets, with $H \subseteq K$, then there exists a continuous projection operator $P : \mathfrak{B}(K) \longrightarrow \mathfrak{B}(H)$, with $Pf = f$ for $f \in \mathfrak{B}(H)$, if and only if $H = \bigcap_{n=1}^{\infty} H_n$ where H_n is open and closed in K. For spaces of distributions [7] such a projection operator exists if and only if H is open in K. It is interesting to see how these results show how some topological conditions are related to the behavior of certain operators, a situation that is also encountered with geometric conditions [5]. It would be interesting to study the existence of restriction operators in the case when H and K are unbounded in the spaces of asymptotic and tempered distributions studied by Schmidt [24].

The plan of the article is as follows. Section 2 presents the basic ideas about hyperfunctions needed in the rest of the paper. In Section 3 we prove the nonexistence of regularization operators in spaces of hyperfunctions. Section 4 contains the study of restriction operators into spaces of hyperfunctions over a closed set.

2. Notation and preliminaries

In this section we state some basic facts about hyperfunctions and about the notation used in this article. For details see [14, 19, 26].

Let U be an open set in $\mathbb{C}$. We denote by $\mathfrak{O}(U)$ the space of analytic functions defined on U. The topology of $\mathfrak{O}(U)$ is that of uniform convergence on compact subsets of U, i.e., the topology generated by the family of seminorms

$$\|f\|_K = \max\left\{|f(z)| : z \in K\right\},\tag{4}$$

for K a compact subset of U. Since we can find a sequence of compact subsets of U, $\{K_n\}_{n=1}^{\infty}$ with $K_n \subset \operatorname{int}(K_{n+1})$, $\bigcup_{n=1}^{\infty} K_n = U$, it follows that $\mathfrak{O}(U)$ is a Fréchet space, actually a strict projective limit of Banach spaces.

A subset S of a topological space X is called locally closed if each $x \in S$ has a neighborhood in X, V_x, such that $S \cap V_x$ is closed in V_x. It can be shown that S is locally closed in X if and only if there exist an open set U and a closed set F such that $S = U \cap F$.

If S is locally closed in X, we say that U is an open neighborhood of S if U is open in X and S is closed in U. We denote the set of open neighborhoods of S as $\mathsf{N}(S)$.

If S is locally closed in $\mathbb{C}$, then $\mathfrak{O}(S)$ is the space of germs of analytic functions defined on S. That is, a function f defined on S belongs to $\mathfrak{O}(S)$ if and only if there exists $U \in \mathsf{N}(S)$ and an analytic function $\widetilde{f} \in \mathfrak{O}(U)$ such that $\pi_S^U(\widetilde{f}) = f$, where π_S^U is the restriction operator from U to S. The system of topological vector spaces $\{\mathfrak{O}(U)\}_{U \in \mathsf{N}(S)}$ with operators $\pi_V^U : \mathfrak{O}(U) \longrightarrow \mathfrak{O}(V)$ for $U \supseteq V$ is actually a directed system and thus we can give $\mathfrak{O}(S)$ the inductive limit topology. When K is compact, then $\mathfrak{O}(K)$ is a strict limit of Banach spaces. If $S \subseteq \mathbb{R}$ is open, then $\mathfrak{O}(S)$ is the space of real analytic functions on S, while if $S \subseteq \mathbb{R}$ is locally closed, then $\mathfrak{O}(S)$ is the space of germs of real analytic functions on S.

If $S \subseteq \mathbb{C}$ is locally closed, then the dual space $\mathfrak{O}'(S)$ is called the space of analytic functionals on S. When $K \subseteq \mathbb{R}$ is compact, then $\mathfrak{O}'(K)$ is isomorphic to the space $\mathfrak{B}(K)$ of hyperfunctions defined on K, as we now explain.

If U is open in $\mathbb{C}$ and S is closed in U, then we define the space $H_S^1(U; \mathfrak{O})$, the first cohomology group of U with support in S and with coefficients in the sheaf $\mathfrak{O}$, as the quotient

$$H_S^1(U; \mathfrak{O}) = \mathfrak{O}(U \setminus S) / \pi_{U \setminus S}^U(\mathfrak{O}(U)),\tag{5}$$

where $\pi^U_{U \setminus S}$ is the canonical restriction. If $U \subset V$, then the canonical homomorphism from $H^1_S(V; \mathfrak{O})$ to $H^1_S(U; \mathfrak{O})$ is an isomorphism as follows from the Mittag-Leffler theorem. Thus we can define $H^1[S; \mathfrak{O}]$, as the inductive limit of the spaces $H^1_S(U; \mathfrak{O})$ as $U \searrow S$. In this case, the space $H^1[S; \mathfrak{O}]$ is actually isomorphic to $H^1_S(U; \mathfrak{O})$ for any $U \in \mathsf{N}(S)$.

If $S \subseteq \mathbb{R}$ is locally closed, then the space of hyperfunctions defined on S is $\mathfrak{B}(S) = H^1[S; \mathfrak{O}]$. If $U \in \mathsf{N}(S)$ and $F \in \mathfrak{O}(U \setminus S)$, then the jump of F across S,

$$[F] = F(x + i0) - F(x - i0),\tag{6}$$

is a hyperfunction on S, and actually, all hyperfunctions over S have this form. Observe that this is true for *all* neighborhoods $U \in \mathsf{N}(S)$. In the case when $S = \mathbb{R}$ or when $S = K$, a compact subset of $\mathbb{R}$, then we can take $U = \mathbb{C}$. Thus

$$\mathfrak{B}(\mathbb{R}) = \mathfrak{O}(\mathbb{C} \setminus \mathbb{R}) / \mathfrak{O}(\mathbb{C}),\tag{7}$$

and

$$\mathfrak{B}(K) = \mathfrak{O}(\mathbb{C} \setminus K) / \mathfrak{O}(\mathbb{C}).\tag{8}$$

If $f \in \mathfrak{B}(K)$, then of all the possible representatives in the space $\mathfrak{O}(\mathbb{C} \setminus K)$, i.e., of all the analytic functions $F \in \mathfrak{O}(\mathbb{C} \setminus K)$ with $[F] = f$, there is exactly one that satisfies

$$\lim_{z \to \infty} F(z) = 0.\tag{9}$$

This representative of f is called the Cauchy or analytic representation of f, and is denoted as

$$F(z) = \mathcal{C}\{f(x); z\}.\tag{10}$$

The operator $\mathcal{C}$ is an isomorphism of $\mathfrak{B}(K)$ onto the subspace $\mathfrak{O}_0(\mathbb{C} \setminus K)$ of $\mathfrak{O}(\mathbb{C} \setminus K)$ formed by those analytic functions that satisfy (9). The operator $\mathcal{C}$ can also be defined as an isomorphism of the space of analytic functionals $\mathfrak{O}'(K)$ onto $\mathfrak{O}_0(\mathbb{C} \setminus K)$ when K is a compact subset of $\mathbb{C}$ that perhaps is not contained in $\mathbb{R}$. It can be shown that the Cauchy representation is given as

$$\mathcal{C}\{f(\omega); z\} = \frac{1}{2\pi i}\left\langle f(\omega), \frac{1}{\omega - z}\right\rangle.\tag{11}$$

The power series expansion of the Cauchy representation at infinity takes the form

$$\mathcal{C}\{f(x); z\} = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{\mu_n(f)}{z^{n+1}}, \quad |z| > \rho,\tag{12}$$

where

$$\mu_n(f) = \langle f(\omega), \omega^n\rangle,\tag{13}$$

are the moments of f and where $\rho = \max\{|z| : z \in K\}$.

If K is a compact subset of $\mathbb{R}$, then the topology of the space of hyperfunctions $\mathfrak{B}(K)$ is given in one of two equivalent ways. First, since $\mathfrak{B}(K) = \mathfrak{O}'(K)$ one can give $\mathfrak{B}(K)$ the topology of the dual space. Alternatively, one could use (8) and consider the quotient topology. In any case, $\mathcal{C}$ is a homeomorphism of the topological vector spaces $\mathfrak{B}(K)$ and $\mathfrak{O}_0(\mathbb{C} \setminus K)$.

If S is a locally closed and bounded subset of $\mathbb{R}$, then [13, 25] one defines the topology of $\mathfrak{B}(S)$ as the quotient topology of $\mathfrak{B}(\overline{S}) / \mathfrak{B}(\overline{S} \setminus S)$. In particular,

$$\mathfrak{B}(a, b] \cong \mathfrak{B}[a, b] / \mathfrak{B}\{a\} \;, \quad \mathfrak{B}(a, b) \cong \mathfrak{B}[a, b] / \mathfrak{B}\{a, b\} \;. \tag{14}$$

Notice that if S_1 is open in S_2, then the canonical projection of $\mathfrak{B}(S_2)$ onto $\mathfrak{B}(S_1)$ is continuous; likewise, if K is compact, and $K \subset S$, then the canonical inclusion from $\mathfrak{B}(K)$ to $\mathfrak{B}(S)$ is continuous. Observe however that if (a, b) is an open interval in $\mathbb{R}$ and V is a region in $\mathbb{C}$ with $V \cap \mathbb{R} = (a, b)$, then $\mathfrak{B}(a, b) \cong \mathfrak{O}(V \setminus (a, b)) / \mathfrak{O}(V)$ algebraically but not topologically, since $\mathfrak{O}(V)$ is not closed but actually *dense* in $\mathfrak{O}(V \setminus (a, b))$.

3. Regularization

Let $a, b \in \mathbb{R}$, $a < b$. Let $\pi : \mathfrak{B}[a, b] \longrightarrow \mathfrak{B}(a, b]$ be the restriction operator. If $g \in \mathfrak{B}(a, b]$, then a regularization of g is a hyperfunction $f \in \mathfrak{B}[a, b]$ such that $\pi f = g$. Since the hyperfunctions form a flabby sheaf [14, 19], the operator π is surjective; this means that any $g \in \mathfrak{B}(a, b]$ admits regularizations. Nevertheless, regularizations are not unique since if f_0 is any regularization of g, then so is

$$f(x) = f_0(x) + \sum_{n=0}^{\infty} c_n \delta^{(n)}(x - a) \;, \tag{15}$$

whenever

$$\sum_{n=0}^{\infty} n! |c_n| r^n < \infty \;, \quad \forall r \geq 0 \;. \tag{16}$$

Our first result shows that f, that is, f_0 and the constants c_n, cannot be chosen to depend continuously on g.

Theorem 1. *There does not exist any continuous regularization operator*

$$R : \mathfrak{B}(a, b] \longrightarrow \mathfrak{B}[a, b] \;, \tag{17}$$

such that

$$\pi R g = g \;, \quad g \in \mathfrak{B}(a, b] \;, \tag{18}$$

where $\pi : \mathfrak{B}[a, b] \longrightarrow \mathfrak{B}(a, b]$ *is the canonical projection.*

Proof. Let us suppose that R exists and find a contradiction. We may suppose without any loss of generality that $a = 0$. Observe that if $f \in \mathfrak{B}[0, b]$, then

$$R\pi f = f, \quad \text{for } x \neq 0, \tag{19}$$

since $\pi(R\pi f) = \pi f$. Therefore, the operator T defined as $Tf = f - R\pi f$ is a continuous operator from $\mathfrak{B}[0, b]$ to $\mathfrak{B}\{0\}$. Hence we can write

$$T\{f(t); x\} = \sum_{n=0}^{\infty} U_n(f)\, \delta^{(n)}(x), \tag{20}$$

where the U_n's are continuous linear functionals on $\mathfrak{B}[0, b]$. Thus there are analytic functions ϕ_n, each defined in a corresponding neighborhood of $[0, b]$, such that

$$U_n(f) = \langle f(x), \phi_n(x) \rangle. \tag{21}$$

We will show that

$$\phi_n(x) = \frac{(-1)^n x^n}{n!} \quad \forall n \in \mathbb{N}. \tag{22}$$

But since $[0, b]$ is connected, it is enough to show that $\phi_n(x)$ and $(-1)^n x^n / n!$ have the same Taylor series at $x = 0$. But, if we observe that $T\{\delta^{(n)}(t); x\} = \delta^{(n)}(x)$ for all $n \in \mathbb{N}$, it follows that $U_n(\delta^{(k)}(x)) = \delta_{kn}$ and consequently,

$$\begin{aligned}
\phi_n^{(k)}(0) &= (-1)^k \langle \delta^{(k)}(x), \phi_n(x) \rangle \\
&= (-1)^k U_n(\delta^{(k)}(x)) \\
&= (-1)^k \delta_{kn} \\
&= \frac{d^k}{dx^k}\left(\frac{(-1)^n x^n}{n!}\right)\Bigg|_{x=0}.
\end{aligned}$$

We conclude that

$$T\{f(t); x\} = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n(f)}{n!} \delta^{(n)}(x) \tag{23}$$

is precisely the *Moment Expansion* of f, since

$$\mu_n(f) = \langle f(x), x^n \rangle = \int_{-\infty}^{\infty} f(x)\, x^n \mathrm{d}x, \tag{24}$$

are the moments of f.

Our result now follows if we observe that the Moment Expansion of $f \in \mathfrak{B}[0, b]$ does not converge in $\mathfrak{B}[0, b]$ unless $f \in \mathfrak{B}\{0\}$. Indeed, if (23) converges, then we may apply the Cauchy representation operator $\mathcal{C}$ to both sides of it. We readily obtain that $\mathcal{C}\{T(f); z\} = \mathcal{C}\{f; z\}$ for $|z| > b$ and consequently for all $z \in \overline{\mathbb{C}} \setminus [0, b]$. But $\mathcal{C}\{T(f); z\}$ is analytic in $\overline{\mathbb{C}} \setminus \{0\}$, while $\mathcal{C}\{f; z\}$ is analytic exactly in $\overline{\mathbb{C}} \setminus \operatorname{supp} f$. Hence $\operatorname{supp} f = \{0\}$.   □

The Theorem 1 applies to the regularization of hyperfunctions g defined in the semiopen interval $(a, b]$. The situation at the endpoint $x = b$ is irrelevant, as is to be expected, and a corresponding result holds in the space $\mathfrak{B}(a, b)$.

Theorem 2. *There does not exist any continuous regularization operator*

$$R : \mathfrak{B}(a, b) \longrightarrow \mathfrak{B}[a, b), \tag{25}$$

such that

$$\pi R g = g, \qquad g \in \mathfrak{B}(a, b), \tag{26}$$

where $\pi : \mathfrak{B}[a, b) \longrightarrow \mathfrak{B}(a, b)$ *is the canonical projection.*

Proof. Suppose R is a continuous extension operator from $\mathfrak{B}(a, b)$ to $\mathfrak{B}[a, b)$. Let $c \in (a, b)$ and let $R_1 = R \circ i$, where $i : \mathfrak{B}(a, c] \longrightarrow \mathfrak{B}(a, b)$ is the canonical injection. Since $R_1(f)(x) = 0$ for $x > c$ whenever $f \in \mathfrak{B}(a, c]$, it follows that $R_1(f) \in \mathfrak{B}[a, c]$. Therefore, R_1 is a continuous regularization operator from $\mathfrak{B}(a, c]$ to $\mathfrak{B}[a, c]$. This contradicts the Theorem 1. $\square$

Several remarks are in order.

Remark 1. The corresponding result for distributions was proved in [7]. The proof is quite different, however. Indeed, that proof is based on the fact that the space $\mathcal{A}_a$ of distributions with support in the one point set $\{a\}$ is a $(\mathcal{DF})$ space, and hence if $T : \mathcal{X} \longrightarrow \mathcal{A}_a$ is any continuous operator from a Fréchet space $\mathcal{X}$ to $\mathcal{A}_a$, then there exists $N \in \mathbb{N}$ such that the order of $T(x)$ is less than or equal to N for all $x \in \mathcal{X}$. On the other hand, the space $\mathfrak{B}\{a\}$ of hyperfunctions concentrated on the one point set $\{a\}$ is a Fréchet space, and thus the result on the boundedness of the orders of all $T(x)$ does not hold in the hyperfunction case.

Remark 2. Let f be a hyperfunction with compact support. Let $K = \operatorname{supp} f$. Then the moment expansion

$$\sum_{n=0}^{\infty} \frac{(-1)^n \mu_n(f)}{n!} \delta^{(n)}(x) \tag{27}$$

cannot be considered as a hyperfunction, since as we demonstrated in the proof of the Theorem 1 the series (27) converges as a hyperfunction if and only if $\operatorname{supp} f = \{0\}$. There are other useful interpretations of the moment expansion, as explained in [11, Chapter 6]. For our present purposes, we may proceed as follows. Let $\rho = \max\{|x| : x \in K\}$, and consider the closed disc $\overline{\mathbb{D}}_\rho$ with center at the origin and radius ρ, which contains K. Then using the Runge's theorem [19], there is an inclusion operator $i : \mathfrak{B}(K) \longrightarrow \mathfrak{O}'\left(\overline{\mathbb{D}}_\rho\right)$. Moreover,

the moment expansion of any analytic functional $g \in \mathfrak{O}'\left(\overline{\mathbb{D}}_\rho\right)$ converges in this space to itself:

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n(g)}{n!} \delta^{(n)}(z), \quad g \in \mathfrak{O}'\left(\overline{\mathbb{D}}_\rho\right). \tag{28}$$

Needless to say, the sum of a moment series in $\mathfrak{O}'\left(\overline{\mathbb{D}}_\rho\right)$ does not have support concentrated on $\{0\}$, in general.

Since i is injective, it follows the moment expansion of $f \in \mathfrak{B}(K)$ determines f uniquely, and it can thus be used in the hyperfunctional analysis of equations [16, 17].

These results do not hold for hyperfunctions of non-compact support, since there are non-zero functions whose moments all vanish [1]. On the other hand, the space of hyperfunctions where the moment *asymptotic* expansion holds was recently identified [24].

4. Restriction to closed sets

In [7] it was shown that if H and K are compact sets, with $H \subseteq K$, then there exists a continuous projection operator $P : \mathcal{E}'[K] \longrightarrow \mathcal{E}'[H]$, with $Pf = f$ for $f \in \mathcal{E}'[H]$, if and only if H is open in K; here $\mathcal{E}'[K]$ is the space of distributions with support contained in the compact set K. It is an interesting fact that this result does not hold for hyperfunction spaces; indeed, we shall show that there is a continuous projection operator $P : \mathfrak{B}(K) \longrightarrow \mathfrak{B}(H)$, with $Pf = f$ for $f \in \mathfrak{B}(H)$, if and only if $H = \bigcap_{n=1}^{\infty} H_n$ where H_n is open and closed in K.

We start with a result on countable intersection of closed-open sets.

Lemma 1. *Let H and K be compact subsets of $\mathbb{R}$ with $H \subseteq K$. Then the following are equivalent:*

a) *There exist a sequence of closed and open subsets H_n of K such that*

$$H = \bigcap_{n=1}^{\infty} H_n. \tag{29}$$

b) $\forall a \in H, \forall \varepsilon > 0,$
$$(a, a+\varepsilon) \nsubseteq K \setminus H, \tag{30}$$

   and
$$(a-\varepsilon, a) \nsubseteq K \setminus H. \tag{31}$$

Proof. Suppose that b) does not hold. Then there exists an $a \in H$ and an $\varepsilon > 0$ such that (30) or (31) does not hold. To fix the ideas, suppose (30) does not

hold, that is, $(a - \varepsilon, a) \subseteq K \setminus H$. Suppose now that $\{H_n\}_{n=1}^{\infty}$ is a sequence of closed and open subsets of K with $H \subseteq H_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} H_n$ cannot be equal to H since $H_n \cap (a, a + \varepsilon)$ is open in $\mathbb{R}$ while $H_n \cap [a, a + \varepsilon]$ is closed and non-empty and this can happen if and only if $[a, a + \varepsilon] \subseteq H_n$.

Conversely, suppose b) holds. Write

$$\mathbb{R} \setminus H = \left(-\infty, \widetilde{b}\right) \cup (\widetilde{a}, \infty) \cup \bigcup_{k \in F} (a_k, b_k), \tag{32}$$

as a disjoint union of open intervals; here F is a set with $|F| \leq \aleph_0$. Construct sequences $\{\widetilde{a}_n\}_{n=1}^{\infty}$, $\{\widetilde{b}_n\}_{n=1}^{\infty}$, $\{a_n^k\}_{n=1}^{\infty}$, and $\{b_n^k\}_{n=1}^{\infty}$ of $\mathbb{R} \setminus K$ such that $\widetilde{a}_n \searrow \widetilde{a}$, $\widetilde{b}_n \nearrow \widetilde{b}$, $a_n^k < b_n^k$, while $a_n^k \searrow a_k$, and $b_n^k \nearrow b_k$. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of finite sets with $\bigcup_{n=1}^{\infty} F_n = F$. Let

$$U_n = \left(-\infty, \widetilde{b}_n\right) \cup (\widetilde{a}_n, \infty) \cup \bigcup_{k \in F_n} (a_n^k, b_n^k). \tag{33}$$

Since U_n is a finite union of open intervals whose endpoints do not belong to K, then $U_n \cap K$ is both open and closed in K. We just need to take $H_n = K \cap (\mathbb{R} \setminus U_n)$ in order to satisfy a). $\qquad\square$

We are now ready to prove the ensuing characterization.

Theorem 3. *Let H and K be compact subsets of $\mathbb{R}$ with $H \subseteq K$. Then the following statements are equivalent:*

a) *There exists a continuous operator*

$$P : \mathfrak{B}(K) \longrightarrow \mathfrak{B}(H), \tag{34}$$

with

$$Pf = f, \quad \forall f \in \mathfrak{B}(H). \tag{35}$$

b) *There exist a sequence of closed and open subsets H_n of K such that*

$$H = \bigcap_{n=1}^{\infty} H_n. \tag{36}$$

Proof. Suppose a) holds but b) does not. Then using the lemma, there exists an $a \in H$ and an $\varepsilon > 0$ such that $(a, a + \varepsilon] \subset K \setminus H$, or $[a - \varepsilon, a) \subset K \setminus H$. Suppose $(a, a + \varepsilon] \subset K \setminus H$. Without loss of generality, suppose $a = 0$. Let i_1 be the injection of $\mathfrak{B}[0, \varepsilon]$ into $\mathfrak{B}(H)$, i_2 the injection of $\mathfrak{B}(H)$ into $\mathfrak{D}'(\mathbb{D}_\rho)$ where $\rho = \sup\{|x| : x \in H\}$, and set

$$T = i_2 \circ P \circ i_1. \tag{37}$$

Then using the fact that $[0, \varepsilon]$ is connected, the same argument used in the proof of the Theorem 1 yields the formula

$$T\left(f\right) = \sum_{n=0}^{\infty} \frac{\left(-1\right)^n \mu_n\left(f\right)}{n!} \delta^{(n)}\left(z\right); \qquad (38)$$

that is, $T\left(f\right)$ is given by the moment expansion of f. But this implies that the Cauchy representation of f and of $T\left(f\right)$, $\mathcal{C}\left\{f\left(x\right); z\right\}$ and $\mathcal{C}\left\{T(f)\left(\omega\right); z\right\}$, coincide for $|z| > \rho$ and thus they have identical analytic continuations to $\overline{\mathbb{C}} \setminus H$. This in turn implies that $f = P(i_1(f))$. However, this is impossible if $\operatorname{supp} f \subset [0, \varepsilon] \setminus H$.

On the other hand, if we assume that b) holds, then we can construct an operator P as follows. Let $\pi_k : \mathfrak{B}\left(K\right) \longrightarrow \mathfrak{B}\left(H_n\right)$ be the projection operator; its existence is derived from the fact that the hyperfunctions are a flabby sheaf. Let $\rho = \sup_{x \in K} |x|$ and define

$$P_0 : \mathfrak{B}\left(K\right) \longrightarrow \mathfrak{O}'\left(\overline{\mathbb{D}}_\rho\right), \qquad (39)$$

as the modified moment series

$$P_0\left(f\right) = \sum_{n=0}^{\infty} \frac{\left(-1\right)^n \mu_n\left(\pi_n(f)\right)}{n!} \delta^{(n)}\left(z\right). \qquad (40)$$

Then P_0 is a continuous operator. The analytic functional $P_0\left(f\right)$ corresponds to a hyperfunction of the space $\mathfrak{B}\left(H_n\right)$ for all $n \in \mathbb{N}$, and thus of the space $\mathfrak{B}\left(H\right)$. Hence there is a continuous operator $P : \mathfrak{B}\left(K\right) \longrightarrow \mathfrak{B}\left(H\right)$ with $i \circ P = P_0$, where $i : \mathfrak{B}\left(H\right) \longrightarrow \mathfrak{O}'\left(\overline{\mathbb{D}}_\rho\right)$ is the injection. Finally we observe that (35) follows from the fact that if $f \in \mathfrak{B}\left(H\right)$, then

$$P_0\left(f\right) = \sum_{n=0}^{\infty} \frac{\left(-1\right)^n \mu_n\left(f\right)}{n!} \delta^{(n)}\left(z\right), \qquad (41)$$

is precisely the Moment Expansion of f. $\qquad \square$

Observe in particular that the theorem shows that if H and K are closed *intervals* with $H \subseteq K$, then there does not exist a restriction operator from $\mathfrak{B}\left(K\right)$ to $\mathfrak{B}\left(H\right)$. The nonexistence of continuous restrictions onto a closed interval is a very important result in the study of integral equations in spaces of hyperfunctions over finite intervals, the so-called finite transforms. Indeed, if $T : \mathfrak{B}\left(\mathbb{R}\right) \to \mathfrak{B}\left(\mathbb{R}\right)$ is an operator, then we can define an associated operator $T_1 : \mathfrak{B}[a, b] \to \mathfrak{B}\left(a, b\right)$ by setting $T_1 = \pi T$, where $\pi : \mathfrak{B}\left(\mathbb{R}\right) \to \mathfrak{B}\left(a, b\right)$ is the canonical projection. Unfortunately, operators of this kind are not well-behaved, specifically, they are not of the Fredholm type. In the ideal situation one would like to have an operator from $\mathfrak{B}[a, b]$ to itself, but in general that

is not possible because of the results of this section. The study of such finite transforms in spaces of distributions by using multiplicative regularizators is given in [8]; several particular equations, as those of the Cauchy type, of the Carleman type, or with logarithmic kernels are considered in [10].

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