

Internal aggregation models with multiple sources and obstacle problems on Sierpiński gaskets

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Abstract. We consider the doubly infinite Sierpiński gasket graph SG_0 , rescale it by factor 2^{-n} , and on the rescaled graphs $SG_n = 2^{-n}SG_0$, for every $n \in \mathbb{N}$, we investigate the limit shape of three aggregation models with initial configuration σ_n of particles supported on multiple vertices. The models under consideration are: *divisible sandpile* in which the excess mass is distributed among the vertices until each vertex is stable and has mass less or equal to one, *internal DLA* in which particles do random walks until finding an empty site, and *rotor aggregation* in which particles perform deterministic counterparts of random walks until finding an empty site. We denote by $SG = \text{cl}(\bigcup_{n=0}^{\infty} SG_n)$ the infinite Sierpiński gasket, which is a closed subset of \mathbb{R}^2 , for which SG_n represents the level- n approximating graph, and we consider a continuous function $\sigma: SG \rightarrow \mathbb{N}$. For σ we solve the obstacle problem, and we describe the noncoincidence set $D \subset SG$ as the solution of a free boundary problem on the fractal SG . If the discrete particle configurations σ_n on the approximating graphs SG_n converge pointwise to the continuous function σ on the limit set SG , we prove that, as $n \rightarrow \infty$, the scaling limits of the three aforementioned models on SG_n starting with initial particle configuration σ_n converge to the deterministic solution D of the free boundary problem on the limit set $SG \subset \mathbb{R}^2$. For D we also investigate boundary regularity properties.

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1. Introduction

Internal DLA (shortly IDLA) on \mathbb{Z}^d , as a special case of the Diaconis–Fulton smash sum [7], was introduced in [18], and it is a random cluster growth model in which m particles are initially located at the origin $o \in \mathbb{Z}^d$, and each of them performs a simple random walk until arriving at a site that has not been visited before, at which site the particle stops and occupies that position forever. The resulting random subset of occupied sites in \mathbb{Z}^d is called *internal DLA cluster* and has, for $m \rightarrow \infty$, a deterministic limit shape as shown in [18]. *Divisible sandpile* is a deterministic growth model introduced in [21] as a potential theoretical tool to approach internal DLA, and relaxes the integrability condition in the similar model, *the Abelian sandpile*. Divisible sandpile uses a continuous amount of mass, and any site with mass more than one is unstable and topples by keeping mass one for itself and distributing the rest equally among the neighbors. As proved in [21], as time goes to infinity, the sequence of mass configurations converges pointwise to a limit configuration in which each site has mass ≤ 1 . By starting with mass m in $o \in \mathbb{Z}^d$ and zero everywhere else, the subset of toppled sites in \mathbb{Z}^d in the limit configuration, called *the divisible sandpile cluster*, has as limit shape a Euclidean ball, like in the internal DLA case. Finally, in the deterministic counterpart of internal DLA, called *rotor aggregation* [20], m particles starting at the origin $o \in \mathbb{Z}^d$ perform rotor walks (deterministic analogues of random walks) until finding sites unvisited previously, where they settle forever. The set of occupied sites is called *rotor cluster* and has on \mathbb{Z}^d the same limit shape as internal DLA and divisible sandpile as shown in [21]. All three models are Abelian, in the sense that, for the resulting final configuration or for the set of occupied sites, it does not matter in which order particles move or in which order sites are being stabilized.

For single point sources of m particles starting at a fixed vertex o in the underlying state space, there has been intensive work that concerns mainly the limit shape and the fluctuations of the cluster around the limit shape. While it is believed that the three models share the same limit shape on any state space, so far the available limit results rely very much on the geometry and the growth of those spaces and, in particular, on the long term behavior of random and rotor walks on them. Other than \mathbb{Z}^d , we mention here a few of the state spaces where the limit shapes for the above models and particles starting at the same vertex have been investigated: on *trees* internal DLA [4] and rotor aggregation [17]; on *comb lattices* [10, 11]; internal DLA on *cylinder graphs* in [13]; on *Sierpiński gasket graphs* internal DLA in [5], divisible sandpile in [12], and rotor aggregation together with Abelian sandpile in [6]. In particular, the Sierpiński gasket graph is the only non-trivial state space (other than \mathbb{Z}) where even a fourth model, *the Abelian sandpile* shares the same limit shape with the other three above introduced models. We do not focus on this model, but we would like to emphasize

that on \mathbb{Z}^d the scaling limit for the Abelian sandpile exists, but a precise description is still an open problem.

While the above results focus on initial mass or particle configurations supported on a single vertex, it is natural to extend these configurations to general functions with bounded support, and to describe the limit shapes and fluctuations. A first step in this direction has been done in [22], where the authors investigate scaling limits for the aggregation models with multiple sources on lattices $\delta_n \mathbb{Z}^d$, with $\delta_n \downarrow 0$, so the processes are run on finer and finer lattices. Given a function σ on \mathbb{R}^d , there is a natural way to construct a corresponding obstacle γ for σ and to investigate its smallest superharmonic majorant s . Then the *noncoincidence set for the obstacle problem* D is the set $D := \{x \in \mathbb{R}^d \mid s(x) > \gamma(x)\}$; for more details on obstacle problems and free boundary problems in \mathbb{R}^d we refer the reader to [9]. In [22], the authors consider the three models of cluster growth on lattices $\delta_n \mathbb{Z}^d$ with starting initial configurations σ_n on $\delta_n \mathbb{Z}^d$ converging in an appropriate way to σ on \mathbb{R}^d , and they prove that the resulting sets of occupied sites converge in the Hausdorff metric to the noncoincidence set $D \subset \mathbb{R}^d$, for reasonable assumptions on the function σ . Their approach is based on potential theoretical analysis in \mathbb{R}^d where most of the needed tools and estimates are well understood.

Our contribution. While it is not clear how to extend such an analysis on any arbitrary state space and how to pass from discrete potential theory to its continuous counterpart on some limiting object similar to \mathbb{R}^d in order to get a precise understanding of the scaling limits for the aggregation models, the focus of this work is on a particular class of fractals: *Sierpiński gaskets* SG as closed subsets of \mathbb{R}^2 . On the discrete graph approximations SG_n of SG, we have a good understanding of the random walk and its limiting process Brownian motion, and they are a rich source of objects with scaling-invariance and self-similar properties on which potential theory is well understood. The limiting object SG which takes the role of \mathbb{R}^d from [22], is from the potential analytical point of view also well understood, even if the analysis on fractals is completely different from the case of \mathbb{R}^d . In order to explore the scaling invariance of the graphs SG_n and the decimation invariance of random walks on them, unlike the case of \mathbb{Z}^d , we cannot rescale the graphs by any arbitrary sequence δ_n converging to 0, and we take $\delta_n = 2^{-n}$; the choice of the scaling constant will be made clear below. The main goal of the underlying work is to study obstacle problems for functions σ on the limit gasket SG and to describe the noncoincidence set $D \subset SG$ and its main properties, by a careful analysis on fractals. On the approximating graphs SG_n with discrete particle configurations σ_n that converge to σ on SG in a way to be described below we show that the divisible sandpile cluster, the internal DLA cluster, and the rotor aggregation cluster share the same scaling limit that converges to the noncoincidence set $D \subset SG$, similar to \mathbb{Z}^d and \mathbb{R}^d from [22]. We would like to emphasize that

the analysis from \mathbb{R}^d does not immediately carry over to fractals, and there are several differences and difficulties to deal with on SG; for instance, on the Sierpiński gasket, there is no known globally defined Green function and this makes the investigation of obstacle problems more subtle. One can overcome this issue on the gasket by defining the solution of the obstacle problem as a limit of solutions of obstacle problems on approximating graphs and showing that this is indeed well-defined.

Main results. In order to state our results, we first refer the reader to Section 2.2, where the level- n approximations SG_n (infinite graphs) as rescalings of the doubly infinite Sierpiński gasket graph SG_0 by the factor 2^{-n} and the “continuous limiting object” SG as a closed subset of \mathbb{R}^2 are introduced. Let $B(0, 2^l)$ be the closed Euclidean ball of radius 2^l around 0 in SG, i.e., two compact triangles with side length 2^l joined together at 0, where $l \in \mathbb{N}$, and set $\text{Tr}_l = B(0, 2^l)$. If we define the Green function G_{Tr_l} on $SG \cap \text{Tr}_l$ as in (2.5), and for a function $f: \text{Tr}_l \rightarrow \mathbb{R}$ we set

$$\mathbf{G}_{\text{Tr}_l} f(x) = \int_{\text{Tr}_l} G_{\text{Tr}_l}(x, y) f(y) d\mu(y),$$

where μ is a multiple of the Hausdorff measure on Tr_l , then we first prove the following.

Proposition 1.1. *Let $\sigma: SG \rightarrow [0, \infty)$ be bounded and continuous almost everywhere, such that $\text{supp}(\sigma)$ is also bounded. We define the obstacles on $\text{Tr}_l \subset SG$ by*

$$\gamma_{\text{Tr}_l} := -\mathbf{G}_{\text{Tr}_l}(\sigma - 1),$$

and the superharmonic majorant of γ_{Tr_l} by

$$s_{\text{Tr}_l} := \inf\{f \mid f \text{ is continuous, superharmonic and } f \geq \gamma_{\text{Tr}_l}\}.$$

Then for the functions $u_{\text{Tr}_l} = s_{\text{Tr}_l} - \gamma_{\text{Tr}_l}$, the pointwise limit $u = \lim_{l \rightarrow \infty} u_{\text{Tr}_l}$ exists and is well-defined. Furthermore, the set $D = \{u > 0\} \subset SG$ is bounded.

Starting with an almost everywhere continuous function σ on SG as in Proposition 1.1, we consider discrete initial particle densities σ_n on SG_n that converge to σ , and for σ_n we perform divisible sandpile, internal DLA, and rotor aggregation on SG_n . We denote the respective clusters by \mathcal{D}_n , \mathcal{I}_n , and \mathcal{R}_n , where the subscript n indicates that we are on the approximating graph SG_n (i.e., we perform divisible sandpile, IDLA and rotor router aggregation with initial density σ_n on SG_n). The next result shows that the scaling limit of these three models converges in the Hausdorff metric to the non-coincidence set $\tilde{D} = D \cup \{\sigma \geq 1\}^\circ$, where D is the set from Proposition 1.1. We say that the sequence $(A_n)_{n \in \mathbb{N}}$ of sets $A_n \subseteq SG_n$ converges to the set $A \subseteq SG$, if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ it holds that

$$A_\varepsilon \cap SG_n \subseteq A_n \subseteq A^\varepsilon,$$

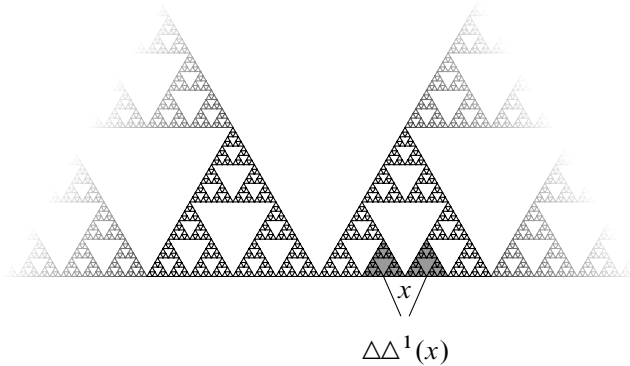


Figure 1. The double sided infinite Sierpiński gasket and $\Delta\Delta^1(x)$ for $x = (1, 0)$.

where A_ε and A^ε are the inner and outer ε -neighborhoods of A , respectively. A special case of the densities σ_n is as in the result below, where $\sigma_n(x)$ can be obtained by averaging σ over a neighborhood of x in SG.

Theorem 1.1. *Let $\sigma: \text{SG} \rightarrow \mathbb{N}_0$ be bounded with compact support and continuous almost everywhere, such that $\{\sigma \geq 1\} = \overline{\{\sigma \geq 1\}^\circ}$. Let \mathcal{D}_n , \mathcal{R}_n and \mathcal{I}_n be the final set of occupied sites (resp. toppled sites in the sandpile) formed by divisible sandpile, rotor aggregation and internal DLA, respectively, started from initial density*

$$\sigma_n(x) = \left[\mu(\Delta\Delta^{n+1}(x))^{-1} \int_{\Delta\Delta^{n+1}(x)} \sigma(y) d\mu(y) \right] \quad (1.1)$$

on SG_n , for every $n \in \mathbb{N}$. Then $\mathcal{D}_n, \mathcal{R}_n \rightarrow \tilde{D}$ and $\mathcal{I}_n \rightarrow \tilde{D}$ with probability one, as $n \rightarrow \infty$.

For the definition of $\Delta\Delta^{n+1}(x)$ see equation (2.3), and for an example of $\Delta\Delta^1(x)$, together with an illustration of SG, see Figure 1. Theorem 1.1 is a summary of Theorem 3.1 (divisible sandpile scaling limit), Theorem 4.2 (rotor aggregation scaling) and Theorem 5.1 (internal DLA scaling limit), for the choice (1.1) of the discrete particle densities σ_n .

The structure of the paper. In Section 2 we introduce the three aggregation models (divisible sandpile, internal DLA, rotor aggregation), the Sierpiński gasket SG as well as its discrete graph approximations SG_n for $n \in \mathbb{N}$, and the potential theoretical tools on fractals. In Section 3 we prove the limit shape result for the divisible sandpile model, by defining the limiting process on SG as a limit of processes on compact subsets of SG. Section 4 investigates the limit shape for rotor aggregation while Section 5 deals with the internal DLA model. Finally, in Appendices A and B we show



Figure 2. Divisible sandpile for initial configuration $\sigma = \mathbb{1}_{B((-2,0),1)} + \mathbb{1}_{B((-1/2,0),2)}$ on SG_2 .

how to calculate the solution of the free boundary problem for a specific starting configuration σ . We also show here that the boundary of the noncoincidence set of the free boundary problem is a null set with respect to the Hausdorff measure on SG , as is needed in the proof of the scaling limit for internal DLA.

2. Preliminaries

2.1. Internal aggregation models

Let $G = (V, E)$ be an infinite, locally finite and regular graph, i.e., every vertex has degree $d \in \mathbb{N}$. If x and y are neighbors in G we write $x \sim y$, and we denote by $d(x, y)$ the graph distance in G .

Divisible sandpile. A function $\nu_0: V \rightarrow \mathbb{R}$ is a sand distribution on G if ν_0 is non-negative and has finite support. The toppling operator at x is

$$T_x \nu_0 = \nu_0 + \max\{\nu_0(x) - 1, 0\} \Delta_G \mathbb{1}_{\{x\}},$$

where $\mathbb{1}_{\{x\}}(\cdot)$ is the function on V taking value 0 everywhere except a single 1 at the vertex x , and Δ_G is *the graph Laplacian of G* , i.e., the operator defined on real functions on V given by

$$(\Delta_G f)(x) = \frac{1}{d} \sum_{z \sim x} f(z) - f(x).$$

The toppling operator T_x takes the mass exceeding 1 at x and distributes it equally among its neighbors. For a sequence of vertices $(x_k)_{k \in \mathbb{N}}$, define the associated toppling process by

$$\nu_n = T_{x_n} \dots T_{x_1} \nu_0,$$

where ν_n is the sand distribution after the first n topplings have been performed. We remark that the total amount of mass is conserved during the toppling procedure. An important tool that will be used in the sequel is the *odometer function*. For an initial sand distribution ν_0 and a toppling sequence $(x_k)_{k \in \mathbb{N}}$ we define the k -th odometer function by

$$u_k(y) = \sum_{j \leq k; x_j = y} (\nu_j(y) - \nu_{j-1}(y)),$$

which keeps track of how much mass has been sent out of each vertex. If we topple along any sequence of vertices that contains each vertex in V infinitely many times, then the sand distributions $(\nu_k)_{k \in \mathbb{N}}$ converge pointwise to some limit distribution ν with $\nu \leq 1$ on G . Also, the sequence of odometer functions $(u_k)_{k \in \mathbb{N}}$ converges to a limit function u called the odometer function of the divisible sandpile started from initial configuration ν_0 . It holds that $\nu = \nu_0 + \Delta_G u$ and the so-called *least action principle* states that u is actually the minimal non-negative function that satisfies

$$\nu_0 + \Delta_G u \leq 1.$$

See [21] for proofs of the pointwise convergence results for the odometer functions and mass distributions, and [22, Lemma 3.2] for the least action principle. The *divisible sandpile cluster* is defined as the set of toppled sites $\mathcal{D} := \{x \in G \mid u(x) > 0\}$. In the literature, it is also common to define the divisible sandpile cluster as the set $\mathcal{D}' := \{x \in G \mid \nu(x) = 1\}$ of fully occupied sites. The sets \mathcal{D} and \mathcal{D}' are related in that $u(x) > 0$ implies $\nu(x) = 1$ so $\mathcal{D} \subseteq \mathcal{D}'$, and $\nu(x) = 1$ implies $u(x) > 0$ or $\nu_0(x) = 1$ or there exists a neighboring vertex $y \sim x$ with $u(y) > 0$. Throughout this paper we will use the definition \mathcal{D} of the sandpile cluster, and the scaling limit of \mathcal{D}' follows immediately from that of \mathcal{D} by the above-mentioned relation between the two sets. For an example of \mathcal{D} on SG_2 see Figure 2.

Internal DLA. For an infinite graph $G = (V, E)$, the initial configuration of particles is a function $\nu_0: V \rightarrow \mathbb{N}$ that is bounded and finitely supported. Denote by $m := \sum_{v \in V} \nu_0(v)$ the total number of particles, and label the particles by integers $i = 1, \dots, m$. Denote by x_i the initial location of the particle labeled i , so $\#\{i \mid x_i = v\} = \nu_0(v)$ represents the number of particles located at v . For each $i = 1, \dots, m$, let $(X_t(i))_{t \in \mathbb{N}}$ be a simple random walk on G starting at position x_i , and we construct inductively the internal DLA cluster as follows. We set the cluster at time 0 to be the empty set, and for $i \geq 1$ if the cluster at time $i - 1$ has already been built, let the particle labeled i start a random walk at its initial position x_i and stop it after first exiting the previous cluster at time $i - 1$. So the cluster at time i is the union between the cluster at time $i - 1$ and $\{X_{\tau_i}(i)\}$, where τ_i denotes the first exit time of the random walk $(X_t(i))_{t \in \mathbb{N}}$ from cluster at time $i - 1$. The stopping time τ_i represents

the number of steps taken by particle labeled i , which is initially located at $x_i \in G$, until stopping at random location $X_{\tau_i}(i)$. Proceeding in this way for all $i = 1, \dots, m$, we obtain the random subset of occupied sites in G that we call *internal DLA cluster*. In the literature sometimes the notation \mathcal{I}_n is used to denote the internal DLA cluster after n particles settled, but throughout this paper, this notation is reserved for internal DLA cluster on the discrete graphs SG_n with particle configuration σ_n .

Rotor aggregation. This model is also called rotor-router aggregation is a deterministic counterpart of internal DLA and it is based on particles performing deterministic walks (rotor walks) instead of random walks. Rotor walks, also called rotor-router walks, were first studied by Priezzhev et al. in [24] under the name *Eulerian Walkers*, and can be defined by means of a rotor configuration and a fixed cyclic ordering of the neighbors for each vertex. Denote the set of neighbors of $v \in V$ by A_v , and fix for any $v \in V$ a cyclic ordering $\text{cyc}(v)$ of the neighbors of v . A rotor configuration is a function $\rho: V \rightarrow V$ such that for all $v \in V$, $\rho(v) \in A_v$, so one can think of $\rho(v)$ as an arrow or a rotor that points to one of the neighbors of v . A particle starting at v performs a *rotor walk* as follows: first it turns the rotor at v from $\rho(v)$ to point to the next neighbor in the cyclic ordering $\text{cyc}(v)$ of the neighbors of v , and then the particle follows this new direction.

For rotor aggregation, we start with the same bounded and finitely supported particle configuration $\nu_0: V \rightarrow \mathbb{N}$ as in the internal DLA. Denoting again by m the total number of available particles and labeling the particles by $i = 1, \dots, m$, and also denoting the starting location of the particle labeled i by x_i , we define rotor aggregation recursively exactly in the same way as internal DLA, with the difference that particles perform rotor walks instead of random walks until finding sites unvisited previously, where they stop. After all m particles have found new sites, we obtain a deterministic subset of occupied sites that we call *the rotor cluster*. While in internal DLA particles perform independent random walks, in rotor aggregation the particle movements are deterministic, but there is interaction between them. If one particle passed through some vertex until finding its final position, on the way there it modified the direction of several rotors and subsequent particles passing through the same vertex will use then the rotor direction left behind by previous visits. If $(Y_t(i))_{t \in \mathbb{N}}$ is a rotor walk starting from position x_i , then the rotor cluster with m particles can be defined as follows: let the rotor cluster at time 0 to be the empty set, and for $i \geq 1$ if the cluster at time $i - 1$ has already been built, let the particle labeled i start a rotor walk with rotor configuration left behind by the previous particles, at its initial position x_i and stop it after first exiting the previous rotor cluster at time $i - 1$. If τ_i denotes the first exit time of the rotor walk $(Y_t(i))_{t \in \mathbb{N}}$ from the previous rotor cluster, then the rotor cluster at time i is the union between the rotor cluster at time $i - 1$ with $\{Y_{\tau_i}(i)\}$.

So τ_i represents the number of rotor steps taken by particle labeled i initially located at $x_i \in G$, which stops at deterministic location $Y_{\tau_i}(i)$.

2.2. Sierpiński gasket

We recall the construction of the Sierpiński gasket as in [3]. Let $a_0 = (0, 0)$, $a_1 = (1, 0)$ and $a_2 = (1/2, \sqrt{3}/2)$ be the corners of an equilateral triangle with side length 1 and let $F_0 = \{a_0, a_1, a_2\}$. Further, let J_0 be the closed convex equilateral triangle with vertices F_0 . Inductively, for $n \geq 1$, we define

$$F_{n+1} := F_n \cup (2^n a_1 + F_n) \cup (2^n a_2 + F_n),$$

so $F_n \subseteq 2^n J_0$. Let $\text{SG}_+ = \bigcup_{n=0}^{\infty} F_n$ and denote by SG_0 the union of SG_+ with its reflection along the y -axis SG_- . For $n \in \mathbb{Z}$, let

$$\delta_n := 2^{-n}, \quad \text{SG}_n := \delta_n \text{SG}_0, \quad \text{SG}_{\infty} := \bigcup_{n=0}^{\infty} \text{SG}_n, \quad \text{SG}_{-\infty} = \{0\}, \quad \text{SG} := \text{cl}(\text{SG}_{\infty}).$$

SG is a closed connected subset of \mathbb{R}^2 called *the double sided infinite Sierpiński gasket* and the graphs SG_n represent the level- n approximating graphs of SG , where $x, y \in \text{SG}_n$ are connected by an edge if and only if $|x - y| = \delta_n$ and the line connecting x to y is contained in SG . For $x \in \text{SG}_{\infty}$, we define its level as

$$\text{level}(x) := \inf\{n \in \mathbb{Z} \mid x \in \text{SG}_n\}.$$

Clearly 0 is the only vertex with $\text{level}(0) = -\infty$. We recall below several properties of SG and the sequence of approximating graphs $(\text{SG}_n)_{n \in \mathbb{N}}$. The *Hausdorff dimension* α and the *walk dimension* β of SG are given by

$$\alpha = \frac{\log 3}{\log 2} \approx 1.58496 \quad \text{and} \quad \beta = \frac{\log 5}{\log 2} \approx 2.32193.$$

Laplacian on SG_n . Recalling that by Δ_{SG_n} we have denoted the graph Laplacian of the infinite graph SG_n , we define the Laplacian on SG_n as a scaled version of Δ_{SG_n} by

$$\Delta_n := \delta_n^{-\beta} \Delta_{\text{SG}_n}.$$

We remark that by our choice of $\delta_n = 2^{-n}$, we have $\delta_n^{-\beta} = 5^n$, which is the scaling used in [26]. We call a function $h: \text{SG}_n \rightarrow \mathbb{R}$ *harmonic* if $\Delta_n h = 0$, *superharmonic* if $\Delta_n h \leq 0$ and *subharmonic* if $\Delta_n h \geq 0$. For a function $\gamma: \text{SG}_n \rightarrow \mathbb{R}$ its *least superharmonic majorant* $s: \text{SG}_n \rightarrow \mathbb{R}$ is defined as

$$s(x) := \inf\{f(x) \mid f: \text{SG}_n \rightarrow \mathbb{R} \text{ is superharmonic and } f \geq \gamma\}.$$

For $n \in \mathbb{N}$, denote by $B^n(x, r) := \{y \in \text{SG}_n \mid d(x, y) \leq r\}$ the ball of radius $r \geq 0$ around $x \in \text{SG}_n$ in the graph metric $d(\cdot, \cdot)$ of SG_n . If $(X_t^n)_{t \in \mathbb{N}}$ is a simple random walk on SG_n , and $B \subset \text{SG}_n$, denote by $\tau_B^n(x)$ its first exit time from B :

$$\tau_B^n := \inf\{t \geq 0 \mid X_t^n \notin B\}. \quad (2.1)$$

For simplicity of notation, if $B = B^n(x, r)$, then we write $\tau_r^n(x)$ instead of $\tau_{B^n(x,r)}^n$. We will use for the rest absolute constants $C, c > 0$ that might change from line to line.

Properties of $\text{SG}_n, n \in \mathbb{N}$. We gather here several important features of SG_n .

- (i) [*Volume growth*] $B^n(x, r)$ has growth of order α : there exists $C > 0$ such that

$$C^{-1}r^\alpha \leq |B^n(x, r)| \leq Cr^\alpha. \quad (V_\alpha)$$

See [1] and [2] for a proof and more details.

- (ii) [*Elliptic Harnack inequality*] There exists $C > 0$ such that for all $x \in \text{SG}_n$, all $r > 0$, and all functions $h > 0$ with $\Delta_n h = 0$ on $B^n(x, 2r)$ we have

$$\sup_{y \in B^n(x, r)} h(y) \leq C \inf_{y \in B^n(x, r)} h(y). \quad (EHI)$$

See [16] for details.

- (iii) [*Expected exit time*] For every $x \in \text{SG}_n$ and $r \geq 0$, there exists $C > 0$ such that

$$C^{-1}r^\beta \leq \mathbb{E}_y[\tau_r^n(x)] \leq Cr^\beta, \quad (E_\beta)$$

for the random walk $(X_t^n)_{t \in \mathbb{N}}$ starting at $y \in B^n(x, r)$; see [2, Corollary 2.3a] for details.

Green functions and operators on SG_n . For any finite subset $T \subseteq \text{SG}_n$, we denote its inner boundary by $\partial T := \{v \in T \mid \exists z \in T^C \text{ with } z \sim v\}$. Consider again the simple random walk $(X_t^n)_{t \in \mathbb{N}}$ on SG_n , and for $x, y \in T$, we define the rescaled stopped Green function as

$$\begin{aligned} g_T^n(x, y) &:= \left(\frac{3}{5}\right)^n \mathbb{E}_x[\#\text{ of visits of } X_t^n \text{ to } y \text{ before hitting } \partial T] \\ &= \delta_n^{\beta-\alpha} \mathbb{E}_x \left[\sum_{k=0}^{\tau_{T \setminus \partial T}^n - 1} \mathbb{1}_{\{X_k^n = y\}} \right], \end{aligned} \quad (2.2)$$

where $\tau_{T \setminus \partial T}^n$ is defined as in (2.1) and $X_0^n = x$. Clearly, $g_T^n(\cdot, y) = 0$ for all $y \in \partial T$. For any function $f: T \rightarrow \mathbb{R}$ we define the Green operator g_T^n as

$$g_T^n f(x) := \delta_n^\alpha \sum_{y \in T} g_T^n(x, y) f(y).$$

Since $\Delta_{\text{SG}_n}(\mathbf{g}_\top^n(\cdot, y)) = -\delta_n^{\beta-\alpha} \mathbb{1}_{\{y\}}(\cdot)$, we have

$$\Delta_n \mathbf{g}_\top^n f(x) = \mathbf{g}_\top^n \Delta_n f(x) = -f(x), \quad \text{for all } x \in \text{Tr} \setminus \partial \text{Tr}.$$

Green functions on the Sierpiński gasket SG. As for its approximating counterpart, there is also no known globally defined Green function on SG. Denote the ball in SG of radius $r \geq 0$ around x in the Euclidean distance by $B(x, r) := \{y \in \text{SG} \mid |x - y| \leq r\}$. We consider

$$\text{Tr} \subset \text{SG} \quad \text{as} \quad \text{Tr} = B(0, 2^l),$$

where the radius 2^l of the ball will be specified later, with $l \in \mathbb{N}$. We want to emphasize here that in introduction we kept the subscript Tr_l , indicating the dependence on the radius 2^l of the ball. As we shall see below, the radius will not be important as long as it is big enough, reason for which we will mostly drop the subscript l . On Tr we define an energy form, a Laplacian, and a corresponding Green function. We first define on Tr a Laplacian weakly via the bilinear energy form \mathcal{E}_{Tr} , which itself is defined via Dirichlet forms on the approximating graphs SG_n . More precisely, for functions $f, g: \text{Tr} \cap \text{SG}_n \rightarrow \mathbb{R}$ let

$$\mathcal{E}_{\text{Tr}}^n(f, g) := \left(\frac{5}{3}\right)^n \frac{1}{2} \sum_{x \sim_n y} \frac{1}{4} (f(x) - f(y))(g(x) - g(y)),$$

where $x \sim_n y$ indicates that x and y are neighbors in SG_n . For the limiting object SG we first define an energy form on the dense subset SG_∞ by

$$\text{Dom}(\mathcal{E}_{\text{Tr}}^\infty) := \{v \mid \text{SG}_\infty \cap \text{Tr} \rightarrow \mathbb{R} \text{ with } \lim_{n \rightarrow \infty} \mathcal{E}_{\text{Tr}}^n(v|_{\text{SG}_n \cap \text{Tr}}, v|_{\text{SG}_n \cap \text{Tr}}) < \infty\},$$

$$\mathcal{E}_{\text{Tr}}^\infty(f, g) := \lim_{n \rightarrow \infty} \mathcal{E}_{\text{Tr}}^n(f|_{\text{SG}_n \cap \text{Tr}}, g|_{\text{SG}_n \cap \text{Tr}}) \quad \text{for } f, g \in \text{Dom}(\mathcal{E}_{\text{Tr}}^\infty).$$

Note that $\text{Dom}(\mathcal{E}_{\text{Tr}}^\infty) \subseteq \mathcal{C}(\text{Tr})$, where $\mathcal{C}(\text{Tr})$ is the set of continuous functions on Tr . We extend this definition to an energy form \mathcal{E}_{Tr} with domain $\text{Dom}(\mathcal{E}_{\text{Tr}}) = \{v \in \mathcal{C}(\text{Tr}) \mid \mathcal{E}_{\text{Tr}}(v, v) < \infty\}$ in a natural way.

Denote by

$$\text{Dom}_0(\mathcal{E}_{\text{Tr}}) := \{f \in \text{Dom}(\mathcal{E}_{\text{Tr}}) \mid f|_{\partial \text{Tr}} = 0\}$$

the set of all functions with finite energy and vanishing at the four boundary points of Tr . We define the set of double sided neighboring triangles for $v \in (\text{SG}_\infty \cap \text{Tr}) \setminus \partial \text{Tr}$ of size 2^{-k} by

$$\Delta \Delta^k(v) := B(v, 2^{-k}) \text{ for } k \geq \text{level}(v). \quad (2.3)$$

Let μ be the rescaled α -dimensional Hausdorff measure on SG such that $\mu(\text{Tr}) = 3^l$. For the existence see [3, Lemma 1.1], with a slightly different rescaling adapted to our setup. The rescaling is chosen such that $\mu(\Delta\Delta^0(0)) = 1$ and therefore

$$\mu(\Delta\Delta^n(0)) = \delta_n^\alpha.$$

Definition 2.1. Let $u \in \text{Dom}(\mathcal{E}_{\text{Tr}})$ and $f: \text{SG} \rightarrow \mathbb{R}$ be continuous. Then we define $u \in \text{Dom}(\Delta_{\text{Tr}})$ with $\Delta_{\text{Tr}}u = f$ if and only if

$$\mathcal{E}_{\text{Tr}}(u, v) = - \int_{\text{Tr}} f v d\mu \quad \text{for all } v \in \text{Dom}_0(\mathcal{E}_{\text{Tr}}).$$

We write $u \in \text{Dom}_{L^2}(\Delta_{\text{Tr}})$ if we replace the assumption on the continuity of f by $f \in L^2(\text{Tr}, \mu)$.

See [26, Def. 2.1.1] for more details on the definition of the weak Laplacian Δ_{Tr} .

Lemma 2.1. *Let $u \in \text{Dom}(\Delta_{\text{Tr}})$. Then the limit holds uniformly on $(\text{SG}_\infty \cap \text{Tr}) \setminus \partial\text{Tr}$:*

$$\Delta_{\text{Tr}}u(x) = \frac{3}{4} \lim_{n \rightarrow \infty} \Delta_n u(x). \quad (2.4)$$

Conversely, if u is a continuous function and the right-hand side of (2.4) converges uniformly to a continuous function on $(\text{SG}_\infty \cap \text{Tr}) \setminus \partial\text{Tr}$, then $u \in \text{Dom}(\Delta_{\text{Tr}})$ and (2.4) holds.

Proof. The claim follows from [26, Theorem 2.2.1], with the minor difference that in our case we have the factor $3/4$ instead of $3/2$, due to our choice of μ , which allocates only half the mass on a unit copy of the gasket SG compared with the unit measure on this copy. Notice that in our definition of \mathcal{E}_{Tr} we have an additional factor of $1/4$ compared to [26] which comes from the fact that we use the probabilistic Laplacian throughout this paper, i.e., this factor is also resembled in Δ_n . ■

For $x, y \in \text{Tr} = B(0, 2^l) \subset \text{SG}$ we define the Green function $G_{\text{Tr}}: \text{Tr} \rightarrow \mathbb{R}$ as in [26]. For $n \in \mathbb{N}$, define the level n Green approximation G_{Tr}^n on $\text{SG} \cap \text{Tr}$ by

$$\begin{aligned} G_{\text{Tr}}^n(x, y) &:= r^{-l} \psi_0^{(-l)}(x) \psi_0^{(-l)}(y) \\ &+ \sum_{m=-l}^{n-1} \sum_{z, z' \in \text{Tr} \cap \text{SG}_{m+1} \setminus \text{SG}_m} g_m(z, z') \psi_z^{(m+1)}(x) \psi_{z'}^{(m+1)}(y) \end{aligned}$$

for

$$g_m(z, z') = \begin{cases} \frac{18}{25} r^m & \text{if } z = z' \in \text{SG}_{m+1} \setminus \text{SG}_m, \\ \frac{6}{25} r^m & \text{if } z \sim_{m+1} z' \in \text{SG}_{m+1} \setminus \text{SG}_m, \\ 0 & \text{otherwise,} \end{cases}$$

and the Green function G_{Tr} on $\text{SG} \cap \text{Tr}$ as:

$$G_{\text{Tr}}(x, y) := \lim_{n \rightarrow \infty} G_{\text{Tr}}^n(x, y), \quad (2.5)$$

where $r := 3/5$ is the energy scaling factor and $\psi_z^{(m)}$ for $z \in \text{SG}_m$ is called a harmonic spline and defined by the unique piecewise harmonic function on SG given by

$$\psi_z^{(m)}(x) = \begin{cases} 1 & \text{if } x = z, \\ 0 & \text{if } x \in \text{SG}_m \setminus \{z\}. \end{cases}$$

Notice that the constants in $g(z, z')$ slightly differ from the ones in [26] and [15], due to us working with the probabilistic Laplacian and the corresponding factor $1/4$ in the definition of \mathcal{E}_{Tr} .

Lemma 2.2. *For $x, y \in \text{SG}_n \cap \text{Tr}$ and for all $m \geq n$ the following holds:*

$$G_{\text{Tr}}^n(x, y) = G_{\text{Tr}}^m(x, y) = G_{\text{Tr}}(x, y).$$

Proof. The key insight is that $\psi_z^{(m+1)}(x)\psi_{z'}^{(m+1)}(y) \neq 0$ only if $x \in \Delta\Delta^{m+1}(z)$ and $y \in \Delta\Delta^{m+1}(z')$. But this cannot hold for $z, z' \in \text{SG}_{m+1} \setminus \text{SG}_m$, where $m \geq n$, and this yields

$$G_{\text{Tr}}(x, y) = G_{\text{Tr}}^n(x, y) + \sum_{m=n}^{\infty} \sum_{z, z' \in \text{SG}_{m+1} \setminus \text{SG}_m} g_m(z, z') \underbrace{\psi_z^{(m+1)}(x)\psi_{z'}^{(m+1)}(y)}_{=0}. \quad \blacksquare$$

Lemma 2.3. *For any $x, y \in \text{SG}_n \cap \text{Tr} \setminus \partial\text{Tr}$ and $n \in \mathbb{N}$ the following holds:*

$$\Delta_n G_{\text{Tr}}(x, \cdot)(y) = -\delta_n^{-\alpha} \mathbb{1}_{\{x\}}(y).$$

Proof. The rescaled discrete Laplacian Δ_n is connected to the energy in the following way. Since $\psi_y^{(m)}$ is piecewise harmonic, we get

$$\mathcal{E}_{\text{Tr}}(u, \psi_y^{(m)}) = \mathcal{E}_{\text{Tr}}^m(u, \psi_y^{(m)}) = -\delta_m^{\alpha-\beta} \sum_{x \in \text{SG}_m \cap \text{Tr}} \Delta_{\text{SG}_m} u(x) \psi_y^{(m)}(x) = -\delta_m^{\alpha} \Delta_m u(y),$$

which, together with [26, Equation (2.6.17)], implies

$$\mathcal{E}_{\text{Tr}}(G_{\text{Tr}}^n(x, \cdot), v) = \sum_{z \in \text{SG}_n \cap \text{Tr} \setminus \partial\text{Tr}} v(z) \psi_z^{(n)}(x) \quad \text{for any } v \in \text{Dom}_0(\mathcal{E}_{\text{Tr}}).$$

Applying the above two equations to $u = G_{\text{Tr}}(x, \cdot)$ and $v = \psi_y^{(n)}$ we finally obtain

$$\begin{aligned} \Delta_n G_{\text{Tr}}(x, \cdot)(y) &= -\delta_n^{-\alpha} \mathcal{E}_{\text{Tr}}(G_{\text{Tr}}(x, \cdot), \psi_y^{(n)}) \\ &= -\delta_n^{-\alpha} \sum_{z \in \text{SG}_n \setminus \partial\text{Tr}} \psi_y^{(n)}(z) \psi_z^{(n)}(x) = -\delta_n^{-\alpha} \mathbb{1}_{\{x\}}(y). \quad \blacksquare \end{aligned}$$

The last claim of this section relates the stopped Green function $g_{\text{Tr}}^n(x, y)$ defined in (2.2) with the Green functions defined in (2.5) in terms of harmonic splines.

Lemma 2.4. *For any $x, y \in \text{SG}_n$ the following holds:*

$$g_{\text{Tr}}^n(x, y) = G_{\text{Tr}}(x, y).$$

Proof. Since $\Delta_n g_{\text{Tr}}^n(x, y) = -\delta_n^{-\alpha} \mathbb{1}_{\{x\}}(y)$ for all $x, y \in \text{SG}_n \cap \text{Tr} \setminus \partial\text{Tr}$ and $g_{\text{Tr}}^n(x, z) = 0$ for $z \in \partial\text{Tr}$, it follows that both functions solve the following Dirichlet problem

$$\begin{cases} \Delta_n u(y) = -\delta_n^{-\alpha} \mathbb{1}_{\{x\}}(y) & \text{if } y \in \text{SG}_n \cap \text{Tr} \setminus \partial\text{Tr}, \\ u(y) = 0 & \text{if } y \in \partial\text{Tr}, \end{cases}$$

which, together with the uniqueness principle, implies $g_{\text{Tr}}^n(x, y) = G_{\text{Tr}}(x, y)$. ■

3. Scaling limit for the divisible sandpile

In this section we investigate odometer functions and obstacle problems on the limiting fractal object SG. There are several difficulties appearing on SG. We could define the pointwise Laplacian on SG and demand the odometer function $u: \text{SG} \rightarrow \mathbb{R}$ to be the smallest function that fulfills $\sigma + \Delta u \leq 1$, where σ is the initial mass on SG. However, this definition is not very handy and it is not clear how to obtain any meaningful properties of it. We choose a different way to define *the continuous divisible sandpile on the gasket* SG and motivate our choice below.

Passing from SG_n to SG and back: some useful notation. Recall that by $\Delta\Delta^k(x)$ we have denoted the double sided triangles of length 2^{-k} with midpoint $x \in \text{SG}_n$. Similarly, the notations provided in Table 1 might be useful when working with points, sets and functions. However, they will only be used when it is absolutely clear on which approximating graph we are working. In the ambiguous case $x \in \text{SG}_{n+1} \setminus \text{SG}_n$ we define x° to be rightmost neighbor.

	SG_n	SG
points	$x^\circ = B(x, 2^{-(n+1)}) \cap \text{SG}_n$	$x^{\Delta\Delta} = \Delta\Delta^{n+1}(x)$
sets	$A^\circ = A \cap \text{SG}_n$	$A^{\Delta\Delta} = \bigcup_{x \in A} x^{\Delta\Delta}$
functions	$f^\circ = f _{\text{SG}_n}$	$f^{\Delta\Delta}(x) = f(x^\circ)$

Table 1. Notation for transitioning between gasket SG and approximating graphs SG_n .

In order to motivate a meaningful definition, we would like to comment on what is needed when solving the divisible sandpile and the corresponding obstacle problem on SG_n , for any $n \in \mathbb{N}$. We consider initial configurations $\sigma: SG \rightarrow [0, \infty)$ and $\sigma_n: SG_n \rightarrow [0, \infty)$ that fulfill the following properties: for a number $M \geq 0$ and a compact subset $T \subseteq SG$ the following holds:

$$0 \leq \sigma, \sigma_n \leq M, \quad (3.1)$$

$$\mu(DC(\sigma)) = 0, \quad (3.2)$$

$$\sigma_n^{\Delta\Delta}(x) \rightarrow \sigma(x), \quad \forall x \notin DC(\sigma) \text{ pointwise}, \quad (3.3)$$

$$\text{supp}(\sigma), \quad \text{supp}(\sigma_n) \subseteq T, \quad (3.4)$$

where by $DC(\sigma)$ we denote the set of discontinuity points of σ , and μ is the (α -rescaled) Hausdorff measure on SG . For the rest of this section we denote the rescaled odometer function u_n of the divisible sandpile on SG_n from initial distribution σ_n by

$$u_n(x) = \delta_n^\beta \cdot (\text{total mass emitted from } x \text{ in the stabilization of } \sigma_n). \quad (3.5)$$

We remark that the rescaled odometer function u_n fulfills the *least action principle* as well, i.e., it is the smallest positive function such that $\sigma_n + \Delta_n u_n \leq 1$. The following lemma provides a ball large enough containing the supports of all u_n , $n \in \mathbb{N}$.

Lemma 3.1. *Let $\mathcal{D}_n = \{u_n > 0\}$ be the divisible sandpile cluster on SG_n started from initial density σ_n , for all $n \in \mathbb{N}$. Then there exists a triangle $\text{Tr} = B(0, 2^L) \subset SG$, $L \in \mathbb{N}$ such that*

$$\bigcup_{n \geq 0} \mathcal{D}_n^{\Delta\Delta} \subseteq \text{Tr}.$$

Proof. Let $B = B(0, 2^l) \subset SG$ and choose $l \in \mathbb{N}$ large enough such that B contains the supports of all σ_n . Let \mathcal{A}_n be the divisible sandpile cluster on SG_n from initial density $\sigma'_n = 2|B \cap SG_n| \mathbb{1}_{\{0\}}$, which, by the inner bound of [12], contains B . Let \mathcal{B}_n be the divisible sandpile cluster on SG_n resulting from initial density $\tau = M \mathbb{1}_B$. By the Abelian property we have $\mathcal{D}_n \subseteq \mathcal{B}_n$, for all $n \in \mathbb{N}$ and also that the clusters \mathcal{C}_n resulting from initial density $M\sigma'_n$ contain \mathcal{B}_n . Finally, choosing $L \in \mathbb{N}$ large enough such that $\text{Tr} = B(0, 2^L)$ has volume at least $3M\mu(B)$, the outer bound of [12] yields that $\mathcal{C}_n \subseteq \text{Tr}$, for all $n \in \mathbb{N}$. ■

For the initial particle configuration σ_n on SG_n , we denote by ν_n the limit mass configuration obtained by stabilizing σ_n , and by u_n the corresponding rescaled odometer function as defined in (3.5) on SG_n . Let Tr be a double sided triangle around 0 as in Lemma 3.1. We define the obstacle function $\gamma_n: SG_n \rightarrow \mathbb{R}$ and its

least superharmonic majorant $s_n: \text{SG}_n \rightarrow \mathbb{R}$ as

$$\gamma_n(x) := - \sum_{y \in \text{SG}_n \cap \text{Tr}} g_{\text{Tr}}^n(x, y)(\sigma_n(y) - 1), \text{ and} \quad (3.6)$$

$$s_n(x) := \inf\{f_n(x) \mid f_n: \text{SG}_n \rightarrow \mathbb{R} \text{ is superharmonic on Tr and } f \geq \gamma_n\}.$$

Then [12, Lemma 3.9] gives for the odometer function u_n on SG_n , $u_n := s_n - \gamma_n$, and the sandpile cluster \mathcal{D}_n on SG_n starting from the initial configuration σ_n

$$\mathcal{D}_n = \{x \in \text{SG}_n \mid s_n(x) > \gamma_n(x)\}, \quad \text{for all } n \in \mathbb{N},$$

which motivates the following definition of divisible sandpile on the Sierpiński gasket SG, by using the ball Tr (double sided triangle around 0) of radius big enough containing all the divisible sandpile clusters \mathcal{D}_n .

Definition 3.1 (continuous divisible sandpile on Tr). Let $\sigma: \text{SG} \rightarrow [0, \infty)$ be bounded with compact support and continuous μ -a.e. such that $\text{supp}(\sigma) \subseteq \text{Tr}$, where Tr is a double sided triangle around 0. We define the obstacle $\gamma_{\text{Tr}}: \text{SG} \rightarrow \mathbb{R}$ by

$$\gamma_{\text{Tr}}(x) := -\mathbf{G}_{\text{Tr}}(\sigma(x) - 1) = - \int_{\text{Tr}} \mathbf{G}_{\text{Tr}}(x, y)(\sigma(y) - 1)d\mu(y)$$

and the corresponding odometer function $u_{\text{Tr}}: \text{SG} \rightarrow \mathbb{R}$ by

$$u_{\text{Tr}} := s_{\text{Tr}} - \gamma_{\text{Tr}}, \text{ with}$$

$$s_{\text{Tr}}(x) := \inf\{f(x) \mid f \text{ continuous, superharmonic on Tr and } f \geq \gamma_{\text{Tr}}\}$$

being the smallest superharmonic majorant of γ_{Tr} . Furthermore, we define the noncoincidence set for the obstacle problem with obstacle γ_{Tr} as the domain

$$D_{\text{Tr}} = \{x \in \text{SG} \mid s_{\text{Tr}}(x) > \gamma_{\text{Tr}}(x)\}.$$

Remark. It is not a priori clear that the smallest superharmonic majorant s_{Tr} of γ_{Tr} as defined above is itself again superharmonic, or even in the domain of the Laplacian. However, using the fact that $\text{Dom}_0(\mathcal{E}_{\text{Tr}})$ can be equipped with an inner product such that it becomes a Hilbert space ($\langle u, v \rangle_{\mathcal{E}} = \mathcal{E}_{\text{Tr}}(u, v) + \langle u, v \rangle_{L^2(\text{Tr})}$) together with the Poincaré-type inequality

$$|u(x) - u(y)|^2 \leq C \mathcal{E}_{\text{Tr}}(u) |x - y|^{\beta/\alpha},$$

for some constant $C > 0$ as in [26], we can deduce the superharmonicity of the superharmonic majorant as on \mathbb{R}^d . We can then use minimization theorems for coercive functionals on Hilbert spaces in order to complete the claim, but since this is very similar to the case of \mathbb{R}^d , we have decided not to include it here. For more details we refer the interested reader to [25] and [9].

We aim at taking the limit $u = \lim_{k \rightarrow \infty} u_{\Delta\Delta^{-k}(0)}$ as the odometer function of the divisible sandpile on the infinite gasket SG, but it is not straightforward why this limit should be well-defined. We show below that this is indeed the case. We first need some auxiliary results, the first one concerning the monotonicity of the model: by starting with more mass, one obtains a larger odometer and a larger noncoincidence set.

Lemma 3.2. *Let $\sigma_1, \sigma_2: \text{SG} \rightarrow [0, \infty)$ be bounded with compact support and $\sigma_1 \leq \sigma_2$. Let Tr be a double sided triangle with $\text{supp}(\sigma_1), \text{supp}(\sigma_2) \subseteq \text{Tr}$ and, for $i = 1, 2$ consider the obstacle*

$$\gamma_{\text{Tr}}^{(i)} = -\mathbf{G}_{\text{Tr}}(\sigma_i - 1).$$

Let $s_{\text{Tr}}^{(i)}$ be the least superharmonic majorant of $\gamma_{\text{Tr}}^{(i)}$ and $u_{\text{Tr}}^{(i)} = s_{\text{Tr}}^{(i)} - \gamma_{\text{Tr}}^{(i)}$. Then $u_{\text{Tr}}^{(1)} \leq u_{\text{Tr}}^{(2)}$.

The proof is similar to [22, Lemma 2.8], applied to our choice of odometers.

Lemma 3.3. *Let $\sigma: \text{SG} \rightarrow [0, \infty)$ as in Definition 3.1, and let $\text{Tr} \subseteq \widetilde{\text{Tr}}$ be double sided triangles. Denote by $u := u_{\text{Tr}}$, $\tilde{u} := u_{\widetilde{\text{Tr}}}$ the corresponding odometer functions as in Definition 3.1. If there exists $\varepsilon > 0$ such that $D_{\text{Tr}}^\varepsilon \subsetneq \text{Tr}$, then $\tilde{u} = u$.*

Proof. Let us denote by $\gamma := \gamma_{\text{Tr}}$ and $s := s_{\text{Tr}}$ (resp. $\tilde{\gamma} := \gamma_{\widetilde{\text{Tr}}}$ and $\tilde{s} := s_{\widetilde{\text{Tr}}}$) the obstacle and its superharmonic majorant on Tr (resp. $\widetilde{\text{Tr}}$). We have $\Delta_{\text{Tr}}\tilde{\gamma} = (1 - \sigma) = \Delta_{\text{Tr}}\gamma$. Notice that γ and s vanish on $\widetilde{\text{Tr}} \setminus \text{Tr}$ and the function $s' = s + \tilde{\gamma} - \gamma$ is superharmonic on $\widetilde{\text{Tr}}$. Since, $s' \geq \tilde{\gamma}$, we obtain that $\tilde{s} \leq s'$ on $\widetilde{\text{Tr}}$. From $D_{\text{Tr}}^\varepsilon \subset \text{Tr}$, it then follows that $s' = \tilde{\gamma}$ in $\widetilde{\text{Tr}} \setminus \text{Tr}$, which implies $\tilde{s} = \tilde{\gamma}$ on $\widetilde{\text{Tr}} \setminus \text{Tr}$, which in turn gives $\tilde{u} = 0$ on $\widetilde{\text{Tr}} \setminus \text{Tr}$. Inside Tr we can define the harmonic function $h = \tilde{\gamma}|_{\text{Tr}} - \gamma|_{\text{Tr}}$. By the definition of s and \tilde{s} we have for $x \in \text{Tr}$

$$\tilde{s} - h = \tilde{s} - \tilde{\gamma} + \gamma \geq \gamma \quad \text{and} \quad s + h = s + \tilde{\gamma} - \gamma \geq \tilde{\gamma},$$

which implies $\tilde{s}(x) \geq s(x) + h(x) \geq \tilde{s}(x)$. Therefore, on Tr it holds that $\tilde{u} = \tilde{s} - \tilde{\gamma} = s - \gamma = u$, and this proves the claim. ■

Lemma 3.4. *Let $\sigma: \text{SG} \rightarrow [0, \infty)$ be a bounded function with compact support and continuous μ -almost everywhere. Denote by $u_{\Delta\Delta^{-k}(0)}$ the odometer of σ on the set $\text{Tr} = \Delta\Delta^{-k}(0)$ as in Definition 3.1. Then the following limit is well-defined:*

$$u = \lim_{k \rightarrow \infty} u_{\Delta\Delta^{-k}(0)}.$$

Proof. We choose $l \in \mathbb{N}$ large enough such that $\text{supp}(\sigma) \subseteq B(0, 2^l)$ and $L \in \mathbb{N}$ large enough such that $\sigma(x) \leq 3^{L-l}$, for all $x \in \text{SG}$. Defining the initial configura-

tion $\sigma': \text{SG} \rightarrow [0, \infty)$ by

$$\sigma' = \frac{\mu(B(0, 2^L))}{\mu(B(0, 2^l))} \mathbb{1}_{B(0, 2^l)} = 3^{L-l} \mathbb{1}_{B(0, 2^l)},$$

we then have $\sigma \leq \sigma'$. Let us denote by $u_{\Delta\Delta^{-k}(0)}$ the odometer of σ in $\Delta\Delta^{-k}(0)$ and by $u'_{\Delta\Delta^{-k}(0)}$ the odometer of σ' in $\Delta\Delta^{-k}(0)$. Appendix A shows the existence of a compact set $\text{Tr} = B(0, 2^{L+1})$ such that for all $k \in \mathbb{N}$ with $\text{Tr} \subseteq \Delta\Delta^{-k}(0)$ we have $u'_{\Delta\Delta^{-k}(0)} = 0$ on $\Delta\Delta^{-k}(0) \setminus \text{Tr}$, which together with Lemma 3.2 implies that also $u_{\Delta\Delta^{-k}(0)} = 0$ on $\Delta\Delta^{-k}(0) \setminus \text{Tr}$. The claim now follows from Lemma 3.3. ■

We remark that Lemma 3.4 is actually a refined version of Proposition 1.1. We now define the *continuous divisible sandpile* on $\text{SG} \subset \mathbb{R}^2$.

Definition 3.2 (Continuous divisible sandpile on SG). Let $\sigma: \text{SG} \rightarrow [0, \infty)$ be a bounded function with compact support and continuous μ -a.e. Denote by $u_{\Delta\Delta^{-k}(0)} = s_{\Delta\Delta^{-k}(0)} - \gamma_{\Delta\Delta^{-k}(0)}$ the odometer function of the continuous divisible sandpile in $\Delta\Delta^{-k}(0)$ as in Definition 3.1. Then the odometer function of the global continuous divisible sandpile is given by

$$u = \lim_{k \rightarrow \infty} u_{\Delta\Delta^{-k}(0)},$$

and we define the corresponding noncoincidence set D by $D := \{x \in \text{SG} \mid u(x) > 0\}$.

3.1. Convergence of odometers in divisible sandpile

We now investigate the convergence of the odometer functions u and u_n with initial distributions $\sigma: \text{SG} \rightarrow \mathbb{R}$ and $\sigma_n: \text{SG}_n \rightarrow \mathbb{R}$, respectively, fulfilling the properties (3.1)–(3.4). In view of Lemma 3.4 and Lemma 3.1, we can fix a large enough double sided triangle Tr that contains the supports of the odometer functions u and u_n for all $n \in \mathbb{N}$. For the rest, in order to simplify the notation we drop the double sided triangle Tr and write

$$G := G_{\text{Tr}}, \quad g^n := g_{\text{Tr}}^n$$

for the Green function G on SG , and for the Green function g stopped when exiting the set Tr in SG_n , for all $n \in \mathbb{N}$.

Lemma 3.5. *Define*

$$G\sigma(x) := \int_{\text{Tr}} G(x, y)\sigma(y)d\mu(y) \quad \text{and} \quad g^n\sigma_n(x) := \delta_n^\alpha \sum_{y \in \text{SG}_n \cap \text{Tr}} g^n(x, y)\sigma_n(y).$$

If σ, σ_n satisfy equations (3.1)–(3.4), then

$$(\mathbf{g}^n \sigma_n)^{\Delta\Delta} \rightarrow \mathbf{G} \sigma \text{ uniformly as } n \rightarrow \infty.$$

Proof. By the triangle inequality we have

$$|(\mathbf{g}^n \sigma_n)^{\Delta\Delta} - \mathbf{G} \sigma| \leq |(\mathbf{g}^n \sigma_n)^{\Delta\Delta} - \mathbf{G} \sigma_n^{\Delta\Delta}| + |\mathbf{G} \sigma_n^{\Delta\Delta} - \mathbf{G} \sigma|.$$

The second term above converges uniformly to zero by the dominated convergence theorem since \mathbf{G} is bounded and by assumptions (3.2) and (3.3) $\sigma_n^{\Delta\Delta}(x) \rightarrow \sigma(x)$, μ -almost everywhere as $n \rightarrow \infty$. The first term is at most

$$|(\mathbf{g}^n \sigma_n)^{\Delta\Delta}(x) - \mathbf{G} \sigma_n^{\Delta\Delta}(x)| = \left| \sum_{y \in \text{SG}_n \cap \text{Tr}} \sigma_n(y) \int_{y^{\Delta\Delta}} (\mathbf{g}^n(x^\circ, y) - \mathbf{G}(x, z)) d\mu(z) \right|. \quad (3.7)$$

For $y \neq x^\circ$ the function $f(z) := \mathbf{g}_n(x^\circ, y) - \mathbf{G}(x^\circ, z)$ is harmonic on $y^{\Delta\Delta}$, and Lemma 2.4 gives $f(y) = 0$ which yields

$$\int_{y^{\Delta\Delta}} (\mathbf{g}^n(x^\circ, y) - \mathbf{G}(x^\circ, z)) d\mu(z) = \int_{y^{\Delta\Delta}} f(z) d\mu(z) = f(y) = 0.$$

So the sum over all $y \neq x^\circ$ in (3.7) can be bounded by

$$\begin{aligned} & \left| \sum_{y \in \text{SG}_n \cap \text{Tr} \setminus \{x^\circ\}} \sigma_n(y) \int_{y^{\Delta\Delta}} (\mathbf{g}^n(x^\circ, y) - \mathbf{G}(x, z)) d\mu(z) \right| \\ &= \left| \sum_{y \in \text{SG}_n \cap \text{Tr} \setminus \{x^\circ\}} \sigma_n(y) \int_{y^{\Delta\Delta}} (\mathbf{G}(x^\circ, z) - \mathbf{G}(x, z)) d\mu(z) \right| < M \varepsilon, \end{aligned}$$

for any $\varepsilon > 0$ and $n \in \mathbb{N}$ large, where the last inequality is due to the uniform continuity of \mathbf{G} together with (3.1). Finally, for n large the summand $y = x^\circ$ in (3.7) can be bounded by

$$\left| \sigma_n(x^\circ) \int_{(x^\circ)^{\Delta\Delta}} (\mathbf{g}^n(x^\circ, x^\circ) - \mathbf{G}(x, z)) d\mu(z) \right| \leq C \mu(y^{\Delta\Delta}) \leq C \delta_n^\alpha < c \varepsilon$$

because \mathbf{G} is bounded and $\mathbf{g}^n(y, y) \sim G(y, y)$ by Lemma 2.4, and this completes the proof. \blacksquare

Corollary 3.1. *If σ, σ_n satisfy (3.1)–(3.4), then $\gamma_n^{\Delta\Delta} \rightarrow \gamma$ uniformly as $n \rightarrow \infty$.*

Lemma 3.6. *If σ, σ_n satisfy (3.1)–(3.4), then $s_n^{\Delta\Delta} \rightarrow s$ as $n \rightarrow \infty$ uniformly on Tr .*

Proof. For $x \in \text{SG}_n$, denote by $\psi_x^{(n)}$ the piecewise harmonic spline of level n which is 1 at x . We then have for the superharmonic majorant s that

$$\Delta_n s(x) = \mathcal{E}_{\text{Tr}}(s, \psi_x^{(n)}) = - \int_{\text{Tr}} \Delta_{\text{Tr}} s \cdot \psi_x^{(n)} d\mu \leq 0,$$

where in the last inequality we have used the superharmonicity of s together with the non-negativity of $\psi_x^{(n)}$. This shows that s restricted to $\text{SG}_n \cap \text{Tr}$ is also superharmonic. By Corollary 3.1 we can find for any $\varepsilon > 0$ an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\gamma_n^{\Delta\Delta} - \varepsilon \leq \gamma.$$

We thus obtain that $s|_{\text{SG}_n} + \varepsilon$ is superharmonic and

$$s|_{\text{SG}_n} + \varepsilon \geq \gamma|_{\text{SG}_n} + \varepsilon \geq \gamma_n,$$

for all $n \geq n_0$. Thus, by the definition of s_n as the least superharmonic majorant of γ_n on SG_n , we obtain for large enough n that

$$s + \varepsilon \geq s_n^{\Delta\Delta}.$$

For the reverse inequality consider v_n the final sandpile distribution on SG_n for the initial density σ_n . Recall that

$$\Delta_n s_n = \Delta_n u_n + \Delta_n \gamma_n = (v_n - \sigma_n) + (\sigma_n - 1) = v_n - 1$$

and therefore $0 \leq \Delta_n s_n \leq 1$. By denoting $\varphi_n := -\Delta_n s_n$ and $\tilde{s} = \mathbf{G} \varphi_n^{\Delta\Delta}(x)$, we obtain for large enough $n \in \mathbb{N}$

$$\begin{aligned} |\tilde{s} - s_n^{\Delta\Delta}(x)| &= |\mathbf{G} \varphi_n^{\Delta\Delta}(x) - \mathbf{g}^n \varphi_n(x^{\cdot\cdot})| \\ &\leq \sum_{y \in \text{SG}_n \cap \text{Tr}} |\varphi_n(y)| \left| \int_{\Delta\Delta^n(y)} \mu(\Delta\Delta^n(y))^{-1} \delta_n^\alpha \mathbf{g}(x^{\cdot\cdot}, y) - \mathbf{G}(x, z) d\mu(z) \right| < \varepsilon, \end{aligned}$$

where in the last step we have used the same arguments as in the proof of Lemma 3.5 in order to show that (3.7) goes to zero together with the fact $0 \leq \varphi_n \leq 1$. Thus,

$$\tilde{s} + 2\varepsilon > s_n^{\Delta\Delta}(x) + \varepsilon > \gamma_n^{\Delta\Delta}(x) + \varepsilon > \gamma$$

and by construction \tilde{s} is superharmonic, from which we can finally conclude that

$$s_n^{\Delta\Delta} + 3\varepsilon \geq \tilde{s} + 2\varepsilon > s,$$

which finishes the proof. ■

Lemma 3.7. *Let u_n be the odometer function for the divisible sandpile on SG_n with initial density σ_n . If σ, σ_n satisfy (3.1)–(3.4), then*

$$u_n^{\Delta\Delta} \rightarrow s - \gamma = u \text{ uniformly on compacts, as } n \rightarrow \infty.$$

Proof. By Corollary 3.1 and Lemma 3.6 we have $\gamma_n^{\Delta\Delta} \rightarrow \gamma$ and $s_n^{\Delta\Delta} \rightarrow s$ uniformly on Tr , as $n \rightarrow \infty$. Using $u_n = s_n - \gamma_n$ and $u = s - \gamma$ gives the result. ■

3.2. Convergence of domains in divisible sandpile

Following [22], we assume additional conditions on the initial density $\sigma: \text{SG} \rightarrow [0, \infty)$:

$$\text{For all } x \in \text{SG} \text{ either } \sigma(x) \leq \lambda \text{ or } \sigma(x) \geq 1 \text{ for some } \lambda < 1, \quad (3.8)$$

$$\{\sigma \geq 1\} = \overline{\{\sigma \geq 1\}^\circ}. \quad (3.9)$$

Moreover, we assume that for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$\text{if } x \in \{\sigma \geq 1\}_\varepsilon, \text{ then } \sigma_n(x) \geq 1 \text{ for all } n \geq N(\varepsilon), \quad (3.10)$$

$$\text{if } x \notin \{\sigma \geq 1\}^\varepsilon, \text{ then } \sigma_n(x) \leq \lambda \text{ for all } n \geq N(\varepsilon). \quad (3.11)$$

We also write

$$\tilde{D} := D \cup \{x \in \text{SG} \mid \sigma(x) \geq 1\}^\circ.$$

In this section we will prove the scaling limit for the divisible sandpile cluster \mathcal{D}_n .

Theorem 3.1. *Let σ and σ_n satisfy (3.1)–(3.4) and (3.8)–(3.11). For $n \geq 1$, let \mathcal{D}_n be the divisible sandpile cluster in SG_n started from initial density σ_n . For any $\varepsilon > 0$ and n large enough, we have*

$$\tilde{D}_\varepsilon \cap \text{SG}_n \subseteq \mathcal{D}_n \subseteq \tilde{D}^\varepsilon \cap \text{SG}_n.$$

Lemma 3.8. *Fix $\varepsilon > 0$ and $x \in \mathcal{D}_n$ with $x \notin \tilde{D}^\varepsilon$. For n large enough, there is a vertex $y \in \text{SG}_n$ with $|x - y| \leq \varepsilon/2 + \delta_n$, such that for some constant $c > 0$ the following holds:*

$$u_n(y) \geq u_n(x) + c(1 - \lambda)\varepsilon^\beta.$$

Proof. From (3.9) we have $\{\sigma \geq 1\} \subseteq \overline{\tilde{D}}$. Thus if $x \notin \tilde{D}^\varepsilon$, the ball $B = B(x, \varepsilon/2)^\circ$, with $B(x, \varepsilon/2) \subset \text{SG}$ is disjoint from $\{\sigma \geq 1\}^{\varepsilon/2}$. Taking $n \geq N(\varepsilon/2)$ gives $\sigma_n \leq \lambda$ on B by assumption (3.11). We define for $y \in \text{SG}_n$

$$\varphi_{x,r}(y) := \mathbb{E}_x[\tau_r^n(x)] - \mathbb{E}_y[\tau_r^n(x)]. \quad (3.12)$$

We then have $\Delta_{\text{SG}_n} \varphi_{x,r}(y) = 1$ for $y \in B^n(x, r)$ and $\varphi_{x,r}(x) = 0$. Since r is the radius in the graph metric, when translating to B we have to rescale it by δ_n . Define $w: \text{SG}_n \rightarrow \mathbb{R}$ by

$$w(y) = u_n(y) - (1 - \lambda) \delta_n^\beta \varphi_{x,\varepsilon/2\delta_n}(y),$$

which is subharmonic on $B \cap \mathcal{D}_n$ and therefore attains its maximum on the boundary. By the choice of $\varphi_{x,\varepsilon/2\delta_n}$, $w(x) > 0$, so the maximum cannot be attained at $\partial \mathcal{D}_n$, where u_n vanishes. Therefore, the maximum is attained at some $y \in \partial B$ and

$$u_n(y) \geq w(y) + (1 - \lambda) \delta_n^\beta \varphi_{x,\varepsilon/2\delta_n}(y) \geq w(y) + (1 - \lambda) c \delta_n^\beta \left(\frac{\varepsilon}{\delta_n} \right)^\beta,$$

where the last inequality follows from (E_β) with $\mathbb{E}_y[\tau_r^n(x)] = 0$ for $y \in \partial B^n(x, r)$. ■

Proof of Theorem 3.1. Fix $\varepsilon > 0$. By the definition of \tilde{D}_ε we have for some $\eta > 0$

$$\tilde{D}_\varepsilon \subseteq D_\eta \cup \{\sigma \geq 1\}_\eta.$$

Since the closure of D_η is compact and contained in D , we have $u \geq m_\eta$ on D_η for some $m_\eta > 0$. From Lemma 3.7 we get

$$u_n > u^\circ - \frac{1}{2} m_\eta > 0 \text{ on } D_\eta \cap \text{SG}_n \text{ for large enough } n,$$

hence $D_\eta \cap \text{SG}_n \subseteq \mathcal{D}_n$, which together with (3.10) gives $\{\sigma \geq 1\}_\eta \cap \text{SG}_n \subseteq \mathcal{D}_n$, and this implies that $\tilde{D}_\varepsilon \cap \text{SG}_n \subseteq \mathcal{D}_n$. For the upper bound, fix $x \in \text{SG}_n$ with $x \notin \tilde{D}_\varepsilon$. Since u vanishes on $B = B(x, \varepsilon/2)$ we get from the convergence of u_n in Lemma 3.7 that

$$u_n < c(1 - \lambda) \varepsilon^\beta \text{ on } B \cap \text{SG}_n \text{ for large enough } n,$$

which together with Lemma 3.8 yields $x \notin \mathcal{D}_n$ and therefore $\mathcal{D}_n \subseteq \tilde{D}_\varepsilon \cap \text{SG}_n$. ■

4. Scaling limit for rotor aggregation

For rotor aggregation, since particles are moving according to a rotor walk on SG_n , the initial densities have to be given as integer valued functions. Similar to [22], we adapt the notion of convergence of densities, and we define the appropriate notions of odometer functions.

Smoothing operation. For more details on the smoothing operation and why it is appropriate to use it in order to define a notion of convergence, we refer the reader to [22, Section 4]. For any $n \in \mathbb{N}$, denote in this section by $(Y_t^n)_{t \in \mathbb{N}}$ the lazy random walk on SG_n , which at each time step stays at the current position with probability

1/2 and with probability 1/8 chooses one of the four possible neighbors and moves there. For any function $f: \text{SG}_n \rightarrow \mathbb{R}$ and $k \in \mathbb{N}$, we define its k -th smoothing by

$$S_k^n f(x) = \sum_{y \in \text{SG}_n} \mathbb{P}_x[Y_k^n = y] f(y) = \mathbb{E}_x[f(Y_k^n)],$$

so the smoothing operation fulfills $S_k S_l = S_{k+l}$ for all $k, l \in \mathbb{N}$ due to the Markov property of the lazy random walk. Furthermore, we have $\Delta_n = 2\delta_n^{-\beta} (S_1 - S_0)$, which implies that the Laplacian and the smoothing operator commute.

Lazy random walk on Sierpiński graphs SG_n . According to [19, Proposition 5.7], the total variation distance between the distribution P^{t+1} at time $t+1$ and the distribution P^t at time t of the lazy random walk $(Y_t^n)_{t \in \mathbb{N}}$ can be bounded by:

$$\|P^{t+1}(x, \cdot) - P^t(x, \cdot)\|_{\text{TV}} \leq 1/\sqrt{t}. \quad (4.1)$$

Next we collect some properties of the lazy random walk on SG_n .

Lemma 4.1. *Let P be the transition matrix of the lazy random walk $(Y_t^n)_{t \in \mathbb{N}}$ on SG_n . Then*

- (1) $\Delta_{\text{SG}_n} P^t(x, y) = 2P^{t+1}(x, y) - 2P^t(x, y)$.
- (2) $P^t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \exp(-c(d(x, y)^\beta/t)^{1/(\beta-1)})$.
- (3) $P^t(x, x) \geq \frac{C'}{t^{\alpha/\beta}}$.

Above $C', C, c > 0$ are suitable constants.

Proof. (1) Using

$$P^t(x, y) = \frac{1}{2} P^{t-1}(x, y) + \frac{1}{8} \sum_{y' \sim y} P^{t-1}(x, y'),$$

we get for the first claim:

$$\begin{aligned} \Delta_{\text{SG}_n} P^t(x, y) &= \frac{1}{4} \sum_{y' \sim y} P^t(x, y') - P^t(x, y) \\ &= 2\left(\frac{1}{2} P^t(x, y) + \frac{1}{8} \sum_{y' \sim y} P^t(x, y')\right) - 2P^t(x, y) = 2P^{t+1}(x, y) - 2P^t(x, y). \end{aligned}$$

(2) For the second and third claim we use the following bounds on the transition probabilities for the simple random walk $(X_t^n)_{t \in \mathbb{N}}$ from [14, Theorem 17]:

$$\mathbb{1}_{\{x=y\}} C_2 \frac{1}{t^{\alpha/\beta}} \leq \mathbb{P}[X_t^n = y \mid X_0^n = x] \leq \frac{C_1}{t^{\alpha/\beta}} \exp\left(-c_1 \left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right),$$

for $C_1, c_1, C_2 > 0$. The t -steps transition probabilities of the lazy random walk $(Y_t^n)_{t \in \mathbb{N}}$ are then

$$\begin{aligned} P^t(x, y) &= \sum_{k=0}^t \binom{t}{k} 2^{-t} \mathbb{P}_x[X_k^n = y] = 2^{-t} \mathbb{1}_{\{x=y\}} + \sum_{k=1}^t \binom{t}{k} 2^{-t} \mathbb{P}_x[X_k^n = y] \\ &\leq 2^{-t} + \exp\left(-c_1 \left(\frac{d(x, y)^\beta}{t}\right)^{1/(\beta-1)}\right) \sum_{k=1}^t \binom{t}{k} 2^{-t} \frac{C_1}{k^{\alpha/\beta}}. \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{k=1}^t \binom{t}{k} 2^{-t} \frac{t^{\alpha/\beta}}{k^{\alpha/\beta}} &= \mathbb{E}\left[\frac{t^{\alpha/\beta}}{Z^{\alpha/\beta}} \mathbb{1}_{\{Z \geq 1\}}\right] \leq \mathbb{E}\left[\frac{t}{Z} \mathbb{1}_{\{Z \geq 1\}}\right] \\ &\leq 2t \mathbb{E}\left[\frac{1}{Z+1}\right] = 2t \frac{2(1-2^{-t+1})}{t+1} \leq 4, \end{aligned}$$

where $Z \sim \text{Binom}(t, 1/2)$, and this proves the second assertion.

(3) Using the lower bound on the transition probabilities $\mathbb{P}_x[X_t^n = x]$ we get

$$\begin{aligned} P^t(x, x) &= \mathbb{P}_x[Y_t^n = x] = \sum_{k=0}^t \binom{t}{k} 2^{-t} \mathbb{P}_x[X_k^n = x] \\ &\geq C_2 \frac{1}{t^{\alpha/\beta}} \sum_{k=0}^t \binom{t}{k} 2^{-t} = C_2 \frac{1}{t^{\alpha/\beta}}, \end{aligned}$$

which proves the final assertion. ■

Proposition 4.1. *For the lazy random walk $(Y_t^n)_{t \geq 0}$ on SG_n , for every $x \in \text{SG}_n$ and $0 < \delta < 1/2\alpha$, we have*

$$\sum_y \sum_{y' \sim y} |P^t(x, y) - P^t(x, y')| \leq \frac{C}{t^{1/4-\delta\alpha/2}}.$$

Proof. Fix any $\delta > 0$ with $\delta < 1/2\alpha$. We bound the sum by considering it for elements inside and outside the ball $B^n(x, t^{1/\beta+\delta})$ separately. We have

$$\begin{aligned} &\sum_{y \in B^n(x, t^{1/\beta+\delta})} \sum_{y' \sim y} |P^t(x, y) - P^t(x, y')| \\ &\leq |B^n(x, t^{1/\beta+\delta})|^{1/2} \left(\sum_{y \in B^n(x, t^{1/\beta+\delta})} \sum_{y' \sim y} |P^t(x, y) - P^t(x, y')|^2 \right)^{1/2} \\ &\leq |B^n(x, t^{1/\beta+\delta})|^{1/2} (\mathcal{E}_{\text{SG}_n}(P^t(x, \cdot), P^t(x, \cdot)))^{1/2}, \end{aligned}$$

where the first inequality holds by Cauchy–Schwarz and the (unscaled) graph energy form $\mathcal{E}_{\text{SG}_n}$ on SG_n is defined as

$$\mathcal{E}_{\text{SG}_n}(f, g) = \frac{1}{2} \sum_{x \sim_n y} \frac{1}{4} (f(x) - f(y))(g(x) - g(y)).$$

By equation (4.1) and Lemma 4.1 we get

$$\begin{aligned} |\mathcal{E}_{\text{SG}_n}(P^t(x, \cdot), P^t(x, \cdot))| &= \left| \sum_y P^t(x, y) \Delta_{\text{SG}_n} P^t(x, y) \right| \\ &\leq \frac{2C}{t^{\alpha/\beta}} \sum_y |P^t(x, y) - P^{t+1}(x, y)| \leq \frac{2C}{t^{\alpha/\beta+1/2}}. \end{aligned}$$

Because $|B^n(x, r)| \leq cr^\alpha$,

$$\sum_{y \in B^n(x, t^{1/\beta+\delta})} \sum_{y' \sim y} |P^t(x, y) - P^t(x, y')| \leq \frac{C''}{t^{-\alpha/2\beta+1/4+\alpha/2\beta-\delta\alpha/2}} = \frac{C''}{t^{-\delta\alpha/2+1/4}}.$$

For the remaining terms we have

$$\begin{aligned} &\sum_{y \notin B^n(x, t^{1/\beta+\delta})} \sum_{y' \sim y} |P^t(x, y) - P^t(x, y')| \\ &\leq C''' \left| B^n(x, t) \setminus B^n(x, t^{1/\beta+\delta}) \right| \frac{1}{t^{\alpha/\beta}} \exp(-c(t^{\delta\beta})^{1/(\beta-1)}). \end{aligned}$$

Since $|B^n(x, t) \setminus B^n(x, t^{1/\beta+\delta})|$ grows polynomially in t , the dominating term is the exponential part in t , which goes to 0 for t large, and this completes the proof. ■

4.1. Convergence of odometers in rotor aggregation

Consider $\sigma: \text{SG} \rightarrow [0, \infty)$ and $\sigma_n: \text{SG}_n \rightarrow \mathbb{N}$ that satisfy equations (3.1), (3.2) and (3.4). As a replacement for (3.3), in order to define a suitable convergence of σ_n to σ , suppose there exist integers $\kappa(n) \rightarrow \infty$ with $\kappa(n) \cdot \delta_n \rightarrow 0$ such that

$$(S_{\kappa(n)}^n \sigma_n)^{\Delta\Delta}(x) \rightarrow \sigma(x), \quad (4.2)$$

for all $x \notin DC(\sigma)$. Since we write $\kappa(n)$ for the smoothing steps on SG_n we drop the superscript and write only $S_{\kappa(n)} = S_{\kappa(n)}^n$. Similar to [22, Section 4], we define *the odometer function for the rotor aggregation on SG_n starting from density σ_n* as the function $u_n: \text{SG}_n \rightarrow \mathbb{N}_0$ with

$$u_n(x) = \delta_n^\beta \cdot \#(\text{particles emitted from } x), \quad (4.3)$$

if $\sigma_n(y)$ particles start at each site y , and we remind that particles perform rotor walks until each of them has found a previously unvisited site where it stops.

Theorem 4.1. *Let u_n be the odometer function for rotor aggregation on SG_n with initial density σ_n as defined in (4.3). If σ, σ_n satisfy (3.1), (3.2), (3.4) and (4.2), then $u_n \rightarrow u$ uniformly as $n \rightarrow \infty$, where u is the odometer for the divisible sandpile on SG starting with σ .*

For a function $f: \text{SG}_n \rightarrow \mathbb{R}$ and a directed edge (x, y) we write $\nabla f(x, y) = \delta_n^{-\beta/2}(f(y) - f(x))$ and for a function ϑ defined on directed edges in SG_n we write

$$\text{div} \vartheta(x) = \frac{1}{4\delta_n^{\beta/2}} \sum_{y \sim x} \vartheta(x, y).$$

With this in mind, the rescaled discrete Laplacian on SG_n is then given by

$$\Delta_n f(x) = \text{div} \nabla f(x).$$

Lemma 4.2. *Denote by $\vartheta(x, y)$ the net number of crossings from x to y performed by particles during a sequence of rotor moves, where (x, y) is some directed edge in SG_n , and let u be the odometer function for this sequence of rotor moves. Then*

$$\nabla u(x, y) = \delta_n^{\beta/2}(-4\vartheta(x, y) + \rho(x, y)),$$

for some function ρ on directed edges of SG_n satisfying $|\rho(x, y)| \leq 6$ for all directed edges.

Proof. If $N(x, y)$ is the number of particles routed over the edge (x, y) in direction y , we have for any $y, z \sim x$

$$\vartheta(x, y) = N(x, y) - N(y, x) \quad \text{and} \quad |N(x, y) - N(x, z)| \leq 1.$$

By definition $u(x) = \delta_n^\beta \sum_{y \sim x} N(x, y)$ and we obtain

$$\delta_n^{-\beta} u(x) - 3 \leq 4N(x, y) \leq \delta_n^{-\beta} u(x) + 3.$$

Finally, this yields

$$\begin{aligned} |\nabla u(x, y) + 4\delta_n^{\beta/2}\vartheta(x, y)| &= \delta_n^{\beta/2} |\delta_n^{-\beta}(u(y) - u(x)) + 4N(x, y) - 4N(y, x)| \\ &\leq 6\delta_n^{\beta/2}. \end{aligned} \quad \blacksquare$$

Let ϑ as in Lemma 4.2 and denote by \mathcal{R}_n the rotor cluster on SG_n with initial particle density σ_n . Clearly, since $\sigma_n(x)$ particles start and $\mathbb{1}_{\mathcal{R}_n}(x)$ particles end up in $x \in \text{SG}_n$, we have

$$4\delta_n^{\beta/2} \text{div} \vartheta = \sigma_n - \mathbb{1}_{\mathcal{R}_n}.$$

Using Lemma 4.2 and taking the divergence of ∇u_n gives

$$\Delta_n u_n = \mathbf{1}_{\mathcal{R}_n} - \sigma_n + \delta_n^{\beta/2} \operatorname{div} \rho.$$

Having in mind that $0 \leq \sigma_n \leq M$ and $|\rho| \leq 6$, we obtain

$$|\Delta_n u_n| \leq M + 6 \quad \text{on } \text{SG}_n. \quad (4.4)$$

The next lemma gives an estimate on the Laplacian of the smoothed odometer $S_k u_n$, and it can be seen as the smoothed counterpart of the sandpile odometer having Laplacian $1 - \sigma_n$.

Lemma 4.3. *For the lazy random walk $(Y_t^n)_{t \in \mathbb{N}}$ on SG_n , there is a constant $C_0 > 0$ such that for every $n \in \mathbb{N}$ for any $0 < \delta < 1/2\alpha$ the following holds:*

$$|\Delta_n S_k u_n(x) - \mathbb{P}_x[Y_k^n \in \mathcal{R}_n] + S_k \sigma_n(x)| \leq \frac{C_0}{k^{1/4 - \delta\alpha/2}}.$$

Proof. Let ϑ and ρ as in Lemma 4.2. Since S_k and Δ_n commute:

$$\Delta_n S_k u_n(x) = \mathbb{P}_x[Y_k \in \mathcal{R}_n] - S_k \sigma_n(x) + \delta_n^{\beta/2} S_k \operatorname{div} \rho(x).$$

Because ϑ and ∇u are antisymmetric, so is ρ by Lemma 4.2. For (Y_t^n) on SG_n , we have

$$\begin{aligned} \delta_n^{\beta/2} S_k \operatorname{div} \rho(x) &= \frac{1}{4} \sum_{y \in \text{SG}_n} \sum_{z \sim y} \mathbb{P}_x[Y_k^n = y] \rho(y, z) \\ &= -\frac{1}{4} \sum_{y \in \text{SG}_n} \sum_{z \sim y} \mathbb{P}_x[Y_k^n = z] \rho(y, z) \\ &= \frac{1}{8} \sum_{y \in \text{SG}_n} \sum_{z \sim y} (\mathbb{P}_x[Y_k^n = y] - \mathbb{P}_x[Y_k^n = z]) \rho(y, z). \end{aligned}$$

Taking the absolute value and using the triangle inequality as well as the fact that ρ is bounded, together with the bound in Proposition 4.1, the claim follows. \blacksquare

Lemma 4.4. *For $0 < \delta < 1/2\alpha$ and the odometer function u_n on SG_n as in (4.3), we have*

$$|S_k u_n - u_n| \leq \delta_n^\beta \left(\frac{1}{2} (M + 6)k + C_0 k^{3/4 + \delta\alpha/2} \right).$$

Proof. It holds that

$$|S_k u_n - u_n| \leq \sum_{j=0}^{k-1} |S_{j+1} u_n - S_j u_n| = \frac{\delta_n^\beta}{2} \sum_{j=0}^{k-1} |\Delta_n S_j u_n|.$$

Lemma 4.3 for $0 < \delta < 1/2\alpha$ and $j \geq 1$ gives

$$|\Delta_n S_j u_n| \leq M + 1 + \frac{C_0}{j^{1/4-\delta\alpha/2}},$$

and for $j = 0$ we have $|\Delta_n S_j u_n| = |\Delta_n u_n| \leq M + 6$. Summing over j yields the result. \blacksquare

Now let $\text{Tr} = B(0, 2^L) \subset \text{SG}$ be the double triangle from Lemma 3.1 that contains the final domains of occupied sites for the divisible sandpiles started from $S_{\kappa(n)}\sigma_n$ on SG_n . In the following, when referring to the Green function, we mean the Green function stopped when exiting Tr° . Consider the following obstacle $\gamma_n: \text{SG}_n \rightarrow \mathbb{R}$ in the rotor aggregation model, similar to (3.6) for the divisible sandpile model

$$\gamma_n(x) = -\mathbf{g}^n(S_{\kappa(n)}\sigma_n(x) - 1)$$

and let s_n be the least superharmonic majorant of γ_n . Then the difference $s_n - \gamma_n$ is the odometer function for the divisible sandpile started from initial density $S_{\kappa(n)}\sigma_n$ on SG_n . In what follows we compare the smoothed odometer for rotor aggregation started from σ_n with the odometer for the divisible sandpile started from initial configuration $S_{\kappa(n)}\sigma_n$.

Lemma 4.5. *Let $\text{Tr} = B(0, 2^L) \subset \text{SG}$ be a double triangle centered at the origin that contains the divisible sandpile clusters started from initial density $S_{\kappa(n)}\sigma_n$ on SG_n . Then there exists a constant m_{Tr} depending on Tr (or L resp.) such that for any $n \in \mathbb{N}$*

$$S_{\kappa(n)}u_n \geq s_n - \gamma_n - C_0 m_{\text{Tr}} \kappa(n)^{-1/4+\delta\alpha/2} \quad \text{on } \text{Tr}^\circ, \quad \text{for } 0 < \delta < 1/2\alpha.$$

Proof. Let $\Phi_{\text{Tr}}: \text{Tr} \rightarrow [0, \infty)$ be the unique function with $\Delta_{\text{Tr}}\Phi_{\text{Tr}} = 3/4$ and $\Phi_{\text{Tr}}(0) = 0$. Since Φ is subharmonic, it attains its maximum in ∂Tr . One can construct Φ_{Tr} as in Appendix A by inductively choosing values such that $\Delta_n \Phi_{\text{Tr}}^\circ = 1$ for all $n \in \mathbb{N}$. We set

$$m_{\text{Tr}} := \sup_{x \in \text{Tr}} \Phi_{\text{Tr}}(x) = 5^L,$$

in view of Appendix A. On Tr° we have $\Delta_n \gamma_n = -1 + S_{\kappa(n)}\sigma_n$, so by Lemma 4.3

$$f(x) = S_{\kappa(n)}u_n(x) + \gamma_n(x) + C_0 \kappa(n)^{-1/4+\delta\alpha/2} (m_{\text{Tr}} - \Phi_{\text{Tr}}(x)), \quad x \in \text{Tr}^\circ$$

is superharmonic. Since $f \geq \gamma_n$, we have $f \geq s_n$ on Tr° and the claim holds on Tr° . \blacksquare

Lemma 4.6. *For $n \in \mathbb{N}$, let $\mathcal{R}_n \subset \text{SG}_n$ be the rotor cluster in SG_n with initial density $\sigma_n: \text{SG}_n \rightarrow \mathbb{N}$. Then there exists a double triangle $\text{Tr} = B(0, 2^L) \subseteq \text{SG}$ such that $\bigcup_{n \geq 0} \mathcal{R}_n \subseteq \text{Tr}$.*

Proof. By assumption (3.4) there is a double triangle $B = B(0, 2^l) \subseteq \text{SG}$ containing the supports of all σ_n for all n . Let \mathcal{A}_n be the rotor cluster in SG_n resulting from initial density $\tau(x) = M \mathbb{1}_{B^\cdot}(x)$, where M is the upper bound of σ, σ_n from equation (3.1). By the Abelian property of the rotor model $\mathcal{R}_n \subseteq \mathcal{A}_n$. By the inner bound [6, Theorem 1], if we start $\lfloor 5\delta_n^{-\alpha} \mu(B) \rfloor \geq |B^\cdot|$ particles at the origin in SG_n , then the resulting cluster contains B^\cdot . So once again, in view of the Abelian property, if one starts $M \lfloor 5\delta_n^{-\alpha} \mu(B \cap \text{SG}_n) \rfloor$ particles at the origin, the resulting set Tr_n of fully occupied sites contains \mathcal{A}_n . Together with the outer bound [6, Theorem 1], Tr_n is contained in a double triangle in SG of volume $6M\mu(B)$. Now choose L large enough to deduce the result. The last factor 6 comes from the fact that the outer bound is slightly bigger than the initial radius (radius + 1). ■

For $k \geq 0$, define

$$\mathcal{R}_n^k = \{x \in \mathcal{R}_n \mid B^n(x, k) \subseteq \mathcal{R}_n\}.$$

Similar to [22, Lemma 2.7], we investigate the growth of the odometer function in rotor aggregation near the boundary of the rotor cluster \mathcal{R}_n .

Lemma 4.7. *Let $k \geq 0$. There exists a constant $c > 0$ such that for all $x \in \text{SG}_n \setminus \mathcal{R}_n^k$*

$$u_n(x) \leq c(M + 6)k^\beta \delta_n^\beta.$$

Proof. Let $x \notin \mathcal{R}_n^k$. The odometer u_n defined in (4.3) has Laplacian bounded by $M + 6$ due to (4.4) and by the definition of \mathcal{R}_n^k we know that there exists $y \in B^n(x, k)$ with $u_n(y) = 0$. Define the functions $f_\pm: \text{SG}_n \rightarrow \mathbb{R}$ as

$$f_\pm(z) = u_n(z) \pm (M + 6)\delta_n^\beta \varphi_{y, 2k}(z),$$

where $\varphi_{y, 2k}(z)$ is as in (3.12). It is easy to check that $\Delta_{\text{SG}_n} \varphi_{y, 2k} = 1$ and $\Delta_n \varphi_{y, 2k} = \delta_n^{-\beta}$ on $B^n(y, 2k)$. Therefore, $f_+ \geq u_n$ is subharmonic, while $f_- \leq u_n$ is superharmonic. Let $B^n = B^n(y, 2k)$ and h_\pm be functions defined on B^n agreeing with f_\pm on ∂B^n and harmonic on $B^n \setminus \partial B^n$. Then by construction and (E $_\beta$) it holds that

$$\begin{aligned} f_+ &\leq h_+ \leq h_- + 2(M + 6)\delta_n^\beta \max_{z \in \partial B^n} \varphi_{y, 2k}(z) \\ &\leq h_- + 2(M + 6)\delta_n^\beta c(2k)^\beta \leq f_- + 2(M + 6)\delta_n^\beta c(2k)^\beta, \end{aligned}$$

which implies, for $c' = c2^{\beta+1}$

$$h_+(y) \leq f_-(y) + c'(M + 6)\delta_n^\beta k^\beta = c'(M + 6)\delta_n^\beta k^\beta.$$

The Elliptic Harnack inequality (EHI) implies the existence of $c'' > 0$ such that $h_+(z) \leq c''h_+(y)$ for all $z \in B^n(y, k)$:

$$f(z) \leq f_+(z) \leq h_+(z) \leq c''h_+(y) \leq c''c'(M + 6)\delta_n^\beta k^\beta. \quad \blacksquare$$

Proof of Theorem 4.1. By Lemma 4.6, there exists a double triangle $\text{Tr} \subset \text{SG}$ containing the cluster of occupied sites for the rotor aggregation started from σ_n and the divisible sandpile cluster started from initial density $S_{\kappa(n)}\sigma_n$ for all n . We enlarge Tr to contain the support of $S_{\kappa(n)}u_n$ for all n , and we define the function

$$\psi(x) = S_{\kappa(n)}u_n(x) - s_n(x) + \gamma_n(x) + C_0\kappa(n)^{-1/4+\delta\alpha/2}\xi_n(x),$$

which in view of Lemma 4.3 is subharmonic for $0 < \delta < 1/2\alpha$ on the set $\mathcal{R}_n^{\kappa(n)}$. Above ξ_n is the function with constant Laplacian 1 on Tr ; see Appendix A for details on how to calculate such a function. For $y \notin \mathcal{R}_n^{\kappa(n)}$, by Lemma 4.7 we have $u_n(y) \leq c(M+6)\kappa(n)^\beta \delta_n^\beta$, which together with Lemma 4.4 gives

$$S_{\kappa(n)}u_n(y) \leq \delta_n^\beta (c(M+6)\kappa(n)^\beta + (M+6)\kappa(n)/2 + C_0\kappa(n)^{3/4+\delta\alpha/2}).$$

The RHS above is at most $C_1\kappa(n)^\beta \delta_n^\beta$ for some $C_1 > 0$. Since $s_n \geq \gamma_n$, we have

$$\psi(y) \leq C_1\kappa(n)^\beta \delta_n^\beta + C_0\kappa(n)^{-1/4+\delta\alpha/2}r \quad \text{for all } y \in \text{Tr} \setminus \mathcal{R}_n^{\kappa(n)},$$

where r is a uniform bound for all ξ_n on Tr , $n \in \mathbb{N}$. By the maximum principle, this bound holds on Tr . Using Lemma 4.5 we finally get

$$-C_0r\kappa(n)^{-1/4+\delta\alpha/2} \leq S_{\kappa(n)}u_n - s_n + \gamma_n \leq C_1\kappa(n)^\beta \delta_n^\beta + C_0r\kappa(n)^{-1/4+\delta\alpha/2}$$

on $\text{Tr} \subset \text{SG}$. By Corollary 3.1 and Lemma 3.6, γ_n and s_n converge uniformly to γ and s on Tr as $n \rightarrow \infty$. Since $\kappa(n) \rightarrow \infty$ and $\kappa(n)\delta_n \rightarrow 0$ as $n \rightarrow \infty$, we can conclude that $S_{\kappa(n)}u_n$ converges uniformly to $s - \gamma$ on Tr . Because both $S_{\kappa(n)}u_n$ and the odometer function for the divisible sandpile vanish outside Tr , this implies that $S_{\kappa(n)}u_n$ has the same limit as the odometer functions of the divisible sandpile on SG_n . Lemma 4.4 implies then that the odometer functions u_n for rotor aggregation as defined in (4.3) converge uniformly to the same odometer u of the divisible sandpile on SG with initial mass σ . \blacksquare

Once we have shown the uniform convergence of the odometer functions for rotor aggregation on SG_n to the odometer function u of the divisible sandpile on SG , it remains to show that the same holds for the corresponding domains of occupied sites by particles doing rotor walks.

4.2. Convergence of domains in rotor aggregation

In addition to the assumptions from Section 4.1, we require for $\sigma: \text{SG} \rightarrow \mathbb{R}$ that for all $x \in \text{SG}$ either $\sigma(x) \geq 1$ or $\sigma(x) = 0$ and

$$\{\sigma \geq 1\} = \overline{\{\sigma \geq 1\}^\circ}. \quad (4.5)$$

Moreover, we assume that for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$\text{if } x \in \{\sigma \geq 1\}_\varepsilon, \text{ then } \sigma_n(x) \geq 1 \text{ for all } n \geq N(\varepsilon), \quad (4.6)$$

and

$$\text{if } x \notin \{\sigma \geq 1\}^\varepsilon, \text{ then } \sigma_n(x) = 0 \text{ for all } n \geq N(\varepsilon). \quad (4.7)$$

Theorem 4.2. *Let σ and σ_n fulfill (4.2) and (4.5)–(4.7) as well as (3.1), (3.2), and (3.4). For $n \geq 1$, let \mathcal{R}_n be the cluster of occupied sites by rotor particles started from initial particle configuration $\sigma_n: \text{SG}_n \rightarrow \mathbb{N}$. Then for any $\varepsilon > 0$ and n large enough*

$$\tilde{D}_\varepsilon \cap \text{SG}_n \subseteq \mathcal{R}_n \subseteq \tilde{D}^\varepsilon,$$

where D is the noncoincidence set started from $\sigma: \text{SG} \rightarrow \mathbb{R}$ and $\tilde{D} = D \cup \{\sigma \geq 1\}^\circ$.

For the rest of this section, we fix again a ball $\text{Tr} = B(0, 2^l) \subset \text{SG}$ of radius 2^l big enough, which contains all rotor clusters $\mathcal{R}_n \subset \text{SG}_n$ started with particle configurations $\sigma_n: \text{SG}_n \rightarrow \mathbb{N}$, for all n , as well as all the divisible sandpile clusters $\mathcal{D}_n \subset \text{SG}_n$ started from initial density $S_{\kappa(n)}\sigma_n$ and its limit shape. Then on $\text{Tr} \subset \text{SG}$ we can write

$$\tilde{D} = \{s > \gamma\} \cup \{\sigma \geq 1\}^\circ,$$

where $\gamma: \text{SG} \rightarrow \mathbb{R}$ is the obstacle associated with σ on Tr as in Definition 3.1 and s its superharmonic majorant. We also assume that Tr is chosen large enough such that it contains any of the inflations of sets used in the upcoming proofs.

Lemma 4.8. *Take $x \notin \tilde{D}^{\varepsilon/2}$, and fix $\varepsilon > 0$ and $n \geq N(\varepsilon/4)$. Denote by $N_r(x)$ the number of sites in \mathcal{R}_n which are contained in the ball $B^n(x, r)$ of radius r around x in SG_n for $1 \leq r \leq \varepsilon/(4\delta_n)$. If $m \geq \delta_n^{-\beta} u_n(x)$ for every $x \in B^n(x, r)$, then the following holds:*

$$N_r(x) \geq (1 + 1/m)N_{r-1}(x).$$

Proof. It is first easy to deduce $\{\sigma \geq 1\}^{\varepsilon/4} \subseteq \tilde{D}^{\varepsilon/4}$ from $\{\sigma \geq 1\} = \overline{\{\sigma \geq 1\}^\circ}$. Also, $B^n(x, r) \cap \tilde{D}^{\varepsilon/4} = \emptyset$, which results in $\sigma_n(y) = 0$ for all $y \in B^n(x, r)$. By the definition of $N_r(x)$, at least $N_{r-1}(x)$ particles have to enter $B^n(x, r-1)$, which implies that for every $n \in \mathbb{N}$:

$$\sum_{y \in \partial B^n(x, r)} u_n(y) \geq \delta_n^\beta N_{r-1}(x).$$

There are at most $N_r(x) - N_{r-1}(x)$ nonzero terms in the LHS of the sum above, each of them being not bigger than $\delta_n^\beta m$, thus

$$m(N_r(x) - N_{r-1}(x)) \geq N_{r-1}(x). \quad \blacksquare$$

Lemma 4.9. *Let $\varepsilon > 0$, then for n large enough and any $x \in \mathcal{R}_n$ with $x \notin \tilde{D}^{\varepsilon/2}$ there exists $y \in B^n(x, \varepsilon/(4\delta_n)) \cap \mathcal{R}_n$ such that*

$$u_n(y) \geq \frac{\varepsilon \delta_n^{\beta-1}}{16 \log(C(\frac{\varepsilon}{4\delta_n})^\alpha)}.$$

Proof. Let $n \geq N(\varepsilon/4)$ be large enough and $B^n = B^n(x, \varepsilon/(4\delta_n))$ giving $\sigma_n(z) = 0$ for all $z \in B^n$. Set $m = \delta_n^{-\beta} \max_{z \in B^n} u_n(z)$ and notice that $N_0(z) = \mathbb{1}_{\mathcal{R}_n}(z)$ for all $z \in \text{SG}_n$. Due to Lemma 4.8, we can bound

$$N_{\varepsilon/(4\delta_n)}(x) \geq (1 + 1/m)^{\lfloor \varepsilon/4\delta_n \rfloor} N_0(x) \geq \exp\left(\frac{\varepsilon}{16m\delta_n}\right).$$

Finally, by (V_α) we have $N_{\varepsilon/(4\delta_n)}(x) \leq |B^n| \leq C(\varepsilon/(4\delta_n))^\alpha$, which yields $\varepsilon/(16m\delta_n) \leq \log(C(\varepsilon/(4\delta_n))^\alpha)$. ■

The next result tells us that, whenever $x \in \mathcal{R}_n$ but $x \notin \tilde{D}^\varepsilon$ with large $S_k u_n(x)$ there has to be a y close to x with even larger $S_k u_n(y)$.

Lemma 4.10. *Let C_0 be the constant from Lemma 4.3 and $0 < \delta < 1/2\alpha$ and let $\varepsilon > 0$ and k large enough such that $C_0 k^{-1/4+\delta\alpha/2} < 1/2$. There exist $c, C_1 > 0$ such that for n large enough and any $x \in \mathcal{R}_n \setminus \tilde{D}^{3\varepsilon/4}$ with $S_k u_n(x) > C_1 k^\beta \delta_n^\beta$, there exists $y \in B^n(x, \varepsilon/(2\delta_n) + 1)$ with*

$$S_k u_n(y) \geq S_k u_n(x) + c\varepsilon^\beta.$$

Proof. Lemma 4.4 gives

$$u_n(x) \geq S_k u_n(x) - \delta_n^\beta (k(M+6)/2 + C_0 k^{3/4+\delta\alpha/2}) > c(M+6)k^\beta \delta_n^\beta$$

for the choice $C_1 = c(M+6) + (M+6)/2 + C_0$. Hence, $x \in \mathcal{R}_n^k$ by Lemma 4.7, and together with Lemma 4.3, we obtain for any $y \in \mathcal{R}_n^k$

$$\Delta_n S_k u_n(y) \geq 1 - S_k \sigma_n(y) - C_0 k^{-1/4+\delta\alpha/2}.$$

By our choice of k , we have $C_0 k^{-1/4+\delta\alpha/2} < 1/2$. Taking n large enough such that $k\delta_n < \varepsilon/8$ and $n > N(\varepsilon/8)$, $S_k \sigma_n$ vanishes on $B^n(x, \varepsilon/(2\delta_n))$ and therefore

$$f(z) = S_k u_n(z) + \frac{\delta_n^\beta}{2} \mathbb{E}_z[\tau_{\varepsilon/(2\delta_n)}^n(x)]$$

is subharmonic on $\mathcal{R}_n^k \cap B^n(x, \varepsilon/(2\delta_n))$ and attains its maximum on the boundary. For $z \in \partial\mathcal{R}_n^k$, Lemma 4.7 gives $u_n(z) \leq c(M+6)k^\beta \delta_n^\beta$. But $u_n(x) > c(M+6)k^\beta \delta_n^\beta$ so the maximum cannot be attained on $\partial\mathcal{R}_n^k$ and instead the maximum has to be attained in some $y \in \partial B^n(x, \varepsilon/(2\delta_n))$. By (E_β) it holds $\mathbb{E}_y[\tau_{\varepsilon/(2\delta_n)}^n(x)] \leq c(\varepsilon/\delta_n)^\beta$, which implies the desired inequality. ■

Now we can finally prove the scaling limit for the rotor aggregation cluster \mathcal{R}_n .

Proof of Theorem 4.2. Take $\varepsilon > 0$ and recall that $u = s - \gamma$, $D = \{u > 0\}$ and finally $\tilde{D} = D \cup \{\sigma \geq 1\}^\circ$. Choose $\eta > 0$ such that $\tilde{D}_\varepsilon \subseteq D_\eta \cup \{\sigma \geq 1\}_\eta$ and $m_\eta > 0$ such that $u \geq m_\eta$ on D_η , which is possible since $\overline{D}_\eta \subsetneq D$. By Theorem 4.1, we get for n large enough

$$u_n(x) > u(x) - m_\eta/2 > 0 \text{ for all } x \in D_\eta \cap \text{SG}_n.$$

On the other hand, $\{\sigma \geq 1\}_\eta \cap \text{SG}_n \subseteq \mathcal{R}_n$ for large n by (4.6), which concludes the lower bound. For the upper bound, we fix $x \in \text{SG}_n$ with $x \notin \tilde{D}^\varepsilon$. As a consequence, $u = 0$ on $B(x, \varepsilon) \subset \text{SG}$. Again, by Theorem 4.1 it holds for large enough n

$$\overline{u}_n(z) < c\varepsilon^\beta/2 \text{ for all } z \in B(x, \varepsilon) \cap \text{SG}_n,$$

where c is the constant from Lemma 4.10. Let $k \geq (2C_0)^{1/(1/4-\delta\alpha/2)}$ as in Lemma 4.10, where C_0 is the constant from Lemma 4.3. Taking n large enough such that $|S_k u_n - u_n| \leq c\varepsilon^\beta/2$, which is possible in view of Lemma 4.4, yields $S_k u_n < c\varepsilon^\beta$ on $B^n(x, \varepsilon/\delta_n) \cap \text{SG}_n$. Lemma 4.10 shows that $S_k u_n \leq C_1 k^\beta \delta_n^\beta$ on $B^n(x, \varepsilon/(4\delta_n))$ or $x \notin \mathcal{R}_n$. In the former case for some constant $C_2 > 0$

$$u_n(y) \leq \frac{S_k u_n(y)}{\mathbb{P}_y[Y_k^n = y]} < C_2 k^{\beta+\alpha/\beta} \delta_n^\beta \text{ for all } y \in B^n(x, \varepsilon/(4\delta_n)),$$

where the first inequality holds in view of the definition of S_k and the last inequality is due to Lemma 4.1(3). The RHS decreases faster than $\varepsilon \delta_n^{\beta-1}/(16 \log(C\varepsilon^\alpha/(4\delta_n)^\alpha))$, so by Lemma 4.9, $x \notin \mathcal{R}_n$ and therefore $\mathcal{R}_n \subset \tilde{D}^\varepsilon$. ■

5. Scaling limit for internal DLA

We investigate here internal DLA. While the proofs follow mostly the lines of [22] and [5] and are based on the ideas introduced in [18], we include some details here since we also have to take care of the fractal structure of the Sierpiński gasket. We follow the usual approach, and we split the proof into two parts. For the *inner estimate*, we show that for any $\varepsilon > 0$ the internal DLA cluster \mathcal{I}_n will eventually contain the ε -deflation of the set $D = \{u > 0\}$, where u is the limit of the odometer for divisible sandpiles started from our smoothed densities. For the *outer estimate* we show that the ε -inflation of D will eventually contain the internal DLA cluster. For the outer estimate it is important that the shells $D \setminus D_\varepsilon$ do not grow too fast, which is the case in SG. We prove the following.

Theorem 5.1. *Let σ and σ_n initial configurations on SG and SG_n , respectively that fulfill (4.2), and (4.5)–(4.7) as well as (3.1), (3.2) and (3.4). Then, for every $\varepsilon > 0$ with probability 1 we have*

$$\tilde{D}_\varepsilon \cap SG_n \subseteq \mathcal{I}_n \subseteq \tilde{D}^\varepsilon$$

for n large enough, where \mathcal{I}_n is the internal DLA cluster started from configuration σ_n on SG_n and $\tilde{D} = \{s > \gamma\} \cup \{\sigma \geq 1\}^\circ$.

Inner estimate. Since we are using the standard approach from [18] and [22], we will not motivate the random variables below. We fix $n \in \mathbb{N}$ and the initial configuration σ_n on SG_n . We label the particles that generate \mathcal{I}_n by $1, \dots, m_n$, with $m_n = \sum_{x \in SG_n} \sigma_n(x)$ being the total number of particles. Denote by x_i the starting position of particle i such that $\#\{i \mid x_i = x\} = \sigma_n(x)$, and for $i = 1, \dots, m_n$ let $(X_t^n(i))_{t \in \mathbb{N}}$ be a sequence of m_n simple independent random walks in SG_n starting in x_i . For $z \in SG_n$ and $\varepsilon > 0$ consider the stopping times

$$\begin{aligned} \tau_z^i &:= \inf\{t \geq 0 \mid X_t^n(i) = z\}, \\ \tau_\varepsilon^i &:= \inf\{t \geq 0 \mid X_t^n(i) \notin D_\varepsilon \cap SG_n\}, \\ \nu^i &:= \inf\{t \geq 0 \mid X_t^n(i) \notin \{X_{\nu^j}^n(j)\}_{j=1}^{i-1}\}, \end{aligned}$$

and we also set $\nu^1 := 0$, and we remark that all three above stopping times depend on the iteration n , but we omit n and write only τ_z^i (resp. τ_ε^i and ν^i) instead of $\tau_z^i(n)$ (resp. $\tau_\varepsilon^i(n)$ and $\nu^i(n)$). If \mathcal{I}_n^j denotes the cluster after j particles settled, then $\mathcal{I}_n^j = \mathcal{I}_n^{j-1} \cup \{X_{\nu^j}^n(j)\}$. Fix $z \in D_\varepsilon \cap SG_n$ and consider the random variables

$$\mathcal{M}_\varepsilon := \sum_{i=1}^{m_n} \mathbb{1}_{\{\tau_z^i < \tau_\varepsilon^i\}} \quad \text{and} \quad L_\varepsilon := \sum_{i=1}^{m_n} \mathbb{1}_{\{\nu^i \leq \tau_z^i < \tau_\varepsilon^i\}}.$$

If $L_\varepsilon < \mathcal{M}_\varepsilon$, then $z \in \mathcal{I}_n$. L_ε being a sum of non-independent random variables, we use another random variable to upper bound it. For each $y \in D_\varepsilon \cap SG_n$, let $(Y_t^n(y))_{t \in \mathbb{N}}$ be a simple random walk on SG_n starting at y , with $Y_t^n(x)$ and $Y_t^n(y)$ independent if $x \neq y$, and

$$\tilde{L}_\varepsilon := \sum_{y \in D_\varepsilon \cap SG_n} \mathbb{1}_{\{\tau_z^y < \tau_\varepsilon^y\}},$$

where $\tau_z^y = \inf\{t \geq 0 \mid Y_t^n(y) = z\}$ and $\tau_\varepsilon^y = \inf\{t \geq 0 \mid Y_t^n(y) \notin D_\varepsilon \cap SG_n\}$. Then indeed $L_\varepsilon \leq \tilde{L}_\varepsilon$, and \tilde{L}_ε is a sum of independent indicators. Define now

$$g^{n,\varepsilon}(y, z) := \mathbb{E}[\#\{t < \tau_\varepsilon^y \mid Y_t^n(y) = z\}] \quad \text{and} \quad f^{n,\varepsilon}(z) := g^{n,\varepsilon}(z, z) \mathbb{E}[\mathcal{M}_\varepsilon - \tilde{L}_\varepsilon].$$

Be aware that $g^{n,\varepsilon}$ is the unscaled version of the Green function defined in (2.2), which is handier for internal DLA. By definition, $f^{n,\varepsilon}$ solves the Dirichlet problem

$$\begin{aligned}\Delta_n f^{n,\varepsilon} &= \delta_n^{-\beta} (1 - \sigma_n), \text{ on } D_\varepsilon \cap \text{SG}_n, \\ f^{n,\varepsilon} &= 0, \text{ on } D_\varepsilon^C \cap \text{SG}_n,\end{aligned}$$

and note that the unscaled divisible sandpile odometer $\delta_n^{-\beta} u_n$ solves the same Dirichlet problem on the set $\mathcal{D}_n = \{u_n > 0\}$.

Lemma 5.1. *Suppose σ_n is bounded by M and has finite support. Let u_n, \tilde{u}_n be the odometer functions of the divisible sandpile with initial densities $\sigma_n, S_k \sigma_n$, respectively. Then*

$$|u_n - \tilde{u}_n| \leq \frac{1}{2} k M \delta_n^\beta \text{ for all } k \in \mathbb{N}.$$

Proof. In view of $S_1 f = f + (\delta_n^\beta/2)\Delta_n f$ and $S_{i+l} = S_i S_l$ we obtain by induction over k

$$S_k \sigma_n = \sigma_n + \Delta_n w_k, \quad \text{with} \quad w_k = \frac{\delta_n^\beta}{2} \sum_{j=0}^{k-1} S_j \sigma_n.$$

We remark that w_k is non-negative and bounded from above by $(1/2)Mk\delta_n^\beta$ due to the boundedness of σ_n and thus also $S_j \sigma_n$ for all $j \in \mathbb{N}$. Let us now denote by v_n (resp. \tilde{v}_n) the limit mass configuration on SG_n for the divisible sandpile started from σ_n (resp. $S_k \sigma_n$). The sum $w_k + \tilde{u}_n$, where \tilde{u}_n is the odometer for divisible sandpile started from $S_k \sigma_n$, solves the inequality

$$\sigma_n + \Delta_n v \leq 1 \text{ and } v \geq 0 \tag{5.1}$$

in view of

$$\tilde{v}_n = S_k \sigma_n + \Delta_n \tilde{u}_n = \sigma_n + \Delta_n (\tilde{u}_n + w_k),$$

and the fact that $\tilde{v}_n \leq 1$. Thus, by the least action principle as introduced in Section 3 (which states that the odometer function is the smallest solution to (5.1)) we obtain $u_n \leq \tilde{u}_n + w_k$. In a similar manner we can see

$$v_n = \sigma_n + \Delta_n u_n = S_k \sigma_n + \Delta_n (u_n - w_k + \frac{1}{2} M k \delta_n^\beta),$$

where again $v_n \leq 1$ and $u_n - w_k + (1/2)Mk\delta_n^\beta \geq 0$, thus again by the least action principle we obtain $\tilde{u}_n \leq u_n - w_k + (1/2)Mk\delta_n^\beta$. ■

Lemma 5.2. *Suppose σ, σ_n fulfill the assumptions in Theorem 5.1 and let u_n be the odometer function for the divisible sandpile on SG_n with initial density σ_n and $\mathcal{D}_n = \{u_n > 0\}$. Then*

$$u_n^{\Delta\Delta} \rightarrow s - \gamma \text{ uniformly as } n \rightarrow \infty.$$

In particular, for all $\varepsilon > 0$ it holds $D_\varepsilon \cap \text{SG}_n \subseteq \mathcal{D}_n$ for n large enough.

Proof. For the odometer function \tilde{u}_n of the divisible sandpile with initial density $S_{\kappa(n)}\sigma_n$ as in (4.2), in view of Theorem 3.7 we have $\tilde{u}_n^{\Delta\Delta} \rightarrow u$ uniformly, as $n \rightarrow \infty$. Together with Lemma 5.1 this gives $u_n^{\Delta\Delta} \rightarrow u$ uniformly as $n \rightarrow \infty$. Finally, let $b > 0$ be the minimum value of u on $\overline{D_\varepsilon}$. For n large enough, we get $u_n > b/2$ on $D_\varepsilon \cap \text{SG}_n$ which yields $D_\varepsilon \cap \text{SG}_n \subseteq \mathcal{D}_n$. ■

Lemma 5.3. *For any $\varepsilon > 0$ let $b = \min\{u(x) \mid x \in \overline{D_\varepsilon}\} > 0$. There exists $0 < \eta < \varepsilon$ such that for all large enough n and all $z \in D_\varepsilon \cap \text{SG}_n$*

$$f^{n,\eta}(z) \geq \frac{1}{2}b\delta_n^{-\beta}.$$

Proof. Let $\eta > 0$ be chosen such that $u < b/6$ outside of D_η , which is possible since u is uniformly continuous on \overline{D} . Then $\eta < \varepsilon$ since $u \geq b$ on $\overline{D_\varepsilon}$ by the definition of b as the minimum of u on $\overline{D_\varepsilon}$. Let u_n be the odometer function for the divisible sandpile on SG_n started from initial density σ_n and let \mathcal{D}_n be the corresponding divisible sandpile cluster. We can now choose $n_0 \in \mathbb{N}$ large enough such that for all $n > n_0$ we have $|u_n - u| < b/6$ on $D_\eta \cap \text{SG}_n$, and this is possible in view of the uniform convergence from Lemma 5.2, hence $u_n \leq b/3$ on $\partial(D_\eta \cap \text{SG}_n)$. Again by Lemma 5.2 we can adjust n_0 such that we also have $D_\eta \cap \text{SG}_n \subseteq \mathcal{D}_n$ for all $n > n_0$. We thus obtain that the function $\delta_n^\beta f^{n,\eta} - u_n$ is harmonic on $D_\eta \cap \text{SG}_n$, and it therefore attains its minimum on the outer boundary. However, the function $f^{n,\eta}$ vanishes outside of $D_\eta \cap \text{SG}_n$ by definition, hence we obtain for any $z \in D_\eta \cap \text{SG}_n$

$$\delta_n^\beta f^{n,\eta}(z) \geq u_n(z) - b/3 \geq u(z) - b/2.$$

By our choice of b , we have $u \geq b$ in $D_\varepsilon \cap \text{SG}_n$, and since $\eta < \varepsilon$ the previous inequality gives $\delta_n^\beta f^{n,\eta} \geq b/2$ on $D_\varepsilon \cap \text{SG}_n$. ■

Let $\tilde{\mathcal{I}}_n := \{X_{v^i}^n(i) \mid v^i < \tilde{\tau}^i\} \subseteq \mathcal{I}_n$ with $\tilde{\tau}^i := \inf\{t \geq 0 \mid X_t^n(i) \notin \tilde{D} \cap \text{SG}_n\}$. The inner estimate follows from the next lemma. Although it suffices to prove the inner estimate for \mathcal{I}_n instead of $\tilde{\mathcal{I}}_n$, we use this stronger estimate in the proof of the outer estimate.

Lemma 5.4. *For any $\varepsilon > 0$, $\mathbb{P}[\tilde{D}_\varepsilon \cap \text{SG}_n \subseteq \tilde{\mathcal{I}}_n \text{ for all but finitely many } n] = 1$.*

Proof. For $z \in \text{SG}_n$, let $E_z(n)$ be the event that z is not contained in the internal DLA cluster on SG_n with initial density σ_n . Using condition (3.10), it suffices to prove that for $\varepsilon' > 0$

$$\sum_{n \geq 1} \sum_{z \in D_{\varepsilon'} \cap \text{SG}_n} \mathbb{P}[E_z(n)] < \infty.$$

Let $b := \inf_{z \in D_{\varepsilon'}} u(z) > 0$ and $0 < \eta < \varepsilon'$ as in Lemma 5.3 such that $f^{n,\eta}(z) \geq b\delta_n^{-\beta}/2$ for $z \in D_{\varepsilon'} \cap \text{SG}_n$ and sufficiently large n . We fix $z \in D_{\varepsilon'} \cap \text{SG}_n$ and choose $R > 0$ such that $D \subseteq B(z, R)$ giving

$$B(z, \varepsilon' - \eta) \subseteq D_\eta \subseteq B(z, R).$$

Notice that R depends on the size of $\text{supp}(\sigma)$ and the supremum over all values of σ . Since $L_\eta \leq \tilde{L}_\eta$ we have

$$\mathbb{P}[E_z(n)] \leq \mathbb{P}[\mathcal{M}_\eta = L_\eta] \leq \mathbb{P}[\mathcal{M}_\eta \leq \tilde{L}_\eta] \leq \mathbb{P}[\mathcal{M}_\eta \leq a] + \mathbb{P}[\tilde{L}_\eta \geq a]$$

for any $a \in \mathbb{R}$. We need an a such that both summands are small. By [18, Lemma 4]

$$\mathbb{P}[\tilde{L}_\eta \geq \mathbb{E}[\tilde{L}_\eta] + \mathbb{E}[\tilde{L}_\eta]^{1/2+\gamma}] < 2 \exp\left(-\frac{1}{4}\mathbb{E}[\tilde{L}_\eta]^{2\gamma}\right), \quad (5.2)$$

$$\mathbb{P}[\mathcal{M}_\eta \leq \mathbb{E}[\mathcal{M}_\eta] - \mathbb{E}[\mathcal{M}_\eta]^{1/2+\gamma}] < 2 \exp\left(-\frac{1}{4}\mathbb{E}[\mathcal{M}_\eta]^{2\gamma}\right) \quad (5.3)$$

for any $0 < \gamma < 1/2$. So we find an appropriate a if the interval

$$I := [\mathbb{E}[\tilde{L}_\eta] + \mathbb{E}[\tilde{L}_\eta]^{1/2+\gamma}, \mathbb{E}[\mathcal{M}_\eta] - \mathbb{E}[\mathcal{M}_\eta]^{1/2+\gamma}] \text{ is non-empty.}$$

For this, note the lower bound

$$\mathbb{E}[\mathcal{M}_\eta] - \mathbb{E}[\tilde{L}_\eta] = \frac{f^{n,\eta}(z)}{g^{n,\eta}(z, z)} \geq \frac{b}{2\delta_n^\beta g^{n,\eta}(z, z)},$$

which immediately gives $\mathbb{E}[\mathcal{M}_\eta] \geq \mathbb{E}[\tilde{L}_\eta]$. So, by choosing $\gamma, \gamma' > 0$ such that

$$\mathbb{E}[\mathcal{M}_\eta]^{1/2+\gamma} := c \left(\frac{b}{R^{\beta-\alpha} \delta_n^\alpha} \right)^{3/4} \leq \frac{b}{4\delta_n^\beta g^{n,\eta}(z, z)},$$

$$\mathbb{E}[\tilde{L}_\eta]^{1/2+\gamma'} := c \left(\frac{b}{R^{\beta-\alpha} \delta_n^\alpha} \right)^{3/4} \leq \frac{b}{4\delta_n^\beta g^{n,\eta}(z, z)},$$

the interval I is non-empty, where the last inequality is due to [5, Lemma 2.10] which states that $g^{n,\eta}(z, z) \leq C(R/\delta_n)^{\beta-\alpha}$. It is not immediately clear that with this choice the condition on $\gamma, \gamma' < 1/2$ is satisfied. Standard theory for Markov chains gives

$$\mathbb{E}[\tilde{L}_\eta] = \sum_{y \in D_\eta \cap \text{SG}_n} \mathbb{P}[\tau_z^y < \tau_\eta^y] = \frac{1}{g^{n,\eta}(z, z)} \sum_{y \in D_\eta \cap \text{SG}_n} g^{n,\eta}(z, y) = \frac{\mathbb{E}[\tau_\eta^z]}{g^{n,\eta}(z, z)}.$$

Applying the lower bounds for expected exit times (E_β) and the upper bound for the stopped Green function [5, Lemma 2.10] yields that $\gamma, \gamma' < 1/2$ since

$$\mathbb{E}[\tilde{L}_\eta] = \frac{\mathbb{E}_z[\tau_{D_\eta \cap \text{SG}_n}]}{g^{n,\eta}(z, z)} \geq C' \left(\frac{\varepsilon' - \eta}{\delta_n} \right)^\beta \frac{1}{(R/\delta_n)^{\beta-\alpha}} = C' \frac{(\varepsilon' - \eta)^\beta}{R^{\beta-\alpha}} \delta_n^{-\alpha}.$$

Ultimately our choice of γ and γ' also gives that the probabilities in (5.2) and (5.3) are exponentially decreasing, because M_η is bounded by $M \sum_{y \in D_\eta \cap \text{SG}_n} \mathbb{P}[\tau_z^y < \tau_\eta^y]$, which gives $\mathbb{E}[M_\eta] \leq MR^\beta / \delta_n^\beta g^{n,\eta}(z, z) \leq MCR^\alpha \delta_n^{-\alpha}$. Therefore, we have

$$\mathbb{E}[\tilde{L}_\eta]^{2\gamma'} = c \left(\frac{b}{R^{\beta-\alpha} \delta_n^\alpha} \right)^{6/4} \mathbb{E}[\tilde{L}_\eta]^{-1} \geq c' M \left(\frac{b}{R^{\beta-\alpha}} \right)^{6/4} R^{-\alpha} \delta_n^{-\alpha/2}.$$

Finally, we obtain

$$\sum_{n \geq 1} \sum_{z \in D_{\varepsilon'} \cap \text{SG}_n} \mathbb{P}[E_z(n)] \leq \sum_{n \geq 1} C'' R^\alpha \delta_n^{-\alpha} \cdot 4 \exp(-c' M b^{3/2} R^{\frac{\alpha-\beta}{2}} \delta_n^{-\alpha/2}) < \infty,$$

which together with the first Borel–Cantelli lemma proves the claim. \blacksquare

Outer estimate. For the outer estimate we rely on [5]. The idea is to argue based on the inner estimate and the fact that the boundary of D is a null set, that not too many particles can leave D . We will then show that the remaining particles will not be able to leave the ε -inflation D^ε of D . For the remainder we use the notation for deflation and inflation for sets $A \subseteq \text{SG}_n$:

$$A_r = \{x \in A \mid B^n(x, r) \subseteq A\} \quad \text{and} \quad A^r = \{x \in \text{SG}_n \mid A \cap B^n(x, r) \neq \emptyset\}.$$

Recall [5, Lemma 3.5, Lemma 3.6]. We first prove the following.

Lemma 5.5. *Let $n \in \mathbb{N}$, then there exist $\rho, \eta \in (0, 1]$ such that for any $D \subseteq \text{SG}_n$ and $r > 0$ the following holds. For all $S \subseteq D^r \setminus D$ with $|S| \leq \rho r^\alpha$ and all $x \in D$ we have*

$$\mathbb{P}_x[\tau_{S \cup D}^n < \tau_{D^r}^n] \geq \eta.$$

Proof. Let us introduce the notation $Y := \partial D^{r/2}$ as the set of all vertices of distance $\lfloor r/2 \rfloor + 1$ from D . By the Markov property it is sufficient to prove the statement for starting points $y \in Y$, since every trajectory of a random walk started inside D hits the set Y . Let $A = B^n(y, r/3) \setminus S$, then, in view of (V_α) with corresponding constant $C \geq 1$, by setting $\rho = 4^{-\alpha}/C \in (0, 1]$, we obtain

$$|A| \geq |B^n(y, r/3)| - |S| \geq \frac{1}{C} \left(\frac{r}{3} \right)^\alpha - \rho r^\alpha \geq \frac{1}{C^2} \left(1 - \left(\frac{3}{4} \right)^\alpha \right) |B^n(y, r/3)|.$$

The choice $\varepsilon = (1/C^2)(1 - (3/4)^\alpha)$ [5, Lemma 3.5] gives the existence of $\eta > 0$ such that $\mathbb{P}_y[\tau_A^n < \tau_{r/3}^n(y)] \geq \eta$. \blacksquare

Next, for ρ as in Lemma 5.5 we estimate the number of particles leaving a given set during the internal DLA process.

Lemma 5.6. *For a bounded set $D \subset \text{SG}_n$ and $r \in \mathbb{N}$, let N be the number of particles that leave D^r during the internal DLA process when starting with all sites in D occupied and k additional particles that start at sites $y_1, \dots, y_k \in D$. If $k \leq \rho r^\alpha$ where ρ is as in Lemma 5.5, then $N \leq \text{Binom}(k, p)$ for some constant $p < 1$.*

Proof. Label the additional k particles with $1, \dots, k$. By \mathcal{I}_n^j we denote the cluster after the first j particles have been settled. By the inductive way of building \mathcal{I}_n^j it holds $|\mathcal{I}_n^j \setminus D| \leq j \leq k \leq \rho r^\alpha$. Denoting by A_j the event that the j -th particle ever leaves D^r , in view of Lemma 5.5 it holds $\mathbb{P}[A_j \mid \mathcal{I}_n^{j-1}] \leq 1 - \eta$, for some $\eta \in (0, 1)$. For $p = 1 - \eta$, by coupling the indicators $\mathbb{1}_{A_j}$ with i.i.d indicators $I_j \geq \mathbb{1}_{A_j}$ of mean p yields $N = \sum_{j=1}^k \mathbb{1}_{A_j} \leq \sum_{j=1}^k I_j = \text{Binom}(k, p)$. ■

The next lemma shows that if too few particles start inside the given set D , then only very few particles can ever reach the outside of D^{2r} .

Lemma 5.7. *For $n \in \mathbb{N}$, let D be a bounded subset of SG_n and let $k, r \in \mathbb{N}$ with $k \leq \rho(1 - p^{1/2\alpha})^\alpha r^\alpha$, where $\rho, p < 1$ as in Lemma 5.6. Let N be the number of particles that ever reach the outside of D^{2r} when starting with all sites inside D already occupied and k additional particles that start at sites $y_1, \dots, y_k \in D$. Then, for constant $c_0, c_1 > 0$ we have*

$$\mathbb{P}[N > 0] \leq c_0 e^{-c_1 r}.$$

Proof. For $j \in \mathbb{N}$ let $r_j := (1 - p^{j/2\alpha})r$, N_j the number of particles that ever leave D^{r_j} and $k_j := p^{j/2}k$. Taking $h_j := r_{j+1} - r_j$, we have

$$k_j \leq p^{j/2} \rho (1 - p^{1/2\alpha})^\alpha r^\alpha = \rho h_j^\alpha.$$

We now apply Lemma 5.6 which conditioned on N_j implies $N_{j+1} \leq \text{Binom}(N_j, p)$. Letting $A_j := \{N_j \leq k_j\}$ yields

$$\mathbb{P}[A_{j+1} \mid A_j] \geq \mathbb{P}[\text{Binom}(k_j, p) \leq k_{j+1}] \geq 1 - 2e^{-ck_j},$$

where the last inequality follows from [22, Lemma 5.8]. Let J be the minimal integer such that $p^{1/2}r \leq k_J < r$. On the event A_J , at most r particles ever leave D^r , hence there are not enough particles left to ever leave D^{2r} . Thus, $\mathbb{P}[N = 0] \geq \mathbb{P}[A_J]$ and

$$\mathbb{P}[N = 0] \geq \mathbb{P}[A_1] \mathbb{P}[A_2 \mid A_1] \dots \mathbb{P}[A_J \mid A_{J-1}] \geq 1 - 2J e^{-cr},$$

and the right-hand side is at least $1 - c_0 e^{-c_1 r}$ for suitable constants $c_0, c_1 > 0$. ■

Proof of Theorem 5.1. Lemma 5.4 gives the inner estimate. For the outer estimate, let $\varepsilon > 0$ and

$$N_n := |\{1 \leq i \leq m_n \mid v^i \geq \tilde{v}^i\}| = m_n - |\tilde{\mathcal{I}}_n|$$

be the number of particles leaving $\tilde{D} \cap \text{SG}_n$ before aggregating to the cluster, where m_n is the number of particles initially in SG_n . Let $p < 1$ as in Lemma 5.6 and set $K_0 := \rho((1 - p^{1/2\alpha})^\alpha)$. Since by Proposition B.2, $\mu(\partial\tilde{D}) = 0$, for sufficiently small $\eta > 0$ we obtain

$$\mu(\tilde{D} \setminus \tilde{D}_{2\eta}) \leq \frac{1}{2} K_0 \varepsilon^\alpha.$$

Clearly, for large enough n the construction gives

$$\delta_n^\alpha |\tilde{D}_\eta \cap \text{SG}_n| = \mu((\tilde{D}_\eta \cap \text{SG}_n)^{\Delta\Delta}) \geq \mu(\tilde{D}_{2\eta}) \geq \mu(\tilde{D}) - \frac{1}{2} K_0 \varepsilon^\alpha,$$

and Proposition B.2 implies

$$|\delta_n^\alpha m_n - \mu(\tilde{D})| < \frac{1}{2} K_0 \varepsilon^\alpha \text{ for } n \text{ large enough,}$$

such that, on the event $\{\tilde{D}_\eta \cap \text{SG}_n \subseteq \tilde{\mathcal{I}}_n\}$, we have for n large enough

$$N_n \leq m_n - |\tilde{D}_\eta \cap \text{SG}_n| \leq m_n - \delta_n^{-\alpha} \mu(\tilde{D}) + \frac{1}{2} K_0 \varepsilon^\alpha \delta_n^{-\alpha} \leq K_0 \varepsilon^\alpha \delta_n^{-\alpha}.$$

So for the event $A_n = \{N_n \leq K_0 \varepsilon^\alpha \delta_n^{-\alpha}\}$ by Lemma 5.4 it clearly holds that

$$\mathbb{P}[\limsup_{n \rightarrow \infty} A_n] \geq \mathbb{P}[\tilde{D}_\eta \cap \text{SG}_n \subseteq \tilde{\mathcal{I}}_n \text{ for all but finitely many}] = 1.$$

When considering the probability that, conditioned on A_n , \mathcal{I}_n has vertices outside the inflation $(\tilde{D} \cap \text{SG}_n)^{\varepsilon/\delta_n}$, Lemma 5.7 then gives

$$\mathbb{P}[\{\mathcal{I}_n \cap ((\tilde{D} \cap \text{SG}_n)^{\varepsilon/\delta_n})^C \neq \emptyset\} \cap A_n] \leq c_0 e^{-c_1(\varepsilon/\delta_n)},$$

and the claim follows from Borel–Cantelli lemma. ■

A. Odometer of the continuous divisible sandpile on SG

We give below a recipe on how to calculate the odometer function of the continuous divisible sandpile on SG as introduced in Section 3, for a concrete choice of the initial configuration $\sigma: \text{SG} \rightarrow [0, \infty)$. More precisely, we take σ to be constant in the ball of

radius $r = 2^l$ for some $l \in \mathbb{N}$ and zero outside, that is, for $N = 2^L > r$ and $L \in \mathbb{N}$ we consider

$$\sigma := \frac{\mu(B(0, N))}{\mu(B(0, r))} \mathbb{1}_{B(0, r)} = 3^{L-l} \mathbb{1}_{B(0, r)},$$

where we further assume that $L > l$. We take $\text{Tr} = B(0, 2N)$ and calculate the obstacle γ , its superharmonic majorant s , and show that the corresponding noncoincidence set D for this obstacle is $D = B(0, N) \subsetneq \text{Tr}$. We summarize first the steps we take when calculating the odometer, in order to make the section easier to follow.

- (1) First, we calculate functions $f_i: B(0, 2^i) \rightarrow \mathbb{R}$ that have constant Laplacian 1, i.e., $\Delta f_i = 1$ and are zero on the boundary, i.e., $f|_{\partial B(0, 2^i)} = 0$.
- (2) In the second step, we construct a function $\eta: \text{Tr} \rightarrow \mathbb{R}$ with $\Delta \eta = \mathbb{1}_{B(0, r)}$. In order to do so, we will take the function f_l (a function with constant Laplacian 1 on $B(0, r)$ and zero on the boundary) and construct a harmonic function $h: B(0, 2N) \setminus B(0, r) \rightarrow \mathbb{R}$ that matches the normal derivative of f_l on the boundary of $B(0, r)$. By then gluing together f_l and h , we obtain the desired function η .
- (3) Next, we construct the obstacle γ by taking a function ξ with constant Laplacian 1 on $B(0, 2N)$ (we can take $\xi = f_{L+1}$) and setting $\gamma = 3^{L-l} \eta - \xi$.
- (4) Finally, we show that the function s defined by

$$s(x) := \begin{cases} \gamma(x) & \text{if } x \in B(0, 2N) \setminus B(0, N), \\ \gamma(y) & \text{if } x \in B(0, N), \end{cases}$$

where $y \in \partial B(0, N)$ and $x \in B(0, 2N)$, is superharmonic and dominates γ .

This shows that for the odometer function u we have $u \leq s - \gamma$. Since $\{s - \gamma > 0\} = B(0, N)$, the conservation of mass in Proposition B.2 yields that $s - \gamma$ has the correct support. Since s is constant on the support of $s - \gamma$, there is no smaller superharmonic function and we have $u = s - \gamma$. Now we follow the steps in the above summary.

The construction of f_i . To construct f_i we use that $(3/4) \cdot 5^n \Delta_{\text{SG}_n} \rightarrow \Delta$ as $n \rightarrow \infty$, in order to calculate f_i on SG_n . The generalized (1/5) – (2/5) rule [12, Theorem A.1] gives $f(0) = -(4/3) \cdot 5^i$, and we can calculate the normal derivatives at the boundary of $B(0, 2^i)$. For this, we calculate the values of f_i at positions a_j, b_j as shown in Figure 3.

Using once again the generalized (1/5) – (2/5) rule, we obtain

$$\begin{pmatrix} f(a_j) \\ f(b_j) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \left(\frac{3}{5}\right)^j + \left(\frac{1}{5}\right)^j & \left(\frac{3}{5}\right)^j - \left(\frac{1}{5}\right)^j \\ \left(\frac{3}{5}\right)^j - \left(\frac{1}{5}\right)^j & \left(\frac{3}{5}\right)^j + \left(\frac{1}{5}\right)^j \end{pmatrix} \begin{pmatrix} f(a_0) \\ f(b_0) \end{pmatrix} - \frac{4}{3} 5^{i-j} (3^j - 1),$$

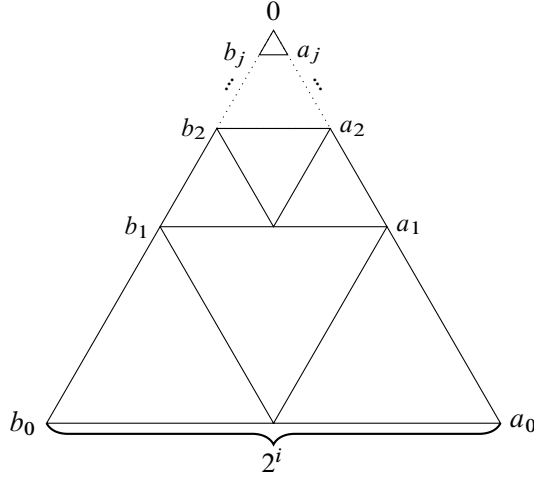


Figure 3. The vertices used in the calculations of the normal derivative of f_i .

which together with $f_i(a_0) = -4/3 \cdot 5^i$ and $f_i(b_0) = 0$ gives

$$f(a_j) = -2\left(\frac{3}{5}\right)^{j-i} 3^i + \frac{2}{3}\left(\frac{1}{5}\right)^j, \quad f(b_j) = -2\left(\frac{3}{5}\right)^{j-i} 3^i + 2\left(\frac{1}{5}\right)^j,$$

which can be used to compute the normal derivative ∂_n as below:

$$\partial_n f_i(x) = \lim_{j \rightarrow \infty} \left(\frac{5}{3}\right)^j \sum_{y \sim_j x} (f_i(x) - f_i(y)) = \lim_{j \rightarrow \infty} (4 \cdot 3^i - 8 \cdot 3^{-j-1}) = 4 \cdot 3^i.$$

The construction of η . We construct the function $\eta: B(0, 2N) \rightarrow \mathbb{R}$ with $\Delta\eta = \mathbb{1}_{B(0,r)}$. For this we take $\eta|_{B(0,r)} = f_l + c$ for some constant $c \in \mathbb{R}$ to be determined. We extend η to the ball $B(0, 2N)$ such that it is harmonic on the annulus $B(0, 2N) \setminus B(0, r)$. For this, we glue a harmonic function $h: B(0, 2N) \setminus B(0, r) \rightarrow \mathbb{R}$ and f_l at $\partial B(0, r)$ such that the resulting function η is in $\text{Dom}_{L^2}(\Delta)$. Following [26, Section 5.1] this results in the two necessary and sufficient conditions

$$f_l|_{\partial B(0,r)} + c = h|_{\partial B(0,r)} \quad \text{and} \quad (\partial_n f)|_{\partial B(0,r)} = -(\partial_n h)|_{\partial B(0,r)}.$$

So it remains to compute $(\partial_n h)|_{\partial B(0,r)}$ which depends on $h|_{\partial B(0,r)} = c$ and to set the constant c accordingly. For the values of h in u_i, v_i for $i = 0, \dots, L-l$ (see Figure 4) we have:

$$h(u_j) = h(u_{L-l}) \frac{1 - \left(\frac{3}{5}\right)^{j+1}}{1 - \left(\frac{3}{5}\right)^{L-l+1}} \quad \text{and} \quad h(v_j) = h(u_{L-l}) \frac{1 - \frac{4}{5}\left(\frac{3}{5}\right)^j}{1 - \left(\frac{3}{5}\right)^{L-l+1}}.$$

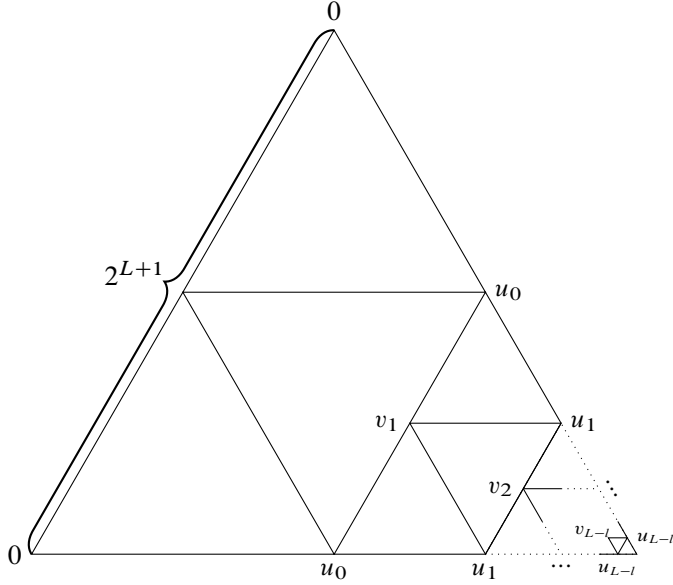


Figure 4. Vertices used in the calculation of the normal derivative of the function h .

The normal derivative of a harmonic function on a triangle of size 2^l is given by

$$\begin{aligned} \partial_n h(u_{L-l}) &= \left(\frac{3}{5}\right)^l (2h(u_{L-l}) - h(u_{L-l+1}) - h(v_{L-l})) \\ &= h(u_{L-l}) \left(\frac{3}{5}\right)^l \frac{1}{\left(\frac{5}{3}\right)^{L-l+1} - 1}. \end{aligned}$$

By the gluing condition we have $\partial_n h(u_{L-l}) = -\partial_n f_l(u_{L-l})$, and we choose

$$c = 4 \cdot 5^l \left(\left(\frac{5}{3}\right)^{L-l+1} - 1 \right).$$

We define $\eta \in \text{Dom}_{L^2}(\Delta)$ as follows:

$$\eta(x) := \begin{cases} f_l(x) + c & \text{if } x \in B(0, r), \\ h(x) & \text{if } x \in B(0, 2N) \setminus B(0, r). \end{cases}$$

Calculating the odometer. Define the obstacle γ and its superharmonic majorant s as

$$\gamma(x) = 3^{L-l} \eta(x) - f_{L+1}(x), \quad s(x) = \begin{cases} \gamma(x) & \text{if } x \in B(0, 2N) \setminus B(0, N), \\ 8/3 \cdot 5^L & \text{if } x \in B(0, N). \end{cases}$$

Notice that s is constant on $B(0, N)$ taking the value $8/3 \cdot 5^L$, which is the value of γ on $\partial B(0, N)$. Clearly, s is superharmonic on $B(0, 2N)$, since $\Delta s = \mathbb{1}_{B(0, 2N) \setminus B(0, N)}$. It remains to show that $s \geq \gamma$. For $x \in B(0, 2N) \setminus B(0, N)$ this is obvious by definition, so it is sufficient to show $8/3 \cdot 5^L \geq \gamma(x)$ for all $x \in B(0, N)$. Since γ is subharmonic on $B(0, r)$ it attains its maximum on the boundary and

$$\gamma|_{\partial B(0, r)} = \frac{4}{3} 5^l (3^{L-l+1} - 1) < \frac{8}{3} \cdot 5^L,$$

it suffices to investigate the values of γ in the annulus $B(0, N) \setminus B(0, r)$. The following result states that it suffices to consider only the junction points approaching $\partial B(0, N)$.

Lemma A.1. *Let $T^n = T^n(A, B, C) \subset \text{SG}_n$, for some $n \in \mathbb{N}$ be the discrete triangle with boundary points A, B, C and a, b, c be the midpoints opposing A, B, C , respectively. Furthermore, let $f: T^n \rightarrow \mathbb{R}$ be a function with $\Delta_n f = -1$ and $f(A) \geq f(B) \geq f(C)$. If*

$$f(A) \geq \max\{f(a), f(b), f(c)\} \quad \text{and} \quad f(A) = \max\{f(x) \mid x \in T^n(A, b, c)\},$$

then $\max_{x \in T(A, B, C)} f(x) = f(A)$.

Proof. Let h be the unique harmonic function on T^n taking boundary values $h|_{\partial T^n} = f|_{\partial T^n}$. Since $\Delta_n f = -1$, we have the decomposition

$$f(x) = h(x) + \sum_{y \in T^n} g_{T^n}^n(x, y),$$

where $g_{T^n}^n$ is the discrete Green function from (2.2). Note that $s' = \sum_{y \in T^n} g_{T^n}^n(x, y)$ is rotationally symmetric, i.e., s' takes the same values on subtriangles $T^n(A, b, c)$, $T^n(a, B, c)$, and $T^n(a, b, C)$. Since $h(A) \geq h(B) \geq h(C)$, after appropriate rotations φ_B, φ_C we get

$$h|_{T^n(A, b, c)} \geq h|_{T^n(a, B, c)} \circ \varphi_B \geq h|_{T^n(a, b, C)} \circ \varphi_C,$$

from which we can conclude $f|_{T^n(A, b, c)} \geq f|_{T^n(a, B, c)} \circ \varphi_B \geq f|_{T^n(a, b, C)} \circ \varphi_C$. ■

We finally calculate γ for all x_k^j, y_k^j, z_k^j , for $k \in \mathbb{N}$ and $j = 1, \dots, L-l-1$, as in Figure 5.

$$\gamma(x_0^j) = \frac{4}{3} 5^{L-j} (3^{j+1} - 1) \leq \gamma(x_0^{j-1}), \quad \gamma(y_0^j) = \frac{4}{3} 5^{L-j} (4 \cdot 3^j - 2) \leq \gamma(x_0^{j-1})$$

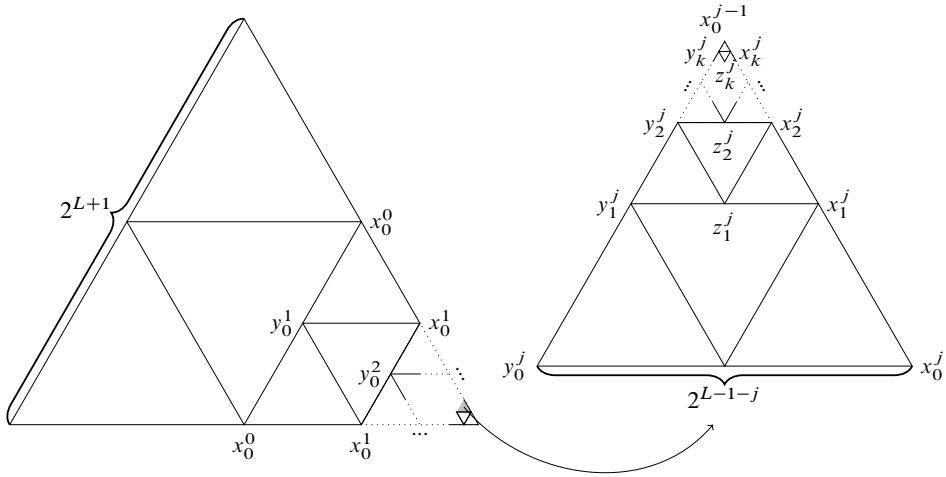


Figure 5. Points needed to show that $8/3 \cdot 5^L \geq \gamma(x)$ for all $x \in B(0, N) \setminus B(0, n)$.

and

$$\begin{aligned} \gamma(x_k^j) &= \gamma(x_0^{j-1}) - \frac{2}{3}(3^{j+1} - 9)\left(\frac{3}{5}\right)^k 5^{L-j} - \frac{2}{3}(3^j + 1)\left(\frac{1}{5}\right)^k 5^{L-j}, \\ \gamma(y_k^j) &= \gamma(x_0^{j-1}) - \frac{2}{3}(3^{j+1} - 9)\left(\frac{3}{5}\right)^k 5^{L-j} + \frac{2}{3}(3^j - 3)\left(\frac{1}{5}\right)^k 5^{L-j}, \\ \gamma(z_k^j) &= \gamma(x_0^{j-1}) - \frac{8}{15}(3^{j+1} - 9)\left(\frac{3}{5}\right)^k 5^{L-j} + \frac{8}{15}\left(\frac{1}{5}\right)^k 5^{L-j}, \end{aligned}$$

which together with Lemma A.1 implies that on all triangles $T^n(x_0^{j-1}, x_0^j, y_0^j) \subseteq SG_n$ the obstacle γ attains its maximum in x_0^{j-1} for all $n \in \mathbb{N}$. Since $\gamma(x_0^j) \leq \gamma(x_0^{j-1})$, γ attains its maximum in $(B(0, N) \setminus B(0, r)) \cap SG_n$ at x_0^0 for any $n \in \mathbb{N}$. Because γ is continuous, we get

$$\frac{8}{3} \cdot 5^L = \gamma(x_0^0) \geq \gamma(x) \text{ for all } x \in B(0, N) \setminus B(0, r).$$

B. Boundary regularity

This final part focuses on properties of the boundary of the noncoincidence set $D \subset SG$ obtained as the solution of the obstacle problem on the Sierpiński gasket $SG \subset \mathbb{R}^2$. We will use a generalization of Lebesgue's density theorem for SG .

We first consider the case when the density function $\sigma: SG \rightarrow [0, \infty)$ is continuous. Recall that by Lemma 3.7 for σ , the odometer function of the obstacle problem is the

limit of odometer functions u_n for the divisible sandpile on SG_n started from initial densities σ_n that fulfill (4.2) and (4.5)–(4.7) as well as the standard assumptions on initial configurations (3.1)–(3.4). A suitable and canonical choice would be to choose $\sigma_n(y)$ as the average value of σ in a double triangle around y as in Theorem 1.1. We will always fix σ_n to be defined this way in the following proofs.

Proposition B.1. *Let $\sigma: \text{SG} \rightarrow [0, \infty)$ be continuous with compact support such that $\mu(\sigma^{-1}(1)) = 0$ and $D = \{u > 0\}$, where u is the odometer function of the obstacle problem for σ . Then*

- (1) $\mu(\partial D) = 0$,
- (2) $\int_D \sigma d\mu = \mu(D)$.

Proof. (1) Fix λ with $0 < \lambda < 1$, and let $x \in \partial D$ with $\sigma(x) \leq \lambda$. We can then find $\varepsilon > 0$ such that σ is less than $(1 + \lambda)/2$ on $B(x, \varepsilon)$. We choose σ_n on SG_n converging to σ as discussed above and denote by u_n the odometer function of the divisible sandpile started from σ_n and by \mathcal{D}_n the corresponding divisible sandpile cluster. By Theorem 3.7, u_n converges to u uniformly, so for n large enough, $\sigma_n(y) \leq (1 + \lambda)/2$ for all $y \in B(x, \varepsilon) \cap \text{SG}_n$. Next we choose a point $x'_n \in \partial \mathcal{D}_n$ that is at a distance of less than $\varepsilon/3$ from x (which we can do for large enough n , since we know that the sets \mathcal{D}_n converge to D) and we then choose $x_n \in \text{SG}_n$ of distance less than $\varepsilon/3$ from x'_n such that $x_n \in \mathcal{D}_n$ (and hence $u_n(x_n) > 0$). Define the function

$$w_n(y) = u_n(y) + \frac{1}{2}(1 - \lambda)\delta_n^\beta \mathbb{E}_y[\tau_{\delta_n^{-1}\varepsilon/3}^n(x_n)],$$

where, as before, $\mathbb{E}_y[\tau_{\delta_n^{-1}\varepsilon/3}^n(x_n)]$ is the expected exit time from the ball $B^n(x_n, \varepsilon/3\delta_n)$ of a simple random walk on SG_n started at $y \in \text{SG}_n$. Since w_n is subharmonic on $\mathcal{D}_n \cap B^n(x_n, \delta_n^{-1}\varepsilon/3)$, it attains its maximum on the boundary, but for $y \in \partial \mathcal{D}_n$, we also have $w_n(y) \leq w_n(x_n)$, hence the maximum is attained in $\partial B^n(x_n, \delta_n^{-1}\varepsilon/3)$. Thus, there exists $y_n \in \partial B^n(x_n, \delta_n^{-1}\varepsilon/3)$ with

$$u_n(y_n) \geq \frac{1}{2}C(1 - \lambda)\varepsilon^\beta$$

for some constant $C > 0$. We remark that the Laplacian of u_n is bounded by some constant $M > 0$ and $u_n = 0$ on $\partial \mathcal{D}_n$, hence, similar to the proof of Lemma 4.7, we conclude

$$u_n(y_n) \leq cMd(y_n, \partial \mathcal{D}_n)^\beta \delta_n^\beta \quad \text{for some } c > 0.$$

By the two previous inequalities and with the choice $C' := (C(1 - \lambda)/(2cM))^{1/\beta}$ we obtain $d(y_n, \partial \mathcal{D}_n) \geq C'\varepsilon\delta_n^{-1}$. Thus, $B^n(y_n, C'\varepsilon\delta_n^{-1}) \subseteq \mathcal{D}_n$, and for any $x \in \partial D \cap$

$\{\sigma \leq \lambda\}$ and suitable $C'' > 0$ we get

$$\mu(B(x, (1 + C')\varepsilon) \cap (\partial D)^C) \geq C'' \varepsilon^\alpha.$$

By Lebesgue's density theorem [8] for the Hausdorff measure on the gasket we have

$$\mu(\partial D \cap \{\sigma \leq \lambda\}) = 0,$$

and by letting $\lambda \rightarrow 1$ and using the fact that $\mu(\{\sigma = 1\}) = 0$ we obtain the first claim.

(2) For the sandpile clusters $\mathcal{D}_n \subset \text{SG}_n$ with initial density σ_n converging to σ we have

$$|\mathcal{D}_n| \leq \sum_{z \in \mathcal{D}_n \cup \partial \mathcal{D}_n} \sigma_n(z) \leq |\mathcal{D}_n| + |\partial \mathcal{D}_n|.$$

For given $\varepsilon > 0$ and n large enough it holds that

$$D_\varepsilon \cap \text{SG}_n \subseteq \mathcal{D}_n \subseteq \mathcal{D}_n \cup \partial \mathcal{D}_n \subseteq D^\varepsilon,$$

hence,

$$\int_{D_\varepsilon} \sigma d\mu \leq \liminf_{n \rightarrow \infty} \int_{D_\varepsilon} \sigma_n^{\Delta\Delta} d\mu \leq \mu((D_n \cup \partial D_n)^{\Delta\Delta}) \leq \mu(D^\varepsilon),$$

and similarly,

$$\int_{D^\varepsilon} \sigma d\mu \geq \mu(D_\varepsilon).$$

Putting together the previous two inequalities we obtain

$$\int_D \sigma d\mu = \int_{D_\varepsilon} \sigma d\mu + \int_{D \setminus D_\varepsilon} \sigma d\mu \leq \mu(D) + \mu(D^\varepsilon \setminus D) + M\mu(D \setminus D^\varepsilon),$$

and

$$\int_D \sigma d\mu \geq \mu(D) - \mu(D \setminus D^\varepsilon) - M\mu(D^\varepsilon \setminus D),$$

where $M > 0$ is as in (3.1). Using $\mu(\partial D) = 0$ and letting $\varepsilon \rightarrow 0$ gives the claim. ■

We also consider densities σ that are not necessarily continuous everywhere, and we investigate the corresponding noncoincidence set D .

Proposition B.2. *Let $\sigma: \text{SG} \rightarrow [0, \infty)$ be continuous almost everywhere and assume that there exists $\lambda > 0$ such that for all $x \in \text{SG}$ we have either $\sigma(x) \leq \lambda$ or $\sigma(x) \geq 1$. If $D = \{u > 0\}$ is the solution to the obstacle problem for σ and $\tilde{D} = D \cup \{\sigma \geq 1\}^\circ$, we then have*

- (1) $\mu(\partial\tilde{D}) = 0$,
- (2) $\mu(\tilde{D}) = \int_{\tilde{D}} \sigma d\mu$.

Proof. (1) We choose $x \in \partial\tilde{D}$. Since the discontinuity points of σ are a null set, we can assume that σ is continuous at x . If $\sigma(x) \leq \lambda$, we can choose $\varepsilon > 0$ small enough such that $\sigma \leq (1 + \lambda)/2$ in $B(x, \varepsilon)$. If we define the discrete particle configurations σ_n as in (1.1) by taking the average of σ around a given point y , then σ_n also eventually are less than $(1 + \lambda)/2$ in $B(x, \varepsilon)$, and from here we can then proceed as in the proof of Proposition B.1 (1). If $\sigma(x) \geq 1$, since σ is continuous at x we can find an $\varepsilon > 0$ such that $\sigma \geq 1$ on $B(x, \varepsilon)$, but this would imply that $x \in \{\sigma \geq 1\}^\circ$, hence x cannot be a point on the boundary, thus completing the proof of the first claim. The second claim works as in the proof of Proposition B.1 (2). ■

Comments and other direction of work. An interesting question to consider is the scaling limit of the Abelian sandpile model, where instead of a continuous amount of mass we only allow whole particles to be distributed among the vertices. In [23], the authors prove that on \mathbb{Z}^d , when starting with n particles at the origin and stabilizing until each site has less than $2d$ particles, as $n \rightarrow \infty$, the set of occupied sites admits a weak* scaling limit on \mathbb{R}^d . The method can by no means be extended to fractal objects, and a completely different approach that makes use of the recursive structure of the gasket should be used in order to prove that also the scaling limit for the Abelian sandpile model with multiple sources is the same as for the other three aggregation models. The method used in the current work can be extended to other nested fractals, but due to the technicalities already appearing on the gasket, we have chosen not to work on the most general case of such self-similar sets.

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