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Fine multibubble analysis in the higher-dimensional Brezis–Nirenberg problem

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Abstract. For a bounded set $\Omega \subset \mathbb{R}^N$ and a perturbation $V \in C^1(\bar{\Omega})$, we analyze the concentration behavior of a blow-up sequence of positive solutions to $-\Delta u_\varepsilon + \varepsilon V u_\varepsilon = N(N-2)u_\varepsilon^{(N+2)/(N-2)}$ for dimensions $N \geq 4$, which are non-critical in the sense of the Brezis–Nirenberg problem. For the general case of multiple concentration points, we prove that concentration points are isolated and characterize the vector of these points as a critical point of a suitable function derived from the Green function of $-\Delta$ on Ω. Moreover, we give the leading-order expression of the concentration speed. This paper, with a recent one by the authors [arXiv:2208.12337, 2022] in dimension $N = 3$, gives a complete picture of blow-up phenomena in the Brezis–Nirenberg framework.

1. Introduction and main results

For $N \geq 4$, let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let u_ε be a sequence of solutions to

$$\begin{cases} -\Delta u_\varepsilon + \varepsilon V u_\varepsilon = N(N-2) u_\varepsilon^{\frac{N+2}{N-2}} & \text{on } \Omega, \\ u_\varepsilon > 0 & \text{on } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

For the perturbation profile V, the canonical choice is $V \equiv -1$, but we will only assume $V \in C^1(\bar{\Omega})$ and $V < 0$ on $\bar{\Omega}$ throughout this paper. The understanding of the behavior of solutions of this equation is pivotal in the Yamabe problem; see for instance [10] and references therein.

Existence and non-existence of solutions to (1.1) is a delicate matter and has been investigated in a famous paper by Brezis and Nirenberg [4]. This is largely due to the Sobolev-critical value of the exponent $\frac{N+2}{N-2} = 2^* - 1$, which allows concentration of a sequence of solutions around one or even several points of Ω. Starting with [1, 6] and particularly an influential paper by Brezis and Peletier [5], in the latter, after studying the behavior of radial solutions, the authors conjecture an asymptotic expression for $\|u_\varepsilon\|_\infty$ in

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the case where (u_ε) has precisely one blow-up point. The present paper, with [21], completely settles this long-standing open question by giving the precise behavior of arbitrary sequences of solutions, notably ones with multiple concentration points.

For one-peak solutions and $N \geq 4$, the location and speed of concentration have been characterized in [19, 28] for $V \equiv -1$ and in [23] for non-constant V. For the related subcritical problem, with $V \equiv 0$ and $u_\varepsilon^{\frac{N+2}{N-2}-\varepsilon}$ on the right-hand side of (1.1), the properties of multi-peak solutions have been analyzed in [3, 29, 30]. In the latter, the authors always assume that the number of concentration points is a priori finite, which is not the case in the present paper and [21].

Conversely, besides the one-peak solutions arising as energy minimizers from [4], we mention that multi-peak solutions with various properties have been constructed e.g. in [9, 24, 25, 27].

When $N = 3$, even in the presence of only one concentration point, the leading order of the speed at which blow-up solutions to (1.1) concentrate is harder to obtain.[1] This is due to a certain cancellation in the energy expansion which forces one to push the asymptotic analysis to a higher degree of precision. Results analogous to [19, 28] for one-peak solutions have been obtained only recently, by the first author and collaborators in a series of papers [14–16]. The full analysis for $N = 3$ comprising multi-peak solutions has been carried out by the authors of the present paper in the recent preprint [21].

Finally, the blow-up of solutions to (1.1) in the case $N \geq 4$ has not been studied in the literature yet, notably because the fine analysis of the concentration points was not available, which is done in Appendix B. The goal of the present paper is to close this gap, using and adapting the new methods of [21]. Remarkably, differently from one-peak solutions in dimension $N \geq 4$, the multi-peak case can also feature a cancellation phenomenon which makes it harder to derive the concentration speed. We will explain this in more detail in the following subsection, where we state our main result.

1.1. Main result

Let us introduce the object that largely governs the asymptotic behavior of (u_ε), namely the Green function $G : \Omega \times \Omega \to \mathbb{R}$. This is the unique function satisfying, for each fixed $y \in \Omega$,

$$\begin{cases} -\Delta_x G(x, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that $G(x, y) > 0$ for every $x, y \in \Omega$. The regular part H of G is defined by

$$H(x, y) := \frac{1}{(N-2)\omega_{N-1}|x-y|^{N-2}} - G(x, y),$$

[1] To be completely precise, for $N = 3$ the relevant equation fulfilled by a blowing-up sequence of solutions is $-\Delta u_\varepsilon + (a + \varepsilon V)u_\varepsilon = 3u_\varepsilon^5$, with a non-zero $a \in C(\overline{\Omega})$ as a consequence of the Brezis–Nirenberg dimensional effect observed in [4].

where ω_{N-1} is the volume of the sphere $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. It is well known that for each $y \in \Omega$ the function $H(\cdot, y)$ is a smooth function in Ω. Thus we may define the *Robin function*

$$\phi(y) := H(y, y).$$

It is known that single-blow-up sequences of solutions to (1.1) must concentrate at critical points x_0 of ϕ when V is constant [5, 19, 28] and of a suitable function depending on ϕ and V when V is non-constant [23].

For any number $n \in \mathbb{N}$ of concentration points, let

$$\Omega_*^n := \{\boldsymbol{x} = (x_1, \dots, x_n) \in \Omega^n : x_i \neq x_j \text{ for all } i \neq j\}.$$

For $\boldsymbol{x} \in \Omega_*^n$ we denote by $M(\boldsymbol{x}) \in \mathbb{R}^{n\times n} = (m_{ij})_{i,j=1}^n$ the matrix with entries

$$m_{ij}(\boldsymbol{x}) := \begin{cases} \phi(x_i) & \text{for } i = j, \\ -G(x_i, x_j) & \text{for } i \neq j. \end{cases} \tag{1.2}$$

Its lowest eigenvalue $\rho(\boldsymbol{x})$ is simple and the corresponding eigenvector can be chosen to have strictly positive components. We denote by $\boldsymbol{\Lambda}(\boldsymbol{x}) \in \mathbb{R}^n$ the unique vector such that

$$M(\boldsymbol{x}) \cdot \boldsymbol{\Lambda}(\boldsymbol{x}) = \rho(\boldsymbol{x})\boldsymbol{\Lambda}(\boldsymbol{x}), \quad (\boldsymbol{\Lambda}(\boldsymbol{x}))_1 = 1.$$

Next, let us define, for $\boldsymbol{\kappa} \in (0, \infty)^n$ and $\boldsymbol{x} \in \Omega_*^n$,

$$F(\boldsymbol{\kappa}, \boldsymbol{x}) := \frac{1}{2}\langle \boldsymbol{\kappa}, M(\boldsymbol{x})\boldsymbol{\kappa}\rangle + d_N \frac{N-2}{4} \sum_i V(x_i)\kappa_i^{\frac{4}{N-2}}, \tag{1.3}$$

where the dimensional constant $d_N > 0$ is given by

$$d_N = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{\Gamma(N-1)\omega_{N-1}(N-2)^2}. \tag{1.4}$$

Moreover, we define the Aubin–Talenti-type bubble function

$$B(x) := (1 + |x|^2)^{-\frac{N-2}{2}}$$

and, for every $\mu > 0$ and $x_0 \in \mathbb{R}^N$, its rescaled and translated versions

$$B_{\mu,x_0}(x) = \mu^{-\frac{N-2}{2}} B\Big(\frac{x - x_0}{\mu}\Big) = \frac{\mu^{\frac{N-2}{2}}}{(\mu^2 + |x - x_0|^2)^{\frac{N-2}{2}}}.$$

We notice that B_{μ,x_0} satisfies $-\Delta B_{\mu,x_0} = N(N-2)B_{\mu,x_0}^{\frac{N+2}{N-2}}$ on $\mathbb{R}^N$, for every $\mu > 0$ and $x_0 \in \mathbb{R}^N$.

Finally, let W be the unique radial solution to

$$-\Delta W - N(N+2)WB^{\frac{4}{N-2}} = -B, \quad W(0) = \nabla W(0) = 0.$$

Here is our main result.

Theorem 1.1. *Let (u_ε) be a sequence of solutions to (1.1), with $V \in C^1(\overline{\Omega})$ and $V < 0$ on $\overline{\Omega}$, such that $\|u_\varepsilon\|_\infty \to \infty$. Then there exists $n \in \mathbb{N}$ and n sequences of points $x_{1,\varepsilon}, \dots, x_{n,\varepsilon} \in \Omega$ such that $x_{i,\varepsilon} \to x_{i,0} \in \Omega$, $\mu_{i,\varepsilon} := u_\varepsilon(x_{i,\varepsilon})^{-\frac{2}{N-2}} \to 0$ as $\varepsilon \to 0$, $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$ for every $\varepsilon > 0$ and $u_\varepsilon \to 0$ uniformly away from $x_1, \dots, x_n$. The ratio $\lambda_{i,\varepsilon} := (\frac{\mu_{i,\varepsilon}}{\mu_{1,\varepsilon}})^{\frac{N-2}{2}}$ has a finite, non-zero limit $\lambda_{i,0} \in (0, \infty)$.*

Moreover, the following hold:

(i) *Refined local asymptotics:* *For any $i = 1, \dots, n$, denote $B_{i,\varepsilon} := B_{\mu_{i,\varepsilon}, x_{i,\varepsilon}}$ and*

$$W_{i,\varepsilon} := \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon}) W(\mu_{i,\varepsilon}^{-1}(x - x_{i,\varepsilon})).$$

Then, for $\delta > 0$ small enough, and every $\nu \in (2,3)$,

$$|(u_\varepsilon - B_{i,\varepsilon} - W_{i,\varepsilon})(x)| \lesssim \left(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\right)|x - x_{i,\varepsilon}|^\nu$$

for all $x \in B(x_{i,\varepsilon}, \delta)$.

(ii) *Blow-up rate:* *The matrix $M(\boldsymbol{x}_0)$ is semi-positive definite with simple lowest eigenvalue $\rho(\boldsymbol{x}_0) \geq 0$.*

(a) *Suppose $\rho(\boldsymbol{x}_0) > 0$. If $N \geq 5$, then*

$$\lim_{\varepsilon \to 0} \varepsilon \mu_{i,\varepsilon}^{-N+4} =: \kappa_{i,0}^{-2\frac{N-4}{N-2}}$$

exists and lies in $(0, \infty)$. Moreover, $(\boldsymbol{\kappa}_0, \boldsymbol{x}_0)$ is a critical point of $F(\boldsymbol{\kappa}, \boldsymbol{x})$ defined in (1.3). If $N \geq 6$, then $\boldsymbol{\kappa}_0$ is the unique critical point of $F(\cdot, \boldsymbol{x}_0)$.

If $N = 4$, then for every i,

$$\lim_{\varepsilon \to 0} \varepsilon \ln(\mu_{i,\varepsilon}^{-1}) = \kappa_0,$$

where $\kappa_0 > 0$ is the unique number such that $M - \kappa_0 \operatorname{diag}(\frac{1}{8\pi^2}|V(x_{i,0})|)$ has its lowest eigenvalue equal to zero. Moreover, $(\boldsymbol{\lambda}_0, \boldsymbol{x}_0)$ is a critical point of

$$\widetilde{F}(\boldsymbol{\lambda}, \boldsymbol{x}) = \frac{1}{2}\langle \boldsymbol{\lambda}, M(\boldsymbol{x})\boldsymbol{\lambda} \rangle + \frac{\kappa_0}{2}\frac{1}{8\pi^2}\sum_i V(x_i)\lambda_i^2. \tag{1.5}$$

(b) *If $\rho(\boldsymbol{x}_0) = 0$, then also $\nabla\rho(\boldsymbol{x}_0) = 0$. Moreover,*

$$\lim_{\varepsilon \to 0} \varepsilon \mu_{i,\varepsilon}^{-N+4} = \mathcal{O}(\mu_\varepsilon^2) \quad \text{if } N \geq 5, \tag{1.6}$$

$$\lim_{\varepsilon \to 0} \varepsilon \ln(\mu_\varepsilon^{-1}) = \mathcal{O}(\mu_\varepsilon^2) \quad \text{if } N = 4, \tag{1.7}$$

and $\Lambda_{i,0} = \lambda_{i,0} = \lim_{\varepsilon \to 0}(\frac{\mu_{i,\varepsilon}}{\mu_{1,\varepsilon}})^{\frac{N-2}{2}}$.

Furthermore, we have the quantitative bounds

$$\rho(\boldsymbol{x}_\varepsilon) = \begin{cases} o(\varepsilon\mu_\varepsilon^{-N+4} + \mu_\varepsilon^2) & \text{if } N \geq 5, \\ o(\varepsilon \ln(\mu_\varepsilon^{-1}) + \mu_\varepsilon^2) & \text{if } N = 4, \end{cases}$$

and, for every $\delta > 0$,

$$|\nabla \rho(\boldsymbol{x}_\varepsilon)| \lesssim \mu_\varepsilon^{2-\delta}.$$

Remarks 1.2. (a) In order to keep the statement of the theorem reasonable, in the refined local asymptotics, we just give the expansion up to the first term after the bubble. But in fact we can go further, as shown by Proposition 2.6. More precisely, our technique, which consists in subtracting recursively a suitable solution of the inhomogeneous linearized equation, will give, if pushed far enough, the estimate

$$\Big|\big(u_\varepsilon - B_{i,\varepsilon} - \sum_{k=1}^{l} W_{i,\varepsilon}^k\big)(x)\Big| \lesssim \big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3+l-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big)|x - x_{i,\varepsilon}|^\nu$$

for all $x \in B(x_{i,\varepsilon}, \delta)$ and $\nu \in (l+1, l+2)$, where

$$W_{i,\varepsilon}^k := \varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+2+k} W^k(\mu_{i,\varepsilon}^{-1}(x - x_{i,\varepsilon})),$$

and W^k is the solution to

$$-\Delta W^k - N(N+2)B^{\frac{4}{N-2}} W^k = f_k(x, W^1, \dots, W^{k-1}), \quad W^k(0) = \nabla W^k(0) = 0.$$

The inhomogeneities f_k, which may depend on V and $W^1, \dots, W^{k-1}$ and their derivatives, are obtained recursively during the expansion.

(b) A remarkable fact about Theorem 1.1 is that in the degenerate case $\rho(\boldsymbol{x}_0) = 0$, the bounds (1.6) and (1.7) are improved in comparison to the case where $\rho(\boldsymbol{x}_0) > 0$. Indeed, in this case (and only then) the first term on the right-hand side of the expansions (3.1), resp. (3.2) cancels, as shown in Section 4. Our analysis of the error terms is fine enough to push the estimates further by a factor of μ_ε^2 in the expansions (3.1) and (3.2).

This should, in particular, be compared with the analysis of the related equation

$$-\Delta u_\varepsilon = u_\varepsilon^{\frac{N+2}{N-2}-\varepsilon}$$

in [3], where in the case $\rho(\boldsymbol{x}_0) = 0$ no improved asymptotics are derived.

(c) We also point out that in the case $n = 1$ of only one concentration point $x_0 \in \Omega$, one simply has $\rho(x_0) = \phi(x_0) > 0$ by the maximum principle. Thus the possibility that $\rho(\boldsymbol{x}_0) = 0$ is indeed particular to the multi-peak case.

In the case where Ω is convex, it is known [17, Theorem 2.7] that no multiple blow-up can happen. Under the weaker assumption that Ω is star-shaped with respect to some

$y_0 \in \Omega$, the same is not known. However, a simple argument shows that if multiple blow-up does happen for Ω star-shaped, we must always be in the non-degenerate case $\rho(\boldsymbol{x_0}) > 0$. Indeed, by Pohožaev's identity we have

$$-\varepsilon \mu_{1,\varepsilon}^{-N+2} \int_\Omega (2V(x) + \nabla V(x) \cdot (x - y_0)) u_\varepsilon^2 \, \mathrm{d}x = \mu_{1,\varepsilon}^{-N+2} \int_{\partial\Omega} \left| \frac{\partial u_\varepsilon}{\partial n} \right|^2 (x - y_0) \cdot n \, \mathrm{d}x.$$

By Proposition 2.1 (v) below, the right-hand side converges to

$$\int_{\partial\Omega} \left| \frac{\partial G_{\boldsymbol{x},\boldsymbol{\nu}}}{\partial n} \right|^2 (x - y_0) \cdot n \, \mathrm{d}x > 0.$$

On the other hand, by standard calculations as in the proof of Proposition 3.1, the left-hand side is equal to

$$\begin{cases} -\varepsilon \mu_{1,\varepsilon}^{-N+2} c_N \sum_j V(x_{j,\varepsilon}) \mu_{j,\varepsilon}^2 + o(\mu_\varepsilon^2) & \text{if } N \geq 5, \\ -\varepsilon \mu_{1,\varepsilon}^{-2} c_4 \sum_j V(x_{j,\varepsilon}) \mu_{j,\varepsilon}^2 \ln(\mu_{j,\varepsilon}^{-1}) & \text{if } N = 4. \end{cases} \tag{1.8}$$

Since $V < 0$ by assumption and all the $\mu_{j,\varepsilon}$ are comparable by Proposition 2.1, the left-hand side is equal to a positive constant times $\varepsilon \mu_\varepsilon^{-N+4}$ if $N \geq 5$, resp. $\varepsilon \ln(\mu_\varepsilon^{-1})$ if $N = 4$. Since we have seen that the right-hand side is strictly positive, the quantities $\varepsilon \mu_\varepsilon^{-N+4}$, resp. $\varepsilon \ln(\mu_\varepsilon^{-1})$, must have a strictly positive limit. In particular, $\rho(\boldsymbol{x_0}) > 0$ by Theorem 1.1.

(d) One may ask whether our hypothesis that $V < 0$ on $\overline{\Omega}$ can be further relaxed. Concerning this question, a few comments are in order. First, if Ω is star-shaped, then by Pohožaev's identity as in Remark 1.2 (c) it is clear that for $V \equiv \text{const.} > 0$ there cannot be a solution u_ε to (1.1). For non-constant V, the situation is less clear. Still for star-shaped Ω, say, the quantity (1.8) seems to suggest that at some blow-up points x_j, positive values $V(x_j) > 0$ might be allowed as long as they are compensated for by others. On the other hand, we are not aware of examples in the literature for a blow-up pattern different from that of Theorem 1.1 (e.g. by exhibiting unbounded energy, clusters of concentration points and/or concentration on the boundary) in a situation where V is not strictly negative. We point out that both our a priori analysis in Appendix B and the proof of our main results in Section 4 require that $V < 0$ everywhere, independently from each other.

(e) Surprisingly, the concentration speed is uniquely determined in terms of Ω, V, n and x_0 in dimensions $N = 4$ and $N \geq 6$, but not $N = 5$. Indeed, in that case we cannot exclude that the function F may fail to be convex.

The structure of the rest of this paper is as follows. In Section 2, starting from some qualitative information about the blow-up of u_ε, we derive very precise pointwise bounds on u_ε near the concentration points, which form the technical core of our method. These are used in turn to derive the main energy expansions in Section 3. Once these are established, the proof of Theorem 1.1 can be concluded in Section 4 by a rather soft argument. We have added several appendices in an attempt to make the analysis self-contained.

2. Asymptotic analysis

We start with some by now classical estimates, which say that a blowing-up sequence can only develop finitely many bubbles and the solutions are controlled by the bubble. Here, the hypothesis $V < 0$ plays a crucial role. This kind of analysis has been initiated by Druet, Hebey and Robert [12] on a manifold. In the domain case, an extra difficulty occurs since we have to avoid concentration near the boundary. This has already been done in dimension $N = 3$ by Druet and the second author [13] in a similar context. In higher dimension $N \geq 4$ the proof is largely analogous. We give it in Appendix B for the sake of completeness and in the hope of providing a useful future reference for the case of a domain Ω.

Proposition 2.1. *Let (u_ε) be a sequence of solutions to (1.1) such that $\|u_\varepsilon\|_\infty \to +\infty$. Then, up to extracting a subsequence, there exists $n \in \mathbb{N}$ and points $x_{1,\varepsilon}, \dots, x_{n,\varepsilon}$ such that the following hold:*

(i) *$x_{i,\varepsilon} \to x_i \in \Omega$ for some $x_i \in \Omega$ with $x_i \neq x_j$ for $i \neq j$.*

(ii) *$\mu_{i,\varepsilon} := u_\varepsilon(x_{i,\varepsilon})^{-\frac{2}{N-2}} \to 0$ as $\varepsilon \to 0$ and $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$ for every i.*

(iii) *$\lambda_{i,0} := \lim_{\varepsilon \to 0} \lambda_{i,\varepsilon} := \lim_{\varepsilon \to 0} \mu_{i,\varepsilon}^{\frac{N-2}{2}} / \mu_{1,\varepsilon}^{\frac{N-2}{2}}$ exists and lies in $(0, \infty)$ for every i.*

(iv) *$\mu_{i,\varepsilon}^{\frac{N-2}{2}} u_\varepsilon(x_{i,\varepsilon} + \mu_{i,\varepsilon} x) \to B$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$.*

(v) *There are $\nu_i > 0$ such that $\mu_{1,\varepsilon}^{-\frac{N-2}{2}} u_\varepsilon \to \sum_{i=1}^n \nu_i G(x_{i,\varepsilon}, \cdot) =: G_{\boldsymbol{x},\boldsymbol{\nu}}$ uniformly in C^1 away from $\{x_1, \dots, x_n\}$.*

(vi) *There is $C > 0$ such that $u_\varepsilon \leq C \sum_{i=1}^n B_{i,\varepsilon}$ on Ω. Moreover, on every compact subset of Ω, there is $C > 0$ such that $\frac{1}{C} \sum_{i=1}^n B_{i,\varepsilon} \leq u_\varepsilon$.*

Up to reordering the $x_{i,\varepsilon}$, we assume that $\mu_{1,\varepsilon} = \max_i \mu_{i,\varepsilon}$ and we set $\mu_\varepsilon = \mu_{1,\varepsilon}$. We also define the small ball

$$\mathsf{b}_{i,\varepsilon} := B(x_{i,\varepsilon}, \delta_0)$$

around $x_{i,\varepsilon}$, with some number $\delta_0 > 0$ independent of ε and chosen so small that $\delta_0 < \frac{1}{2} \min_{i \neq j} |x_{i,\varepsilon} - x_{j,\varepsilon}|$.

The main result of this section consists of quantitative bounds on the remainder

$$r_{i,\varepsilon} := u_{i,\varepsilon} - B_{i,\varepsilon}, \tag{2.1}$$

as well as the improved remainders

$$q_{i,\varepsilon} := r_{i,\varepsilon} - \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon}) W\Big(\frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) \tag{2.2}$$

and

$$p_{i,\varepsilon} := q_{i,\varepsilon} - \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+4} W_2\Big(\frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) \nabla V(x_{i,\varepsilon}) \cdot \frac{x - x_{i,\varepsilon}}{|x - x_{i,\varepsilon}|} \tag{2.3}$$

on $\mathsf{b}_{i,\varepsilon}$. Here, the functions W and W_2 are solutions to the inhomogeneous ODEs

$$-W''(r) - \frac{N-1}{r} W'(r) - N(N+2) B(r)^{\frac{4}{N-2}} W(r) = -B,$$

$$-W_2''(r) - \frac{N-1}{r} W_2'(r) + \frac{N-1}{r^2} W_2(r) - N(N+2) B(r)^{\frac{4}{N-2}} W_2(r) = -B(r) r,$$

respectively. These bounds are stated in the subsections below as Propositions 2.4, 2.5 and 2.6.

An important ingredient in the proof of Theorem 1.1 will be a non-degeneracy property of the bubble B. Namely, consider the linearized equation

$$-\Delta u = N(N+2) B^{\frac{4}{N-2}} u \quad \text{on } \mathbb{R}^N. \tag{2.4}$$

Then the behavior of non-trivial solutions to (2.4) is restricted by the following proposition [21, Corollary A.2].

Proposition 2.2. *Let u be a solution to (2.4) and suppose that $|u(x)| \lesssim |x|^\tau$ on $\mathbb{R}^N$ for some $\tau \in (1, \infty) \setminus \mathbb{N}$. Then $u \equiv 0$.*

Before we go on, let us note a simple a priori estimate which will simplify the following estimates on $r_{i,\varepsilon}$ and $q_{i,\varepsilon}$.

Lemma 2.3. *Suppose that $V < 0$. If $N \geq 5$, then $\varepsilon \lesssim \mu_\varepsilon^{N-4}$. If $N = 4$, then $\varepsilon \lesssim \frac{1}{\ln(\mu_\varepsilon^{-1})}$.*

Proof. By Pohožaev's identity (see Appendix E), we have, for any i,

$$\begin{aligned} &-2\varepsilon \int_{\mathsf{b}_{i,\varepsilon}} V u_\varepsilon^2 - \varepsilon \int_{\mathsf{b}_{i,\varepsilon}} u_\varepsilon^2 \nabla V(x) \cdot (x - x_{i,\varepsilon})\, \mathrm{d}x \\ &\quad = 2 \int_{\partial \mathsf{b}_{i,\varepsilon}} \Big(\delta_0 (\partial_\nu u_\varepsilon)^2 - \delta_0 \Big(|\nabla u_\varepsilon|^2 + \frac{2 u_\varepsilon^{p+1}}{p+1} - \varepsilon V u_\varepsilon^2 \Big) + (N-2) u_\varepsilon \partial_\nu u_\varepsilon \Big). \end{aligned}$$

Since $V < 0$, by using Proposition 2.1 (iv), the left-hand side is proportional to $\varepsilon \mu_\varepsilon^2$ if $N \geq 5$ and to $\varepsilon \mu_\varepsilon^2 \ln(\mu_\varepsilon^{-1})$ if $N = 4$.

On the other hand, by Proposition 2.1 (v) the modulus of the right-hand side is bounded by a constant times μ_ε^{N-2}. This concludes the proof. ∎

2.1. The bound on $r_{i,\varepsilon}$

Proposition 2.4. *Let $i = 1, \dots, n$ and let $r_{i,\varepsilon}$ be defined by (2.1). As $\varepsilon \to 0$, for every $\theta \in (0,1) \cup (1,2)$,*

$$|r_{i,\varepsilon}(x)| \lesssim (\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta} + \mu_\varepsilon^{\frac{N-2}{2}}) |x - x_{i,\varepsilon}|^\theta \quad \text{on } \mathsf{b}_{i,\varepsilon}.$$

Moreover, for $\theta = 0$, we have

$$r_{i,\varepsilon}(x) \lesssim \begin{cases} \varepsilon\mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}} & \text{if } N \geq 5, \\ \varepsilon\mu_\varepsilon \ln(\mu_\varepsilon^{-1}) + \mu_\varepsilon & \text{if } N = 4. \end{cases}$$

Proof. We first assume that $\theta \in (0,1) \cup (1,2)$. The case $\theta = 0$ will be treated below by a separate argument.

Recall that $r_{i,\varepsilon} = u_\varepsilon - B_{i,\varepsilon}$. We denote

$$R_{i,\varepsilon}(x) := \frac{r_{i,\varepsilon}(x)}{|x - x_{i,\varepsilon}|^\theta}.$$

Fix some $z_{i,\varepsilon} \in \mathsf{b}_{i,\varepsilon}$ such that

$$R_{i,\varepsilon}(z_{i,\varepsilon}) \geq \tfrac{1}{2}\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}. \tag{2.5}$$

Moreover, we denote $d_{i,\varepsilon} := |x_{i,\varepsilon} - z_{i,\varepsilon}|$. Let us define the rescaled and normalized version

$$\bar{r}_{i,\varepsilon}(x) := \frac{r_{i,\varepsilon}(x_{i,\varepsilon} + d_{i,\varepsilon}x)}{r_{i,\varepsilon}(z_{i,\varepsilon})}, \quad x \in B(0, d_{i,\varepsilon}^{-1}\delta_0).$$

By the choice of $B_{i,\varepsilon}$, and observing (2.5), we have

$$\bar{r}_{i,\varepsilon}(0) = \nabla\bar{r}_{i,\varepsilon}(0) = 0, \quad \bar{r}_{i,\varepsilon}(x) \lesssim |x|^\theta, \ \ x \in B(0, d_{i,\varepsilon}^{-1}\delta_0), \tag{2.6}$$

in particular $\bar{r}_\varepsilon$ is uniformly bounded on compacts of $\mathbb{R}^N \setminus \{0\}$.

On $B(0, d_{i,\varepsilon}^{-1}\delta_0)$, we have

$$-\Delta\bar{r}_{i,\varepsilon} - \bar{r}_{i,\varepsilon} d_{i,\varepsilon}^2 Q(\bar{u}_{i,\varepsilon}, \bar{B}_{i,\varepsilon}) = -\varepsilon d_{i,\varepsilon}^2 \bar{V}_{i,\varepsilon} \frac{\bar{u}_{i,\varepsilon}}{r_{i,\varepsilon}(z_{i,\varepsilon})}, \tag{2.7}$$

where

$$Q(u,v) := N(N-2)\frac{u^{\frac{N+2}{N-2}} - v^{\frac{N+2}{N-2}}}{u - v}.$$

Moreover, we wrote $\bar{u}_{i,\varepsilon}(x) := u_\varepsilon(x_{i,\varepsilon} + d_{i,\varepsilon}x)$ and likewise $\bar{a}_{i,\varepsilon}(x) := a_\varepsilon(x_{i,\varepsilon} + d_{i,\varepsilon}x)$ and $\bar{B}_{i,\varepsilon}(x) := B_{i,\varepsilon}(x_{i,\varepsilon} + d_{i,\varepsilon}x) = \mu_{i,\varepsilon}^{-\frac{N-2}{2}} B(\mu_{i,\varepsilon}^{-1} d_{i,\varepsilon}x)$.

We treat three cases separately, depending on the ratio of μ_ε and $d_{i,\varepsilon}$. It will be useful to observe the bounds

$$\bar{B}_{i,\varepsilon}(x) = \Big(\frac{\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2 + d_{i,\varepsilon}^2|x|^2}\Big)^{\frac{N-2}{2}} \lesssim \begin{cases} \mu_{i,\varepsilon}^{-\frac{N-2}{2}} & \text{if } \mu_{i,\varepsilon} \gtrsim d_{i,\varepsilon}, \\ \mu_{i,\varepsilon}^{\frac{N-2}{2}} d_{i,\varepsilon}^{-N+2} & \text{if } \mu_{i,\varepsilon} \lesssim d_{i,\varepsilon}, \end{cases} \tag{2.8}$$

uniformly for x in compacts of $\mathbb{R}^N \setminus \{0\}$.

Case 1*:* $\mu_\varepsilon \gg d_{i,\varepsilon}$ *as* $\varepsilon \to 0$. Since $\bar{u}_{i,\varepsilon} \lesssim \bar{B}_{i,\varepsilon}$ on $\mathsf{b}_{i,\varepsilon}$ and $|Q(u,v)| \lesssim |u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}}$, the second summand on the left-hand side of (2.7) tends to zero uniformly on compacts by (2.8), because $d_{i,\varepsilon}^2 \mu_\varepsilon^{-2} \to 0$.

Using $\bar{u}_{i,\varepsilon} \lesssim \bar{B}_{i,\varepsilon} \lesssim \mu_{i,\varepsilon}^{-\frac{N-2}{2}}$ and $\frac{1}{r_{i,\varepsilon}(z_{i,\varepsilon})} \lesssim d_{i,\varepsilon}^{-\theta} \frac{1}{\|R_{i,\varepsilon}\|_\infty}$ by (2.5), the right-hand side of (2.7) is bounded by

$$\left| \varepsilon d_{i,\varepsilon}^2 \bar{V}_{i,\varepsilon} \frac{\bar{u}_{i,\varepsilon}}{r_{i,\varepsilon}(z_{i,\varepsilon})} \right| \lesssim \frac{\varepsilon d_{i,\varepsilon}^{2-\theta} \mu_{i,\varepsilon}^{-\frac{N-2}{2}}}{\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}} \lesssim \frac{\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}}{\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}}.$$

Now suppose for contradiction that $\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} \gg \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}$ as $\varepsilon \to 0$. Then this term goes to zero uniformly. Thus, by elliptic estimates, we have convergence on any compact of $\mathbb{R}^N \setminus \{0\}$, and the limit $\bar{r}_{i,0} := \lim_{\varepsilon \to 0} \bar{r}_{i,\varepsilon}$ satisfies

$$-\Delta \bar{r}_{i,0} = 0 \quad \text{on } \mathbb{R}^N \setminus \{0\}.$$

By Bôcher's and Liouville's theorems, the growth bound (2.6) implies that $\bar{r}_{i,0} \equiv 0$. But by the choice of $d_{i,\varepsilon}$, there is $\xi_{i,\varepsilon} := \frac{z_{i,\varepsilon} - x_{i,\varepsilon}}{d_{i,\varepsilon}} \in \mathbb{S}^{N-1}$ such that $\bar{r}_{i,\varepsilon}(\xi_{i,\varepsilon}) = 1$. Up to a subsequence, $\xi_{i,0} := \lim_{\varepsilon \to 0} \xi_{i,\varepsilon} \in \mathbb{S}^{N-1}$ exists and satisfies $\bar{r}_{i,0}(\xi_{i,0}) = 1$. This contradicts $\bar{r}_{i,0} \equiv 0$.

Thus we must have $\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}$, i.e. $r_{i,\varepsilon}(x) \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta} |x - x_{i,\varepsilon}|^\theta$.

Case 2(a)*:* $\mu_\varepsilon \ll d_{i,\varepsilon} \ll 1$ *as* $\varepsilon \to 0$. In this case, we have

$$\bar{r}_{i,\varepsilon} d_{i,\varepsilon}^2 F(\bar{u}_{i,\varepsilon}, \bar{B}_{i,\varepsilon}) \lesssim d_{i,\varepsilon}^2 \bar{B}_{i,\varepsilon}^{\frac{4}{N-2}} \lesssim \mu_\varepsilon^2 d_{i,\varepsilon}^{-2} \to 0$$

and

$$\left| \varepsilon d_{i,\varepsilon}^2 \bar{V}_{i,\varepsilon} \frac{\bar{u}_{i,\varepsilon}}{r_{i,\varepsilon}(z_{i,\varepsilon})} \right| \lesssim \frac{\varepsilon d_{i,\varepsilon}^{-N+4-\theta} \mu_\varepsilon^{\frac{N-2}{2}}}{\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}} \lesssim \frac{\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}}{\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}}$$

uniformly on compacts of $\mathbb{R}^N \setminus \{0\}$. If $\|R_{i,\varepsilon}\|_\infty \gg \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}$, then, using that $d_{i,\varepsilon} \to 0$ still, $\bar{r}_{i,0} := \lim_{\varepsilon \to 0} \bar{r}_{i,\varepsilon}$ satisfies

$$-\Delta \bar{r}_{i,0} = 0 \quad \text{on } \mathbb{R}^N \setminus \{0\}.$$

Using the Bôcher and Liouville theorems again, $\bar{r}_{i,0} \equiv 0$. As in Case 1, we can now derive a contradiction.

Thus we must have $\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}$, i.e. $r_{i,\varepsilon}(x) \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta} |x - x_{i,\varepsilon}|^\theta$ in this case as well.

Case 2(b)*:* $d_{i,\varepsilon} \sim 1$ *as* $\varepsilon \to 0$. In this case there is no need for a blow-up argument. Instead, we can simply bound, by the definition of $z_{i,\varepsilon}$,

$$\frac{|r_{i,\varepsilon}(x)|}{|x - x_{i,\varepsilon}|^\theta} \lesssim \frac{|r_{i,\varepsilon}(z_{i,\varepsilon})|}{d_{i,\varepsilon}^\theta} \lesssim |r_{i,\varepsilon}(z_{i,\varepsilon})| \lesssim \mu_{i,\varepsilon}^{\frac{N-2}{2}},$$

where the last inequality simply comes from the bound $|u_\varepsilon| \lesssim B_{i,\varepsilon}$ on $\mathsf{b}_{i,\varepsilon}$ and the observation that $d_{i,\varepsilon} \sim 1$ implies $B_{i,\varepsilon}(z_{i,\varepsilon}) \lesssim \mu_\varepsilon^{\frac{N-2}{2}}$. Thus

$$|r_{i,\varepsilon}(x)| \lesssim \mu_\varepsilon^{\frac{N-2}{2}} |x - x_{i,\varepsilon}|^\theta,$$

which completes the discussion of this case.

Case 3: *$\mu_\varepsilon \sim d_{i,\varepsilon}$ as $\varepsilon \to 0$.* This is the most delicate case because the second summand on the left-hand side of (2.7) now tends to a non-trivial limit. Indeed, $\beta_{i,0} := \lim_{\varepsilon\to 0} \beta_{i,\varepsilon} := \lim_{\varepsilon\to 0} \frac{\mu_{i,\varepsilon}}{d_{i,\varepsilon}}$ exists and $\beta_{i,0} \in (0, \infty)$. Then

$$d_{i,\varepsilon}^{\frac{N-2}{2}} \bar{B}_{i,\varepsilon} = \frac{\beta_{i,\varepsilon}^{\frac{N-2}{2}}}{(\beta_{i,\varepsilon}^2 + |x|^2)^{\frac{N-2}{2}}} \to \frac{\beta_{i,0}^{\frac{N-2}{2}}}{(\beta_{i,0}^2 + |x|^2)^{\frac{N-2}{2}}} = B_{0,\beta_{i,0}}.$$

By the convergence of u_ε from Proposition 2.1, we also have $d_{i,\varepsilon}^{\frac{N-2}{2}} \bar{u}_{i,\varepsilon} \to B_{0,\beta_{i,0}}$ uniformly on compacts of $\mathbb{R}^N$. Thus $d_{i,\varepsilon}^2 Q(\bar{u}_{i,\varepsilon}, \bar{B}_{i,\varepsilon}) \to N(N+2)B_{0,\beta_{i_0}}$ uniformly on compacts of $\mathbb{R}^N$.

On the other hand,

$$\left|\varepsilon d_{i,\varepsilon}^2 \bar{V}_{i,\varepsilon} \frac{\bar{u}_{i,\varepsilon}}{r_{i,\varepsilon}(z_{i,\varepsilon})}\right| \lesssim \frac{\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}}{\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}}.$$

If $\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} \gg \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}$, we therefore recover the limit equation

$$-\Delta \bar{r}_{i,0} = N(N+2)\bar{r}_{i,0} B_{0,\beta_{i,0}}^{\frac{4}{N-2}} \quad \text{on } \mathbb{R}^N,$$

which is precisely the linearized equation (2.4). By (2.6), we have $|r_{i,0}(x)| \lesssim |x|^\theta$ for all $x \in \mathbb{R}^N$. Thus by the classification, see [21, Proposition A.1], and the fact that $\bar{r}_{i,0}(0) = \nabla \bar{r}_{i,0}(0) = 0$, we conclude $\bar{r}_{i,0} \equiv 0$. This contradicts $\bar{r}_{i,0}(\xi_{i,0}) = 1$, as desired.

Thus we have shown $\|R_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta}$, i.e. $r_{i,\varepsilon}(x) \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta} |x - x_{i,\varepsilon}|^\theta$, also in the third and final case. This finishes the proof for $\theta \in (0,1) \cup (1,2)$.

Let us finally prove the assertion in the case $\theta = 0$, i.e.

$$r_{i,\varepsilon}(x) \lesssim \begin{cases} \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}} & \text{if } N \geq 5, \\ \varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1}) + \mu_\varepsilon & \text{if } N = 4, \end{cases} \quad \text{for } x \in \mathsf{b}_{i,\varepsilon}. \tag{2.9}$$

To prove (2.9), we consider Green's formula

$$\begin{aligned} r_{i,\varepsilon}(x) &= \int_\Omega (-\Delta r_{i,\varepsilon})(y) G(x,y)\, \mathrm{d}y - \int_{\partial\Omega} r_{i,\varepsilon}(y) \frac{\partial G(x,y)}{\partial \nu}\, \mathrm{d}\sigma(y) \\ &= \int_\Omega \big(u_\varepsilon^{\frac{N+2}{N-2}}(y) - B_{i,\varepsilon}^{\frac{N+2}{N-2}}(y) - \varepsilon V(y) u_\varepsilon(y)\big) G(x,y)\, \mathrm{d}y \\ &\quad - \int_{\partial\Omega} r_{i,\varepsilon}(y) \frac{\partial G(x,y)}{\partial \nu}\, \mathrm{d}\sigma(y). \end{aligned}$$

Since $r_{i,\varepsilon} \lesssim \sum_j B_{j,\varepsilon} \lesssim \mu_\varepsilon^{\frac{N-2}{2}}$ on $\partial\Omega$, the second term is bounded by

$$\left| \int_{\partial\Omega} r_{i,\varepsilon}(y) \frac{\partial G(x,y)}{\partial \nu} \, \mathrm{d}\sigma(y) \right| \lesssim \mu_\varepsilon^{\frac{N-2}{2}}.$$

A similar bound, which we do not detail, gives

$$\left| \int_{\Omega \setminus \bigcup_j \mathsf{b}_{j,\varepsilon}} (-\Delta r_{i,\varepsilon})(y) G(x,y) \, \mathrm{d}y \right| \lesssim \varepsilon \mu_\varepsilon^{\frac{N-2}{2}} + \mu_\varepsilon^{\frac{N+2}{2}} \lesssim \begin{cases} \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}} & \text{if } N \geq 5, \\ \varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1}) + \mu_\varepsilon & \text{if } N = 4. \end{cases}$$

To evaluate the remaining integral over $\mathsf{b}_{i,\varepsilon}$, we use

$$|-\Delta r_{i,\varepsilon}| = \left| u_\varepsilon^{\frac{N+2}{N-2}} - B_{i,\varepsilon}^{\frac{N+2}{N-2}} - \varepsilon V u_{i,\varepsilon} \right| \lesssim B_{i,\varepsilon}^{\frac{4}{N-2}} r_{i,\varepsilon} + \varepsilon B_{i,\varepsilon} \quad \text{on } \mathsf{b}_{i,\varepsilon}.$$

The term containing ε is bounded by

$$\varepsilon \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon} \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y \leq \varepsilon \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon} \frac{1}{|x_{i,\varepsilon}-y|^{N-2}} \, \mathrm{d}y \lesssim \begin{cases} \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} & \text{if } N \geq 5, \\ \varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1}) & \text{if } N = 4. \end{cases}$$

Here, the first inequality follows by the Hardy–Littlewood rearrangement inequality (see e.g. [22, Theorem 3.4]), because both B and $z \mapsto |z|^{-N+2}$ are symmetric-decreasing functions.

To control the last remaining term, we choose some $\theta \in (0,1) \cup (1,2)$ and reinsert the bound already proved for this θ. This yields

$$\begin{aligned}
&\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}}(y) |r_{i,\varepsilon}(y)| \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y \\
&\lesssim \left(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3-\theta} + \mu_\varepsilon^{\frac{N-2}{2}} \right) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}}(y) |x_{i,\varepsilon} - y|^\theta \frac{1}{|x-y|^{N-2}} \, \mathrm{d}y \\
&= \left(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}+\theta} \right) \int_{B(0,\delta_0 \mu_{i,\varepsilon}^{-1})} B^{\frac{4}{N-2}}(z) |z|^\theta \frac{1}{|z - \frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}|^{N-2}} \, \mathrm{d}z \\
&\leq \left(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}+\theta} \right) \int_{\mathbb{R}^N} (1+|z|^2)^{-2+\frac{\theta}{2}} \frac{1}{|z - \frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}|^{N-2}} \, \mathrm{d}z \\
&\leq \left(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}+\theta} \right) \int_{\mathbb{R}^N} (1+|z|^2)^{-2+\frac{\theta}{2}} \frac{1}{|z|^{N-2}} \, \mathrm{d}z \\
&\lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+3} + \mu_\varepsilon^{\frac{N-2}{2}+\theta}.
\end{aligned}$$

The second-to-last inequality follows again from the Hardy–Littlewood rearrangement inequality, because $z \mapsto (1+|z|^2)^{-2+\frac{\theta}{2}}$ and $z \mapsto |z|^{-N+2}$ are symmetric-decreasing functions. Combining all the above estimates, the proof in the case $\theta = 0$ is complete. ■

2.2. The bound on $q_{i,\varepsilon}$

Proposition 2.5. *Let $i = 1, \dots, n$ and let $q_{i,\varepsilon}$ be defined by (2.2). As $\varepsilon \to 0$, for all $\nu \in (2,3)$,*

$$|q_{i,\varepsilon}(x)| \lesssim \big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big)|x - x_{i,\varepsilon}|^\nu \quad \textit{for all } x \in \mathsf{b}_{i,\varepsilon}.$$

Proof. Let $Q_{i,\varepsilon}(x) := \frac{q_{i,\varepsilon}(x)}{|x-x_{i,\varepsilon}|^\nu}$, fix a point $z_{i,\varepsilon}$ with $Q_{i,\varepsilon}(z_{i,\varepsilon}) \geq \frac{1}{2}\|Q_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}$ and let $d_{i,\varepsilon} := |x_{i,\varepsilon} - z_{i,\varepsilon}|$. When $d_{i,\varepsilon} \gtrsim 1$, we have

$$\begin{aligned} Q_{i,\varepsilon}(x) \lesssim \frac{q_{i,\varepsilon}(z_{i,\varepsilon})}{d_{i,\varepsilon}^\nu} &\lesssim |B_{i,\varepsilon}(z_{i,\varepsilon})| + \Big|\varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3} W\Big(\frac{z_{i,\varepsilon}-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\Big| \\ &\lesssim \mu_\varepsilon^{\frac{N-2}{2}} + \varepsilon\mu_\varepsilon^{-\frac{N}{2}+3} \lesssim \mu_\varepsilon^{\frac{N-2}{2}}, \end{aligned} \tag{2.10}$$

where we used Lemma 2.3 and the fact that W is bounded by Lemma A.1. So it remains to treat the case $d_{i,\varepsilon} = o(1)$ in the following.

In the following, let us assume $N \geq 6$. Then $\frac{N+2}{N-2} \leq 2$ and the equation satisfied by $q_{i,\varepsilon}$ can be written as

$$-\Delta q_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}} q_{i,\varepsilon} = \varepsilon B_{i,\varepsilon}(V(x_{i,\varepsilon}) - V(x)) - \varepsilon V r_{i,\varepsilon} + \mathcal{O}\big(r_{i,\varepsilon}^{\frac{N+2}{N-2}}\big) \quad \text{on } \mathsf{b}_{i,\varepsilon}.$$

(When $N = 4, 5$, and hence $\frac{N+2}{N-2} > 2$, the last term needs to be replaced by $\mathcal{O}(r_{i,\varepsilon}^2 B_{i,\varepsilon}^{\frac{6-N}{N-2}})$.)

Then $\bar{q}_{i,\varepsilon}(x) := \frac{q_{i,\varepsilon}(x_{i,\varepsilon}+d_{i,\varepsilon}x)}{q_{i,\varepsilon}(z_{i,\varepsilon})}$ satisfies

$$\begin{aligned} &-\Delta\bar{q}_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}}\bar{q}_{i,\varepsilon} \\ &\quad = \frac{d_{i,\varepsilon}^{2-\nu}}{\|Q_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}}\big(\varepsilon\bar{B}_{i,\varepsilon}(V(x_{i,\varepsilon}) - \bar{V}) - \varepsilon\bar{V}\bar{r}_{i,\varepsilon} + \mathcal{O}\big(\bar{r}_{i,\varepsilon}^{\frac{N+2}{N-2}}\big)\big) \end{aligned}$$

and

$$\bar{q}_{i,\varepsilon}(0) = \nabla\bar{q}_{i,\varepsilon}(0) = 0, \quad |\bar{q}_{i,\varepsilon}(x)| \leq |x|^\nu \ \text{ on } B(d_{i,\varepsilon}^{-1}\delta, 0).$$

By Proposition 2.4 with $\theta = \nu - 1$,

$$|\bar{r}_{i,\varepsilon}(x)| \leq \big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big)d_\varepsilon^{\nu-1}|x|.$$

Then, by Lemma 2.3, and using $N \geq 6$ and $d_{i,\varepsilon} \lesssim 1$, we see that $|\bar{r}_\varepsilon| \leq 1$ for ε small, which gives

$$\begin{aligned} &-\Delta\bar{q}_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}}\bar{q}_{i,\varepsilon} \\ &\quad = \frac{1}{\|Q_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}}\mathcal{O}(\varepsilon\bar{B}_{i,\varepsilon}d_{i,\varepsilon}^{3-\nu} + d_{i,\varepsilon}^{2-\nu}|\bar{r}_{i,\varepsilon}|) \\ &\quad = \frac{1}{\|Q_{i,\varepsilon}\|_{L^\infty(\mathsf{b}_{i,\varepsilon})}}\mathcal{O}\big(\varepsilon\bar{B}_{i,\varepsilon}d_{i,\varepsilon}^{3-\nu} + \varepsilon\mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big). \end{aligned} \tag{2.11}$$

For completeness, we show how to bound the term $\mathcal{O}(r_{i,\varepsilon}^2 B_{i,\varepsilon}^{\frac{N+2}{N-2}-2})$ that occurs for $N = 4, 5$. We have, by Proposition 2.4 with $2\theta \in [\nu-2, \nu-2+6-N]$,

$$d_{i,\varepsilon}^{2-\nu} r_{i,\varepsilon}^2 B_{i,\varepsilon}^{\frac{N+2}{N-2}-2} \lesssim \begin{cases} \big(\varepsilon^2 \mu_\varepsilon^{-N+6-2\theta} + \mu_\varepsilon^{\frac{N-2}{2}}\big) d_{i,\varepsilon}^{2-\nu+2\theta} \mu_\varepsilon^{-\frac{6-N}{2}} & \text{if } d_{i,\varepsilon} \lesssim \mu_{i,\varepsilon}, \\ \big(\varepsilon^2 \mu_\varepsilon^{-N+6-2\theta} + \mu_\varepsilon^{\frac{N-2}{2}}\big) d_{i,\varepsilon}^{2-\nu+2\theta+N-6} \mu_\varepsilon^{\frac{6-N}{2}} & \text{if } d_{i,\varepsilon} \gtrsim \mu_{i,\varepsilon}, \end{cases}$$
$$\lesssim \varepsilon^2 \mu_\varepsilon^{-\frac{N}{2}+5-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}.$$

Let us now estimate the remaining first term on the right-hand side of (2.11). By (2.8) and the fact that $2 < \nu < 3$, we have

$$\varepsilon \bar{B}_{i,\varepsilon} d_{i,\varepsilon}^{3-\nu} \lesssim \begin{cases} \varepsilon \mu_\varepsilon^{-\frac{N-2}{2}} d_\varepsilon^{3-\nu} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\nu} & \text{if } d_{i,\varepsilon} \lesssim \mu_{i,\varepsilon}, \\ \varepsilon \mu_\varepsilon^{\frac{N-2}{2}} d_{i,\varepsilon}^{-N+5-\nu} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\nu} & \text{if } \mu_{i,\varepsilon} \lesssim d_{i,\varepsilon} \ll 1. \end{cases}$$

In both the cases $d_\varepsilon \lesssim \mu_\varepsilon$ and $o(1) = d_\varepsilon \gtrsim \mu_\varepsilon$, the blow-up argument detailed in the proof of Proposition 2.4 now yields that $Q_{i,\varepsilon}$ is bounded by a constant times $\varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}$. Taking (2.10) into account, we get the conclusion. ∎

2.3. The bound on $p_{i,\varepsilon}$

Proposition 2.6. *Let $i = 1, \dots, n$ and let $p_{i,\varepsilon}$ be defined by (2.3). As $\varepsilon \to 0$, for all $\nu \in (3, 4)$,*

$$|p_{i,\varepsilon}(x)| \lesssim \big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+5-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big)|x - x_{i,\varepsilon}|^\nu \quad \textit{for all } x \in \mathrm{b}_{i,\varepsilon}.$$

Proof. The proof works exactly the same as those of Propositions 2.4 and 2.5. There is only one subtlety that we point out; the rest is exactly the same. Let $P_{i,\varepsilon}(x) := \frac{p_{i,\varepsilon}(x)}{|x-x_{i,\varepsilon}|^\nu}$, fix a point $z_{i,\varepsilon}$ with $P_{i,\varepsilon}(z_{i,\varepsilon}) \geq \frac{1}{2}\|P_{i,\varepsilon}\|_{L^\infty(\mathrm{b}_{i,\varepsilon})}$ and let $d_{i,\varepsilon} := |x_{i,\varepsilon} - z_{i,\varepsilon}|$. When $d_{i,\varepsilon} \gtrsim 1$, we have

$$P_{i,\varepsilon}(x) \lesssim \mu_\varepsilon^{\frac{N-2}{2}}.$$

So it remains to treat the case $d_{i,\varepsilon} = o(1)$ in the following. We also assume $N \geq 6$. Then the equation satisfied by $p_{i,\varepsilon}$ can be written as

$$-\Delta p_{i,\varepsilon} - N(N+2) B_{i,\varepsilon}^{\frac{4}{N-2}} p_{i,\varepsilon} = \varepsilon B_{i,\varepsilon}(V(x_{i,\varepsilon}) + \nabla V(x_{i,\varepsilon}) \cdot x - V(x))$$
$$- \varepsilon V(W_{i,\varepsilon} + q_{i,\varepsilon}) + \mathcal{O}\big(r_{i,\varepsilon}^{\frac{N+2}{N-2}}\big) \quad \text{on } \mathrm{b}_{i,\varepsilon}.$$

(When $N = 4, 5$, and hence $\frac{N+2}{N-2} > 2$, the last term needs to be replaced by $\mathcal{O}(r_{i,\varepsilon}^2 B_{i,\varepsilon}^{\frac{N+2}{N-2}-2})$. This term can be estimated identically to the proof of Proposition 2.5. Notice that the range $2\theta \in [\nu-2, \nu-2+6-N]$ is still compatible with $\theta \in (0, 2)$ and $\nu \in (3, 4)$, and that the resulting bound $\varepsilon^2 \mu_\varepsilon^{-\frac{N}{2}+5-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}$ is strong enough for the present case as well.)

Then $\bar{p}_{i,\varepsilon}(x) := \frac{p_{i,\varepsilon}(x_{i,\varepsilon}+d_{i,\varepsilon}x)}{p_{i,\varepsilon}(z_{i,\varepsilon})}$ satisfies

$$\begin{aligned}&-\Delta \bar{p}_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}}\bar{p}_{i,\varepsilon}\\ &\quad = \frac{d_{i,\varepsilon}^{2-\nu}}{\|P_{i,\varepsilon}\|_{L^\infty(\mathrm{b}_{i,\varepsilon})}}\big(\varepsilon \bar{B}_{i,\varepsilon}(\bar{V}(0) + \nabla \bar{V}(0)\cdot x - \bar{V}) - \varepsilon \bar{V}\bar{W}_{i,\varepsilon} - \varepsilon \bar{V}\bar{q}_{i,\varepsilon}\\ &\qquad + \mathcal{O}(\bar{W}_{i,\varepsilon} + \bar{q}_{i,\varepsilon})^{\frac{N+2}{N-2}}\big)\end{aligned}$$

and

$$\bar{p}_{i,\varepsilon}(0) = \nabla \bar{p}_{i,\varepsilon}(0) = 0, \quad |\bar{p}_{i,\varepsilon}(x)| \le |x|^\nu \ \text{ on } B(d_{i,\varepsilon}^{-1}\delta, 0).$$

By Proposition 2.5 applied with exponent $\nu - 1 \in (2,3)$,

$$|\bar{q}_{i,\varepsilon}(x)| \lesssim \big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+5-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big)d_\varepsilon^{\nu-1}. \tag{2.12}$$

Since $N \ge 6$, this implies by Lemma 2.3 that $|\bar{q}_{i,\varepsilon}| \lesssim 1$. Using Lemmas 2.3 and A.1, we also have $\bar{W}_{i,\varepsilon} \lesssim 1$. Then

$$-\Delta \bar{p}_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}}\bar{p}_{i,\varepsilon} = \frac{1}{\|P_{i,\varepsilon}\|_{L^\infty(\mathrm{b}_{i,\varepsilon})}}\mathcal{O}\big(\varepsilon d_{i,\varepsilon}^{4-\nu}\bar{B}_{i,\varepsilon} + d_{i,\varepsilon}^{2-\nu}(\bar{W}_{i,\varepsilon} + \bar{q}_{i,\varepsilon})\big).$$

Moreover, we easily check, since $W(0) = \nabla W(0) = 0$, that

$$d_{i,\varepsilon}^{2-\nu}|\bar{W}_{i,\varepsilon}| = \mathcal{O}\big(\varepsilon \mu^{-\frac{N}{2}+5-\nu}\big)$$

which gives with (2.12),

$$-\Delta \bar{p}_{i,\varepsilon} - N(N+2)B_{i,\varepsilon}^{\frac{4}{N-2}}\bar{p}_{i,\varepsilon} = \frac{1}{\|P_{i,\varepsilon}\|_{L^\infty(\mathrm{b}_{i,\varepsilon})}}\mathcal{O}\big(\varepsilon d_{i,\varepsilon}^{4-\nu}\bar{B}_{i,\varepsilon} + \varepsilon \mu_\varepsilon^{-\frac{N}{2}+5-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}\big).$$

Let us now estimate the remaining first term on the right-hand side of this equation. By (2.8) and the fact that $3 < \nu < 4$, we have

$$\varepsilon \bar{B}_{i,\varepsilon} d_{i,\varepsilon}^{4-\nu} \lesssim \begin{cases} \varepsilon \mu_\varepsilon^{-\frac{N-2}{2}} d_\varepsilon^{4-\nu} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+5-\nu} & \text{if } d_{i,\varepsilon} \lesssim \mu_{i,\varepsilon}, \\ \varepsilon \mu_\varepsilon^{\frac{N-2}{2}} d_{i,\varepsilon}^{-N+6-\nu} \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+5-\nu} & \text{if } \mu_{i,\varepsilon} \lesssim d_{i,\varepsilon} \ll 1. \end{cases}$$

In both the cases $d_\varepsilon \lesssim \mu_\varepsilon$ and $o(1) = d_\varepsilon \gtrsim \mu_\varepsilon$, the blow-up argument detailed in the proof of Proposition 2.4 now yields that $P_{i,\varepsilon}$ is bounded by a constant times $\varepsilon \mu_\varepsilon^{-\frac{N}{2}+5-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}$. ∎

3. The main expansions

We will also need the matrix $\widetilde{M}^l(\boldsymbol{x}) \in \mathbb{R}^{n\times n} = (\widetilde{m}_{ij}^l(\boldsymbol{x}))_{i,j=1}^n$ with entries

$$\widetilde{m}_{ij}^l(\boldsymbol{x}) := \begin{cases} \partial_l \phi(x_i) & \text{for } i = j, \\ -2\partial_l^x G(x_i, x_j) & \text{for } i \ne j. \end{cases}$$

Recall that the matrix $M(\boldsymbol{x})$ was defined in (1.2).

The main results of this section are collected in the following two propositions.

Proposition 3.1. *If $N \geq 5$, as $\varepsilon \to 0$,*

$$\sum_j m_{ij}(\boldsymbol{x}_\varepsilon)\mu_{j,\varepsilon}^{\frac{N-2}{2}} = -d_N(V(x_{i,\varepsilon}) + o(1))\varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3} + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big), \tag{3.1}$$

where d_N is given by (1.4).

If $N = 4$, as $\varepsilon \to 0$,

$$\sum_j m_{ij}(\boldsymbol{x}_\varepsilon)\mu_{j,\varepsilon} = -\frac{1}{8\pi^2}(V(x_{i,\varepsilon}) + o(1))\varepsilon\mu_{i,\varepsilon}\ln(\mu_{i,\varepsilon}^{-1}) + \mathcal{O}(\mu_\varepsilon^3). \tag{3.2}$$

Proposition 3.2. *If $N \geq 5$, as $\varepsilon \to 0$, for every $l = 1, \dots, N$ and every $\delta > 0$,*

$$\sum_j \tilde{m}_{ij}^l(\boldsymbol{x}_\varepsilon)\mu_{j,\varepsilon}^{\frac{N-2}{2}} = -d_N\frac{N-2}{2}\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}(\partial_{x_l}V(x_{i,\varepsilon}) + o(1)) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}-\delta}\big), \tag{3.3}$$

where d_N is given by (1.4).

If $N = 4$, as $\varepsilon \to 0$, for every $l = 1, \dots, N$ and every $\delta > 0$,

$$\sum_j \tilde{m}_{ij}^l(\boldsymbol{x}_\varepsilon)\mu_{j,\varepsilon}^{\frac{N-2}{2}} = -\frac{1}{8\pi^2}(\partial_{x_l}V(x_{i,\varepsilon}) + o(1))\varepsilon\mu_\varepsilon\ln(\mu_\varepsilon^{-1}) + \mathcal{O}(\mu_\varepsilon^{3-\delta}).$$

Proof of Proposition 3.1. We multiply equation (1.1) by $G(x, x_{i,\varepsilon})$ and integrate over x. Then the left-hand side becomes

$$\begin{aligned}
&\int_\Omega(-\Delta u_\varepsilon + \varepsilon V u_\varepsilon)G(x, x_{i,\varepsilon})\,\mathrm{d}x \\
&\quad = u_\varepsilon(x_{i,\varepsilon}) + \mu_{i,\varepsilon}^{-\frac{N}{2}+3}\varepsilon V(x_{i,\varepsilon})\int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} B\frac{1}{\omega_{N-1}(N-2)|z|^{N-2}}\,\mathrm{d}z \\
&\qquad + o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big).
\end{aligned} \tag{3.4}$$

The right-hand side is

$$\begin{aligned}
&N(N-2)\int_\Omega u_\varepsilon^{\frac{N+2}{N-2}}G(x, x_{i,\varepsilon})\,\mathrm{d}x \\
&\quad = N(N-2)\sum_j\int_{\mathsf{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{N+2}{N-2}}G(x, x_{j,\varepsilon})\,\mathrm{d}x \\
&\qquad + \varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3}V(x_{i,\varepsilon})N(N+2)\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}}W\Big(\frac{x - x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)G(\cdot, x_{i,\varepsilon})\,\mathrm{d}x \\
&\qquad + \mathcal{O}\bigg(\int_{\mathsf{b}_{i,\varepsilon}}(B_{i,\varepsilon}^{\frac{4}{N-2}}q_{i,\varepsilon} + |r_{i,\varepsilon}|^{\frac{N+2}{N-2}})G(x, x_{i,\varepsilon})\,\mathrm{d}x \\
&\qquad\quad + \sum_{j\neq i}\int_{\mathsf{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{4}{N-2}}|r_{j,\varepsilon}|G(\cdot, x_{i,\varepsilon})\,\mathrm{d}x + \int_{\Omega\setminus\bigcup_j\mathsf{b}_{j,\varepsilon}} u_\varepsilon^{\frac{N+2}{N-2}}G(\cdot, x_{i,\varepsilon})\,\mathrm{d}x\bigg)
\end{aligned} \tag{3.5}$$

When $N = 4, 5$, similarly to the remark in the proof of Proposition 2.5, the term $r_{i,\varepsilon}^{\frac{N+2}{N-2}}$ in the above error term needs to be replaced by $B_{i,\varepsilon}^{\frac{N+2}{N-2}-2} r_{i,\varepsilon}^2$. The ensuing estimates are very similar to the case $N \geq 6$ presented below and we leave the details to the reader.

Let us first evaluate the two main terms in (3.5). We have

$$\sum_j \int_{\mathsf{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{N+2}{N-2}} G(x, x_{j,\varepsilon})\,\mathrm{d}x = \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{N+2}{N-2}} \Big(\frac{1}{\omega_{N-1}(N-2)|x-x_{i,\varepsilon}|^{N-2}} - H(x, x_{i,\varepsilon})\Big)\,\mathrm{d}x$$
$$+ \sum_{j\neq i} \int_{\mathsf{b}_{i,\varepsilon}} B_{j,\varepsilon}^{\frac{N+2}{N-2}} G(x, x_{i,\varepsilon})\,\mathrm{d}x.$$

We compute the terms on the right-hand side separately. First, by direct computation,

$$\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{N+2}{N-2}} \frac{N}{\omega_{N-1}|x-x_{i,\varepsilon}|^{N-2}}\,\mathrm{d}x = \mu_{i,\varepsilon}^{-\frac{N-2}{2}} + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big).$$

Next, by the radial symmetry of B and the mean value property of the harmonic function $x \mapsto H(x, x_{i,\varepsilon})$, it is easy to see that

$$-N(N-2) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{N+2}{N-2}} H(x, x_{i,\varepsilon})\,\mathrm{d}x = -N(N-2)\phi(x_{i,\varepsilon}) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{N+2}{N-2}}\,\mathrm{d}x$$
$$= -\omega_{N-1}(N-2)\mu_{i,\varepsilon}^{\frac{N-2}{2}} \phi(x_{i,\varepsilon}) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big).$$

Finally, by a similar argument, using that $G(x, x_{i,\varepsilon})$ is harmonic for $x \in \mathsf{b}_{j,\varepsilon}$, for every $j \neq i$ we have

$$N(N-2) \int_{\mathsf{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{N+2}{N-2}} G(x, x_{i,\varepsilon})\,\mathrm{d}x = N(N-2)G(x_{j,\varepsilon}, x_{i,\varepsilon}) \int_{\mathsf{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{N+2}{N-2}}\,\mathrm{d}x$$
$$= \omega_{N-1}(N-2)\mu_{j,\varepsilon}^{\frac{N-2}{2}} G(x_{j,\varepsilon}, x_{i,\varepsilon}) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big).$$

This completes the computation of the first main term of (3.5).

Using that $N(N+2)B^{\frac{4}{N-2}}W = -\Delta W + B$ by the equation satisfied by W, the second main term of (3.5) can be rewritten as

$$\varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon})N(N+2) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} W\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big) G(x, x_{i,\varepsilon})\,\mathrm{d}x$$
$$= \varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon})N(N+2) \int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} B^{\frac{4}{N-2}} W \frac{1}{\omega_{N-1}(N-2)|z|^{N-2}}\,\mathrm{d}z$$
$$+ o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big)$$
$$= \varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon}) \int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} (-\Delta W) \frac{1}{\omega_{N-1}(N-2)|z|^{N-2}}\,\mathrm{d}z$$
$$+ \varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon}) \int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} B \frac{1}{\omega_{N-1}(N-2)|z|^{N-2}}\,\mathrm{d}z + o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big).$$

The second term cancels precisely with the corresponding term in (3.4). The term containing ΔW can be evaluated as follows. By Green's formula and $W(0) = 0$, for every $R > 0$,

$$\int_{B_R} (-\Delta W(z))|z|^{-N+2} \, dz = \int_{\partial B_R} W \frac{\partial |z|^{-N+2}}{\partial \nu} - \frac{\partial W}{\partial \nu} |z|^{-N+2}$$
$$= -\omega_{N-1}(W'(R)R + W(R)(N-2)).$$

By Lemma A.1 we have

$$\begin{cases} W'(R) = o(R^{-1}), & W(R) \to \frac{c_N}{N-2} \text{ as } R \to \infty \quad \text{if } N \geq 5, \\ W'(R) = o(R^{-1} \ln R), & W(R) = \frac{1}{2} \ln(R) + o(\ln R) \quad \text{if } N = 4, \end{cases}$$

with $c_N = \frac{\Gamma(N/2)\Gamma((N-4)/2)}{\Gamma(N-1)}$. Thus

$$\frac{\varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} V(x_{i,\varepsilon})}{\omega_{N-1}(N-2)} \int_{B(0,\delta_0 \mu_{i,\varepsilon}^{-1})} (-\Delta W) \frac{1}{|z|^{N-2}} \, dz$$
$$= \begin{cases} -\frac{c_N}{N-2} \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} (V(x_{i,\varepsilon}) + o(1)), & N \geq 5, \\ -\frac{1}{2} \varepsilon \mu_{i,\varepsilon} \ln \mu_{i,\varepsilon}^{-1} (V(x_{i,\varepsilon}) + o(1)), & N = 4. \end{cases}$$

Putting everything together and observing that the divergent terms $u(x_{i,\varepsilon}) = \mu_{i,\varepsilon}^{-\frac{N-2}{2}}$ cancel precisely, we obtain the assertion, provided that we can prove that the error terms from above are negligible, i.e.

$$\int_{b_{i,\varepsilon}} (B_{i,\varepsilon}^{\frac{4}{N-2}} q_{i,\varepsilon} + r_{i,\varepsilon}^{\frac{N+2}{N-2}}) G(x, x_{i,\varepsilon}) \, dx + \sum_{j \neq i} \int_{\mathrm{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{4}{N-2}} r_{j,\varepsilon} G(x, x_{i,\varepsilon}) \, dx$$
$$+ \int_{\Omega \setminus \bigcup_j \mathrm{b}_{j,\varepsilon}} u_\varepsilon^{\frac{N+2}{N-2}} G(x, x_{i,\varepsilon}) \, dx$$
$$= o\big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big). \tag{3.6}$$

To bound the first error term, we apply Proposition 2.5 with $2 < \nu < 3$. Then

$$\left| \int_{b_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} q_{i,\varepsilon} G(x, x_{i,\varepsilon}) \, dx \right| \lesssim (\varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\nu} + \mu_\varepsilon^{\frac{N-2}{2}}) \mu_\varepsilon^\nu \int_{B(0,\delta_0 \mu_\varepsilon^{-1})} B^{\frac{4}{N-2}} |x|^{-N+2+\nu} \, dx$$
$$\lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+6-\nu} + \mu_\varepsilon^{\frac{N+2}{2}}$$
$$= o\big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big)$$

because $\nu < 3$. For the next term, we observe that

$$|r_{i,\varepsilon}|^{\frac{N+2}{N-2}} \lesssim (\varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3})^{\frac{N+2}{N-2}} W(\mu_\varepsilon^{-1}(x - x_{i,\varepsilon}))^{\frac{N+2}{N-2}} + |q_{i,\varepsilon}|^{\frac{N+2}{N-2}}$$
$$\lesssim \mu_\varepsilon^{\frac{N+2}{2}} + B_{i,\varepsilon}^{\frac{4}{N-2}} |q_{i,\varepsilon}|,$$

where we used Lemma 2.3, $|q_{i,\varepsilon}| \lesssim B_{i,\varepsilon}$ and the fact that W is bounded by Lemma A.1. Thus

$$\int_{b_{i,\varepsilon}} r_{i,\varepsilon}^{\frac{N+2}{N-2}} G(x, x_{i,\varepsilon})\,\mathrm{d}x \lesssim \mu_\varepsilon^{\frac{N+2}{2}} \int_{\mathsf{b}_{i,\varepsilon}} \frac{1}{|x - x_{i,\varepsilon}|^{N-2}}\,\mathrm{d}x + \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} q_{i,\varepsilon} G(x, x_{i,\varepsilon})\,\mathrm{d}x$$
$$= o\big(\mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big),$$

by the bound we have already proved.

Next, for any $j \neq i$, by Proposition 2.4, for fixed $\theta \in (0, 2)$ we estimate

$$\left|\int_{\mathsf{b}_{j,\varepsilon}} B_{j,\varepsilon}^{\frac{4}{N-2}} r_{j,\varepsilon} G(x, x_{i,\varepsilon})\,\mathrm{d}x\right| \lesssim \big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3-\theta} + \mu_\varepsilon^{\frac{N-2}{2}}\big)\mu_\varepsilon^{-2+N+\theta} \int_{B(0,\delta_0\mu_\varepsilon^{-1})} B^{\frac{4}{N-2}} |x|^\theta\,\mathrm{d}x$$
$$\lesssim \varepsilon\mu_\varepsilon^{-\frac{N}{2}+5-\theta} + \mu_\varepsilon^{\frac{N+2}{2}}$$
$$= o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big)$$

because $\theta < 2$. Finally, to estimate the last remaining term in (3.6), we simply recall $u_\varepsilon \lesssim \sum_j B_{j,\varepsilon} \lesssim \mu_\varepsilon^{\frac{N-2}{2}}$ as well as $G(x, x_{i,\varepsilon}) \lesssim 1$ on $\Omega \setminus \bigcup_j \mathsf{b}_{j,\varepsilon}$, so that

$$\int_{\Omega\setminus\bigcup_j \mathsf{b}_{j,\varepsilon}} u_\varepsilon^{\frac{N+2}{N-2}} G(x, x_{i,\varepsilon})\,\mathrm{d}x \lesssim \mu_\varepsilon^{\frac{N+2}{2}}.$$

This completes the proof of (3.6), and hence of the proposition. ∎

Proof of Proposition 3.2. The overall strategy and the nature of the multiple estimates needed is very similar to the preceding proof of Proposition 3.1, which is why in the following we will be briefer in places.

We multiply equation (1.1) against $\partial_{y_l} G(x, x_{i,\varepsilon})$ and integrate over $\mathrm{d}x$. Since by definition of G and $x_{i,\varepsilon}$,

$$\int_\Omega u_\varepsilon \nabla_y G(x, x_{i,\varepsilon})\,\mathrm{d}x = \nabla u_\varepsilon(x_{i,\varepsilon}) = 0,$$

the resulting identity is (for any fixed $l = 1, \dots, N$)

$$\varepsilon \int_\Omega V u_\varepsilon \partial_{y_l} G(x, x_{i,\varepsilon})\,\mathrm{d}x = N(N-2) \int_\Omega u_\varepsilon^{\frac{N+2}{N-2}} \partial_{y_l} G(x, x_{i,\varepsilon})\,\mathrm{d}x. \tag{3.7}$$

In the following, we will repeatedly decompose

$$\nabla_y G(x, x_{i,\varepsilon}) = \frac{1}{\omega_{N-1}} \frac{x - x_{i,\varepsilon}}{|x - x_{i,\varepsilon}|^N} - \nabla_y H(x, x_{i,\varepsilon})$$

and use that $\nabla H(\cdot, x_{i,\varepsilon})$ is bounded on Ω.

We first evaluate the left-hand side. Since $u_\varepsilon \lesssim \sum_j B_{j,\varepsilon}$, clearly

$$\int_{\Omega\setminus\bigcup \mathsf{b}_{j,\varepsilon}} \varepsilon V u_\varepsilon \partial_{y_l} G(x, x_{i,\varepsilon})\,\mathrm{d}x = \mathcal{O}\big(\varepsilon\mu_\varepsilon^{\frac{N-2}{2}}\big) = \begin{cases} o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big) & \text{if } N \geq 5, \\ o(\varepsilon\mu_\varepsilon \ln(\mu_\varepsilon^{-1})) & \text{if } N = 4. \end{cases}$$

On $\mathsf{b}_{i,\varepsilon}$, we have

$$\varepsilon \int_{\mathsf{b}_{i,\varepsilon}} V u_\varepsilon \nabla_y H(x, x_{i,\varepsilon}) \, \mathrm{d}x = \mathcal{O}\big(\varepsilon \mu_\varepsilon^{\frac{N-2}{2}}\big) = \begin{cases} o\big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}\big) & \text{if } N \geq 5, \\ o(\varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1})) & \text{if } N = 4. \end{cases}$$

To evaluate the integrals involving the singular term of ∇G, we also decompose $u_\varepsilon = B_{i,\varepsilon} + r_{i,\varepsilon}$ and $V(x) = V(x_{i,\varepsilon}) + \nabla V(x_{i,\varepsilon}) \cdot (x - x_{i,\varepsilon}) + o(|x - x_{i,\varepsilon}|)$.

Then, by antisymmetry the main term vanishes, namely

$$\varepsilon V(x_{i,\varepsilon}) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon} \frac{(x - x_{i,\varepsilon})_l}{|x - x_{i,\varepsilon}|^N} \, \mathrm{d}x = 0.$$

The gradient term, for every $l = 1, \dots, N$, yields, if $N \geq 5$,

$$\begin{aligned} &\frac{\varepsilon}{\omega_{N-1}} \partial_{x_l} V(x_{i,\varepsilon}) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon} \frac{(x - x_{i,\varepsilon})_l^2}{|x - x_{i,\varepsilon}|^N} \, \mathrm{d}x \\ &\qquad = \frac{1}{\omega_{N-1}} \varepsilon \mu_{i,\varepsilon}^{-\frac{N}{2}+3} \partial_{x_l} V(x_{i,\varepsilon}) \int_{B(0, \delta_0 \mu_{i,\varepsilon}^{-1})} B \frac{z_l^2}{|z|^N} \, \mathrm{d}z. \end{aligned}$$

If $N = 4$, this gives

$$\begin{aligned} &\frac{\varepsilon}{\omega_{N-1}} \partial_{x_l} V(x_{i,\varepsilon}) \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon} \frac{(x - x_{i,\varepsilon})_l^2}{|x - x_{i,\varepsilon}|^N} \, \mathrm{d}x \\ &\qquad = \frac{1}{4} \varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1}) (\partial_{x_l} V(x_{i,\varepsilon}) + o(1)). \end{aligned}$$

If $N \geq 5$, this term will exactly cancel with another contribution coming from the error term in $q_{i,\varepsilon}$ on the right-hand side.

Finally, by the bound for $\theta = 0$ from Proposition 2.4 and Lemma 2.3,

$$\begin{aligned} &\varepsilon \int_{\mathsf{b}_{i,\varepsilon}} V |r_{i,\varepsilon}| \frac{x - x_{i,\varepsilon}}{|x - x_{i,\varepsilon}|^N} \, \mathrm{d}x \\ &\quad \lesssim \begin{cases} \varepsilon^2 \mu_\varepsilon^{-\frac{N}{2}+3} + \varepsilon \mu_\varepsilon^{\frac{N-2}{2}} = o\big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}\big) & \text{if } N \geq 5, \\ \varepsilon^2 \mu_\varepsilon \ln(\mu_\varepsilon^{-1}) + \varepsilon \mu_\varepsilon \lesssim \varepsilon \mu_\varepsilon = o(\varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1})) & \text{if } N = 4. \end{cases} \end{aligned}$$

Let us now turn to evaluating the right-hand side of (3.7). Since

$$\int_{\Omega \setminus \bigcup \mathsf{b}_{j,\varepsilon}} u_\varepsilon^{\frac{N+2}{N-2}} \nabla_y G(x, x_{i,\varepsilon}) \, \mathrm{d}x = \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big),$$

we only need to consider integrals over the balls $\mathsf{b}_{j,\varepsilon}$. On $\mathsf{b}_{i,\varepsilon}$, we split

$$\nabla_y G(x, x_{i,\varepsilon}) = \frac{1}{\omega_{N-1}(N-2)} \frac{x - x_{i,\varepsilon}}{|x - x_{i,\varepsilon}|^N} - \nabla_y H(x, x_{i,\varepsilon}).$$

To treat the singular term, write $u_\varepsilon = B_{i,\varepsilon} + r_{i,\varepsilon} = B_{i,\varepsilon} + W_{i,\varepsilon} + q_{i,\varepsilon}$. By antisymmetry, the terms involving $B_{i,\varepsilon}^{\frac{N+2}{N-2}}$ and $B_{i,\varepsilon}^{\frac{4}{N-2}} W_{i,\varepsilon}$ vanish. Thus

$$\begin{aligned}
&\left|\int_{\mathsf{b}_{i,\varepsilon}} u_\varepsilon^{\frac{N+2}{N-2}} \frac{x-x_{i,\varepsilon}}{|x-x_{i,\varepsilon}|^N}\,\mathrm{d}x\right| \\
&\quad = \int_{\mathsf{b}_{i,\varepsilon}} \big(B_{i,\varepsilon}^{\frac{4}{N-2}}|q_{i,\varepsilon}| + |r_{i,\varepsilon}|^{\frac{N+2}{N-2}}\big)|x-x_{i,\varepsilon}|^{-N+1} \\
&\quad = \int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} q_{i,\varepsilon} \frac{x-x_{i,\varepsilon}}{|x-x_{i,\varepsilon}|^N}\,\mathrm{d}x \\
&\qquad + \begin{cases} \mathcal{O}\big((\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3})^{\frac{N+2}{N-2}} + \mu_\varepsilon^{\frac{N+2}{2}}\big) = o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big) & \text{if } N \geq 5, \\ \mathcal{O}\big(+\varepsilon^3\mu_\varepsilon^3(\ln(\mu_\varepsilon^{-1}))^3 + \mu_\varepsilon^3\big) = o(\varepsilon\mu_\varepsilon\ln(\mu_\varepsilon^{-1})) + \mathcal{O}(\mu_\varepsilon^3) & \text{if } N = 4, \end{cases}
\end{aligned}$$

by Proposition 2.4 with $\theta = 0$.

Let us extract the contribution from the term in $q_{i,\varepsilon}$. When $N = 4$, Proposition 2.5 yields

$$\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} |q_{i,\varepsilon}| \frac{1}{|x-x_{i,\varepsilon}|^{N-1}}\,\mathrm{d}x \lesssim \mu_\varepsilon = o(\varepsilon\mu_\varepsilon\ln(\mu_\varepsilon^{-1})).$$

So for $N = 4$ the term is negligible. Let us now look at $N \geq 5$. By Proposition 2.6 with any $\nu \in (3,4)$, we have

$$\begin{aligned}
\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} |p_{i,\varepsilon}| \frac{1}{|x-x_{i,\varepsilon}|^{N-1}}\,\mathrm{d}x &\lesssim \varepsilon\mu_\varepsilon^{-\frac{N}{2}+7-\nu} + \mu_\varepsilon^{\frac{N+2}{2}} \\
&= o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big).
\end{aligned}$$

Finally, using $N(N+2)B^{\frac{4}{N-2}}W_2 = -\Delta W_2 + B|x|$, we get

$$\begin{aligned}
&\frac{N(N+2)}{\omega_{N-1}}\varepsilon\mu_{i,\varepsilon}^{-\frac{N}{2}+4}\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} W_2\Big(\frac{x-x_{i,\varepsilon}}{\mu_{i,\varepsilon}}\Big)\nabla V(x_{i,\varepsilon})\cdot\frac{x-x_{i,\varepsilon}}{|x-x_{i,\varepsilon}|^N} \\
&\quad = \frac{1}{\omega_{N-1}}\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\partial_l V(x_{i,\varepsilon})\int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} N(N+2)B^{\frac{4}{N-2}}W_2\frac{z_l^2}{|z|^{N+1}}\,\mathrm{d}z \\
&\quad = \frac{1}{\omega_{N-1}}\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\partial_l V(x_{i,\varepsilon})\int_{B(0,\delta_0\mu_{i,\varepsilon}^{-1})} (-\Delta W_2 + B|x|)\frac{z_l^2}{|z|^{N+1}}\,\mathrm{d}z.
\end{aligned}$$

The term in $B|x|$ cancels precisely with the term from the left-hand side pointed out above. The term in $-\Delta W_2$, arguing as in the proof of Proposition 3.1, gives

$$\begin{aligned}
&\frac{1}{\omega_{N-1}}\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\partial_l V(x_{i,\varepsilon})\int_{B(0,R)} (-\Delta W_2 + B|x|)\frac{z_l^2}{|z|^{N+1}}\,\mathrm{d}z \\
&\quad = -\frac{\partial_l V(x_{i,\varepsilon})}{N}\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}((N-1)W_2(R)R^{-1} + W_2'(R)) \\
&\quad = -\partial_l V(x_{i,\varepsilon})\frac{a_N}{N}\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3} + o\big(\varepsilon\mu_\varepsilon^{-\frac{N}{2}+3}\big),
\end{aligned}$$

with $R = \delta_0\mu_{i,\varepsilon}^{-1}$ and a_N as in Lemma A.2.

This finishes the discussion of the term in $q_{i,\varepsilon}$.

Now we evaluate the integral over $\mathsf{b}_{i,\varepsilon}$ against $\nabla_y H(x, x_{i,\varepsilon})$, for which we again decompose $u_\varepsilon = B_{i,\varepsilon} + r_{i,\varepsilon}$. Taylor expanding,

$$\partial_{y_l} H(x, x_{i,\varepsilon}) = \partial_{y_l} H(x_{i,\varepsilon}, x_{i,\varepsilon}) + \nabla_x \partial_{y_l} H(x_{i,\varepsilon}, x_{i,\varepsilon}) \cdot (x - x_{i,\varepsilon}) + \mathcal{O}(|x - x_{i,\varepsilon}|^2),$$

and using that the gradient term cancels by antisymmetry, we find

$$\begin{aligned} -\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{N+2}{N-2}} \partial_{y_l} H(x, x_{i,\varepsilon}) \, \mathrm{d}x &= -\frac{\omega_{N-1}}{N} \mu_{i,\varepsilon}^{\frac{N-2}{2}} \partial_{y_l} H(x_{i,\varepsilon}, x_{i,\varepsilon}) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}} \ln(\mu_\varepsilon^{-1})\big) \\ &= -\frac{\omega_{N-1}}{2N} \mu_{i,\varepsilon}^{\frac{N-2}{2}} \partial_{x_l} \phi(x_{i,\varepsilon}) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2} - \delta}\big), \end{aligned}$$

which is (the diagonal part of) the main term we desired to extract. On the other hand, since $\nabla_y H(x, x_{i,\varepsilon})$ is bounded, the principal remainder term in $r_{i,\varepsilon}$, by Proposition 2.4 with $\theta \in [0, 1)$, is bounded by

$$\int_{\mathsf{b}_{i,\varepsilon}} B_{i,\varepsilon}^{\frac{4}{N-2}} |r_{i,\varepsilon}| \, \mathrm{d}x \lesssim \varepsilon \mu_\varepsilon^{-\frac{N}{2}+4-\theta} + \mu_\varepsilon^{\frac{N+2}{2}} = \begin{cases} o\big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}-\delta}\big) & \text{if } N \geq 5, \\ o(\varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1})) + \mathcal{O}(\mu_\varepsilon^{3-\delta}) & \text{if } N = 4. \end{cases}$$

Finally, on $\mathsf{b}_{j,\varepsilon}$ with $j \neq i$, analogous computations permit us to extract the remaining (off-diagonal) part of the main term as

$$\begin{aligned} \int_{\mathsf{b}_{j,\varepsilon}} u_\varepsilon^{\frac{N+2}{N-2}} \partial_{y_l} G(x, x_{i,\varepsilon}) \, \mathrm{d}x = {} & \frac{\omega_{N-1}}{N} \partial_{y_l} G(x_{j,\varepsilon}, x_{i,\varepsilon}) \mu_{j,\varepsilon}^{\frac{N-2}{2}} \\ & + \begin{cases} o\big(\varepsilon \mu_\varepsilon^{-\frac{N}{2}+3}\big) + \mathcal{O}\big(\mu_\varepsilon^{\frac{N+2}{2}}\big) & \text{if } N \geq 5, \\ o(\varepsilon \mu_\varepsilon \ln(\mu_\varepsilon^{-1})) + \mathcal{O}(\mu_\varepsilon^{3-\delta}) & \text{if } N = 4. \end{cases} \end{aligned}$$

Combining everything, and observing that $\frac{2N}{N(N-2)\omega_{N-1}} \frac{a_N}{N} = d_N \frac{N-2}{2}$, with d_N given by (1.4), the proof is complete. ∎

4. Proof of Theorem 1.1

We now show how the expansions (3.1) and (3.3) can be used to conclude the proof of Theorem 1.1.

We introduce the vector $\boldsymbol{\lambda}_\varepsilon \in (0, \infty)^n$ with components

$$(\boldsymbol{\lambda}_\varepsilon)_i := \lambda_{i,\varepsilon} := \left(\frac{\mu_{i,\varepsilon}}{\mu_{1,\varepsilon}} \right)^{\frac{N-2}{2}},$$

and note that $\lambda_{i,\varepsilon}$ is bounded away from 0 and ∞ by Proposition 2.1.

Let us rewrite (3.1) and (3.2) as

$$(M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\lambda}_\varepsilon)_i+\mathcal{O}(\mu_\varepsilon^2)=\begin{cases}-d_N\varepsilon\mu_{i,\varepsilon}^{-N+4}(V(x_{i,\varepsilon})+o(1))\lambda_{i,\varepsilon} & \text{if } N\geq 5,\\ -(8\pi^2)^{-1}\varepsilon\ln\mu_{i,\varepsilon}^{-1}(V(x_{i,\varepsilon})+o(1)) & \text{if } N=4.\end{cases} \tag{4.1}$$

By Perron–Frobenius theory (see [3]), the lowest eigenvalue $\rho(\boldsymbol{x}_\varepsilon)$ of $M(\boldsymbol{x}_\varepsilon)$ is simple and the associated eigenvector $\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$, normalized so that $(\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon))_1=1$, has strictly positive entries.

Taking the scalar product of (4.1) with $\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$ shows

$$\begin{aligned}\rho(\boldsymbol{x}_\varepsilon)\langle\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon),\boldsymbol{\lambda}_\varepsilon\rangle&=\langle\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon),M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\lambda}_\varepsilon\rangle\\&=\begin{cases}-d_N\varepsilon\sum_i\mu_{i,\varepsilon}^{-N+4}(V(x_{i,\varepsilon})+o(1))\lambda_{i,\varepsilon}(\Lambda(x_\varepsilon))_i+o(1) & \text{if } N\geq 5,\\ -(8\pi^2)^{-1}\varepsilon\sum_i\ln\mu_{i,\varepsilon}^{-1}(V(x_{i,\varepsilon})+o(1))\lambda_{i,\varepsilon}(\Lambda(x_\varepsilon))_i+o(1) & \text{if } N=4.\end{cases}\end{aligned} \tag{4.2}$$

Since $\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$ and $\boldsymbol{\lambda}_\varepsilon$ both have strictly positive entries, and since $V<0$ by assumption, this shows that $0<\rho(\boldsymbol{x}_\varepsilon)$ for all $\varepsilon>0$. For the limit $\rho(\boldsymbol{x}_0)$, two cases are possible.

Case 1*:* $\rho(\boldsymbol{x}_0)>0$. Assume $N\geq 5$ first. In this case, (4.2) shows that $\lim_{\varepsilon\to 0}\varepsilon\mu_{i,\varepsilon}^{-N+4}>0$. (Note that this limit always exists up to a subsequence and is finite as a consequence of Lemma 2.3.)

Introducing the variable

$$\kappa_{i,\varepsilon}:=\left(\varepsilon^{\frac{1}{-N+4}}\mu_{i,\varepsilon}\right)^{\frac{N-2}{2}}$$

we can write (3.1) as

$$(M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\kappa}_\varepsilon)_i=-d_NV(x_{i,\varepsilon})\kappa_{i,\varepsilon}^{-q},$$

with

$$q:=\frac{N-6}{N-2}.$$

Moreover, (3.3) can be written in terms of $\boldsymbol{\kappa}_\varepsilon$ as

$$(\widetilde{M}^l(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\kappa}_\varepsilon)_i=-d_N\frac{N-2}{2}\partial_{x_l}V(x_{i,\varepsilon})\kappa_{i,\varepsilon}^{-q}.$$

Since $\partial_{\kappa_k}\langle\boldsymbol{\kappa},M(\boldsymbol{x})\boldsymbol{\kappa}\rangle=(M(\boldsymbol{x})\cdot\kappa)_k$ and $\partial_{(x_k)_l}\langle\boldsymbol{\kappa},M(\boldsymbol{x})\boldsymbol{\kappa}\rangle=\frac{1}{2}\kappa_k(\widetilde{M}^l(\boldsymbol{x})\cdot\kappa)_k$, the point $(\boldsymbol{\kappa}_0,\boldsymbol{x}_0)$ is a critical point of F defined in (1.3).

Since $\rho(\boldsymbol{x}_0)>0$ in this case, $M(\boldsymbol{x}_0)$ is strictly positive definite. If additionally $q\geq 0$ (i.e. $N\geq 6$), then $D^2_{\boldsymbol{\kappa}}F(\kappa,\boldsymbol{x}_0)$ is strictly positive definite for every $\boldsymbol{\kappa}$. We obtain that $F(\boldsymbol{\kappa},\boldsymbol{x}_0)$ is convex in the variable $\boldsymbol{\kappa}$ on $(0,\infty)$, hence it has a unique critical point. This is the desired characterization of κ_0, and hence of $\lim_{\varepsilon\to 0}\varepsilon\mu_{i,\varepsilon}^{-N+4}=\kappa_{i,0}^{-2\frac{N-4}{N-2}}$.

If $N=4$, we find in a similar way that $\lim_{\varepsilon\to 0}\varepsilon\ln(\mu_\varepsilon^{-1})>0$. To characterize the limit, we argue slightly differently. Since $\varepsilon\ln\mu_{i,\varepsilon}^{-1}=\varepsilon\ln\mu_{1,\varepsilon}^{-1}+o(1)=:\kappa_0+o(1)$, passing to the limit in (4.1) gives

$$(M(\boldsymbol{x}_0)\cdot\boldsymbol{\lambda}_0)_i=\frac{1}{8\pi^2}|V(x_{i,0})|\kappa_0\lambda_{i,0}.$$

Similarly, the identity from Proposition 3.2 reads

$$(\widetilde{M}^l(\boldsymbol{x}_0)\cdot\boldsymbol{\lambda}_0)_i=\frac{1}{8\pi^2}|V(x_{i,0})|\kappa_0\lambda_{i,0}.$$

This shows that $(\boldsymbol{\lambda}_0,\boldsymbol{x}_0)$ is a critical for $\widetilde{F}(\boldsymbol{\lambda},\boldsymbol{x})$ as given in (1.5).

Finally. let us discuss the property of κ_0. If we define

$$M_1(\kappa):=M(\boldsymbol{x}_0)-\frac{\kappa}{8\pi^2}\operatorname{diag}(|V(x_{i,0})|),$$

this can be written as $M_1(\kappa_0)\cdot\boldsymbol{\lambda}_0=0$, i.e. $\boldsymbol{\lambda}_0$ is a zero eigenvalue of $M_1(\kappa_0)$. Since $M_1(\kappa)$ differs from $M(\boldsymbol{x}_0)$ only on the diagonal, the Perron–Frobenius arguments used above can still be applied to $M_1(\kappa)$. Thus $\boldsymbol{\lambda}_0$ must be the lowest eigenvector of $M_1(\kappa_0)$, because it has strictly positive entries. Since $V<0$, the lowest eigenvalue of $M_1(\kappa)$ is clearly a strictly monotonic function of κ, so κ_0 is indeed unique with the property that the lowest eigenvalue of $M_1(\kappa_0)$ equals zero.

This completes the proof of Theorem 1.1 in the case $\rho(\boldsymbol{x}_0)>0$.

Case 2: $\rho(\boldsymbol{x}_0)=0$. In this case, (4.2) shows that $\lim_{\varepsilon\to 0}\varepsilon\mu_\varepsilon^{-N+4}=0$ and that $\boldsymbol{\lambda}_0$ is an eigenvector with eigenvalue 0. Since $(\boldsymbol{\lambda}_0)_1=1=(\boldsymbol{\Lambda}(\boldsymbol{x}_0))_1$ and $\rho(\boldsymbol{x}_0)$ is simple, we have in fact $\boldsymbol{\lambda}_0=\boldsymbol{\Lambda}(\boldsymbol{x}_0)$, i.e. $\boldsymbol{\lambda}_0$ is precisely the lowest eigenvector of $M(\boldsymbol{x}_0)$, with eigenvalue $\rho(\boldsymbol{x}_0)=0$.

For the following analysis, we decompose $\boldsymbol{\lambda}_\varepsilon=\alpha_\varepsilon\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)+\boldsymbol{\delta}(x_\varepsilon)$, where $\alpha_\varepsilon\in\mathbb{R}$, $\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$ is the lowest eigenvalue of $M(\boldsymbol{x}_\varepsilon)$ and $\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)\perp\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$. Notice that $\alpha_\varepsilon\to 1$ as a consequence of $\boldsymbol{\lambda}_\varepsilon\to\boldsymbol{\Lambda}(\boldsymbol{x}_0)$.

Here is the central piece of information which we need to conclude in this case.

Proposition 4.1. *As $\varepsilon\to 0$,*

$$|\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)|=\begin{cases}\mathcal{O}(\varepsilon\mu_\varepsilon^{-N+4}+\mu_\varepsilon^2\ln(\mu_\varepsilon^{-1})+|\rho(\boldsymbol{x}_\varepsilon)|) & \text{if } N\geq 5,\\ \mathcal{O}(\varepsilon\ln(\mu_\varepsilon^{-1})+\mu_\varepsilon^2\ln(\mu_\varepsilon^{-1})+|\rho(\boldsymbol{x}_\varepsilon)|) & \text{if } N=4.\end{cases}\tag{4.3}$$

Suppose moreover that $\rho(\boldsymbol{x}_0)=0$. Then, as $\varepsilon\to 0$,

$$\rho(\boldsymbol{x}_\varepsilon)=\begin{cases}o(\varepsilon\mu_\varepsilon^{-N+4}+\mu_\varepsilon^2) & \text{if } N\geq 5,\\ o(\varepsilon\ln(\mu_\varepsilon^{-1})+\mu_\varepsilon^2) & \text{if } N=4.\end{cases}\tag{4.4}$$

Before we prove Proposition 4.1, let us use it to conclude the proof of Theorem 1.1 in the present case $\rho(\boldsymbol{x}_0)=0$.

Taking the scalar product of identity (4.1) with $\lambda_{i,\varepsilon}$ and using the properties of $\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$ and $\boldsymbol{\delta}_\varepsilon$, we obtain

$$\begin{aligned}&\rho(\boldsymbol{x}_\varepsilon)|\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)|^2\alpha_\varepsilon^2 + \langle \boldsymbol{\delta}(\boldsymbol{x}_\varepsilon), M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)\rangle + \mathcal{O}(\mu_\varepsilon^2)\\ &\quad = \begin{cases} -d_N\varepsilon \sum_i \mu_{i,\varepsilon}^{-N+4}(V(x_{i,\varepsilon})+o(1))\lambda_{i,\varepsilon}^2 & \text{if } N\geq 5,\\ -(8\pi^2)^{-1}\varepsilon\sum_i \ln \mu_{i,\varepsilon}^{-1}(V(x_{i,\varepsilon})+o(1)) & \text{if } N=4.\end{cases}\end{aligned}$$

The crucial information given by Proposition 4.1 is now that the terms in $\rho(\boldsymbol{x}_\varepsilon)$ and in $\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)$ on the left-hand side are negligible. Since $V<0$ and $\lambda_{i,\varepsilon}\sim 1$, the above identity then implies $\varepsilon\mu_{i,\varepsilon}^{-N+4} = \mathcal{O}(\mu_\varepsilon^2)$ if $N\geq 5$, resp. $\varepsilon\ln(\mu_{i,\varepsilon}^{-1}) = \mathcal{O}(\mu_\varepsilon^2)$ if $N=4$, as claimed. This completes the proof of Theorem 1.1.

Proof of Proposition 4.1. Arguing as in [20, Lemma 5.5], we get

$$\begin{aligned}\partial_l^{x_i}\rho(\boldsymbol{x}_\varepsilon) &= \partial_l^{x_i}\langle\boldsymbol{\lambda}_\varepsilon, M_a(\boldsymbol{x})\cdot\boldsymbol{\lambda}_\varepsilon\rangle|_{\boldsymbol{x}=\boldsymbol{x}_\varepsilon} + \mathcal{O}(|\rho(\boldsymbol{x}_\varepsilon)|+|\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)|)\\ &= \lambda_{i,\varepsilon}(\widetilde{M}_a^l(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\lambda}_\varepsilon)_i + \mathcal{O}(|\rho(\boldsymbol{x}_\varepsilon)|+|\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)|).\end{aligned}$$

Inserting the bound from Proposition 3.2, we thus get, for every $\delta>0$,

$$|\nabla\rho(\boldsymbol{x}_\varepsilon)| = \begin{cases}\mathcal{O}(\varepsilon\mu_\varepsilon^{-N+4}+\mu_\varepsilon^{2-\delta}+|\rho(\boldsymbol{x}_\varepsilon)|+|\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)|) & \text{if } N\geq 5,\\ \mathcal{O}(\varepsilon\ln(\mu_\varepsilon^{-1})+\mu_\varepsilon^{2-\delta}+|\rho(\boldsymbol{x}_\varepsilon)|+|\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)|) & \text{if } N=4.\end{cases} \tag{4.5}$$

On the other hand, writing $M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\lambda}_\varepsilon = \alpha_\varepsilon\rho(\boldsymbol{x}_\varepsilon)\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)+M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)$, (4.1) implies

$$M(\boldsymbol{x}_\varepsilon)\cdot\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon) = \begin{cases}\mathcal{O}(\varepsilon\mu_\varepsilon^{-N+4}+\mu_\varepsilon^2\ln(\mu_\varepsilon^{-1})+|\rho(\boldsymbol{x}_\varepsilon)|) & \text{if } N\geq 5,\\ \mathcal{O}(\varepsilon\ln(\mu_\varepsilon^{-1})+\mu_\varepsilon^2\ln(\mu_\varepsilon^{-1})+|\rho(\boldsymbol{x}_\varepsilon)|) & \text{if } N=4.\end{cases}$$

Since $\rho(\boldsymbol{x}_\varepsilon)$ is simple, $M(\boldsymbol{x}_\varepsilon)$ is uniformly coercive on the subspace orthogonal to $\boldsymbol{\Lambda}(\boldsymbol{x}_\varepsilon)$, which contains $\boldsymbol{\delta}(\boldsymbol{x}_\varepsilon)$. Hence (4.3) follows.

Moreover, with (4.3) we can simplify (4.5) to

$$|\nabla\rho(\boldsymbol{x}_\varepsilon)| = \begin{cases}\mathcal{O}(\varepsilon\mu_\varepsilon^{-N+4}+\mu_\varepsilon^{2-\delta}+|\rho(\boldsymbol{x}_\varepsilon)|) & \text{if } N\geq 5,\\ \mathcal{O}(\varepsilon\ln(\mu_\varepsilon^{-1})+\mu_\varepsilon^{2-\delta}+|\rho(\boldsymbol{x}_\varepsilon)|) & \text{if } N=4.\end{cases} \tag{4.6}$$

Now we claim that there is $\sigma>1$ such that

$$\rho(\boldsymbol{x}_\varepsilon) \lesssim |\nabla\rho(\boldsymbol{x}_\varepsilon)|^\sigma. \tag{4.7}$$

If we choose $\delta>0$ so small that $(2-\delta)\sigma>2$, together with (4.6) this yields

$$\rho(\boldsymbol{x}_\varepsilon) = \begin{cases} o(\varepsilon\mu_\varepsilon^{-N+4}+\mu_\varepsilon^2)+\mathcal{O}(\rho(\boldsymbol{x}_\varepsilon)^\sigma) & \text{if } N\geq 5,\\ o(\varepsilon\ln(\mu_\varepsilon^{-1})+\mu_\varepsilon^2)+\mathcal{O}(\rho(\boldsymbol{x}_\varepsilon)^\sigma) & \text{if } N=4.\end{cases}$$

Here we used that the assumption $\rho(\boldsymbol{x}_0) = 0$ implies $\varepsilon\mu_\varepsilon^{-N+4} = o(1)$, resp. $\varepsilon \ln(\mu_\varepsilon^{-1}) = o(1)$, as observed above. Hence we have $(\varepsilon\mu_\varepsilon^{-N+4})^\sigma = o(\varepsilon\mu_\varepsilon^{-N+4})$ and $(\varepsilon \ln(\mu_\varepsilon^{-1}))^\sigma = o(\varepsilon \ln(\mu_\varepsilon^{-1}))$.

In the same way, since $\rho(\boldsymbol{x}_\varepsilon) = o(1)$, we can absorb $\mathcal{O}(\rho(\boldsymbol{x}_\varepsilon)^\sigma) = o(\rho(\boldsymbol{x}_\varepsilon))$ into the left-hand side and (4.4) follows, as desired. With this information, we can return to (4.6) to deduce the bound on $|\nabla\rho(x_\varepsilon)|$ claimed in Theorem 1.1.

So it remains only to justify (4.7). This follows by arguing as in [20, proof of Theorem 2.1] once we note that $\rho(\boldsymbol{x})$ is an analytic function of $\boldsymbol{x}$. Indeed, $\rho(\boldsymbol{x})$ is a simple eigenvalue of the matrix $M(\boldsymbol{x})$. Hence it depends analytically on $\boldsymbol{x}$ if the entries of $M(\boldsymbol{x})$ do so. But this is clearly the case: $G_0(\cdot, y)$ is harmonic, hence analytic on $\Omega \setminus \{y\}$, and $H_0(\cdot, y)$ is harmonic, hence analytic on all of Ω, hence so is $\phi(x) = H(x,x)$. The proof is therefore complete. ∎

A. Some computations

Lemma A.1. *Let W be the unique radial solution to*

$$-\Delta W - N(N+2)B^{\frac{4}{N-2}} W = -B, \quad W(0) = \nabla W(0) = 0.$$

Then, as $R \to \infty$,

$$W(R) = \begin{cases} \dfrac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{(N-2)\Gamma(N-1)} + o(1) & \text{if } N \geq 5, \\ \frac{1}{2}\ln R + o(\ln R) & \text{if } N = 4, \end{cases}$$

and

$$W'(R) = \begin{cases} o(R^{-1}) & \text{if } N \geq 5, \\ o(R^{-1}\ln R) & \text{if } N = 4. \end{cases}$$

Proof. By the variation of constants ansatz, we write $W = v\varphi$ with

$$v(x) = \frac{1-|x|^2}{(1+|x|^2)^{N/2}},$$

which solves $-\Delta v = vB^{\frac{4}{N-2}}$. Then $\psi := \varphi'$ solves

$$\psi'(r) + \Big(\frac{N-1}{r} + \frac{2v'}{v}\Big)\psi = \frac{B}{v}.$$

Again by the variation of constants, we may write $\psi = \eta\psi_0$, with

$$\psi_0(r) := \exp\Big(-\int_1^r \Big(\frac{N-1}{s} + \frac{2v'}{v}\Big)\,\mathrm{d}s\Big) = \frac{1}{r^{N-1}v^2}.$$

Since $\psi_0'(r) + (\frac{N-1}{r} + \frac{2v'}{v})\psi_0 = 0$, it remains to solve

$$\eta' = \frac{B}{v\psi_0} = Bvr^{N-1},$$

which gives

$$\eta(r) = \int_0^r Bs^{N-1}v \,\mathrm{d}s = \int_0^r \frac{s^{N-1}(1-s^2)}{(1+s^2)^{N-1}} \,\mathrm{d}s.$$

If $N \geq 5$, this integral remains finite as $r \to \infty$ and we find, using the integral representation of the beta function,

$$\lim_{r\to\infty} \eta(r) = -\frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{\Gamma(N-1)}.$$

On the other hand, if $N = 4$, the integral diverges and we have

$$\eta(r) = (-1 + o(1)) \ln r \quad \text{as } r \to \infty.$$

Using $v(r) \sim -r^{-N+2}$, we moreover find

$$\psi_0(r) \sim r^{N-3}$$

and hence

$$\psi(r) = \eta(r)\psi_0(r) \sim \begin{cases} -\dfrac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{\Gamma(N-1)} r^{N-3} & \text{if } N \geq 5, \\ -r \ln r & \text{if } N = 4, \end{cases}$$

respectively

$$\varphi(r) = \int_0^r \psi(s) \,\mathrm{d}s \sim \begin{cases} -\dfrac{1}{(N-2)} \dfrac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{\Gamma(N-1)} r^{N-2} & \text{if } N \geq 5, \\ -\frac{1}{2} r^2 \ln r & \text{if } N = 4. \end{cases}$$

By recalling $W = v\varphi$ the claimed asymptotic behavior of W follows.

Similarly, using $v'(r) \sim (N-2)^2 r^{-N+1}$ and the above asymptotics for φ and ψ, we get

$$W'(r) = v'(r)\varphi(r) + v(r)\psi(r) = o(r^{-1}),$$

because the terms of size r^{-1} cancel precisely, and similarly for $N = 4$. This completes the proof. ∎

A very similar argument, whose details we omit, yields the asymptotics of W_2 arising as the main term of $q_{i,\varepsilon}$.

Lemma A.2. *Let W_2 solve*

$$-W_2''(r) - \frac{N-1}{r} W_2'(r) + \frac{N-1}{r^2} W_2(r) - N(N+2) B(r)^{\frac{4}{N-2}} W_2(r) = -B(r) r \quad \text{on } (0,\infty)$$

with $W_2(0) = W_2'(0) = 0$. Then

$$\lim_{R\to\infty} W_2(R) R^{-1} = \lim_{R\to\infty} W_2'(R) = \frac{a_N}{N},$$

with

$$a_N = \frac{N}{4} \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{\Gamma(N-1)}.$$

B. Classical asymptotic analysis

In this section we generalize the result of [13] to $N \geq 3$ under appropriate assumptions. The proof is globally the same except at the level of Claim B.4 where some refined analysis is needed when $N \geq 4$. As already mentioned in [13], the proof follows [11].

Proposition B.1. *Consider a sequence (u_ε) of C^2 solutions to*

$$\begin{cases} -\Delta u_\varepsilon + h_\varepsilon u_\varepsilon = N(N-2) u_\varepsilon^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \\ u_\varepsilon > 0 & \text{in } \Omega, \end{cases} \tag{B.1}$$

where Ω is some smooth domain of $\mathbb{R}^N$ and

$$h_\varepsilon \to h_0 \quad \text{in } C^{0,\eta}(\Omega) \text{ as } \varepsilon \to 0$$

if $N = 3$, or

$$h_\varepsilon = \varepsilon V$$

where $V \in C^1(\Omega) \cup C(\overline{\Omega})$ with $V < 0$ on $\overline{\Omega}$ if $N \geq 4$.

Then either $\|u_\varepsilon\|_\infty$ is bounded or, up to extracting a subsequence, there exists $n \in \mathbb{N}$ and points $x_{1,\varepsilon}, \dots, x_{n,\varepsilon}$ such that the following hold:

(i) $x_{i,\varepsilon} \to x_i \in \Omega$ *for some $x_i \in \Omega$ with $x_i \neq x_j$ for $i \neq j$.*

(ii) $\mu_{i,\varepsilon} := u_\varepsilon(x_{i,\varepsilon})^{-\frac{2}{N-2}} \to 0$ *as $\varepsilon \to 0$ and $\nabla u_\varepsilon(x_{i,\varepsilon}) = 0$ for every i.*

(iii) $\lambda_{i,0} := \lim_{\varepsilon\to 0} \lambda_{i,\varepsilon} := \lim_{\varepsilon\to 0} \mu_{i,\varepsilon}^{\frac{N-2}{2}} / \mu_{1,\varepsilon}^{\frac{N-2}{2}}$ *exists and lies in $(0,\infty)$ for every i.*

(iv) $\mu_{i,\varepsilon}^{\frac{N-2}{2}} u_\varepsilon(x_{i,\varepsilon} + \mu_{i,\varepsilon} x) \to B$ *in $C^1_{\text{loc}}(\mathbb{R}^n)$.*

(v) *There are $\nu_i > 0$ such that $\mu_{1,\varepsilon}^{-\frac{N-2}{2}} u_\varepsilon \to \sum_i \nu_i G(x_{i,\varepsilon}, \cdot) =: \widetilde{\mathcal{G}}$ uniformly in C^1 away from $\{x_1, \dots, x_n\}$, where G is the Green function of $-\Delta + h_0$.*

(vi) *There is $C > 0$ such that $u_\varepsilon \leq C \sum_i B_{i,\varepsilon}$ on Ω.*

The proof is divided into many steps. The first one consists in transforming a weak estimate such as (B.3) into a strong one such as (B.4) around a concentration point, that is to say, at a certain scale u_ε behaves like a bubble. So we consider a sequence u_ε which satisfies the hypotheses of Proposition B.1 and we also assume that we have a sequence (x_ε) of points in Ω and a sequence (ρ_ε) of positive real numbers with $0 < 3\rho_\varepsilon \le d(x_\varepsilon, \partial\Omega)$ such that

$$\nabla u_\varepsilon(x_\varepsilon) = 0$$

and

$$\rho_\varepsilon \Big[\sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon(x) \Big]^{\frac{2}{N-2}} \to +\infty \quad \text{as } \varepsilon \to 0. \tag{B.2}$$

First, we prove that, under this extra assumption, the following holds:

Proposition B.2. *If there exists $C_0 > 0$ such that*

$$|x_\varepsilon - x|^{\frac{N-2}{2}} u_\varepsilon \le C_0 \quad \text{in } B(x_\varepsilon, 3\rho_\varepsilon), \tag{B.3}$$

then there exists $C_1 > 0$ such that

$$\begin{aligned} u_\varepsilon(x_\varepsilon) u_\varepsilon(x) &\le C_1 |x_\varepsilon - x|^{2-N} \quad \text{in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}, \\ u_\varepsilon(x_\varepsilon) |\nabla u_\varepsilon(x)| &\le C_1 |x_\varepsilon - x|^{1-N} \quad \text{in } B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}. \end{aligned} \tag{B.4}$$

Moreover, if $\rho_\varepsilon \to 0$, then

$$\rho_\varepsilon^{N-2} u_\varepsilon(x_\varepsilon) u_\varepsilon(x_\varepsilon + \rho_\varepsilon x) \to \frac{1}{|x|^{N-2}} + b \quad \text{in } C^1_{\mathrm{loc}}(B(0,2) \setminus \{0\}) \text{ as } \varepsilon \to 0,$$

where b is some harmonic function in $B(0,2)$ with $b(0) \le 0$ and $\nabla b(0) = 0$.

B.1. Proof of Proposition B.2

We divide the proof of the proposition into several claims. The first one gives the asymptotic behavior of u_ε around x_ε at an appropriate small scale.

Claim B.1. *After passing to a subsequence, we have*

$$\mu_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(x_\varepsilon + \mu_\varepsilon x) \to B \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^3) \text{ as } \varepsilon \to 0,$$

where $\mu_\varepsilon = u_\varepsilon(x_\varepsilon)^{\frac{2}{2-N}}$.

Proof. Let $\tilde{x}_\varepsilon \in \overline{B(x_\varepsilon, \rho_\varepsilon)}$ and $\tilde{\mu}_\varepsilon > 0$ be such that

$$u_\varepsilon(\tilde{x}_\varepsilon) = \sup_{B(x_\varepsilon,\rho_\varepsilon)} u_\varepsilon = \tilde{\mu}_\varepsilon^{\frac{2-N}{2}}.$$

Thanks to (B.2), we have

$$\tilde{\mu}_\varepsilon \to 0 \text{ and } \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon} \to +\infty \quad \text{as } \varepsilon \to 0. \tag{B.5}$$

Thanks to (B.3), we also have

$$|x_\varepsilon - \tilde{x}_\varepsilon| = O(\tilde{\mu}_\varepsilon). \tag{B.6}$$

We set, for $x \in \Omega_\varepsilon = \{x \in \mathbb{R}^N \text{ s.t. } \tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x \in \Omega\}$,

$$\tilde{u}_\varepsilon(x) = \tilde{\mu}_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x),$$

which verifies

$$\begin{aligned} -\Delta \tilde{u}_\varepsilon + \tilde{\mu}_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon &= N(N-2)\tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \quad \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon(0) &= \sup_{B\left(\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon}, \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon}\right)} \tilde{u}_\varepsilon = 1, \end{aligned} \tag{B.7}$$

where $\tilde{h}_\varepsilon = h(\tilde{x}_\varepsilon + \tilde{\mu}_\varepsilon x)$. Thanks to (B.5) and (B.6), we get

$$B\Big(\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon}, \frac{\rho_\varepsilon}{\tilde{\mu}_\varepsilon}\Big) \to \mathbb{R}^N \quad \text{as } \varepsilon \to 0. \tag{B.8}$$

Now, thanks to (B.7), (B.8), and by standard elliptic theory, we get that, after passing to a subsequence, $\tilde{u}_\varepsilon \to B$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $\varepsilon \to 0$, where B satisfies

$$-\Delta B = N(N-2)B^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N \quad \text{and} \quad 0 \le B \le 1 = U(0).$$

Thanks to the work of Caffarelli, Gidas and Spruck [7], we know that

$$B(x) = (1 + |x|^2)^{-\frac{N-2}{2}}.$$

Moreover, thanks to (B.6), we know that, after passing to a new subsequence, $\frac{x_\varepsilon - \tilde{x}_\varepsilon}{\tilde{\mu}_\varepsilon} \to x_0$ as $\varepsilon \to 0$ for some $x_0 \in \mathbb{R}^N$. Hence, since x_ε is a critical point of u_ε, x_0 must be a critical point of U, namely $x_0 = 0$. We deduce that $\frac{\mu_\varepsilon}{\tilde{\mu}_\varepsilon} \to 1$, where μ_ε is as in the statement of the claim. Claim B.1 follows. ∎

For $0 \le r \le 3\rho_\varepsilon$, we set

$$\psi_\varepsilon(r) = \frac{r^{\frac{N-2}{2}}}{\omega_{N-1} r^{N-1}} \int_{\partial B(x_\varepsilon, r)} u_\varepsilon \, d\sigma,$$

where $d\sigma$ denotes the Lebesgue measure on the sphere $\partial B(x_\varepsilon, r)$ and ω_{N-1} is the volume of the unit $(N-1)$-sphere. We easily check, thanks to Claim B.1, that

$$\begin{aligned} \psi_\varepsilon(\mu_\varepsilon r) &= \Big(\frac{r}{1+r^2}\Big)^{\frac{N-2}{2}} + o(1), \\ \psi_\varepsilon'(\mu_\varepsilon r) &= \frac{N-2}{2}\Big(\frac{r}{1+r^2}\Big)^{\frac{N}{2}}\Big(\frac{1}{r^2} - 1\Big) + o(1). \end{aligned} \tag{B.9}$$

We define r_ε by

$$r_\varepsilon = \max\big\{r \in [2\mu_\varepsilon, \rho_\varepsilon] \text{ s.t. } \psi_\varepsilon'(s) \le 0 \text{ for } s \in [2\mu_\varepsilon, r]\big\}.$$

Thanks to (B.9), the set on which the maximum is taken is not empty for ε small enough and, moreover,

$$\frac{r_\varepsilon}{\mu_\varepsilon} \to +\infty \quad \text{as } \varepsilon \to 0. \tag{B.10}$$

We now prove the following:

Claim B.2. *There exists $C > 0$, independent of ε, such that*

$$u_\varepsilon(x) \le C\mu_\varepsilon^{\frac{N-2}{2}} |x_\varepsilon - x|^{2-N} \qquad \text{in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\},$$
$$|\nabla u_\varepsilon(x)| \le C\mu_\varepsilon^{\frac{N-2}{2}} |x_\varepsilon - x|^{1-N} \qquad \text{in } B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\}.$$

Proof. We first prove that for any given $0 < \nu < \frac{1}{2}$, there exists $C_\nu > 0$ such that

$$u_\varepsilon(x) \le C_\nu\Big(\mu_\varepsilon^{\frac{N-2}{2}(1-2\nu)}|x - x_\varepsilon|^{(2-N)(1-\nu)} + \alpha_\varepsilon\Big(\frac{r_\varepsilon}{|x - x_\varepsilon|}\Big)^{(N-2)\nu}\Big) \tag{B.11}$$

for all $x \in B(x_\varepsilon, 2r_\varepsilon)$ and ε small enough, where

$$\alpha_\varepsilon = \sup_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon. \tag{B.12}$$

First of all, we can use (B.3) and apply the Harnack inequality (see Lemma D.1) to get the existence of some $C > 0$ such that

$$\frac{1}{C}\max_{\partial B(x_\varepsilon, r)}(u_\varepsilon + r|\nabla u_\varepsilon|) \le \frac{1}{\omega_{N-1} r^{N-1}}\int_{\partial B(x_\varepsilon, r)} u_\varepsilon \, d\sigma \le C \min_{\partial B(x_\varepsilon, r)} u_\varepsilon \tag{B.13}$$

for all $0 < r < \frac{5}{2}\rho_\varepsilon$ and all $\varepsilon > 0$. Hence, thanks to (B.9) and (B.10), we have

$$|x - x_\varepsilon|^{\frac{N-2}{2}} u_\varepsilon(x) \le C\psi_\varepsilon(r) \le C\psi_\varepsilon(R\mu_\varepsilon) = C\Big(\frac{R}{1 + R^2}\Big)^{\frac{N-2}{2}} + o(1)$$

for all $R \ge 2$, all $r \in [R\mu_\varepsilon, r_\varepsilon]$, all ε small enough and all $x \in \partial B(x_\varepsilon, r)$. Thus we get

$$\sup_{B(x_\varepsilon, r_\varepsilon)\setminus B(x_\varepsilon, R\mu_\varepsilon)} |x - x_\varepsilon|^{\frac{N-2}{2}} u_\varepsilon(x) = e(R) + o(1), \tag{B.14}$$

where $e(R) \to 0$ as $R \to +\infty$. Let $\mathcal{G}(x, y) = \frac{1}{(N-2)\omega_{N-1}}\frac{1}{|x-y|^{N-2}}$, in particular

$$-\Delta\mathcal{G}(\,, y) = \delta_y \quad \text{on } \mathbb{R}^N.$$

We fix $0 < \nu < \frac{1}{2}$ and we set

$$\Phi_{\varepsilon,\nu} = \mu_\varepsilon^{\frac{N-2}{2}(1-2\nu)}\mathcal{G}(x_\varepsilon, x)^{1-\nu} + \alpha_\varepsilon(r_\varepsilon^{N-2}\mathcal{G}(x_\varepsilon, x))^\nu.$$

Then (B.11) reduces to proving that

$$\sup_{B(x_\varepsilon, 2r_\varepsilon)} \frac{u_\varepsilon}{\Phi_{\varepsilon,\nu}} = \mathcal{O}(1).$$

We let $y_\varepsilon \in \overline{B(x_\varepsilon, 2r_\varepsilon) \setminus \{x_\varepsilon\}}$ be such that

$$\sup_{B(x_\varepsilon, 2r_\varepsilon)} \frac{u_\varepsilon}{\Phi_{\varepsilon,\nu}} = \frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)}.$$

We are going to consider the various possible behaviors of the sequence (y_ε).

First of all, assume that there is $R < \infty$ such that

$$\frac{|x_\varepsilon - y_\varepsilon|}{\mu_\varepsilon} \to R \quad \text{as } \varepsilon \to 0.$$

Thanks to Claim B.1, we have in this case that

$$\mu_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(y_\varepsilon) \to (1 + R^2)^{-\frac{N-2}{2}} \quad \text{as } \varepsilon \to 0.$$

On the other hand, we can write

$$\begin{aligned}
\mu_\varepsilon^{\frac{N-2}{2}} \Phi_{\varepsilon,\nu}(y_\varepsilon) &= \Big(\frac{\mu_\varepsilon^{N-2}}{(N-2)\omega_{N-1}|x_\varepsilon - y_\varepsilon|^{N-2}} \Big)^{1-\nu} + \mathcal{O}\Big(\alpha_\varepsilon \mu_\varepsilon^{\frac{N-2}{2}} \Big(\frac{r_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \Big)^{(N-2)\nu} \Big) \\
&= ((N-2)R^{N-2}\omega_{N-1})^{\nu-1} + \mathcal{O}\big((r_\varepsilon^{\frac{N-2}{2}} \alpha_\varepsilon) \mu_\varepsilon^{\frac{N-2}{2}(1-2\nu)} r_\varepsilon^{\frac{1}{2}(2\nu-1)} \big) \\
&= ((N-2)R^{N-2}\omega_{N-1})^{\nu-1} + o(1),
\end{aligned}$$

if $R > 0$, and $\mu_\varepsilon^{\frac{N-2}{2}} \Phi_{\varepsilon,\nu}(y_\varepsilon) \to +\infty$ as $\varepsilon \to 0$ if $R = 0$. In any case, $(\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)})$ is bounded.

Assume now that there exists $\delta > 0$ such that $y_\varepsilon \in B(x_\varepsilon, r_\varepsilon) \setminus B(x_\varepsilon, \delta r_\varepsilon)$. Thanks to Harnack's inequality (B.13), we get that $u_\varepsilon(y_\varepsilon) = \mathcal{O}(\alpha_\varepsilon)$, which easily gives that $\frac{u_\varepsilon(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)} = \mathcal{O}(1)$.

Hence, we are left with the following situation:

$$\frac{|x_\varepsilon - y_\varepsilon|}{r_\varepsilon} \to 0 \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|}{\mu_\varepsilon} \to +\infty \quad \text{as } \varepsilon \to 0. \tag{B.15}$$

Thanks to the definition of y_ε, we can then write

$$\frac{-\Delta u_\varepsilon(y_\varepsilon)}{u_\varepsilon(y_\varepsilon)} \geq \frac{-\Delta \Phi_{\varepsilon,\nu}(y_\varepsilon)}{\Phi_{\varepsilon,\nu}(y_\varepsilon)}.$$

Thanks to the definition of $\Phi_{\varepsilon,\nu}$ and multiplying by $|x_\varepsilon - y_\varepsilon|^2$, this gives

$$\begin{aligned}
&|x_\varepsilon - y_\varepsilon|^2 \big(-h_\varepsilon(y_\varepsilon) + N(N-2) u_\varepsilon(y_\varepsilon)^{\frac{4}{N-2}} \big) \\
&\geq \nu(1-\nu) \frac{|x_\varepsilon - y_\varepsilon|^2}{\Phi_{\varepsilon,\eta}(y_\varepsilon)} \Big(\alpha_\varepsilon r_\varepsilon^{(N-2)\nu} \frac{|\nabla \mathcal{G}(x_\varepsilon, y_\varepsilon)|^2}{\mathcal{G}(x_\varepsilon, y_\varepsilon)^2} \mathcal{G}(x_\varepsilon, y_\varepsilon)^\nu \\
&\qquad + \mu_\varepsilon^{\frac{N-2}{2}(1-2\nu)} \frac{|\nabla \mathcal{G}(x_\varepsilon, y_\varepsilon)|^2}{\mathcal{G}(x_\varepsilon, y_\varepsilon)^2} \mathcal{G}(x_\varepsilon, y_\varepsilon)^{1-\nu} \Big).
\end{aligned}$$

Thanks to (B.14), the left-hand side goes to 0 as $\varepsilon \to 0$. Then, thanks to (B.15), we get

$$o(1) \geq (N-2)^2 \nu(1-\nu) + o(1),$$

which is a contradiction, and shows that this last case cannot occur. This ends the proof of (B.11).

We now claim that there exists $C > 0$, independent of ε, such that

$$u_\varepsilon(x) \le C\big(\mu_\varepsilon^{\frac{N-2}{2}}|x-x_\varepsilon|^{2-N} + \alpha_\varepsilon\big) \quad \text{in } B(x_\varepsilon, r_\varepsilon). \tag{B.16}$$

Thanks to Claim B.1 and (B.13), this holds for all sequences $y_\varepsilon \in B(x_\varepsilon, r_\varepsilon) \setminus \{x_\varepsilon\}$ such that $|y_\varepsilon - x_\varepsilon| = \mathcal{O}(\mu_\varepsilon)$ or $\frac{|y_\varepsilon - x_\varepsilon|}{r_\varepsilon} \not\to 0$. Thus we may assume from now that

$$\frac{|y_\varepsilon - x_\varepsilon|}{\mu_\varepsilon} \to +\infty \quad \text{and} \quad \frac{|y_\varepsilon - x_\varepsilon|}{r_\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

Let us consider $\mathcal{G}_\varepsilon$, the Green function of the operator $-\Delta + h_\varepsilon$. This function exists since, by Appendix C, the operator is coercive; moreover, one has the classical estimate (see [2] or the nice notes [31])

$$\sup_{x \ne y} |x-y|^{n-2}|\mathcal{G}_\varepsilon(x,y)| + |x-y|^{n-1}|\nabla \mathcal{G}_\varepsilon(x,y)| = \mathcal{O}(1). \tag{B.17}$$

Thanks to the Green representation formula, we have

$$\begin{aligned} u_\varepsilon(y_\varepsilon) = {} & \int_{B(x_\varepsilon, r_\varepsilon)} \mathcal{G}_\varepsilon(y_\varepsilon, \cdot)(-\Delta u_\varepsilon + h_\varepsilon u_\varepsilon)\, \mathrm{d}x \\ & + \mathcal{O}\bigg(r_\varepsilon^{-(N-2)} \int_{\partial B(x_\varepsilon, r_\varepsilon)} |\partial_\nu u_\varepsilon|\, \mathrm{d}\sigma + r_\varepsilon^{-(N-1)} \int_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon \, \mathrm{d}\sigma\bigg). \end{aligned}$$

This gives with (B.12), (B.13) and (B.17) that

$$u_\varepsilon(y_\varepsilon) = \mathcal{O}\bigg(\int_{B(x_\varepsilon, r_\varepsilon)} |x-y_\varepsilon|^{-(N-2)} u_\varepsilon^{\frac{N+2}{N-2}}\, \mathrm{d}x\bigg) + \mathcal{O}(\alpha_\varepsilon). \tag{B.18}$$

Using (B.11) with $\nu = \frac{1}{N+2}$, and $1 < p < \frac{N}{N-2}$ we can write

$$\begin{aligned} & \int_{B(x_\varepsilon, r_\varepsilon)} |x-y_\varepsilon|^{2-N} u_\varepsilon^{\frac{N+2}{N-2}}\, \mathrm{d}x \\ & \quad = \int_{B(x_\varepsilon, \mu_\varepsilon)} \frac{u_\varepsilon^{\frac{N+2}{N-2}}}{|x-y_\varepsilon|^{N-2}}\, \mathrm{d}x + \int_{B(x_\varepsilon, r_\varepsilon)\setminus B(x_\varepsilon, \mu_\varepsilon)} \frac{u_\varepsilon^{\frac{N+2}{N-2}}}{|x-y_\varepsilon|^{N-2}}\, \mathrm{d}x \\ & \quad = \mathcal{O}\big(\mu_\varepsilon^{\frac{N-2}{2}} |y_\varepsilon - x_\varepsilon|^{2-N}\big) \\ & \qquad + \alpha_\varepsilon^{\frac{N+2}{N-2}} r_\varepsilon \int_{B(x_\varepsilon, r_\varepsilon)\setminus B(x_\varepsilon, \mu_\varepsilon)} \frac{1}{|x-y_\varepsilon|^{N-2}} \frac{1}{|x-x_\varepsilon|}\, \mathrm{d}x \\ & \qquad + \mu_\varepsilon^{\frac{N}{2}} \int_{B(x_\varepsilon, r_\varepsilon)\setminus B(x_\varepsilon, \mu_\varepsilon)} \frac{1}{|x-y_\varepsilon|^{N-2}} \frac{1}{|x-x_\varepsilon|^{N+1}}\, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}\big(\mu_\varepsilon^{\frac{N-2}{2}} |y_\varepsilon - x_\varepsilon|^{2-N}\big) \\
&\quad + \alpha_\varepsilon^{\frac{N+2}{N-2}} r_\varepsilon \bigg(\int_{B(x_\varepsilon, r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} \frac{1}{|x-y_\varepsilon|^{p(N-2)}} \,\mathrm{d}x \bigg)^{\frac{1}{p}} \\
&\qquad \times \bigg(\int_{B(x_\varepsilon, r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon)} \frac{1}{|x-x_\varepsilon|^{p'}} \,\mathrm{d}x \bigg)^{\frac{1}{p'}} \\
&\quad + \mathcal{O}\bigg(\frac{\mu_\varepsilon^{\frac{N}{2}}}{|y_\varepsilon - x_\varepsilon|^{N+1}} \int_{(B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon))\cap B(y_\varepsilon, \frac{|x_\varepsilon - y_\varepsilon|}{2})} \frac{1}{|x-y_\varepsilon|^{N-2}} \,\mathrm{d}x \bigg) \\
&\quad + \mathcal{O}\bigg(\frac{\mu_\varepsilon^{\frac{N}{2}}}{|x_\varepsilon - y_\varepsilon|^{N-2}} \int_{(B(x_\varepsilon,r_\varepsilon)\setminus B(x_\varepsilon,\mu_\varepsilon))\setminus B(y_\varepsilon, \frac{|x_\varepsilon - y_\varepsilon|}{2})} \frac{1}{|x-x_\varepsilon|^{N+1}} \,\mathrm{d}x \bigg) \\
&= \mathcal{O}\big(\mu_\varepsilon^{\frac{N-2}{2}} |y_\varepsilon - x_\varepsilon|^{2-N}\big) + \mathcal{O}\big(\alpha_\varepsilon^{\frac{N+2}{N-2}} r_\varepsilon^2\big).
\end{aligned}
$$

Thanks to (B.10) and to (B.14), this leads to

$$
\int_{B(x_\varepsilon, r_\varepsilon)} |x-y_\varepsilon|^{2-N} |-\Delta u_\varepsilon| \,\mathrm{d}x = \mathcal{O}\big(\mu_\varepsilon^{\frac{N-2}{2}} |y_\varepsilon - x_\varepsilon|^{2-N} + \alpha_\varepsilon\big).
$$

which, thanks to (B.18), proves (B.16).

In order to end the proof of the first part of Claim B.2, we just have to prove that

$$
\alpha_\varepsilon = \sup_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon = \mathcal{O}\big(\mu_\varepsilon^{\frac{N-2}{2}} r_\varepsilon^{2-N}\big). \tag{B.19}
$$

For that purpose, we use the definition of r_ε to write that

$$
\psi_\varepsilon(\beta r_\varepsilon) \geq \psi_\varepsilon(r_\varepsilon)
$$

for all $0 < \beta < 1$. Using (B.13), this leads to

$$
r_\varepsilon^{\frac{N-2}{2}} \Big(\sup_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon \Big) \leq C (\beta r_\varepsilon)^{\frac{N-2}{2}} \Big(\sup_{\partial B(x_\varepsilon, \beta r_\varepsilon)} u_\varepsilon \Big).
$$

Thanks to (B.16), we obtain

$$
\sup_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon \leq C \beta^{\frac{N-2}{2}} \Big(\mu_\varepsilon^{\frac{N-2}{2}} (\beta r_\varepsilon)^{2-N} + \sup_{\partial B(x_\varepsilon, r_\varepsilon)} u_\varepsilon \Big).
$$

Choosing β small enough clearly gives (B.19) and thus the pointwise estimate on u_ε of Claim B.2. The estimate on ∇u_ε then follows from standard elliptic theory. ∎

We now prove the following:

Claim B.3. *If $r_\varepsilon \to 0$ as $\varepsilon \to 0$, then, up to passing to a subsequence,*

$$
r_\varepsilon^{N-2} u_\varepsilon(x_\varepsilon) u_\varepsilon(x_\varepsilon + r_\varepsilon x) \to \frac{1}{|x|^{N-2}} + b \quad \text{in } C^1_{\mathrm{loc}}(B(0,2)\setminus\{0\}) \text{ as } \varepsilon \to 0,
$$

where b is some harmonic function in $B(0,2)$. Moreover, if $r_\varepsilon < \rho_\varepsilon$, then $b(0) = 1$.

Proof. We set, for $x \in B(0,2)$,

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{2-N}{2}} r_\varepsilon^{N-2} u_\varepsilon(x_\varepsilon + r_\varepsilon x),$$

which verifies

$$-\Delta \tilde{u}_\varepsilon + r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = N(N-2)\Big(\frac{\mu_\varepsilon}{r_\varepsilon}\Big)^2 \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \quad \text{in } B(0,2), \tag{B.20}$$

where $\tilde{h}_\varepsilon = h(x_\varepsilon + r_\varepsilon x)$. Thanks to Claim B.2, there exists $C > 0$ such that

$$\tilde{u}_\varepsilon(x) \le \frac{C}{|x|^{N-2}} \quad \text{in } B(0,2) \setminus \{0\}. \tag{B.21}$$

Then, thanks to standard elliptic theory, we get that, after passing to a subsequence, $\tilde{u}_\varepsilon \to U$ in $C^1_{\mathrm{loc}}(B(0,2) \setminus \{0\})$ as $\varepsilon \to 0$, where U is a non-negative solution of

$$-\Delta U = 0 \quad \text{in } B(0,2) \setminus \{0\}.$$

Then, thanks to the Bôcher theorem on singularities of harmonic functions, we get

$$U(x) = \frac{\lambda}{|x|^{N-2}} + b(x),$$

where b is some harmonic function in $B(0,2)$ and $\lambda \ge 0$. Now, integrating (B.20) on $B(0,1)$, we get

$$\int_{\partial B(0,1)} \partial_\nu \tilde{u}_\varepsilon \, d\sigma = \int_{B(0,1)} \Big(r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon - N(N-2)\Big(\frac{\mu_\varepsilon}{r_\varepsilon}\Big)^2 \tilde{u}_\varepsilon^{\frac{N+2}{N-2}}\Big) \, dx.$$

Thanks to (B.21), and since $r_\varepsilon \to 0$ by hypothesis,

$$\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon \, dx \to 0 \quad \text{as } \varepsilon \to 0,$$

and, thanks to (B.21) and Claim B.1,

$$\begin{aligned} &N(N-2)\int_{B(0,1)} \Big(\frac{\mu_\varepsilon}{r_\varepsilon}\Big)^2 \tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \, dx \\ &\quad \to N(N-2)\int_{\mathbb{R}^N} B^{\frac{N+2}{N-2}} \, dx = (N-2)\omega_{N-1} \quad \text{as } \varepsilon \to 0. \end{aligned}$$

On the other hand, we have

$$\int_{\partial B(0,1)} \partial_\nu \tilde{u}_\varepsilon \, d\sigma \to (2-N)\omega_{N-1}\lambda \quad \text{as } \varepsilon \to 0.$$

We deduce that $\lambda = 1$, which proves the first part of Claim B.3.

Now, if $r_\varepsilon < \rho_\varepsilon$, we have thanks to the definition of r_ε that

$$\psi_\varepsilon'(r_\varepsilon) = 0.$$

Setting $\tilde{\psi}_\varepsilon(r) = (\frac{r_\varepsilon}{\mu_\varepsilon})^{\frac{N-2}{2}} \psi_\varepsilon(r_\varepsilon r)$ for $0 < r < 2$, we see that

$$\tilde{\psi}_\varepsilon(r) \to \frac{r^{\frac{N-2}{2}}}{\omega_{N-1} r^{N-1}} \int_{\partial B(0,r)} U \, \mathrm{d}\sigma = r^{-\frac{N-2}{2}} + r^{\frac{N-2}{2}} b(0).$$

We deduce that $b(0) = 1$, which ends the proof of Claim B.3. ∎

Finally, we prove the following:

Claim B.4. *Using the notation of Claim B.3, we have that* $b(0) \leq 0$ *and* $\nabla b(0) = 0$.

Proof. We use the notation of the proof of Claim B.3. Let us apply the Pohožaev identity (E.1) to $\tilde{u}_\varepsilon$ in $B(0,1)$. We obtain

$$\frac{1}{2} \int_{B(0,1)} r_\varepsilon^2 ((N-2)\tilde{h}_\varepsilon \tilde{u}_\varepsilon^2 + \tilde{h}_\varepsilon \langle x, \nabla \tilde{u}_\varepsilon^2 \rangle) \, \mathrm{d}x = \tilde{B}_1^\varepsilon + \tilde{B}_2^\varepsilon,$$

where

$$\tilde{B}_1^\varepsilon = \int_{\partial B(0,1)} (\partial_\nu \tilde{u}_\varepsilon)^2 + \frac{N-2}{2} \tilde{u}_\varepsilon \partial_\nu \tilde{u}_\varepsilon - \frac{|\nabla \tilde{u}_\varepsilon|^2}{2} \, \mathrm{d}\sigma,$$

$$\tilde{B}_2^\varepsilon = \frac{(N-2)^2}{2} \int_{\partial B(0,1)} \left(\frac{\mu_\varepsilon}{r_\varepsilon} \right)^2 \tilde{u}_\varepsilon^{2^*} \, \mathrm{d}\sigma.$$

Thanks to Claim B.3, we can pass to the limit to obtain that the right-hand side is equal to

$$\int_{\partial B(0,1)} (\partial_\nu U)^2 + \frac{N-2}{2} U \partial_\nu U - \frac{|\nabla U|^2}{2} \, \mathrm{d}\sigma.$$

Since b is harmonic, it is easily checked that it is just $-\frac{(N-2)^2 \omega_{N-1} b(0)}{2}$. Moreover, when $N = 3$, thanks to (B.21) and the dominated convergence theorem, the left-hand side goes to zero, which proves that $b(0) = 0$. If $N \geq 4$, we have to make a more precise expansion of the left-hand side. First, integrating by parts we get

$$\begin{aligned}
&\frac{1}{2} \int_{B(0,1)} r_\varepsilon^2 ((N-2)\tilde{h}_\varepsilon \tilde{u}_\varepsilon^2 + \tilde{h}_\varepsilon \langle x, \nabla \tilde{u}_\varepsilon^2 \rangle) \, \mathrm{d}x \\
&\quad = -\int_{B(0,1)} r_\varepsilon^2 (\tilde{h}_\varepsilon \tilde{u}_\varepsilon^2 + \tfrac{1}{2} \tilde{u}_\varepsilon^2 \langle x, \nabla \tilde{h}_\varepsilon \rangle) \, \mathrm{d}x + o(1) \\
&\quad = -\tilde{h}_\varepsilon(0) r_\varepsilon^2 \int_{B(0,1)} \tilde{u}_\varepsilon^2 \, \mathrm{d}x - r_\varepsilon^2 \int_{B(0,1)} (\tilde{h}_\varepsilon - \tilde{h}_\varepsilon(0)) \tilde{u}_\varepsilon^2 + \tfrac{1}{2} \tilde{u}_\varepsilon^2 \langle x, \nabla \tilde{h}_\varepsilon \rangle \, \mathrm{d}x + o(1) \\
&\quad = \varepsilon r_\varepsilon^2 \left(-V(x_\varepsilon) \int_{B(0,1)} \tilde{u}_\varepsilon^2 \, \mathrm{d}x + \mathcal{O}\left(r_\varepsilon \int_{B(0,1)} |x| \tilde{u}_\varepsilon^2 \, \mathrm{d}x \right) \right) + o(1).
\end{aligned}$$

Then, thanks to Claims B.1 and B.2, we have easily for $N \geq 5$ that

$$\int_{B(0,1)} \tilde{u}_\varepsilon^2 \,\mathrm{d}x = \Big(\frac{r_\varepsilon}{\mu_\varepsilon}\Big)^{N-4}\Big(\int_{\mathbb{R}^N} B^2 \,\mathrm{d}x + o(1)\Big),$$
$$\int_{B(0,1)} |x|\tilde{u}_\varepsilon^2 \,\mathrm{d}x = \mathcal{O}\Big(\Big(\frac{r_\varepsilon}{\mu_\varepsilon}\Big)^{N-4-1}\Big).$$

In particular,

$$\lim_{\varepsilon\to 0} -\varepsilon r_\varepsilon^2 V(x_\varepsilon)\int_{B(0,1)} \tilde{u}_\varepsilon^2 \,\mathrm{d}x = -\frac{(N-2)^2\omega_{N-1}b(0)}{2}. \tag{B.22}$$

Hence, using the fact that $V < 0$, we obtain that $b(0) \leq 0$ for $N \geq 5$. Similarly, for $N = 4$,

$$\int_{B(0,1)} \tilde{u}_\varepsilon^2 \,\mathrm{d}x = (1 + o(1))\log\Big(\frac{r_\varepsilon}{\mu_\varepsilon}\Big)$$

and

$$\int_{B(0,1)} |x|\tilde{u}_\varepsilon^2 \,\mathrm{d}x = \mathcal{O}(1),$$

which also proves that $b(0) \leq 0$. In order to prove the second part of Claim B.4, we apply the Pohožaev identity (E.2) to $\tilde{u}_\varepsilon$ in $B(0,1)$. We obtain

$$\int_{\partial B(0,1)}\Big(\frac{|\nabla\tilde{u}_\varepsilon|^2}{2}\nu - \partial_\nu\tilde{u}_\varepsilon\nabla\tilde{u}_\varepsilon\Big)\,\mathrm{d}\sigma$$
$$= -\int_{B(0,1)} r_\varepsilon^2\tilde{h}_\varepsilon\frac{\nabla\tilde{u}_\varepsilon^2}{2}\,\mathrm{d}x + \int_{\partial B(0,1)}\frac{(N-2)^2}{2}\Big(\frac{\mu_\varepsilon}{r_\varepsilon}\Big)^2\tilde{u}_\varepsilon^{2^*}\nu\,\mathrm{d}\sigma. \tag{B.23}$$

It is clear that

$$\int_{\partial B(0,1)}\Big(\frac{|\nabla\tilde{u}_\varepsilon|^2}{2}\nu - \partial_\nu\tilde{u}_\varepsilon\nabla\tilde{u}_\varepsilon\Big)\,\mathrm{d}\sigma \to \int_{\partial B(0,1)}\Big(\frac{|\nabla U|^2}{2}\nu - \partial_\nu U\nabla U\Big)\,\mathrm{d}\sigma \quad \text{as } \varepsilon\to 0.$$

Moreover, thanks to the fact that b is harmonic, we easily get

$$\int_{\partial B(0,1)}\Big(\frac{|\nabla U|^2}{2}\nu - \nabla U\partial_\nu U\Big)\,\mathrm{d}\sigma = (N-2)\omega_{N-1}\nabla b(0).$$

It remains to deal with the right-hand side of (B.23). It is clear that

$$\int_{\partial B(0,1)}\Big(\frac{\mu_\varepsilon}{r_\varepsilon}\Big)^2\tilde{u}_\varepsilon^{2^*}\nu\,\mathrm{d}\sigma \to 0 \quad \text{as } \varepsilon\to 0.$$

Then we rewrite the first term of the right-hand side of (B.23) as

$$\int_{B(0,1)} r_\varepsilon^2\tilde{h}_\varepsilon\frac{\nabla\tilde{u}_\varepsilon^2}{2}\,\mathrm{d}x = -\int_{B(0,1)} r_\varepsilon^2\frac{\nabla\tilde{h}_\varepsilon}{2}\tilde{u}_\varepsilon^2\,\mathrm{d}x + o(1) = \mathcal{O}\Big(\varepsilon r_\varepsilon^3\int_{B(0,1)}\tilde{u}_\varepsilon^2\,\mathrm{d}x\Big).$$

Then, thanks to (B.22), we have

$$\lim_{\varepsilon\to 0}\int_{B(0,1)} r_\varepsilon^2 \tilde{h}_\varepsilon \frac{\nabla\tilde{u}_\varepsilon^2}{2}\,\mathrm{d}x = 0.$$

Finally, collecting the above information, and passing to the limit $\varepsilon \to 0$ in (B.23), we get that $\nabla b(0) = 0$, which achieves the proof of Claim B.4. ■

We are now in a position to end the proof of Proposition B.2.

Proof of Proposition B.2. If $\rho_\varepsilon \to 0$ as $\varepsilon \to 0$, then we deduce the proposition from Claims B.3 and B.4. If $\rho_\varepsilon \not\to 0$ as $\varepsilon \to 0$, then Claims B.3 and B.4 give that $r_\varepsilon \not\to 0$ as $\varepsilon \to 0$. Then, using the Harnack inequality (B.13), one can extend the result of Claim B.2 to $B(x_\varepsilon, 2\rho_\varepsilon) \setminus \{x_\varepsilon\}$, which proves the first part of Proposition B.2 when $\rho_\varepsilon \not\to 0$. ■

B.2. Proof of Proposition B.1

Let us now turn to the proof of Proposition B.1. This is done in two steps. In Claim B.5, mimicking [11], we exhaust a family of critical points of u_ε, $(x_{1,\varepsilon}, \dots, x_{N_\varepsilon,\varepsilon})$, such that each sequence $(x_{i_\varepsilon,\varepsilon})$ satisfies the assumptions of Proposition B.2 with

$$\rho_\varepsilon = \min_{1\le i\le N_\varepsilon, i\neq i_\varepsilon} \left\{|x_{i,\varepsilon} - x_{i_\varepsilon,\varepsilon}|, d(x_{i_\varepsilon,\varepsilon}, \partial\Omega)\right\}.$$

In Claim B.6, we prove that these concentration points are in fact isolated. In particular, this shows that (u_ε) develops only finitely many concentration points.

First of all, we extract sequences (whose number is a priori not bounded) of critical points of u_ε which are candidates to be the blow-up points.

Claim B.5. *There exists $D > 0$ such that for all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^*$ and N_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{n_\varepsilon,\varepsilon})$ such that*

$$d(x_{i,\varepsilon}, \partial\Omega) u_\varepsilon(x_{i,\varepsilon})^{\frac{2}{N-2}} \ge 1 \quad \text{for all } i \in [1, n_\varepsilon],$$
$$|x_{i,\varepsilon} - x_{j,\varepsilon}| u_\varepsilon(x_{i,\varepsilon})^{\frac{2}{N-2}} \ge 1 \quad \text{for all } i \neq j \in [1, n_\varepsilon],$$

and

$$\Big(\min_{i\in[1,n_\varepsilon]} |x_{i,\varepsilon} - x|\Big) u_\varepsilon(x)^{\frac{2}{N-2}} \le D$$

for all $x \in \Omega$ and all $\varepsilon > 0$.

Proof. First of all, we claim that

$$\left\{x \in \Omega \text{ s.t. } \nabla u_\varepsilon(x) = 0 \text{ and } d(x, \partial\Omega) u_\varepsilon(x)^{\frac{2}{N-2}} \ge 1\right\} \neq \emptyset \tag{B.24}$$

for ε small enough. Let us prove (B.24). Let $y_\varepsilon \in \Omega$ be a point where u_ε achieves its maximum. We set $\mu_\varepsilon = u_\varepsilon(y_\varepsilon)^{-\frac{2}{N-2}} \to 0$ as $\varepsilon \to 0$. Also, we set for all $x \in \Omega_\varepsilon = \{x \in \mathbb{R}^N \text{ s.t. } y_\varepsilon + \mu_\varepsilon x \in \Omega\}$,

$$\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(y_\varepsilon + \mu_\varepsilon x),$$

which verifies

$$-\Delta \tilde{u}_\varepsilon + \mu_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = N(N-2)\tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \text{ in } \Omega_\varepsilon,$$

where $\tilde{h}_\varepsilon = h(y_\varepsilon + \mu_\varepsilon x)$. Note that $0 \le \tilde{u}_\varepsilon \le \tilde{u}_\varepsilon(0) = 1$. Thanks to standard elliptic theory, we get that $\tilde{u}_\varepsilon \to U$ in $C^1_{\text{loc}}(\Omega_0)$, where U satisfies

$$-\Delta U = U^{\frac{N+2}{N-2}} \quad \text{in } \Omega_0 \quad \text{and} \quad 0 \le U \le 1,$$

and where $\Omega_0 = \lim_{\varepsilon\to 0} \Omega_\varepsilon$. Moreover, $U \not\equiv 0$ by Harnack's inequality; see [18, Theorem 4.17]. Then, thanks to [8, Theorem 2], we have $\Omega_0 = \mathbb{R}^N$, which proves that $d(y_\varepsilon, \partial\Omega)u_\varepsilon(y_\varepsilon)^{\frac{2}{N-2}} \to +\infty$ as $\varepsilon \to 0$. This ends the proof of (B.24).

Now, applying Lemma F.1, for ε small enough, there exist $n_\varepsilon \in \mathbb{N}^*$ and n_ε critical points of u_ε, denoted by $(x_{1,\varepsilon}, \dots, x_{n_\varepsilon,\varepsilon})$, such that

$$\begin{aligned} d(x_{i,\varepsilon}, \partial\Omega)u_\varepsilon(x_{i,\varepsilon})^{\frac{2}{N-2}} &\ge 1 \quad \text{for all } i \in [1, n_\varepsilon], \\ |x_{i,\varepsilon} - x_{j,\varepsilon}|u_\varepsilon(x_{i,\varepsilon})^{\frac{2}{n-2}} &\ge 1 \quad \text{for all } i \ne j \in [1, n_\varepsilon], \end{aligned}$$

and

$$\Big(\min_{i\in[1,n_\varepsilon]} |x_{i,\varepsilon} - x|\Big)u_\varepsilon(x)^{\frac{2}{N-2}} \le 1 \tag{B.25}$$

for every critical point x of u_ε such that $d(x, \partial\Omega)u_\varepsilon(x)^{\frac{2}{N-2}} \ge 1$. It remains to show that there exists $D > 0$ such that

$$\Big(\min_{i\in[1,n_\varepsilon]} |x_{i,\varepsilon} - x|\Big)u_\varepsilon(x)^{\frac{2}{N-2}} \le D$$

for all $x \in \Omega$. We proceed by contradiction, assuming that

$$\sup_{x\in\Omega}\Big(\Big(\min_{i\in[1,n_\varepsilon]} |x_{i,\varepsilon} - x|\Big)u_\varepsilon^{\frac{2}{N-2}}(x)\Big) \to +\infty \tag{B.26}$$

as $\varepsilon \to 0$. Let $z_\varepsilon \in \Omega$ be such that

$$\Big(\min_{i\in[1,n_\varepsilon]} |x_{i,\varepsilon} - z_\varepsilon|\Big)u_\varepsilon(z_\varepsilon)^{\frac{2}{N-2}} = \sup_{x\in\Omega}\Big(\Big(\min_{i\in[1,n_\varepsilon]} |x_{i,\varepsilon} - x|\Big)u_\varepsilon(x)^{\frac{2}{N-2}}\Big).$$

We set $\hat{\mu}_\varepsilon = u_\varepsilon(z_\varepsilon)^{-\frac{2}{N-2}}$ and $S_\varepsilon = \{x_{1,\varepsilon}, \dots, x_{n_\varepsilon,\varepsilon}\}$. Thanks to (B.26), we check that

$$\hat{\mu}_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0$$

and that

$$\frac{d(S_\varepsilon, z_\varepsilon)}{\hat{\mu}_\varepsilon} \to +\infty \quad \text{as } \varepsilon \to 0. \tag{B.27}$$

Then we set, for all $x \in \widehat{\Omega}_\varepsilon = \{x \in \mathbb{R}^3 \text{ s.t. } z_\varepsilon + \hat{\mu}_\varepsilon x \in \Omega\}$,

$$\hat{u}_\varepsilon(x) = \hat{\mu}_\varepsilon^{\frac{N-2}{2}} \hat{u}_\varepsilon(z_\varepsilon + \hat{\mu}_\varepsilon x),$$

which verifies

$$-\Delta \hat{u}_\varepsilon + \hat{\mu}_\varepsilon^2 \hat{h}_\varepsilon \hat{u}_\varepsilon = N(N-2)\hat{u}_\varepsilon^{\frac{N+2}{N-2}} \quad \text{in } \Omega_\varepsilon,$$

where $\hat{h}_\varepsilon = h(z_\varepsilon + \hat{\mu}_\varepsilon x)$. Note that $\hat{u}_\varepsilon(0) = 1$ and also that

$$\lim_{\varepsilon \to 0} \sup_{B(0,R)\cap\Omega_\varepsilon} \hat{u}_\varepsilon = 1$$

for all $R > 0$, thanks to (B.26) and (B.27). Standard elliptic theory then gives that $\hat{u}_\varepsilon \to \widehat{U}$ in $C^1_{\text{loc}}(\widehat{\Omega}_0)$, where U satisfies

$$-\Delta \widehat{U} = N(N-2)\widehat{U}^{\frac{N+2}{N-2}} \text{ in } \widehat{\Omega}_0 \quad \text{and} \quad 0 \le \widehat{U} \le 1,$$

with $\widehat{\Omega}_0 = \lim_{\varepsilon\to 0} \widehat{\Omega}_\varepsilon$. As above, we deduce that $\widehat{\Omega}_0 = \mathbb{R}^N$, which gives

$$\lim_{\varepsilon \to 0} d(z_\varepsilon, \partial\Omega) u_\varepsilon^{\frac{2}{N-2}}(z_\varepsilon) \to +\infty. \tag{B.28}$$

Moreover, thanks to [7], we know that

$$\widehat{U}(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}.$$

Since $\widehat{U}$ has a strict local maximum at 0, there exists $\hat{x}_\varepsilon$, a critical point of u_ε, such that $|z_\varepsilon - \hat{x}_\varepsilon| = o(\hat{\mu}_\varepsilon)$ and $\hat{\mu}_\varepsilon u_\varepsilon(\hat{x}_\varepsilon)^2 \to 1$ as $\varepsilon \to 0$. Thanks to (B.27) and (B.28), this contradicts (B.25) and proves Claim B.5. ■

We define

$$d_\varepsilon = \min\left\{ d(x_{i,\varepsilon}, x_{j,\varepsilon}), d(x_{i,\varepsilon}, \partial\Omega) \text{ s.t. } 1 \le i < j \le n_\varepsilon \right\}$$

and prove the following claim:

Claim B.6. *There exists $d > 0$ such that $d_\varepsilon \ge d$.*

Proof. Assume that $d_\varepsilon \to 0$ as $\varepsilon \to 0$. There are two cases to consider: either the distance between two critical points goes to 0, or one of them goes to the boundary.

Up to reordering the concentration points, we can assume that

$$d_\varepsilon = d(x_{1,\varepsilon}, x_{2,\varepsilon}) \text{ or } d(x_{1,\varepsilon}, \partial\Omega).$$

For $x \in \Omega_\varepsilon = \{x \in \mathbb{R}^3 \text{ s.t. } x_{1,\varepsilon} + d_\varepsilon x \in \Omega\}$, we set

$$\tilde{u}_\varepsilon(x) = d_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(x_{1,\varepsilon} + d_\varepsilon x),$$

which verifies

$$-\Delta \tilde{u}_\varepsilon + d_\varepsilon^2 \tilde{h}_\varepsilon \tilde{u}_\varepsilon = N(N-2)\tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \quad \text{in } \Omega_\varepsilon,$$

where $\tilde{h}_\varepsilon = h(x_{1,\varepsilon} + d_\varepsilon x)$. We have, up to a harmless rotation,

$$\lim_{\varepsilon \to 0} \Omega_\varepsilon = \Omega_0 = \mathbb{R}^N \text{ or }]-\infty; d[\times \mathbb{R}^{N-1}, \quad \text{where } d \geq 1.$$

We also set

$$\tilde{x}_{i,\varepsilon} = \frac{x_{i,\varepsilon} - x_{1,\varepsilon}}{d_\varepsilon}.$$

We claim that, for any sequence $i_\varepsilon \in [1, n_\varepsilon]$ such that

$$\tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon}) = \mathcal{O}(1), \tag{B.29}$$

we have

$$\sup_{B(\tilde{x}_{i_\varepsilon,\varepsilon}, \frac{1}{2})} \tilde{u}_\varepsilon = \mathcal{O}(1). \tag{B.30}$$

Indeed, let $y_\varepsilon \in \overline{B(\tilde{x}_{i_\varepsilon,\varepsilon}, \frac{1}{2})}$ be such that $\sup_{B(\tilde{x}_{i_\varepsilon,\varepsilon}, \frac{1}{2})} \tilde{u}_\varepsilon = \tilde{u}_\varepsilon(y_\varepsilon)$ and assume by contradiction that

$$\tilde{u}_\varepsilon(y_\varepsilon)^{\frac{2}{N-2}} \to +\infty \quad \text{as } \varepsilon \to 0. \tag{B.31}$$

Thanks to the definitions of d_ε and y_ε and to the last assertion of Claim B.5, we can write that

$$|d_\varepsilon(y_\varepsilon - \tilde{x}_{i_\varepsilon,\varepsilon})| u_\varepsilon(x_{1,\varepsilon} + d_\varepsilon y_\varepsilon)^{\frac{2}{N-2}} \leq D$$

so that

$$|y_\varepsilon - \tilde{x}_{i_\varepsilon,\varepsilon}| = o(1).$$

For $x \in B(0, \frac{1}{3\hat{\mu}_\varepsilon})$ and ε small enough, we set

$$\hat{u}_\varepsilon(x) = \hat{\mu}_\varepsilon^{\frac{N-2}{2}} \tilde{u}_\varepsilon(y_\varepsilon + \hat{\mu}_\varepsilon x),$$

where $\hat{\mu}_\varepsilon = u_\varepsilon(y_\varepsilon)^{-\frac{2}{N-2}}$. It satisfies

$$-\Delta \hat{u}_\varepsilon + (\hat{\mu}_\varepsilon d_\varepsilon)^2 \hat{h}_\varepsilon \hat{u}_\varepsilon = \hat{u}_\varepsilon^{\frac{N+2}{N-2}} \text{ in } B(0, \tfrac{1}{3\hat{\mu}_\varepsilon}) \quad \text{and} \quad \hat{u}_\varepsilon(0) = \sup_{B(0, \frac{1}{3\hat{\mu}_\varepsilon})} \hat{u}_\varepsilon = 1,$$

where $\hat{h}_\varepsilon = \tilde{h}_\varepsilon(y_\varepsilon + \hat{\mu}_\varepsilon x)$. Thanks to (B.31), $B(0, \frac{1}{3\hat{\mu}_\varepsilon}) \to \mathbb{R}^N$ as $\varepsilon \to +\infty$. Then $(\hat{u}_\varepsilon)$ is uniformly locally bounded and, by standard elliptic theory, $\hat{u}_\varepsilon$ converges to $\hat{U}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where $\hat{U}$ satisfies

$$-\Delta \hat{U} = \hat{U}^{\frac{N+2}{N-2}} \text{ in } \mathbb{R}^N \quad \text{and} \quad 0 \leq \hat{U} \leq 1 = \hat{U}(0).$$

Thanks to the classification of Caffarelli–Gidas–Spruck [7] and to the fact that $\frac{\tilde{x}_{i_\varepsilon,\varepsilon} - y_\varepsilon}{\hat{\mu}_\varepsilon}$ is bounded, we can write that

$$\liminf_{\varepsilon \to 0} \frac{\tilde{u}_\varepsilon(x_{i_\varepsilon,\varepsilon})}{\tilde{u}_\varepsilon(y_\varepsilon)} > 0$$

which is a contradiction with (B.29) and (B.31), and achieves the proof of (B.30).

For $R > 0$, we set $S_{R,\varepsilon} = \{\tilde{x}_{i,\varepsilon} \mid \tilde{x}_{i,\varepsilon} \in B(0,R)\}$. Thanks to the definition of d_ε, up to a subsequence, $S_{R,\varepsilon} \to S_R$ as $\varepsilon \to 0$, where S_R is a non-empty finite set; then up to performing a diagonal extraction, we can define the countable set

$$S = \bigcup_{R>0} S_R.$$

Thanks to the previous definition, we are ready to prove the following assertion:

$$\forall i_\varepsilon \in [1, n_\varepsilon] \text{ s.t. } d(x_{i_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = \mathcal{O}(d_\varepsilon), \quad \tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon}) \to +\infty \text{ as } \varepsilon \to 0. \tag{B.32}$$

Assume that there exists i_ε such that $d(x_{i_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = \mathcal{O}(d_\varepsilon)$ with $\tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon})$ bounded. Then, for all sequences j_ε such that $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = \mathcal{O}(d_\varepsilon)$, we have that $\tilde{u}_\varepsilon(\tilde{x}_{j_\varepsilon,\varepsilon})$ is bounded. Indeed, if there exists a sequence j_ε such that $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = \mathcal{O}(d_\varepsilon)$ and $\tilde{u}_\varepsilon(\tilde{x}_{j_\varepsilon,\varepsilon}) \to +\infty$ as $\varepsilon \to 0$, thanks to Claim B.5, we can apply Proposition B.2 with $x_\varepsilon = \tilde{x}_{j_\varepsilon,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$. We obtain that up to a subsequence $\tilde{u}_\varepsilon \to 0$ in $C^1_{\mathrm{loc}}(B(\tilde{x}, \frac{2}{3})) \setminus \{\tilde{x}\}$, where $\tilde{x} = \lim_{\varepsilon\to 0} \tilde{x}_{j_\varepsilon,\varepsilon}$. But $(\tilde{u}_\varepsilon)$ is uniformly bounded in $B(\tilde{y}, \frac{1}{2})$, where $\tilde{y} = \lim_{\varepsilon\to 0} \tilde{x}_{i_\varepsilon,\varepsilon}$. We thus obtain, thanks to Harnack's inequality, that $\tilde{u}_\varepsilon(\tilde{x}_{i_\varepsilon,\varepsilon}) \to 0$ as $\varepsilon \to 0$, which is a contradiction with the first or the second assertion of Claim B.5.

Thus we have proved that for every sequence j_ε such that $d(x_{j_\varepsilon,\varepsilon}, x_{1,\varepsilon}) = \mathcal{O}(d_\varepsilon)$, $\tilde{u}_\varepsilon(\tilde{x}_{j_\varepsilon,\varepsilon})$ is bounded. This proves that $(\tilde{u}_\varepsilon)$ is uniformly bounded in a neighborhood of any finite subset of S. But thanks to Claim B.5, $\tilde{u}_\varepsilon$ is bounded in any compact subset of $\Omega_0 \setminus S$. This clearly proves that $\tilde{u}_\varepsilon$ is uniformly bounded on any compact of Ω_0. Then, by standard elliptic theory, $\tilde{u}_\varepsilon \to U$ in $C^1_{\mathrm{loc}}(\Omega_0)$ as $\varepsilon \to 0$, where U is a non-negative solution of

$$-\Delta U = U^{\frac{N+2}{N-2}} \quad \text{in } \Omega_0.$$

But, thanks to the first or second assertion of Claim B.5, we know that $U(0) \geq 1$, hence we have necessarily that $\Omega_0 = \mathbb{R}^N$, and thus U possesses at least two critical points, namely 0 and $\check{x}_2 = \lim_{\varepsilon\to 0} \check{x}_{2,\varepsilon}$. Thanks to the classification of Caffarelli–Gidas–Spruck [7], this is impossible. This ends the proof of (B.32).

We are now going to consider two cases, depending on Ω_0.

Case 1*:* $\Omega_0 = \mathbb{R}^N$. In this case, up to a subsequence, $d_\varepsilon = d(x_{1,\varepsilon}, x_{2,\varepsilon})$ and $S = \{0, \tilde{x}_2 = \lim_{\varepsilon\to 0} \tilde{x}_{2,\varepsilon}, \dots\}$ contains at least two points. Applying Proposition B.2 with $x_\varepsilon = \tilde{x}_{i,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$, we obtain

$$\tilde{u}_\varepsilon(0)\tilde{u}_\varepsilon(x) \to H = \frac{1}{|x|^{N-2}} + \frac{\lambda_2}{|x - \tilde{x}_2|^{N-2}} + \tilde{b} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^N \setminus S) \text{ as } \varepsilon \to 0,$$

where $\tilde{b}$ is a harmonic function in $\Omega_0 \setminus \{S \setminus \{0, \check{x}_2\}\}$, and $\lambda_2 > 0$. Moreover, $\tilde{b}(0) \leq -\lambda_2$. We prove in the following that $\tilde{b}$ is non-negative, which will give a contradiction and end the study of this case. To check that $\tilde{b}$ is non-negative, for any positive number r, we rewrite H as

$$H = \sum_{\tilde{x}_i \in S \cap B(0,r)} \frac{\lambda_i}{|x - \tilde{x}_i|^{N-2}} + \hat{b}_r,$$

where $\lambda_i > 0$. Then, taking $R > r$ large enough, we get that $\hat{b}_r > \frac{-1}{r^{N-2}}$ on $\partial B(0, R)$. Moreover, for any $\tilde{x}_j \in B(0, R) \setminus B(0, r)$, there exists a neighborhood $V_{j,r}$ of $\tilde{x}_j$ such that $\hat{b}_r > 0$ on $V_{j,r}$. Thanks to the maximum principle, $\hat{b}_r > \frac{-1}{r^{N-2}}$ on $B(0, R)$, hence it is decreasing and lower bounded, then $\hat{b}_r \to \hat{b}$ on every compact set as $r \to +\infty$, we get that

$$H = \sum_{\tilde{x}_i \in S} \frac{\lambda_i}{|x - \tilde{x}_i|^{N-2}} + \hat{b},$$

with $\hat{b} \geq 0$, which proves that $\tilde{b} \geq 0$. This is the contradiction we were looking for, and this ends the proof of Claim B.6 in this first case.

Case 2: $\Omega_0 =]-\infty, d[\times \mathbb{R}^{N-1}$. We still denote $S = \{0 = \tilde{x}_1, \tilde{x}_2, \dots\}$ and we apply Proposition B.2 with $x_\varepsilon = x_{i,\varepsilon}$ and $\rho_\varepsilon = \frac{d_\varepsilon}{3}$ to get

$$\tilde{u}_\varepsilon(0)\tilde{u}_\varepsilon(x) \to H = \sum_{\tilde{x}_i \in S} \frac{\lambda_i}{|x - \tilde{x}_i|^{N-2}} + \tilde{b} \quad \text{in } C^1_{\text{loc}}(\Omega_0 \setminus S),$$

where $\lambda_i > 0$, and $\tilde{b}$ is some harmonic function in Ω_0. We extend H to $\mathbb{R}^N$ by setting

$$\hat{H}(x) = \begin{cases} H(x) & \text{if } x_1 \leq d, \\ -H(s(x)) & \text{otherwise,} \end{cases}$$

where s is the reflection with respect to the hyperplane $\{d\} \times \mathbb{R}^{N-1}$. We also extend $\tilde{b}$ by setting

$$\hat{H} = \sum_{\tilde{x}_i \in S} \Big(\frac{\lambda_i}{|x - \tilde{x}_i|^{N-2}} - \frac{\lambda_i}{|s(x) - \tilde{x}_i|^{N-2}} \Big) + \hat{b}.$$

It is clear that $\hat{b}$ is harmonic on $\mathbb{R}^N$ and satisfies $\hat{b} \geq 0$ in Ω_0 and $\hat{b} \leq 0$ in $\mathbb{R}^N \setminus \Omega_0$. This can be proved as in Case 1. For $\mathcal{G}_R$ the Green function of the Laplacian on the ball $B(0, R)$ centered at 0 with radius R, we get, thanks to the Green representation formula, that

$$\hat{b}(x) = \int_{\partial B(0,R)} \partial_\nu \mathcal{G}_R(x, y) \hat{b}(y) \, d\sigma.$$

Since

$$\partial_\nu \mathcal{G}_R(x, y) = \frac{R^2 - |x|^2}{\omega_{N-1} R |x - y|^N} \quad \text{on } \partial B(0, R),$$

this gives

$$\partial_1 \hat{b}(0) = \frac{N}{\omega_{N-1} R^N} \int_{\partial B(0,R)} y_1 \hat{b}(y) \, d\sigma.$$

Now we decompose $\partial B(0, R)$ into three sets, namely

$$A = \big\{ y \in \partial B(0, R) \text{ s.t. } y_1 \geq d \big\},$$
$$B = \big\{ y \in \partial B(0, R) \text{ s.t. } 0 \leq y_1 \leq d \big\},$$
$$C = \big\{ y \in \partial B(0, R) \text{ s.t. } y_1 \leq 0 \big\}.$$

In A and B, we have that $y_1\hat{b}(y) \leq d\hat{b}(y)$, and in C, we have that $y_1\hat{b}(y) \leq 0$. Since $\hat{b} \geq 0$ in C, we arrive at

$$\partial_1\hat{b}(0) \leq \frac{Nd}{\omega_{N-1}R^N}\int_{A\cup B}\hat{b}(y)\,\mathrm{d}\sigma \leq \frac{Nd}{\omega_2 R^N}\int_{\partial B(0,R)}\hat{b}(y)\,\mathrm{d}\sigma = \frac{Nd\hat{b}(0)}{R}.$$

Passing to the limit $R \to +\infty$ gives $\partial_1\hat{b}(0) \leq 0$. In order to obtain a contradiction, we rewrite H in a neighborhood of 0 as

$$H(x) = \frac{1}{|x|^{N-2}} + \check{b}(x),$$

where

$$\check{b}(x) = \hat{b}(x) - \frac{1}{|s(x)|^{N-2}} + \sum_{\check{x}_i\in S\setminus\{0\}}\lambda_i\Big(\frac{1}{|x-\check{x}_i|^{N-2}} - \frac{1}{|s(x)-\check{x}_i|^{N-2}}\Big).$$

As is easily checked, $\partial_1\check{b}(0) < 0$, which is a contradiction with Proposition B.2. This ends the proof of Claim B.6 in this second case. ∎

Proof of Proposition B.1. It only remains to prove (v) and (vi) of Proposition B.1. Assertion (vi) is true locally around each concentration point by applying the first part of Proposition B.2, and extending it to the whole domain using Harnack's inequality. Finally, (v) follows directly from (vi). Indeed, all the $\mu_{i,\varepsilon}$ are comparable by Harnack's inequality. Then multiplying the equation by $\mu_{1,\varepsilon}^{-\frac{N-2}{2}}$ and passing to the limit thanks to (vi) gives the desired result. ∎

C. Necessity of coercivity

In this section we briefly recall why the operator $-\Delta + h$ is necessarily coercive as soon as there exists a blowing-up sequence satisfying (B.1).

Lemma C.1. *If there exists $u \in C_0^{2,\eta}(\Omega)$ such that $u > 0$ and $-\Delta u + hu > 0$ on Ω, then $-\Delta + h$ is coercive.*

Proof. See [12, Appendix B] for the case where Ω is a compact manifold. The proof applies verbatim for a domain with Dirichlet boundary condition. ∎

In particular, the operator $-\Delta + h_\varepsilon$ must be coercive for every $\varepsilon > 0$. But in fact, $-\Delta + h$ must also be coercive under our assumption. Indeed, this is proved in [12, Appendix B], when Ω is a compact manifold and under the assumption that there exists a finite number of sequences $(x_i^\varepsilon)_{1\leq i\leq k} \in \Omega$ and $\mu_i^\varepsilon \to 0$ such that

$$\frac{1}{C}\sum_{i=1}^{k} B_{i,\varepsilon} \leq u_\varepsilon \leq C\sum_{i=1}^{k} B_{i,\varepsilon}$$

for some $C > 0$, where $B_{i,\varepsilon}(x) = B(\frac{x-x_\varepsilon^i}{\mu_\varepsilon^i})$. This hypothesis is clearly verified thanks to Proposition B.1. Now the proof in the domain case with Dirichlet boundary data follows verbatim from the one presented in [12, Appendix B].

D. Harnack's inequality

Lemma D.1. *Let u_ε satisfy the hypotheses of Proposition B.1. Then there exists $C > 0$ depending only on C_0 and $\|h\|_\infty$ such that*

$$\frac{1}{C}\max_{\partial B(x_\varepsilon,r)}(u_\varepsilon + r|\nabla u_\varepsilon|) \le \frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B(x_\varepsilon,r)} u_\varepsilon \, d\sigma \le C \min_{\partial B(x_\varepsilon,r)} u_\varepsilon$$

for all $r \in [0, \frac{5}{2}\rho_\varepsilon]$ and all $\varepsilon > 0$.

The proof follows [11, Lemma 1.3].

Proof of Lemma D.1. Let $0 < r_\varepsilon < \frac{5}{2}\rho_\varepsilon$. We set

$$\tilde{u}_\varepsilon(x) = r_\varepsilon^{\frac{N-2}{2}} u_\varepsilon(\tilde{x}_\varepsilon + r_\varepsilon x)$$

which verifies

$$-\Delta\tilde{u}_\varepsilon + r_\varepsilon^2\tilde{h}_\varepsilon\tilde{u}_\varepsilon = N(N-2)\tilde{u}_\varepsilon^{\frac{N+2}{N-2}} \quad \text{in } B(0, \tfrac{\rho_\varepsilon}{r_\varepsilon}),$$

where $\tilde{h}_\varepsilon = h(\tilde{x}_\varepsilon + r_\varepsilon x)$. Thanks to (B.3), we have

$$\tilde{u}_\varepsilon \le \frac{C_0}{|x|^{N-2}},$$

in particular $\tilde{u}_\varepsilon$ is uniformly bounded on $B(0,2) \setminus B(0,\frac{1}{2})$. Hence, applying the Moser–Harnack inequality [18, Theorem 4.17], we have for all $x \in B(0,3/2) \setminus B(0,\frac{2}{3})$ and $0 < r < \frac{1}{6}$ that

$$\max_{B(x,r)} \tilde{u}_\varepsilon \le C\Big(\min_{B(x,r/2)} \tilde{u}_\varepsilon + r\|\tilde{u}_\varepsilon\|_\infty\| - r_\varepsilon^2\tilde{h}_\varepsilon + N(N-2)\tilde{u}_\varepsilon^{\frac{4}{N-2}}\|_N\Big),$$

with $C > 0$ depending only on N. Then, taking r small enough depending only on C_0 and $\|h_\infty\|_\infty$, we have

$$\max_{B(x,r)} \tilde{u}_\varepsilon \le C \min_{B(x,r/2)} \tilde{u}_\varepsilon.$$

Then, using a covering argument, we get

$$\max_{B(0,5/4)\setminus B(0,4/5)} \tilde{u}_\varepsilon \le C \min_{B(0,5/4)\setminus B(0,4/5)} \tilde{u}_\varepsilon.$$

Finally, using standard elliptic theory,

$$\max_{B(0,7/6)\setminus B(0,6/7)} |\nabla\tilde{u}_\varepsilon| \le C \max_{B(0,7/6)\setminus B(0,6/7)} \tilde{u}_\varepsilon,$$

which achieves the proof. ∎

E. General Pohožaev's identities

For the sake of completeness, we derive here several forms of the classical Pohožaev identity [26] that we used in this paper. Assume that u is a C^2 solution of

$$-\Delta u = N(N-2)u^{\frac{N+2}{N-2}} - hu \quad \text{in } \Omega.$$

Multiplying this equation by $\langle x, \nabla u\rangle$ and integrating by parts, one easily gets that

$$\frac{1}{2}\int_\Omega ((N-2)hu^2 + h\langle x, \nabla u^2\rangle)\, \mathrm{d}x = B_1 + B_2, \tag{E.1}$$

where

$$B_1 = \int_{\partial\Omega} \Big(\langle x, \nabla u\rangle \partial_\nu u + \frac{N-2}{2} u \partial_\nu u - \langle x, \nu\rangle \frac{|\nabla u|^2}{2}\Big)\, \mathrm{d}\sigma,$$

$$B_2 = \frac{(N-2)^2}{2}\int_{\partial\Omega} \langle x, \nu\rangle \frac{u^{2^*}}{2^*}\, \mathrm{d}\sigma.$$

Hence, if $u = 0$ on $\partial\Omega$, we get

$$\int_\Omega h((N-2)u^2 + \langle x, \nabla u^2\rangle)\, \mathrm{d}x = \int_{\partial\Omega} \langle x, \nu\rangle (\partial_\nu u)^2\, \mathrm{d}\sigma.$$

Integrating by parts again, we get the Pohožaev identity in its usual form:

$$\int_\Omega \Big(h + \frac{\langle x, \nabla h\rangle}{2}\Big) u^2\, \mathrm{d}x = -\frac{1}{2}\int_{\partial\Omega} \langle x, \nu\rangle (\partial_\nu u)^2\, \mathrm{d}\sigma.$$

In a similar way, multiplying the equation by ∇u and integrating by parts, one can derive the following Pohožaev identity:

$$\int_{\partial\Omega} \Big(\frac{|\nabla u|^2}{2}\nu - \partial_\nu u \nabla u - \frac{(N-2)^2}{2} u^{2^*} \nu\Big)\, \mathrm{d}\sigma = -\int_\Omega h \frac{\nabla u^2}{2}\, \mathrm{d}x. \tag{E.2}$$

F. A general simple lemma on functions

Lemma F.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and $u \in C_0^1(\Omega)$ positive on Ω. Assume that*

$$K_u := \Big\{x \in \Omega \text{ s.t. } \nabla u(x) = 0 \text{ and } d(x, \partial\Omega) u^{\frac{2}{N-2}}(x) \ge 1\Big\}$$

is non-empty.

Then there exist $n \in \mathbb{N}^*$ and n points of K_u, denoted by $(x_1, \dots, x_n)$, such that

$$|x_i - x_j| u(x_i)^{\frac{2}{N-2}} \ge 1 \quad \text{for all } i \ne j \in [1, n]$$

and

$$\Big(\min_{i \in [1,n]} |x_i - x|\Big) u(x)^{\frac{2}{N-2}} \le 1 \quad \text{for all } x \in K_u.$$

Proof. Let $K_0 := K_u$. By assumption, K_0 is non-empty. Moreover, it is clear that K_0 is compact. We let $x_1 \in K_0$ and $K_1 \subset K_0$ be such that

$$u(x_1) = \max_{K_0} u$$

and

$$K_1 = \big\{x \in K_0 \text{ s.t. } |x_1 - x| u(x)^{\frac{2}{N-2}} \geq 1\big\}.$$

Then we proceed by induction. Assume that we have constructed $K_0 \supset \cdots \supset K_p$ and $x_1, \ldots, x_p$ such that $x_i \in K_{i-1}$ for all $i \in [1, p]$. If $K_p \neq \emptyset$, we let $x_{p+1} \in K_p$ be such that

$$u(x_{p+1}) = \max_{K_p} u$$

and we define $K_{p+1} \subset K_p$ by

$$K_{p+1} = \big\{\, x \in K_p \text{ s.t. } \min_{i\in[1,p+1]} |x - x_i| u(x)^{\frac{2}{N-2}} \geq 1\big\}. \tag{F.1}$$

We claim that for any $x_1, \ldots, x_p$ constructed in this way, we have

$$|x_i - x_j| u(x_i)^{\frac{2}{N-2}} \geq 1 \quad \text{for all } i \neq j \in [1, p]. \tag{F.2}$$

We prove (F.2) by induction. For $p = 1$, there is nothing to prove. Suppose now that (F.2) is true for some $p \geq 1$ and that $K_p \neq \emptyset$. Since $x_{p+1} \in K_p$, by definition of K_p, we have

$$|x_{p+1} - x_i| u(x_{p+1})^{\frac{2}{N-2}} \geq 1 \quad \text{for all } i \in [1, p]. \tag{F.3}$$

Moreover, for any $i \in [1, p]$, we have $K_{i-1} \supset K_p$, and hence $u(x_i) \geq u(x_{p+1})$, since x_i and x_{p+1} are defined to be the maxima of u over these sets. In particular, $u(i) \geq u(x_{p+1})$. Thus (F.3) implies

$$|x_{p+1} - x_i| u(x_i)^{\frac{2}{N-2}} \geq 1 \quad \text{for all } i \in [1, p].$$

By the induction assumption, (F.2) is already true when both i and j are in $[1, p]$. Thus we have proved (F.2) for all $i \neq j \in [1, p+1]$.

Next we observe that (F.2) implies the lower bound $|x_i - x_j| \geq \frac{1}{\|u\|_{L^\infty(\Omega)}} > 0$. Hence, the construction of the x_p must stop after finitely many steps because Ω is bounded.

Thus, there is $n \in \mathbb{N}^*$ such that $K_n = \emptyset$. Fix any $x \in K_u$. We claim that

$$\Big(\min_{i\in[1,n]} |x_i - x|\Big) u^{\frac{2}{N-2}}(x) \leq 1. \tag{F.4}$$

Together with (F.2), this will end the proof of the lemma. Since $K_n = \emptyset$, there exists $p \in [1, n]$ such that $x \in K_{p-1}$ and $x \notin K_p$. By the definition (F.1) of the set K_p, we must have

$$\min_{i\in[1,p]} |x - x_i| u(x)^{\frac{2}{N-2}} < 1.$$

Since trivially $\min_{i\in[1,n]} |x - x_i| \leq \min_{i\in[1,p]} |x - x_i|$, inequality (F.4) follows. As already explained, this proves the lemma. ∎

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