

Symmetry and asymmetry in a multi-phase overdetermined problem

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Abstract. A celebrated theorem of Serrin asserts that one overdetermined condition on the boundary is enough to obtain radial symmetry in the so-called one-phase overdetermined torsion problem. It is also known that imposing just one overdetermined condition on the boundary is not enough to obtain radial symmetry in the corresponding multi-phase overdetermined problem. In this paper we show that, in order to obtain radial symmetry in the two-phase overdetermined torsion problem, two overdetermined conditions are needed. Moreover, it is noteworthy that this pattern does not extend to multi-phase problems with three or more layers, for which we show the existence of nonradial configurations satisfying countably infinitely many overdetermined conditions on the outer boundary.

1. Introduction

Let \mathbb{N} denote the set of positive integers. For some fixed number $m \in \mathbb{N}$, let us introduce the problem setting and notation related to multi-phase (m -layered) elliptic overdetermined problems (see Figure 1).

Let Ω_k ($k \in \{0, 1, \dots, m\}$) be a collection of bounded domains of \mathbb{R}^N ($N \geq 2$) satisfying

$$\emptyset =: \Omega_0 \subset\subset \Omega_1 \subset\subset \dots \subset\subset \Omega_m =: \Omega,$$

where $A \subset\subset B$ means that “ A is compactly contained in B ”, that is, $\bar{A} \subset B$. Also, for $k \in \{1, \dots, m\}$, we will assume that the sets $D_k := \Omega_k \setminus \bar{\Omega}_{k-1}$ are connected. Moreover, let σ be the piece-wise constant function defined as

$$\sigma := \sum_{k=1}^m \sigma_k \chi_{D_k},$$

where σ_k are positive constants satisfying

$$\sigma_{k-1} \neq \sigma_k \quad \text{for } k \in \{2, \dots, m\}. \tag{1.1}$$

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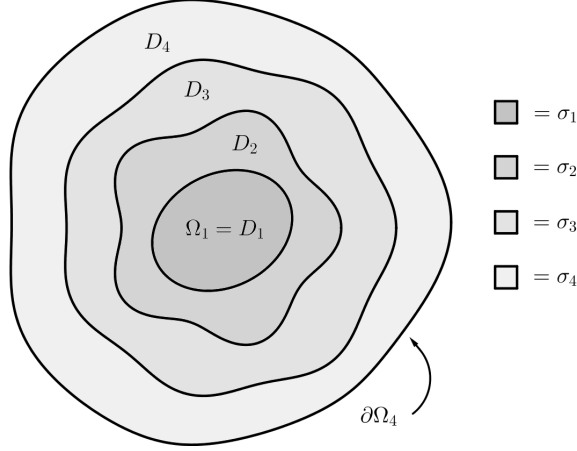


Figure 1. Problem setting for $m = 4$.

Finally, let n denote the outward unit normal vector to Ω_k ($k = 1, \dots, m$) and let ∂_n be the corresponding normal derivative. Similarly, let $(\partial_n)^j$ denote the so-called j th-order normal derivative, which is given by the following expression:

$$(\partial_n)^j u(x) := \sum_{|\beta|=j} \partial^\beta u(x) n^\beta(x),$$

where the summation is taken over all multi-indices $\beta = (\beta_1, \dots, \beta_N)$ of length k and

$$\partial^\beta := \frac{\partial^j}{\partial x_1^{\beta_1} \dots \partial x_N^{\beta_N}}, \quad n^\beta = (n_1, \dots, n_N)^\beta := n_1^{\beta_1} \dots n_N^{\beta_N}.$$

In this paper, we will consider the following boundary value problem, which will be referred to as the multi-phase torsion problem:

$$\begin{cases} -\operatorname{div}(\sigma \nabla u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

We recall that even when no additional smoothness assumptions are imposed on Ω_k , the weak solution to (1.2) is defined as the unique function $u \in H_0^1(\Omega)$ satisfying

$$\int_{\Omega} \sigma \nabla u \cdot \nabla \phi = \int_{\Omega} \phi \quad \text{for all } \phi \in H_0^1(\Omega).$$

It is a well-known fact (see [4]) that, under suitable smoothness assumptions, the boundary value problem in (1.2) can be rewritten as the following *transmission problem*

(the name *diffraction problem* is also used, see [15, Chapter 16]):

$$\begin{cases} -\sigma_k \Delta u = 1 & \text{in } D_k \ (k = 1, \dots, m), \\ \llbracket u \rrbracket = 0 & \text{on } \partial\Omega_k \ (k = 1, \dots, m-1), \\ \llbracket \sigma \partial_n u \rrbracket = 0 & \text{on } \partial\Omega_k \ (k = 1, \dots, m-1), \\ u = 0 & \text{on } \partial\Omega_m. \end{cases} \quad (1.3)$$

Here, the quantity $\llbracket \cdot \rrbracket$, called the *jump* through the interface $\partial\Omega_k$, is defined as follows: for any function $f \in H^1(\Omega_m)$, we set

$$\llbracket f \rrbracket := f_{k+1}|_{\partial\Omega_k} - f_k|_{\partial\Omega_k} \quad \text{on } \partial\Omega_k,$$

where $f_j := f|_{D_j}$ ($j = 1, \dots, m$).

If the domains Ω_k are concentric balls then, by unique solvability, there exist real constants $\{c_k\}_{k \in \mathbb{N}}$ such that the solution u of (1.2) satisfies $(\partial_n)^k u \equiv c_k$ on $\partial\Omega$ for all k . The aim of this paper is to investigate to what extent the reverse implication holds. It may not be surprising to know that the answer depends on the number of layers m .

The case $m = 1$ was solved by Serrin. By adapting the famous *reflection principle* of Alexandrov (see [1] and Theorem C), he showed the following symmetry result:

Theorem A ([18]). *Let $m = 1$. Problem (1.2) admits a solution $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfying $\partial_n u \equiv c$ on $\partial\Omega$ for some constant $c \in \mathbb{R}$ if and only if Ω is a ball.*

We refer the interested reader to the survey paper [16] for an overview of some qualitative and quantitative results related to Theorems A and C.

As shown by the author and Yachimura by making use of a perturbation method relying on the implicit function theorem, *one* overdetermined condition is not enough to obtain symmetry when $m = 2$.

Theorem B ([10]). *Let $m = 2$. Then, there exist infinitely many pairs of domains $\Omega_1 \subset\subset \Omega_2$ that are not concentric balls but such that the solution u of (1.2) satisfies $\partial_n u \equiv c$ on $\partial\Omega_2$ for some constant $c \in \mathbb{R}$.*

Regarding the study of nontrivial solutions to the above two-phase overdetermined problem, further analysis has been carried out concerning local bifurcation (see [11]) and stability (see [9]).

The difference in behaviors between one-phase and two-phase elliptic overdetermined problems presented by Theorems A and B admits the following heuristic interpretation: the one-phase overdetermined problem of Theorem A has one constraint (the overdetermined condition) and one degree of freedom (the shape of Ω). On the other hand, the two-phase overdetermined problem of Theorem B also has one constraint (the overdetermined condition) but two degrees of freedom (the shapes of both Ω_1 and Ω_2). In other words, Theorem B shows that, when the number of degrees of freedom exceeds that of constraints, the overdetermined problem admits nontrivial solutions.

By combining the soap bubble theorem of Alexandrov [1] and a symmetry result by Sakaguchi [17], we obtain the following result:

Theorem I. *Let $m = 2$ and let $\Omega_1 \subset\subset \Omega_2$ be bounded domains of class C^2 such that $D_1 := \Omega_2 \setminus \overline{\Omega_1}$ is connected. Then, problem (1.2) admits a solution u of class C^2 in a neighborhood of $\partial\Omega_2$ satisfying $(\partial_n)^k u \equiv c_k$ on $\partial\Omega_2$ ($k = 1, 2$) for some constants $c_1, c_2 \in \mathbb{R}$ if and only if (Ω_1, Ω_2) are concentric balls.*

In light of this result, together with Theorem A, one might be tempted to formulate the following (false!) conjecture:

Conjecture 1.1 (False). *Let $m \in \mathbb{N}$ and let $\Omega_k, k \in \{1, \dots, m\}$, be as in the introduction. Then, problem (1.2) admits a solution u of class C^m in a neighborhood of $\partial\Omega_m$ satisfying*

$$(\partial_n)^k u \equiv c_k \quad \text{on } \partial\Omega_m \quad (k = 1, 2, \dots, m)$$

for some constants $c_k \in \mathbb{R}$ if and only if the sets Ω_k are concentric balls.

We will show that Conjecture 1.1 does not hold for $m \geq 3$. As a matter of fact, we are able to exhibit a counterexample as follows:

Theorem II. *Let $m \geq 3$. Then, for all $\sigma_1, \dots, \sigma_m > 0$ satisfying (1.1), there exist infinitely many domains $\Omega_1 \subset\subset \dots \subset\subset \Omega_m$ where (Ω_1, Ω_2) are not concentric balls but such that the solution u of (1.2) satisfies*

$$(\partial_n)^k u \equiv c_k \quad \text{on } \partial\Omega_m, \quad \forall k \in \mathbb{N} \quad (1.4)$$

for some constants $c_k \in \mathbb{R}$.

This paper is organized as follows: in Section 2, we show Theorem I, while the subsequent sections are devoted to the proof of Theorem II. In Section 3, we introduce some preliminary results concerning the shape differentiability of state functions in two-phase problems. In Section 4, we study the invertibility properties of a linearized operator (Dirichlet-to-Neumann map). Then, in Section 5 we combine the results of the two preceding sections to give a proof of Theorem II by means of the implicit function theorem. Finally, Section 6 is devoted to some comments on the proof of Theorem II and how it relates to the existing literature.

2. Proof of Theorem I

In this section, we will give a simple proof of Theorem I. To this end, we will need the following symmetry results:

Theorem C ([1]). *A compact hypersurface, embedded in \mathbb{R}^N , that has constant mean curvature must be a sphere.*

Theorem D ([17]). *Let $m = 2$. Let Ω_2 be an open ball and let $\Omega_1 \subset\subset \Omega_2$ be a bounded open set of class C^2 with finitely many connected components such that $D_1 := \Omega_2 \setminus \overline{\Omega_1}$ is connected. Then, problem (1.2) admits a solution u satisfying $\partial_n u \equiv c_1$ on $\partial\Omega_2$ for some real constant c_1 if and only if (Ω_1, Ω_2) are concentric balls.*

We remark that both theorems above were originally stated in a more general setting (see also [2]), but the formulations above are enough for our purposes.

Theorem I now follows by combining the two theorems above.

Proof of Theorem I. Let $m = 2$ and let the sets Ω_1 and Ω_2 satisfy the hypotheses of the theorem. In what follows, we will assume that problem (1.2) admits a solution u of class C^2 in a neighborhood of $\partial\Omega_2$ satisfying

$$\partial_n u \equiv c_1, \quad (\partial_n)^2 u \equiv c_2 \quad \text{on } \partial\Omega_2 \quad (2.1)$$

for some real constants c_1, c_2 and then show that (Ω_1, Ω_2) must be concentric balls. The reverse implication is trivial and therefore omitted.

Since the solution u is of class C^2 in a neighborhood of $\partial\Omega_2$, the decomposition formula for the Laplace operator ([13, Proposition 5.4.12]) combined with (2.1) yields:

$$\frac{-1}{\sigma_2} = \Delta u = (\partial_n)^2 u + H \partial_n u + \Delta_\tau u = c_2 + H c_1 \quad \text{on } \partial\Omega_2, \quad (2.2)$$

where $\Delta_\tau = \text{div}_\tau \circ \nabla_\tau$ denotes the tangential Laplacian (that is, the tangential divergence of the tangential gradient, also known as the ‘‘Laplace–Beltrami operator’’; see [13, Definitions 5.4.5, 5.4.6 and 5.4.11]) on $\partial\Omega_2$ and H is the (additive) mean curvature given by the tangential divergence of the outward unit normal n (notice that, under this definition, the mean curvature of a ball of radius R is $\frac{N-1}{R}$).

The terms in (2.2) can be rearranged to show that the mean curvature H is constant on the entire $\partial\Omega_2$. Thus, by applying Theorem C to each connected component of $\partial\Omega_2$, we obtain that $\partial\Omega_2$ is the disjoint union of a finite number of spheres with the same radius and orientation. This leaves us with just one possibility, that is, $\partial\Omega_2$ is a sphere and Ω_2 is a ball. The conclusion readily follows from Theorem D. ■

Remark 2.1. Notice that in the proof above, we did not use the connectedness of Ω_1 (nor of $\partial\Omega_2$), but just that of Ω_2 and $D_1 = \Omega_2 \setminus \overline{\Omega_1}$.

In [5, 6], the authors showed the radial symmetry of the solutions to a similar multi-phase overdetermined problem in \mathbb{R}^N (in the elliptic and parabolic settings, respectively), where the overdetermined condition considered requires the solution u to be constant of each interface. In our setting, the following analogous result holds:

Corollary 2.2. *Let $m \in \mathbb{N}$ and let $\Omega_k, k \in \{1, \dots, m\}$, be as in the introduction. If the solution u to (1.3) satisfies*

$$\begin{aligned} u &\equiv \alpha_k \quad \text{on } \partial\Omega_k \quad (k = 1, \dots, m-1), \\ \partial_n u &\equiv c \quad \text{on } \partial\Omega_m, \end{aligned} \quad (2.3)$$

for some real constants $\alpha_1, \dots, \alpha_{m-1}$ and c , then the sets Ω_k are concentric balls and the function u is radial.

Proof. We will show the claim by induction on the number of layers m . The base case $m = 1$ is exactly Serrin's result (Theorem A). In what follows, let us assume that the claim holds when the number of layers is strictly less than m and then show that the claim holds for m as well. Let u be the solution to (1.3) with m layers $\Omega_1 \subset\subset \dots \subset\subset \Omega_m$ and assume that u satisfies the overdetermined conditions given by (2.3). Let

$$\tilde{u} := \begin{cases} \frac{\sigma_1}{\sigma_2}(u - \alpha_1) + \alpha_1 & \text{in } \Omega_1, \\ u & \text{in } \Omega_m \setminus \Omega_1. \end{cases}$$

By construction, \tilde{u} solves transmission problem (1.3) with $m - 1$ layers $\Omega_2 \subset\subset \dots \subset\subset \Omega_m$. Moreover, \tilde{u} also satisfies the overdetermined conditions given by (2.3) (starting from the "first" interface $\partial\Omega_2$). Thus, by the inductive hypothesis, $\Omega_2, \dots, \Omega_m$ are concentric balls and \tilde{u} is radial. As a consequence, u is also radial and, since $\partial\Omega_1$ is a level set of u , the remaining set Ω_1 is also a ball concentric with Ω_m . This concludes the proof. ■

3. Preliminaries on shape derivatives

In this section, we are going to introduce the main definitions and known results concerning shape calculus for two-phase problems that are going to be useful in the proof of Theorem II. The experienced reader might therefore skip this section.

In what follows, let Ω_1 and Ω_2 be concentric balls of radii $R \in (0, 1)$ and 1 respectively. Also, without loss of generality, suppose that $\sigma_2 := 1$. Let $\alpha \in (0, 1)$. For sufficiently small $\eta \in C^{2,\alpha}(\partial\Omega_1)$ and $\xi \in C^{2,\alpha}(\partial\Omega_2)$, let D_η and Ω_ξ be the bounded domains whose boundaries are given by

$$\partial D_\eta := \{x + \eta(x)n(x) \mid x \in \partial\Omega_1\}, \quad \partial\Omega_\xi := \{x + \xi(x)n(x) \mid x \in \partial\Omega_2\}. \quad (3.1)$$

Let $v_{\xi,\eta}$ be the solution to the following two-phase boundary value problem associated to the pair (D_η, Ω_ξ) :

$$\begin{cases} -\operatorname{div}(\sigma_{\xi,\eta} \nabla v_{\xi,\eta}) = 1, & \text{in } \Omega_\xi, \\ v_{\xi,\eta} = f & \text{on } \partial\Omega_\xi, \end{cases} \quad (3.2)$$

where $\sigma_{\xi,\eta} := \sigma_1 \mathcal{X}_{D_\eta} + \sigma_2 \mathcal{X}_{\Omega_\xi \setminus D_\eta}$ and $f \in C^{2,\alpha}(\mathbb{R}^N)$ is a given function.

The machinery of shape derivatives is the right tool to give a quantitative description of how $v_{\xi,\eta}$ depends on the perturbations (ξ, η) . The main technical difficulties lie in the following two points: firstly, the functions $v_{\xi,\eta}$ depend on two parameters, and secondly, each $v_{\xi,\eta}$ lies in a different function space depending on the pair (ξ, η) . To overcome these difficulties, let $\Theta := C^{2,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ and consider the following construction: For small $\theta \in \Theta$, set

$$D_\theta := (\operatorname{Id} + \theta)(\Omega_1), \quad \Omega_\theta := (\operatorname{Id} + \theta)(\Omega_2), \quad \sigma_\theta := \sigma_1 \mathcal{X}_{D_\theta} + \mathcal{X}_{\Omega_\theta \setminus D_\theta}$$

and let v_θ be the unique solution to (3.2) with respect to the pair $(D_\theta, \Omega_\theta)$. Moreover, set

$$V(\theta) := v_\theta \circ (\text{Id} + \theta) \in H^1(\Omega_2) \quad \text{for small } \theta \in \Theta. \quad (3.3)$$

Then, the (first-order) *shape derivative* of v_θ at $\theta = 0$ is defined as

$$v'[\theta] := V'(0)[\theta] - \nabla V(0) \cdot \theta \quad \text{for } \theta \in \Theta, \quad (3.4)$$

where $V'(0)[\theta]$ denotes the Fréchet derivative of V at $\theta = 0$ in the direction $\theta \in \Theta$. Notice that the definition in (3.4) is given in such a way as to be compatible with a formal application of partial differentiation with respect to θ in (3.3).

Lemma 3.1. *The following statements hold:*

- (i) *The map $\theta \mapsto V(\theta) \in H^1(\Omega_2)$ is Fréchet differentiable in a neighborhood of $0 \in \Theta$.*
- (ii) *Let U be a neighborhood of $\partial\Omega_2$ that does not intersect $\overline{\Omega}_1$ and set $K := \overline{U} \cap \overline{\Omega}_2$. Then, $\theta \mapsto V(\theta)|_K \in C^{2,\alpha}(K)$ is Fréchet differentiable in a neighborhood of $0 \in \Theta$.*
- (iii) *Let $E : C^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1) \rightarrow \Theta$ be a bounded linear extension operator such that*

$$E(\xi, \eta)|_{\partial\Omega_1} = \eta n, \quad E(\xi, \eta)|_{\partial\Omega_2} = \xi n \quad \text{for } (\xi, \eta) \in C^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1).$$

Following (3.3), set $V(\xi, \eta) := V(E(\xi, \eta))$. Then, the mappings $(\xi, \eta) \mapsto V(\xi, \eta) \in H^1(\Omega_2)$ and $(\xi, \eta) \mapsto V(\xi, \eta)|_K \in C^{2,\alpha}(K)$ are Fréchet differentiable in a neighborhood of $(0, 0) \in C^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1)$.

- (iv) *Following (3.4), let $v'[\xi] := v'[E(\xi, 0)]$ denote the shape derivative of v_θ with respect to the outer perturbation ξ only. Then, $v'[\xi]$ is independent of the extension operator E and can be characterized as the unique solution to the following boundary value problem:*

$$\begin{cases} -\operatorname{div}(\sigma \nabla v'[\xi]) = 0 & \text{in } \Omega_2, \\ v'[\xi] = (\partial_n f - \partial_n V(0))\xi & \text{on } \partial\Omega_2. \end{cases} \quad (3.5)$$

Sketch of the proof. In the case of the Laplace operator ($\sigma \equiv 1$) with homogeneous Dirichlet boundary conditions ($f \equiv 0$), claims (i)–(iv) are well-known results which can be obtained by a standard procedure that combines the implicit function theorem (see Theorem E) and the Schauder regularity theory [12, Chapter 6]; see, for instance, [13, Section 5.3] and the final remark therein. Also, the case of two-phase boundary value problems with homogeneous Dirichlet boundary conditions has been dealt with in [8, Appendix], while the case of general boundary conditions for the Laplace operator has been briefly covered in [13, Section 5.6] and the references therein. Finally, as far as the extension operator E is concerned, we refer the interested reader to [12, Section 6.9].

Here, we will just limit to showing a simple trick that allows us to reduce to the cases mentioned above. For small $\theta \in \Theta$, let v_θ be the solution to (3.2) with respect to the pair $(D_\theta, \Omega_\theta)$ and let

$$w_\theta := v_\theta - f \in H_0^1(\Omega_\theta) \cap C^{2,\alpha}(K_\theta), \quad W(\theta) := w_\theta \circ (\text{Id} + \theta) \in H_0^1(\Omega_2) \cap C^{2,\alpha}(K),$$

where $K_\theta := (\text{Id} + \theta)(K)$. Notice that, by construction, w_θ is a weak solution to

$$\begin{cases} -\operatorname{div}(\sigma_\theta \nabla w_\theta) = F_\theta & \text{in } \Omega_\theta, \\ w_\theta = 0 & \text{on } \partial\Omega_\theta, \end{cases}$$

where the function $F_\theta \in H^{-1}(\Omega_\theta) \cap C^{2,\alpha}(K_\theta)$ is given by $F_\theta := 1 + \operatorname{div}(\sigma_\theta \nabla f)$. Also, notice that, by construction, F_θ admits an extension to \mathbb{R}^N that is independent of ξ . Therefore, results (i)–(iv) hold for w_θ . Since $v_\theta = w_\theta + f$ by definition, it is clear that (i)–(iii) hold for v_θ as well. To conclude, we just need to check that $v'[\xi]$ solves (3.5). To this end, notice that, by definition, we have $w_\theta = v_\theta - f$. As a result, $w'[\xi] = v'[\xi]$ and thus, $v'[\xi]$ satisfies the equation in (3.5). On the other hand, notice that $W(0) = V(0) - f$ holds by construction, while, by (iv), $w'[\xi]$ satisfies the boundary condition

$$w'[\xi] = -\partial_n W(0)\xi \quad \text{on } \partial\Omega_2.$$

Claim (iv) for v_θ follows by combining the identities above. ■

4. The two-phase Dirichlet-to-Neumann map

As in the previous section, let Ω_1 and Ω_2 be concentric balls of radii $R \in (0, 1)$ and 1, respectively, and let $\sigma_2 := 1$. Let us introduce the following two-phase Dirichlet-to-Neumann map $\mathcal{N} : C^{2,\alpha}(\partial\Omega_2) \rightarrow C^{1,\alpha}(\partial\Omega_2)$ defined as $\xi \mapsto \partial_n w[\xi]$, where $w[\xi]$ is the unique solution to the following boundary value problem:

$$\begin{cases} -\operatorname{div}(\sigma \nabla w) = 0 & \text{in } \Omega_2, \\ w = \xi & \text{on } \partial\Omega_2. \end{cases} \quad (4.1)$$

Let $\{Y_{k,i}\}_{k,i}$ ($k \in \{0, 1, \dots\}$, $i \in \{1, 2, \dots, d_k\}$) be a maximal family of linearly independent solutions to the eigenvalue problem

$$-\Delta_\tau Y_{k,i} = \lambda_k Y_{k,i} \quad \text{on } \partial\Omega_2,$$

with k th eigenvalue $\lambda_k = k(N + k - 2)$ of multiplicity d_k and normalized in such a way that $\|Y_{k,i}\|_{L^2(\partial\Omega_2)} = 1$. The functions $\{Y_{k,i}\}$ are usually referred to as spherical harmonics. By the method of separation of variables, it can be shown that the spherical harmonics form an orthonormal basis of eigenfunctions of \mathcal{N} in $L^2(\partial\Omega_2)$. The eigenvalues of \mathcal{N} have been computed in [10].

Lemma 4.1. For $k \in \mathbb{N} \cup \{0\}$ and $i \in \{1, \dots, d_k\}$, we have

$$\mathcal{N}(Y_{k,i}) = k \frac{(2 - N - k)(1 - \sigma_1) + (N - 2 + k + k\sigma_1)R^{2-N-2k}}{F} Y_{k,i}, \quad (4.2)$$

where

$$F := k(1 - \sigma_1) + (N - 2 + k + k\sigma_1)R^{2-N-2k} > 0.$$

Moreover, $\mathcal{N}(Y_{k,i}) = 0$ if and only if $k = 0$.

Proof. Since the eigenvalues of (4.2) have been computed in [10], in what follows we only need to check that the right-hand side in (4.2) vanishes if and only if $k = 0$. Furthermore, since $\mathcal{N}(Y_{0,i}) = 0$ by construction, it will suffice to show that $\mathcal{N}(Y_{k,i}) \neq 0$ for $k \in \mathbb{N}$. To this end, let ϕ denote the numerator in the right-hand side of (4.2), that is

$$\phi(R) := (2 - N - k) - \sigma_1(2 - N - k) + (N - 2 + k + k\sigma_1)R^{2-N-2k}.$$

We will show that $\phi(R) > 0$ for all $R \in (0, 1]$ and $k \in \mathbb{N}$, proving the claim. First, notice that ϕ is a decreasing function of R . Thus, for all $R \in (0, 1]$,

$$\begin{aligned} \phi(R) &\geq \phi(1) = (2 - N - k) - \sigma_1(2 - N - k) + (N - 2 + k + k\sigma_1) \\ &= \sigma_1(2k - 2 + N) > 0, \end{aligned}$$

which is what we wanted to show. \blacksquare

Let $C_*^{i,\alpha}(\partial\Omega_2)$ denote the set of all functions in $C^{i,\alpha}(\partial\Omega_2)$ with zero average over $\partial\Omega_2$ ($i \in \{1, 2\}$). Notice that, by Lemma 4.1, \mathcal{N} fixes the eigenspaces of the Laplace–Beltrami operator; hence, \mathcal{N} is a well-defined operator from $C_*^{2,\alpha}(\partial\Omega_2)$ into $C_*^{1,\alpha}(\partial\Omega_2)$. Also by Lemma 4.1, \mathcal{N} is injective. Actually, it can be shown that $\mathcal{N} : C_*^{2,\alpha}(\partial\Omega_2) \rightarrow C_*^{1,\alpha}(\partial\Omega_2)$ is a bijection. In order to show this, we first need the following lemma:

Lemma 4.2. The map $\text{Id} + \mathcal{N} : C^{2,\alpha}(\partial\Omega_2) \rightarrow C^{1,\alpha}(\partial\Omega_2)$ is a bijection.

Proof. We will show that, for all $\eta \in C^{1,\alpha}(\partial\Omega_2)$, there exists a unique $\xi \in C^{2,\alpha}(\partial\Omega_2)$ that satisfies

$$\mathcal{N}\xi + \xi = \eta.$$

First of all, let us consider the Sobolev space $H^1(\Omega_2)$ endowed with the (equivalent) norm $\|\psi\|_{H^1(\Omega_2)} := \|\nabla\psi\|_{L^2(\Omega_2)} + \|\psi|_{\partial\Omega_2}\|_{L^2(\partial\Omega_2)}$ and consider the bilinear form $\mathcal{B} : H^1(\Omega_2) \times H^1(\Omega_2) \rightarrow \mathbb{R}$ given by

$$\mathcal{B}(\psi, \phi) := \int_{\Omega_2} \sigma \nabla\psi \cdot \nabla\phi + \int_{\partial\Omega_2} \psi\phi.$$

Notice that, by construction, \mathcal{B} is bilinear, continuous, and coercive. Now fix an element $\eta \in C^{1,\alpha}(\partial\Omega_2) \subset L^2(\partial\Omega_2)$. By the Lax–Milgram theorem, there exists a unique function $w \in H^1(\Omega_2)$ such that

$$\mathcal{B}(w, \phi) = \langle \eta, \phi|_{\partial\Omega_2} \rangle_{L^2(\partial\Omega_2)} \quad \text{for all } \phi \in H^1(\Omega_2). \quad (4.3)$$

Now, if we restrict the identity above to ϕ in $H_0^1(\Omega_2)$, then we realize that w must satisfy

$$\int_{\Omega_2} \sigma \nabla w \cdot \nabla \phi = 0 \quad \text{for all } \phi \in H_0^1(\Omega_2).$$

In other words, w satisfies the equation

$$-\Delta w = 0 \quad \text{in } \Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1}) \quad (4.4)$$

and the transmission conditions

$$[[w]] = 0, \quad [[\sigma \partial_n w]] = 0 \quad \text{on } \partial\Omega_1. \quad (4.5)$$

Moreover, integration by parts in (4.3) and the arbitrariness of the trace of $\phi \in H^1(\Omega_2)$ on $\partial\Omega_2$ yield

$$\partial_n w + w = \eta \quad \text{on } \partial\Omega_2. \quad (4.6)$$

Now, since w is the solution to the transmission problem given by (4.4), (4.5), and (4.6), we can inductively bootstrap its regularity in a classical way by means of the standard elliptic regularity estimates [12, Chapter 8] and the Schauder boundary estimates [12, Chapter 6] (see, for example, the argument in the proof of [14, Proposition 5.2] after (5.7)). We obtain that w is of class $C^{2,\alpha}$ in an open neighborhood of $\partial\Omega_2$ (whose closure does not intersect $\overline{\Omega_1}$). In particular, the function

$$\xi := w|_{\partial\Omega_2} \quad (4.7)$$

is a well-defined element of $C^{2,\alpha}(\partial\Omega_2)$. This, together with (4.4), (4.5), and (4.6), implies that w is the solution to (4.1). In particular, by (4.7) and (4.6),

$$\mathcal{N}\xi + \xi = \partial_n w + w = \eta \quad \text{on } \partial\Omega_2.$$

By the arbitrariness of $\eta \in C^{1,\alpha}(\partial\Omega_2)$, the above shows that $\text{Id} + \mathcal{N} : C^{2,\alpha}(\partial\Omega_2) \rightarrow C^{1,\alpha}(\partial\Omega_2)$ is surjective. Injectivity follows from the coercivity of \mathcal{B} . As a result, $\text{Id} + \mathcal{N} : C^{2,\alpha}(\partial\Omega_2) \rightarrow C^{1,\alpha}(\partial\Omega_2)$ is a bijection (whose inverse is continuous, by the bounded inverse theorem). ■

Lemma 4.3. *The operator \mathcal{N} is a bijection from $C_*^{2,\alpha}(\partial\Omega_2)$ into $C_*^{1,\alpha}(\partial\Omega_2)$.*

Proof. Let $K : C^{1,\alpha}(\partial\Omega_2) \rightarrow C^{2,\alpha}(\partial\Omega_2) \hookrightarrow C^{1,\alpha}(\partial\Omega_2)$ denote the inverse of $\text{Id} + \mathcal{N}$ (that exists by Lemma 4.2). Notice that, by the compactness of the embedding $C^{2,\alpha}(\partial\Omega_2) \hookrightarrow C^{1,\alpha}(\partial\Omega_2)$, K is a compact operator from $C^{1,\alpha}(\partial\Omega_2)$ into itself. By Lemma 4.1 and the Fredholm alternative (Riesz–Schauder theory) [7, Theorem 6.6 (c)], $\text{Id} - K$ admits a continuous inverse function $T : C_*^{1,\alpha}(\partial\Omega_2) \rightarrow C_*^{1,\alpha}(\partial\Omega_2)$. Thus, for $(\xi, \eta) \in C_*^{2,\alpha}(\partial\Omega_2) \times C_*^{1,\alpha}(\partial\Omega_2)$ we have

$$\begin{aligned} \mathcal{N}\xi = \eta &\iff \xi + \mathcal{N}\xi = \xi + \eta \iff \xi = K(\xi + \eta) \iff (\text{Id} - K)\xi = K\eta \\ &\iff \xi = TK\eta. \end{aligned}$$

In other words, the operator $\mathcal{N} : C_*^{2,\alpha}(\partial\Omega_2) \rightarrow C_*^{1,\alpha}(\partial\Omega_2)$ admits a continuous inverse, given by $\mathcal{N}^{-1} = T \circ K$. ■

5. Proof of Theorem II

Let $m \geq 3$. For $k = 1, \dots, m$, let Ω_k be the open ball of radius $R_k > 0$ centered at the origin. Also assume that $0 < R_k < R_{k+1}$ for $k = 1, \dots, m-1$. Moreover, unless otherwise specified, we will always assume $R_2 := 1$, $\sigma_2 := 1$ (notice that this does not result in a loss of generality).

In what follows, we will show Theorem II. In particular, we will find a nontrivial collection of domains

$$D_\eta \subset\subset \Omega_\xi \subset\subset \Omega_3 \subset\subset \dots \subset\subset \Omega_m \quad (5.1)$$

such that (1.4) is satisfied.

To this end, we will employ the following version of the implicit function theorem for Banach spaces (see [3, Theorem 2.3, page 38] for a proof):

Theorem E (Implicit function theorem). *Let $\Psi \in \mathcal{C}^k(X \times \Lambda, Z)$, $k \in \mathbb{N}$, where Z is a Banach space and X (resp. Λ) is an open set of a Banach space \tilde{X} (resp. $\tilde{\Lambda}$). Suppose that $\Psi(x^*, \lambda^*) = 0$ and that the partial derivative $\partial_x \Psi(x^*, \lambda^*)$ is a bounded invertible linear transformation from X to Z .*

Then, there exist neighborhoods Λ' of λ^ in $\tilde{\Lambda}$ and X' of x^* in \tilde{X} , and a map $\xi \in \mathcal{C}^k(\Lambda', X')$ such that the following hold:*

- (i) $\Psi(\xi(\lambda), \lambda) = 0$ for all $\lambda \in \Lambda$,
- (ii) If $\Psi(x, \lambda) = 0$ for some $(x, \lambda) \in X' \times \Lambda'$, then $x = \xi(\lambda)$,
- (iii) $\xi'(\lambda) = -(\partial_x \Psi(p))^{-1} \circ \partial_\lambda \Psi(p)$, where $p = (\xi(\lambda), \lambda)$ and $\lambda \in \Lambda'$.

Before giving the proof of Theorem II, let us first define some auxiliary functions. Set $R_0 := 0$ and let u_0 denote the following radial function:

$$u_0(x) := \begin{cases} \frac{1}{2N} \left(\sum_{j=k+1}^{m-1} \frac{1}{\sigma_{j+1}} (R_{j+1}^2 - R_j^2) + \frac{1}{\sigma_{k+1}} (R_{k+1}^2 - |x|^2) \right) & \\ \text{if } |x| \in [R_k, R_{k+1}], \quad (k = 0, \dots, m-1), & \end{cases} \quad (5.2)$$

where the value of the sum $\sum_{j=k+1}^{m-1}$ is set to be zero if $m-1 < k+1$ (empty sum). It is easy to check that u_0 solves transmission problem (1.3). The following auxiliary function will also play a crucial role in our construction:

$$v_0(x) := \begin{cases} \frac{1}{2N} \left(\sum_{j=3}^{m-1} \frac{1}{\sigma_{j+1}} (R_{j+1}^2 - R_j^2) + \frac{1}{\sigma_3} (R_3^2 - |x|^2) \right) & \text{if } |x| \in [0, R_3], \\ u_0(x) & \text{if } |x| \in (R_3, R_m]. \end{cases} \quad (5.3)$$

This is nothing but the solution to problem (1.3) in the case $\sigma_1 = \sigma_2 = \sigma_3$.

Let $\alpha \in (0, 1)$. For sufficiently small $\eta \in C^{2,\alpha}(\partial\Omega_1)$ and $\xi \in C^{2,\alpha}(\partial\Omega_2)$, let D_η and Ω_ξ be the bounded domains whose boundaries are given by (3.1). Moreover, let $v_{\xi,\eta}$ be the

solution to the following boundary value problem associated to the pair (D_η, Ω_ξ) :

$$\begin{cases} -\operatorname{div}(\sigma_{\xi,\eta} \nabla v_{\xi,\eta}) = 1 & \text{in } \Omega_\xi, \\ v_{\xi,\eta} = v_0 & \text{on } \partial\Omega_\xi, \end{cases} \quad (5.4)$$

where $\sigma_{\xi,\eta} := \sigma_1 \mathcal{X}_{D_\eta} + \sigma_2 \mathcal{X}_{\Omega_\xi \setminus \overline{D_\eta}}$. Notice that $v_{\xi,\eta}|_{(\xi,\eta)=(0,0)} = u_0|_{\Omega_2}$.

Lemma 3.1 yields

$$\begin{cases} -\operatorname{div}(\sigma \nabla v'[\xi]) = 0 & \text{in } \Omega_2, \\ v'[\xi] = (\partial_n v_0 - \partial_n u_0)\xi = \frac{1}{N}(1 - \frac{1}{\sigma_3})\xi & \text{on } \partial\Omega_2. \end{cases} \quad (5.5)$$

Proof of Theorem II. We construct nontrivial domains D_η and Ω_ξ with $\eta \in C^{2,\alpha}(\partial\Omega_1)$ and $\xi \in C^{2,\alpha}(\partial\Omega_2)$ such that the solution $v_{\xi,\eta}$ to (5.4) satisfies

$$\partial_{n_\xi} v_{\xi,\eta} = \sigma_3 \partial_{n_\xi} v_0 \quad \text{on } \partial\Omega_\xi, \quad (5.6)$$

where ∂_{n_ξ} is the normal derivative in the outward direction on $\partial\Omega_\xi$. In other words, if (5.6) holds, then the function

$$v := \begin{cases} v_{\xi,\eta} & \text{in } \Omega_\xi, \\ v_0 & \text{in } \Omega_m \setminus \Omega_\xi \end{cases}$$

solves transmission problem (1.3) with respect to (5.1). Moreover, since the function v defined above is radial in $\Omega_m \setminus \Omega_\xi$, in particular, it satisfies (1.4). We are therefore left with the problem of finding a nontrivial pair of functions (ξ, η) such that (5.6) holds. To this end, consider the following mapping:

$$\begin{aligned} \Psi : C_*^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1) &\rightarrow C_*^{1,\alpha}(\partial\Omega_2), \\ (\xi, \eta) &\mapsto ((\partial_{n_\xi} v_{\xi,\eta} - \sigma_3 \partial_{n_\xi} v_0) \circ (\operatorname{Id} + \xi n)) J_\tau(\xi). \end{aligned} \quad (5.7)$$

Let us first clarify the notation employed in the definition of Ψ . Here, n stands for the outward unit normal vector to the unperturbed boundary $\partial\Omega_2$ (we recall that the outer normal to $\partial\Omega_\xi$ is denoted by n_ξ) and $\operatorname{Id} + \xi n$ is the natural pullback mapping from $\partial\Omega_\xi$ to $\partial\Omega_2$. Moreover, $J_\tau(\xi)$ is the tangential Jacobian associated with the mapping $\operatorname{Id} + \xi n$ (that is, it is the multiplicative term that appears in the integrand of a surface integral after the corresponding change of variables; see [13, (5.67)–(5.68)]). It is known that both n_ξ and $J_\tau(\xi)$ are differentiable with respect to ξ at $\xi = 0$ (see [13, Proposition 5.4.14 and Lemma 5.4.15]). We remark that in [13, Proposition 5.4.14], only Gâteaux differentiability is shown. This notwithstanding, the Fréchet differentiability of the normal can be shown analogously or by noticing that a smooth extension of n can be written as the normalized gradient of some subharmonic function vanishing on the boundary). These facts together with Lemma 3.1 imply that Ψ is a well-defined and Fréchet differentiable mapping from a neighborhood of $(0, 0) \in C_*^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1)$ into $C^{1,\alpha}(\partial\Omega_2)$. We just need to check that, for small $(\xi, \eta) \in C_*^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1)$, the image $\Psi(\xi, \eta)$ is indeed a function of zero average over $\partial\Omega_2$. To this end, notice that

$$\operatorname{div}(\sigma_{\xi,\eta} \nabla v_{\xi,\eta}) = -1 = \sigma_3 \Delta v_0 \quad \text{in } \Omega_\xi,$$

and thus, the divergence theorem and a change of variables yield the desired identity:

$$\begin{aligned}
\int_{\partial\Omega_2} \partial_{n_\xi} v_{\xi,\eta} \circ (\text{Id} + \xi n) J_\tau(\xi) &= \int_{\partial\Omega_\xi} \partial_{n_\xi} v_{\xi,\eta} = \int_{\Omega_\xi} \text{div}(\sigma_{\xi,\eta} \nabla v_{\xi,\eta}) \\
&= -|\Omega_\xi| = \int_{\Omega_\xi} \sigma_3 \Delta v_0 = \int_{\partial\Omega_\xi} \sigma_3 \partial_{n_\xi} v_0 \\
&= \int_{\partial\Omega_2} \sigma_3 \partial_{n_\xi} v_0 \circ (\text{Id} + \xi n) J_\tau(\xi).
\end{aligned}$$

Moreover, since the tangential Jacobian J_τ never vanishes, it is clear by definition that $\Psi(\xi, \eta) = 0$ if and only if $v_{\xi,\eta}$ satisfies (5.6).

In what follows, we will apply the implicit function theorem (Theorem E) to the mapping $\Psi : C_*^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1) \rightarrow C_*^{1,\alpha}(\partial\Omega_2)$. To this end, it will be sufficient to show that the partial Fréchet derivative of Ψ with respect to ξ at $\xi = 0$ is a bijection between $C_*^{2,\alpha}(\partial\Omega_2)$ and $C_*^{1,\alpha}(\partial\Omega_2)$. A direct computation with (5.2), (5.3), (5.5), and (4.1) at hand yields:

$$\begin{aligned}
\partial_\xi \Psi(0, 0)[\xi] &= \underbrace{(\partial_n u_0 - \sigma_3 \partial_n v_0)}_{=0} J'_\tau(0)[\xi] + \underbrace{((\partial_n)^2 u_0 - \sigma_3 (\partial_n)^2 v_0)}_{=0} \xi \underbrace{J_\tau(0)}_{=1} \\
&= \partial_n v'[\xi] = \frac{1}{N} \left(1 - \frac{1}{\sigma_3}\right) \mathcal{N}(\xi).
\end{aligned}$$

We remark that the computation above is also dramatically simplified, because the Fréchet derivative of $n_\xi \circ (\text{Id} + \xi n)$ at $\xi = 0$ is tangent to $\partial\Omega_2$, that is, orthogonal to n (as a matter of fact, it is equal to $-\nabla_\tau \xi$ by [13, Proposition 5.4.14]). Finally, since $\sigma_3 \neq \sigma_2 = 1$, the conclusion follows from Lemma 4.3. ■

6. Some final comments

In this section, we give some comments on the various topological and regularity assumptions used in this paper.

On the topological assumptions in Theorem D

Theorem D ensures spherical symmetry in a two-phase setting under the topological assumption that Ω_1 has finitely many connected components and $D_1 := \Omega_2 \setminus \overline{\Omega}_1$ is connected. Notice that, by Theorem II, we know that the connectedness of D_1 is necessary. Indeed, since Theorem II holds in the “two-phase-three-layer” case (that is, $m = 3$ and $\sigma_1 = \sigma_3$), there exists a nontrivial triplet of domains $D_\eta \subset \subset \Omega_\xi \subset \subset \Omega_3$ such that (1.4) holds. Renaming the domains as

$$\Omega_2 := \Omega_3, \quad \Omega_1 := \Omega_\xi \setminus \overline{D}_\eta, \quad D_2 := \Omega_2 \setminus \overline{\Omega}_1 = D_\eta \dot{\cup} (\Omega_3 \setminus \overline{\Omega}_\xi)$$

gives the desired counterexample to Theorem D for disconnected $D_2 = \Omega_2 \setminus \overline{\Omega}_1$. To the best of my knowledge, it is still an open question whether there exists a counterexample to Theorem D where Ω_1 has infinitely many connected components without developing a microstructure.

On the regularity

In Theorem II we constructed a pair of nontrivial domains $D_\eta \subset \subset \Omega_\xi$ of class $C^{2,\alpha}$ such that (1.4) is satisfied. We remark that this particular choice of regularity has been made only to simplify the exposition. Indeed, one could have chosen higher regularity spaces such as $C^{k,\alpha}$ ($k \geq 3$) in (5.7), or even different regularities altogether for η and ξ . The latter can be done by the “simultaneous asymmetric perturbation method” introduced in [8].

On the choice of the mapping Ψ

We remark that the choice of the mapping Ψ used in the proof of Theorem II is not by chance. The reader might wonder why we opted for such a convoluted approach (cutting the solution at the second phase, deforming it, and then gluing it back together with the radial unperturbed solution) instead of the more direct approach given by a simple Neumann-tracking on $\partial\Omega_m$. In what follows, we aim to give an intuitive explanation of why such a naive method fails. Instead of the one defined in (5.7), let Ψ be the following Neumann-tracking-type operator:

$$\begin{aligned} \Psi : C_*^{2,\alpha}(\partial\Omega_2) \times C^{2,\alpha}(\partial\Omega_1) &\rightarrow C_*^{k,\alpha}(\partial\Omega_m), \\ (\xi, \eta) &\mapsto \partial_n u_{\xi,\eta} - c_1, \end{aligned}$$

where $u_{\xi,\eta}$ is the solution to (1.2) with respect to (5.1). First of all, notice that, by the Schauder regularity theory, $u_{\xi,\eta}$ is of class $C^{k,\alpha}$ (for all $k \geq 2$) in a neighborhood of $\partial\Omega_m$ and so the map Ψ is well defined. Accordingly, one has to replace the Dirichlet-to-Neuman map \mathcal{N} with the following “jump-to-Neumann” map $\xi \mapsto \mathcal{J}(\xi) := \partial_n w[\xi]$, where $w[\xi]$ is the solution to the following transmission problem:

$$\begin{cases} -\Delta w = 0 & \text{in } D_1 \cup \dots \cup D_m, \\ \llbracket w \rrbracket = \llbracket \sigma \partial_n w \rrbracket = 0 & \text{on } \partial\Omega_i \ (i = 1, 3, 4, \dots, m-1), \\ \llbracket w \rrbracket = \xi & \text{on } \partial\Omega_2, \\ \llbracket \sigma \partial_n w \rrbracket = 0 & \text{on } \partial\Omega_2, \\ w = 0 & \text{on } \partial\Omega_m. \end{cases}$$

As briefly mentioned before, notice that the function $\partial_n w[\xi]$ is arbitrarily smooth, irrespective of the regularity of ξ . In other words, in passing from ξ to $\mathcal{J}(\xi)$, all information about the regularity of ξ gets lost, and thus, solving the equation

$$\mathcal{J}(\xi) := \partial_n w[\xi] = \eta$$

in the appropriate Banach spaces becomes an ill-conditioned problem. As a result, Fredholmness is lost and the proofs of the analogs of Lemmas 4.2–4.3 fail. As a rule of thumb, we can say that this sort of ill-conditioning usually happens when the “free boundary” (in this case, $\partial\Omega_\varepsilon$) and the “overdetermined boundary” (that is, the boundary where the tracking takes place—in this case, $\partial\Omega_m$) do not coincide.

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