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Iterated monodromy groups of exponential maps

Bernhard Reinke

Abstract. This paper introduces iterated monodromy groups for transcendental functions and discusses them in the simplest setting, for post-singularly finite exponential functions. These groups are self-similar groups in a natural way, based on an explicit construction in terms of kneading sequences. We investigate the group theoretic properties of these groups, and show in particular that they are amenable, but they are not elementary subexponentially amenable.

1. Introduction

In the iteration theory of rational maps, iterated monodromy groups are self-similar groups associated with post-singularly finite dynamical systems. These groups encode the Julia set of a rational function from the point of view of symbolic dynamics [13]. Conversely, many classical examples of self-similar groups with exotic geometric properties, such as the Fabrykowski–Gupta [6] and the Basilica group [7], arise in a natural way as iterated monodromy groups of rational maps.

Much of the study of symbolic dynamics of quadratic polynomials has been done in terms of dynamic rays, as well as in terms of kneading sequences [4,12,22], before iterated monodromy groups were introduced as a new and powerful tool [1,13]. The relationships between these groups and kneading sequences were developed in [2]. This paper is a first in a series of papers that study iterated monodromy groups of entire functions. Here, we focus on a particularly fundamental class of functions, the exponential family, motivated by the well-known strong analogy between the combinatorics of quadratic polynomials and exponential maps (see, e.g., [3]). Like polynomials, exponential maps have so far only been studied in terms of rays and kneading sequences (see, e.g., [20]) resulting in a complete classification in [10], based on [8].

In this paper, we introduce iterated monodromy groups for exponential maps and compare them to self-similar groups defined just in terms of formal kneading sequences. For an exponential map f, we show that the iterated monodromy action of f is conjugate to the self-similar group action defined by the kneading sequence of f. For all kneading sequences, we show that the obtained group is a left-orderable amenable group that is residually solvable but not residually finite.

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We give a short background in holomorphic dynamics in Section 2, with a special focus on the exponential family. Next, in Section 3, we provide the algebraic and graph theoretic background to define the iterated monodromy group of a post-singularly finite entire function. We give an explicit description of the iterated monodromy group in terms of kneading automata in Section 4, see Theorem 4.6. The structure of the orbital Schreier graphs is investigated in Section 5, where we show in Theorem 5.5 that every component of the (reduced) orbital Schreier graph is a tree with countably many ends. This result together with the work in [16] is then used in Section 6, where we collect group theoretic properties of the iterated monodromy groups of exponential functions, in particular amenability (see Theorem 6.5).

This paper is based on the second chapter of the author's PhD thesis [17].

2. Dynamics of exponential maps

2.1. General entire dynamics

We give a very short introduction into transcendental dynamics relevant to our needs, see [19] for a survey. We start with the definition of a post-singularly finite entire function.

Definition 2.1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. A *critical value* is the image of a critical point, i.e., $f(c)$ where $f'(c) = 0$. An *asymptotic value* is a limit $\lim_{t\to\infty} f(\gamma(t))$ where $\gamma\colon [0,\infty) \to \mathbb{C}$ is a path with $\lim_{t\to\infty} \gamma(t) = \infty$. The set of singular values is defined as

$$\mathbf{S}(f) = \overline{\{\text{critical values}\}} \cup \overline{\{\text{asymptotic values}\}}$$

and the set of post-singular values is

$$\mathbf{P}(f) = \overline{\bigcup_{n\geq 0} f^n(\mathbf{S}(f))}.$$

The map f is called *post-singularly finite* if $\mathbf{P}(f)$ is finite.

The following lemma is the basis of our consideration.

Lemma 2.2 ([19, Theorem 1.13]). *Let f be an entire function. Then, f restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(\mathbf{S}(f))$ to $\mathbb{C} \setminus \mathbf{S}(f)$.*

In fact, an alternative definition of $\mathbf{S}(f)$ is that $\mathbf{S}(f)$ is the smallest closed subset S such that f restricts to an unbranched covering over $\mathbb{C} \setminus S$. As $\mathbf{P}(f)$ is closed and contains $\mathbf{S}(f)$, we see that f also restricts to an unbranched covering from $\mathbb{C} \setminus f^{-1}(\mathbf{P}(f))$ to $\mathbb{C} \setminus \mathbf{P}(f)$. As $\mathbf{P}(f)$ is forward invariant, we have that

$$\mathbf{P}(f) \subset f^{-1}(\mathbf{P}(f)) \subset f^{-2}(\mathbf{P}(f)) \subset \cdots$$

is an increasing chain of closed subsets. From this, we can show by induction that f^n restricts to an unbranched covering from $\mathbb{C} \setminus f^{-n}(\mathbf{P}(f)) \to \mathbb{C} \setminus \mathbf{P}(f)$, using the fact that compositions of coverings of manifolds are again coverings. The *escaping set* $\mathbf{I}(f)$ is the set of points that escape to infinity under the iteration of f, i.e.,

$$\mathbf{I}(f) = \left\{z\colon \lim_{n\to\infty} f^n(z) = \infty\right\}.$$

Definition 2.3. A *dynamic ray* is a maximal injective curve $\gamma\colon (0,\infty) \to \mathbf{I}(f)$ with $\gamma(t) \to \infty$ as $t \to \infty$. We say that γ *lands* at a if $\gamma(t) \to a$ for $t \to 0$.

We should note that this definition is not the precise standard definition given in [19], however, it is appropriate in the study of post-singularly finite exponential maps as done in [10]. We will only use dynamic rays for exponential maps. So, this is not an issue for us.

2.2. Combinatorics of exponential maps

The *exponential family* is the family of functions $E_\lambda(z) = \lambda \exp(z)$ for $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The only singular value of $\lambda \exp(z)$ is 0. It is the limiting value along the negative real axis. It is also an omitted value. For the exponential family, Lemma 2.2 specializes to the well-known fact that every function in the exponential family is a covering from $\mathbb{C}$ to $\mathbb{C}^*$. This is in fact a universal covering, and the group of deck transformations are given by translations of $2\pi i$. In the following, we will often consider collections that form a free orbit under translations with multiples of $2\pi i$. A prime example is the set of preimages $E_\lambda^{-1}(z)$ of any point $z \in \mathbb{C}^*$. As $\mathbf{S}(E_\lambda(z)) = \{0\}$, we have

$$\mathbf{P}(E_\lambda(z)) = \left\{E_\lambda^n(0)\colon n \geq 0\right\} = \left\{0, \lambda, E_\lambda(\lambda), \ldots\right\}.$$

In this section, f will always denote a post-singularly finite function in the exponential family. In this setting, 0 is a strictly preperiodic point, as it is an omitted value and has finite forward orbit. We denote the preperiod of 0 as k and the period of 0 as p, so

$$\mathbf{P}(f) = \left\{0, f(0), \ldots, f^{k+p-1}(0)\right\} \quad \text{with } f^{k+p}(0) = f^k(0).$$

The dynamics of post-singularly finite exponential maps can be studied via dynamic rays, as seen in the following theorem.

Theorem 2.4 ([20]). *Let $f(z) = \lambda \exp(z)$ be a post-singularly finite function in the exponential family. Then, there is a dynamic ray landing at 0 that is preperiodic.*

We collect some facts about dynamic rays of exponential maps that are all discussed in [10, 20].

Fact 2.5. (1) Two different dynamic rays do not intersect, but they might land at the same point.

(2) The preimage of a dynamic ray is a family of dynamic rays forming a free orbit under translations with multiples of $2\pi i$.

(3) If γ lands at a, then for every $b \in f^{-1}(a)$ there is a unique preimage component of γ landing at b.

(4) If γ lands at 0, then all preimage components separate the plane, the connected components of $\mathbb{C} \setminus f^{-1}(\gamma)$ also form a free orbit under translations with multiples of $2\pi i$.

Definition 2.6. A *ray spider* is a (disjoint) family $\mathbb{S} = (\gamma_a)_{a \in \mathbf{P}(f)}$ such that γ_a is a dynamic ray landing at a for each $a \in \mathbf{P}(f)$.

Remark 2.7. In this definition, we do not require any invariance properties. As we only consider spiders consisting of dynamic rays, the elements of a spider (also called *spider legs*) are always disjoint. Our notion of a ray spider is a special case of the general notion of spiders given in [20]. By Theorem 2.4, there exists a ray spider: if γ is a dynamic ray landing at 0, then $\gamma_{f^i(0)} = f^i(\gamma), 0 \leq i < k + p$ is a ray spider. This spider is not necessarily forward invariant, as it might happen that $f^k(\gamma) \neq f^{k+p}(\gamma)$ (the period of the rays may be a multiple of the period of the landing point). This is not an issue in our construction as we will consider the family of pullbacks of a given spider.

Definition 2.8. Let $\mathbb{S} = (\gamma_a)_{a \in \mathbf{P}(f)}$ be a ray spider. The *pullback* of $\mathbb{S}$ is the ray spider $(\tilde{\gamma}_a)$ where $\tilde{\gamma}_a$ is the unique preimage of $\gamma_{f(a)}$ landing at a. The *dynamical partition* associated to $\mathbb{S}$ is the partition of $\mathbb{C} \setminus f^{-1}(\gamma_0)$ into its connected components. We denote the connected component of 0 by $\mathbb{U}_0$ and define $\mathbb{U}_n = \mathbb{U}_0 + 2\pi i n$. Note that the dynamical partition only depends on the ray landing at 0. The kneading sequence of f is the sequence $(k_n)_{n \in \mathbb{N}}$ so that $f^n(0) \in \mathbb{U}_{k_n}$. The kneading sequence is in fact independent of $\mathbb{S}$, see [10] for a more detailed discussion.

Example 2.9. Let $k \in \mathbb{Z} \setminus \{0\}$, and consider $f(z) = 2k\pi i \exp(z)$. For this map, 0 is mapped to $2k\pi i$, which is a fixed point of f. Hence, f is post-singularly finite with $\mathbf{P}(f) = \{0, 2k\pi i\}$. Let γ_0 be a dynamic ray landing at 0, and let $\mathbb{U}$ be the associated dynamical partition. Then, $0 \in \mathbb{U}_0$ by definition of $\mathbb{U}_0$ and $2k\pi i \in \mathbb{U}_k = \mathbb{U}_0 + 2k\pi i$, so the kneading sequence of f is $0\bar{k}$.

3. Iterated monodromy groups

3.1. The dynamical preimage tree $\mathcal{T}$

Let f be a post-singularly finite entire function and $t \in \mathbb{C} \setminus \mathbf{P}(f)$.

Definition 3.1. Choose a base point $t \in \mathbb{C} \setminus \mathbf{P}(f)$. Let $L_n := f^{-n}(t)$ be the preimage of t under the nth iterate of f. The *dynamical preimage tree* $\mathcal{T}$ is a rooted tree with vertex set

$$\bigsqcup_{n \geq 0} L_n,$$

where $\bigsqcup$ denotes disjoint union and edges $(f(w), w)$ for $w \in L_{n+1}, f(w) \in L_n$. Its root is t.

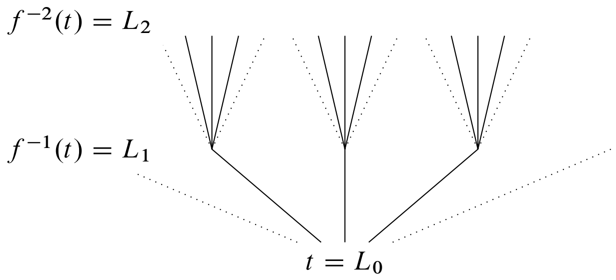


Figure 3.1. Dynamical preimage tree.

The dynamical preimage tree is always a regular rooted tree, i.e., all vertices have the same number of children. For polynomials, this number is the degree of the polynomial. For transcendental entire functions, every vertex has countably infinite many children (see Figure 3.1). We will show in Section 3.3 that for postsingularly finite exponential maps, the dynamical preimage tree has an extra regularity based on the periodicity of the exponential map.

3.2. Iterated monodromy action

Each level of $\mathcal{T}$ is the preimage of t under a covering map, namely, $f^n : \mathbb{C} \setminus f^{-n}(\mathbf{P}(f)) \to \mathbb{C} \setminus \mathbf{P}(f)$. Hence, $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ acts on L_n via path lifting, if $\gamma : [0,1] \to \mathbb{C} \setminus \mathbf{P}(f)$ is a loop based on t and $v \in L_n$ is a nth preimage of t, then there is a unique lift γ^v making the following diagram commute:

$$\begin{array}{ccc} & & (\mathbb{C} \setminus f^{-n}(\mathbf{P}(f)), v) \\ & \overset{\gamma^v}{\nearrow} & \big\downarrow f^n \\ ([0,1], 0) & \xrightarrow{\ \gamma\ } & (\mathbb{C} \setminus \mathbf{P}(f), t) \end{array}$$

So, $\gamma^v(0) = v$, and $\gamma^v(1) \in L_n$ might be another nth preimage. We define $[\gamma](v) := \gamma^v(1)$. Using the homotopy lifting properties of coverings, we can see that this defines an action of $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ on L_n. If $w \in L_{n+1}$ is a child of v, then the following diagram commutes (by uniqueness of lifts):

$$\begin{array}{ccc} & & (\mathbb{C} \setminus f^{-n-1}(\mathbf{P}(f)), w) \\ & \overset{\gamma^w}{\nearrow} & \big\downarrow f^n \\ & & (\mathbb{C} \setminus f^{-n}(\mathbf{P}(f)), v) \\ & \overset{\gamma^v}{\nearrow} & \big\downarrow f^n \\ ([0,1], 0) & \xrightarrow{\ \gamma\ } & (\mathbb{C} \setminus \mathbf{P}(f), t) \end{array}$$

By commutativity of the diagram $f(\gamma^w(1)) = \gamma^v(1)$. So, $[\gamma](w)$ is also a child of $[\gamma](v)$. This means that actions on the levels are compatible and give rise to an action of $\pi_1(\mathbb{C} \setminus \mathbf{P}(f))$ on $\mathcal{T}$. This is the *iterated monodromy action*.

Definition 3.2. Let f be a post-singularly finite entire function, $t \in \mathbb{C} \setminus \mathbf{P}(f)$. Let ϕ: $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t) \to \operatorname{Aut}(\mathcal{T})$ be the group homomorphism induced by the iterated monodromy action. The iterated monodromy group of f with base point t is the image of ϕ. By the first factor theorem, we have

$$\operatorname{IMG}(f) \cong {}^{\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)}\!\big/_{\ker \phi}.$$

This definition depends a priori on the base point $t \in \mathbb{C} \setminus \mathbf{P}(f)$. For a different base point t', every path from t to t' gives rise to an isomorphism of preimage trees over t and over t', so we can identify the groups up to inner automorphisms. See [13, Proposition 5.1.2] for a detailed discussion in the rational case.

3.3. $\mathbb{Z}$-regular rooted trees

We use the following definition of rooted trees.

Definition 3.3. A *rooted tree* is a tuple $T = (V, E, r)$ such that (V, E) forms a tree (with vertex set V and edge set E) and $r \in V$, which we call the *root* of T. We endow T with the unique orientation so that all vertices are reachable from the root, i.e., for every vertex v, there is directed path from the root to v. If (v, w) is a directed edge for this orientation, we say that w is a *child* of v and v is the *parent* of w. If v has no children, we call it a *leaf*. If w is reachable form v, we say that w is a descendant of v and v is an ancestor of w. We denote by T_v the rooted tree that is the induced subgraph on the set of descendants of v together with v as the new root. An end of a rooted tree T is a sequence v_n so that v_0 is the root of T and v_{n+1} is a child of v_n. We denote by ∂T the set of ends of T.

We will mainly consider countably infinite trees without leaves. In fact, ∂T can be defined without fixing a root of T: one way is by considering equivalence classes of geodesic rays, where two geodesic rays are equivalent if they have a common tail. Given a root r and a geodesic ray γ, there is always a unique geodesic ray starting at r equivalent to γ. Also, ∂T is a totally disconnected Hausdorff space with clopen subset $\partial T_v \subset \partial T$. The topology is also independent of the root. If T is a locally finite tree without leaves, then ∂T is compact.

Definition 3.4. A *$\mathbb{Z}$-regular rooted tree* T is a tuple (V, E, r, η), where (V, E, r) is a rooted tree and η is a right $\mathbb{Z}$-action η: $V \times \mathbb{Z} \to V$ such that for all vertices $v \in V$, the set of its children forms a free orbit under the action. Note that this implies that the root is fixed by the action, as it is the only vertex without a parent. Also, the tree has no leaves, as the empty set is not a free orbit under a $\mathbb{Z}$-action. An isomorphism between $\mathbb{Z}$-regular rooted trees is a tree isomorphism that preserves the root and commutes with the additional right $\mathbb{Z}$-actions. We denote by $\operatorname{Aut}_{\mathbb{Z}}(T)$ the group of automorphisms of T as a $\mathbb{Z}$-regular rooted tree. Every element of $\mathrm{Aut}_{\mathbb{Z}}(T)$ preserves the root of T and acts by a translation on the first level. We denote by $\rho\colon \mathrm{Aut}_{\mathbb{Z}}(T) \to \mathbb{Z}$ the group homomorphism given by the first level action. The kernel of ρ is the stabilizer of the first level, as every element of $\mathrm{Aut}_{\mathbb{Z}}(T)$ acts by translation, this is also the stabilizer of any vertex on the first level. For a vertex $v \in V$ and a subgroup $G \subset \mathrm{Aut}_{\mathbb{Z}}(T)$ we denote the stabilizer of v in G by $\mathrm{Stab}_G(v)$. We denote the stabilizer of the first level as Stab_G.

Note that $\mathrm{Aut}_{\mathbb{Z}}(T)$ also acts on ∂T. This action is in fact faithful, as every vertex is part of a sequence defining an end.

Example 3.5. The standard $\mathbb{Z}$-regular tree has as vertex set $\mathbb{Z}^*$, the set of finite words in $\mathbb{Z}$. Its root is the empty word $\emptyset$. Its edges are all pairs of the form (v, vn) for $v \in \mathbb{Z}^*, n \in \mathbb{Z}$ (here vn denotes the word v concatenated with the letter n). So, for each vertex v, the set of its children are all words obtained by concatenating one letter to it. Also, the set of ends can be identified with the set of right-infinite words, which we denote by $\mathbb{Z}^\omega$. The right action is given by

$$\eta(vn, m) = v(n + m).$$

So, the action is by translation on the last letter. By abuse of notation, we will denote the standard $\mathbb{Z}$-regular tree also by $\mathbb{Z}^*$.

The subgroups of $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ were studied in [14] under the name of *ZC-groups*. Note that if T is a $\mathbb{Z}$-regular rooted tree and v is a vertex of T, then T_v is also a $\mathbb{Z}$-regular rooted tree. However, in general, we have no canonical choice of an isomorphism between T and T_v. This is different for the standard $\mathbb{Z}$-regular tree.

Definition 3.6. For $g \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$, $v \in \mathbb{Z}^*$, let $g_{|v}$ denote the unique element in $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ such that

$$g(vw) = g(v) g_{|v}(w).$$

We say that $g_{|v}$ is the *section* of g at v.

We will use the following set of easily verifiable cocycle equations:

$$(g_{|v})_{|w} = g_{|vw}, \tag{3.1}$$

$$(gh)_{|v} = g_{|h(v)} h_{|v}. \tag{3.2}$$

We say that $g \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is of finite activity on level n if the set $\{v \in \mathbb{Z}^n \colon g_{|v} \neq \mathbf{1}\}$ is finite. We define $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ as the group of automorphisms that have finite activity on every level. We will mainly work with subgroups of $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$. As we work with an infinite alphabet, we have to distinguish direct products and direct sums for the wreath recursion. The wreath recursion for $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is

$$\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*) \cong \left(\prod_{x \in \mathbb{Z}} \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^*) \right) \rtimes \mathbb{Z},$$

$$g \mapsto (x \mapsto g_{|x}, \rho(g)).$$

Since an automorphism in $\operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ has only finitely many nontrivial sections on the first level, the wreath recursion for $\operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is given by

$$\operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*) \cong \left(\bigoplus_{x\in\mathbb{Z}} \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*)\right) \rtimes \mathbb{Z},$$
$$g \mapsto (x \mapsto g_{|x}, \rho(g)).$$

So, we take the direct sum for $\operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ in the wreath recursion instead of the direct product as for $\operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$. For $g \in \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*), n \in \mathbb{Z}$, we denote by $g@n \in \operatorname{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ the automorphism that acts trivially on the first level and satisfies

$$(g@n)_{|n} = g, \quad (g@n)_{|m} = \mathbf{1} \quad \text{for } m \neq n.$$

We say a subgroup $G \subset \operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is self-similar if $g_{|v} \in G$ for all $g \in G$ and $v \in \mathbb{Z}^*$. A subgroup $G \subset \operatorname{Aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ is self-replicating if its action on every level is transitive and for all $v \in \mathbb{Z}^*$ and $g \in G$ there exists an $h \in \operatorname{Stab}_G(v)$ with $h_{|v} = g$. It is easy to see that is enough to check this on the first level.

Lemma 3.7. *Let f be a post-singularly finite exponential function, $t \in \mathbb{C} \setminus \mathbf{P}(f)$. Then, the dynamical preimage tree of f with base point t is a $\mathbb{Z}$-regular tree and* $\operatorname{IMG}(f)$ *is a subgroup of* $\operatorname{Aut}_{\mathbb{Z}}(\mathcal{T})$.

Proof. The $\mathbb{Z}$-regular structure is given by translation by multiples of $2\pi i$. As two complex numbers have the same value under the exponential map if and only if they differ by a multiple of $2\pi i$, it is clear that this really defines a $\mathbb{Z}$-regular structure. Also, if w is an nth preimage of t, and γ is a loop on t, for the lift γ^w, the $2\pi i$ translate of γ^w is also a lift of γ by the $2\pi i$ periodicity of f^n. This shows that the iterated monodromy action commutes which the $\mathbb{Z}$ action given by the $\mathbb{Z}$-regular structure, so $\operatorname{IMG}(f) \subset \operatorname{Aut}_{\mathbb{Z}}(\mathcal{T})$. ∎

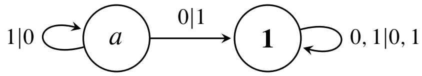
4. Combinatorial description

4.1. Automata

Definition 4.1. An automaton **A** is a map $\tau\colon Q \times X \to X \times Q$. We call Q the *state set* and X the *alphabet*. We will write the components of $\tau(a,x)$ often as $(a(x), a|x)$. Here, $a(x) \in X$ is called the image of x under a, and $a|x$ is the restriction of a at x. A *group automaton* is an automaton such that for all $a \in Q$, the map $x \mapsto a(x)$ is a bijection on Q. If the alphabet is $\mathbb{Z}$, that automaton is a $\mathbb{Z}$-automaton if for all $a \in Q$, the map $x \mapsto a(x)$ is a translation on $\mathbb{Z}$, i.e., equal to the map $x \mapsto x + n$ for some $n \in \mathbb{Z}$.

We will only consider automata which have a distinguished identity state $\mathbf{1}$, i.e., a state such that $\tau(\mathbf{1}, x) = (x, \mathbf{1})$ for all $x \in X$. We can draw automata using Moore diagrams. As vertices we take the state set Q, and if $\tau(a,x) = (y,b)$, we draw an edge from a to

b labeled $x|y$. Here is an example of a Moore diagram of the so-called binary adding machine:



Definition 4.2. Let $\mathbf{A}$ be an automaton given by $\tau\colon Q \times X \to X \times Q$. We extend τ to a map $Q \times X^* \to X^* \times Q$ recursively via

$$\tau(a, xv) = (a(x)a|x(v), a|x|v).$$

If $\mathbf{A}$ is a group automaton, then for each $a \in A$, the extended map $X^* \to X^*$ induces a tree automorphism of the regular X-tree. If $\mathbf{A}$ is a $\mathbb{Z}$-automaton, it is an automorphism preserving the regular $\mathbb{Z}$-tree structure.

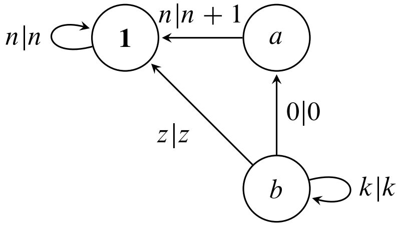
4.2. Kneading automata

Definition 4.3. Given two words $x_1, \dots, x_k, y_1, \dots, y_p \in \mathbb{Z}^*$ with $x_k \neq y_p$, the automaton $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ has alphabet $\mathbb{Z}$ and states $a_1, \dots, a_k, b_1, \dots, b_p$ (and the identity state $\mathbf{1}$) and the following transitions:

$$\begin{aligned} \tau(a_1, z) &= (z + 1, \mathbf{1}), \\ \tau(a_{i+1}, x_i) &= (x_i, a_i), \\ \tau(b_1, x_k) &= (x_k, a_k), \\ \tau(b_1, y_p) &= (y_p, b_p), \\ \tau(b_{i+1}, y_i) &= (y_i, b_i), \\ \tau(q, z) &= (z, \mathbf{1}) \quad \text{for all other cases.} \end{aligned}$$

We note that $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ is a $\mathbb{Z}$-automaton; indeed, a_1 acts on $\mathbb{Z}$ by the translation by one, and all other states act on $\mathbb{Z}$ as the identity. Figure 4.1 shows a reduced Moore diagram of $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$, where labels with only one letter z are abbreviations for the label $z|z$ and all trivial arrows ending in the identity state have been omitted.

Example 4.4. The automaton $\mathbf{K}(0, k)$ with $k \in \mathbb{Z} \setminus \{0\}$ has the following (non-reduced) Moore diagram:



Here, n stands for any element of $\mathbb{Z}$, and z for any element of $\mathbb{Z} \setminus \{0, k\}$.

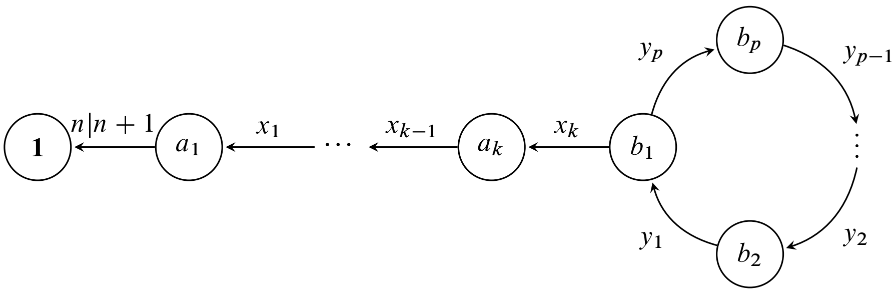


Figure 4.1. Moore diagram of kneading automata.

Remark 4.5. We see that every non-trivial state has exactly one edge ending in it, so for every non-trivial state, there is a unique left-infinite path ending in it. This implies that $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ is a bounded activity automaton in the sense of [21]; for any length m, there are $k + p$ paths of length m ending in a non-trivial state in the Moore diagram, so for any q, the set $\{v \in \mathbb{Z}^m : q|v \neq \mathbf{1}\}$ has cardinality bounded by $k + p$.

We denote by $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ the group of automorphisms of $\mathbb{Z}^*$ generated by $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$.

Theorem 4.6. *Let f be a post-singularly finite exponential function with kneading sequence $x_1, \dots, x_k \overline{y_1, \dots, y_p} \in \mathbb{Z}^\omega$. Then, the iterated monodromy action of f is conjugate to the action of $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ on $\mathbb{Z}^*$.*

In particular, for functions of the form $2\pi i k \exp(z)$ with $k \in \mathbb{Z} \setminus \{0\}$, the iterated monodromy action is conjugate to the action of the automata group $\mathcal{K}(0, k)$ discussed in Example 4.4.

Proof. We choose a ray spider $\mathbb{S}_0$ for f and consider the sequence $\mathbb{S}_n$, where $\mathbb{S}_{n+1}$ is the pullback of $\mathbb{S}_n$. We denote by $\gamma_{z,n}$ the ray in $\mathbb{S}_n$ landing at z; also, let $\mathbb{U}_{*,n}$ be the dynamical partition induced by $\mathbb{S}_n$. Choose a base point $t \in \mathbb{C} \setminus \bigcup_{z \in \mathbf{P}(f)} \gamma_{z,0}$. We recursively define an isomorphism $\Phi : \mathcal{T} \to \mathbb{Z}^*$ from the dynamical preimage tree $\mathcal{T}$ to the standard $\mathbb{Z}$-tree $\mathbb{Z}^*$. We send the root t to the empty word $\emptyset$. We denote the inverse of Φ by Ψ. Suppose we already defined Φ on $L_n \subset \mathcal{T}$, and let $w \in L_n$ be mapped to $v \in \mathbb{Z}^n$. Then, for the dynamical partition $\mathbb{U}_{*,n}$, there is exactly one child of w in each component. We send the child lying in $\mathbb{U}_{m,n}$ to vm. As t is disjoint to $\mathbb{S}_0$, we see that $\Psi(w) \notin \mathbb{C} \setminus \bigcup_{z \in \mathbf{P}(f)} \gamma_{z,n}$ for $w \in \mathbb{Z}^n$. By construction, this defines an isomorphism of $\mathbb{Z}$-trees. The complement of each ray spider is a simply connected domain. For words $w_1, w_2 \in \mathbb{Z}^n$, $g_n(w_1, w_2)$ be a path from $\Psi(w_1)$ to $\Psi(w_2)$ crossing no ray of $\mathbb{S}_n$ and for $z \in \mathbf{P}(z)$, let $g_{z,n}(w_1, w_2)$ be a path from w_1 to w_2 crossing only the ray of $\gamma_{z,n}$ once in a positive sense (so that $g_n(w_1, w_2)$ composed with $g_{z,n}(w_1, w_2)$ has winding number 1 around z) and no other ray of $\mathbb{S}_n$. We let $g_{z,n}(w_1, w_2)$ denote $g_n(w_1, w_2)$ for $z \in f^{-1}(\mathbf{P}(z)) \setminus \mathbf{P}(z)$. The homotopy classes of g_n and $g_{z,n}$ are well defined in the fundamental groupoid $\Pi_1(\mathbb{C} \setminus \mathbf{P}(f))$. For $m \in \mathbb{Z}$, we denote by $g^m_{z,n}(v_1, v_2)$ the lift of $g_{z,n}(v_1, v_2)$ starting at $\Psi(v_1 m)$, similarly for $g^m_n(v_1, v_2)$. We will show the following lifting properties, where $v, v' \in \mathbb{Z}^n$, $m \in \mathbb{Z}$, $z \in \mathbf{P}(z) \setminus 0$ and $\bar{z}$ is the unique preimage of z in $\mathbb{U}_{m,n}$:

$$g^m_n(v, v') \cong g_{n+1}(vm, v'm), \tag{4.1}$$
$$g^m_{0,n}(v, v') \cong g_{n+1}(vm, v'(m+1)), \tag{4.2}$$
$$g^m_{z,n}(v, v') \cong g_{\bar{z},n+1}(vm, v'm). \tag{4.3}$$

$g^m_n(v, v') \cong g_{n+1}(vm, v'm)$: the lift $g^m_n(v, v')$ is a path in $\mathbb{C}$ meeting no preimage of $\gamma_{z,n}$ for $z \in \mathbf{P}(z)$. So, it meets no element of $\mathbb{S}_{n+1}$ nor a boundary of the components $\mathbb{U}_{*,n}$. As $g^m_n(v, v')$ starts in $\Psi(vm)$, it must end in $\Psi(v'm)$. So, $g^m_n(v, v')$ must be homotopic to $g^m_{n+1}(vm, v'm)$.

$g^m_{0,n}(v, v') \cong g_{n+1}(vm, v'(m+1))$: we see similarly that $g^m_{0,n}(v, v)$ is a path that does not cross any ray of $\mathbb{S}_{n+1}$, and as $g_{0,n}(v, v)$ has winding number 1 around 0, the lift $g^v_{0,n}(v, v)$ must end in $\Psi(vm) + 2\pi i = \Psi(v(m+1))$. Hence,

$$g^v_{0,n}(v, v) \cong g_{n+1}(vm, v(m+1)),$$

and by composition,

$$g^v_{0,n}(v, v') \cong g_{n+1}(vm, v'(m+1)).$$

$g^m_{z,n}(v, v') \cong g_{\bar{z},n+1}(vm, v'm)$: let $z \in \mathbf{P}(f) \setminus 0$. Then, $g^m_{z,n}(v, v')$ crosses no boundary of $\mathbb{U}_{*,n}$, so it must end in $\Psi(v'm)$. Also, $g^m_{z,n}(v, v')$ crosses the preimage of $\gamma_{z,n}$ landing at $\bar{z}$. So,

$$g^v_{z,n}(v, v') \cong g_{\tilde{z},n+1}(vm, v'm),$$

with the understanding that

$$g_{\tilde{z},n+1}(vm, v'm) \cong g_{n+1}(vm, v'm)$$

for $\bar{z} \in z \in f^{-1}(\mathbf{P}(z)) \setminus \mathbf{P}(z)$.

Now, $\pi_1(\mathbb{C} \setminus \mathbf{P}(f), t)$ is freely generated by $(g_{z,0}(\emptyset, \emptyset)_{z \in \mathbf{P}(f)})$. Numerate $\mathbf{P}(f)$ by $z_1 = 0, z_{i+1} = f(z_i), 1 \le i \le k + p - 1$. We claim that the group homomorphism given by

$$\begin{aligned} g_{z_i,0}(\emptyset, \emptyset) &\mapsto a_i, & 1 \le i \le k \\ g_{z_i,0}(\emptyset, \emptyset) &\mapsto b_{i-k}, & k+1 \le i \le k+p \end{aligned}$$

conjugates the iterated monodromy action of f to the action of the group $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$. From the pullback behavior given by (4.1)–(4.3), we can iteratively determine the action of $g_{z_i,0}(\emptyset, \emptyset)$ on every level. By definition of the kneading sequence, we have that $z_i \in \mathbb{U}_{m,n}$ if the ith entry of the kneading sequence is m, independently of n. From this, we see that the recursion given by (4.1)–(4.3) agrees with the recursion for $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$, so their actions are conjugated. ∎

5. Schreier graphs

For this section, we fix $x_1, \dots, x_k, y_1, \dots, y_p$ with $x_k \neq y_p$. We will give a combinatorial description of the action of $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ on the standard $\mathbb{Z}$-tree $\mathbb{Z}^*$. We will work in this section with the generating set $S := \{a_1, \dots, a_k, b_1, \dots, b_p\}$ of $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$.

Definition 5.1. Let $n \in \mathbb{N}$. The nth level *Schreier graph* has vertex set $\mathbb{Z}^n$ and edges $v \to s(v)$ for $v \in \mathbb{Z}^n, s \in S$. The *orbital Schreier graph* Γ_ω has the ends of the standard $\mathbb{Z}$-tree as vertex set (which can be identified with $\mathbb{Z}^\omega$) and also has edges $v \to s(v)$ for $v \in \mathbb{Z}^\omega, s \in S$.

The *reduced Schreier graph* $\overline{\Gamma}_n$ and reduced orbital Schreier graph $\overline{\Gamma}_\omega$ are obtained by deleting all loops of Γ_n, respectively, Γ_ω.

Let $w_m \in \mathbb{Z}^m$ be the reverse of the length m prefix of $x_1, \dots, x_k \overline{y_1, \dots, y_p}$. In the Moore diagram in Figure 4.1, we see that w_m is the concatenation of the labels of the unique path p of length m ending in a_1. Let c_m be the starting state of p (so $c_m = a_m$ for $m \leq k$, and $c_m = b_{m'}$ for appropriate m' otherwise). Then, $c_m|w_m = a_1$ and $s|v \neq a_1$ for all other pairs of a state s and $v \in \mathbb{Z}^m$. As a_1 is the only state which acts non-trivially on the first level, we have

$$\begin{aligned} c_m|w_m(i) &= i + 1, \\ s|v(i) &= i \qquad \text{for other pairs.} \end{aligned}$$

Since additionally a_1 only restricts to the identity state, we also have that if $s(v) = w$ with $v \neq w \in \mathbb{Z}^m$ for some state s, then $s(vi) = wi$ for all $i \in \mathbb{Z}$. In fact, v and w must differ in exactly one position. This discussion can be summarized in the following lemma.

Lemma 5.2. *The Schreier graph $\overline{\Gamma}_{m+1}$ can be obtained from $\overline{\Gamma}_m$ in the following way: take as vertex set vx, where $v \in \mathbb{Z}^m, x \in \mathbb{Z}$. For edges, we have the following two construction rules:*

- $v \to v'$ *edge in* $\overline{\Gamma}_m \Rightarrow vi \to v'i$ *is an edge in* $\overline{\Gamma}_{m+1}$ *for all* $i \in \mathbb{Z}$,
- $w_m i \to w_m(i+1)$ *for all* $i \in \mathbb{Z}$.

See Figure 5.1 for a visualization of the construction rules.

Example 5.3. We can use this construction to produce the first few $\overline{\Gamma}_m$ for the group $\mathcal{K}(0, 1)$. As in Example 4.4, we name the generators a and b instead of a_1 and b_1. Note that a acts by translation on the first level, and b acts trivially on the first level, so $\overline{\Gamma}_1$ is just a bi-infinite line. To use the construction rule, we note that $w_1 = 0$, so we obtain $\overline{\Gamma}_2$ as a comb in Figure 5.2. The loops at $1, 0$ and $1, 1$ are of course not present in the reduced Schreier graph, but we did include them here for they are the loops which “split up” in the further generations: as b restricts to a at $1, 0$, we obtain $\overline{\Gamma}_3$ by connecting $\mathbb{Z}$ many copies of $\overline{\Gamma}_2$ by a bi-infinite line going through the copies of $1, 0$.

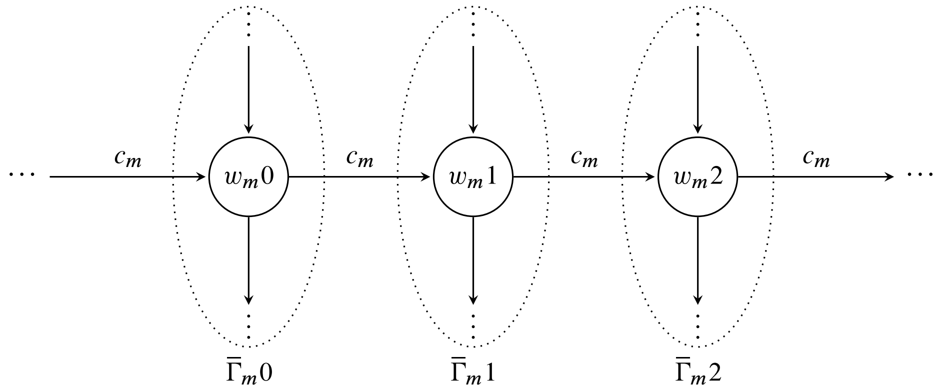


Figure 5.1. Inductive construction of Schreier graphs.

With this inductive description, we can prove the following lemma.

Lemma 5.4. *For all $m \in \mathbb{N}$, the reduced Schreier graph $\overline{\Gamma}_m$ is a tree with countably (or finitely) many ends.*

Proof. We do induction over m. For $m = 1$, the Schreier graph $\overline{\Gamma}_1$ is a bi-infinite line, so it is in particular a tree with finitely many ends. Now, by Lemma 5.2, $\overline{\Gamma}_{m+1}$ is the union of countably many copies of $\overline{\Gamma}_m$ and a bi-infinite line intersecting each copy in one point. So, it is again a tree. We claim that they have the following inductive description of the space of ends:

$$\partial\overline{\Gamma}_{m+1} \cong \mathbb{Z} \times \partial\overline{\Gamma}_m \cup \{-\infty, +\infty\}. \tag{5.1}$$

Here, the right-hand space is a compactification of $\mathbb{Z} \times \partial\overline{\Gamma}_m$, where $-\infty$ has the open sets

$$U_{<n} := \{z \in \mathbb{Z} : z < n\} \times \partial\overline{\Gamma}_m \cup \{-\infty\}$$

as neighborhood basis, and similarly, $+\infty$ has the open sets

$$U_{>n} := \{z \in \mathbb{Z} : z > n\} \times \partial\overline{\Gamma}_m \cup \{+\infty\}$$

as neighborhood basis. The identification in (5.1) works as follows: we take w_m as our root of $\overline{\Gamma}_m$ and w_m0 as the root of $\partial\overline{\Gamma}_{m+1}$. Then, we have the following identifications.

- We send $-\infty$ to the end $(w_m(-i))_{i\in\mathbb{N}}$; i.e., we walk the bi-infinite line in the negative direction.
- We send $+\infty$ to the end $(w_m(+i))_{i\in\mathbb{N}}$; i.e., we walk the bi-infinite line in the positive direction.
- Given a pair $(z, v) \in \mathbb{Z} \times \partial\overline{\Gamma}_m$, we identify it with the end which is given by the concatenation of the path from w_m0 to w_mz together with the sequence v_nm. This means that first walk to the root of the copy of $\overline{\Gamma}_m$ labeled by z and then go to the end defined by the sequence v in this copy.

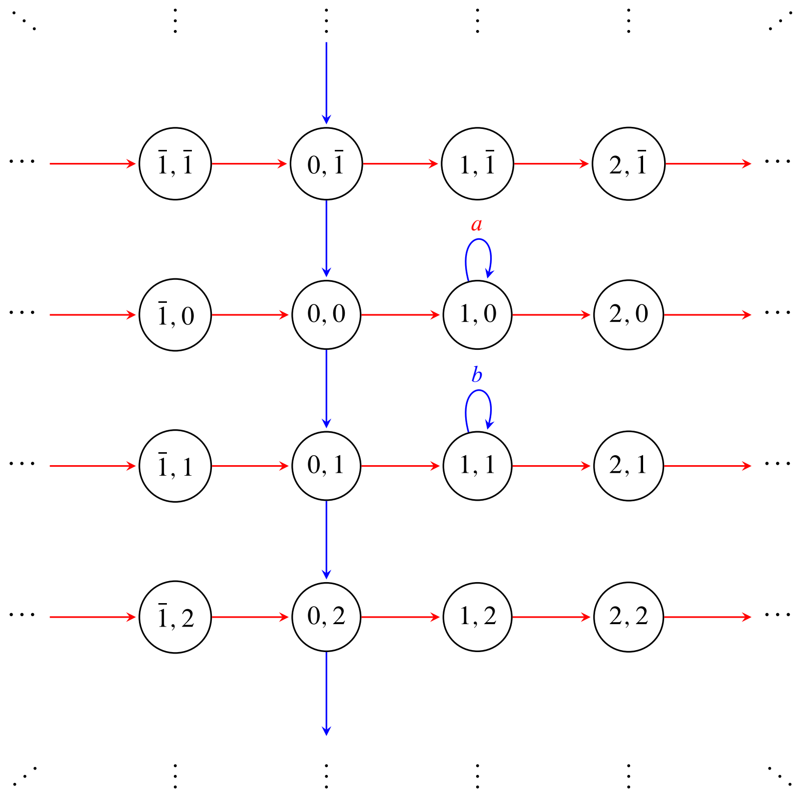


Figure 5.2. Second level Schreier graph of $\mathcal{K}(0, 1)$ (where $\bar{1} = -1$ for notational convenience).

Using Lemma 5.2, it is easy to check that this indeed defines a homeomorphism as given in (5.1). Now, $\mathbb{Z} \times \partial\overline{\Gamma}_m \cup \{-\infty, +\infty\}$ is a countable union of countable sets, so $\partial\overline{\Gamma}_{m+1}$ is countable. ∎

Let us fix some notation related to the orbital Schreier graph $\overline{\Gamma}_\omega$. For $u \in \mathbb{Z}^\omega$, let $T_m(u)$ be the induced subgraph of $\overline{\Gamma}_\omega$ on the set

$$\{u' \in \mathbb{Z}^\omega : u_i = u'_i \text{ for all } i > m\}.$$

We denote the union $\bigcup_{m\in\mathbb{N}} T_m(u)$ by $T(u)$.

Theorem 5.5. *The connected component of u in $\overline{\Gamma}_\omega$ is $T(u)$. It is a tree with countably many ends.*

Proof. The projection to the prefix of length m is a bijection from the vertex set of $T_m(u)$ to $\mathbb{Z}^m$. It gives rise to graph isomorphism from $T_m(u)$ to $\overline{\Gamma}_m$, as the generating set acts by changing at most one letter at once. So, $T(u)$ is an increasing union of trees; hence, it is also a tree. As there are no edges in $\overline{\Gamma}_\omega$ leaving $T(u)$, we have that $T(u)$ is indeed the connected component of u in $\overline{\Gamma}_\omega$. Each end of $T(u)$ either stays in some $T_m(u)$ or leaves all $T_m(u)$. The first kind is a countable union of countable sets; hence, we only need to consider ends leaving all $T_m(u)$. Let E_m be the set of edges in $T(u)$ leaving $T_m(u)$. We have a map $E_m \to E_{m-1}$ which sends an edge e leaving $T_m(u)$ to the unique edge leaving $T_{m-1}(u)$ on the geodesic from u to e. It is possible that an edge is sent to itself, if it leaves multiple subtrees at once. Now, the set of ends leaving all $T_m(u)$ is isomorphic to $\varprojlim E_m$. So, the sets E_m have uniform bounded cardinality. This can be seen as follows: let w be the m-suffix of u. Then, an edge in E_m corresponds to a pair $v \in \mathbb{Z}^m, q \in S \cup S^{-1}$ with $q|v(w) \neq w$, in particular, the restriction $q|v$ is not trivial. But $\mathbf{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ is a bounded activity automaton, so the number of pairs $(v, q) \in \mathbb{Z}^m \times (S \cup S^{-1})$ with $q|v \neq \mathbf{1}$ is uniformly bounded, and so, are the sets E_m. Hence, the inverse limit has finite cardinality, so in total, we have countably many ends. ∎

6. Group theoretic properties

The groups $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$ are examples of ZC-groups defined as [14]. In particular, they are left-orderable residually solvable groups. In this section, we will always work with a fixed pair of sequences $x_1, \dots, x_k$, $y_1, \dots, y_p$ and we will just write $\mathcal{K}$ instead of $\mathcal{K}(x_1, \dots, x_k, y_1, \dots, y_p)$. We still use $S := \{a_1, \dots, a_k, b_1, \dots, b_p\}$ as our generating set.

Lemma 6.1. *The abelianization of $\mathcal{K}$ is the free abelian group on S.*

Proof. We have a family of group homomorphisms:

$$\bar{\rho}_n \colon \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*) \to \mathbb{Z},$$
$$g \mapsto \sum_{v \in \mathbb{Z}^n} \rho(g_{|v}).$$

Note that the sum is defined as $g_{|v}$ is trivial for almost all v, so almost all summands are 0. By the cocycle equations (3.2) and (3.1), we see that $\bar{\rho}_n$ is indeed a group homomorphism, and for all $g \in \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$, we have $\bar{\rho}_{n+1}(g) = \sum_{v \in \mathbb{Z}} \bar{\rho}_n(g_{|v})$. The transitions given in the Definition 4.3 translate to

$$\begin{aligned} \bar{\rho}_0(a_1) &= 1, \\ \bar{\rho}_0(s) &= 0 \quad \text{for all } s \in S \setminus \{a_0\}, \\ \bar{\rho}_{n+1}(a_{i+1}) &= \bar{\rho}_n(a_i), \\ \bar{\rho}_{n+1}(b_1) &= \bar{\rho}_n(a_k) + \bar{\rho}_n(b_p), \\ \bar{\rho}_{n+1}(b_{j+1}) &= \bar{\rho}_n(a_j). \end{aligned}$$

If we collect $\bar{\rho}_0, \dots, \bar{\rho}_{k+p-1}$ to a group homomorphism $\bar{\rho}\colon \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*) \to \mathbb{Z}^{k+p}$, we can show row by row that $(\bar{\rho}(a_1), \dots, \bar{\rho}(a_k), \bar{\rho}(b_1), \dots, \bar{\rho}(b_p)) \in \mathbb{Z}^{(k+p)\times(k+p)}$ is the identity matrix. So, $\bar{\rho}$ induces an isomorphism between the abelianization of $\mathcal{K}$ and $\mathbb{Z}^{k+p}$. ■

Lemma 6.2. *$\mathcal{K}$ surjects onto the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$. In particular, $\mathcal{K}$ is of exponential growth.*

Proof. The action on $\mathbb{Z}^2$ gives a map to $\mathbb{Z} \wr \mathbb{Z}$. We see that a_1 and a_2, respectively, b_1 if $k = 1$ are mapped to the standard generating set of $\mathbb{Z} \wr \mathbb{Z}$, so we have a surjection. As $\mathbb{Z} \wr \mathbb{Z}$ has exponential growth (see [5,15] for a detailed discussion), $\mathcal{K}$ also has exponential growth. ■

Lemma 6.3. *The group $\mathcal{K}$ is level-transitive and self-replicating. For the derived subgroup $\mathcal{K}' \subset \mathrm{Stab}_{\mathcal{K}}$, we have the following: under the map $\mathrm{Stab}_{\mathcal{K}} \hookrightarrow \bigoplus_{x\in\mathbb{Z}} \mathcal{K}$ induced by the wreath recursion, the image of $\mathcal{K}'$ contains $\bigoplus_{x\in\mathbb{Z}} \mathcal{K}'$ and the composition*

$$\mathcal{K}' \hookrightarrow \mathrm{Stab}_{\mathcal{K}} \hookrightarrow \bigoplus_{x\in\mathbb{Z}} \mathcal{K} \to \mathcal{K} \tag{6.1}$$

is surjective, where the last map is the projection map to any summand.

Proof. Note that $a := a_1$ acts just by translations on the first level, and every generator is the section of another generator that acts trivially on the first level. This already implies level-transitivity and self-replication. To show that the composition (6.1) is surjective, it is easy to see that every generator of $\mathcal{K}$ is a section of a commutator of a generator and a sufficiently large power of a_1. So, it is easy to see that $\mathcal{K}'$ surjects geometrically onto $\mathcal{K}$. As a_1 is just the first-level shift and $\mathcal{K}'$ is a normal subgroup of $\mathcal{K}$, to show that $\bigoplus_{x\in\mathbb{Z}} \mathcal{K}' \subset \mathcal{K}'$, it is enough to show that $\mathcal{K}'@0 \subset \mathcal{K}'$. Since $\mathcal{K}$ is self-replicating, it is enough to show that $[s,t]@0 \in \mathcal{K}'$ for every commutator of two generators $s, t \in S$. Now, if c and d are the generators which have s and t as sections at z and w, then a straight forward calculation shows that $[a^{-z}ca^{z}, a^{-w}da^{w}] = [s,t]@0$. ■

Lemma 6.4. *The group $\mathcal{K}$ is not residually finite.*

Proof. By the previous lemma, $\mathrm{Stab}_{\mathcal{K}}$ surjects onto $\mathcal{K}$, and since $\mathcal{K}$ is not abelian (it surjects onto an non-abelian group), neither is $\mathrm{Stab}_{\mathcal{K}}$. Let $x, y \in \mathrm{Stab}_{\mathcal{K}}$ be a non-commuting pair. Suppose $\mathcal{K}$ is residually finite, then there exists a group homomorphism $\phi\colon \mathcal{K} \to F$ to a finite group F such that $\phi([x,y])$ is non-trivial. But F is finite, so $\phi(a_1)$ has finite order. So, there is an $n > 0$ with $\phi(a_1^{mn}) = 1$ for all m. Then, $\phi([x,y]) = \phi([a_1^{-mn}xa_1^{mn}, y])$. Now, under the wreath recursion, x and y have finite support in the direct sum

$$\bigoplus_{\mathbb{Z}} \mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}),$$

so for m large enough, the support of $a_1^{-mn}xa_1^{mn}$ and y will be disjoint; hence, they commute. So, $\phi([x,y]) = \phi([a_1^{-mn}xa_1^{mn}, y])$ is trivial, so we arrive at a contradiction. ■

Theorem 6.5. *The group $\mathcal{K}$ is amenable but not elementary subexponentially amenable.*

Proof. We first show that the group $\mathcal{K}$ is amenable. We already observed in Remark 4.5 that $\mathcal{K}$ is generated by bounded activity automata. Hence $\mathcal{K}$ is a subgroup of $\mathrm{Aut}^{\mathrm{f.s.}}_{\mathcal{B}}(\mathbb{Z}^*;\mathbb{Z})$ in the notation of [16]. As the left action of $\mathbb{Z}$ on itself is recurrent, by [16, Theorem B] the group $\mathrm{Aut}^{\mathrm{f.s.}}_{\mathcal{B}}(\mathbb{Z}^*;\mathbb{Z})$ is amenable, and so is the subgroup $\mathcal{K}$. The group $\mathcal{K}$ have exponential growth by Lemma 6.2. Lemma 6.3 together with [9, Corollary 3] implies that $\mathcal{K}$ is not elementary subexponentially amenable. ■

We should note that [9] only deals with finite alphabets. The proof can be easily modified to deal with subgroups of $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$.

Remark 6.6. A key step in the proof of Theorem B of [16] is to show that the simple random walk on the connected components of Γ_ω is recurrent. By Theorem 5.5, we can conclude this directly in our setting, as the simple random walk on a tree with countably many ends is always recurrent (see, for example, [23, Chapter I, Corollary 6.15]).

7. Outlook

This paper is the beginning of our study of iterated monodromy groups for entire transcendental maps and a stepping stone towards a more general discussion. The regularity of the monodromy of the exponential map simplifies the discussion and has consequences that are special to the exponential case. In particular, the left order on the dynamical preimage tree heavily uses this regularity. For other entire transcendental functions, we should expect torsion elements in the monodromy group and torsion elements for some iterated monodromy groups of functions in that parameter space. In an upcoming paper [18] (see also [17, Chapter 4]), we discuss the general structure of iterated monodromy groups of entire maps. In particular, we also apply the results of [16] to show that the iterated monodromy groups of entire functions are amenable if and only if their monodromy group is. For polynomials and the exponential family, the condition is trivially satisfied, as finite groups and abelian groups are amenable. However, there are entire maps with virtually free monodromy groups, so we have to impose this condition. Moreover, we can also try to generalize from entire functions to meromorphic functions. Here, a good starting family would be the functions of the form $M \circ \exp$ including tangent, where M is a Möbius transform. We should think of this as the analogy to the family of bicritical rational maps, see also [11, Appendix D]. In this case, we can also define iterated monodromy group for post-singularly finite maps and show that they are ZC-groups. So, the class of ZC-groups, in particular subgroups of $\mathrm{aut}_{\mathbb{Z}}(\mathbb{Z}^*)$ has many examples of self-similar groups coming from complex dynamics. This warrants a further general investigation of ZC-groups. Outside of this family $M \circ \exp$, we should not expect to have the left orderability of all IMGs in one parameter space, as it might be a special phenomenon due to the very rigid monodromy groups of exponential maps.

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Bernhard Reinke
Institut de Mathématiques (I2M UMR CNRS7373), Aix-Marseille Université, Campus de Luminy, 163 avenue de Luminy – Case 907, 13288 Marseille, France; current address: Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103 Leipzig, Germany;
bernhard.reinke@mis.mpg.de