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Ricci curvature in dimension 2

Abstract. We prove that in two dimensions the synthetic notions of lower bounds on sectional and on Ricci curvature coincide.

Keywords. RCD spaces, Alexandrov spaces, synthetic Ricci curvature, conformal parametrization

1. Introduction

1.1. The result

In this note we provide an affirmative answer to a well-known conjecture in the theory of spaces with synthetic bounds on Ricci curvature, formulated in print by Cédric Villani [50, Open Problem 9].

Theorem 1.1. *Let (X, d) be a metric space and let $\mathcal{H}^2$ be the 2-dimensional Hausdorff measure on X. If $(X, d, \mathcal{H}^2)$ is an* RCD$(\kappa, 2)$ *space then X is an Alexandrov space of curvature at least κ.*

The converse of our theorem is due to Anton Petrunin [43]. A combination of several recent results [6, 15, 21, 26, 32, 41] implies that the claim of Theorem 1.1 extends to all *compact* RCD$(\kappa, 2)$ spaces (X, d, μ) with arbitrary measures μ.

As a consequence of Theorem 1.1, all RCD$(\kappa, 2)$ spaces $(X, d, \mathcal{H}^2)$ are topological surfaces, possibly with boundary. As another consequence, any RCD$(\kappa, 2)$ space $(X, d, \mathcal{H}^2)$ is a metric-measure Gromov–Hausdorff limit of a sequence of 2-dimensional smooth Riemannian manifolds with convex boundaries and curvature at least κ.

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1.2. Theory of RCD spaces

The synthetic theory of spaces with lower Ricci curvature bounds, initiated in [49] and [35], has experienced a tremendous growth over the last 15 years. We refer the reader to the surveys [50] and [1] for an overview of the current state of the theory and the huge bibliography.

The analytic aspects of the theory are highly developed. Most results, previously known on Riemannian manifolds with lower bounds on Ricci curvature, now possess natural generalizations to the synthetic framework: see for instance [3,9,18]. Moreover, there is a sophisticated understanding of measure-theoretic, or rather "almost everywhere", properties of such spaces [6,41].

On the other hand, gaining information on the "everywhere" metric structure, or even less ambitiously, on the topological structure of RCD spaces, seems to be very difficult. Only a few results in this direction are known, all of which require either a rigid setting [14, 17], or strong a priori assumptions on the geometry [29, 32]. Our approach is based on recent advances of minimal surface theory in metric spaces [36, 37, 39, 40, 48].

1.3. Central steps in the proof

The proof is a combination of analytic RCD-techniques, some basic results about Alexandrov spaces, the canonical uniformization of singular surfaces introduced in [40] and Reshetnyak's analytic theory of surfaces with integral curvature bounds [46].

In the first step, we use Bishop–Gromov comparison and the cone-rigidity theorem [14] to study the infinitesimal structure of a space X as in Theorem 1.1. More precisely, it is shown that unique tangent spaces exist at all points and are Euclidean cones over either intervals or circles. In analogy with the theory of Alexandrov spaces, we call the set of points at which the tangent space is homeomorphic to a half-plane the *boundary* of X and denote it by ∂X. Another application of Bishop–Gromov comparison implies that ∂X is a closed subset of X. This is the content of Section 2.

In Section 3, we consider, for any $\delta > 0$, the set X^δ of points at which the density of X is δ-close to the density of the Euclidean plane. The set X^δ is open, dense, disjoint from ∂X, and by the theorem of Cheeger–Colding–Reifenberg [11], it is a topological surface without boundary once δ is small enough. Due to [10], X^δ satisfies an isoperimetric inequality of Euclidean type.

The next two sections are the central pieces of the proof. In Section 4, we use [40] to find for any point $x \in X^\delta$ a small closed neighborhood $\bar{\Omega}$ and a canonical parametrization $\phi : \bar{D} \to \bar{\Omega}$ obtained by solving a Plateau problem. This parametrization is a *quasisymmetric map*, which in addition is *infinitesimally conformal*. This implies, in particular, that there exists a function $f \in L^2(D)$, the *conformal factor*, such that the lengths of *almost all curves* in $\bar{\Omega}$ are obtained by integrating f along the corresponding curve in $\bar{D}$. In other words, the metric in $\bar{\Omega}$ appears to be obtained from the flat metric in $\bar{D}$ by a "singular conformal change with factor f".

In Section 5, we identify the Laplace operator of Ω in terms of D and f and use the Bakry–Émery property of RCD spaces to derive an analytic condition on f. This condition implies log-subharmonicity in the case $\kappa = 0$ and a related property for general κ.

Using [46], we deduce in Section 6 that the metric on Ω is *everywhere* controlled by f, and has curvature bounded below by κ in the sense of Alexandrov.

In order to finish the proof of the main theorem we show that X^δ is a convex subset of X, a statement which is expected to be true in much greater generality [12]. Once the convexity is verified, Toponogov's globalization theorem, in the form of [44], provides that last step. The proof of this convexity in Section 7 relies on the non-branching property of RCD spaces [45]. Assuming that a geodesic between two non-boundary points passes through a point x on the boundary ∂X, we easily obtain many branching geodesics, once we know that X is a topological surface with boundary near x. While, in general, topological control in RCD spaces is very difficult to achieve, here we obtain the required statement by a twofold application of the Cheeger–Colding–Reifenberg theorem.

2. Basic structure

2.1. Notation

We denote by D the open unit disc in $\mathbb{R}^2$ and by $\bar{D}$ its closure. By $\mathcal{H}^k$ or $\mathcal{H}_X^k$ we denote the k-dimensional Hausdorff measure on a metric space X.

In a metric space X we denote by d the distance and by $B_r(x)$ the open ball of radius r around the point x. By $\ell(\gamma) = \ell_X(\gamma)$ we denote the length of a curve γ in X.

2.2. Setting

We assume some familiarity with the synthetic theory of lower Ricci curvature bounds. In particular, we assume that the reader is familiar with the notion of $\mathrm{RCD}(\kappa, N)$ spaces, which we will not define.

In the rest of the paper we fix a space X satisfying the assumptions of Theorem 1.1. Thus X is a geodesic, locally compact metric space. By assumption, the space X is $\mathrm{RCD}(\kappa, 2)$ with respect to the reference measure $\mathcal{H}^2$, whose support is X. In the terminology introduced in [15], this means that X is a *non-collapsed* $\mathrm{RCD}(\kappa, 2)$ space.

For $x \in X$ and $r > 0$ we set

$$b(x, r) = \mathcal{H}^2(B_r(x)). \tag{2.1}$$

Recall that the balls satisfy the *Bishop–Gromov property*, thus, for all $s < r$, the quotient $b(x, r)/b(x, s)$ is bounded from above by the corresponding quotient in the 2-dimensional simply connected Riemannian manifold M_κ^2 of constant curvature κ [49].

Therefore, for any $x \in X$ we have a well-defined positive *density*

$$0 < b(x) := \lim_{r \to 0} \frac{b(x, r)}{\pi r^2} \le 1$$

(see [15, Corollary 2.13]).

Again by the Bishop–Gromov property, the space X is *locally* 2-*Ahlfors regular*: for every compact subset K of X there exists some $L \geq 1$ such that for all $x \in K$ and all $0 < r < 1$ we have

$$\frac{1}{L} r^2 \leq \mathcal{H}^2(B_r(x)) \leq L r^2.$$

2.3. Convergence and blow-ups

Any sequence (Y_i, y_i) of non-collapsed $\mathrm{RCD}(\kappa, 2)$ spaces has a subsequence converging in the pointed measured Gromov–Hausdorff topology to a limit space (Y, y). This limit space Y is an $\mathrm{RCD}(\kappa, 2)$ space with respect to a limiting measure μ [3, 19]. If there is $\epsilon > 0$ such that $\mathcal{H}^2(B_1(y_i)) \geq \epsilon$ for all i, then the limit space Y is non-collapsed and $\mu = \mathcal{H}^2$ [15, Theorem 1.2].

Moreover, in this situation the densities behave semicontinuously:

$$b(y) \leq \liminf b(y_i).$$

In particular, the density function $b : X \to (0, 1]$ is lower semicontinuous.

As a special case of [15, Theorem 1.2], for any sequence $x_i \in X$ converging to a point $x \in X$ and any sequence r_i of positive numbers converging to 0, there is a subsequence such that the rescaled spaces $(\frac{1}{r_i} X, x_i)$ converge to a non-collapsed $\mathrm{RCD}(2, 0)$ space (Y, y). We call any such space (Y, y) a *blow-up* of X.

We claim that for all $z \in Y$ and all $s \geq 0$,

$$\mathcal{H}^2(B_s(z)) \geq \pi b(x) s^2. \tag{2.2}$$

Indeed, by the Bishop–Gromov inequality in X and continuity of $\mathcal{H}^2$ with respect to convergence, we get for every $t > 0$,

$$\frac{\mathcal{H}^2(B_t(y))}{\pi t^2} = \lim_{i \to \infty} \frac{\mathcal{H}^2(B_{r_i t}(x_i))}{\pi (r_i t)^2} \geq b(x).$$

On the other hand, by the Bishop–Gromov inequality in Y,

$$\frac{\mathcal{H}^2(B_s(z))}{\pi s^2} \geq \lim_{t \to \infty} \frac{\mathcal{H}^2(B_t(y))}{\pi t^2}.$$

2.4. Tangent cones

A blow-up for a constant sequence of base points $x_i = x$ is called a *tangent cone* of X at the point x. Relying on the volume rigidity of metric cones [14], it has been shown in [15, Proposition 2.7] that any such tangent cone T of X at the point x is isometric to a Euclidean cone $C(Z)$ over a compact space Z. Moreover, Z is an $\mathrm{RCD}(0, 1)$ space of diameter at most π [31].

As has been proved in [32], such a space Z is a closed interval or a circle. Due to the stability of $\mathcal{H}^2$ and the definition of $b(x)$, the density $b(x)$ coincides with the density

of $C(Z)$ at the vertex 0_x of the cone $C(Z)$. Therefore,

$$b(x) = \frac{1}{2\pi} \mathcal{H}^1(Z).$$

Thus, at any point $x \in X$ at most two different tangent cones can exist.

However, the following lemma, valid for all doubling metric measure spaces and probably well-known to experts, shows that there cannot exist exactly two different tangent cones.

Lemma 2.1 ([19]). *The space $\mathcal{T}^x$ of all metric measure tangent cones $(Y, y, \mathcal{H}^2)$ at the point $x \in X$ is a connected subset of the space $\mathbb{X}$ of isomorphism classes of all pointed, proper metric measure spaces.*

Proof. Consider the map $F : (0, 1] \to \mathbb{X}$ which sends a number $r > 0$ to the pointed metric measure space $(\frac{1}{r} X, x, \mathcal{H}^2)$. The map is continuous and the image of F has compact closure in $\mathbb{X}$.

The set $\mathcal{T}^x$ consists of all limit points $\lim_{r_i \to 0} F(r_i)$ of this continuous curve. Thus,

$$\mathcal{T}^x = \bigcap_{n \in \mathbb{N}} \overline{F(0, 1/n]}$$

is the intersection of a nested sequence of compact sets. Each element of the sequence is the closure of a curve, hence is connected. Since the intersection of any nested sequence of compact connected metric spaces is connected, so is $\mathcal{T}^x$. ∎

Thus, we have shown

Corollary 2.2. *At any $x \in X$ there exists a unique tangent cone. This tangent cone is a Euclidean cone $C(Z)$, where Z is a circle or an interval. Moreover, $\mathcal{H}^1(Z) = b(x) \cdot 2\pi$ and the diameter of Z is at most π.*

We denote the unique tangent cone at $x \in X$ by $T_x = T_x X$. By the above, T_x is homeomorphic to a plane or to a half-plane.

Remark 2.3. We mention that an application of the analog of Lemma 2.1 for 3-dimensional RCD spaces, together with [15, 26], Perelman's stability theorem [8, 28] and our main theorem, implies the following stability result for weakly non-collapsed RCD$(\kappa, 3)$ spaces Y:

- For any $y \in Y$ all metric measure tangent cones of Y at y are pairwise homeomorphic.

For non-collapsed limits of Riemannian manifolds this confirms [13, Conjecture 1.2] in dimension 3, which can also be deduced from [47].

2.5. The boundary

We define the *boundary* ∂X of X to be the set of all points $x \in X$ for which the tangent space $T_x X$ is homeomorphic to a half-plane.

Lemma 2.4. *The boundary ∂X is a closed subset of X which contains only points x with $b(x) \leq 1/2$.*

Proof. By definition, $x \in \partial X$ if and only if T_x is isometric to $C(Z)$, where Z is an interval. Since Z has length $b(x) \cdot 2\pi$ and diameter at most π, this implies $b(x) \leq 1/2$.

Let $x \in X \setminus \partial X$ be arbitrary. Then T_x is a cone over a circle, thus for any point z in T_x but the vertex 0_x, the density of T_x at z is 1. For any sequence $x_i \in X \setminus \{x\}$ converging to x, we choose $r_i = d(x, x_i)$. Then (possibly after choosing a subsequence) under the convergence of $(\frac{1}{r_i} X, x)$ to $(T_x, 0_x)$ the points x_i converge to a point z at distance 1 from 0_x. The semicontinuity of densities implies that $\lim_{i \to \infty} b(x_i) = 1$. In particular, x_i is not in ∂X for i large enough.

Hence, $X \setminus \partial X$ must be open. ∎

As a consequence of the splitting theorem we obtain

Lemma 2.5. *Let $x \in X$ be a point which is an interior point of a geodesic. If $x \in \partial X$ then T_x is isometric to the Euclidean half-plane. If $x \in X \setminus \partial X$ then $T_x X$ is isometric to $\mathbb{R}^2$.*

Proof. The assumption implies that $T_x X$ contains a line (the tangent space to the geodesic through x). By the splitting theorem [17], the space T_x splits off a line. This implies the claim, since T_x is a cone over an interval or a circle. ∎

3. Almost regular parts

For any $\delta > 0$, we call a point $x \in X$ a δ-*regular point* if $b(x) > 1 - \delta$. We denote by X^δ the set of all δ-regular points in X. We have the following discreteness statement:

Lemma 3.1. *The set X^δ is open in X for any $\delta > 0$. For any $\delta < 1/2$, the set X^δ is disjoint from ∂X and the complement $(X \setminus \partial X) \setminus X^\delta$ is discrete in $X \setminus \partial X$.*

Proof. The semicontinuity of the density function shows that X^δ is open. Due to Lemma 2.4, the set X^δ is disjoint from ∂X for $\delta < 1/2$.

Finally, the last argument in the proof of Lemma 2.4 implies that any point in $X \setminus \partial X$ has a punctured neighborhood completely contained in X^δ. This implies the last claim. ∎

The following observation is a very special and rough case of the results obtained in [11] and [10] (see also [30]).

Lemma 3.2. *There exist $\delta, C > 0$ with the following property. Every point $x \in X^\delta$ has a neighborhood U_x, homeomorphic to D, such that for any subset K of U_x homeomorphic to $\bar{D}$,*

$$\mathcal{H}^2(K) \leq C \cdot (\mathcal{H}^1(\partial K))^2. \tag{3.1}$$

Proof. Due to [15, Theorem 1.2] and the Cheeger–Colding–Reifenberg theorem [11, Theorem A.1], there exists some $\delta > 0$ such that any point $x \in X^\delta$ has a neighborhood U_x homeomorphic to D.

It remains to prove that, for sufficiently small δ, we have the isoperimetric inequality (3.1) for any closed topological disc K in some neighborhood of any point in X^δ.

This statement is proved in [10, Corollary 1.6] (in a much more general and precise form) with $\mathcal{H}^1(\partial K)^2$ on the right hand side of (3.1) replaced by $m(K)^2$, where $m(K)$ is the *outer Minkowski content*

$$m(K) := \liminf_{r \to 0} \frac{\mathcal{H}^2(B_r(K)) - \mathcal{H}^2(K)}{r}.$$

Here $B_r(K)$ denotes the set of points with distance at most r to K.

It suffices to prove $4\pi \cdot \mathcal{H}^1(\partial K) \geq m(K)$ for any subset K of X homeomorphic to a closed disc (cf. [16, Theorem 3.2.39]).

If ∂K is not rectifiable there is nothing to prove. Otherwise, for any natural n, we set $r = \frac{1}{2n}\mathcal{H}^1(\partial K)$ and find an r-dense subset A_n in ∂K with n points. Then $B_r(\partial K) \subset B_{2r}(A_n)$.

By the Bishop–Gromov property, we deduce, for r small enough,

$$\mathcal{H}^2(B_r(\partial K)) \leq \mathcal{H}^2(B_{2r}(A_n)) \leq n \cdot 2 \cdot \pi \cdot (2r)^2 = 4\pi r \mathcal{H}^1(\partial K).$$

This implies $m(K) \leq 4\pi \mathcal{H}^1(\partial K)$ and finishes the proof.   ∎

4. Conformal parametrization

4.1. Choice of a domain

Let $\delta, C > 0$ be as in Lemma 3.2, let $x_0 \in X^\delta$ be arbitrary and let U_{x_0} be an open neighborhood of x_0 in X provided by Lemma 3.2.

Any Jordan curve Γ in U_{x_0} determines a *Jordan domain* $\Omega \subset U_{x_0}$, homeomorphic to D, such that $\bar{\Omega} = \Omega \cup \Gamma$ is homeomorphic to $\bar{D}$.

Starting with any Jordan curve Γ in U_{x_0} whose Jordan domain Ω contains x_0, we can replace Γ by a nearby curve and assume that Γ is biLipschitz to the round circle S^1 [40, Lemma 4.3].

We fix this curve Γ and domain Ω for the rest of the section.

4.2. Metric properties of Ω

Consider the set $\bar{\Omega}$ with its intrinsic metric d_Ω. Clearly, $d_X \leq d_\Omega$ on $\bar{\Omega}$. Moreover, on Ω the metrics d_X and d_Ω coincide locally.

Since Γ is a biLipschitz embedding of a circle, the metric d_Ω is biLipschitz to the induced metric d_X on $\bar{\Omega}$. Indeed, for any curve γ in X connecting a pair of points in $\bar{\Omega}$, we can replace the part of γ between its first and last intersection point with Γ by the shorter part of Γ between these intersection points. This new curve $\hat{\gamma}$ is contained in $\bar{\Omega}$. Moreover, its length is at most a multiple of the length of γ where the factor $c \geq 1$ is the biLipschitz constant of Γ. Hence, $d_\Omega \leq c d_X$ on $\bar{\Omega}$.

In order to apply the parametrization results of [40], we need to make sure that $\bar{\Omega}$ is Ahlfors 2-regular and is *linearly locally connected* in the following sense.

A *continuum* is a compact connected space. A metric space Y is called *linearly locally connected* if there exists a constant $0 < C < 1$ such that for all $y \in Y$ and all $0 < r <$ diam(Y) the following holds true. Any pair of points $z_1, z_2 \in B_{Cr}(y)$ is contained in a continuum $P \subset B_r(y)$; any pair of points $z_1, z_2 \in Y \setminus B_r(y)$ is contained in a continuum $P \subset Y \setminus B_{Cr}$.

If the space Y is geodesic, the first condition is always satisfied. Linear local connectedness is preserved under biLipschitz transformations.

Lemma 4.1. *The space* $(\bar{\Omega}, d_\Omega)$ *is* 2-*Ahlfors regular and linearly locally connected.*

Proof. Since the $\mathcal{H}^2$-measures with respect to d_X and d_Ω coincide on $\bar{\Omega}$, a quadratic upper bound on the $\mathcal{H}^2$-area of balls in Ω follows from the corresponding upper bound on the area of balls in X.

The existence of a lower quadratic bound on the area of balls in $\bar{\Omega}$ is essentially proved in [38, Theorem 9.4], as a consequence of the quadratic isoperimetric inequality (3.1). We provide a simplified version of the argument here.

Relying on the lower bound on the area of balls in X, it is sufficient to find a constant $C_0 < 1$ with the following property. For all small r and any $z \in \Gamma$ the ball $B_{2r}(z)$ contains a point $y \in \Omega$ with distance at least $C_0 r$ to Γ.

For topological reasons, the distance sphere of radius r around z must contain a continuum P, joining two points on Γ, locally separated on Γ by z. From the biLipschitz property of the boundary curve Γ, we now deduce the existence of $C_0 < 1$ such that the continuum P contains a point y as claimed above.

This finishes the proof of the Ahlfors 2-regularity of $\bar{\Omega}$.

In order to prove that $\bar{\Omega}$ is linearly locally connected, recall first from [38, Theorem 8.6] that the isoperimetric property (3.1) implies the following statement. There exists a constant $C_1 > 1$ such that, for any $y \in \bar{\Omega}$ and any $r > 0$, the ball $B_r(y)$ is contractible inside $B_{C_1 r}(y)$.

The space $\bar{\Omega}$ is geodesic and the boundary Γ is linearly locally connected. Thus it suffices to prove the following claim. There exists some $C_2 < 1$ such that, for every $y \in \bar{\Omega}$, every $r < $ diam$(\bar{\Omega})$ and every $z \in \bar{\Omega} \setminus B_r(y)$ there is a curve connecting z with Γ outside of $B_{C_2 r}(y)$.

Assume the contrary. By the Jordan curve theorem, there exists a Jordan curve T in $B_{C_2 r}(y)$ whose Jordan domain contains the point z. But this implies that $B_{C_2 r}(y)$ is not contractible in $B_r(y)$, which contradicts the result of [38, Theorem 8.6], mentioned above, once C_2 is sufficiently small.

This finishes the proof of the lemma. ∎

4.3. Moduli of families of curves and Newton–Sobolev maps

For the convenience of the reader, we recall the notions of moduli of families of curves and of Newton–Sobolev maps with values in a metric space [25, Sections 5–7] in the special case used here.

Let Y be a metric spaces with finite $\mathcal{H}^2(Y)$. For a family $\mathcal{C}$ of curves in Y, a Borel function $\sigma : Y \to [0, \infty]$ is called *admissible for $\mathcal{C}$* if $\int_\gamma \sigma \geq 1$ for every locally rectifiable curve in $\mathcal{C}$. The *modulus* (more precisely the 2-modulus) of the family is defined as

$$\mathrm{mod}(\mathcal{C}) := \inf_\sigma \int_Y \sigma^2 \, d\mathcal{H}^2,$$

where the infimum is taken over all Borel functions admissible for $\mathcal{C}$.

A statement holds for *almost every curve* in Y if the family $\mathcal{C}$ of all curves in Y for which the statement does not hold has modulus 0.

A measurable map $u : Y \to Z$ into a separable metric space Z is in the *Newton–Sobolev space* $N^{1,2}(Y, Z)$ if for some $z \in Z$ the composition $d_z \circ u : Y \to \mathbb{R}$ is in $L^2(Y)$ and the following statement holds true. There exists a function $\rho \in L^2(Y)$, called a *weak upper gradient* of u, such that for almost any curve γ in Y the composition $u \circ \gamma$ is absolutely continuous and

$$\ell(u \circ \gamma) \leq \int_\gamma \rho. \tag{4.1}$$

There is a unique *minimal weak upper gradient* $\rho_u \in L^2(Y)$ of u such that $\rho_u \leq \rho$ almost everywhere for any weak upper gradient ρ of u. The quantity

$$\mathrm{Ch}_Y(u) := \frac{1}{2} \int_Y \rho_u^2 \, d\mathcal{H}^2$$

is called the *Cheeger energy* of u. In [37, 40] the equivalent notion of *Reshetnyak* energy $E_+^2(u) = 2\,\mathrm{Ch}(u)$ has been used.

4.4. Canonical parametrization

We will not recall the definition of *quasisymmetric maps* [23]. Instead we will use the theory of quasisymmetric maps as a black-box, providing references for each required statement.

From [40, Theorem 1.1] and Lemma 4.1 above we deduce

Corollary 4.2. *Among all homeomorphisms $u : \bar{D} \to \bar{\Omega}$ there exists a homeomorphism $\phi \in N^{1,2}(\bar{D}, \bar{\Omega})$ with minimal Cheeger energy $\mathrm{Ch}_{\bar{\Omega}}(\phi)$. The homeomorphism ϕ is quasisymmetric.*

The inverse $\phi^{-1} : \bar{\Omega} \to \bar{D}$ is then quasisymmetric as well [22, Proposition 10.6]. Since the disc $\bar{D}$ satisfies the 1-*Poincaré inequality* [25, Section 8], the space $\bar{\Omega}$ satisfies the q-Poincaré inequality for some $q < 2$ [33, Theorem 2.3].

The map ϕ and its inverse have *Luzin's property N*, thus ϕ and ϕ^{-1} preserve the class of sets of $\mathcal{H}^2$-measure 0 [24, Theorem 8.12].

Due to [23, Theorem 9.3], $\phi \in N^{1,2}(\bar{D}, \bar{\Omega})$ and $\phi^{-1} \in N^{1,2}(\bar{\Omega}, \bar{D})$. By [24, Theorem 9.8], a family $\mathcal{C}$ of curves in $\bar{D}$ has modulus 0 if and only if the image $\phi \circ \mathcal{C}$ of this family has modulus 0 in $\bar{\Omega}$.

4.5. *Conformality and its consequences*

No tangent space of the space X contains a non-Euclidean normed vector space. Due to [37, Proposition 11.2], X has property ET, introduced in [37, Definition 11.1]. The metric on Ω locally coincides with d_X. Since Γ has vanishing $\mathcal{H}^2$-measure the space $\bar{\Omega}$ has property ET as well.

Due to [37, Theorem 11.3], the energy minimizer ϕ is a conformal map in the sense of [37, Definition 6.1], meaning that almost all *approximate metric differentials* of ϕ are multiples of the (fixed) Euclidean norm on $\mathbb{R}^2$. Using [38, Lemma 3.1] this reads

$$\ell_\Omega(\phi \circ \gamma) = \int_\gamma \rho_\phi \tag{4.2}$$

for almost all curves γ in D. Here, as before, ρ_ϕ denotes the minimal weak upper gradient of ϕ.

For any Borel subset $E \subset \bar{D}$ we have [38, Lemma 3.3]

$$\mathcal{H}_X^2(\phi(E)) = \int_E \rho_\phi^2 \, d\mathcal{H}_{\bar{D}}^2. \tag{4.3}$$

Since the inverse ϕ^{-1} has Luzin's property N, the minimal weak upper gradient ρ_ϕ must be positive almost everywhere.

From (4.2), [38, Lemma 3.1] and the absolute continuity on almost all curves of the Sobolev maps ϕ, ϕ^{-1} [25, Proposition 6.3.2], we deduce that for any non-negative Borel function $h : \bar{D} \to \mathbb{R}$ and almost every curve γ in $\bar{D}$,

$$\int_{\phi \circ \gamma} h \circ \phi^{-1} = \int_\gamma \rho_\phi \cdot h. \tag{4.4}$$

Therefore, the Borel function

$$g := \frac{1}{\rho_\phi \circ \phi^{-1}} : \bar{\Omega} \to [0, \infty]$$

satisfies

$$\int_\eta g = \ell_{\bar{D}}(\phi^{-1} \circ \eta)$$

for almost every curve η in $\bar{\Omega}$. This implies that g is the minimal weak upper gradient of $\phi^{-1} \in N^{1,2}(\bar{\Omega}, \bar{D})$.

5. Conformal factor

5.1. Setting and aim

We continue to use the notations from the previous section. Thus, we have a domain $\Omega \subset X$ and a conformal homeomorphism $\phi : \bar{D} \to \bar{\Omega}$ which is contained in $N^{1,2}(\bar{D}, \bar{\Omega})$. We let $f := \rho_\phi : \bar{D} \to [0, \infty]$ be the minimal weak upper gradient of ϕ.

The aim of this section is the following result and its consequence, Corollary 5.2.

Proposition 5.1. *The function f^{-2} is in $L^\infty_{\mathrm{loc}}(D)$. For any harmonic function $v : D \to \mathbb{R}$ the distributional Laplacian $\Delta_D(f^{-2}|\nabla v|^2)$ is a Radon measure on D and satisfies as a measure*

$$\Delta_D(f^{-2}|\nabla v|^2) \geq 2\kappa \cdot |\nabla v|^2 \cdot \mathcal{H}^2_D. \tag{5.1}$$

The proof will be a direct consequence of the Bakry–Émery inequality on the $\mathrm{RCD}(\kappa, 2)$ space X, once we have identified via the conformal homeomorphism ϕ the Sobolev spaces and Laplacians on D with the corresponding objects on X.

5.2. Identifications

Whenever no confusion is possible we will use the homeomorphism ϕ^{-1} to identify $\bar{\Omega}$ with $\bar{D}$.

Due to (4.3), under this identification we have

$$\mathcal{H}^2_X|_\Omega = \mathcal{H}^2_\Omega = f^2 \cdot \mathcal{H}^2_D. \tag{5.2}$$

We are going to identify the space $N^{1,2}(\Omega) = N^{1,2}(\Omega, \mathbb{R})$ of Sobolev functions with the "classical" space $N^{1,2}(D) = W^{1,2}(D)$.

From (4.4) we draw the following conclusion. Let $u : \Omega \to \mathbb{R}$ be measurable. A Borel function $\rho : \Omega \to [0, \infty]$ is a weak upper gradient of u, thus satisfies (4.1) for almost every curve γ in Ω, if and only if $(\rho \circ \phi) \cdot f : D \to [0, \infty]$ is a weak upper gradient of the composition $u \circ \phi : D \to \mathbb{R}$.

Due to (5.2), $\rho \in L^2(\Omega)$ if and only if $(\rho \circ \phi) \cdot f \in L^2(D)$.

Since $\bar{D}$ and $\bar{\Omega}$ satisfy the 2-Poincaré inequality, the 2-integrability of a weak upper gradient implies that the function itself is 2-integrable [25, Lemma 8.1.5, Theorem 9.1.2].

This shows that a map $v : D \to \mathbb{R}$ is in $N^{1,2}(D)$ if and only if $u := v \circ \phi^{-1}$ is in $N^{1,2}(\Omega)$. Moreover, in this case the minimal weak upper gradients satisfy

$$\rho_u = \frac{\rho_v}{f} \circ \phi^{-1} = \frac{|\nabla v|}{f} \circ \phi^{-1}. \tag{5.3}$$

Combining this with (5.2) and identifying D and Ω shows that

$$|\nabla v|^2 \cdot \mathcal{H}^2_D = \rho^2_u \cdot \mathcal{H}^2_\Omega \quad \text{and} \quad \mathrm{Ch}_\Omega(u) = \mathrm{Ch}_D(v). \tag{5.4}$$

5.3. Laplacians

By the RCD property, the spaces X and $\mathbb{R}^2$ are *infinitesimally Hilbertian*, meaning that Ch_X and $\mathrm{Ch}_{\mathbb{R}^2}$ are quadratic forms on $N^{1,2}(X)$ and $N^{1,2}(\mathbb{R}^2)$, respectively. Thus, D and Ω are infinitesimally Hilbertian as well [18, Proposition 4.22].

For $Y = X, \Omega, D, \mathbb{R}^2$ we consider the corresponding bilinear forms $\mathcal{E}_Y : N^{1,2}(Y) \times N^{1,2}(Y) \to \mathbb{R}$, called *Dirichlet forms*,

$$\mathcal{E}_Y(u_1, u_2) = \tfrac{1}{2}(\mathrm{Ch}_Y(u_1 + u_2)^2 - \mathrm{Ch}_Y(u_1 - u_2)^2).$$

From (5.4) we see that for $v_{1,2} \in N^{1,2}(D)$ and $u_{1,2} = v_{1,2} \circ \phi^{-1} \in N^{1,2}(\Omega)$,

$$\mathcal{E}_\Omega(u_1, u_2) = \mathcal{E}_D(v_1, v_2). \tag{5.5}$$

For $Y = D, \Omega$, a function $u \in N^{1,2}(Y)$ is in the domain of the (measure-valued) Laplacian on Y if there exists a Radon measure ν on Y with the following property [18, Definition 4.4, Proposition 4.7, Lemma 4.26]. For all $w \in N^{1,2}(Y)$ continuous and with compact support in Y,

$$\mathcal{E}_Y(u, w) = -\int_Y w \, d\nu.$$

In this case, we set $\mathbf{\Delta}_Y(u) := \nu$.

From (5.5), a function $u \in N^{1,2}(\Omega)$ is in the domain of the Laplacian if and only if $v = u \circ \phi$ is in the domain of the Laplacian on D. Moreover, in this case

$$\mathbf{\Delta}_D(v) = \mathbf{\Delta}_\Omega(u), \tag{5.6}$$

where we identify Radon measures on D and Ω via ϕ.

5.4. The proof of Proposition 5.1

Due to (5.6), for any harmonic function $v : \mathbb{R}^2 \to \mathbb{R}$, the composition $u = v \circ \phi^{-1}$ satisfies $\mathbf{\Delta}_\Omega(u) = 0$, thus u is a *harmonic function on Ω*.

Due to the regularity of harmonic functions on RCD spaces, the function $u : \Omega \to \mathbb{R}$ is locally Lipschitz ([27, Theorem 1.1], which is a combination of [3, Theorem 6.2], [4, Corollary 2.3] and [34, Theorem 1.1, Proposition 5.1]). Applying this observation to the coordinate functions $v_{1,2}$ we deduce that $\phi^{-1} : \Omega \to D$ is locally Lipschitz. Since $\frac{1}{f} \circ \phi^{-1}$ is the minimal weak upper gradient of $\phi^{-1} : \Omega \to D$, we deduce that f^{-1} is locally bounded on D.

Let now $v : D \to \mathbb{R}$ be a harmonic function. Consider the composition $u = v \circ \phi^{-1} \in N^{1,2}(\Omega)$. By (5.6), $\mathbf{\Delta}_\Omega(u) = 0$. Due to (5.3) and (5.4) the right hand side of (5.1) is given by $2\kappa \cdot \rho_u^2 \cdot \mathcal{H}_\Omega^2$, where ρ_u is the minimal weak upper gradient of u. Using (5.3) again, it remains to show the following claim for any open subset $O \subset \bar{O} \subset \Omega$. A representative of ρ_u^2 is in the domain of the Laplacian in $N^{1,2}(O)$ and we have the comparison of measures

$$\mathbf{\Delta}_O(\rho_u^2) \geq 2\kappa \cdot \rho_u^2 \cdot \mathcal{H}_O^2. \tag{5.7}$$

The proof of (5.7) follows from the Bochner inequality [18, Proposition 4.36], [3, Remark 6.3], [2] by localization, as follows.

If the function u is a restriction to O of a *test function* $\hat{u} \in N^{1,2}(X)$ in the sense of [1, (7.2)] then (5.7) is precisely [18, Proposition 4.36], since $\mathbf{\Delta}_O(u) = 0$. In general, we multiply u by a test function which is constant 1 on O and has support in Ω (the existence of such cut-off functions has been verified in [5, Lemma 6.7]). This provides a function $\hat{u} \in N^{1,2}(X)$ which restricts to u on O and is a test function on X [18, Proposition 4.17, Theorem 4.29].

This finishes the proof of Proposition 5.1.

5.5. An analytic conclusion

By a combination of a smoothing argument and a pointwise computation we are going to deduce

Corollary 5.2. *The function* $\log(f^2)$ *is contained in* $L^1_{\mathrm{loc}}(D)$. *The distributional Laplacian* $\Delta(\log(f^2))$ *is a Radon measure on* D *and satisfies*

$$\Delta(\log(f^2)) \leq -2\kappa \cdot f^2 \cdot \mathcal{H}^2_D. \tag{5.8}$$

For any domain O *compactly contained in* D *there exists a sequence of smooth functions* $f_n : O \to (0, \infty)$ *that satisfy* (5.8) *on* O, $\log(f_n)$ *converges to* $\log(f)$ *in* $L^1(O)$ *and* $\Delta(\log(f_n^2))$ *weakly converges to* $\Delta(\log(f^2))$ *as measures on* O.

Proof. Consider the function $h = f^{-2}$. Due to Proposition 5.1, the function h is locally bounded on D. Moreover, for any harmonic function v on $\mathbb{R}^2$, the function h satisfies

$$\Delta(h \cdot |\nabla v|^2) \geq 2\kappa \cdot |\nabla v|^2 \cdot \mathcal{H}^2_D \tag{5.9}$$

in the sense of measures on D. We fix a ball $O = B_{1-\delta}(0) \subset D$ for the rest of the proof. For all $\delta > 2\epsilon > 0$ consider the mollifications $h_\epsilon : O \to \mathbb{R}$ obtained by convolution of h with the usual smooth mollifiers $\rho_\epsilon : \mathbb{R}^2 \to [0, \infty)$ supported in $B_\epsilon(0) \subset \mathbb{R}^2$. Since h is positive almost everywhere, the smooth functions h_ϵ are positive on O. By a direct computation (or using the observation that (5.9) is a system of linear inequalities on the function h, which is moreover equivariant with respect to translations), we see that h_ϵ satisfies (5.9) on O for all harmonic functions v on $\mathbb{R}^2$.

The function h is bounded from above on the ϵ-neighborhood of O, hence so is $\log(h)$. On the other hand, $-\log(h) = \log(1/h) \leq 1/h$. Thus, the integrability of $f^2 = 1/h$ shows $\log(h) \in L^1(O)$.

The convexity of the function $t \mapsto 1/t$ and Jensen's inequality imply

$$\frac{1}{h_\epsilon} \leq \left(\frac{1}{h}\right)_\epsilon,$$

where on the right side we have the mollifications of the function $1/h$. Since $(1/h)_\epsilon$ converges in $L^1(O)$ to $1/h$ as ϵ goes to 0, we deduce that $1/h_\epsilon$ converges to $1/h$ in $L^1(O)$ as well.

Similarly, $t \mapsto -\log(t) = \log(1/t)$ is convex and arguing as above with Jensen's inequality we see that $\log(h_\epsilon)$ converges in $L^1(O)$ to $\log(h)$ as ϵ goes to 0.

For all $n > 2/\delta$ consider the smooth function f_n on O such that $f_n^{-2} = h_{1/n}$. As above, $\log(f_n) = -2\log(h_{1/n})$ converges to $\log(f)$ in $L^1(O)$. The remaining statements follow by a continuity argument, once we have verified $\Delta(\log(f_n^2)) \leq -2\kappa \cdot f_n^2$ on O.

As seen above, the smooth positive functions $h_{1/n}$ satisfy (5.9) on O for all harmonic functions $v : \mathbb{R}^2 \to \mathbb{R}$. Therefore, we only need to verify that for a *smooth positive* function h on a domain U in $\mathbb{R}^2$ the validity of (5.9) for all harmonic functions $v : \mathbb{R}^2 \to \mathbb{R}$ implies

$$\Delta(\log(h)) = \frac{\Delta(h)}{h} - \frac{|\nabla h|^2}{h^2} \geq \frac{2\kappa}{h}. \tag{5.10}$$

It is sufficient to verify this pointwise statement at a single point $z \in U$ which we may assume to be $z = 0$.

Set $e := \nabla h(0)$. If $e = 0$ we choose v to be a linear non-zero function on $\mathbb{R}^2$. Then (5.9) implies $\Delta(h)(0) \geq 2\kappa$, hence (5.10) holds.

If $e \neq 0$, fix $\lambda \in \mathbb{R}$ and consider the uniquely determined symmetric traceless matrix $A : \mathbb{R}^2 \to \mathbb{R}^2$ with $A(e) = \lambda \cdot e$. The function

$$v(z) := \langle z, A(z) + e \rangle$$

is harmonic on $\mathbb{R}^2$ and satisfies $\nabla v(z) = e + 2A(z)$. Therefore,

$$\nabla(|\nabla v|^2)(z) = 4\lambda \cdot e + 8A^2(z) \quad \text{and} \quad \Delta(|\nabla v|^2)(z) = 16\lambda^2.$$

Thus, the right hand side of (5.9) at $z = 0$ is just $2\kappa \cdot |e|^2$.

For the left hand side of (5.9) at the point $z = 0$ we compute

$$\Delta(h) \cdot |\nabla v|^2 + 2\langle \nabla h, \nabla(|\nabla v|^2) \rangle + h \cdot \Delta(|\nabla v|^2)$$
$$= \Delta(h)(0) \cdot |e|^2 + 8\lambda \cdot |e^2| + 16 \cdot \lambda^2 \cdot h(0).$$

For $\lambda = -\frac{|e|^2}{4h(0)}$ the inequality (5.9) now reads

$$|e|^2 \cdot \left(\Delta(h)(0) - \frac{|e|^2}{h(0)} \right) \geq 2\kappa |e|^2.$$

Dividing by $h(0)|e|^2$ we deduce (5.10).

This finishes the proof of Corollary 5.2. ∎

6. Curvature bound in the regular part

6.1. Preliminaries from Alexandrov geometry

We refer to [7] for the basics of Alexandrov geometry and just agree on notation here, following [44]. Let κ be a fixed real number as before. For points p, x_1, x_2 in a met-

ric space Y, we denote by $\widetilde{\angle}^\kappa(p_{x_2}^{x_1})$ the κ-*comparison angle*, whenever it exists. Thus, $\widetilde{\angle}^\kappa(p_{x_2}^{x_1})$ is the angle in the constant curvature surface M_κ^2 at the vertex $\widetilde{p}$ of a triangle $\widetilde{p}\widetilde{x}_1\widetilde{x}_2$ with the same side-lengths as px_1x_2.

A subset O of a metric space Y satisfies the $(1+3)$-*point comparison* if for any quadruple of points $p, x_1, x_2, x_3 \in O$ the inequality

$$\widetilde{\angle}^\kappa(p_{x_2}^{x_1}) + \widetilde{\angle}^\kappa(p_{x_3}^{x_2}) + \widetilde{\angle}^\kappa(p_{x_1}^{x_3}) \leq 2\pi$$

is true or one of the κ-comparison angles is not defined.

A metric space Y has *curvature* $\geq \kappa$ if every point $y \in Y$ has a neighborhood O which satisfies the $(1+3)$-point comparison and every pair of points in O is connected in Y by a geodesic.

A complete geodesic metric space of curvature $\geq \kappa$ is called an *Alexandrov space* of curvature $\geq \kappa$.

If Y has curvature $\geq \kappa$, the ball $\bar{B}_{2R}(y) \subset Y$ is compact and any pair of points in $B_R(y)$ is connected in Y by a geodesic, then $B_R(y)$ satisfies the $(1+3)$-point comparison, [44, p. 3].

Note finally that the $(1+3)$-point comparison property (of subsets) is stable under Gromov–Hausdorff convergence.

The aim of this section is to prove

Proposition 6.1. *The subspace X^δ of δ-regular points in X has curvature $\geq \kappa$.*

6.2. Reshetnyak's theory

We keep the notation from Sections 4 and 5. For a homeomorphism $\phi : D \to \Omega \subset X$ the length $\ell_X(\phi \circ \gamma)$ is given by (4.2) for almost all curves γ in D. The conformal factor $f = \rho_\phi$ satisfies the conclusion of Corollary 5.2.

By Corollary 5.2, the function $\Delta(\log(f))$ is a Radon measure. Thus, on any compactly contained domain $O \subset D$ we can canonically represent $\log(f)$ as a sum of a harmonic function and a *Riesz potential* [46, p. 99]. Using this representation, the f-length $\ell_f(\gamma)$ of *any* rectifiable curve γ on O is defined in [46, p. 100] by the formula (4.2).

This induces a new metric d_f on O by letting $d_f(z_1, z_2)$ be the infimum of all f-lengths of rectifiable curves connecting z_1 and z_2. The metric d_f induces the original Euclidean topology on O [46, Theorem 7.1.1]. We define the metric space $O_f = (O, d_f)$ to be the domain O equipped with the metric d_f.

The following statement is implicitly contained in [46, p. 140]. For convenience of the reader, we reduce the result to other more explicit statements in [46].

Lemma 6.2. *The space O_f has curvature $\geq \kappa$.*

Proof. We find smooth positive functions $f_n : O \to \mathbb{R}$ approximating f as in Corollary 5.2. By [46, Theorem 7.3.1], the distance functions d_{f_n} converge on O locally uniformly to d_f.

On the other hand, d_{f_n} is a smooth Riemannian metric. Its curvature can be computed pointwise by the classical formula $-\frac{\Delta(\log(f_n))}{f_n^2}$ [46, p. 40]. Thus, (5.8) shows that the Riemannian manifold O_{f_n} has sectional curvature $\geq \kappa$.

Hence, O_{f_n} has curvature $\geq \kappa$ in the sense of Alexandrov [7, Theorem 6.5.6]. Therefore, for any compact metric ball B in (O, d_f) and the concentric ball B' with one-third the radius of B, the set $B' \subset O_{f_n}$ satisfies the $(1 + 3)$-point comparison for all large n. By continuity, B' satisfies the $(1 + 3)$-point comparison when considered as a subset of O_f. This completes the proof. ∎

6.3. Sobolev-to-Lipschitz

By construction, for *almost all* curves γ in O, the length of $\phi \circ \gamma$ in X is equal to the f-length of γ, hence to the length of γ in the metric space O_f. The easiest way to upgrade the equality statement from *almost all* curves to *all* curves is via an application of the *Sobolev-to-Lipschitz property* of RCD spaces, stated as follows [17, p. 48], [4]:

For any RCD$(\kappa, 2)$ space X, any open subset W of X and any $u \in N^{1,2}(W)$ for which the constant function $\rho = 1$ is a weak upper gradient, the function u has a locally 1-Lipschitz representative.

In fact, the Sobolev-to-Lipschitz property is defined and verified in [17, p. 48] only in the global case $W = X$, but the proof presented there covers the local version formulated above.

Lemma 6.3. *The open embedding $\phi : O_f \to X$ is a local isometry.*

Proof. Set $W = \phi(O_f)$ and consider the inverse map $\psi = \phi^{-1} : W \to O_f$. By construction, ψ preserves the length of almost every curve γ in W. Therefore, the map ψ is contained in the Sobolev space $N^{1,2}(W, O_f)$ and the constant function 1 is a weak upper gradient of ψ.

Therefore, for any point $y \in O_f$, the composition $\psi_y \in N^{1,2}(W)$ of ψ with the distance function in O_f to the point y has the constant function 1 as a weak upper gradient. By the Sobolev-to-Lipschitz property, this implies that ψ_y is locally 1-Lipschitz. Since X is a geodesic space, and y was arbitrary, this implies that ψ is locally 1-Lipschitz.

In order to prove that ϕ is locally 1-Lipschitz, we apply the same argument. (Alternatively, this can be seen directly, as in [39, Lemma 9.3].) Firstly, $\mathcal{H}^2_{O_f} = f^2 \cdot \mathcal{H}^2_O$ and the same computation as in Section 4 shows that any family of curves of modulus 0 in O has modulus 0 in O_f. Thus, the map ϕ preserves the lengths of almost all curves in O_f. Therefore, $\phi \in N^{1,2}(O_f, X)$ and the constant function 1 is a weak upper gradient of ϕ. Arguing as above we deduce that ϕ is locally 1-Lipschitz, once the Sobolev-to-Lipschitz property has been verified locally in O_f.

But any point in O_f has a compact neighborhood isometric to an Alexandrov space, by [42, Theorem 7.1.3]. This implies the Sobolev-to-Lipschitz property, as a consequence of [43] and [17, p. 40].

Since ϕ and ϕ^{-1} are locally 1-Lipschitz, ϕ is a local isometry. ∎

For any δ-regular point $x_0 \in X$, we choose a domain Ω containing x_0 as in Section 4. Lemmas 6.3 and 6.2 imply the existence of a neighborhood of x_0 in Ω which has curvature $\geq \kappa$. This finishes the proof of Proposition 6.1.

7. Extension to the singular points

7.1. Topological statement

Most of this rather long section is devoted to the proof of the following topological statement.

Proposition 7.1. *Any point $z \in \partial X$ with $b(z) = 1/2$ has an open neighborhood U in X homeomorphic to the closed half-plane H such that z lies on the boundary line ∂H.*

Before we embark on the proof of this proposition, we explain why this statement is sufficient to finish the proof of our main theorem.

Lemma 7.2. *The validity of Proposition 7.1 implies that the set X^δ is strongly convex in X. Thus any geodesic γ with endpoints $x, y \in X^\delta$ is completely contained in X^δ.*

Proof. Assume the contrary and consider a point $z \in X \setminus X^\delta$ on γ closest to x.

Due to Lemma 2.5, the point z is in ∂X and satisfies $b(z) = 1/2$. Applying Proposition 7.1 we find a neighborhood U of z homeomorphic to H such that z lies on ∂H.

The part γ^+ of the geodesic γ between x and z is contained in X^δ (up to the point z) hence it intersects ∂H only in z. By choosing the neighborhood U smaller we can therefore assume that γ^+ separates U into two components.

Fix a point q on the part of γ between z and y sufficiently close to z and let U^+ be the component of $U \setminus \gamma^+$ which does not contain q. Thus, for any $m \in U^+$ sufficiently close to z, any geodesic between q and m intersects γ^+. In particular, z lies on a geodesic between m and q.

Thus, we have found points $q \neq z \in X$ and an open non-empty set $\widetilde{U}^+$ in X such that for any $m \in \widetilde{U}^+$ the point z lies on a geodesic connecting q and m. This contradicts the essentially non-branching property of X [20], as we are going to explain now.

Indeed, due to [20, Corollary 1.4], for almost every $m \in \widetilde{U}^+$ there is only one geodesic between q and m. Hence this geodesic must contain z. In particular, the geodesic η between q and z has to be unique.

By [20, Theorem 1.1], there exists a unique *optimal geodesic plan* e_t from the normalized restriction $\mu := \frac{1}{\mathcal{H}^2(\widetilde{U}^+)} \cdot \mathcal{H}^2|_{\widetilde{U}^+}$ to the Dirac measure δ_q. By construction, for all sufficiently small ϵ the push-forward $(e_{1-\epsilon})_{\#}\mu$ is concentrated on the geodesic η. However, by [20, Corollary 1.6], the probability measure $(e_t)_{\#}\mu$ is absolutely continuous with respect to $\mathcal{H}^2$. This contradiction finishes the proof. ∎

Therefore, Proposition 7.1 implies that X^δ is a geodesic space. Since X is the completion of X^δ, we would deduce from [44] that X is an Alexandrov space with curvature $\geq \kappa$ and finish the proof of Theorem 1.1.

7.2. Interior points

Consider a point $x \in X \setminus \partial X$. Assume that x is not contained in X^δ. By Lemma 3.1, we find some $r > 0$ such that $B_{2r}(x) \setminus X^\delta$ contains only the point x. By Proposition 6.1, $B_{2r}(x) \setminus \{x\}$ has curvature $\geq \kappa$ in the sense of Alexandrov.

Due to Lemma 2.5, for any $y, z \in B_r(x) \setminus \{x\}$, any geodesic connecting y and z in X is contained in $B_{2r}(x) \setminus \{x\}$. Toponogov's globalization theorem in the version of [44] now shows that $B_r(x) \setminus \{x\}$ satisfies the $(1 + 3)$-point comparison. By continuity, $B_r(x)$ satisfies the $(1 + 3)$-point comparison as well.

We have just verified

Corollary 7.3. *The subspace $X \setminus \partial X$ of X has curvature $\geq \kappa$.*

7.3. Setting

We now fix a point $z \in \partial X$ with $b(z) = 1/2$. We consider a sequence of points $x_i \in X$ converging to z and a sequence r_i of positive numbers converging to 0. After choosing a subsequence we may and will assume that the blow-up $(Y, y) = \lim \left(\frac{1}{r_i} X, x_i\right)$ exists.

Furthermore, we consider the sequence of non-negative numbers $s_i = d(x_i, \partial X)$. By choosing a further subsequence, we may and will assume that the following limit exists:

$$A := \lim_{i \to \infty} \frac{s_i}{r_i} \in [0, \infty].$$

As we have seen in Section 2, any ball of any radius t in the blow-up Y has $\mathcal{H}^2$-measure at least $\frac{\pi}{2} t^2$. From the volume-cone rigidity [14] and Lemma 2.4 we deduce

Lemma 7.4. *The $\mathrm{RCD}(0, 2)$ space Y can have non-empty boundary only if Y is isometric to the flat half-plane H.*

Our next aim is to show that (non-) boundary points of X converge to (non-) boundary points in the blow-up Y. Then we will show that the blow-up Y is isometric to a plane or a half-plane.

7.4. Stability of the boundary

In the previous notation we are going to show

Lemma 7.5. *If $A > 0$ then the point y is not contained in ∂Y.*

Proof. Assume the contrary. By Lemma 7.4, Y is isometric to the half-plane H and $y \in \partial H$.

By rescaling, we may assume that $A > 9$. Thus, for all i large enough, the compact ball $\hat{B}_i$ in $X_i = \frac{1}{r_i} X$ of radius 6 around x_i does not contain boundary points. Due to Corollary 7.3, the open ball B_i of radius 3 around x_i in X_i satisfies the $(1 + 3)$-point comparison (see Section 6 and [44]).

A contradiction to the fact that the surfaces B_i without boundary converge to a surface with boundary (the ball in H around the limit point y) can now be deduced in several ways. We choose a way relying on (the simplest case of) Perelman's topological stability theorem [8, 28].

From [42, Theorem 7.1.3] we deduce the existence of some $\delta > 0$ and closed compact convex subsets $C_i \subset B_i$ containing the ball $B_\delta(x_i) \subset B_i$. The spaces C_i are compact 2-dimensional Alexandrov spaces converging (after choosing a subsequence) to an Alexandrov space $C \subset H$ which contains the ball $B_\delta(y) \subset H$. By Perelman's stability theorem [8, 28], for all large i, there exists a homeomorphism $\Phi_i : C_i \to C$ close to the identity. Since $B_\delta(x_i)$ is a 2-manifold without boundary, we deduce that y must have in Y a neighborhood homeomorphic to a 2-manifold without boundary, which is impossible.

This contradiction finishes the proof. ∎

Lemma 7.6. *If $A = 0$ then Y is isometric to the half-plane H and y is on ∂H.*

Proof. Consider points $z_i \in \partial X$ with $d(z_i, x_i) = s_i = d(\partial X, x_i)$. The assumption $A = 0$ implies that the points $z_i \in X_i = \frac{1}{r_i} X$ converge to the same point $y \in Y$.

Since $z_i \in \partial X$, we see that the density $b(y)$ at Y is at most $1/2$. The volume-cone rigidity argument implies that $b(y) = 1/2$ and Y is isometric to the Euclidean cone $T_y Y$. It remains to show that $T_y Y$ cannot be the Euclidean cone over the circle of length π.

We assume that Y is a Euclidean cone over a circle and are going to derive a contradiction.

For all i large enough, there exist no points p_i in ∂X with $r_i < d(p_i, z_i) < 2r_i$, since $Y \setminus \{y\}$ is locally Euclidean. Denote by K_i the set $\partial X \cap \bar{B}_{r_i}(z_i)$ and by $\hat{K}_i$ the complement $\partial X \setminus K_i$. Then, for all i large enough, K_i is compact and $\hat{K}_i$ is closed in ∂X. Moreover,

$$d(z, z_i) \geq t_i := d(K_i, \hat{K}_i) \geq r_i.$$

Consider points $k_i \in K_i$ and $\hat{k}_i \in \hat{K}_i$ realizing the distance between K_i and $\hat{K}_i$ and take the blow-up (choosing a subsequence) $(\hat{Y}, k) = \lim(\frac{1}{t_i} X, k_i)$.

As before, the volume-cone rigidity implies that $\hat{Y}$ is either a half-plane or a cone over a circle. However, the points $\hat{k}_i$ converge (after taking a subsequence) to a non-Euclidean point $\hat{k} \in \hat{Y}$ with distance 1 to k. Therefore, $\hat{Y}$ must be isometric to H and the geodesic between k and $\hat{k}$ must lie on ∂H.

Hence, the midpoints m_i of any geodesic between k_i and $\hat{k}_i$ in X converge to a point on the boundary of $\hat{Y}$.

But, by construction, the point m_i in X has distance $t_i/2$ from ∂X. We obtain a contradiction with Lemma 7.5 and finish the proof. ∎

Lemma 7.7. *If $A = \infty$ then Y is isometric to $\mathbb{R}^2$.*

Proof. Due to Lemmas 7.6 and 7.5, the sequence $(\frac{1}{s_i} X, x_i)$ converges to a half-plane $(H, \hat{y})$ where $\hat{y}$ has distance 1 from ∂H. Therefore, $\mathcal{H}^2(B_1(\hat{y})) = \pi$. Stability of the

Hausdorff measures implies

$$\frac{\mathcal{H}^2(B_{s_i}(x_i))}{s_i^2} \to \pi.$$

Applying the Bishop–Gromov inequality and the assumption $A = \infty$, we see that for any fixed $t > 0$,

$$\frac{b(x_i, t r_i)}{r_i^2} \to \pi t^2.$$

Thus, the ball $B_t(y)$ in Y has $\mathcal{H}^2$-measure πt^2. Since t is arbitrary, the volume-cone rigidity implies that Y is isometric to $T_y Y = \mathbb{R}^2$. ∎

Combining the last three statements we easily arrive at

Corollary 7.8. *The space Y is either $\mathbb{R}^2$ or the half-plane H. The case $Y = H$ happens if and only if $A < \infty$. Moreover, in this case ∂H coincides with the limit of the boundary ∂X.*

Proof. The only statement not directly contained in Lemmas 7.5–7.7 is that for $0 < A < \infty$ the space Y is isometric to H. However, if $A < \infty$ we can replace the base points x_i by closest points $z_i \in \partial X$ and apply Lemma 7.6 to deduce the statement. ∎

7.5. Reifenberg's lemma twice

We can now finish

Proof of Proposition 7.1. Due to Corollary 7.8, for any sequence $z_i \in \partial X$ converging to z and any sequence $r_i \to 0$ the sequence $(\frac{1}{r_i} \partial X, z_i)$ converges to the line $\partial H = \mathbb{R}$. Thus, for any $\epsilon > 0$ there exists some $\delta > 0$ with the following properties. For any $z' \in B_\delta(z) \cap \partial X$ and any $0 < r < \delta \epsilon$, the ball of radius 1 in $\frac{1}{r} \partial X$ around z' has Gromov–Hausdorff distance less than ϵ to the interval of length 2.

Applying the Cheeger–Colding–Reifenberg Lemma [11, Theorem A1], we find that a neighborhood of z in ∂X is homeomorphic to an interval.

Consider now the doubling $W = X \cup_{\partial X} X$ of X with its natural length metric [8, Section 5].

We claim that for any sequence $x_i \in W$ converging to z and any sequence of positive numbers r_i converging to 0 the blow-up $\lim(\frac{1}{r_i} W, x_i)$ is isometric to $\mathbb{R}^2$.

Choosing a subsequence and using symmetry we may assume that $x_i \in X \subset W$ and that for $s_i := d(x_i, \partial X)$ the quotients s_i / r_i converge to a number $A \in [0, \infty]$. Applying Corollary 7.8, we see that in the case $A = \infty$, the blow-up coincides with $\lim(\frac{1}{r_i} X, x_i) = \mathbb{R}^2$.

On the other hand, if $A < \infty$, we can change the base points x_i and assume $x_i \in \partial X$, without changing the isometry class of the blow-up. Then the limit $\lim(\frac{1}{r_i} W, x_i)$ is the doubling of $H = \lim(\frac{1}{r_i} W, x_i)$ along the boundary $\partial H = \lim(\frac{1}{r_i} \partial X, z_i)$.

Having proved the claim, we can now apply the Cheeger–Colding–Reifenberg Lemma a second time and deduce that a neighborhood V of z in W is homeomorphic to an open

disc. Passing to a smaller subdisc if necessary, we obtain a homeomorphism between V and the plane which takes $V \cap \partial X$ to a line.

By connectedness we see that $X \cap V$ must be homeomorphic to a half-plane. ∎

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