Rend. Sem. Mat. Univ. Padova (Online first)
DOI 10.4171/RSMUP/141

Galois lines for a canonical curve of genus 4, II: Skew cyclic lines

Jiryo Komeda (*) – Takeshi Takahashi (**)

Abstract – Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field k of characteristic zero. For a line l, we consider the projection $\pi_l: C \to \mathbb{P}^1$ with center l and the extension of the function fields $\pi_l^*: k(\mathbb{P}^1) \hookrightarrow k(C)$. A line l is referred to as a *cyclic line* if the extension $k(C)/\pi_l^*(k(\mathbb{P}^1))$ is cyclic. A line $l \subset \mathbb{P}^3$ is said to be *skew* if $C \cap l = \emptyset$. We prove that the number of skew cyclic lines is equal to 0, 1, 3 or 9. We determine curves that have nine skew cyclic lines.

Mathematics Subject Classification (2020) – Primary 14H50; Secondary 14H37, 14H45.

Keywords – Canonical curve of genus 4, cyclic line, projection, automorphism group.

1. Introduction and the main theorem

Yoshihara [10] investigated various properties of skew Galois lines (for the definition, see below) for nondegenerate nonsingular curves C in $\mathbb{P}^3$. He proved that the number of skew Galois lines for an irrational C is finite, and that the number of skew Galois lines for C is at most one if $\deg C$ is a prime and $\deg C \geq 5$. He also studied the defining equations of curves C of low degrees that have skew Galois lines. In addition, Yoshihara et al. [2, 7, 11], studied the number and arrangement of skew Galois lines for elliptic space curves. Fukasawa and Higashine [4] and subsequent work by Fukasawa [3] determined the arrangement of all the Galois lines for the Giulietti–Korchmáros

(*) *Indirizzo dell'A.*: Department of Mathematics, Kanagawa Institute of Technology, 1030 Shimo-Ogino, Atsugi 243-0292, Japan; komeda@gen.kanagawa-it.ac.jp
(**) *Indirizzo dell'A.*: Education Center for Engineering and Technology, Faculty of Engineering, Niigata University, 8050 Ikarashininocho Nishi-ku, Niigata 950-2181, Japan; takeshi@eng.niigata-u.ac.jp

curve and for the Artin–Schreier–Mumford curve, respectively. More recently, in [8], we studied the number of non-skew cyclic lines for canonical curves of genus 4. As a continuation of this work [8], in this study, we investigate the number of skew cyclic lines for canonical curves of genus 4. We would like to note Kuribayashi et al. [9], however we will not use it in the present paper. By giving generators with respect to linear representations in the vector space of holomorphic differentials, they presented a complete classification of automorphism groups for compact Riemann surfaces of genera 3 and 4.

Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field k of characteristic 0, which is a $(2,3)$-complete intersection in $\mathbb{P}^3$. A line $l \subset \mathbb{P}^3$ is said to be *skew* if $C \cap l = \emptyset$. For a line l, we consider the projection $\pi_l \colon C \to \mathbb{P}^1$ with center l and the extension of the function fields $\pi_l^* \colon k(\mathbb{P}^1) \hookrightarrow k(C)$. Because $\deg C = 6$, we have $\deg \pi_l \leq 6$, and if l is skew, then we have $\deg \pi_l = 6$. We refer to a line l as a *Galois line* if the extension is Galois. We refer to the Galois line l as a C_6*-line* (resp. S_3*-line*) if the Galois group is isomorphic to the cyclic group C_6 of order 6 (resp. the symmetric group S_3 on 3 letters). We note that l is a skew cyclic line if and only if l is a C_6-line, in the setting of this paper. In [8], we explicitly gave the equations of C in the particular case in which C has two cyclic trigonal morphisms; we prove that the number of cyclic lines with $\deg \pi_l = 4$ is at most 1; and the number of cyclic lines with $\deg \pi_l = 5$ is at most 1. Our main theorem of the present paper is as follows.

Theorem. *Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4 over an algebraically closed field of characteristic 0. Then, the number of C_6-lines equals $0, 1, 3$ or 9. Moreover, if there exist nine C_6-lines for C, then C is projectively equivalent to the curve defined by one of the following:*

$$\begin{cases} XY - Z^2 = 0, \\ X^3 + Y^3 + W^3 = 0, \end{cases} \tag{1}$$

or

$$\begin{cases} X^2 + Y^2 + Z^2 = 0, \\ XYZ + W^3 = 0, \end{cases} \tag{2}$$

where $(X : Y : Z : W)$ are homogeneous coordinates on $\mathbb{P}^3$.

In Section 2, we present selected preliminary results. The proof of the theorem is provided in Section 3. In Sections 4 and 5, we determine all the C_6-lines for curves defined by equations (1) and (2). Section 6 presents examples of curves that have only one or three C_6-lines.

In the present paper, we assume that the base field k is algebraically closed and $\mathrm{char}(k) = 0$. For a line l, "skew" means "skew with respect to C", and also C_6-line means "with respect to C", and the reference to C will always be tacitly assumed. For the Galois line l, we denote $\{\sigma \in \mathrm{Aut}(C) \mid \pi_l \circ \sigma = \pi_l\}$ by G_l, which is isomorphic to the Galois group. We denote by C_m the cyclic group of order m; by D_m the dihedral group of order $2m$; by A_m the alternating group on m letters; by S_m and the symmetric group on m letters.

2. Preliminaries

Let $C \subset \mathbb{P}^3$ be a canonical curve of genus 4. Let $(X : Y : Z : W)$ be homogeneous coordinates on $\mathbb{P}^3$. The following are well-known facts.

Proposition 2.1 ([1, page 118], [6, page 298]). *The curve C is a $(2, 3)$-complete intersection; that is, the homogeneous ideal $I(C) \subset k[X, Y, Z, W]$ of C is generated by a quadratic form Q and cubic form F. The degree of C is 6. The surface $Q = 0$ is a unique quadric surface that contains C. The gonality $\mathrm{gon}(C)$ of C is equal to 3. If $\mathrm{rank}\, Q = 3$, then C has a unique trigonal morphism $C \to \mathbb{P}^1$, which is given by the projection from the vertex of the surface $Q = 0$. If $\mathrm{rank}\, Q = 4$, then C has exactly two trigonal morphisms $C \to \mathbb{P}^1$.*

Let $l \subset \mathbb{P}^3$ be a line and $\pi_l : C \to \mathbb{P}^1$ the projection with center l. Because $\deg C = 6$ and C is not hyperelliptic, we have $3 \le \deg \pi_l \le 6$. A line l is skew if and only if $\deg \pi_l = 6$. If $\deg \pi_l \ge 4$, then π_l uniquely determines the center l.

Proposition 2.2 ([8]). *Assume $\deg \pi_l \ge 4$. Then, $\pi_l = \pi_{l'}$ (up to an isomorphism of the codomains $\mathbb{P}^1$ of π_l and $\pi_{l'}$), if and only if $l = l'$.*

We have a canonical representation $\mathrm{Aut}(C) \hookrightarrow \mathrm{GL}(\Gamma(C, \Omega^1)) \cong \mathrm{GL}(4, k)$, where Ω^1 is the sheaf of regular 1-forms on C. As $C \subset \mathbb{P}^3$ is a canonical curve, we also have $\mathrm{Aut}(C) \hookrightarrow \mathrm{Aut}(\mathbb{P}^3) \cong \mathrm{PGL}(4, k)$. That is, for every $\sigma \in \mathrm{Aut}(C)$, there exists a unique projective transformation $T : \mathbb{P}^3 \to \mathbb{P}^3$ such that $T(C) = C$ and $T|_C = \sigma$. We express the elements in $\mathrm{Aut}(C)$ as the projective transformations of $\mathbb{P}^3$.

Proposition 2.3. *There exist a quadratic form $Q \in k[X, Y, Z, W]$ and a cubic form $F \in k[X, Y, Z, W]$ with $I(C) = (Q, F)$ such that $\sigma(Q = 0) = (Q = 0)$ and $\sigma(F = 0) = (F = 0)$ for any $\sigma \in \mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbb{P}^3)$.*

Proof. There exists a unique quadric $Q = 0$ that contains C. Clearly, $\sigma(Q = 0) = (Q = 0)$. Let

$$I_3 := \{F \in k[X, Y, Z, W] \mid F \text{ is a cubic form}, C \subset (F = 0)\} \cup \{0\},$$
$$J := \{(aX + bY + cZ + dW)Q \mid a, b, c, d \in k\},$$

and let $G \subset \mathrm{GL}(4, k)$ be a finite group isomorphic to $\mathrm{Aut}(C)$ via the natural quotient map $\mathrm{GL}(4, k) \twoheadrightarrow \mathrm{PGL}(4, k)$. Then, $\dim_k I_3 = 5$, $\dim_k J = 4$, $J \subsetneq I_3$, and G acts linearly on I_3 and J. Because $\mathrm{char}(k) = 0$, according to Maschke's theorem, the representation $G \to \mathrm{GL}(I_3)$ is completely reducible. Thus, there exists $F \in I_3 \setminus J$ such that $(A^* F)/F \in k \setminus \{0\}$ for any $A \in G$. ■

Proposition 2.4 ([10]). *Assume that there exists a C_6-line l. Then, by taking a suitable projective transformation of $\mathbb{P}^3$, we may assume that l is defined by $X = Y = 0$, and a generator σ of $G_l \subset \mathrm{Aut}(\mathbb{P}^3)$ is expressed by a diagonal matrix with diagonal components $1, 1, \alpha, \beta$ ($\alpha, \beta \in k \setminus \{0\}$), and $(\mathrm{ord}(\alpha), \mathrm{ord}(\beta)) = (3, 6), (2, 3)$, or $(2, 6)$. That is, we may assume*

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix},$$

where ζ is a primitive sixth root of the unity.

Proof. Most of the claims are proved in the proof of [10, Theorem 4.5] (see Claim 7 on pages 466-467 of [10]). We only have to verify the following: the diagonal matrix with diagonal components $1, 1, \zeta^4, \zeta$ is unsuitable for a generator σ of G_l. Indeed, if σ is such an automorphism, then using Proposition 2.3, Q will be reducible. ■

In Proposition 2.4, we note that the position of the line l and the form of the generator σ are specified simultaneously. From the following argument we see that this is possible: first, we fix the position of the line l to be $X = Y = 0$; next, from $\pi_l = \pi_l \circ \sigma$, we find the conditions that the representation matrix of σ must satisfy; finally by using a projective transformation that does not change the position of l, we diagonalize the representation matrix of σ.

Definition 2.5. We say that a C_6-line l is *of type* $(3, 6)$ (resp. *of type* $(2, 3)$, *of type* $(2, 6)$) if a generator of $G_l \subset \mathrm{Aut}(\mathbb{P}^3)$ can be represented as a matrix with eigenvalues $1, 1, \alpha, \beta$ with $(\mathrm{ord}(\alpha), \mathrm{ord}(\beta)) = (3, 6)$ (resp. $(2, 3)$, $(2, 6)$).

Corollary 2.6. *We assume that there exists a C_6-line l. Let $Q \in k[X, Y, Z, W]$ be a quadratic form such that the quadric surface $Q = 0$ contains C. Then,* $\operatorname{rank} Q = 3$. *Hence, there exists only one trigonal morphism $g_3^1 \colon C \to \mathbb{P}^1$, which is given by the projection from the vertex of $Q = 0$.*

Proof. The quadric $Q = 0$ containing C satisfies $\sigma(Q = 0) = (Q = 0)$ for any $\sigma \in G_l$. From Proposition 2.4, we see that $\operatorname{rank} Q = 3$. ■

For σ as stated in Proposition 2.4, we note that $\operatorname{Fix}(\sigma) := \{P \in \mathbb{P}^3 \mid \sigma(P) = P\}$ consists of a line $Z = W = 0$ and two points $(0 : 0 : 1 : 0), (0 : 0 : 0 : 1)$, and $l : X = Y = 0$ passes through these two points. Hence, we can immediately see the following.

Proposition 2.7. *Let l_1 and l_2 be distinct C_6-lines for C. Then, $G_{l_1} \neq G_{l_2}$ as subgroups of* $\operatorname{Aut}(C)$.

On S_3-lines, we have the following proposition. Proposition 2.8 is not used in the proof of our main theorem, but is required for the calculations in Sections 4 and 5. In Sections 4 and 5, we will determine not only C_6-lines but also S_3-lines for curves concretely defined by Equations (1) and (2).

Proposition 2.8 (Proof of [10, Theorem, 4.5]). *Let l be an S_3-line for C. Then, by taking a suitable projective transformation, we may assume that $l : X = Y = 0$, and G_l is generated by the following two elements:*

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and} \quad \tau := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

where ω is a primitive cubic root of the unity.

Since the proof of Proposition 2.8 is not stated in [10] as it is obvious, we present it here.

Proof. Let σ and τ be automorphisms of C such that $G_l = \langle \sigma, \tau \rangle$, where $\sigma^3 = \tau^2 = \mathrm{id}_C$ and $\tau\sigma\tau = \sigma^2$. By taking a suitable projective transformation, we may assume that l is defined by $X = Y = 0$. Because $\pi_l \circ \sigma = \pi_l$ and $\pi_l \circ \tau = \pi_l$, we have that σ and τ are represented as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

Because $\sigma^3 = \mathrm{id}_C$, σ is diagonalizable. We may assume that

$$\sigma = \begin{pmatrix} I & O \\ O & A \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} I & O \\ L & M \end{pmatrix},$$

where L and M are some 2×2 matrices,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \text{ or } \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

By using $\tau^2 = \mathrm{id}_C$ and $\tau\sigma\tau = \sigma^2$, we infer that $L + ML = O$, $M^2 = I$, $L + MAL = O$ and $MAM = A^2$. We have

$$A = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad L = O, \quad \text{and} \quad M = \begin{pmatrix} 0 & c \\ 1/c & 0 \end{pmatrix}$$

for some $c \in k \setminus \{0\}$. By taking the projective transformation that is represented by the diagonal matrix with diagonal elements 1, 1, c and 1, we have the representations of σ and τ as stated in the proposition. ■

For σ and τ as stated as in Proposition 2.8, we note that $\mathrm{Fix}(\sigma) := \{P \in \mathbb{P}^3 \mid \sigma(P) = P\}$ consists of a line $Z = W = 0$ and two points $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$, and l passes through these two points. The set $\mathrm{Fix}(\tau) := \{P \in \mathbb{P}^3 \mid \tau(P) = P\}$ consists of a hyperplane $Z - W = 0$ and a point $(0 : 0 : -1 : 1)$, and l passes through the point.

Assume that rank $Q = 3$, where the quadric $Q = 0$ contains C. Because the trigonal morphism $g^1_3\colon C \to \mathbb{P}^1$ is unique, for any $\sigma \in \mathrm{Aut}(C)$, there exists $A_\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ such that $g^1_3 \circ \sigma = A_\sigma \circ g^1_3$. Let G be a subgroup of $\mathrm{Aut}(C)$. Let $\varphi\colon G \to \mathrm{Aut}(\mathbb{P}^1)$ be the map $\sigma \mapsto A_\sigma$, which is a homomorphism between the groups. Let $\mathrm{Ker}\,\varphi$ and $\mathrm{Im}\,\varphi$ be the kernel and image of φ, respectively. We denote the inclusion $\mathrm{Ker}\,\varphi \hookrightarrow G$ as ψ. We have a short exact sequence

$$1 \longrightarrow \mathrm{Ker}\,\varphi \xrightarrow{\psi} G \xrightarrow{\varphi} \mathrm{Im}\,\varphi \longrightarrow 1. \tag{3}$$

The short exact sequence (3) and Proposition 2.9 play central roles in the proof of our main theorem.

PROPOSITION 2.9. *We have the following:*

(I) *The group* $\operatorname{Im}\varphi$ *is isomorphic to one of the following groups:* C_m $(m \in \mathbb{Z}_{>0})$, D_m $(m \in \mathbb{Z}_{>0})$, A_4, S_4 *or* A_5.

(II) *The three conditions "*$\operatorname{Ker}\varphi \neq 1$*", "*$\operatorname{Ker}\varphi \cong C_3$*", and "*$g^1_3$ *is cyclic" are equivalent.*

PROOF. Because $\operatorname{Im}\varphi \subset \operatorname{Aut}(\mathbb{P}^1)$ is finite, (I) is well known. As $\operatorname{Ker}\varphi = \{\sigma \in G \mid g^1_3 \circ \sigma = g^1_3\}$, we see that (II) holds. ∎

On automorphism groups of a plane quadric curve, we have the following proposition. Proposition 2.10 is required in the proof of our main theorem.

PROPOSITION 2.10. *Let* $V \subset \mathbb{P}^2$ *be the curve defined by* $XY = Z^2$, *which is isomorphic to* $\mathbb{P}^1$.

(I) *Let* $S_4 \subset \operatorname{Aut}(V) \subset \operatorname{Aut}(\mathbb{P}^2)$ *be the symmetric group on four letters. Then, by taking a suitable projective transformation, we can assume that* $S_4 = \langle \rho, \tau \rangle$,

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & i \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{pmatrix},$$

where i *is a primitive fourth root of the unity.*

(II) *Let* $D_m \subset \operatorname{Aut}(V) \subset \operatorname{Aut}(\mathbb{P}^2)$ $(m \geq 2)$ *be the dihedral group of order* $2m$. *Then, by taking a suitable projective transformation, we can assume that* $D_m = \langle \rho, \tau \rangle$,

$$\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_m^2 & 0 \\ 0 & 0 & \zeta_m \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ζ_m *is a primitive* m*th root of the unity.*

PROOF. We may assume that the group $S_4 \subset \operatorname{Aut}(\mathbb{P}^1)$ (resp. $D_m \subset \operatorname{Aut}(\mathbb{P}^1)$) is generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \left(\text{resp. } \begin{pmatrix} 1 & 0 \\ 0 & \zeta_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

The form of the matrices in the proposition comes from the images of these generators via the embedding $\mathbb{P}^1 \ni (x_0 : x_1) \mapsto (x_0^2 : x_1^2 : x_0 x_1) \in \mathbb{P}^2$. ∎

3. Proof of the main theorem

In this section, we prove the main theorem. Note that, if there exists a C_6-line, then C has a unique trigonal morphism $g^1_3 : C \to \mathbb{P}^1$ by Corollary 2.6. Let us consider the short exact sequence (3) for

$$G := \langle \sigma \in \mathrm{Aut}(C) \mid \sigma \in G_l \text{ for some } C_6\text{-line } l \rangle.$$

The map φ defined just before the sequence (3) will be used many times with the group G defined here.

We give an overview of the proof. We will assume that there exist at least two C_6-lines, and discuss the proof in the following two cases: there exists at least one C_6-line of type $(3,6)$; there does not exist a C_6-line of type $(3,6)$. It will be important that g^1_3 is cyclic in both cases (Propositions 3.1 and 3.2). In the case that there exists a C_6-line of type $(3,6)$, we can determine the defining equations of the curve C concretely (Lemma 3.4). Once the curve C is given by the concrete equations, it is possible to find all the Galois lines completely (Section 4). In the case that there does not exist a C_6-line of type $(3,6)$, we will consider the short exact sequence (3). The group $\mathrm{Ker}\,\varphi$ and homomorphisms φ and ψ are easy to understand, and it is known what groups can be isomorphic to the group $\mathrm{Im}\,\varphi$ (Proposition 2.9). We will discuss the proof for each group that may be $\mathrm{Im}\,\varphi$, and we will find $\mathrm{Im}\,\varphi \cong D_2, D_3$ or S_4 (Lemmas 3.6–3.10). In the case that $\mathrm{Im}\,\varphi \cong S_4$, we can determine the defining equations of the curve C concretely (Lemma 3.12), and find all the Galois lines completely (Section 5). In the cases that $\mathrm{Im}\,\varphi \cong D_2, D_3$, we can determine the defining equations of C roughly, and we will see that the number of C_6-lines is equal to 3 (Lemma 3.13).

The two propositions below provide sufficient conditions for g^1_3 to be cyclic.

PROPOSITION 3.1. *Assume that there exists a C_6-line l of type $(2,3)$ or $(2,6)$. Let σ_l be a generator of G_l. Then, $\mathrm{Ker}\,\varphi = \langle \sigma_l^2 \rangle$, and $\mathrm{ord}(\varphi(\sigma_l)) = 2$. In particular, the trigonal morphism g^1_3 is cyclic.*

PROOF. By Proposition 2.4, using a suitable projective transformation, we may assume that σ_l is expressed as the diagonal matrix with diagonal components $1, 1, -1, \zeta^2$ or $1, 1, -1, \zeta$, where ζ is a primitive sixth root of the unity. The quadric $Q = 0$ that contains C has the vertex $R := (0 : 0 : 0 : 1)$. The trigonal morphism g^1_3 is given by the projection π_R with center R. Because $\pi_R \circ \sigma_l^2 = \pi_R$, we have $\sigma_l^2 \in \mathrm{Ker}\,\varphi$. Use Proposition 2.9. ■

PROPOSITION 3.2. *Assume that there exist two C_6-lines. Then, the trigonal morphism g^1_3 is cyclic.*

Proof. Let l_1 and l_2 be two C_6-lines for C. We assume that $\operatorname{Ker}\varphi = 1$. Then, $G \cong \operatorname{Im}\varphi \cong C_m, D_m, A_4, S_4$, or A_5. This contradicts the fact that G includes two cyclic groups, G_{l_1} and G_{l_2}, of order 6. Therefore, $\operatorname{Ker}\varphi \neq 1$. Use Proposition 2.9. ■

We assume that there exist two C_6-lines for C. Let $P_1, \ldots, P_6$ be all the ramification points of the cyclic trigonal morphism g^1_3.

Lemma 3.3. *There exists a hyperplane $H \subset \mathbb{P}^3$ such that $\{P_1, \ldots, P_6\} \subset H$.*

Proof. By [8, Proposition 3.1], there exist $x, y \in k(C)$ such that $k(C) = k(x, y)$ and $y^3 = \prod_{j=1}^{5}(x - c_j)$. We can assume that $x(P_j) = c_j$ $(j = 1, \ldots, 5)$ and $x(P_6) = \infty$. Then, $(x - c_j) = 3P_j - 3P_6$ $(j = 1, \ldots, 5)$ and $(y) = P_1 + \cdots + P_5 - 5P_6$. By using the Riemann–Roch theorem, it is clear that $K_C \sim 6P_6$. Thus, $K_C \sim P_1 + \cdots + P_6$. Because $C \subset \mathbb{P}^3$ is a canonical curve, this concludes the lemma. ■

Lemma 3.4. *Assume that there exists a C_6-line of type $(3, 6)$ and that the trigonal morphism g^1_3 is cyclic. Then, C is projectively equivalent to the curve defined by equations (1).*

Proof. Let l be a C_6-line of type $(3, 6)$. We assume that $G_l = \langle \sigma_l \rangle$ and

$$\sigma_l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \zeta^2 & 0 \\ 0 & 0 & 0 & \zeta \end{pmatrix},$$

where ζ denotes a primitive sixth root of the unity. By using Proposition 2.3 and considering a suitable projective transformation, we can determine the defining equation of C as follows:

$$\begin{cases} Q = b(X, Y)Z + W^2 = 0, \\ F = X^3 + Y^3 + Z^3 = 0, \end{cases} \tag{4}$$

where $b(X, Y) = X - aY$ $(a \in k)$ or Y. If $b(X, Y) = Y$, then C is projectively equivalent to the curve defined by equations (1). Assume that $b(X, Y) = X - aY$. Let us show $a = 0$. The vertex of quadric $Q = 0$ is $R := (a : 1 : 0 : 0)$. The trigonal morphism $g^1_3 \colon C \to \mathbb{P}^1$ is given by the projection $\pi_R \colon (X : Y : Z : W) \mapsto (X - aY : Z : W)$. Let $P \in C$ be a ramification point of g^1_3. Then, $Z(P) \neq 0$. Indeed, if $Z(P) = 0$, then $P = (\zeta^{2j+1} : 1 : 0 : 0)$, where $j = 0, 1$ or 2. However, $(\zeta^{2j+1} : 1 : 0 : 0)$ is not a ramification point of g^1_3. Let $\pi_R(P) = (c : 1 : \sqrt{-c})$, where $c \in k$. A point in $C \cap \pi_R^{-1}(\pi_R(P))$ is $(ay + c : y : 1 : \sqrt{-c})$, where $y \in k$ satisfies

$$(ay + c)^3 + y^3 + 1 = 0. \tag{5}$$

Note that $a^3+1 \neq 0$, because C is nonsingular. As P is a total ramification point of g^1_3, equation (5) has a triple root. In other words, there exists $\beta \in k$ such that

$$(a^3+1)(y-\beta)^3 = (a^3+1)y^3 + 3a^2cy^2 + 3ac^2y + c^3 + 1.$$

Then, we have

$$\begin{cases} -3\beta(1+a^3) = 3a^2c, \\ 3\beta^2(1+a^3) = 3ac^2, \\ -\beta^3(1+a^3) = c^3+1. \end{cases} \tag{6}$$

If $a \neq 0$, then equations (6) do not have a root β. Hence, $a=0$ and C is projectively equivalent to the curve defined by equations (1). ■

We note that as in the proof of Lemma 3.4, for the curve defined by equations (1), there exists a C_6-line of type $(3,6)$ and g^1_3 is cyclic. The number of C_6-lines of the curve defined by equations (1) will be calculated later in Section 4. In the discussion of Section 4 we do not use the results in Section 3. From Proposition 3.2, Lemma 3.4, and Section 4, we have the following result.

PROPOSITION 3.5. *Assume that there exist two C_6-lines and one of them is of type* $(3,6)$. *Then, C is projectively equivalent to the curve defined by equations* (1). *There are exactly nine C_6-lines and exactly one S_3-line for C. We have that* $\mathrm{Aut}(C) \cong C_3 \times D_6$.

PROOF. From the assumption that there exist two C_6-lines, by using Proposition 3.2, the trigonal morphism g^1_3 is cyclic. Combining this with the assumption that there exists a C_6-line of type $(3,6)$, by using Lemma 3.4, we have that C is projective equivalent to the curve defined by equations (1). By the results in Section 4, we have $\mathrm{Aut}(C)$ and the number of skew Galois lines. ■

Hereafter, in this section, we continue to prove our main theorem, except in the case that C is projectively equivalent to the curve defined by equations (1). That is, we assume that there exist at least two C_6-lines for C, and every C_6-line is not of type $(3,6)$.

LEMMA 3.6. *We have that* $\mathrm{Im}\,\varphi \not\cong A_5$.

PROOF. Assume that $\mathrm{Im}\,\varphi \cong A_5$. Then, $|G| = 180$. However, the Hurwitz theorem states $|G| = 84(g-1), 48(g-1), 40(g-1), \ldots = 252, 144, 120, \ldots$; thus, this is a contradiction. ■

LEMMA 3.7. *We have that* $\mathrm{Im}\,\varphi \not\cong A_4$ *or* C_m.

Proof. From Proposition 3.1, $\operatorname{Im}\varphi$ is generated by some elements of order 2. However, A_4 and C_m $(m \geq 3)$ are not generated by elements of order 2. If $\operatorname{Im}\varphi \cong C_2$, then, G does not include two C_6 subgroups, because the order of G equals 6. ■

Lemma 3.8. *If* $\operatorname{Im}\varphi \cong D_m$, *then* $m \leq 6$.

Proof. Let $Q = 0$ be the quadric that contains C, where the rank of the quadratic Q equals 3, and R be the vertex of the quadric $Q = 0$. Then, the cyclic trigonal morphism g^1_3 is given by the projection π_R with center R. All the ramification points $P_1, \dots, P_6$ of g^1_3 are on a hyperplane $H = 0$. Because $g^1_3 = \Phi_{|3P_j|}$ $(j = 1, \dots, 6)$, for any $\sigma \in \operatorname{Aut}(C)$, $\sigma(\{P_1, \dots, P_6\}) = \{P_1, \dots, P_6\}$. Thus, $\sigma((Q = H = 0)) = (Q = H = 0)$, where $Q = H = 0$ is a plane quadric curve. We can regard that $g^1_3 = \pi_R|_C : C \to (Q = H = 0) \cong \mathbb{P}^1$ and $\varphi : G \ni \sigma \mapsto \sigma|_{Q=H=0} \in \operatorname{Im}\varphi \subset \operatorname{Aut}(Q = H = 0)$. Because $\operatorname{Im}\varphi$ acts on the set $\{P_1, \dots, P_6\} \subset (Q = H = 0)$ faithfully, we determine that the order of each element in $\operatorname{Im}\varphi$ is at most 6. This concludes that $m \leq 6$. ■

By Lemmas 3.6, 3.7, and 3.8, we have $\operatorname{Im}\varphi \cong D_m$ $(2 \leq m \leq 6)$ or S_4.

Lemma 3.9. *The maximum number of* C_6*-lines is nine. If there exist nine* C_6*-lines, then* $\operatorname{Im}\varphi \cong S_4$.

Proof. Let $l_1, l_2, \dots$ be all the C_6-lines for C, which are of type $(2,3)$ or $(2,6)$. Let σ_j $(j = 1, 2, \dots)$ be a generator of G_{l_j}. By Propositions 2.7 and 3.1, $\varphi(\sigma_1), \varphi(\sigma_2), \dots$ are mutually distinct elements in $\operatorname{Im}\varphi$ and are of order 2. The number of elements of order 2 in S_4 (resp. D_6, D_5, D_4, D_3, D_2) equals 9 (resp. 7, 5, 5, 3, 3). This now concludes the lemma. ■

Lemma 3.10. *We have that* $\operatorname{Im}\varphi \not\cong D_4$, D_5, D_6.

Proof. Assume that $\operatorname{Im}\varphi \cong D_4$. Because the rank of the quadric $Q = 0$ that contains C equals 3, by taking a suitable projective transformation, we may assume that $Q = XY - Z^2$. From Lemma 3.3, all the ramification points $P_1, \dots, P_6$ of the cyclic trigonal morphism g^1_3 are on some hyperplane $H = 0$. By taking a suitable projective transformation that does not change Q, we may assume $H = W$. Note that we can take such a projective transformation because $(0:0:0:1) \notin H$. By using Proposition 2.10 and the same argument as in the proof of Lemma 3.8, we may assume that:

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

where ω (resp. i) is a primitive cubic (resp. fourth) root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By using Proposition 2.3, we find a cubic form $F \in k[X, Y, Z, W] \setminus \{0\}$, such that the cubic surface $F = 0$ contains C. By the condition $\sigma(F = 0) = (F = 0)$ for any $\sigma \in G$, we have $F = a(X^2 + Y^2)Z + W^3$, $F = a(X^2 - Y^2)Z + W^3$, or $F = aXYZ + bZ^3 + W^3$, where $a, b \in k$. The curves defined by $Q = XY - Z^2 = 0$ and $F = a(X^2 + Y^2)Z + W^3 = 0$ are projectively equivalent to the curve defined by equations (2), and thus, $\operatorname{Im}\varphi \cong S_4$. The curves defined by $Q = XY - Z^2 = 0$ and $F = a(X^2 - Y^2)Z + W^3 = 0$ are also projectively equivalent to the curve defined by equations (2). The curves defined by $Q = XY - Z^2 = 0$ and $F = aXYZ + bZ^3 + W^3 = 0$ have singular points $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$. Hence, we see that $\operatorname{Im}\varphi \not\cong D_4$.

By using the same argument as above, we also see that $\operatorname{Im}\varphi \not\cong D_5$.

Assume that $\operatorname{Im}\varphi \cong D_6$. From the same argument as above, we see that C must be projectively equivalent to the curve defined by equations (1). Then, there exists a C_6-line for C of type $(3, 6)$. However, this is a contradiction. This concludes $\operatorname{Im}\varphi \not\cong D_6$. ■

REMARK 3.11. To prove our main theorem, we have discussed the proof above with the assumption that there is no C_6-line of type $(3, 6)$, which is stated just after Proposition 3.5. If we allow the existence of C_6-lines of type $(3, 6)$, then by the same argument as in the proof of Lemma 3.10, we see the following: if a canonical curve $C \subset \mathbb{P}^3$ of genus 4 satisfies the conditions "there exists a unique trigonal morphism g^1_3", "g^1_3 is cyclic", and "$\operatorname{Im}\varphi \cong D_6$", then C is projectively equivalent to the curve defined by equations (1).

Hence, $\operatorname{Im}\varphi \cong D_2$, D_3, or S_4.

LEMMA 3.12. *Assume that* $\operatorname{Im}\varphi \cong S_4$. *Then,* C *is projectively equivalent to the curve defined by equations* (2). *Hence, there exist nine* C_6*-lines (see Section* 5*).*

PROOF. We may assume that the ramification points $P_1, \ldots, P_6$ of the trigonal morphism g^1_3 are on the hyperplane $W = 0$ and the quadric $Q = 0$ that contains C is $XY - Z^2 = 0$. By using Proposition 2.10, we can assume that

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

where ω (resp. i) is a primitive cubic (resp. fourth) root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By the same argument as in the proof of Lemma 3.10, C must be defined by

$$\begin{cases} XY - Z^2 = 0, \\ c(X^2 - Y^2)Z + W^3 = 0, \end{cases}$$

where $c \in k$. Then, C is projectively equivalent to the curve defined by equations (2). ■

Note that we do not use the results in Section 3 in the discussion of Section 5.

LEMMA 3.13. *If* $\operatorname{Im}\varphi \cong D_2$ *or* D_3, *then the number of* C_6*-lines equals* 3.

PROOF. If $\operatorname{Im}\varphi \cong D_2$ or D_3, then the number of C_6-lines is at most three because the group $\operatorname{Im}\varphi$ contains only three elements of order 2.

Assume that $\operatorname{Im}\varphi \cong D_2$. We may assume that all the ramification points $P_1, \ldots, P_6$ of the trigonal morphism g_3^1 are on the hyperplane $W = 0$ and the quadric $Q = 0$ that contains C is $XY - Z^2 = 0$. By using Proposition 2.10, we can assume that

$$G = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} \right\rangle,$$

where ω is a primitive cubic root of the unity and $\lambda_1, \lambda_2 \in k \setminus \{0\}$. By the same argument as in the proof of Lemma 3.10, C must be projectively equivalent to the curve defined by

$$\begin{cases} XY - Z^2 = 0, \\ (X^3 + Y^3) + c(X + Y)Z^2 + W^3 = 0, \end{cases} \tag{7}$$

or

$$\begin{cases} XY - Z^2 = 0, \\ (X^2 + Y^2)Z + cZ^3 + W^3 = 0, \end{cases} \tag{8}$$

where $c \in k$. Then, the three lines $X = Y = 0$, $X + Y = Z = 0$, $X - Y = Z = 0$ are C_6-lines. Indeed, if C is defined by equations (7) (resp. equations (8)), then we have automorphisms of order 6 as follows:

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&-1&0\\0&0&0&\omega\end{pmatrix}$$

$$\text{(resp. }\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix}\text{).}$$

Thus, the number of C_6-lines is at least three.

Assume that $\operatorname{Im}\varphi \cong D_3$. According to the above argument, C must be projectively equivalent to the curve defined by

$$\begin{cases} XY - Z^2 = 0, \\ (X^3 + Y^3) + cZ^3 + W^3 = 0, \end{cases} \tag{9}$$

where $c \in k$. Then, the three lines $X + Y = Z = 0$, $X + \omega Y = Z = 0$, $X + \omega^2 Y = Z = 0$ are C_6-lines. Indeed, we have automorphisms of order 6 as follows:

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&\omega&0&0\\\omega^2&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix},\begin{pmatrix}0&\omega^2&0&0\\\omega&0&0&0\\0&0&1&0\\0&0&0&\omega\end{pmatrix}.$$

Thus, the number of C_6-lines is at least three. ∎

The proof of our main theorem is now complete.

4. Example: Galois lines for the curve defined by equations (1)

In this section, let C be the nonsingular projective curve such that $k(C) = k(x, y)$, and

$$x^6 + y^3 + 1 = 0. \tag{10}$$

The polynomial on the left-hand side of equation (10) is irreducible. Let $g^1_3 \colon C \to \mathbb{P}^1$ be the trigonal morphism given by the function x. Then, g^1_3 is a cyclic triple covering, and there exist 6 branch points. By using the Riemann–Hurwitz formula, we have that the genus of C is equal to 4. Let $(x)_\infty = D$ be the divisor of poles of x. Then, $(x^2)_\infty = (y)_\infty = 2D$. Therefore, $\dim_k H^0(C, \mathcal{O}_C(2D)) \geq 4$. By using the Riemann–Roch theorem, we have that $K_C \sim 2D$. The morphism $C \ni P \mapsto (1 : x^2(P) : x(P) : y(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this canonical embedding is expressed as equations (1). We regard C as the canonical curve defined by equations (1).

We can identify nine C_6-lines and one S_3-line, as indicated in Tables 1 and 2. Because $\deg \pi_{l_j} = 6$, $\sigma_j \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma_j) = 6$, and $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ $(j = 1, \ldots, 9)$, it is clear that the lines $l_1, \ldots, l_9$ are C_6-lines. As $\deg \pi_{l_{10}} = 6$, $\sigma_{10}, \tau_{10} \in \mathrm{Aut}(C)$, $\langle \sigma_{10}, \tau_{10} \rangle \cong S_3$, $\pi_{l_{10}} \circ \sigma_{10} = \pi_{l_{10}}$, and $\pi_{l_{10}} \circ \tau_{10} = \pi_{l_{10}}$, the line l_{10} is clearly an S_3-line.

Let $R := (0 : 0 : 0 : 1)$, which is the vertex of the quadric $XY - Z^2 = 0$. The projection $\pi_R \colon C \to (XY - Z^2 = W = 0) \cong \mathbb{P}^1 \subset (W = 0) \cong \mathbb{P}^2$ yields the unique trigonal morphism g^1_3. We have that g^1_3 is cyclic,

$$\rho := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \in \mathrm{Aut}(C),\ \mathrm{ord}(\rho) = 3 = \deg g^1_3, \text{ and } \pi_R \circ \rho = \pi_R.$$

The ramification points of g^1_3 are

$$\begin{aligned} P_1 &:= (1 : -1 : i : 0), & P_2 &:= (1 : -1 : -i : 0), \\ P_3 &:= (1 : -\omega : i\omega^2 : 0), & P_4 &:= (1 : -\omega : -i\omega^2 : 0), \\ P_5 &:= (1 : -\omega^2 : i\omega : 0), & P_6 &:= (1 : -\omega^2 : -i\omega : 0), \end{aligned}$$

where ω (resp. i) is a primitive cubic (resp. fourth) root of the unity. Because $g^1_3 = \Phi_{|3P_j|}$ $(j = 1, \ldots, 6)$, we have $\mathrm{Aut}(C)$ acts on $\{P_1, \ldots, P_6\}$. Thus, $\sigma(W = 0) = (W = 0)$ for any $\sigma \in \mathrm{Aut}(C) \subset \mathrm{Aut}(\mathbb{P}^3)$.

Because g^1_3 is a unique trigonal morphism, a unique $A_\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ exists for any $\sigma \in \mathrm{Aut}(C)$ such that $g^1_3 \circ \sigma = A_\sigma \circ g^1_3$. We denote the map $\sigma \mapsto A_\sigma$ as $\varphi \colon \mathrm{Aut}(C) \to \mathrm{Aut}(\mathbb{P}^1)$, which is a homomorphism between the groups. Note that $\sigma(W = 0) = (W = 0)$, and g^1_3 is obtained by using the projection $\pi_R \colon (X : Y : Z : W) \mapsto (X : Y : Z)$. By considering $\varphi(\sigma) = A_\sigma$ as an automorphism of the quadric plane curve $(XY - Z^2 = W = 0) \subset (W = 0) \cong \mathbb{P}^2$, we see that φ is expressed as follows:

$$\sigma = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \longmapsto \sigma' = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

Line l	Defining equation of l	G_l	Generators of G_l
l_1	$X = Y = 0$	C_6	$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_2	$X + Y = Z = 0$	C_6	$\sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_3	$X + \omega Y = Z = 0$	C_6	$\sigma_3 = \begin{pmatrix} 0 & \omega & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_4	$X + \omega^2 Y = Z = 0$	C_6	$\sigma_4 = \begin{pmatrix} 0 & \omega^2 & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_5	$X - Y = Z = 0$	C_6	$\sigma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_6	$X - \omega Y = Z = 0$	C_6	$\sigma_6 = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ -\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_7	$X - \omega^2 Y = Z = 0$	C_6	$\sigma_7 = \begin{pmatrix} 0 & -\omega^2 & 0 & 0 \\ -\omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_8	$X = W = 0$	C_6	$\sigma_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
l_9	$Y = W = 0$	C_6	$\sigma_9 = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
ω is a primitive cubic root of the unity.			

TABLE 1. C_6-lines for the curve defined by Equations (1)

Line l	Defining equation of l	G_l	Generators of G_l
l_{10}	$Z = W = 0$	S_3	$\sigma_{10} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,
			$\tau_{10} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
ω is a primitive cubic root of the unity.			

TABLE 2. S_3-lines for the curve defined by equations (1)

where σ' is regarded as an element of $\mathrm{Aut}(XY - Z^2 = W = 0) \subset \mathrm{Aut}(\mathbb{P}^2)$. Let $\mathrm{Ker}\,\varphi$ and $\mathrm{Im}\,\varphi$ be the kernel and image of φ, respectively. We have the short exact sequence (3) for $\mathrm{Aut}(C)$, and $\mathrm{Ker}\,\varphi = \langle \rho \rangle$.

CLAIM 4.1. We have that $\mathrm{Im}\,\varphi \cong D_6$, which is the dihedral group of order 12.

PROOF. From Proposition 2.9, $\mathrm{Im}\,\varphi$ is isomorphic to C_m, D_m, A_4, S_4, or A_5. Let σ_j $(j = 1, \dots, 10)$ be the automorphism provided in Tables 1 and 2. Because the order of $\varphi(\sigma_8)$ is equal to 6, we see that $\mathrm{Im}\,\varphi \cong C_m$ or D_m, where m is a multiple of 6. Because $\varphi(\sigma_1) \neq \varphi(\sigma_2)$, and the orders of both $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ are equal to 2, we have $\mathrm{Im}\,\varphi \cong D_m$. Note that $\mathrm{Aut}(C)$ acts on the set $\{P_1, \dots, P_6\}$. Let $\sigma \in \mathrm{Aut}(C)$. If $\sigma(P_j) = P_j$ for every P_j $(j = 1, \dots, 6)$, then $\varphi(\sigma)$ is the identity. Thus, the order of $\varphi(\sigma)$ is at most 6. This concludes that $\mathrm{Im}\,\varphi \cong D_6$. ■

We have an exact sequence $1 \to C_3 \xrightarrow{\psi} \mathrm{Aut}(C) \xrightarrow{\varphi} D_6 \to 1$. The order of $\mathrm{Aut}(C)$ is 36. Let $G := \langle \rho, \sigma_2, \sigma_8 \rangle$.

CLAIM 4.2. We have that $\mathrm{Aut}(C) = G \cong C_3 \times D_6$.

PROOF. Because of the exact sequence $1 \to C_3 \xrightarrow{\psi} G \xrightarrow{\varphi} D_6 \to 1$, $G = \mathrm{Aut}(C)$. We show that there is a left-inverse of ψ. For $\sigma \in G$, we have a unique matrix representation M_σ such that $M_\sigma^*(XY - Z^2) = XY - Z^2$ and the $(4, 4)$-component of M_σ is $1, \omega$, or ω^2. We denote the $(4, 4)$-component of M_σ as λ_σ. Let $\psi' \colon G \to \mathrm{Ker}\,\varphi \cong C_3$ be as follows:

$$\sigma = M_\sigma \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_\sigma \end{pmatrix}.$$

Because ψ' is a homomorphism between groups, and $\psi' \circ \psi = \mathrm{id}$, this concludes that $G \cong C_3 \times D_6$. ∎

The group $\mathrm{Aut}(C) \cong C_3 \times D_6$ has only ten C_6 subgroups:

$$\langle \sigma_1 \rangle, \dots, \langle \sigma_9 \rangle, \left\langle \overline{\sigma} := \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

As $\overline{\sigma}$ has no multiple eigenvalues, $\langle \overline{\sigma} \rangle$ is not a Galois group associated with a Galois line. Therefore, the number of C_6-lines is equal to 9. The group $\mathrm{Aut}(C) \cong C_3 \times D_6$ has only six S_3 subgroups: $\langle \sigma_m, \tau_n \rangle$ for $(m, n) = (0,0), (0,1), (1,0), (1,1), (2,0)$ and $(2,1)$, where

$$\sigma_m = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega^m \end{pmatrix} \quad \text{and} \quad \tau_n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & (-1)^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 2.8, the lines that might be S_3-lines are $l_8 : X = W = 0$, $l_9 : Y = W = 0$, and $l_{10} : Z = W = 0$. However, l_8 and l_9 are C_6-lines. The line l_{10} is the only one S_3-line.

Remark 4.3. Let $P' := (1 : 0 : 0 : 0)$, which is the point at which lines l_9 and l_{10} intersect. By the projection $\pi_{P'}: (X : Y : Z : W) \mapsto (Y : Z : W)$ with center P', we have a singular plane curve $T_6 : Y^6 + Z^6 + Y^3W^3 = 0$ as the image $\pi_{P'}(C)$. The points $(0:1:0) = \pi_{P'}(l_9)$ and $(1:0:0) = \pi_{P'}(l_{10})$ are outer Galois points for T_6 with Galois groups C_6 and S_3, respectively. The plane curves $T_{2m} : Y^{2m} + Z^{2m} + Y^mW^m = 0$ are examples of curves that are known to have two outer Galois points with Galois groups C_{2m} and D_m (See [5]).

5. Example: Galois lines for the curve defined by equations (2)

In this section, let C be the nonsingular projective curve such that $k(C) = k(x, y)$, and

$$y^6 + x^2(x^2 + 1) = 0. \tag{11}$$

The polynomial on the left-hand side of equation (11) is irreducible. Let $g_6^1 : C \to \mathbb{P}^1$ be the cyclic morphism of degree 6 given by the function x. By using the Riemann–Hurwitz formula, we have that the genus of C is equal to 4. Let P_∞, $P_{\infty'}$, P_0, $P_{0'}$,

P_i, P_{-i} be six points on C such that $x(P_\infty) = x(P_{\infty'}) = \infty$, $x(P_0) = x(P_{0'}) = 0$, $x(P_i) = i$, and $x(P_{-i}) = -i$, where i is a primitive fourth root of the unity. Because $(x) = 3P_0 + 3P_{0'} - 3P_\infty - 3P_{\infty'}$, $(y) = P_0 + P_{0'} + P_i + P_{-i} - 2P_\infty - 2P_{\infty'}$, and $(x - i) = 6P_i - 3P_\infty - 3P_{\infty'}$, we have

$$\left(\frac{y^3}{x(x-i)}\right)_\infty = 3P_i \quad \text{and} \quad \left(\frac{y}{x-i}\right)_\infty = 5P_i.$$

Hence, the Weierstrass semigroup of P_i is $H(P_i) = \langle 3, 5\rangle$ (for the definition of Weierstrass semigroup, see [8, Equation (1)]). Thus, C is not hyperelliptic and $K_C \sim 6P_i \sim 3P_\infty + 3P_{\infty'}$. Because $1, y^3/(x(x-i)), y/(x-i), 1/(x-i)$ are linearly independent over k, the morphism $C \ni P \mapsto (x^2(P) : y^3(P) : x(P) : -x(P)y(P)) \in \mathbb{P}^3$ is a canonical embedding. The image of this embedding is expressed as equations (2). We regard C as the canonical curve defined by equations (2).

We can find nine C_6-lines and four S_3-lines, as in Tables 3 and 4. Because $\deg \pi_{l_j} = 6$, $\sigma_j \in \mathrm{Aut}(C)$, $\mathrm{ord}(\sigma_j) = 6$ and $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ $(j = 1, \ldots, 9)$, we see that the lines $l_1, \ldots, l_9$ are C_6-lines. As $\deg \pi_{l_j} = 6$, $\sigma_j, \tau_j \in \mathrm{Aut}(C)$, $\langle \sigma_j, \tau_j\rangle \cong S_3$, $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ and $\pi_{l_j} \circ \tau_j = \pi_{l_j}$ $(j = 10, \ldots, 13)$, we see that the lines $l_{10}, \ldots, l_{13}$ are S_3-lines.

Claim 5.1. We have that $\mathrm{Aut}(C) \cong C_3 \times S_4$.

Proof. We have the following automorphisms of C:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where ω is a primitive cubic root of the unity. The group generated by these four elements, which is a subgroup of $\mathrm{Aut}(C)$, is isomorphic to $C_3 \times S_4$. By considering the short exact sequence (3) for $G = \mathrm{Aut}(C)$, we have that $1 \to C_3 \to \mathrm{Aut}(C) \xrightarrow{\varphi} \mathrm{Im}\,\varphi \to 1$ and $\mathrm{Im}\,\varphi \cong C_m, D_m, A_4, S_4$, or A_5. By using the same argument as in the proof of Lemma 3.8 or Claim 4.1, if $\mathrm{Im}\,\varphi \cong C_m$ or D_m, then $m \leq 6$. Because $C_3 \times S_4 \subset \mathrm{Aut}(C)$, we see that $\mathrm{Im}\,\varphi \cong S_4$ and $\mathrm{Aut}(C) \cong C_3 \times S_4$. ∎

Because the group $C_3 \times S_4$ contains exactly nine C_6 subgroups and exactly four S_3 subgroups, this concludes that the lines in Tables 3 and 4 are all the C_6-lines and all the S_3-lines, respectively.

Line l	Defining equation of l	G_l	Generators of G_l
l_1	$X = Y = 0$	C_6	$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_2	$Y = Z = 0$	C_6	$\sigma_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_3	$X = Z = 0$	C_6	$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\omega \end{pmatrix}$
l_4	$X + Y = Z = 0$	C_6	$\sigma_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_5	$X - Y = Z = 0$	C_6	$\sigma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_6	$X + Z = Y = 0$	C_6	$\sigma_6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_7	$X - Z = Y = 0$	C_6	$\sigma_7 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_8	$X = Y + Z = 0$	C_6	$\sigma_8 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
l_9	$X = Y - Z = 0$	C_6	$\sigma_9 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$
ω is a primitive cubic root of the unity.			

TABLE 3. C_6-lines for the curve defined by equations (2)

Line l	Defining equation of l	G_l	Generators of G_l
l_{10}	$X+Y+Z=W=0$	S_3	$\sigma_{10}=\begin{pmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{10}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
l_{11}	$X-Y+Z=W=0$	S_3	$\sigma_{11}=\begin{pmatrix}0&-1&0&0\\0&0&-1&0\\1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{11}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
l_{12}	$-X+Y+Z=W=0$	S_3	$\sigma_{12}=\begin{pmatrix}0&-1&0&0\\0&0&1&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{12}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
l_{13}	$X+Y-Z=W=0$	S_3	$\sigma_{13}=\begin{pmatrix}0&1&0&0\\0&0&-1&0\\-1&0&0&0\\0&0&0&1\end{pmatrix}$, $\tau_{13}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

TABLE 4. S_3-lines for the curve defined by equations (2)

6. Other examples

In this section, we present two examples of canonical curves of genus 4, which have exactly one C_6-line and exactly three C_6-lines, respectively.

EXAMPLE 6.1. Let $C \subset \mathbb{P}^3$ be the curve defined by

$$\begin{cases} Q := YZ - W^2 = 0, \\ F := X^3 - X^2Y - XY^2 + Z^3 = 0. \end{cases}$$

Then, C is a canonical curve of genus 4. The line $l : X = Y = 0$ is a C_6-line. Indeed,

$$\sigma := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & -\omega^2 \end{pmatrix}$$

(where ω is a primitive cubic root of the unity) satisfies $\sigma \in \mathrm{Aut}(C)$, $\pi_l \circ \sigma = \pi_l$, and $\mathrm{ord}(\sigma) = 6 = \deg \pi_l$. Because rank $Q = 3$, the trigonal morphism g_3^1 is unique, and g_3^1 is obtained by the projection π_R with center $R := (1 : 0 : 0 : 0)$, which is the vertex of $Q = 0$. Because $\pi_R^{-1}((1 : 1 : 1))$ consists of only two points $(1 : 1 : 1 : 1)$ and $(-1 : 1 : 1 : 1)$, we see that g_3^1 is not Galois. From Proposition 3.2, the number of C_6-lines equals one.

EXAMPLE 6.2. Let $C \subset \mathbb{P}^3$ be the curve defined by

$$\begin{cases} Q := XY - Z^2 = 0, \\ F := X^3 + Y^3 + Z^3 + W^3 = 0. \end{cases}$$

Then, C is a canonical curve of genus 4. The lines $l_1 : X + Y = Z = 0$, $l_2 : X + \omega Y = Z = 0$, and $l_3 : X + \omega^2 Y = Z = 0$ are C_6-lines of type $(2, 3)$. Indeed,

$$\sigma_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & \omega & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \ \text{and } \sigma_3 := \begin{pmatrix} 0 & \omega^2 & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

(where ω is a primitive cubic root of the unity) satisfy $\sigma_j \in \mathrm{Aut}(C)$, $\pi_{l_j} \circ \sigma_j = \pi_{l_j}$ and $\mathrm{ord}(\sigma_j) = 6 = \deg \pi_{l_j}$ $(j = 1, 2, 3)$.

We show that all the C_6-lines for C of types $(2,3)$ or $(2,6)$ are the three lines l_1, l_2 and l_3. Here, we explain how to find the C_6-lines of types $(2,3)$ or $(2,6)$. Let

$$P_1 := (1 : \zeta^2 : \zeta : 0), \qquad P_2 := (1 : \zeta^4 : \zeta^2 : 0),$$
$$P_3 := (1 : \zeta^8 : \zeta^4 : 0), \qquad P_4 := (1 : \zeta : \zeta^5 : 0),$$
$$P_5 := (1 : \zeta^5 : \zeta^7 : 0), \qquad P_6 := (1 : \zeta^7 : \zeta^8 : 0),$$

where ζ is a primitive ninth root of the unity. Because rank $Q = 3$, from Proposition 2.1, there exists a unique trigonal morphism $g^1_3 : C \to \mathbb{P}^1$. From Proposition 3.1 (or 3.2), g^1_3 is cyclic. Points $P_1, \ldots, P_6$ are all the ramification points of g^1_3. Let $H(3P_m + 3P_n) \subset \mathbb{P}^3$ (resp. $H(6P_m) \subset \mathbb{P}^3$) ($P_m, P_n \in \{P_1, \ldots, P_6\}$) be the hyperplane that defines the divisor $3P_m + 3P_n$ (resp. $6P_m$) on C. Let l be a C_6-line of type $(2,3)$ or $(2,6)$. From Proposition 3.1, the projection $\pi_l : C \to \mathbb{P}^1$ is the composition of $g^1_3 : C \to \mathbb{P}^1$ and some morphism $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2. Thus, $P_1, \ldots, P_6$ are ramification points of π_l. At least two fibers of π_l are formed as $3P_m + 3P_n$, where $P_m \neq P_n$ and $P_m, P_n \in \{P_1, \ldots, P_6\}$. In other words, there exist four mutually distinct points $P_{m_1}, P_{m_2}, P_{m_3}, P_{m_4} \in \{P_1, \ldots, P_6\}$ such that $H(3P_{m_1} + 3P_{m_2}) \cap H(3P_{m_3} + 3P_{m_4}) = l$. Moreover, we have $l \subset H(3P_{m_5} + 3P_{m_6})$ or $l \subset H(6P_{m_5}) \cap H(6P_{m_6})$, where $\{P_{m_1}, \ldots, P_{m_6}\} = \{P_1, \ldots, P_6\}$. By using this fact, we search for lines that might be C_6-lines of types $(2,3)$ or $(2,6)$.

For example, let $l_{1234} \subset \mathbb{P}^3$ be the line $H(3P_1 + 3P_2) \cap H(3P_3 + 3P_4)$. Because $H(3P_1 + 3P_2)$ and $H(3P_3 + 3P_4)$ are defined by $\zeta^3 X + Y - (\zeta + \zeta^2) Z = 0$ and $X + Y - (\zeta^4 + \zeta^5) Z = 0$, respectively, we have

$$R_{1234} := (-\zeta(1+\zeta) : \zeta(1+\zeta)(1+\zeta^3) : 1 : 0) \in l_{1234}.$$

The hyperplanes $H(3P_5 + 3P_6)$, $H(6P_5)$, and $H(6P_6)$ are defined by $\zeta^6 X + Y - (\zeta^7 + \zeta^8) Z = 0$, $\zeta^5 X + Y - 2\zeta^7 Z = 0$, and $\zeta^7 X + Y - 2\zeta^8 Z = 0$, respectively. We see that $R_{1234} \notin H(3P_5 + 3P_6)$, $R_{1234} \notin H(6P_5)$, and $R_{1234} \notin H(6P_6)$. Thus, $l_{1234} \not\subset H(3P_5 + 3P_6)$, $l_{1234} \not\subset H(6P_5)$, and $l_{1234} \not\subset H(6P_6)$. This concludes that l_{1234} is not a C_6-line of type $(2,3)$ or $(2,6)$. By using the same argument as above and computer calculations, we check whether $l_{m_1 m_2 m_3 m_4}$ can be a C_6-line of type $(2,3)$ or $(2,6)$ for every line $l_{m_1 m_2 m_3 m_4} := H(3P_{m_1} + 3P_{m_2}) \cap H(3P_{m_3} + 3P_{m_4})$. Then, we see that only three lines l_{1236}, l_{1423}, l_{1625} might be C_6-lines of types $(2,3)$ or $(2,6)$, which are C_6-lines l_3, l_2, l_1, respectively.

According to Sections 4 and 5, seven C_6-lines of types $(2,3)$ or $(2,6)$ exist for the curve defined by equations (1), and nine C_6-lines of types $(2,3)$ or $(2,6)$ exist for the curve defined by equations (2). Thus, C is not projectively equivalent to the curves defined by equations (1) or (2). From our main theorem, we see that all the C_6-lines for C are the three lines l_1, l_2, and l_3.

Acknowledgments – The authors wish to thank Professor Akira Ohbuchi for his valuable advice and wish to express their gratitude to Professor Satoru Fukasawa for his contribution in the form of Remark 4.3. The authors thank the reviewer for their helpful comments to improve the readability of this paper. The authors would like to thank Editage (www.editage.com) for English language editing.

Funding – This work was supported by JSPS KAKENHI Grant Numbers JP18K03228 and JP19K03441.

References

[1] E. Arbarello – M. Cornalba – P. A. Griffiths – J. Harris, *Geometry of algebraic curves. Vol. I*. Grundlehren Math. Wiss. 267, Springer, New York, 1985. Zbl 0559.14017 MR 770932

[2] M. C. L. Duyaguit – H. Yoshihara, Galois lines for normal elliptic space curves. *Algebra Colloq.* **12** (2005), no. 2, 205–212. Zbl 1076.14036 MR 2127245

[3] S. Fukasawa, Galois lines for the Artin–Schreier–Mumford curve. *Finite Fields Appl.* **75** (2021), article no. 101894. Zbl 1470.14061 MR 4281893

[4] S. Fukasawa – K. Higashine, Galois lines for the Giulietti–Korchmáros curve. *Finite Fields Appl.* **57** (2019), 268–275. Zbl 1420.14067 MR 3921290

[5] S. Fukasawa – P. Speziali, Plane curves possessing two outer Galois points. 2018, arXiv:1801.03198.

[6] P. Griffiths – J. Harris, *Principles of algebraic geometry*. Wiley Classics Lib., John Wiley & Sons, Inc., New York, 1994. Zbl 0836.14001 MR 1288523

[7] M. Kanazawa – H. Yoshihara, Galois lines for space elliptic curve with $j = 12^3$. *Beitr. Algebra Geom.* **59** (2018), no. 3, 431–444. Zbl 1396.14027 MR 3844636

[8] J. Komeda – T. Takahashi, Galois lines for a canonical curve of genus 4, I: non-skew cyclic lines. *Rend. Semin. Mat. Univ. Padova*, to appear.

[9] I. Kuribayashi – A. Kuribayashi, Automorphism groups of compact Riemann surfaces of genera three and four. *J. Pure Appl. Algebra* **65** (1990), no. 3, 277–292. Zbl 0709.30036 MR 1072285

[10] H. Yoshihara, Galois lines for space curves. *Algebra Colloq.* **13** (2006), no. 3, 455–469. Zbl 1095.14030 MR 2233104

[11] H. Yoshihara, Galois lines for normal elliptic space curves, II. *Algebra Colloq.* **19** (2012), Special Issue no. 1, 867–876. Zbl 1294.14014 MR 2999240

Manoscritto pervenuto in redazione il 2 febbraio 2022.