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Exponent of self-similar finite p-groups

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Abstract. Let p be a prime and G a pro-p group of finite rank that admits a faithful, self-similar action on the p-ary rooted tree. We prove that if the set $\{g \in G \mid g^{p^n} = 1\}$ is a nontrivial subgroup for some n, then G is a finite p-group with exponent at most p^n. This applies, in particular, to power abelian p-groups.

1. Introduction

A group G is self-similar of some integer degree m if it has a faithful representation on an infinite regular one-rooted m-tree $\mathcal{T}_m$ such that the representation is state-closed and is transitive on the tree's first level. Equivalently, a group G is self-similar of degree m if there exists a subgroup H of G of index m and a homomorphism f from H to G such that the only subgroup of H, normal in G and f-invariant, is the trivial subgroup.

Faithful self-similar representations are known for many particular finitely generated groups ranging from the torsion groups of Grigorchuk and Gupta–Sidki to free groups. Such representations have also been studied for the family of abelian groups, finitely generated nilpotent groups, wreath product of abelian groups, as well as for arithmetic groups, see [2–5, 9] for more references.

Let p be a prime and G a finite p group. Consider the following subgroups of G: $\Omega_n(G) = \langle g \in G \mid g^{p^n} = 1\rangle$ and $G^{p^n} = \langle g^{p^n} \mid g \in G\rangle$. We say that G is power abelian if it satisfies the following three conditions for all n:

(1) $\Omega_n(G) = \{g \in G \mid g^{p^n} = 1\}$,

(2) $G^{p^n} = \{g^{p^n} \mid g \in G\}$,

(3) $|G^{p^n}| = |G : \Omega_n(G)|$ for all n.

In [8], P. Hall introduced the notion of regular groups and showed that they are power abelian. Recently, it was proved in [7] that if p is an odd prime, then potent p-groups are also power abelian. Recall that a finite p-group G is potent if $[G, G] \leq G^4$ for $p = 2$ or $\gamma_{p-1}(G) \leq G^p$ for $p > 2$. Note that the family of potent p-groups includes powerful p-group ($[G, G] \leq G^4$ for $p = 2$ or $[G, G] \leq G^p$ for $p > 2$). More information on finite p-groups satisfying property (1), (2) or (3) can be found in [10].

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Not much is known about self-similarity of finite p-groups. Nekrashevych and Sidki [13] showed that a finite abelian 2-group has a self-similar action on $\mathcal{T}_2$ if and only if it is elementary abelian. We can mention the work of Sunic [14], he proved that a finite p-group G is self-similar of degree p with an abelian first level stabilizer if and only if G is a split extension of an elementary abelian group by a cyclic group of order p. Actually, in [1, Theorem 2.2] it was shown that this characterization extends to all finite p-groups with an abelian maximal subgroup, without needing to specify any properties of its action on the tree. In [1], it was also proved that if G is a self-similar finite p-group of rank r (self-similar of degree p), then its order is bounded by a function of p and r. In other words, they proved that, for every prime p, there are only finitely many self-similar p-groups of a given rank.

In the present article, we address the case where G is a self-similar pro-p group of degree p. We show that the exponent of G is determined by its "power structure". Recall that a pro-p group has finite rank if there exists a positive integer r such that any closed subgroup has a topological generating set with no more than r elements.

Theorem 1.1. *Let p be a prime and G a pro-p group of finite rank such that $\{g \in G \mid g^{p^n} = 1\}$ is a nontrivial subgroup of G for some n. If G is self-similar of degree p, then G is a finite p-group and has exponent at most p^n.*

If G is a finite p-group, then of course $\Omega_n(G) = \{g \in G \mid g^{p^n} = 1\}$ for some n. Using Theorem 1.1, we obtain that the exponent of a self-similar finite p-group satisfying property (1) is equal to p. In particular, the exponent of a finite self-similar power abelian p-group is equal to p. In some sense, Theorem 1.1 generalizes the following results:

- A self-similar torsion abelian group of degree p has exponent p, as proved by Brunner and Sidki [3].
- If a finitely generated nilpotent group G is self-similar of degree p, then G is either free abelian or a finite p-group, as proved by Berlatto and Sidki [2, Corollary 2].

It should be mentioned that if G is self-similar of degree p and the set $\{g \in G \mid g^{p^n} = 1\}$ is not a subgroup, then G need not be a torsion free group. For example, the pro-2 group $\mathbb{Z}_2 \rtimes C_2$ (dihedral pro-2 group) is self-similar of degree 2 and the set of elements of order 2 is not a subgroup.

We use Theorem 1.1 to obtain a generalization of the above mentioned result of [14].

Theorem 1.2. *Let p be a prime. If G is a self-similar finite p-group of degree p such that the first level stabilizer is power abelian, then G is a split extension of a p-group of exponent p by a cyclic group of order p.*

In general, p-groups with a maximal power abelian subgroup are not self-similar. In particular, we have the following example. Let G be an extra-special p-group of exponent p and order at least p^5. Consider a maximal subgroup H of G. In this case, we have that $[G, G] = [H, H]$. Then the subgroup $[H, H]$ is normal and f-invariant for any homomorphism f from H to G and G is not self-similar. On the other hand, in Section 4 we provide examples of self-similar finite p-groups with nonabelian but power abelian first level stabilizer.

2. Preliminaries

Self-similar groups. Let $\mathcal{A}_m = \mathrm{Aut}(\mathcal{T}_m)$ be the group of automorphisms of the infinite regular one-rooted m-ary tree $\mathcal{T}_m$. The group $\mathcal{A}_m$ is isomorphic to $\mathcal{A}_m \wr S_m$, where S_m is the symmetric group of degree m. Then an element α of $\mathcal{A}_m$ is given by

$$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1})\sigma(\alpha),$$

here $\alpha_i \in \mathcal{A}_m$ and $\sigma(\alpha)$ is the action of α on the first level of $\mathcal{T}_m$. A subgroup G of $\mathcal{A}_m$ is self-similar (or state-closed) if for all α, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1})\sigma(\alpha) \in G$ implies $\alpha_i \in G$. Also, let $P(G)$ denote the permutation group induced by G on the first level of the tree. A subgroup G of $\mathcal{A}_m$ is said to be transitive self-similar provided it is a state-closed and $P(G)$ is transitive on the first level of the tree. Note that if a pro-p group G of $\mathcal{A}_p$ is self-similar, then G is transitive self-similar.

Virtual endomorphism. Let G be a group, and let H be a subgroup of G of index m. A homomorphism $f\colon H \to G$ is called a virtual endomorphism of G. Let

$$T = \{t_0, t_1, \ldots, t_{m-1}\}$$

be a transversal of H in G with $t_0 \in H$. Thus

$$\varphi\colon g \mapsto g^{\varphi} = ([t_i g t_{(i)\sigma(g)}^{-1}]^{f\varphi})_{0 \le i \le m-1}\sigma(g) \quad \forall g \in G,$$

is a homomorphism of G in $\mathcal{A}_m$, where $\sigma(\alpha)$ is the permutation of T induced by g. The kernel of φ is the subgroup f-core(H) given by

$$f\text{-}\mathrm{core}(H) = \langle K \le H \mid K \lhd G,\ K^f \le K\rangle.$$

If f-core$(H) = \{1\}$, then f is called simple endomorphism. Note that $H^{\varphi} = Fix_{G^{\varphi}}(0) = \{g^{\varphi} \in G^{\varphi} \mid (0)\sigma(g) = 0\}$. The next theorem is well known (see [11,12] for further details).

Theorem 2.1. *A group G is transitive self-similar of degree m if and only if there exists a simple endomorphism $f\colon H \to G$, where H is a subgroup of G of index m.*

Example 2.2. Consider the finite p-group G with the presentation

$$\langle a, b \mid [a,b] = c, [a,c] = [b,c] = a^p = b^p = c^p = 1\rangle,$$

H is its subgroup $\langle a, c\rangle$ of index p and define $f\colon H \to G$ the homomorphism that extends the map $a \mapsto c$, $c \mapsto b$. Note that f is a simple virtual endomorphism. Hence G is a self-similar group of degree p. If $T = \{1, b, \ldots, b^{p-1}\}$, then H, f and T induce a representation $\varphi\colon G \to \mathcal{A}_p$ such that

$$G^{\varphi} = \langle \alpha = (\gamma, \gamma\beta^{p-1}, \gamma\beta^{p-2}, \ldots, \gamma\beta),\ \gamma = (\beta, \ldots, \beta),\ \beta = (0\,1\,\ldots\,p-1)\rangle.$$

The following lemma was proved in [14, Lemma 3]. For the reader's convenience, we provide a proof.

Lemma 2.3. *Let G be a pro-p group and $f\colon H \to G$ a simple virtual endomorphism of degree p. If H contains elements of order p, then $H^f \setminus H$ contains elements of order p. In particular, $G = H \rtimes \langle a\rangle$ for some element a of order p.*

Proof. Let P be the set of elements in H of order dividing p. The set P contains nontrivial elements and the subgroup $\langle P\rangle$ is a nontrivial, characteristic subgroup of H. Therefore, $\Omega_1(H)$ is nontrivial, normal subgroup of G. The elements of P are mapped under f to elements of order dividing p. If all elements of P are mapped inside H, then they are mapped to other elements in P. In that case $\Omega_1(H)$ would be a nontrivial, normal, f-invariant subgroup of G, a contradiction. Therefore, there exists an element in P that is mapped under f outside of H. The image of such an element has order p and this completes the proof. ■

A more general version of the next lemma is proved in [2, Proposition 2].

Lemma 2.4. *Let G be a pro-p group and $f\colon H \to G$ a simple virtual endomorphism of degree p of G. Then $f\colon H \cap H^f \to H^f$ is also a simple virtual endomorphism of degree p.*

Proof. Since $[G : H] = p$, we have that $H \lhd G$ and $G = HH^f$. Moreover, since

$$\Big|\frac{H^f}{H \cap H^f}\Big| = \Big|\frac{HH^f}{H}\Big|,$$

we have that $H \cap H^f$ has index p in H^f. Let K be a subgroup of $H \cap H^f$, normal in H^f, with $K^f \leq K$. Note that K^H is a normal subgroup of G. Let $k^h \in K^H$, where $k \in K$ and $h \in H$. Applying f on k^h, we obtain that $(k^h)^f = (k^f)^{h^f} \in K^{H^f} = K$. Then $(K^H)^f \leq K$. Since f is a simple endomorphism, we conclude that $K = 1$, as desired. ■

Theorem 2.5. *Let G be a pro-p group and $f\colon H \to G$ a simple virtual endomorphism of degree p of G. Then $H^{p^n} = 1$ if and only if $(H^f)^{p^n} = 1$.*

Proof. It is clear that if $H^{p^n} = 1$, then $(H^f)^{p^n} = 1$. Suppose that $(H^f)^{p^n} = 1$ and let K be the kernel of f. Thus $H^{p^n} \leq K$ since $(H^{p^n})^f = 1$. On the other hand, H^{p^n} is a characteristic subgroup of H and then it is a normal subgroup of G. Therefore, $H^{p^n} = 1$. ■

3. Proof of Theorems 1.1 and 1.2

Assume the hypothesis of Theorem 1.1. Thus G is a pro-p group of finite rank such that the set $\{g \in G \mid g^{p^n} = 1\}$ is a nontrivial subgroup for some n and there exists a simple virtual endomorphism $f\colon H \to G$ of degree p. We want to show that G has exponent p^n. Clearly, in this case we have $\Omega_n(G) = \{g \in G \mid g^{p^n} = 1\}$.

A pro-p group of finite rank has a characteristic torsion free subgroup of finite index, see, for example, [6, Corollary 4.3]. Then G contains a torsion free subgroup F of finite index. Since $\Omega_n(G) \cap F = 1$, we have that $\Omega_n(G)$ is finite. Note that G^{p^i} will be torsion free for some i. In this case, $[G^{p^i}, \Omega_n(G)] = 1$ since both are normal subgroups.

First we will prove that G is finite. By Lemma 2.3 and taking into account that $\Omega_n(G) \neq 1$, we have that $G = H \rtimes \langle a \rangle$, where a is an element of order p. Recall that for any two elements x, y in an arbitrary group, we have $(xy)^t = x^t(y^{x^{t-1}}) \dots y^x y$. Now, suppose that G^{p^i} is torsion free and $|\langle a \rangle^G| = p^j$. We will prove that $G^{p^{i+j}} = H^{p^{i+j}}$.

Let $g \in G$. Without loss of generality, we can consider that $g = ha$ where $h \in H$. Thus,

$$g^{p^{i+j}} = (ha)^{p^{i+j}} = h^{p^{i+j}} a^{h^{p^{i+j}-1}} \dots a^h a.$$

Since $[h^{p^i}, a] = 1$, we obtain that

$$(ha)^{p^{i+j}} = h^{p^{i+j}} (a^{h^{p^i-1}} \dots a^h a)^{p^j} = h^{p^{i+1}}.$$

Thus, $(H^{p^{i+j}})^f = (H^f)^{p^{i+j}} \leq G^{p^{i+j}} = H^{p^{i+j}}$. Therefore, $H^{p^{i+j}} = 1$ and so G is finite.

Now, we use induction on $|G|$. Using Lemma 2.4, we obtain that H^f has a simple virtual endomorphism of degree p and since $|H^f| < |G|$ we conclude that H^f has exponent at most p^n. Now, by Lemma 2.5 H has exponent at most p^n. Therefore, $G = \{g \in G \mid g^{p^n} = 1\}$ and this completes the proof of Theorem 1.1.

Now, assume the hypothesis of Theorem 1.2. Thus G is a finite p-group, H is power abelian and $f: H \to G$ a simple virtual endomorphism of degree p. Thus $\Omega_1(H) = \{h \in H \mid h^p = 1\}$ and then $\Omega_1(H^f) = \{g \in H^f \mid g^p = 1\}$. Now, using Lemma 2.4 and Theorem 1.1 we conclude that H^f has exponent p. Therefore, H has exponent p by Lemma 2.5 and the proof is complete.

4. Discussion

By [14], an extra-special p-group of order p^3 and exponent p and the wreath product $C_p \wr C_p$ are self-similar p-groups of degree p with abelian first level stabilizer, since they have a maximal elementary abelian subgroup. It is clear that they have exponent p and p^2, respectively. In [12, Proposition 2.9.3], it was proved that the direct power of a self-similar group is again self-similar. In particular, it is easy to see that a direct power of extra-special p-groups of order p^3 and exponent p is self-similar with a nonabelian but power abelian first level stabilizer. The next result shows that it is also possible to construct self-similar p-groups of exponent p^2 with a nonabelian but power abelian first level stabilizer.

Theorem 4.1. *If G is a self-similar finite p-group of degree p, then so is $P = G \wr C_p$.*

Proof. By Theorem 2.1 and Lemma 2.4, there exist a subgroup H of index p in G, a simple endomorphism $f: H \to G$ and an element a in G of order p such that $G = H \rtimes \langle a \rangle$.

Let $T = \{1, a, a^2, \dots, a^{p-1}\}$ be a transversal of H in G. Then H, f and T induce the representation $\varphi\colon G \to \mathcal{A}_p$ that extends the following map:

$$\begin{aligned} a &\mapsto a^\varphi = (0\ 1\ \dots\ p-1) = \sigma, \\ h &\mapsto h^\varphi = (h^{f\varphi}, h^{a^{p-1}f\varphi}, \dots, h^{af\varphi}) = (k_0, k_1, \dots, k_{p-1}), \end{aligned}$$

where $h \in H$. Consider the homomorphism

$$\theta\colon G^\varphi \to \mathcal{A}_p, \quad g^\varphi \mapsto (g^\varphi, e, \dots, e).$$

We assert that the group $P = \langle G^{\varphi\theta}, \sigma\rangle$ is state-closed. In fact, the states of P are the elements of G^φ. Since that $a^\varphi = \sigma \in P$, we need only show that $h^\varphi \in P$, for all $h \in H$. Note that

$$h^\varphi = (k_0, k_1, \dots, k_{p-1}) = k_0^\theta k_1^{\theta\sigma} \dots k_{p-1}^{\theta\sigma^{p-1}} \in \langle G^{\varphi\theta}, \sigma\rangle.$$

The assertion follows. As P is transitive and isomorphic to $G \wr C_p$, the results follow. ■

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