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Proper proximality among various families of groups

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Abstract. In this paper, the notion of proper proximality (introduced by Boutonnet, Ioana, and Peterson [Ann. Sci. Éc. Norm. Supér. (4) 54 (2021), 445–482]) is studied and classified in various families of groups. We show that if a group acts non-elementarily by isometries on a tree such that, for any two edges, the intersection of their edge stabilizers is finite, then G is properly proximal. We show that the wreath product $G \wr H$ is properly proximal if and only if H is non-amenable. We then completely classify proper proximality among graph products of non-trivial groups. Our results generalize the recent work of Duchesne, Tucker-Drob, and Wesolek classifying inner amenability for these families of groups. Our results also recover some rigidity results associated to the group von Neumann algebras by virtue of being properly proximal. A key idea in the proofs of our theorems is a technique to upgrade from relative proper proximality using computations in the double dual of the small at infinity boundary.

1. Introduction and statements of main results

The goal of this paper is to provide several new examples of *properly proximal* groups. The authors of [1] who introduced this property were motivated by the program of classifying group von Neumann algebras. Proper proximality is a dynamical/geometric property by nature, so it is independently of interest to group theorists and geometers. One advantage of proper proximality is that it applies to a robust family of non-amenable groups, including all lattices in non-compact semi-simple Lie groups. This, in particular, allowed the authors of [1] to demonstrate the first W^*-strong rigidity results for compact actions of higher-rank lattices.

Theorem 1.1 ([1, Theorem 1.1]). *For all properly proximal groups G, the group von Neumann algebra $L(G)$ has no weakly compact Cartan subalgebras in the sense of Ozawa and Popa* [22]. *Moreover, for any free ergodic probability measure preserving (p.m.p.) action $\sigma : G \curvearrowright (X, \mu)$, the crossed product $L^\infty(X, \mu) \rtimes G$ admits a weakly compact Cartan subalgebra A if and only if σ is weakly compact, and, in this case, A is unitary conjugate with $L^\infty(X, \mu)$.*

Our first main result proves proper proximality for a family of groups acting on trees. Unless otherwise mentioned, all groups in this paper are *countable*.

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Theorem 1.2. *Let an infinite countable G act on a countably infinite tree T such that*

(1) *the action is non-elementary on the Bowditch compactification ΔT;*

(2) *for any pairs of distinct edges $e, f \in E(T)$, one has $\mathrm{Stab}(e) \cap \mathrm{Stab}(f)$ is finite.*

Then, G is properly proximal.

The most natural examples of groups with the above phenomenon arise from fundamental groups of graphs of groups via Bass–Serre theory. Particular cases of these include amalgamated free products and HNN extensions. Recall that a subgroup H is almost malnormal in G if for any $g \in G \setminus H$ one has $|gHg^{-1} \cap H| < \infty$.

Corollary 1.3. *If $G = G_1 *_H G_2$ is a countable group such that H is almost malnormal in G and $[G_1 : H] \geq 3$, $[G_2 : H] \geq 2$, then G is properly proximal.*

Corollary 1.4. *If $G*_{H\sim K}$ is an HNN extension of a countable group G over almost malnormal subgroups H and K and $[G : H] \geq 3$, then $G*_{H\sim K}$ is properly proximal.*

Remark 1.5. A group is said to be a *convergence group* if it admits a non-elementary convergence action (see [2]). In [9,21], many amalgamated products and HNN extensions are shown to be convergence groups and hence properly proximal (see [1, Example 4.6]). However, the above corollaries include examples that are not non-elementary convergence groups. For instance, consider $G = (G_1 \wr \Gamma) *_\Gamma (G_2 \wr \Gamma)$, where G_1 and G_2 are non-trivial amenable groups and Γ is any infinite countable group. Suppose that there exists a non-elementary convergence action $G \curvearrowright K$ on some compact space K; then as $\bigoplus_\Gamma G_i$ is infinite amenable for $i = 1, 2$, there must exist $a_i, b_i \in K$ such that $\{a_i, b_i\}$ is fixed by $\bigoplus_\Gamma G_i$ set-wise. Since the normalizer of $\bigoplus_\Gamma G_i$ is $G_i \wr \Gamma$, we have $G_i \wr \Gamma$ fixing $\{a_i, b_i\}$ as well. Now, taking any infinite sequence in Γ, we see from north-south dynamics that

$$\{a_1, b_1\} = \{a_2, b_2\};$$

hence, G fixes $\{a_1, a_2\}$ which contradicts the fact that the action is non-elementary.

Remark 1.6. We would like to point out for the particular cases of groups in Corollaries 1.3 and 1.4, the consequence of Theorem 1.1 is weaker than what is already known in the literature due to work of Ioana in [17] and subsequently Vaes in [29]. Indeed, they show the absence of any Cartan subalgebras for these examples, as opposed to just weakly compact Cartan subalgebras. However, there are additional applications to showing proper proximality that do not follow from existing results, such as classifying the W*-equivalence classes of inner amenable groups (see [19]).

Our second main result is the following theorem.

Theorem 1.7. *Let G be a non-trivial group and H an infinite group. Then, $G \wr H$ is properly proximal if and only if H is non-amenable.*

As a consequence of the above theorem (in combination with Theorem 1.1), we deduce the absence of weakly compact Cartan subalgebras in Bernoulli shift crossed products of non-amenable groups. This result is in the flavor of the well-known open question [18, Problem III] of whether such crossed products have unique Cartan subalgebra up to unitary conjugation.

The proofs of the above two results use the double dual characterization of proper proximality to upgrade from proper proximality relative to certain malnormal subgroups.

Our third main result is a complete classification of proper proximality for graph products of non-trivial groups. We provide the following algorithm to decide if a graph product of groups is properly proximal.

Theorem 1.8. *Let Γ be a finite graph, and denote by $V(\Gamma)$ the vertex set of Γ.*

If $|V(\Gamma)| = 2$, then $\Gamma(G)$ is properly proximal if and only if

(1) *v_1 is disconnected to v_2 and $|G_{v_i}| \geq 3$ for some i;*

(2) *v_1 is connected to v_2 and G_{v_i} are properly proximal for each $i = 1, 2$.*

If $|V(\Gamma)| \geq 3$, $\Gamma(G)$ is properly proximal if and only if one of the following holds:

(1) *Γ is irreducible (i.e., there does not exist a non-empty strict subgraph S such that for all $v \in V(S)$ and $u \in V(S^c)$, $(v, u) \in E(\Gamma)$).*

(2) *Γ is reducible, and S is a non-empty strict subgraph S such that, for all $v \in V(S)$ and $u \in V(S^c)$, $(v, u) \in E(\Gamma)$, and $\Gamma_S(G)$ and $\Gamma_{S^c}(G)$ are both properly proximal.*

Note that since proper proximality implies non-inner amenability, the above theorem generalizes [13, Theorem 4.14] where the same classification result is obtained in the context of inner amenability.

Note also that from Theorem 1.8 we immediately deduce the following corollary.

Corollary 1.9. *Arbitrarily finite graph products of properly proximal groups are properly proximal.*

Rigidity properties of graph products of groups have gathered considerable interest recently. For instance, [7] shows primeness results for the group von Neumann algebras associated to many graph products of groups. Also, [6] shows strong solidity and/or absence of Cartan for classes of Hecke von Neumann algebras associated to some graph products. As a consequence of Theorem 1.8, we obtain several new examples of groups satisfying the conclusion of Theorem 1.1.

Moreover, if one has a combination of weak amenability and proper proximality, we can deduce the absence of Cartan subalgebras and $\mathcal{C}$-rigidity in the sense of [23][1] (see [1, Theorem 1.5]). In [24], Reckwerdt shows that graph products of weakly amenable groups with Cowling–Haagerup constant 1 continue to be weakly amenable. Therefore, we obtain the following corollary.

[1] By $\mathcal{C}$-rigidity for G, we mean that $L^\infty(X, \mu) \rtimes G$ admits a unique Cartan subalgebra up to unitary conjugation for any free ergodic p.m.p. action $\sigma : G \curvearrowright (X, \mu)$. See [23] for more details.

Corollary 1.10. *Let $\Gamma(G)$ be a graph product of groups that is properly proximal, and further assume that the vertex group G_v has Cowling–Haagerup constant 1 for all $v \in V(\Gamma)$. Then, $L(\Gamma(G))$ admits no Cartan subalgebras. Moreover, $\Gamma(G)$ is $\mathscr{C}$-rigid.*

We ask the following general question in the context of our result above.

Question 1.11. Which graph products of groups do not admit Cartan subalgebras in their group von Neumann algebras?

After circulating an initial version of this preprint, the second author and I. Chifan answered the above question in [8].

2. Preliminaries

2.1. Graph products of groups

Throughout this paper, we denote by Γ a non-empty finite simple (every edge connects two distinct vertices) undirected graph, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For an edge e, for convenience, denote by e_1, e_2 the endpoints of the edge. For $v \in V(\Gamma)$, its link is $S_v = \{w \in V(\Gamma) \mid (v, w) \in E(\Gamma)\}$. For any two vertices $v, w \in V(\Gamma)$, let $d(v, w) \in \mathbb{N} \cup \{\infty\}$ be the length of the shortest path between v and w. For any subset $T \subset V(\Gamma)$, denote the induced subgraph on T by Γ_T.

The *radius* of a graph Γ is given by

$$r(\Gamma) = \inf_{v \in V(\Gamma)} \sup_{w \in V(\Gamma)} d(u, v).$$

And a graph Γ is called *irreducible* if the complement graph Γ^c (i.e., the graph consisting of the same vertices, and $(u, v) \in E(\Gamma^c)$ if and only if $(u, v) \notin E(\Gamma)$) is connected.

Given Γ as a graph and $\{G_v\}_{v \in V(\Gamma)}$ as a family of countable groups labeled by the vertex set of Γ, the *graph product* denoted by $G = \Gamma(G)$ is the quotient group of the free product $*_{v \in V(\Gamma)} G_v$, with relations $[g, h] = 1$ for all $g \in G_u$ and $h \in G_v$ with $(u, v) \in E(\Gamma)$.

Definition 2.1. Consider an amalgamated free product group $G_1 *_H G_2$. Choose T_i as a transversal for the cosets $\{Hx : x \in G_i\}$. A normal word is a word $g = ht_1 \cdots t_k$, where $h \in H$, $k \geq 0$, and $t_j \in T_{i_j} \setminus \{1\}$ for some $i_j \in \{1, 2\}$ and $i_j \neq i_{j+1}$ for $1 \leq j \leq k - 1$.

In amalgamated free products, it is well known that every element can also be represented by a unique normal word (see [4, Theorem 3.7]). As is the case for free products, elements in graph products also admit normal forms.

Definition 2.2 ([15, Definition 3.5]). Let $G = \Gamma\{G_v\}_{v \in V(\Gamma)}$ be a graph product of groups. A word $g_1 g_2, \ldots, g_n \in G$ is said to be *reduced* if the following hold.

(1) $g_i \in G_{v_i}$ for all $i \in \{1, 2, \ldots, n\}$, where $v_i \in V(\Gamma)$.

(2) $g_i \neq 1$ for all $i \in \{1, 2, \ldots, n\}$.

(3) For any $i \leq k < j$, if

$$[g_i, g_{i+1}] = [g_i, g_{i+2}] = \cdots = [g_i, g_k] = 1$$

and

$$[g_{k+1}, g_j] = [g_{k+2}, g_j] = \cdots = [g_{j-1}, g_j] = 1,$$

then $v_i \neq v_j$.

Theorem 2.3 ([15, Theorem 3.9]). *Let $G = \Gamma\{G_v\}_{v \in V(\Gamma)}$ be a graph products of groups. Then, each non-trivial element $g \in G$ can be uniquely (up to commuting segments) expressed as a product*

$$g = g_1 \cdots g_n,$$

where $g_1 \cdots g_n$ is a reduced word.

Graph products decompose naturally as amalgamated free products. We record this below in the following lemma.

Lemma 2.4. *Let $\Gamma(G)$ be a graph product, and let $v \in V(\Gamma)$. Then, $\Gamma(G)$ splits as an amalgamated free product:*

$$\Gamma(G) \cong \Gamma_{S_v \cup \{v\}}(G) *_{\Gamma_{S_v}(G)} \Gamma_{V(\Gamma) \setminus \{v\}}(G).$$

2.2. Proper proximality

As stated in the introduction, the family of properly proximal groups is a robust family including the following classes of groups:

(1) Non-amenable bi-exact groups.

(2) Non-elementary convergence groups.

(3) Lattices in non-compact semi-simple Lie groups.

(4) Groups admitting a proper 1-cocycle into a non-amenable representation.

(5) Groups acting properly non-elementarily by isometries on proper CAT(0) spaces.

(6) Non-elementary mapping class groups.

(7) Groups measure equivalent to any of the above.

Items (1) to (4) are results of [1], items (5) and (6) are results of [16], and item (7) is due to [19].

We say a sequence $\{g_n\}_{n \in \mathbb{N}} \in G$ goes to infinity *relative* to a countable family of subgroups $\{H_i\}_{i \in I}$, denoted by $g_n \to \infty / \{H_i\}_{i \in I}$, if for any $t_1,\ t_2 \in G$ and any $i \in I$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $g_n \notin t_1 H_i t_2$.

Consider the C^*-subalgebra $c_0(G, \mathcal{S}) \subset \ell^\infty(G)$ consisting of functions f such that for all $g_n \to \infty / \mathcal{S}$, $f(g_n) \to 0$, where $\mathcal{S} = \{H_i\}_{i \in I}$. It contains the ideal $c_0(G)$ consisting of functions vanishing at infinity. Observe also that $c_0(G, \mathcal{S})$ is globally left and right G invariant. Therefore, $\ell^\infty(G)/c_0(G, \mathcal{S})$ is isomorphic to $C(X_{\mathcal{S}})$, where $X_{\mathcal{S}}$ is a closed left and right invariant subset of the Stone–Cech boundary $\Delta G \setminus G$ and is called the *boundary piece* associated to a collection of subgroups $\mathcal{S}$.

Definition 2.5 ([1, Theorem 4.3]). Given a group G and a countable family of subgroups $\mathcal{S}$, we say G is properly proximal relative to $\mathcal{S}$ if one of the following equivalent conditions holds:

(1) There is an action $G \curvearrowright K$ by homeomorphisms on a compact Hausdorff space K such that there is no G-invariant probability measure on K and there exists a probability measure $\eta \in \mathrm{Prob}(K)$ with

$$X_{\mathcal{S}} = \partial_\eta G := \Big\{\omega \in \Delta G \setminus G \mid \lim_{g\to\omega}^{wk^*} g\cdot\eta - (gh)\cdot\eta = 0 \text{ for any } h \in G\Big\}.$$

(2) There are actions by homeomorphisms $G \curvearrowright K_i$, $i = 1,\dots,k$ on compact Hausdorff spaces K_i such that there is no G-invariant measure on any K_i and there exists a probability measure $\eta_i \in \mathrm{Prob}(K_i)$ with $X_{\mathcal{S}} \subset \bigcup_{i=1}^k \partial_{\eta_i} G$.

(3) There is no left-G invariant state on $(\ell^\infty(G)/c_0(G,\mathcal{S}))^{G_r}$ (i.e., the right-G invariant subspace of $\ell^\infty(G)/c_0(G,\mathcal{S})$).

(4) There is no left-G invariant state on $((\ell^\infty(G)/c_0(G,\mathcal{S}))^{**})^{G_r}$.

The group G is properly proximal if it is properly proximal relative to the trivial subgroup.

Note that, from the above definition/theorem, if one has a finite collection of subgroups $\{H_i\}_{i=1}^n$ of G such that G is properly proximal relative to $\{H_i\}$ for each $i = 1,\dots,n$, where $\bigcup_{i=1}^n X_{H_i} = \Delta G \setminus G$, then G is properly proximal.

2.3. Groups acting on trees

We say that an action of G on a set X is non-elementary if it does not preserve any set of cardinality at most 2.

Given a simplicial tree T, let ΔT denote the compactification introduced in [3] (see [5, Section 5.2] for details). The compactification ΔT is defined to be $V(T) \sqcup \partial T$ as a set, where ∂T is the Gromov boundary. For each $x \in \Delta T$ and each finite set $F \subset V(T)$, set

$$U(x,F) = \{y \in \Delta T \mid [x,y] \cap F = \emptyset\} \cup \{x\},$$

where $[x,y]$ denotes the unique geodesic path between x and y. Then,

$$\{U(x,F) \mid F \subset V(T) \text{ finite}\}_{x\in\Delta T}$$

forms a basis for the *Bowditch topology* on ΔT. ΔT equipped with this topology is compact and Hausdorff, and any isometric action $G \curvearrowright T$ extends to an action by homeomorphisms on ΔT (see [5, Proposition 5.2.5]).

If G is an amalgamated free product or HNN extension, then we have an action by homeomorphisms of $G \curvearrowright \Delta T$, where T is the Bass–Serre tree associated to G (see [25]).

Also note that if T is countable, then the topology on ΔT is second countable, hence metrizable, as the basis may be taken as $\{U(x, F) \mid F \subset V(T) \text{ finite}\}_{x \in V(T)}$.

3. Proofs of main theorems

3.1. Proofs of Theorems 1.2 and 1.7

The proof of Theorem 1.2 splits into two steps. The first step is to establish proper proximality relative to the edge stabilizers, and the second step is to upgrade this to proper proximality using the malnormality condition. The way we accomplish step (1) is by obtaining a relative north-south dynamics for the action on the Bowditch compactification of the tree. For step (2), we use the double dual proximal space $((\ell^\infty(G)/c_0(G, \mathcal{S}))^{**})^{G_r}$ as a filler space to create several right invariant projections that are indexed by left cosets of the edge stabilizers in $\mathcal{S}$. Under the malnormality assumption, we see that these are orthogonal projections. From step (1), we see that any G-invariant state vanishes on the complement of the sum of these projections. To conclude, we G-equivariantly embed $C(\Delta T)$ into $((\ell^\infty(G)/c_0(G, \mathcal{S}))^{**})^{G_r}$ along these projections. This contradicts the fact that $G \curvearrowright \Delta T$ is non-elementary. A modification of the above step (2), in combination with a technical lemma adapted from [5], gives us Theorem 1.7.

We begin by showing the relative north-south dynamics result described above. We remark that a result in this flavor in the setting of graphs of convergence groups was recently obtained by R. Tomar in [26, Section 4].

Proposition 3.1. *Let G be a group acting on a tree T with $|V(T)| = \infty$ by isometries. Set $\mathcal{S} = \{\mathrm{Stab}(e) \mid e \in E(T)\}$, and suppose that $g_n \to \infty/\mathcal{S}$. Then, there exists a subsequence (g_{n_k}) and points a, b in ΔT such that, for any $x \in \Delta T \setminus \{b\}$, we have $lim_{k\to\infty} g_{n_k} x = a$. Furthermore, if the action $G \curvearrowright \Delta T$ is non-elementary, then G is properly proximal relative to $\mathcal{S}$.*

Proof. For any such a sequence $\{g_n\}$, one may pick a vertex $o \in V(T)$ such that $|\{g_n o\}|$ and $|\{g_n^{-1} o\}|$ are infinite. Indeed, for each edge e of T there are only finitely many $n \in \mathbb{N}$ with $e \in g_n \cdot [u, v]$, since $n \mapsto g_n e$ is finite to one for any edge e between two vertices u and v. Thus, there exists a subsequence $\{g_{n_k}\}_k$ and distinct vertices u and v such that $k \mapsto g_{n_k} u$ and $k \mapsto g_{n_k} v$ are both one to one. Since we also have $g_{n_k}^{-1} \to \infty/\mathcal{S}$, the same argument shows that either $|\{g_{n_k}^{-1} u\}_k| = \infty$ or $|\{g_{n_k}^{-1} v\}_k| = \infty$.

Now, pick a subsequence $\{g_{n_k}\}$ such that $\lim g_{n_k} o = a$ and $\lim g_{n_k}^{-1} o = b$.

Case 1. $a \in \Delta T \setminus T$.

Since $d(o, g_{n_k} o) = d(g_{n_k}^{-1} o, o)$, b is also in $\Delta T \setminus T$. The argument in this case proceeds similarly to the argument in [28]. Let $U_a = U(a, \{e\})$ and $U_b = U(b, \{f\})$ be basic open sets in ΔT, where e and f are edges with endpoints e_1, e_2 and f_1, f_2, respectively. For $v \in \Delta T$, denote by β_v the geodesic from o to v. Denote by $\langle x, y \rangle_o$ the Gromov product, given by the distance between o and the center of the unique geodesic tripod formed by x, y, and o. For all $c \notin U_b$, i.e., satisfying $[b,c] \cap [f_1, f_2] \neq \emptyset$, we have

$$\liminf_{m,n\to\infty} \langle \beta_b(m), \beta_c(n)\rangle_o \leq \max\{d(o, f_1), d(o, f_2)\} =: d_1.$$

Define $V(b; 2d_1) = \{x \in \Delta T \mid \liminf_{m,n\to\infty} \langle \beta_x(m), \beta_b(n)\rangle_o \geq 2d_1\}$. From the above, we have $U_b^c \subset V(b, 2d_1)^c$. Now, let $c \in V(b, 2d_1)$. We have

$$\liminf_{m,n\to\infty} \langle \beta_a(m), g_{n_k}\beta_c(n)\rangle_o \geq \min\{\liminf_{m\to\infty} \langle \beta_a(m), g_{n_k} o\rangle_o, \liminf_{n\to\infty} \langle g_{n_k} o, g_{n_k}\beta_c(n)\rangle_o\}.$$

Note that $\lim_{m\to\infty} \langle \beta_a(m), g_{n_k} 0\rangle_o \to \infty$ as $k \to \infty$; also,

$$\langle g_{n_k} o, g_{n_k}\beta_c(n)\rangle_o = \langle o, \beta_c(n)\rangle_{g_{n_k}^{-1} o} \to \infty$$

as $k \to \infty$ since $c \notin V(b, 2d_1)$. Thus, $\forall R > 0$, there exists $K_0(R)$ such that, for all $k \geq K_0(R)$, we have $g_{n_k}(V(b, 2d_1)) \subset V(a, R)$.

Now, suppose that, for all $k > 0$, there exists $k_0 > k$ and $c \in U_b^c$ such that $g_{n_k}(c) \notin U(a, \{e\})$; that is,

$$\liminf_{m,n\to\infty} \langle \beta_a(n), g_{n_k}\beta_c(m)\rangle_o \leq \min\{d(o, e_1), d(o, e_2)\} =: d_2.$$

Take $R = 2d_2$. Then, $g_{n_k}(c) \notin U(a, \{f\})$, but $g_{n_k}(c) \in V(a, 2d_2)$ for all $k > K(R)$, which is a contradiction. Hence, we have the conclusion in the case that $a \in \Delta T \setminus T$.

Case 2. $a \in T$. Let $x \in \Delta T$. We claim that $\lim_{k\to\infty} g_{n_k}(x) = a$.

First, suppose that $x \in T$. For any open set $U(a, \{e\})$, we have the following:

$$|\{n \in \mathbb{N} : g_n[o, x] \ni e\}| < \infty.$$

Indeed, if the above set was infinite, then there would be an infinite set $I \subset \mathbb{N}$ with $g_{n_i}^{-1} e \in [o, x]$ for $i \in I$. Since $[o, x]$ is a fixed finite-length path, this contradicts the fact that g_{n_i} escapes all edge stabilizers. Now, since $g_n(o) \to a$, we have that $\exists N > 0$ such that for all $n \geq N$, $g_n o \in U(a, \{e\})$. Hence, for n sufficiently large, we have that e is not in the segment $[g_n o, g_n x]$ and e is not in the segment $[g_n o, a]$; hence, e is not in the segment $[g_n x, a]$ which gives us what is required.

Suppose that $x \in \partial T$. We have that $\{g_n o\}$ is bounded, and hence, let

$$d_0 = \sup_n d(a, g_n o).$$

Given an open set $U(a, \{e\})$, let $d_1 = d(a, e_1) + 1$, where d is the combinatorial metric on T. Let $N = d_0 + d_1 + 1$. For any $n \geq N$, we apply the previous argument to the point $\beta_x(n)$ to obtain that $e \notin g_m[\alpha(0), \beta_x(n)]$ for large m. Moreover, due to the bound on $d(a, e)$, we have that $e \notin [\beta_x(n), x]$.

To show relative proper proximality, consider a sequence $\{g_n\}$ with $g_n \to \infty/\mathcal{S}$. As mentioned in [2, Section 2], if $G \curvearrowright \Delta T$ is non-elementary, then the limit set is perfect;

i.e., there exists a compact Hausdorff perfect set inside ΔT, which implies the existence of some diffuse measure. Fix a diffuse probability measure η on ΔT, and let $a, b \in \Delta T$ and $\{g_{n_k}\}$ be as in the above arguments. By [14, Lemma 8.3], for any $h \in G$, $\lim_{k\to\infty} g_{n_k} h\eta = \delta_a$, and thus, $X_{\mathcal{S}} = \partial_\eta G$. Moreover, there is no G-invariant probability measure on ΔT. Indeed, if there were a G-invariant probability measure μ, then we see from the north-south dynamics that $\mu(\{a, b\}) = 1$, where a and b are some north and south poles. This contradicts the fact that the action is non-elementary. ∎

Applying the above proposition to the Bass–Serre tree associated with an amalgamated free product, we obtain the following corollary.

Corollary 3.2. *Given G_1, G_2, and H with $H \leq G_i$, $i = 1, 2$. Then, $G = G_1 *_H G_2$ is properly proximal relative to H if $[G_i : H] \geq 3$ and $[G_j : H] \geq 2$ for $\{i, j\} = \{1, 2\}$.*

Proof. This follows by considering the standard action on the compactification of the Bass–Serre tree associated to the amalgamated free product. This action is non-elementary on ΔT because the boundary is infinite, and orbits of the action on the boundary are infinite. Moreover, the edge stabilizers are precisely conjugates of H. ∎

The following is a direct proof of the above corollary without involving Bass–Serre theory. We would like to point out that the technical difference between non-inner amenability and proper proximality is clearly seen in this proof. Indeed, for proper proximality, one is required to establish a paradoxical decomposition on the small at infinity boundary as opposed to the entire Stone–Cech boundary for non-inner amenability.

Alternative proof of Corollary 3.2. Fix a choice of transversals

$$T = \{e\} \cup \{t_i\}_{i\in I}$$

for cosets $\{Hx : x \in G_1\}$ and $S = \{e\} \cup \{s_j\}_{j\in J}$ for cosets $\{Hx : x \in G_2\}$. Denote by p_1 the characteristic function on the set of elements g whose normal form begins with ht_i, where $h \in H$, $i \in I$; similarly, let p_2 be characteristic function on the set of elements g whose normal form begins with hs_j, where $h \in H$, $j \in J$.

Claim. $p_i \in (\ell^\infty(G)/c_0(G, H))^{G_r}$ for $i = 1, 2$.

Suppose that there exist some $g \in G$ and a sequence $g_n \to \infty/H$ such that

$$\lim_{n\to\infty} p_1(g_n) - p_1(g_n g) \neq 0;$$

then we may extract a subsequence, still denoted by (g_n), with

$$\lim_{n\to\infty} p_1(g_n) - p_1(g_n g) = 1.$$

Let $g_n = h_n t_{1,n} s_{1,n} \cdots t_{k_n,n} s_{k_n,n}$ be the normal form of each g_n, where $h_n \in H$, $t_{i,n} \in T \setminus \{e\}$, and $S_{i,n} \in S \setminus \{e\}$, with $t_{1,n}$ and $s_{k_n,n}$ possibly being e. Note that since $g_n \to \infty/H$, $\{h_n^{-1} g_n \mid n \in \mathbb{N}\}$ is an infinite set. Set $g = ht_1 s_1 \cdots t_m s_m$ to be its normal form.

As $\lim_{n\to\infty} p_1(g_n) - p_1(g_n g) = 1$, for large enough n, we have $t_{1,n} \in T \setminus \{e\}$ while $g_n g \notin \operatorname{supp}(p_1)$. Note that

$$\begin{aligned} g_n g &= h_n t_{1,n} s_{1,n} \cdots t_{k_n,n} s_{k_n,n} h t_1 s_1 \cdots t_m s_m \\ &= h_n h' t'_{1,n} s'_{1,n} \cdots t'_{k_n,n} s'_{k_n,n} t_1 s_1 \cdots t_m s_m, \end{aligned}$$

where $t'_{i,n} \in T \setminus \{e\}$, $s'_{i,n} \in S$ are obtained as we move h to the front. By our assumption, $g_n g \notin \operatorname{supp}(p_1)$; i.e., $t'_{1,n}$ would disappear after possible cancelations for all large enough n; however, g is given, while $\{h_n^{-1} g_n \mid n \in \mathbb{N}\}$ is an infinite set, we arrive at a contradiction. This proves the claim.

Now, take $g_1, g_2 \in G_i \setminus H$ and $g_3 \in G_j \setminus H$, which is allowed by the assumption that $[G_i : H] \geq 3$ and $[G_j : H] \geq 2$. Notice that $g_1 p_1 + g_2 p_1 \leq p_2$, while $g_3 p_2 \leq p_1$ and $p_1 + p_2 = 1$. If there exists a left G-invariant state φ on $(\ell^\infty(G)/c_0(G, H))^{G_r}$, then

$$2\varphi(p_1) \leq \varphi(p_2) \leq \varphi(p_1);$$

i.e., $\varphi(p_1) = \varphi(p_2) = 0$, while $1 = \varphi(p_1 + p_2)$, which is a contradiction. ∎

Proof of Theorem 1.2. Suppose that G is not properly proximal. Then, there exists a left G-invariant state φ on the space $((\ell^\infty(G)/c_0(G))^{**})^{G_r}$. Let $\{H_i\}_{i\in I}$ be a family of subgroups of G such that

$$\bigsqcup_{i\in I} \{g H_i g^{-1} \mid g \in G\} = \{\operatorname{stab}(e) \mid e \in E(t)\}.$$

Set $p_i = \bigvee_{t,s\in G} 1_{tH_i s}$ to be a projection in $((\ell^\infty(G)/c_0(G))^{**})^{G_r}$. From Proposition 3.1, we obtain that $\varphi(p) = 1$, where

$$p = \bigvee_{i\in I} p_i.$$

Indeed, we have that G is properly proximal relative to $\mathcal{S} = \{\operatorname{stab}(e) \mid e \in E(t)\}$, and so, there is no left invariant state on $((\ell^\infty(G)/c_0(G, \mathcal{S}))^{**})^{G_r}$. But observe that

$$((\ell^\infty(G)/c_0(G, \mathcal{S}))^{**})^{G_r} = (1 - p)(\ell^\infty(G)/c_0(G))^{G_r},$$

and hence, $\varphi(1 - p) = 0$. For $t \in G$ and $i \in I$, put

$$p_{t,i} = \bigvee_{s\in G} 1_{tH_i s} \in ((\ell^\infty(G)/c_0(G))^{**})^{G_r},$$

and we have $\bigvee_{t\in G} p_{t,i} = p_i$. Let $\{F_n\}$ be a sequence of increasing finite subsets such that $\bigcup_n F_n = G$. For each $t \in G$, consider projections $p_{t,i,n} = 1_{tH_i F_n} \in (\ell^\infty(G)/c_0(G))^{**}$, and then $\lim_{n\to\infty} p_{t,i,n} = p_{t,i}$ in SOT.

Note that $|tH_i s \cap H_j| < \infty$ for any $i, j \in I$ (not necessarily distinct), $s \in G$, and $t \in G$ such that $tH_i t^{-1} \neq H_j$; indeed, suppose that there exists some $ths \in tH_i s \cap H_j$. Then, for any $th's \in tH_i s \cap H_j$, we have $ths(th's)^{-1} = thh'^{-1}t^{-1} \in tH_i t^{-1} \cap H_j$, which is finite as for any pair of distinct edges $e, f \in E(T)$, $\mathrm{Stab}(e) \cap \mathrm{Stab}(f)$ is finite. Therefore, we have $p_{t,i,n} p_{s,j,n} = 0$ for all $t, s \in G$ such that $tH_i t^{-1} \neq sH_j s^{-1}$. Since $(p_{t,i,n})$ and $(p_{s,j,n})$ are bounded, $\lim_{n\to\infty} p_{t,i,n} p_{s,j,n} = p_{t,i} p_{s,j}$, i.e., $p_{t,i} p_{s,j} = 0$.

Consider a G-equivariant unital embedding

$$\psi : \ell^\infty(V(T)) \ni \delta_v \mapsto \sum_{e\in E_v} \delta_e/2 \in \ell^\infty(E(T)),$$

where $E_v = \{e \in E(T) \mid v \in e\}$. Given $e \in E(T)$, there exists a unique $i \in I$ and some $t \in G$ such that $\mathrm{stab}(e) = tH_i t^{-1}$. Thus, $p_{t,i}$ is a unique projection corresponding to e, and we denote it by $p(e)$. Therefore, we obtain an embedding

$$\iota : \ell^\infty(E(T)) \to ((\ell^\infty(G)/c_0(G))^{**})^{G_r}$$

given by $\iota(\delta_e) = p(e)$, and it is easy to check that this embedding is unital and G-equivariant. Finally, as $C(\Delta T) \subset \ell^\infty(V(T))$, we have a G-invariant state $\varphi \circ \iota \circ \psi$ on $C(\Delta T)$, which contradicts the assumption that $G \curvearrowright \Delta T$ is non-elementary. ∎

In the above proof, we actually showed the following useful tool.

Lemma 3.3. *Let H be a subgroup of G such that G is properly proximal relative to H. Suppose that H is almost malnormal; then G is properly proximal.*

For the convenience of the reader, we extract the argument here: if G is not properly proximal, there exists a left G-invariant state φ on the space $((\ell^\infty(G)/c_0(G))^{**})^{G_r}$. Setting $p = \bigvee_{t,s\in G} 1_{tHs}$ to be a projection in $((\ell^\infty(G)/c_0(G))^{**})^{G_r}$, we see that

$$((\ell^\infty(G)/c_0(G,H))^{**})^{G_r} = (1-p)(\ell^\infty(G)/c_0(G))^{G_r},$$

and hence, $\varphi(1-p) = 0$ from the relative proper proximality hypothesis. For $t \in G$, setting

$$p_t = \bigvee_{s\in G} 1_{tHs} \in ((\ell^\infty(G)/c_0(G))^{**})^{G_r},$$

we have $\bigvee_{t\in G} p_t = p$. As in the above proof, from the almost malnormality of H, we see that p_g is orthogonal to p_e for any $g \notin H$. Therefore, we obtain an embedding

$$\iota : \ell^\infty(G/H) \to p((\ell^\infty(G)/c_0(G))^{**})^{G_r}$$

given by $\iota(\delta_g) = p_g$, and it is easy to check that this embedding is unital and G-equivariant. Composing with φ, we obtain the co-amenability of $H < G$, which contradicts Lemma 3.3 of [27].

Lemma 3.4. *If G is a non-trivial group and H is non-amenable, then*

$$K = G \wr H$$

is properly proximal relative to H.

Proof. We use a trick due to Ozawa which he uses to study bi-exactness of wreath products (see [5, Section 15.3]). Fix a proper length function $|\cdot|_H$ on H and a proper length function $|\cdot|_G$. For $yt \in G \wr H$, where $y \in \bigoplus_H G$ and $t \in H$, define $\zeta : K \to \ell^1(H)$, given by

$$\zeta(yt)(p) = \begin{cases} \min\{|p|_H, |t^{-1}p|_H\} + |y(p)|_G & \text{if } p \in \operatorname{supp}(y), \\ 0 & \text{if } p \notin \operatorname{supp}(y). \end{cases}$$

From [5, Lemmas 15.3.7 and 15.3.8], we see that ζ satisfies

$$\lim_{x \to \infty/H} \frac{\|\zeta(sxt) - s\zeta(x)\|}{\|\zeta(x)\|} = 0$$

for all $s, t \in K$. Hence, by the first computation in the proof of [5, Lemma 15.2.6], we see that there exists a K-equivariant u.c.p. map from $\ell^\infty(K/\bigoplus_H G)$ to the relative proximal space $(\ell^\infty(K)/c_0(K,H))^{K_r}$. Hence, if K is not properly proximal relative to H, by composing with this u.c.p. map, we obtain an H invariant state on $\ell^\infty(K/\bigoplus_H G) \cong \ell^\infty(H)$, which contradicts the non-amenability of H. ■

Proof of Theorem 1.7. Observe that since G is non-trivial, H is almost malnormal inside $G \wr H$. Moreover, if H is non-amenable, from Lemma 3.4 and Lemma 3.3, we have that $G \wr H$ is properly proximal, since H is never co-amenable inside $G \wr H$. Indeed, we may first identify $(G \wr H)/H$ with $\bigoplus_G H$ as H-sets. Then, consider the following map $\rho : \ell^\infty(H) \to \ell^\infty(\bigoplus_G H)$ given by

$$\rho(f)(\xi) = \frac{\sum_{t \in \operatorname{supp}(\xi)} f(t)}{|\operatorname{supp}(\xi)|}$$

for any $f \in \ell^\infty(H)$ and $\xi \in \bigoplus_G H$, and clearly, ρ is unital and H equivariant.

Conversely, if H is amenable and infinite, then from [13, Theorem 3.9 (1)], we see that $G \wr H$ is inner amenable and hence not properly proximal. ■

3.2. Proof of Theorem 1.8

The main idea in proving Theorem 1.8 is to consider various amalgamated product decompositions in a graph product and obtain relative proper proximality relative to each of these amalgams. Then, we show that the Stone–Cech boundary of the graph product is filled by each of these boundary pieces coming from the amalgams, thereby showing proper proximality. The other key step is to show that if a product of groups is properly proximal, then each of the groups has to be properly proximal. This is obtained by establishing a natural isomorphism of the proximal spaces at the level of the double dual. The classification result is then obtained by a careful analysis of some cases, given by the radius of the graphs.

Given two subgraphs Γ_1, Γ_2 of a graph Γ, denote by $\Gamma_1 \cap \Gamma_2$ the subgraph of Γ generated by the vertex set $V(\Gamma_1) \cap V(\Gamma_2)$. As a convention, $\Gamma(G)$ is set to be the trivial group if $V(\Gamma) = \emptyset$.

Lemma 3.5. *Let $\Gamma(G)$ be a graph product group with graph Γ and generating groups $\{G_v \mid v \in V(\Gamma)\}$. For any subgraphs Γ_1, Γ_2 of Γ and $g, h \in \Gamma(G)$, we have*

$$\Gamma_1(G) \cap g\Gamma_2(G)h \subset \bigcup_{i=1}^{n} c_i(\Gamma_1 \cap \Gamma_2)(G)d_i$$

for some finite subset $F = \{c_i, d_i \mid 1 \leq i \leq n\} \subset \Gamma(G)$.

Proof. Given $g, h \in \Gamma(G)$ with $g = g_1 \cdots g_k$ and $h = h_1 \cdots h_m$ as their reduced words, respectively, let $F = \{c_i, d_i\}$ be the finite set consisting of words of the form $c_i = \prod_j g_{i_j}$ and $d_i = \prod_j h_{i_k}$, where (i_j) and (i_k) are increasing. We claim that F is the required set. It suffices to show that, for every i, we have

$$\Gamma_1(G) \cap c_i\Gamma_2(G)d_i \in \bigcup_{j=1}^{n} c_j(\Gamma_1 \cap \Gamma_2)(G)d_j.$$

We show this by induction on $n(t) = |\{t_j \mid 1 \leq j \leq l,\ t_j \in G_v \text{ for some } v \notin \Gamma_1 \cap \Gamma_2\}|$, where $t = t_1 \cdots t_l$ is a reduced word in $\Gamma_2(G)$ such that $gth \in \Gamma_1(G)$. The claim is true for $n = 0$, as $c_itd_i \in c_i(\Gamma_1 \cap \Gamma_2)(G)d_i$.

Suppose that the claim is true for $n \leq K$, and let $n = K + 1$. As

$$n = \Bigg| \bigcup_{v \notin V(\Gamma_1 \cap \Gamma_2)} \{t_j \mid 1 \leq j \leq l,\ t_j \in G_v\} \Bigg|,$$

we may assume that $|\{t_j \mid 1 \leq j \leq l, t_j \in G_{v_0}\}| \geq 1$ for some $v_0 \notin V(\Gamma_1 \cap \Gamma_2)$. For a word w, define $\hat{v}_0(w)$ to be the ordered set of letters in w (ordered by the left-to-right order in the word w) that do not belong to G_{v_0}. For an ordered set of letters u, denote by $u_{\Gamma(G)}$ the product of letters of u in $\Gamma(G)$. Denote by w^* the reduced word of c_itd_i, and note that $\hat{v}_0(w^*)_{\Gamma(G)} = w^*$ as $c_itd_i \in \Gamma_1(G)$.

Claim. $\hat{v}_0(c_itd_i)_{\Gamma(G)} = c_itd_{i\,\Gamma(G)} = w^*$.

Denote the word c_itd_i by $p_1p_2 \cdots p_n$, and suppose that $p_{i_1}, \ldots, p_{i_d}$ are letters that belong to G_{v_0}. Define words $w_1 = p_1 \cdots p_{i_1-1}$, $w_j = p_{i_{j-1}+1} \cdots p_{i_j-1}$ for $1 < j \leq d$ and $w_{d+1} = p_{i_d+1} \cdots p_n$. Upon individually reducing these words, we obtain the words w_i^*. Now, we have

$$c_itd_{i\,\Gamma(G)} = w_1^* p_{i_1} w_2^* p_{i_2} \cdots p_{i_d} w^*_{d+1\,\Gamma(G)}.$$

We may assume that $w^*_{j\,\Gamma(G)}$ does not commute with G_{v_0} for some $2 \leq j \leq d$. Then, $w^*_{j\,\Gamma(G)}$ neither commute with $a = w_1^* p_{i_1} \cdots p_{i_{j-1}\,\Gamma(G)}$ nor with

$$b = p_{i_j} w^*_{j+1} \cdots w^*_{d+1\,\Gamma(G)}.$$

It follows that $p_jp_{j+1} \cdots p_d = 1$ and $p_1 \cdots p_{j-1} = 1$. Suppose otherwise, and then

$$c_itd_{i\,\Gamma(G)} = aw_j^*b_{\Gamma(G)},$$

while $a, b \notin \ker(\pi_{v_0})$, where π_{v_0} is the canonical surjection from $\Gamma(G) \to G_{v_0}$. Since w_j^* does not commute with G_{v_0}, we see that $c_i t d_{i\,\Gamma(G)} \notin \Gamma_1(G)$, which is a contradiction. We proceed recursively to conclude the claim.

Now, it suffices to show $\hat{v}_0(c_i t d_i) \in \bigcup_{j=1}^n c_j(\Gamma_1 \cap \Gamma_2)(G) d_j$, and this follows from the inductive hypothesis as

$$\widehat{v_0}(c_i t d_i) = c_{i'} \widehat{v_0}(t) d_{i'}$$

for some other i', and $n(\widehat{v_0}(t)) < n(t) = K + 1$. ∎

Lemma 3.6. *Let Γ be a graph. Then,*

$$\bigcup_{v \in V(\Gamma)} X_{\Gamma_{S_v}(G)} = \Delta\Gamma(G).$$

Proof. Set $V(\Gamma) = \{v_i \mid i = 1, \ldots, n\}$ and $\Gamma_i = \Gamma_{S_{V_i}}$ for each i. Note that this statement is equivalent to

$$\bigcap_{i=1}^n c_0(\Gamma(G), \Gamma_i(G)) = c_0(\Gamma(G)),$$

and thus, it suffices to show that $|\bigcap_{i=1}^n s_i \Gamma_i(G) t_i| < \infty$ for any $s_i, t_i \in \Gamma(G)$.

By Lemma 3.5, $\bigcap_{i=1}^n s_i \Gamma_i(G) t_i \subset \bigcup_{j=1}^m g_j (\bigcap_{i=1}^n \Gamma_i)(G) h_j$ for some finite subset $\{g_j, h_j \mid 1 \leq j \leq m\}$ of $\Gamma(G)$. Note that $\bigcap_{i=1}^n S_{V_i} = \emptyset$, and thus,

$$\left(\bigcap_{i=1}^n \Gamma_i\right)(G) = \{e\};$$

i.e., $\bigcap_{i=1}^n s_i \Gamma_i(G) t_i$ is finite. ∎

Proposition 3.7. *Let Γ be a finite graph with $|V(\Gamma)| \geq 3$. The following are equivalent.*

(1) *Γ satisfies the following condition: $r(\Gamma)$ is at least 2 and there does not exist a pair of vertices $v_1, v_2 \in V(\Gamma)$ such that $(u, v_i) \in E(\Gamma)$ for all $u \in V(\Gamma) \setminus \{v_1, v_2\}$, and $i = 1, 2$.*

(2) *For all choices of non-trivial groups G_v, $v \in V(\Gamma)$, the graph product $\Gamma(G)$ is properly proximal.*

Proof of Proposition 3.7. First, we show that (2)⇒(1). Suppose that there is a radius 2 graph such that there is a pair of vertices v_1, v_2 satisfying $(u, v_i) \in E(\Gamma)$ for all $u \in V(\Gamma) \setminus \{v_1, v_2\}$, and $i = 1, 2$, then one can simply consider

$$G_{v_1} = \mathbb{Z}/2\mathbb{Z} = G_{v_2}.$$

Since $(v_1, v_2) \notin E(\Gamma)$, we see that

$$\Gamma(G) = \mathbb{Z}_2 * \mathbb{Z}_2 \times \Gamma_{V(\Gamma)\setminus\{v_1,v_2\}}(G)$$

is inner amenable and therefore not properly proximal by [1, Proposition 4.11].

For (1)⇒(2), observe that, by Lemma 2.4, we have a natural amalgamated free product decomposition: $\Gamma(G) \cong G_1 *_H G_2$, using the same notation. Since Γ satisfies (1), we have

$$\max\{[G_1 : H], [G_2 : H]\} = \infty,$$

and H is neither G_1 nor G_2. Then, it follows immediately from Lemma 3.6 and Corollary 3.2. ∎

It follows directly that, for an arbitrary choice of non-trivial groups indexed by vertices, $\Gamma(G)$ is properly proximal, provided that Γ is an irreducible graph or a graph satisfying $r(\Gamma) \geq 3$.

Proposition 3.8. *Let Γ be such that $r(\Gamma) \geq 2$. Then, $\Gamma(G)$ either is properly proximal or contains an infinite amenable Cartesian factor.*

Proof. The only case to check is if Γ has radius 2 and it does not satisfy condition (1) of Proposition 3.7. That is, there exist vertices v_1, v_2 such that $S = V(\Gamma) \setminus \{v_1, v_2\}$ is connected to v_1 and v_2. Then, since v_1 is not connected to v_2 (else it violates the radius 2 condition), we have

$$\Gamma(G) \cong (G_{v_1} * G_{v_2}) \times \Gamma_S(G).$$

The left summand is either infinite amenable ($\mathbb{Z}_2 * \mathbb{Z}_2$) or properly proximal (since they are non-trivial free products). The above decomposition can be iterated on S and further on, using the hypothesis. The result follows from the fact that properly proximal groups cannot have infinite amenable Cartesian factors because then they are inner amenable [1, Proposition 4.11]. ∎

We thank J. Peterson for suggesting the proof of the following proposition. A generalization of the result appears in [11].

Proposition 3.9. *A direct product $G = G_1 \times G_2$ is properly proximal if and only if G_1 and G_2 are properly proximal.*

Proof. One direction is proved in [1, Proposition 4.10]. Define the projection

$$p = \bigvee_{g \in G_2} 1_{G_1 \times \{g\}} \in (\ell^\infty(G)/c_0(G))^{**}.$$

We claim that the following isomorphism holds:

$$\bigoplus_{G_2}^{\infty} (\ell^\infty G_1/c_0(G_1))^{**} \cong p(\ell^\infty G/c_0(G))^{**},$$

where the direct sum is equipped with ℓ^∞-norm. Indeed, for any finite subset $F \subset G_2$, consider the isomorphism

$$\Theta_F : \bigoplus_F \ell^\infty G_1 \ni (f_h)_{h \in F} \mapsto \tilde{f}_F \in 1_{G_1 \times F} \ell^\infty G,$$

where $\tilde{f}_F(g,h) = f_h(g)$ for $g \in G_1$, $h \in F$. Composing with the quotient map, one has an isomorphism

$$\Theta_F : \bigoplus_F (\ell^\infty G_1/c_0(G_1)) \to 1_{G_1\times F}(\ell^\infty G/c_0(G)).$$

Furthermore, we may extend this map to a weak* continuous isomorphism

$$\widetilde{\Theta}_F : \bigoplus_F (\ell^\infty G_1/c_0(G_1))^{**} \to 1_{G_1\times F}(\ell^\infty G/c_0(G))^{**}.$$

For each $(f_h)_{h\in G_2} \in \bigoplus_{G_2}(\ell^\infty G_1/c_0(G_1))^{**}$, define $\widetilde{\Theta}((f_h))$ to be the SOT limit point of the net $\{\widetilde{\Theta}_F((f_h)_{h\in F})\}_F$ in $p(\ell^\infty G/c_0(G))^{**}$, where F ranges over all finite subsets of G_2; conversely, for $pg \in p(\ell^\infty G/c_0(G))^{**}$, set $\widetilde{\Theta}^{-1}(pg)$ to be the weak* limit of $\{\widetilde{\Theta}_F^{-1}(1_{G_1\times F}\, g)\}_F$ in $\bigoplus_{G_2}(\ell^\infty G_1/c_0(G_1))^{**}$. Thus, $\widetilde{\Theta}$ implements the required isomorphism, which is also G-equivariant by construction.

Now, by taking the G-right invariant subspaces in both sides, we have

$$\big((\ell^\infty G_1/c_0(G_1))^{**}\big)^{G_{1r}} = \left(\bigoplus_{G_2}^{\infty}(\ell^\infty G_1/c_0(G))^{**}\right)^{G_r} \cong p\big((\ell^\infty G/c_0(G))^{**}\big)^{G_r}.$$

Now, suppose that there is a left G_1-invariant state φ on $((\ell^\infty G_1/c_0(G_1))^{**})^{G_{1r}}$, then $\varphi \circ \widetilde{\Theta}^{-1}$ is a left G-invariant state on $p((\ell^\infty G/c_0(G))^{**})^{G_r}$. Set

$$\psi(f) = \varphi \circ \widetilde{\Theta}^{-1}(pf)$$

for $f \in ((\ell^\infty G/c_0(G))^{**})^{G_r}$, and we obtain a G-invariant state, which contradicts the proper proximality of G. ■

Proof of Theorem 1.8. In the case $|\Gamma| = 2$, it follows directly from the fact that non-amenable convergence groups are properly proximal [1] and Proposition 3.9.

In the case $|V(\Gamma)| \geq 3$, note that if Γ is irreducible, then $r(\Gamma) \geq 2$. The rest follows from Propositions 3.7 and 3.9. ■

Remark 3.10. In [16] (see also [12, Remark 4.12]), it is shown that proper CAT(0) cubical complex groups are properly proximal. Graph products admit natural actions on CAT(0) cube complexes due to [10, 20]; however, these actions are in general, not proper. Thus, Theorem 1.8 cannot be deduced directly from [16].

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