

Hölder Inequalities and Sharp Embeddings in Function Spaces of B_{pq}^s and F_{pq}^s Type

W. Sickel and H. Triebel

Abstract. Besides assertions on sharp embeddings of B_{pq}^s and F_{pq}^s we give necessary and sufficient conditions on the parameters $s, p, q, p_1, q_1, p_2, q_2$ for the Hölder type inequalities

$$\|f \cdot g|B_{pq}^s\| \leq c \|f|B_{p_1q_1}^s\| \|g|B_{p_2q_2}^s\| \quad \text{and} \quad \|f \cdot g|F_{pq}^s\| \leq c \|f|F_{p_1q_1}^s\| \|g|F_{p_2q_2}^s\|$$

to hold.

Keywords: *Function spaces, atoms, embeddings, paramultiplication, pointwise multiplication, multiplication algebras*

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1. Introduction and motivation

1.1 Introduction. The classical Hölder inequality for the Lebesgue spaces on \mathbb{R}^n is given by

$$L_{r_1} L_{r_2} \subset L_r \tag{1.1.1}$$

where

$$1 \leq r_1 \leq \infty, \quad 1 \leq r_2 \leq \infty, \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} \leq 1. \tag{1.1.2}$$

Of course (1.1.1) is a short version of the pointwise multiplication inequality

$$\|fg|L_r(\mathbb{R}^n)\| \leq c \|f|L_{r_1}(\mathbb{R}^n)\| \cdot \|g|L_{r_2}(\mathbb{R}^n)\| \tag{1.1.3}$$

where in that special case $c = 1$ may be chosen. With exception of Subsection 1.2, all spaces in this paper are defined on \mathbb{R}^n . This justifies to omit \mathbb{R}^n in the sequel. One of the main aims of the paper is to study the appropriate counterparts of (1.1.1) and (1.1.2) for the spaces B_{pq}^s and F_{pq}^s . That means for a given smoothness s we are looking for

$$B_{p_1q_1}^s B_{p_2q_2}^s \subset B_{pq}^s \tag{1.1.4}$$

$$F_{p_1q_1}^s F_{p_2q_2}^s \subset F_{pq}^s \tag{1.1.5}$$

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interpreted similarly as in (1.1.3). Recall that the spaces B_{pq}^s and F_{pq}^s cover some well-known spaces, such as the (fractional) Sobolev spaces, the classical Besov spaces, the Hölder-Zygmund spaces and the (inhomogeneous) Hardy spaces. These spaces have been studied systematically in [24, 26]. Our interest in inequalities of type (1.1.4) and (1.1.5) comes from some recent work on eigenvalue distributions of degenerate elliptic differential operators, where (1.1.1), (1.1.4) and (1.1.5) play a decisive role. We refer to [5]. In Subsection 1.2 we outline roughly this motivation for (1.1.4) and (1.1.5).

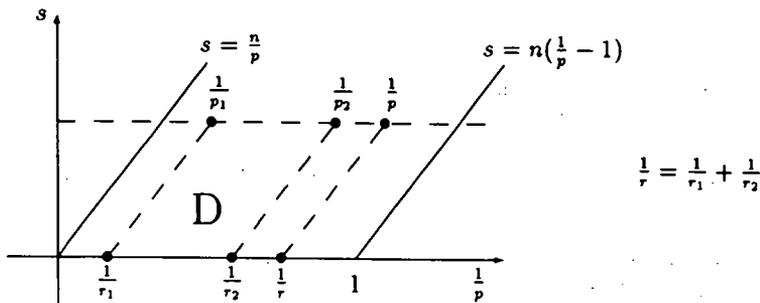


Figure 1

If we plot s against $1/p$ (see Figure 1), then the distinguished strip

$$D: \quad 0 < p < \infty, \quad n \left(\frac{1}{p} - 1 \right)_+ < s < \frac{n}{p} \tag{1.1.6}$$

plays a crucial role. The lines of slope n indicate embeddings with constant differential dimension $s - n/p$. One of the main results of this paper reads as follows:

Let s, p_1, p_2 and p be given as indicated in Figure 1 and let $0 < q_1 \leq \infty, 0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Then

- (i) (1.1.4) holds if and only if $0 < q_1 \leq r_1, 0 < q_2 \leq r_2$ and $q \geq \max(q_1, q_2)$
- (ii) (1.1.5) holds if and only if $q \geq \max(q_1, q_2)$.

In other words, the classical Hölder inequality (1.1.1) which corresponds to the bottom line $s = 0$ of the strip D in the way indicated in Figure 1 is shifted along the lines of slope n to the level of smoothness s . This situation justifies to denote (1.1.4) and (1.1.5) under the just sketched values of the involved parameters as *Hölder inequalities*. As a special case of (1.1.5) we have

$$H_{p_1}^s H_{p_2}^s \subset H_p^s$$

under the conditions indicated in Figure 1, where $H_p^s = F_{p,2}^s$ are the (fractional) Sobolev-Hardy spaces. In accordance with the limiting cases in (1.1.1), (1.1.2) we pay some attention to related limiting cases with respect to the strip D , that means

$$s = \frac{n}{p_1} \quad \text{and} \quad s = n \left(\frac{1}{p} - 1 \right)_+ \tag{1.1.7}$$

including especially the bottom line $s = 0$. These two limiting cases are connected with L_∞ and L_1 , respectively. Inevitable linked with Hölder inequalities of type (1.1.4) and (1.1.5) are sharp embeddings with constant smoothness s of type

$$B_{pu}^s \subset F_{pq}^s \subset B_{pv}^s$$

and with constant differential dimension $s - n/p$. These embeddings correspond to the horizontal lines and the lines of slope n in Figure 1, respectively. We give final answers in these cases. In connection with the above sketched limiting cases (1.1.7) where L_∞ and L_1 make their natural appearance we complement the just mentioned two sharp embeddings by sharp assertions under which the spaces B_{pq}^s and F_{pq}^s are embedded in L_∞ (which is known) and in L_1^{loc} , respectively. Of course, the latter can be rephrased as the search for sharp conditions such that B_{pq}^s and F_{pq}^s consist solely of regular distributions. Some of the incorporated sharp embeddings are known, especially the “if”-parts. But we seal several gaps, mostly related to the “only if”-parts.

The plan of the paper is the following. As mentioned above we provide in Subsection 1.2 motivations for inequalities of type (1.1.4) and (1.1.5). Section 2 contains the necessary definitions and some preparations about paramultiplication. In Section 3 we present the results about sharp embeddings: with constant smoothness, with constant differential dimension, in L_∞ , and in L_1^{loc} . Section 4 deals with Hölder inequalities. In Subsection 4.1 we describe the necessary conditions for s and $1/p$ such that we have (1.1.4) and (1.1.5) (see Figures 3a and 3b). The cases of our interest correspond to the heavy lines. In Subsection 4.2 we formulate the Hölder inequalities, whereas Subsection 4.3 deals with the indicated limiting cases. Finally Subsection 4.4 contains further results connected with the shaded areas in Figure 3 covering the region of necessity for the inequalities (1.1.4) and (1.1.5) treated in Subsection 4.1. It comes out that this is also the region of sufficiency with some peculiarities on the border lines. Proofs are presented in Section 5.

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1.2 Motivation. Let Ω be a bounded smooth domain in \mathbb{R}^n . Let Δ be the Laplacian and let

$$\begin{aligned} Au(x) &= a(x)(id - \Delta)a(x)u(x) && \text{in } \Omega \\ u(x) &= 0 && \text{on } \partial\Omega \end{aligned}$$

be a degenerate elliptic differential operator with non-smooth coefficients related to the Dirichlet problem. Assume that

$$Bu(x) = b(x)(id - \Delta)^{-1}b(x)u(x) \quad \text{with } b(x) = a^{-1}(x) \in L_r(\Omega),$$

where $2 \leq n < r \leq \infty$, makes sense as the inverse of A . In accordance with well-known classical assertions we obtained in [5] sharp assertions for the distribution of the eigenvalues λ_k of A of type $\lambda_k \approx k^{2/n}$ based on two ingredients:

(i) Sharp assertions for the entropy numbers of the compact embeddings

$$id: B_{p_1q_1}^{s_1}(\Omega) \longrightarrow B_{p_0q_0}^{s_0}(\Omega), \quad s_1 - \frac{n}{p_1} > s_0 - \frac{n}{p_0}, \quad s_1 > s_0$$

(and similarly with $F_{pq}^s(\Omega)$).

(ii) Sharp embeddings of type (1.1.1), (1.1.4) and (1.1.5).

To describe the flavour of this approach we start with $L_2(\Omega)$, multiplication by $b(x)$ brings us to $L_w(\Omega)$, where we used (1.1.1). Then we apply $(id - \Delta)^{-1}$. We arrive at $H_v^2(\Omega)$. The embedding of $H_v^2(\Omega)$ into $L_w(\Omega)$ is compact since the slope of the corresponding line is steeper than n . Finally a second multiplication by $b(x)$ brings us back to $L_2(\Omega)$.

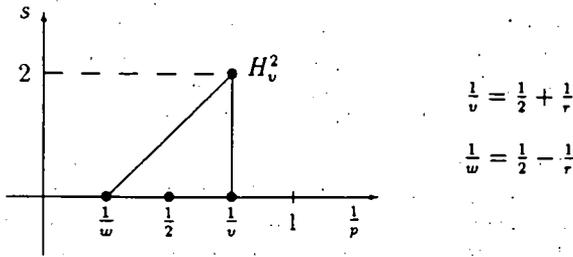


Figure 2

The compact embedding is the point, where the entropy numbers come in, whereas for the multiplications by $b(x)$ one needs inequalities of type (1.1.1) in the outlined case as sharp as possible. The interplay between the two ingredients is clear. Necessary explanations and details, especially about the role played by the entropy numbers, may be found in [5]. It is not necessary to begin with $L_2(\Omega)$ as the basic space. One can start with other suitable spaces in Figure 2. Then the triangle in Figure 2 is shifted, say in the distinguished strip D in Figure 1. Instead of the classical inequality (1.1.1) one has to work with the Hölder inequalities (1.1.4) and (1.1.5).

2. Definitions and preparations

2.1 Definitions. In general all functions, spaces, etc. are defined on the Euclidean n -space \mathbb{R}^n . So we omit \mathbb{R}^n in notations. Further we shall use \mathbb{N} to denote the set of natural numbers, \mathbb{N}_0 to denote $\mathbb{N} \cup \{0\}$, and a_+ instead of $\max(a, 0)$.

Let S be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions. By S' we denote its topological dual, the space of tempered distributions. If $\varphi \in S$, then

$$\mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(\xi) d\xi \quad (x \in \mathbb{R}^n)$$

denotes the Fourier transform $\mathcal{F}\varphi$ of φ . As usual, $\mathcal{F}^{-1}\varphi$ means the inverse Fourier transform of φ . Both, \mathcal{F} and \mathcal{F}^{-1} are extended to S' in the standard way.

Let $\psi \in S$ be a non-negative function with

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 3/2. \end{cases} \tag{2.1.1}$$

We define

$$\begin{aligned} \varphi_0(x) &= \psi(x) \\ \varphi_1(x) &= \psi(x/2) - \psi(x) \\ &\vdots \\ \varphi_k(x) &= \varphi_1(2^{-k+1}x) \end{aligned} \quad (x \in \mathbb{R}^n; k = 2, 3, \dots). \tag{2.1.2}$$

For $M \in \mathbb{N}_0$ it follows

$$\begin{aligned} \sum_{j=0}^M \varphi_j(x) &= \psi(2^{-M}x) \quad \text{and} \quad \sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad (x \in \mathbb{R}^n) \\ \text{supp } \varphi_j &\subset \{x : 2^{j-1} \leq |x| \leq 3 \cdot 2^{j-1}\} \quad (j = 1, \dots, n) \end{aligned} \tag{2.1.3}$$

and the implication

$$\frac{3}{2} \cdot 2^{j-1} \leq |x| \leq 2^j \implies \varphi_k(x) = \delta_{k,j} \quad (j \in \mathbb{N}; k \in \mathbb{N}_0). \tag{2.1.4}$$

L_p are the usual Lebesgue spaces on \mathbb{R}^n .

Definition 2.1.1. Let $-\infty < s < \infty$ and $0 < q \leq \infty$.

(i) If $0 < p < \infty$, we put

$$F_{pq}^s = \left\{ f \in S' : \|f\|_{F_{pq}^s} = \left\| \left(\sum_{j=0}^{\infty} 2^{sjq} |\mathcal{F}^{-1}[\varphi_j \mathcal{F}f](\cdot)|^q \right)^{1/q} \right\|_{L_p} < \infty \right\}$$

(usual modification if $q = \infty$).

(ii) If $0 < p \leq \infty$, we put

$$B_{pq}^s = \left\{ f \in S' : \|f\|_{B_{pq}^s} = \left(\sum_{j=0}^{\infty} 2^{sjq} \left\| \mathcal{F}^{-1}[\varphi_j \mathcal{F}f](\cdot) \right\|_{L_p}^q \right)^{1/q} < \infty \right\}$$

(usual modification if $q = \infty$).

Remark 2.1.1. These types of spaces are studied systematically in [24, 26]. We always assume that the reader is familiar with it. Recall some special cases:

- $F_{p,2}^0 = L_p \quad (1 < p < \infty) - \text{Lebesgue spaces}$
- $F_{p,2}^s = W_p^s \quad (1 < p < \infty, s \in \mathbb{Z}) - \text{Sobolev spaces}$
- $F_{p,2}^s = H_p^s \quad (1 < p < \infty, s \in \mathbb{R}) - \text{fractional Sobolev spaces}$
- $F_{p,2}^0 = h_p \quad (0 < p < \infty) - \text{inhomogeneous Hardy spaces}$
- $B_{pq}^s \quad (1 < p < \infty, 1 \leq q \leq \infty, s > 0) - \text{classical Besov spaces}$
- $B_{\infty\infty}^s = C^s \quad (s > 0) - \text{Hölder-Zygmund spaces.}$

2.2 Pointwise multiplication. Let ψ be the function defined in (2.1.1) and let $\{\varphi_j\}_{j=0}^\infty$ be the corresponding system (cf. (2.1.2)). For brevity we put

$$\begin{aligned} f_j(x) &= \mathcal{F}^{-1} [\varphi_j(\xi) \mathcal{F}f(\xi)](x) \\ f^j(x) &= \mathcal{F}^{-1} [\psi(2^{-j}\xi) \mathcal{F}f(\xi)](x) \quad (j \in N_0). \end{aligned} \tag{2.2.1}$$

It is easily checked that $\lim_{j \rightarrow \infty} f^j =_{S'} f$ for any $f \in S'$. Moreover, f^j is an entire analytic function of exponential type. Hence, the product $f^j \cdot g^j$ makes sense for any j and any $f, g \in S'$. We define

$$f \cdot g = \lim_{j \rightarrow \infty} f^j \cdot g^j \tag{2.2.2}$$

whenever this limit exists. Note that

$$\begin{aligned} f \cdot g &= \lim_{l \rightarrow \infty} \left(\sum_{j=0}^l f_j \right) \left(\sum_{k=0}^l g_k \right) \\ &= \sum_{j=2}^\infty f^{j-2} g_j + \sum_{k=2}^\infty f_k g^{k-2} + \sum_{k=0}^\infty \sum_{j=k-1}^{k+1} f_k g_j \\ &= \sum' (f, g) + \sum'' (f, g) + \sum''' (f, g) \quad (\text{put } g_{-1} \equiv 0). \end{aligned}$$

The advantage of such a decomposition is based on the inclusions

$$\text{supp } \mathcal{F}(f^{k-2} g_k) \subset \left\{ \xi : 3 \cdot 2^{k-3} \leq |\xi| \leq 11 \cdot 2^{k-3} \right\} \tag{2.2.3}$$

$$\text{supp } \mathcal{F} \left(\sum_{j=k-1}^{k+1} f_k g_j \right) \subset \left\{ \xi : |\xi| \leq 9 \cdot 2^{k-1} \right\}. \tag{2.2.4}$$

Remark 2.2.1. Recall the Fatou property of the underlying spaces. Let A_{pq}^s denote either B_{pq}^s or F_{pq}^s . If $\{f^j g^j\}_j$ is a Cauchy sequence in S' with limit h and if

$$\sup_j \|f^j g^j\|_{A_{p,q}^s} = A < \infty,$$

then it follows $h \in A_{pq}^s$ and $\|h\|_{A_{pq}^s} \leq c A$, where c is independent of f and g (cf. [7]).

Remark 2.2.2. The operator

$$\Pi_f : g \longrightarrow \sum_{j=2}^\infty f^{j-2} g_j$$

is called *paramultiplication operator*. Estimates for this operator are the heart of several contributions to the problem of pointwise multiplication (see, e.g., [14, 20, 21, 24, 30 - 32]). Further they are of importance in microlocal analysis and in the theory of Calderon-Zygmund singular integral operators (see, e.g., [3, 16, 30 - 32]).

The essence of the needed estimates are formulated in the following proposition, where we make use of the abbreviation $h_\infty = L_\infty$.

Proposition 2.2.1. Let $0 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $0 < q \leq \infty$.

(i) Let $0 < p < \infty$. Then

$$\begin{aligned} & \left\| \left(\sum_{k=0}^{\infty} \left| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum' (f, g) \right] (\cdot) \right|^q \right)^{1/q} \Big|_{L_p} \right\| \\ & \leq c \|f\|_{h_{p_2}} \left\| \left(\sum_{k=0}^{\infty} |g_k(\cdot)|^q \right)^{1/q} \Big|_{L_{p_1}} \right\| \end{aligned} \tag{2.2.5}$$

where c is independent of f and g (usual modification if $q = \infty$).

(ii) Let $0 < p \leq \infty$. Then

$$\begin{aligned} & \left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum''' (f, g) \right] (\cdot) \Big|_{L_p} \right\| \\ & \leq c \max_{-1 \leq j \leq 1} \sum_{l=-2}^{\infty} \|f_{k+l}\|_{L_{p_1}} \|g_{k+l+j}\|_{L_{p_2}} \end{aligned} \tag{2.2.6}$$

if $p \geq 1$ and

$$\begin{aligned} & \left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum''' (f, g) \right] (\cdot) \Big|_{L_p} \right\| \\ & \leq c \max_{-1 \leq j \leq 1} \left(\sum_{l=-2}^{\infty} 2^{ln(1-p)} \|f_{k+l}\|_{L_{p_1}}^p \|g_{k+l+j}\|_{L_{p_2}}^p \right)^{1/p} \end{aligned}$$

if $p < 1$, where c is independent of f , g and $k \in \mathbb{N}_0$ (put $f_r = g_r = 0$ if $r < 0$).

(iii) Let $0 < p < \infty$ and $s > n(\frac{1}{p} - 1)_+$. Then

$$\begin{aligned} & \left\| \sup_k 2^{ks} \left| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum''' (f, g) \right] (\cdot) \right| \Big|_{L_p} \right\| \\ & \leq c \max_{-1 \leq j \leq 1} \left\| \sup_k 2^{ks/2} |f_k| \Big|_{L_{p_1}} \right\| \left\| \sup_k 2^{ks/2} |g_{k+j}| \Big|_{L_{p_2}} \right\| \end{aligned}$$

where c is independent of f and g .

Remark 2.2.3. In the scalar case of (i), given by

$$\left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum' (f, g) \right] (\cdot) \Big|_{L_p} \right\| \leq c \|f\|_{h_{p_2}} \max_{-1 \leq j \leq 1} \|g_{k+j}\|_{L_{p_1}},$$

also $p = \infty$ is admissible. Part (iii) is taken from [30: Theorem 3.7], complemented by the use of the Hölder inequality with respect to $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Proofs of statements (i) and (ii) will be given in Subsection 5.5.

3. Sharp embeddings

3.1 Embeddings with constant smoothness. Here the sign \subset stands for continuous embedding. Recall that all spaces are defined on \mathbb{R}^n .

Theorem 3.1.1. *The following statements are true.*

(i) *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, $0 < u \leq \infty$ and $0 < v \leq \infty$. Then*

$$B_{pu}^s \subset F_{pq}^s \subset B_{pv}^s \tag{3.1.1}$$

if and only if

$$0 < u \leq \min(p, q) \quad \text{and} \quad \max(p, q) \leq v \leq \infty. \tag{3.1.2}$$

(ii) *Let $0 < u \leq \infty$ and $0 < v \leq \infty$. Then*

$$B_{1,u}^0 \subset L_1 \subset B_{1,v}^0 \tag{3.1.3}$$

if and only if $0 < u \leq 1$ and $v = \infty$.

(iii) *Let $0 < u \leq \infty$. Then*

$$F_{1,u}^0 \subset L_1 \tag{3.1.4}$$

if and only if $0 < u \leq 2$. Furthermore

$$L_1 \not\subset F_{1,\infty}^0. \tag{3.1.5}$$

(iv) *Let $0 < u \leq \infty$ and $0 < v \leq \infty$. Then*

$$B_{\infty,u}^0 \subset L_\infty \subset B_{\infty,v}^0 \tag{3.1.6}$$

if and only if $0 < u \leq 1$ and $v = \infty$.

Remark 3.1.1. Let C be the space of all complex-valued bounded and uniformly continuous functions on \mathbb{R}^n normed in the usual way. In (3.1.6) one can replace L_∞ by C .

Remark 3.1.2. By (3.1.1) we know $F_{1,\infty}^0 \subset B_{1,\infty}^0$. The assertion (3.1.5) shows that the second inequality in (3.1.3) can not be improved by replacing $B_{1,\infty}^0$ by $F_{1,\infty}^0$.

Remark 3.1.3. The “if”-parts of Theorem 3.1.1 are known (see [24: Proposition 2.3.2/2, p.47, Proposition 2.5.7, p.89 and Theorem 2.5.8/1, p.92]). In other words, we have to complement these known assertions by the “only if”-parts and the proof of (3.1.5).

3.2 Embeddings with constant differential dimension. Recall that $s - n/p$ is called the *differential dimension* both of B_{pq}^s and F_{pq}^s . It is a characteristic number which plays a crucial role in the theory of these spaces (see, for instance, Figure 1 and the accompanying remarks).

Theorem 3.2.1. *The following statements are true.*

(i) *Let $0 < p_0 < p < p_1 \leq \infty$, $s \in \mathbb{R}$, $s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}$, $0 < q \leq \infty$, $0 < u \leq \infty$ and $0 < v \leq \infty$. Then*

$$B_{p_0 u}^{s_0} \subset F_{pq}^s \subset B_{p_1 v}^{s_1} \tag{3.2.1}$$

if and only if

$$0 < u \leq p \leq v \leq \infty. \tag{3.2.2}$$

(ii) *Let $0 < p < p_1 < \infty$, $s \in \mathbb{R}$,*

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1} \tag{3.2.3}$$

and $0 < q \leq \infty$. Then

$$F_{p\infty}^s \subset F_{p_1 q}^{s_1}. \tag{3.2.4}$$

(iii) *Let $0 < p < 1$ and $0 < q \leq \infty$. Then*

$$B_{pq}^{n(1/p-1)} \subset L_1 \tag{3.2.5}$$

if and only if $0 < q \leq 1$.

Remark 3.2.1. The “if”-part of statement (i) is due to Jawerth and Franke (see [7, 10], [24: p. 131] and [25: p. 191]). Furthermore, statement (ii) is mentioned here for the sake of completeness (see [24: Theorem 2.7.1, p. 129]). Of course, by the monotonicity of the F_{pq}^s -spaces, ∞ in (3.2.4) can be replaced by any positive number. As mentioned in the Introduction, embeddings in L_1 and L_∞ deserve special attention. The L_∞ -counterpart of (3.2.5) will be described in the next subsection.

3.3 Embeddings in L_∞ and in L_1^{loc} . The space C has been defined in Remark 3.1.1.

Theorem 3.3.1. *The following statements are true.*

(i) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Then the following three assertions are equivalent :*

- (a) $F_{pq}^s \subset L_\infty$
- (b) $F_{pq}^s \subset C$
- (c) *either $s > \frac{n}{p}$ or $s = \frac{n}{p}$ and $0 < p \leq 1$.*

(ii) *Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Then the following three assertions are equivalent :*

- (a) $B_{pq}^s \subset L_\infty$
- (b) $B_{pq}^s \subset C$
- (c) *either $s > \frac{n}{p}$ or $s = \frac{n}{p}$ and $0 < q \leq 1$.* (3.3.1)

Remark 3.3.1. This theorem is known. We incorporate it both for sake of completeness and because it will be of great service later on in this paper. A proof of

statement (i) may be found in [7]. As far as statement (ii) is concerned we refer to [24: Theorem 1, p. 133] (see also [24: Item 2.8.3, p. 146] and [25: Item 2.8.3, p. 211]).

Remark 3.3.2. Let A_{pq}^s be either B_{pq}^s or F_{pq}^s . Then A_{pq}^s is called a *multiplication algebra* if $A_{pq}^s A_{pq}^s \subset A_{pq}^s$ where the multiplication of two distributions is given by (2.2.2). Part (i) of the theorem can be complemented by statement

(d) F_{pq}^s is a multiplication algebra.

A corresponding assertion for B_{pq}^s is only "almost" true. More precisely : B_{pq}^s is a multiplication algebra if and only if

(d) either $s > \frac{n}{p}$ with $0 < p \leq \infty$ or $s = \frac{n}{p}$ with $0 < p < \infty$ and $0 < q \leq 1$.

This assertion differs from (3.3.1) by the case $s = 0, p = \infty$. We refer to [7] and [23] (see also [24: Item 2.8.3], with the indicated correction as far as the case $s = 0, p = \infty$ is concerned, and [20: p. 56]). The case $B_{\infty q}^0$ will be established in Remark 4.3.5 in the indicated way: It is not a multiplication algebra. Although the study of multiplication algebras fits quite well in the framework of our paper we shall not stress this point in the sequel. We are mostly interested in multiplication with essentially unbounded function.

Of course, L_1^{loc} stands for the collection of all complex-valued functions which are locally integrable in \mathbb{R}^n . It is interpreted here as the set of all regular distributions on \mathbb{R}^n .

Theorem 3.3.2. *The following statements are true.*

(i) Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leq \infty$. Then the following two assertions (i₁) and (i₂) are equivalent:

(i₁) $F_{pq}^s \subset L_1^{\text{loc}}$

(i₂) either $0 < p < 1, s \geq n \left(\frac{1}{p} - 1 \right), 0 < q \leq \infty$ (3.3.2)

or $1 \leq p < \infty, s > 0, 0 < q \leq \infty$

or $1 \leq p < \infty, s = 0, 0 < q \leq 2$. (3.3.3)

(ii) Let $s \in \mathbb{R}, 0 < p \leq \infty$ and $0 < q \leq \infty$. Then the following two assertions (ii₁) and (ii₂) are equivalent:

(ii₁) $B_{pq}^s \subset L_1^{\text{loc}}$

(ii₂) either $0 < p \leq \infty, s > n \left(\frac{1}{p} - 1 \right)_+, 0 < q \leq \infty$

or $0 < p \leq 1, s = n \left(\frac{1}{p} - 1 \right), 0 < q \leq 1$

or $1 < p \leq \infty, s = 0, 0 < q \leq \min(p, 2)$. (3.3.4)

Remark 3.3.3. If $s > n \left(\frac{1}{p} - 1 \right)_+$, then it is well-known that B_{pq}^s and F_{pq}^s consist of regular distributions. In other words, the interesting part of Theorem 3.3.2 is the final classification what happens in the limiting case $s = n \left(\frac{1}{p} - 1 \right)_+$.

We compare the above Theorem 3.3.2 with the sharp embeddings described in Subsections 3.1 and 3.2. The case $p = \infty$ plays a special role. Without going in details we mention

$$B_{\infty, 2}^0 \subset F_{\infty, 2}^0 = bmo \tag{3.3.5}$$

(see [24: pp. 37, 50 and 93] for definitions and explanations; as far as the spaces $F_{\infty,q}^0$ are concerned we refer also to [8, 14]).

Corollary 3.3.1. *The following conclusions are true.*

(i) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. Let A_{pq}^s be either B_{pq}^s or F_{pq}^s . Then the following two assertions (i₁) and (i₂) are equivalent :*

$$\begin{aligned} \text{(i}_1\text{)} \quad & A_{pq}^s \subset L_1^{\text{loc}} \\ \text{(i}_2\text{)} \quad & A_{pq}^s \subset L_{\bar{p}} \quad \text{with } \bar{p} = \max(1, p). \end{aligned} \tag{3.3.6}$$

(ii) *Let $s \in \mathbb{R}$ and $0 < q \leq \infty$. Then the following two assertions (ii₁) and (ii₂) are equivalent :*

$$\begin{aligned} \text{(ii}_1\text{)} \quad & B_{\infty q}^s \subset L_1^{\text{loc}} \\ \text{(ii}_2\text{)} \quad & B_{\infty q}^s \subset \text{bmo}. \end{aligned} \tag{3.3.7}$$

Proof. Part (ii) is covered by (3.3.5) and the above Theorem 3.3.2, especially (3.3.4). The F -case of part (i) related to (3.3.2) follows from (3.2.4), (3.2.3) and (3.1.3):

$$F_{pq}^{n(1/p-1)} \subset B_{1,1}^0 \subset L_1 \quad (p < 1).$$

The F -case related to (3.3.3) is clear since $F_{p,2}^0 = L_p$ ($1 < p < \infty$) and

$$F_{1,2}^0 = h_1 \subset L_1. \tag{3.3.8}$$

The B -case of part (i) follows immediately from the above Theorem 3.3.2, (3.2.5), (3.1.1) and (3.3.8). ■

Remark 3.3.4. Let again A_{pq}^s be either B_{pq}^s or F_{pq}^s and let

$$A_{pq}^s \subset L_1. \tag{3.3.9}$$

Let ψ be a C^∞ -function with $\psi(x) = 0$ near the origin and $\psi(y) = 1$ if, say, $|y| \geq 1$. Let

$$r_j : f \longrightarrow \mathcal{F}^{-1} \left[\psi(\xi) \frac{\xi_j}{|\xi|} \mathcal{F}f(\xi) \right] (\cdot) \quad (j = 1, \dots, n)$$

be the inhomogeneous Riesz transforms. Recall that the (inhomogeneous) Hardy spaces $h_1 = F_{1,2}^0$ can also be characterized as the collection of all $f \in L_1$ with $r_j f \in L_1$ if $j = 1, \dots, n$ (see [24: pp. 93/94]). Since $r_j A_{pq}^s \subset A_{pq}^s$ we can improve (3.3.9) by $A_{pq}^s \subset h_1$. In other words, if $p = 1$, then (3.3.6) can be strengthened by $A_{1,q}^s \subset h_1$. Now (3.3.7) looks a little bit more natural since $\text{bmo} = h_1'$.

Remark 3.3.5. For better reference we formulate one consequence of Theorems 3.1.1, 3.2.1 and 3.3.2, Remark 2.1.1 (first item) and (3.3.8) once again. Let $s > 0$, $0 < p < \infty$ and $0 < q \leq \infty$. If $1 \geq \frac{1}{r} = \frac{1}{p} - \frac{s}{n} > 0$, then

$$B_{pq}^s \subset L_r \iff 0 < q \leq r.$$

A corresponding assertion for F_{pq}^s holds without restrictions on q .

4. Hölder inequalities

4.1 Necessary conditions for s and p . Let $s \in \mathbb{R}$, $0 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 < p \leq \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. We ask under which conditions

$$B_{p_1 q_1}^s B_{p_2 q_2}^s \subset B_{p q}^s \tag{4.4.1}$$

$$F_{p_1 q_1}^s F_{p_2 q_2}^s \subset F_{p q}^s \tag{4.4.2}$$

hold, where in case of (4.1.2) we assume, in addition, $p_1 \neq \infty$, $p_2 \neq \infty$ and $p \neq \infty$.

Theorem 4.1.1. *If either (4.1.1) or (4.1.2) hold under the indicated general conditions for the parameters, then*

$$\max\left(\frac{1}{p_1}, \frac{1}{p_2}\right) \leq \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \tag{4.1.3}$$

$$2s \geq n \left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right)_+ \tag{4.1.4}$$

$$s \geq n \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \tag{4.1.5}$$

Remark 4.1.1. Of course, (4.1.5) can be rewritten as

$$s - \frac{n}{p} \leq s - \frac{n}{p_1} + s - \frac{n}{p_2} \tag{4.1.6}$$

Recall that $s - \frac{n}{p}$ is the differential dimension both of $B_{p q}^s$ and $F_{p q}^s$. In other words, the differential dimension of the target spaces in (4.1.1) or (4.1.2) has to be less than or equal to the sum of the differential dimension of the spaces on the left-hand sides of (4.1.1) and (4.1.2). In the Figures 3a and 3b we summarized the above restrictions in dependence on whether $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ or $\frac{1}{p_1} + \frac{1}{p_2} > 1$.

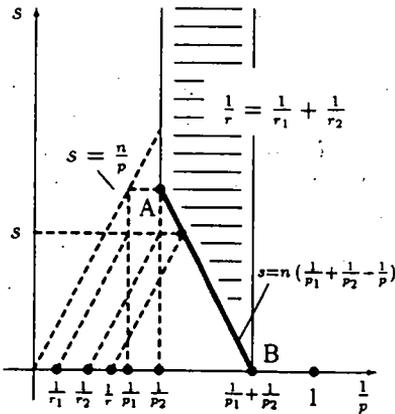


Figure 3a

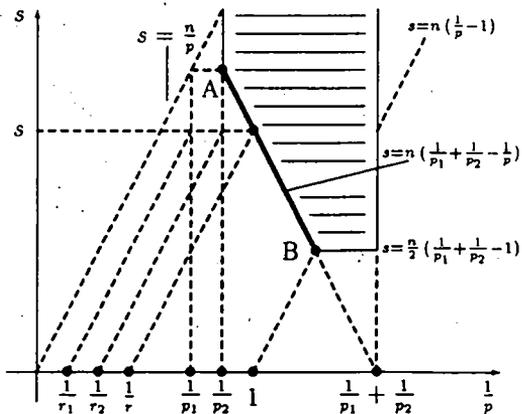


Figure 3b

If one compares Figures 1, 3a and 3b, then we are mainly interested in those cases, where we have equality in (4.1.5) and (4.1.6). This corresponds to the heavy lines in Figures 3a and 3b and in that case (4.1.6) is identical with $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ in Figure 1. The limiting cases (1.1.7) correspond to the points A and B in Figures 3a and 3b, respectively. However in the final Subsection 4.4 we sketch briefly what happens inside of the shaded regions.

4.2 The main results. Recall that all the spaces are defined on \mathbb{R}^n . Furthermore, the Hölder inequalities we are looking for are characterized by the situation sketched in Figure 1 and indicated by the heavy lines in Figures 3a and 3b. There is a significant difference between B -spaces and F -spaces acting as pointwise multiplier spaces which can be clearly seen by the theorem below and which will be prepared by the following proposition.

Proposition 4.2.1. *Let $s \in \mathbb{R}$, $0 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < p < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let*

$$\frac{1}{r_1} = \frac{1}{p_1} - \frac{s}{n} > 0 \quad \text{and} \quad \frac{1}{p_2} + \frac{1}{r_1} = \frac{1}{p}. \tag{4.2.1}$$

Let independently

$$A_{p_2 q_2}^s \text{ be either } B_{p_2 q_2}^s \text{ or } F_{p_2 q_2}^s \quad \text{and} \quad A_{pq}^s \text{ be either } B_{pq}^s \text{ or } F_{pq}^s.$$

If

$$B_{p_1 q_1}^s A_{p_2 q_2}^s \subset A_{pq}^s, \tag{4.2.2}$$

then

$$q_1 \leq r_1. \tag{4.2.3}$$

Remark 4.2.1. If one replaces the pointwise multiplier space $B_{p_1 q_1}^s$ by $F_{p_1 q_1}^s$, then the restriction of type (4.2.3) simply does not occur (see the Theorem 4.2.1 below). As far as r_1 is concerned we refer to Figure 1. If r_2 and r are defined in a similar way, then the second part of (4.2.1) can be reformulated as $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ which coincides with Figures 1, 3a and 3b. However it is not assumed that the involved spaces are characterized by the points within \bar{D} , where D is given by (1.1.6).

Theorem 4.2.1. *Let $s > 0$, $0 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < p < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let*

$$\frac{1}{r_1} = \frac{1}{p_1} - \frac{s}{n} > 0, \quad \frac{1}{r_2} = \frac{1}{p_2} - \frac{s}{n} > 0, \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r} = \frac{1}{p} - \frac{s}{n} < 1. \tag{4.2.4}$$

(i) *Then there holds*

$$B_{p_1 q_1}^s B_{p_2 q_2}^s \subset B_{pq}^s \tag{4.2.5}$$

if and only if

$$0 < q_1 \leq r_1, \quad 0 < q_2 \leq r_2, \quad \infty \geq q \geq \max(q_1, q_2). \tag{4.2.6}$$

(ii) Then there holds

$$F_{p_1 q_1}^s F_{p_2 q_2}^s \subset F_{pq}^s \tag{4.2.7}$$

if and only if

$$\infty \geq q \geq \max(q_1, q_2). \tag{4.2.8}$$

Remark 4.2.2. Both (4.2.5) and (4.2.7) are the Hölder inequalities in the distinguished strip D in (1.1.6) we are looking for. The situation described in Figure 1 is the same as in (4.2.4). Compared with Figures 3a and 3b condition (4.2.4) corresponds to the heavy lines where the endpoints A and B are excluded. Furthermore, (4.2.4) is connected with embeddings with constant differential dimensions, see Theorem 3.2.1 and the broken lines in the Figures 3a and 3b ending at $\frac{1}{r_1}$, $\frac{1}{r_2}$, and $\frac{1}{r}$.

Remark 4.2.3. One can try to mix B -spaces and F -spaces in (4.2.5) and (4.2.7). We do not go into detail. By the above proposition it is quite clear what can be expected.

4.3 Two limiting cases. We discuss two limiting cases connected with the point A in Figures 3a and 3b and point B in Figure 3a. First we assume $s = \frac{n}{p_1}$. In agreement with (4.2.1) we have $r_1 = \infty$ and $p_2 = p$. However it comes out that (4.2.3) is no longer the natural condition. In contrast to Proposition 4.2.1 we have now to handle the B -spaces and the F -spaces separately.

Proposition 4.3.1. *The following statements are true.*

(i) Let $0 < p_1 \leq \infty, 0 < p \leq \infty, 0 < q_1 \leq \infty, 0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let

$$s = \frac{n}{p_1}. \tag{4.3.1}$$

If

$$B_{p_1 q_1}^s B_{p q_2}^s \subset B_{pq}^s, \tag{4.3.2}$$

then

$$q_1 \leq 1. \tag{4.3.3}$$

(ii) Let $0 < p_1 < \infty, 0 < p < \infty, 0 < q_1 \leq \infty, 0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let s be given by (4.3.1). If

$$F_{p_1 q_1}^s F_{p q_2}^s \subset F_{pq}^s, \tag{4.3.4}$$

then

$$p_1 \leq 1. \tag{4.3.5}$$

Proof. If (4.3.2) holds, then it follows by the same arguments as in [7: pp. 38/39] that $B_{p_1 q_1}^s \subset L_\infty$. Similarly, if (4.3.4) holds, then we have necessarily $F_{p_1 q_1}^s \subset L_\infty$. Now (4.3.3), respectively (4.3.5) follow immediately from Theorem 3.3.1. ■

Theorem 4.3.1. *Let $0 < p_1 < \infty, 0 < p < \infty, 0 < q_1 \leq \infty, 0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let*

$$s = \frac{n}{p_1} \quad \text{and} \quad 0 < \frac{1}{p} - \frac{s}{n} = \frac{1}{r} < 1.$$

(i) Then there holds

$$B_{p_1 q_1}^s B_{p_2 q_2}^s \subset B_{pq}^s$$

if and only if

$$0 < q_1 \leq 1, 0 < q_2 \leq r \quad \text{and} \quad \infty \geq q \geq \max(q_1, q_2).$$

(ii) Then there holds

$$F_{p_1 q_1}^s F_{p_2 q_2}^s \subset F_{pq}^s$$

if and only if

$$0 < p_1 \leq 1 \quad \text{and} \quad \infty \geq q \geq \max(q_1, q_2).$$

Remark 4.3.1. This theorem is connected with the points A in the Figures 3a and 3b. The formulation is chosen in such a way that it can be compared immediately with Theorem 4.2.1. Instead of (4.2.6) with the expected $0 < q_1 \leq \infty$ we have now $0 < q_1 \leq 1$, and (4.2.8) must now be complemented by $0 < p_1 \leq 1$.

The second limiting case is connected with the point B in Figure 3a, that means with $s = 0$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1$.

Theorem 4.3.2. Let $0 < q_1 \leq \infty, 0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let

$$1 \leq p_1 \leq \infty, \quad 1 \leq p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1.$$

Let independently

$$\begin{aligned} A_{p_1 q_1}^0 & \text{ be either } B_{p_1 q_1}^0 \quad \text{or} \quad F_{p_1 q_1}^0 \\ A_{p_2 q_2}^0 & \text{ be either } B_{p_2 q_2}^0 \quad \text{or} \quad F_{p_2 q_2}^0 \\ A_{pq}^0 & \text{ be either } B_{pq}^0 \quad \text{or} \quad F_{pq}^0 \end{aligned}$$

(we assume $A_{p_1 q_1}^0 = B_{p_1 q_1}^0$ if $p_1 = \infty$). Then there holds

$$A_{p_1 q_1}^0 A_{p_2 q_2}^0 \subset A_{pq}^0 \tag{4.3.6}$$

if and only if

$$A_{p_1 q_1}^0 \subset L_{p_1}, \quad A_{p_2 q_2}^0 \subset L_{p_2}, \quad L_p \subset A_{pq}^0. \tag{4.3.7}$$

Remark 4.3.2. We compare the above assertion with the classical Hölder inequality $L_{p_1} L_{p_2} \subset L_p$. Then Theorem 4.3.1 states that, within the scales B_{pq}^s and F_{pq}^s with $s = 0$, the classical Hölder inequality is not improvable.

As a consequence of Theorems 4.3.2 and 3.1.1 and $F_{p,2}^0 = L_p$ ($1 < p < \infty$) we obtain the following corollary.

Corollary 4.3.1. Let $0 < q_1 \leq \infty, 0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let

$$1 < p_1 < \infty, \quad 1 < p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} < 1.$$

(i) Then there holds

$$B_{p_1 q_1}^0 B_{p_2 q_2}^0 \subset B_{pq}^0$$

if and only if $q_1 \leq \min(p_1, 2)$, $q_2 \leq \min(p_2, 2)$ and $q \geq \max(p, 2)$.

(ii) Then there holds

$$F_{p_1 q_1}^0 F_{p_2 q_2}^0 \subset F_{pq}^0 \tag{4.3.8}$$

if and only if $q_1 \leq 2$, $q_2 \leq 2$ and $q \geq 2$.

Remark 4.3.3. In case $1 \leq p_1 < \infty$, $1 \leq p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = 1$ Theorem 4.3.2 yields : there do not exist q_1, q_2 and q such that (4.3.8) holds. This follows from (3.1.5).

Remark 4.3.4. Of special interest is also the situation in case $p_1 = \infty$. Let $1 \leq p < \infty$, $0 < q_1 \leq \infty$ and $0 < q \leq \infty$. By (4.3.7) and Theorem 3.1.1 we have

$$B_{\infty q_1}^0 B_{pq}^0 \subset B_{pq}^0 \tag{4.3.9}$$

if and only if $0 < q_1 \leq 1$ and $p = q = 2$ (that means $B_{2,2}^0 = L_2$) and

$$B_{\infty q_1}^0 F_{pq}^0 \subset F_{pq}^0$$

if and only if $0 < q_1 \leq 1$, $1 < p < \infty$ and $q = 2$ (that means $F_{p,2}^0 = L_p$). Hence, with the obvious exception of L_p the space L_∞ is not contained in the set of pointwise multipliers of these spaces, which was proved earlier in [8]. Inclusion (4.3.9) improves also some results of Bourdaud [4].

Not as a consequence of Theorem 4.3.2 but as a consequence of the proof of this theorem one obtains the following corollary. Here b_{pq}^s denotes the closure of S in B_{pq}^s , equipped with the same quasi-norm as B_{pq}^s .

Corollary 4.3.2. Let $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Then there holds

$$b_{\infty q_1}^0 b_{\infty q_2}^0 \subset b_{\infty q}^0$$

if and only if $b_{\infty q_1}^0 \subset L_\infty$, $b_{\infty q_2}^0 \subset L_\infty$ and $L_\infty \subset b_{\infty q}^0$.

Remark 4.3.5. Because of $b_{\infty q}^0 \neq L_\infty$ one consequence of Corollary 4.3.2 is the fact that $b_{\infty q}^0$ is not a multiplication algebra (cf. Remark 3.3.2). But this implies that also $B_{\infty q}^0$ can not be an algebra with respect to pointwise multiplication.

4.4 Further results. We complement our previous considerations by collecting some further results, mostly connected with the shaded areas in the Figures 3a and 3b. We refer also to Subsections 3.3 and 4.3, where we characterized the conditions under which B_{pq}^s or F_{pq}^s are multiplication algebras.

Theorem 4.4.1. Let $0 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < q_1 \leq \infty$, $0 < q_2 \leq \infty$ and $0 < q \leq \infty$. Let $(s, \frac{1}{p})$ be a point in the interior of the shaded areas in Figures 3a and 3b, that means

$$\max\left(\frac{1}{p_1}, \frac{1}{p_2}\right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2} \tag{4.4.1}$$

$$2s > n \left(\frac{1}{p_1} + \frac{1}{p_2} - 1 \right)_+ \tag{4.4.2}$$

$$s > n \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \right) \tag{4.4.3}$$

Then there holds

$$B_{p_1 q_1}^s B_{p_2 q_2}^s \subset B_{pq}^s \tag{4.4.4}$$

if and only if

$$q \geq \max(q_1, q_2) \tag{4.4.5}$$

and

$$F_{p_1 q_1}^s F_{p_2 q_2}^s \subset F_{pq}^s \tag{4.4.6}$$

if and only if

$$q \geq \max(q_1, q_2). \tag{4.4.7}$$

In case of the B -spaces $p_1 = \infty$ and $p_2 = \infty$ are admitted in (4.4.4).

Remark 4.4.1. We used the same notations as in Theorem 4.2.1 and it is immediately clear that the above assertion complements that theorem, where we are now not restricted to the strip D . The necessity of (4.4.5) and respectively (4.4.7) is proved in [7].

Remark 4.4.2. The above theorem has a lot of forerunners. Without going into detail we refer to [1, 2, 7, 9, 12 - 15, 17, 20 - 24, 29 - 34].

Remark 4.4.3. Again let us cast a look on the Figures 3a and 3b. The results given so far answer the question of the existence of inequalities of type (4.4.4) and (4.4.6) in the interior of the shaded area (yes) and partly also on the boundary. Whereas on the vertical lines the answer is again yes in any case, except maybe the points A and B itself (these claims can be proved by suitable modifications of the proof of the above theorem given in Subsection 5.7 (cf. Remark 5.7.1)) the answer on the horizontal line $2s = n(\frac{1}{p_1} + \frac{1}{p_2} - 1)$ may be yes or no. A partly positive answer for the existence of (4.4.6) on this line is given by [7]. A negative answer for the existence of (4.4.4) one obtains in case $\frac{1}{q_1} + \frac{1}{q_2} < 1$ by replacing the simple counterexample used in proof of (4.1.4) by a more sophisticated one (cf. [11] or [18: Lemma 4.3.1/3]).

5. Proofs

5.1 Proofs of the assertions in Subsection 3.1.

Proof of Theorem 3.1.1. *Step 1.* Proofs of the “if”-parts of statements (i), (ii) and (iv) may be found in [24: Proposition 2.3.2/2, p. 46 and Proposition 2.5.7, p. 89]. The “if”-part of statement (iii) is an immediate consequence of the identity $F_{1,2}^0 = h_1$ (cf. [24: Theorem 1, p. 92]). So in what follows we restrict ourselves to the “only if”-parts and to the proof of (3.1.5).

Step 2 (Proof of (3.1.1)). First note that it will be sufficient to prove (3.1.1) in case of a fixed s . By well-known lifting properties (cf. [24: Theorem 2.3.8, p. 58]) it can be extended to arbitrary s afterwards.

Substep 2.1. Let $\psi \in S$ such that

$$\text{supp } \mathcal{F}\psi \subset \left\{ \xi : \xi_1 \leq 0, \frac{7}{4} \leq |\xi| \leq 2 \right\}. \tag{5.1.1}$$

For given complex numbers a_j and $e = (1, 0, \dots, 0)$ we put

$$f(x) = \sum_{j=3}^{\infty} a_j e^{i\lambda_j \langle e, x \rangle} \psi(x).$$

Then

$$\mathcal{F}f(\xi) = \sum_{j=3}^{\infty} a_j (\mathcal{F}\psi)(\xi - \lambda_j e).$$

Choosing $\lambda_j = 2^j - 2$ ($j = 3, 4, \dots$) we arrive at

$$\mathcal{F}^{-1}[\varphi_j \mathcal{F}f](x) = a_j e^{i\lambda_j \langle e, x \rangle} \psi(x)$$

where we have used (2.1.3), (2.1.4) and (5.1.1). Consequently we have

$$\|f\|_{F_{p,q}^0} = c \left(\sum_{j=3}^{\infty} |a_j|^q \right)^{1/q} \quad \text{and} \quad \|f\|_{B_{p,u}^0} = c \left(\sum_{j=3}^{\infty} |a_j|^u \right)^{1/u}$$

The monotonicity of the l_q -norms gives

$$u \leq q \leq v. \tag{5.1.2}$$

Substep 2.2. We wish to prove the counterpart of (5.1.2) with p instead of q . For this purpose we use local means (cf. [26: Subsections 1.8.4 and 2.5.3]). Let $k_0, k^0 \in S$ such that

$$\mathcal{F}k_0(0) \neq 0, \quad \mathcal{F}k^0(0) \neq 0$$

$$\text{supp } k_0 \subset \{y : |y| \leq 1\}, \quad \text{supp } k^0 \subset \{y : |y| \leq 1\}.$$

Define

$$k(y) = \Delta^N k^0(y) = \left(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} \right)^N k^0(y)$$

with $N \in \mathbb{N}$. We introduce the local means by

$$k(t, f)(x) = \int k(y) f(x + ty) dy \quad (t > 0) \tag{5.1.3}$$

and similarly $k_0(t, f)$. Recall, for N large enough we have

$$\|f\|_{B_{p,u}^s} \approx \|k_0(1, f)(\cdot)\|_{L_p} + \left(\sum_{m=1}^{\infty} 2^{msu} \|k(2^{-m}, f)(\cdot)\|_{L_p}^u \right)^{1/u} \tag{5.1.4}$$

Let $\varphi \in S$ be non-trivial with a compact support near the origin. Let l be an integer. Then we have

$$\begin{aligned} k(2^l, \Delta^N \varphi)(x) &= \int \Delta^N k^0(y) (\Delta^N \varphi)(x + 2^l y) dy \\ &= 2^{2lN} \int k^0(y) (\Delta^{2N} \varphi)(x + 2^l y) dy \\ &= 2^{-2lN} \int \Delta^{2N} k^0(y) \varphi(x + 2^l y) dy. \end{aligned}$$

In other words, we have

$$|k(2^l, \Delta^N \varphi)(x)| \leq c_N 2^{-2|l|N} \quad (l \in \mathbb{Z}, N \in \mathbb{N}) \tag{5.1.5}$$

and similarly

$$|k_0(2^l, \Delta^N \varphi)(x)| \leq c_N 2^{-2lN} \quad (l \in \mathbb{N}_0, N \in \mathbb{N}). \tag{5.1.6}$$

Now we put

$$f = \sum_{j=0}^{\infty} a_j (\Delta^N \varphi)(2^j(x - x^j)) \tag{5.1.7}$$

with $a_j \in \mathbf{C}$ and, say, $x^j = c(j, 0, \dots, 0)$ where $c > 0$ is a suitable positive number. We insert f into (5.1.3) and calculate

$$\|k(2^{-m}, f)(\cdot)|L_p\|^p = \sum_{j=0}^{\infty} |a_j|^p \|k(2^{-m}, (\Delta^N \varphi)(2^j(\cdot - x^j)))|L_p\|^p \tag{5.1.8}$$

where we used the construction of the local means and the fact that the supports of the terms of f have a sufficiently large distance from each other. For the term with $j = m \in \mathbb{N}$ we have

$$\|k(2^{-m}, (\Delta^N \varphi)(2^m \cdot - 2^m x^m))|L_p\|^p = 2^{-mn} \|k(1, \Delta \varphi)|L_p\|^p \tag{5.1.9}$$

where we may assume that the last factor on the right-hand side is positive. Of course we have an obvious counterpart of (5.1.8) with $k_0(1, f)$ instead of $k(2^{-m}, f)$ and of (5.1.9) for the term $j = m = 0$. Hence, by (5.1.4) we have

$$\|f|B_{pu}^s\|^u \geq c \sum_{m=0}^{\infty} 2^{msu} |2^{-mn/p} a_m|^u \tag{5.1.10}$$

for some $c > 0$. To prove the converse estimate we apply (5.1.5) with $l = j - m$ to the corresponding term in (5.1.8), and (5.1.6) with $l = j$ to the k_0 -counterpart of (5.1.8). For sake of convenience we put $a_j = 0$ for $-j \in \mathbb{N}$. Then (5.1.5) and (5.1.8) yield

$$\begin{aligned} \|k(2^{-m}, f)(\cdot)|L_p\|^p &\leq c \sum_j |2^{-jn/p} a_j|^p 2^{-2N|j-m|p} \\ &\leq c' \sup_j |2^{-jn/p} a_j|^p 2^{-\alpha|j-m|p} \\ &\leq c' \sum_j (|2^{-jn/p} a_j|^u 2^{-\alpha|j-m|u})^{p/u} \end{aligned}$$

for $0 < \alpha < 2N$ and appropriate $c > 0$ and $c' > 0$. We choose N and afterwards α sufficiently large such that

$$2^{msu} \|k(2^{-m}, f)(\cdot)|L_p\|^u \leq c \sum_j 2^{jsu} |2^{-jn/p} a_j|^u 2^{-\beta|j-m|u} \tag{5.1.11}$$

with $\beta = \alpha - |s| > 0$. Now we insert (5.1.11) and its k_0 -counterpart into (5.1.4), change the order of summation and arrive at the converse of (5.1.10). Hence we have

$$\|f|B_{pu}^s\|^u \approx \sum_{m=0}^{\infty} 2^{msu} |2^{-mn/p} a_m|^u. \tag{5.1.12}$$

On the other hand, by the localization property of F_{pq}^s (cf. [26: Subsection 2.4.7]) we have

$$\|f|F_{pq}^s\|^p \approx \sum_{j=0}^{\infty} |a_j|^p \|(\Delta^N \varphi)(2^j \cdot -2^j x^j)|F_{pq}^s\|^p. \tag{5.1.13}$$

To estimate the last factors we use the counterpart of (5.1.4) for the F_{pq}^s -space (see again [26: Subsection 2.4.6]). By (5.1.5), (5.1.6) and the same technique as above we have

$$\|(\Delta^N \varphi)(2^j \cdot -2^j x^j)|F_{pq}^s\| \approx 2^{j(s-n/p)}.$$

We insert this result into (5.1.13) and arrive at

$$\|f|F_{pq}^s\|^p \approx \sum_{m=0}^{\infty} 2^{msp} |2^{-mn/p} a_m|^p. \tag{5.1.14}$$

Now (3.1.1), (5.1.12) and (5.1.14) yield $u \leq p \leq v$. Together with (5.1.2) this proves (3.1.2).

Step 3 (Proof of (3.1.6)). Taking the characteristic function χ_Q of the cube $Q = \{x : |x_l| \leq 1 \ (l = 1, \dots, n)\}$ it is well-known that

$$\chi_Q \in B_{\infty q}^0 \iff q = \infty \tag{5.1.15}$$

(cf. [23: pp. 142 - 145]). There the one-dimensional case is treated only but the general result can be deduced by using some tensorproduct arguments. From the equivalence (5.1.15) it follows $v = \infty$. The remaining implication

$$B_{\infty u}^0 \subset L_\infty \implies u \leq 1$$

can be derived from the existence of essentially unbounded functions in $B_{\infty u}^0$ ($u > 1$) (cf. [23: pp. 134/135]). This proves (3.1.6).

Step 4 (Proof of (3.1.3)). The proof of (3.1.3) ("only if"-part) can be reduced to (3.1.6) by using duality arguments. Suppose

$$B_{1,u}^0 \subset L_1 \subset B_{1,v}^0 \quad \text{for some } 1 < u < \infty \text{ and/or } v < \infty.$$

Then this would imply (cf. [24: Subsection 2.11.2, p.178])

$$B_{\infty v'}^0 \subset L_\infty \subset B_{\infty u'}^0 \quad \left(\frac{1}{u} + \frac{1}{u'} = \frac{1}{v} + \frac{1}{v'} = 1 \right).$$

This contradicts (3.1.6). Hence, (3.1.3) is proved.

Step 5 (Proof of (3.1.4)). The proof of $F_{1,q}^0 \not\subset L_1$ ($q > 1$) we postpone to the proof of the stronger implication $F_{1,q}^0 \subset L_1^{loc} \implies q \leq 2$ (cf. Subsection 5.3).

Step 6 (Proof of (3.1.5)). To prove (3.1.5) we apply again duality arguments. Let $f_{1,\infty}^0$ be the closure of S in $F_{1,\infty}^0$. Assuming $L_1 \subset F_{1,\infty}^0$ we conclude $L_1 \subset f_{1,\infty}^0$ using the density of S in L_1 . This yields

$$(f_{1,\infty}^0)' = F_{\infty,1}^0 \subset L_\infty \tag{5.1.16}$$

(cf. [14]). But this is false since $F_{\infty,1}^0$ contains essentially unbounded functions. This can be derived from the embedding

$$B_{p\infty}^{n/p} \subset F_{\infty,1}^0 \quad (0 < p < \infty) \tag{5.1.17}$$

(cf. again [14] and Theorem 3.3.1) ■

Remark 5.1.1. To avoid the technical difficulties occurring in Substep 2.2 one can use the following elegant argumentation. Let $n \geq 2$. If s is large enough, then the trace on \mathbb{R}^{n-1} ($\approx x_n = 0$) makes sense. Suppose (3.1.1). Then it follows

$$B_{pu}^{s-1/p}(\mathbb{R}^{n-1}) \subset B_{pp}^{s-1/p}(\mathbb{R}^{n-1}) \subset B_{pv}^{s-1/p}(\mathbb{R}^{n-1})$$

(cf. [24: Theorem 2.7.2, p. 132]). This proves $u \leq p \leq v$, unfortunately in case $n \geq 2$ only.

5.2 Proofs of the assertions in Subsection 3.2.

Proof of Theorem 3.2.1. Step 1. The proof of (3.2.5) (“only if”-part) will be postponed to the proof of the stronger implication $B_{pq}^{n(1/p-1)} \subset L_1^{loc} \implies q \leq 1$ given in Subsection 5.3. The “if”-part of (3.2.5) follows from $B_{pq}^{n(1/p-1)} \subset B_{1,q}^0$ (see [24: Theorem 2.7.1]) and (3.1.3). Furthermore, as pointed out in Remark 3.2.1, both statement (ii) and the “if”-part of statement (i) are known.

Step 2 (Proof of the “only if”-part of (3.2.1)). Let f be given by (5.1.7). We put $b_j = 2^{j(s-n/p)} a_j$. Since $s - \frac{n}{p} = s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$ we find by (5.1.12) and (5.1.14)

$$\|f|_{B_{p_0 u}^{s_0}}\|^u \approx \sum_{j=0}^{\infty} |b_j|^u, \quad \|f|_{B_{p_1 v}^{s_1}}\|^v \approx \sum_{j=0}^{\infty} |b_j|^v, \quad \|f|_{F_{pq}^s}\|^p \approx \sum_{j=0}^{\infty} |b_j|^p.$$

Then (3.2.1) implies $u \leq p$ and $p \leq v$ ■

5.3 Proofs of the assertions in Subsection 3.3.

Proof of Theorem 3.3.2. *Step 1.* The implications $(i_2) \Rightarrow (i_1)$ and $(ii_2) \Rightarrow (ii_1)$ are known and follow directly from the sharper embeddings $B_{pq}^s \subset L_{\bar{p}}$ and $F_{pq}^s \subset L_{\bar{p}}$ with $\bar{p} = \max(1, p)$ (see Theorems 3.1.1 and 3.2.1, Remark 2.1.1 and the proof of Corollary 3.3.1).

Step 2. Let $p \leq 1$ and $s = n(1/p - 1)$. We shall prove the implication

$$B_{pq}^{n(1/p-1)} \subset L_1^{\text{loc}} \implies q \leq 1. \tag{5.3.1}$$

Let φ be a non-vanishing C^∞ -function with support near to the origin and $\mathcal{F}\varphi(0) = 0$. Let

$$f = \sum_{j=0}^{\infty} \lambda_j 2^{jn} \varphi(2^j x - x^j) \quad (x^j \in \mathbb{R}^n, |x^j| \leq 1) \tag{5.3.2}$$

where we assumed that the functions $\varphi(2^j x - x^j)$ have disjoint supports. Then (5.3.2) is an atomic representation of f with

$$\|f|B_{pq}^{n(1/p-1)}\| \leq c \left(\sum_{j=0}^{\infty} |\lambda_j|^q \right)^{1/q} < \infty \tag{5.3.3}$$

(see [8] or [26: Subsection 1.9.2]). On the other hand we have

$$\left\| \sum_{j=0}^M \lambda_j 2^{jn} \varphi(2^j \cdot - x^j) \Big| L_1 \right\| = c \sum_{j=0}^M |\lambda_j| \quad (c \neq 0). \tag{5.3.4}$$

If $1 < q < \infty$, then we find numbers λ_j with (5.3.3) such that (5.3.4) diverges. Hence f does not belong to L_1 . This proves (5.3.1).

Step 3. Let $1 \leq p < \infty$. We shall prove that $F_{pq}^0 \subset L_1^{\text{loc}}$ implies $q \leq 2$. Assume $q > 2$. For technical reasons we switch temporarily to the one-dimensional periodic case. Let T^1 be the 1-torus. Let $\{a_k\}_k \notin l_2$. Immediately it follows:

(i) The lacunary series $\sum_k a_k e^{i2^k t}$ belongs to $F_{pq}^0(T^1)$ if and only if $\{a_k\}_k \in l_q$ (cf. [19: Subsections 3.5.1 and 6.4.2]).

(ii) $f \notin L_1(T^1)$, hence f is not a regular distribution on the 1-torus (cf. [6: Subsections 15.3.1 and 15.3.2]).

This yields the result in the one-dimensional periodic case. The same argumentation works in the general non-periodic case if we start with

$$g(x) = f(x_1) \cdot \chi(x) \quad (x = (x_1, \dots, x_n)).$$

Here χ denotes a compactly supported C^∞ -function in \mathbb{R}^n which is identically 1 in the cube $[-\pi, \pi]^n$. One can prove this claim by using the characterization of F_{pq}^s -spaces via local means with kernels having a product structure (cf. [26: Subsection 1.8.4]).

Step 4. Let $1 \leq p \leq \infty$. The same machinery as in the preceding step can be applied to prove that $B_{pq}^0 \subset L_1^{loc}$ implies $q \leq 2$.

Step 5. Let $1 \leq p \leq \infty$. It remains to check the implication

$$B_{pq}^0 \subset L_1^{loc} \implies 0 < q \leq p. \tag{5.3.5}$$

We shall prove the existence of a singular distribution in B_{pq}^0 ($1 \leq p < q \leq \infty$).

Substep 5.1 (Preparations). Let f be a smooth function, non-trivial, supported around the origin, $|f(x)| \leq 1$ and $\int f(x) dx = 0$. Let $\sigma > 1$. Define

$$\kappa_0 = 0 \quad \text{and} \quad \kappa_j = \sum_{l=1}^j l^{-1} (\log(l+1))^\sigma \quad (j \in \mathbb{N}).$$

Since $\sigma > 1$, there exists a real number κ with $\kappa_j \rightarrow \kappa$ if $j \rightarrow \infty$. Further we put

$$R_j = \left\{ x = (x_1, \dots, x_n) : \kappa_{j-1} < x_1 \leq \kappa_j, 0 < x_l < 1 \ (l = 2, \dots, n) \right\}$$

for $j \in \mathbb{N}$. Next we subdivide R_j by

$$N_j = 2^{j(n-1)} \left[2^j j^{-1} (\log(j+1))^{-\sigma} \right]$$

where $[\]$ denotes the integer part into cubes of side-length 2^{-j} and centered at $x^{j,r}$.

Substep 5.2. The announced singular distribution is given by

$$g = \sum_{j=1}^{\infty} \sum_{r=1}^{N_j} (\log(j+1))^\sigma f(2^{j+1}(x - x^{j,r})).$$

To see this, first note that $2^{jn/p} f(2^{j+1}(x - x^{j,r}))$ is an atom. More exactly, it is an $(Q_{j,r}, 0, p)$ -atom (cf. [26: p. 62]) where $Q_{j,r}$ is an appropriate cube with volume $\approx 2^{-jn}$ and located around $x^{j,r}$. Using the characterization of Besov spaces via atoms, due to Frazier and Jawerth (cf. [26: Theorem 1.9.2, p.63]), we obtain

$$\begin{aligned} \|g\|_{B_{pq}^0}^q &\leq c \sum_{j=1}^{\infty} 2^{-j(n/p)q} (\log(j+1))^{\sigma q} \left(\sum_{r=1}^{N_j} 1 \right)^{q/p} \\ &\leq c \sum_{j=1}^{\infty} j^{-q/p} (\log(j+1))^{\sigma q(1-1/p)} \\ &< \infty \end{aligned}$$

since $q > p$. Hence,

$$g \in B_{pq}^0 \quad \text{if } 1 \leq p < q < \infty. \tag{5.3.6}$$

By construction g has compact support. Furthermore

$$\int |g(x)| dx = \sum_{j=1}^{\infty} \int_{R_j} |g(x)| dx \approx \sum_{j=1}^{\infty} (\log(j+1))^\sigma |R_j| = \sum_{j=1}^{\infty} j^{-1} = \infty.$$

Hence,

$$g \notin L_1^{loc}. \tag{5.3.7}$$

The formulas (5.3.6) and (5.3.7) prove that g has the required properties, which gives (5.3.5). ■

5.4 Proofs of the assertions in Subsection 4.1.

Proof of Theorem 4.1.1. *Step 1* (Proof of (4.1.3)). The right-hand side of this inequality was proved in [20: p. 51]. We prove the left-hand side and assume that (4.1.1) holds. Then we have

$$\|f|B_{pq}^s\| \leq c \|f|B_{p_1q_1}^s\| \tag{5.4.1}$$

for all $f \in B_{p_1q_1}^s$ with compact support, say, in the unit cube. Assume that f is non-trivial and smooth and $\int x^\beta f(x) dx = 0$ for $|\beta| \leq L$. If L is sufficiently large, then $2^{j(n/p-s)} f(2^j x)$ are atoms in $B_{p_1q_1}^s$ (see [26: Subsection 1.9.2]). Then $\|f(2^j \cdot)|B_{pq}^s\| \approx 2^{j(s-n/p)}$. Similarly, for $B_{p_1q_1}^s$. Then (5.4.1) yields $\frac{1}{p_1} \leq \frac{1}{p}$. By (3.1.1) the assumption that (4.1.2) holds yields a corresponding assertion of type (4.1.1). Hence we have again $\frac{1}{p_1} \leq \frac{1}{p}$.

Step 2 (Proof of 4.1.4)). The necessity of $s \geq 0$ is proved in [7]. So it remains to check $2s \geq \frac{n}{p_1} + \frac{n}{p_2} - n$. Let φ_j be the functions defined in (2.1.2). Consider the sequences

$$\begin{aligned} H_j(x) &= 2^{-j\alpha_1} \mathcal{F}^{-1} \varphi_j(x) \\ G_j(x) &= 2^{-j\alpha_2} \mathcal{F}^{-1} \varphi_j(x) \end{aligned} \quad (j \in \mathbb{N}). \tag{5.4.2}$$

Obviously,

$$\|H_j|B_{p_1q_1}^s\| \|G_j|B_{p_2q_2}^s\| \approx 2^{j(2s-(\alpha_1+\alpha_2)+2n-\frac{n}{p_1}-\frac{n}{p_2})}. \tag{5.4.3}$$

Let us assume $2s < \frac{n}{p_1} + \frac{n}{p_2} - n$. Then we can choose $\alpha_1 + \alpha_2 < n$ such that

$$2s - (\alpha_1 + \alpha_2) + 2n - \frac{n}{p_1} - \frac{n}{p_2} = 0. \tag{5.4.4}$$

Next we consider the sequence $(G_j \cdot H_j)(\varphi)$, where φ is taken from S . We find

$$\begin{aligned} (G_j \cdot H_j)(\varphi) &= \int G_j(x) H_j(x) \varphi(x) dx \\ &= 2^{-j(\alpha_1+\alpha_2)} 2^{2jn} \iint \varphi_1(\tau) \varphi_1(\xi - \tau) d\tau \mathcal{F}\varphi(2^j \xi) d\xi. \end{aligned}$$

We choose φ such that $\mathcal{F}\varphi \geq 0, \xi \in \mathbb{R}$ and $\mathcal{F}\varphi(\xi) = 1$ ($|\xi| \leq 1$). Then it follows

$$|(G_j \cdot H_j)(\varphi)| \geq c 2^{j(n-(\alpha_1+\alpha_2))}$$

for some appropriate positive constant c . From the continuous embedding $B_{pq}^s \subset S'$ (cf. [24: Theorem 2.3.2, p. 48]) and (5.4.3), (5.4.4) we obtain a contradiction to (4.1.1). Since (5.4.3) remains true if we replace the B -spaces by F -spaces the same argumentation works also in this case.

Step 3 (Proof of (4.1.5)). Again we can make use of the sequences defined in (5.4.2). Now it will be sufficient to take $\alpha_1 = \alpha_2 = 0$. Observe that

$$\begin{aligned} \|G_j \cdot H_j|B_{p\infty}^s\| &\geq 2^{js} \left\| \mathcal{F}^{-1} \left[\varphi_j \mathcal{F} (\mathcal{F}^{-1} \varphi_j \cdot \mathcal{F}^{-1} \varphi_j) \right] (x) \right\|_{L_p} \\ &\geq 2^{js} 2^{2jn} 2^{-jn/p} \left\| \mathcal{F}^{-1} \left[\varphi_1(\varphi_1 * \varphi_1) \right] (x) \right\|_{L_p}. \end{aligned} \tag{5.4.5}$$

Comparing (5.4.5) with (5.4.3) the necessity of (4.1.5) in case (4.1.1) follows. As in the preceding step the same proof can be taken over to the case of F -spaces. ■

Remark 5.4.1. Let $A_{p_1q_1}^s$ denote either $B_{p_1q_1}^s$ or $F_{p_1q_1}^s$ and similar $A_{p_2q_2}^s$ and A_{pq}^s . Without any changes the above proof can be taken over to the more general problem whether

$$A_{p_1q_1}^s A_{p_2q_2}^s \subset A_{pq}^s \tag{5.4.6}$$

holds or not. Then as above from (5.4.6) the necessity of (4.1.3) - (4.1.5) will follow.

Remark 5.4.2. There is a difference between (4.1.4) and (4.1.5). Whereas (4.1.5) is necessary to keep the product in $B_{p\infty}^s$, (4.1.4) saves the membership of the product to S' .

5.5 Proofs of the assertions in Subsection 4.2.

Proof of Proposition 4.2.1. For sake of simplicity we always assume $n = 1$. Otherwise one has to modify the following in an obvious way.

Step 1 (Preparations). We construct a smooth counterpart of Rademacher functions. To this end, let ϱ_0 be a C^∞ -function supported near 1 and identical 1 in a certain neighbourhood of 1. Then we put $\varrho_1(x) = \varrho_0(x) - \varrho_0(x - 1)$. Consequently $\int \varrho_1(x) dx = 0$. Next we define $\varrho_2(x) = \varrho_1(x) - \varrho_1(x - 2)$. This function ϱ_2 has now two vanishing moments

$$\int \varrho_2(x) dx = 0$$

$$\int x \varrho_2(x) dx = \int x \varrho_1(x) dx - \int (x - 2) \varrho_1(x - 2) dx = 0.$$

Iteration of this construction yields a family of functions ϱ_k having the following properties:

$$\text{supp } \varrho_k \subset [0, 2^k + 1]$$

and

$$|\mathcal{F}\varrho_k(\xi)| \leq c_k |\xi|^k \quad (|\xi| \leq 1) \tag{5.5.1}$$

$$|\mathcal{F}\varrho_k(\xi)| \leq c_{k,K} |\xi|^{-K} \quad (|\xi| \geq 1) \tag{5.5.2}$$

for arbitrary K and suitable constants c_k and $c_{k,K}$. Both k and K are at our disposal.

Step 2. We fix some k and denote the corresponding function ϱ_k simply by ϱ . In what follows we investigate linear combinations of some scaled versions of this function. Let $\varrho^l(x) = \varrho(2^l x)$ ($l \in \mathbb{N}$). Recall that $\{\varphi_k\}_k$ denotes the decomposition of unity defined in (2.1.2) and ψ the function from (2.1.1). Let $\Phi \in S$ be a function with

$$\text{supp } \Phi \subset \left[-4, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 4\right] \quad \text{and} \quad \Phi(x) = 1 \quad \text{on } \text{supp } \varphi_1.$$

From (5.5.1) and (5.5.2) one derives

$$\|\Phi(\cdot)(\mathcal{F}\varrho)(2^{j-l}\cdot)\|_{W_2^m} \leq c_m 2^{-a|j-l|} \tag{5.5.3}$$

$$\|\psi(\cdot)(\mathcal{F}\varrho)(2^{-l}\cdot)\|_{W_2^m} \leq c_m 2^{-al} \tag{5.5.4}$$

for any $m \geq 0$ and c_m does not depend on $j \in \mathbb{N}_0$ and $l \in \mathbb{N}_0$. Here $a > 0$ is at our disposal. The Fourier multiplier theorem (see [24: Subsection 1.5.2]) and (5.5.3) yield in case $j > 0$

$$\begin{aligned} \left\| \mathcal{F}^{-1} [\varphi_j \mathcal{F} \varrho^l] (\cdot) \Big|_{L_p} \right\| &= 2^{-l+j(1-1/p)} \left\| \mathcal{F}^{-1} \left[\varphi_1(\xi) \Phi(\xi) \mathcal{F} \varrho(2^{j-l}\xi) \right] (\cdot) \Big|_{L_p} \right\| \\ &\leq c 2^{-l+j(1-1/p)} \left\| \Phi(\cdot) (\mathcal{F} \varrho)(2^{j-l}\cdot) \Big|_{W_2^m} \right\| \left\| \mathcal{F}^{-1} \varphi_1 \Big|_{L_p} \right\| \quad (5.5.5) \\ &\leq c' 2^{-j/p} 2^{-a|j-l|} \end{aligned}$$

where m has to be sufficiently large. In case $j = 0$ one has to apply (5.5.4) instead of (5.5.3) and obtains

$$\left\| \mathcal{F}^{-1} [\varphi_0 \mathcal{F} \varrho^l] (\cdot) \Big|_{L_p} \right\| \leq c 2^{-a l} \quad (5.5.6)$$

Both (5.5.5) and (5.5.6) lead to

$$2^{js} \left\| \mathcal{F}^{-1} [\varphi_j \mathcal{F} \varrho^l] (\cdot) \Big|_{L_p} \right\| \leq c 2^{j(s-1/p)} 2^{-a|j-l|} \quad (5.5.7)$$

which may be assumed to be an equivalence if $j = l$. We introduce

$$\lambda^L(x) = \sum_{l=0}^L a_l \varrho^l(x - x_l) \quad (5.5.8)$$

where the points x_l are chosen such that $\varrho^l(\cdot - x_l)$ have disjoint supports (that is not important in this step but it will be used later on). Next we wish to calculate $\|\lambda^L\|_{B_{p,q}^s}$. We have

$$\begin{aligned} &\left\| \mathcal{F}^{-1} [\varphi_j \mathcal{F} \lambda^L] (\cdot) \Big|_{L_p} \right\|^{\min(1,p)} \\ &\geq |a_j|^{\min(1,p)} \left\| \mathcal{F}^{-1} [\varphi_j \mathcal{F} \varrho^j] (\cdot) \Big|_{L_p} \right\|^{\min(1,p)} \\ &\quad - \sum_{l \neq j} |a_l|^{\min(1,p)} \left\| \mathcal{F}^{-1} [\varphi_j \mathcal{F} \varrho^l] (\cdot) \Big|_{L_p} \right\|^{\min(1,p)} \end{aligned}$$

if $L \geq j$. Using (5.5.7) this leads to

$$\begin{aligned} &2^{js \min(1,p)} \left\| \mathcal{F}^{-1} [\varphi_j \mathcal{F} \lambda^L] (\cdot) \Big|_{L_p} \right\|^{\min(1,p)} \\ &\geq c_1 |a_j|^{\min(1,p)} 2^{j(s-1/p) \min(1,p)} \\ &\quad - c_2 \sum_{l \neq j} |a_l|^{\min(1,p)} 2^{l(s-1/p) \min(1,p)} 2^{-a|j-l| \min(1,p)} \end{aligned}$$

where c_1 and c_2 are positive constants independent of j and l . This implies

$$\begin{aligned} & \|\lambda^L |B_{pq}^s|\|^{\min(1,p,q)} \\ & \geq c_1' \left(\sum_{j=0}^L 2^{j(s-1/p)q} |a_j|^q \right)^{(1/q)\min(1,p,q)} \\ & \quad - c_2' \left(\sum_{j=0}^{\infty} \left| \sum_{l \neq j} |a_l|^{\min(1,p)} 2^{l(s-1/p)\min(1,p)} \right. \right. \\ & \quad \left. \left. \times 2^{-a|l-j|\min(1,p)} \right|^{\frac{q}{\min(1,p)}} \right)^{(1/q)\min(1,p,q)} \end{aligned}$$

Choosing a large we may assume that the second term on the right-hand side can be estimated from above by, say,

$$\frac{c_1'}{2} \left(\sum_j 2^{j(s-1/p)q} |a_j|^q \right)^{(1/q)\min(1,p,q)}$$

Then we obtain

$$\|\lambda^L |B_{pq}^s|\| \geq c \left(\sum_{j=0}^L 2^{j(s-1/p)q} |a_j|^q \right)^{1/q} \tag{5.5.9}$$

for an appropriate positive constant c . The reverse inequality to (5.5.9) can be derived in a similar way, again based on (5.5.7). Hence we have

$$\|\lambda^L |B_{pq}^s|\| \approx \left(\sum_{j=0}^L 2^{j(s-1/p)q} |a_j|^q \right)^{1/q} \tag{5.5.10}$$

and the corresponding constants do not depend on L and the sequences $\{a_j\}_j$ and $\{x_l\}_l$.

Step 3. Let $l = 0, \dots, L$ and $j > L$. Then we can choose points t^r ($r = 1, \dots, c2^{j-l}$) such that $\text{supp } \varrho^j(\cdot - t^r) \subset \{t : \varrho^l(t) = 1\}$ and the supports of $\varrho^j(\cdot - t^r)$ and $\varrho^j(\cdot - t^m)$ ($r \neq m$) have a mutual distance of at least $c'2^{-j}$ for some positive numbers c and c' . We put

$$\mu_l^j(x) = \sum_{r=1}^{c2^{j-l}} \varrho^j(x - t^r).$$

Such functions are studied in [28] (for partial results see also [27]). By the theorem in [28: p. 183] it follows

$$\|\mu_l^j |F_{p_2q_2}^s|\| \approx \|\mu_l^j |B_{p_2q_2}^s|\| \approx 2^{js-l/p_2}.$$

Step 4. Of course, the results of Step 3 remains unchanged replacing ρ^l by $\rho^l(\cdot - x^l)$. We put

$$\mu^j(x) = \sum_{l=0}^L b_l \mu_l^j(x - x^l). \tag{5.5.11}$$

Then all the ingredients have mutually disjoint supports. Moreover, from [28] we know that

$$\|\mu^j|F_{p_2q_2}^s\| \approx \|\mu^j|B_{p_2q_2}^s\| \approx 2^{js} \left(\sum_{l=0}^L |2^{-l/p_2} b_l|^{p_2} \right)^{1/p_2}. \tag{5.5.12}$$

Step 5. We multiply λ^L from (5.5.8) by μ^j from (5.5.11). By construction

$$\mu^j(x) \lambda^L(x) = \sum_{l=0}^L b_l a_l \mu_l^j(x - x^l).$$

By (5.5.12) it follows

$$\|\mu^j \lambda^L|F_{pq}^s\| \approx \|\mu^j \lambda^L|B_{pq}^s\| \approx 2^{js} \left(\sum_{l=0}^L |2^{-l/p} b_l a_l|^p \right)^{1/p}. \tag{5.5.13}$$

Step 6. Assume (4.2.2) holds. Then by (5.5.13), (5.5.12) and (5.5.10) (with p_1 instead of p and q_1 instead of q)

$$\begin{aligned} & \left(\sum_{l=0}^L |2^{-l/p} b_l a_l|^p \right)^{1/p} \\ & \leq c \left(\sum_{l=0}^L |2^{-l/p_2} b_l|^{p_2} \right)^{1/p_2} \left(\sum_{l=0}^L 2^{l(s-1/p_1)q_1} |a_l|^{q_1} \right)^{1/q_1} \end{aligned} \tag{5.5.14}$$

with c independent of L and a_l, b_l . Let $\beta_l = 2^{-l/p_2} b_l$ and $\alpha_l = 2^{-l/r_1} a_l$ (see (4.2.1)). Applying (4.2.1) then (5.5.14) yields

$$\left(\sum_{l=0}^L |\beta_l \alpha_l|^p \right)^{1/p} \leq c \left(\sum_{l=0}^L |\beta_l|^{p_2} \right)^{1/p_2} \left(\sum_{l=0}^L |\alpha_l|^{q_1} \right)^{1/q_1}$$

Let $\beta_l = \alpha_l = 1$. Then $L^{1/p} \leq c L^{1/p_2+1/q_1}$ and hence $\frac{1}{p_2} + \frac{1}{r_1} = \frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{q_1}$ from which (4.2.3) follows. ■

To prove Theorem 4.2.1 we need Proposition 2.2.1. Therefore we start to prove Proposition 2.2.1 first.

Proof of Proposition 2.2.1. *Step 1* (Proof of statement (i)). We shall use (2.2.3), (2.1.1), (2.1.3) together with a Fourier multiplier assertion (cf. [24: Theorem 1.6.3, p.31]). This gives

$$\left\| \left(\sum_{k=0}^{\infty} \left| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum' (f, g) \right] (\cdot) \right|^q \right)^{1/q} \right\|_{L_p} \leq c \left\| \left(\sum_{k=2}^{\infty} |f^{k-2} g_k|^q \right)^{1/q} \right\|_{L_p}.$$

Applying Hölder's inequality and $\| \sup_k |f^k(x)| \|_{L_{p_2}} \leq c \|f\|_{h_{p_2}}$ (cf. [24: p.37]) we arrive at (2.2.5).

Step 2 (Proof of statement (ii)). We shall apply the identity

$$\mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \left(\sum''' (f, g) \right) \right] (x) = \sum_{l=-2}^{\infty} \sum_{j=-1}^1 \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} (f_{k+l} \cdot g_{k+l+j}) \right] (x). \quad (5.5.15)$$

Here we have used (2.2.4), (2.1.1) and (2.1.3). First, let $p \geq 1$. By the Michlin-Hörmander-Fourier multiplier theorem, the triangle inequality and Hölder's inequality we get

$$\begin{aligned} \left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum''' (f, g) \right] (\cdot) \right\|_{L_p} &\leq c \left\| \sum_{l=-2}^{\infty} \sum_{j=-1}^1 \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} (f_{k+l} \cdot g_{k+l+j}) \right] (\cdot) \right\|_{L_p} \\ &\leq c \max_{-1 \leq j \leq 1} \sum_{l=-2}^{\infty} \|f_{k+l}\|_{L_{p_1}} \|g_{k+l+j}\|_{L_{p_2}}. \end{aligned}$$

This proves (2.2.6).

Let $p < 1$. Let $l \geq -2$. Proposition 1.5.1 and Remark 1.5.2/3 in [24: p. 25/28] give

$$\left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} (f_{k+l} \cdot g_{k+l}) \right] (\cdot) \right\|_{L_p} \leq c 2^{ln(1/p-1)} \|f_{k+l} \cdot g_{k+l}\|_{L_p}.$$

Again we use (5.5.15). Hence

$$\begin{aligned} &\left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} \sum''' (f, g) \right] (\cdot) \right\|_{L_p}^p \\ &\leq c \max_{-1 \leq j \leq 1} \sum_{l=-2}^{\infty} \left\| \mathcal{F}^{-1} \left[\varphi_k \mathcal{F} (f_{k+l} \cdot g_{k+l+j}) \right] (\cdot) \right\|_{L_p}^p \\ &\leq c \max_{-1 \leq j \leq 1} \sum_{l=-2}^{\infty} 2^{ln(1/p-1)p} \|f_{k+l}\|_{L_{p_1}}^p \|g_{k+l+j}\|_{L_{p_2}}^p. \end{aligned}$$

This completes the proof of Proposition 2.2.1. ■

Proof of Theorem 4.2.1. Necessity part. After application of Proposition 4.2.1 it remains to prove $q \geq \max(q_1, q_2)$. But this is stated in a more general context in [7].

Sufficiency part. Step 1 (Proof of 4.2.5). Therefore we use the preparations made in Subsection 2.2.

Substep 1.1 (Estimate of \sum', \sum''). Our assumptions in (4.2.4) and (4.2.6) imply the embeddings $B_{p_1 q_1}^s \subset L_{r_1}$ and $B_{p_2 q_2}^s \subset L_{r_2}$ (cf. Remark 3.3.5). Hence, we may use Proposition 2.2.1/(i) to obtain

$$\|2^{js} \mathcal{F}^{-1} [\varphi_j \mathcal{F} \sum' (f, g)](\cdot) |L_p\| \leq c (2^{js} \|g_j |L_{p_2}\|) \|f |h_{r_1}\|. \tag{5.5.16}$$

$$\|2^{js} \mathcal{F}^{-1} [\varphi_j \mathcal{F} \sum'' (f, g)](\cdot) |L_p\| \leq c (2^{js} \|f_j |L_{p_1}\|) \|g |h_{r_2}\|. \tag{5.5.17}$$

Taking the q -th power and summing up the desired estimates of \sum' and \sum'' follow, where we used $h_{r_1} = L_{r_1} \supset B_{p_1 q_1}^s$ and $h_{r_2} = L_{r_2} \supset B_{p_2 q_2}^s$.

Substep 1.2 (Estimate of \sum'''). Let $\frac{1}{t} = \frac{1}{p_1} + \frac{1}{p_2}$. We assume $0 < q \leq \min(1, t) = u$. Using Proposition 2.2.1/(ii) we derive

$$\begin{aligned} & \left\| \sum''' (f, g) |B_{tq}^{2s} \right\|^q \\ & \leq c \max_{-1 \leq j \leq 1} \sum_{k=0}^{\infty} 2^{2ksq} \left(\sum_{l=-2}^{\infty} 2^{l(n(1/u-1)u} \|f_{k+l} \cdot g_{k+l+j} |L_t\|^u \right)^{q/u} \\ & \leq c \max_{-1 \leq j \leq 1} \sum_{l=-2}^{\infty} 2^{l(n(1/u-1)-2s)q} \\ & \quad \times \sum_{k=0}^{\infty} 2^{2(k+l)sq} \|f_{k+l} |L_{p_1}\|^q \|g_{k+l+j} |L_{p_2}\|^q. \end{aligned} \tag{5.5.18}$$

Because of

$$n \left(\frac{1}{t} - 1 \right) = n \left(\frac{1}{p_1} + \frac{1}{p_2} - 1 \right) = n \left(\frac{1}{p} + \frac{s}{n} - 1 \right) \tag{5.5.19}$$

(cf. (4.2.4)) and $s > 0$ we have $2s > n \left(\frac{1}{u} - 1 \right)$ and hence the right-hand side can be estimated from above by $c \|f |B_{p_1 q_1}^s\|^q \|g |B_{p_2 q_2}^s\|^q$. In addition we have the embedding $B_{tq}^{2s} \subset B_{pq}^s$. This follows from Theorem 3.2.1 and (4.2.4) (see also (5.5.19)). This proves (4.2.5) in case $0 < q \leq \min(1, t)$. If $q > \min(1, t)$, one has to modify the above estimate by using the triangle inequality in $l_{q/u}$. Estimates (5.5.16) - (5.5.18) together with (2.2.2) prove (4.2.5).

Step 2 (Proof of (4.2.7)). Substep 2.1 (Estimate of \sum', \sum''). Because of $F_{p_1 q_1}^s \subset L_{r_1} = h_{r_1}$ and $F_{p_2 q_2}^s \subset L_{r_2} = h_{r_2}$ without restrictions on q_1, q_2 (cf. Theorem 3.2.1/(ii) and Remark 2.1.1/first item) we may apply Proposition 2.2.1/(i) to obtain

$$\left\| \sum' (f, g) |F_{pq}^s \right\| \leq c \|g |F_{p_2 q}^s\| \|f |h_{r_1}\| \leq c' \|g |F_{p_2 q_2}^s\| \|f |F_{p_1 q_1}^s\|. \tag{5.5.20}$$

Here we have used in addition $q \geq q_2$. Similar one derives

$$\left\| \sum'' (f, g) |F_{pq}^s| \right\| \leq c \|g|F_{p_2q_2}^s\| \|f|F_{p_1q_1}^s\|. \tag{5.5.21}$$

Substep 2.2 (Estimate of \sum'''). Again we put $\frac{1}{i} = \frac{1}{p_1} + \frac{1}{p_2}$. As in Substep 1.2 we know that $2s > n(\frac{1}{i} - 1)$ (cf. (4.2.4) and (5.5.19)). This ensures the applicability of Proposition 2.2.1/(iii) and we arrive at

$$\left\| \sum''' (f, g) |F_{i\infty}^{2s}| \right\| \leq c \|f|F_{p_1q_1}^s\| \|g|F_{p_2q_2}^s\|. \tag{5.5.22}$$

The embedding $F_{i\infty}^{2s} \subset F_{pq}^s$ (cf. Theorem 3.2.1/(ii) and (4.2.4)) complements the estimate of \sum''' . Now (4.2.7) follows from (5.5.20) - (5.5.22). ■

5.6 Proofs of the assertions in Subsection 4.3.

Proof of Theorem 4.3.1. Concerning the sufficiency part one can follow the same arguments as in proof of Theorem 4.2.1, where one has to use now Theorem 3.3.1. The necessity part is covered by Proposition 4.3.1 and [7], where the last reference is used to prove $q \geq \max(q_1, q_2)$ ■

Proof of Theorem 4.3.2. We have to prove the “only if”-part only (cf. Remark 4.3.2).

Step 1. Let $f \in A_{p_1q_1}^0$ and assume to have (4.3.6). Then the operator $T_f : g \rightarrow f \cdot g$ is bounded from $A_{p_2q_2}^0$ into A_{pq}^0 . Using Fourier multiplier assertions (cf. [24: Theorem 2.3.7; p. 57]) one derives that T_{f^j} yields a uniformly bounded family of those operators, where f^j is given by (2.2.1).

Let $g \in L_{p_2}$. Then $g^k \in L_{p_2}$ and $\|g^k|L_{p_2}\| \leq c\|g|L_{p_2}\|$ with c independent of g and k . Note that the Fourier image of $g^k e^{ic2^k x_1}$ is concentrated near $|\xi| \approx 2^k$ if c is large enough (cf. (2.2.1) and (2.1.3)). Let $k > j$. Then the same is true in case of $f^j g^k e^{ic2^k x_1}$. Hence

$$\begin{aligned} \|f^j g^k |L_p\| &= \|f^j g^k e^{ic2^k x_1} |L_p\| \\ &\approx \|f^j g^k e^{ic2^k x_1} |A_{pq}^0\| \\ &\leq c \|f^j |A_{p_1q_1}^0\| \|g^k e^{ic2^k x_1} |A_{p_2q_2}^0\| \\ &\leq c \|f |A_{p_1q_1}^0\| \|g^k e^{ic2^k x_1} |L_{p_2}\|. \end{aligned} \tag{5.6.1}$$

Consequently we have $\|f^j g^k |L_p\| \leq c(f) \|g |L_{p_2}\|$ for all $g \in L_{p_2}$. Let j be fixed. Then for $k \rightarrow \infty$ we get $\|f^j g |L_p\| \leq c(f) \|g |L_{p_2}\|$ using the Fatou lemma. By standard arguments we conclude that $f^j \in L_{p_1}$ and moreover, by (5.6.1) we conclude that $\|f^j |L_{p_1}\| \leq c(f) \leq c \|f |A_{p_1q_1}^0\|$. Let $1 < p_1 < \infty$. Then using again Fatou’s lemma we obtain $f \in L_{p_1}$. The case $p_1 = \infty$ can be covered by a Lebesgue point argument. If

$p_1 = 1$, then $\|f^j|L_1\|$ in the above inequality can be replaced by $\|f^j|h_1\|$ (cf. Remark 3.3.4). Then again by Fatou's lemma we obtain that $f \in h_1 \subset L_1$. Hence

$$A_{p_1q_1}^0 \subset L_{p_1}. \tag{5.6.2}$$

Step 2. It remains to check

$$L_p \subset A_{pq}^0. \tag{5.6.3}$$

We may assume $q < \infty$. In case $q = \infty$ nothing more is to prove with the exception of $A_{pq}^0 = F_{1,\infty}^0$. The latter case will be considered in Substep 2.5. Further, observe that Step 1 implies $q_2 < \infty$.

Substep 2.1. Let $A_{p_2q_2}^0 = B_{p_2q_2}^0$ and $A_{pq}^0 = B_{pq}^0$. By duality (cf. [24: Theorem 2.11.2, p.178]) (4.3.6) leads to $A_{p_1q_1}^0 B_{p'q'}^0 \subset B_{p_2q_2}^0$ (we put $q' = \infty$ if $q \leq 1$). As in Step 1 this yields

$$B_{p'q'}^0 \subset L_{p'}. \tag{5.6.4}$$

Using either $B_{p'q'}^0 \subset L_{p'}$ if and only if $q' \leq \min(p', 2)$, $p' \neq 1$ or $B_{\infty q'}^0 \subset L_\infty$ if and only if $q' \leq 1$, (5.6.4) gives $L_p \subset B_{pq}^0$ (cf. Theorems 3.1.1 and 3.3.2) which is (5.6.3) in our case here.

Substep 2.2. Let $A_{p_2q_2}^0 = F_{p_2q_2}^0$ and $A_{pq}^0 = F_{pq}^0$ and assume $p > 1$ and $p_2 > 1$. To use the duality argument is a little bit more complicated than in Substep 2.1. Temporarily we restrict us to $1 < q_2 < \infty$ and $1 < q < \infty$. Under these conditions the duality argument works (cf. again [24: Theorem 2.11.2, p.178]) and we arrive at $A_{p_1q_1}^0 F_{p'q'}^0 \subset F_{p_2q_2}^0$. By Step 1 this implies $F_{p'q'}^0 \subset L_{p'}$ which gives (5.6.3) by using again duality. If $q_2 \leq 1$ and/or $q \leq 1$, then (4.3.6) yields $A_{p_1q_1}^0 B_{p_2q_2}^0 \subset F_{pq}^0$ (cf. Theorem 3.1.1). By duality we find $F_{p_1q_1}^0 (F_{pq}^0)' \subset B_{p_2q_2}^0$. Step 1 gives $(F_{pq}^0)' \subset L_{p'}$. Using the monotonicity of the F -spaces with respect to q , (5.6.3) follows.

Substep 2.3. Let $A_{p_2q_2}^0 = F_{p_2q_2}^0$ and $A_{pq}^0 = F_{1,q}^0$. The proof runs the same way as in Step 1 and Substep 2.2 if one takes into account $(F_{1,q}^0)' = F_{\infty,q'}^0$ ($1 \leq q < \infty$) and $(F_{1,q}^0)' = B_{\infty\infty}^0$ ($0 < q \leq 1$) (cf. [8, 14] and [24: p.180]). Furthermore

$$\|g^k e^{i c 2^k x_1} |F_{\infty,q'}^0\| \approx \|g^k |B_{\infty\infty}^0\| \approx \|g^k |L_\infty\|$$

where the latter one follows from $B_{\infty q'}^0 \subset F_{\infty q'}^0 \subset B_{\infty\infty}^0$.

Substep 2.4. Let $A_{p_2q_2}^0 = F_{1,q_2}^0$ and $A_{pq}^0 = F_{1,q}^0$ with $q < \infty$. Then necessarily we have $A_{p_1q_1}^0 = B_{\infty,q_1}^0$. We use that (4.3.6) implies

$$B_{\infty q_1}^0 B_{1,\min(1,q_2)}^0 \subset F_{1,q}^0. \tag{5.6.5}$$

Now we can argue as in Substep 2.3.

Substep 2.5. It remains (5.6.5) with $q = \infty$. Restricting to completions of S in the involved spaces we may replace $F_{1,\infty}^0$ by $f_{1,\infty}^0$ (see the end of Subsection 5.1). Then we can use the arguments in (5.1.16) and (5.1.17) which disprove this possibility. ■

Proof of Corollary 4.3.1. Theorem 3.1.1 and Corollary 3.3.1 imply

$$\begin{aligned} B_{pq}^0 \subset L_p &\iff q \leq \min(p, 2) \\ F_{pq}^0 \subset L_p &\iff q \leq 2 \end{aligned} \quad (1 \leq p < \infty).$$

In view of these equivalences the corollary is a reformulation of Theorem 4.3.2. ■

Proof of Corollary 4.3.2. Our method in proving Theorem 4.3.2 depends on the duality procedure. If we switch from B_{pq}^0 to b_{pq}^0 , the closure of S in B_{pq}^0 , then the restriction $p_2 < \infty$ becomes superfluous (cf. [24: Remark 2.11.2/2, p. 180]). The result is formulated as a consequence of (5.6.2) and (5.6.3). ■

5.7 Proofs of the assertions in Subsection 4.4.

Proof of Theorem 4.4.1. Step 1 (Proof of (4.4.4)/sufficiency part). *Substep 1.1* (Estimate of $\sum'(f, g)$ and $\sum''(f, g)$). Thanks to Proposition 2.2.1 we know

$$\left\| 2^{js} \mathcal{F}^{-1} \left[\varphi_j \mathcal{F} \sum'(f, g) \right] (\cdot) \Big|_{L_p} \right\| \leq c (2^{js} \|g_j\|_{L_{p_2}}) \|f\|_{h_{r_1}} \quad (5.7.1)$$

where $\frac{1}{p} = \frac{1}{p_2} + \frac{1}{r_1}$. Theorems 3.1.1 and 3.2.1 tell us that we can apply (5.7.1) with

$$\frac{1}{n} \left(\frac{n}{p_1} - s \right)_+ < \frac{1}{r_1} < \frac{1}{p_1} \quad (5.7.2)$$

in the same way as in proof of Theorem 4.2.1 given in Subsection 5.5. Similar we obtain

$$\left\| 2^{js} \mathcal{F}^{-1} \left[\varphi_j \mathcal{F} \sum''(f, g) \right] (\cdot) \Big|_{L_p} \right\| \leq c (2^{js} \|f_j\|_{L_{p_1}}) \|g\|_{h_{r_2}} \quad (5.7.3)$$

and now $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{r_2}$ holds. As above we derive the restrictions

$$\frac{1}{n} \left(\frac{n}{p_2} - s \right)_+ < \frac{1}{r_2} < \frac{1}{p_2} \quad (5.7.4)$$

for $1/r_2$. Relations (5.7.2) and (5.7.4) yield that (5.7.1) and (5.7.3) are applicable simultaneously if

$$\max_{i \neq j} \left(\frac{1}{n} \left(\frac{n}{p_i} - s \right)_+ + \frac{1}{p_j} \right) < \frac{1}{p} < \frac{1}{p_1} + \frac{1}{p_2} \quad (5.7.5)$$

But this is ensured by (4.4.1) and (4.4.3).

Substep 1.2 (Estimate of $\sum'''(f, g)$). Let $\frac{1}{t} = \frac{1}{p_1} + \frac{1}{p_2}$. We put $\min(1, t) = u$. Since $2s > n(\frac{1}{u} - 1)$ (cf. (4.4.2)) we derive as in (5.5.18)

$$\left\| \sum'''(f, g) \Big|_{B_{tq}^{2s}} \right\| \leq c \|f\|_{B_{p_1 q_1}^s} \|g\|_{B_{p_2 q_2}^s}.$$

Because of $B_{tq}^{2s} \subset B_{pq}^s$ (cf. Theorem 3.2.1 and (4.4.3); see also (5.5.19)) this proves the desired estimate in case of $\sum'''(f, g)$. This completes the proof of (4.4.4).

Step 2 (Proof of (4.4.6)/sufficiency part). The proof is similar to that given in Step 1 (cf. also the proof of Theorem 4.2.1/(ii) in Subsection 5.5). Proposition 2.2.1/(i) and $q \geq \max(q_1, q_2)$ yield

$$\begin{aligned} \left\| \sum' (f, g) |F_{pq}^s| \right\| &\leq c \|g|F_{p_2q_2}^s\| \|f|h_{r_1}\| \leq c \|g|F_{p_2q_2}^s\| \|f|F_{p_1q_1}^s\| \\ \left\| \sum'' (f, g) |F_{pq}^s| \right\| &\leq c \|f|F_{p_1q_1}^s\| \|g|h_{r_2}\| \leq c \|f|F_{p_1q_1}^s\| \|g|F_{p_2q_2}^s\| \end{aligned}$$

if (5.7.5) holds, which is guaranteed by (4.4.1) and (4.4.3). To estimate \sum''' we may apply (5.5.22) once again. ■

Remark 5.7.1. The proof shows that (4.4.4) and (4.4.6) remain valid not only in the interior of the shaded area in Figures 3a and 3b, they are true also on the vertical lines of the boundary, may be with exception of the endpoints (note that (4.4.1) is used only to establish (5.7.5)). With the help of Theorems 3.1.1 and 3.2.1 one has to check under which conditions $<$ in (5.7.2), (5.7.4) and (5.7.5) can be replaced by \leq (cf. [11] and [18] for a more detailed explanation).

References

- [1] Amann, H.: *Multiplication in Sobolev and Besov spaces*. Estrado da Nonlinear Analysis, Scuola Normale Superiore, Pisa 1991, 27 - 50.
- [2] Bachelot, A.: *Formes quadratiques A-compatibles dans les espace de type L^p* . Publ. Anal. Appl. Univ. Bourdaux I. n° 8410 (1984), 1 - 37.
- [3] Bony, J. M.: *Calcul symbolique et propagation des singularites pour les equation aux derivees partielles non lineaires*. Ann Sci. Ecole Norm. Sup. 14 (1981), 209 - 246.
- [4] Bourdaud, G.: *Personal communication*. Jena 1992.
- [5] Edmunds, D. and H. Triebel: *Eigenvalue distributions of some degenerate elliptic differential operators : an approach via entropy numbers*. Math. Ann. 299 (1994), 311 - 340.
- [6] Edwards, R. E.: *Fourier Series. A Modern Introduction*. Vol. II. New York: Springer - Verlag 1982.
- [7] Franke, J.: *On the spaces $F_{p,q}^s$ of Triebel-Lizorkin type: Pointwise multipliers and spaces on domains*. Math. Nachr. 125 (1986), 29 - 68.
- [8] Frazier, M. and B. Jawerth: *A discrete transform and decomposition of distribution spaces*. J. Funct. Anal. 93 (1990), 34 - 170.
- [9] Hanouzet, B.: *Applications bilineaires compatibles avec un systeme a coefficients variables continuite dans les espaces de Besov*. Comm. Part. Diff. Equ. 10 (1985), 433 - 465.
- [10] Jawerth, B.: *Some observations on Besov and Lizorkin-Triebel spaces*. Math. Scand. 40 (1977), 94 - 104.
- [11] Johnsen, J.: *Pointwise multiplication of Besov and Triebel-Lizorkin spaces*. Math. Nachr. (to appear).

- [12] Kaljabin, G. A.: *Multiplier conditions of function spaces of Besov- and Lizorkin-Triebel type* (in Russian). Dokl. Akad. Nauk SSSR 251 (1980), 25 - 26.
- [13] Kaljabin, G. A.: *Criteria of the multiplication property and the embedding in C of spaces of Besov-Lizorkin-Triebel type* (in Russian). Mat. Zametki 30 (1981), 517 - 526.
- [14] Marschall, J.: *Some remarks on Triebel spaces*. Studia Math. 87 (1987), 79 - 92.
- [15] Maz'ya, V. G. and T. O. Shaposnikova: *Theory of multipliers in spaces of differentiable functions*. Boston: Pitman 1985.
- [16] Meyer, Y.: *Remarques sur un theoreme de J. M. Bony*. Suppl. Rend. Circ. Mat. Palermo 1 (1981), 1 - 20.
- [17] Runst, Th.: *Mapping properties of non-linear operators in spaces of Triebel-Lizorkin and Besov type*. Anal. Math. 12 (1986), 313 - 346.
- [18] Runst, T. and W. Sickel: *Sobolev Spaces of Fractional Order, Nemytzkij Operators, and Nonlinear Partial Differential Equations* (forthcoming book.)
- [19] Schmeisser, H.-J. and H. Triebel: *Topics in Fourier Analysis and Function Spaces*. Leipzig: Geest & Portig 1987, and Chichester: Wiley 1987.
- [20] Sickel, W.: *On pointwise multipliers in Besov-Triebel-Lizorkin spaces* (Teubner-Texte zur Mathematik: Vol. 96). Leipzig: B. G. Teubner Verlagsges. 1987.
- [21] Sickel, W.: *Pointwise multiplication in Triebel-Lizorkin spaces*. Forum Math. 5 (1993), 73 - 91.
- [22] Strichartz, R. S.: *Multipliers on fractional Sobolev spaces*. J. Math. Mechanics 16 (1967), 1031 - 1060.
- [23] Triebel, H.: *Spaces of Besov-Sobolev-Hardy type* (Teubner-Texte zur Mathematik: Vol. 9). Leipzig: B. G. Teubner Verlagsges 1978.
- [24] Triebel, H.: *Theory of Function Spaces*. Leipzig: Akad. Verlagsges. Geest & Portig K.G. 1983, and Basel-Boston-Stuttgart: Birkhäuser Verlag 1983.
- [25] Triebel, H.: *Theory of Function Spaces* (Extended Russian translation of [24]). Moskva: Mir 1986.
- [26] Triebel, H.: *Theory of Function Spaces*. Part II. Basel-Boston-Stuttgart: Birkhäuser Verlag 1992.
- [27] Triebel, H.: *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces*. Proc. London Math. Soc. 66 (1993), 589 - 618.
- [28] Triebel, H.: *A localization property for B_{pq}^s and F_{pq}^s spaces*. Studia Math. 109 (1994), 183 - 195.
- [29] Valent, V.: *A property of multiplication in Sobolev spaces. Some applications*. Rend. Sem. Mat. Univ. Padova 74 (1985), 63 - 73.
- [30] Yamazaki, M.: *A quasi-homogeneous version of paradifferential operators. Part I: Boundedness on spaces of Besov type*. J. Fac. Sci. Univ. Tokyo (Sect. IA Math.) 33 (1986), 131 - 174.
- [31] Yamazaki, M.: *A quasi-homogeneous version of paradifferential operators. Part II: A symbol calculus*. J. Fac. Sci. Univ. Tokyo (Sect. IA Math.) 33 (1986), 311 - 345.
- [32] Yamazaki, M.: *A quasi-homogeneous version of microlocal analysis for nonlinear partial differential equations*. Japan J. Math. 14 (1988), 225 - 260.
- [33] Youssfi, A.: *Localisation des espaces de Lizorkin-Triebel homogenes*. Math. Nachr. 147 (1990), 107 - 121.

- [34] Zolesio, J. L.: *Multiplication dans les espaces de Besov*. Proc. Royal Soc. Math. Edinburgh 78 A (1977), 113 - 117.

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