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Monodromy of double elliptic logarithms

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Abstract – We determine the relative monodromy group of abelian logarithms with respect to periods in the cases of fibered products of elliptic schemes. This gives rise to a result stronger than a theorem due to Y. André and implies in particular the algebraic independence of the logarithm of any non-torsion section and the periods. We then conjecture an analogous result for the general case of an abelian scheme of arbitrary relative dimension. This generalizes a theorem of Corvaja and Zannier which determines the said group in the case of a single elliptic scheme.

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1. Introduction

The following paper is devoted to the study of the monodromy of *double elliptic logarithms*, i.e. a generalized notion of logarithm defined on fibered products of elliptic schemes.

Let us consider an abelian scheme $\mathcal{A} \to B$ over an algebraic curve and a section $\sigma\colon B \to \mathcal{A}$. Period functions, abelian logarithms of σ and the Betti map can always be globally defined on the universal cover of B, but they cannot in general be well defined on the whole of B. We are interested in studying the minimal unramified cover on which abelian logarithm and periods become well defined in the case of products of elliptic schemes.

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This analysis starts from a paper by Corvaja and Zannier (see [4]), where they study the monodromy problem in the case of a non-isotrivial elliptic scheme $\mathcal{E} \to B$. In that case, we can consider the minimal unramified cover $B^* \to B$ on which periods become well defined and the minimal unramified cover $B_\sigma \to B^*$ on which an elliptic logarithm of σ becomes well defined. They proved the following:

Theorem 1.1. *Given a non-torsion (rational) section $\sigma \colon B \to \mathcal{E}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^2$.*

In the context of abelian schemes of arbitrary relative dimension, a theorem due to André [1] provides, under suitable assumptions, the best possible information about the Zariski closure of the image of the monodromy representation of the fundamental group, associated to a section; here, we want to determine the relative monodromy group, which is more than the information provided passing through the Zariski closure of the monodromy group (see Remark 2.4). This paper aims at extending Theorem 1.1 to all fibered products of two non-isotrivial elliptic schemes, which are abelian schemes of relative dimension 2. First we consider an abelian scheme $\mathcal{A} \to B$ and a section $\sigma \colon B \to \mathcal{A}$. In analogy with the case of elliptic schemes, we can consider the minimal unramified cover $B^* \to B$ on which periods become well defined and the minimal unramified cover $B_\sigma \to B^*$ on which an abelian logarithm of σ becomes well defined. We begin by stating the following conjecture (which is beyond our aims since it concerns abelian schemes of arbitrary relative dimension) and proving that it is invariant under isogeny:

Conjecture (Conjecture 3.1). *Let $\pi \colon \mathcal{A} \to B$ be an abelian scheme of relative dimension g which has no fixed part. If the image of $\sigma \colon B \to \mathcal{A}$ is not contained in any proper group-subscheme, then the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^{2g}$.*

Observe that, as in the case of elliptic schemes, the previous conjecture is stronger than André's theorem and implies in particular the algebraic independence of the logarithm of any non-torsion section and the periods.

Then we will consider the case where $\mathcal{A}$ is a fibered product of the form $\mathcal{E}_1 \times_B \mathcal{E}_2 \to B$, where an elliptic scheme is always assumed to be non-isotrivial.

Remark 1.2. If we consider a trivial elliptic scheme $E \times B \to B$, where E is an elliptic curve defined over $\mathbb{C}$, then the cover $B_\sigma \to B^*$ can be trivial for some non-torsion section σ. First, let us observe that $B^* = B$, since periods can be defined on the whole of B. Moreover, let us consider a non-torsion point $P \in E$ and let us define the non-torsion section

$$\sigma \colon b \mapsto (P, b) \quad \text{for each } b \in B.$$

In this case, the logarithm $\log_\sigma$ is a constant function. Thus, it is well defined on the whole of B and we have $B_\sigma = B^* = B$.

If $\mathcal{E} \to B$ is an isotrivial but non-trivial elliptic scheme then periods can be defined on the whole of B, so that we again have $B^* = B$. In this case the Mordell–Weil theorem for function fields predicts that the group of $\mathbb{C}(B)$-rational points of the generic fiber of $\mathcal{E} \to B$ is finitely generated. Thus, we can prove that a logarithm of a non-trivial section cannot be defined on $B^* = B$ by arguing in the following way: if a non-zero rational section admits an elliptic logarithm which is well defined on the whole of $B(\mathbb{C})$, then we may divide it, and hence the section, by any prescribed positive integer and again we have maps well defined on $B(\mathbb{C})$. Thus the section would be infinitely divisible on $B(\mathbb{C})$ (since the submultiples of the sections would be algebraic and well defined on $B(\mathbb{C})$, hence rational on $B(\mathbb{C})$). But this violates the Mordell–Weil theorem for the generic fiber of $\mathcal{E} \to B$ over the function field of $B(\mathbb{C})$.

This remark shows that the case of an isotrivial elliptic scheme is well understood. This is why we only focus on non-isotrivial elliptic schemes in what follows.

Let us go back to considering fibered products of two elliptic schemes. We will distinguish two cases:

- the elliptic schemes $\mathcal{E}_1 \to B$ and $\mathcal{E}_2 \to B$ are isogenous;
- the elliptic schemes $\mathcal{E}_1 \to B$ and $\mathcal{E}_2 \to B$ are not isogenous.

Distinguishing the two cases, a careful analysis of the relationship between periods and logarithms of sections of the two factors leads us to the following two results:

THEOREM (Theorem 4.7). *Let $\sigma_1 \colon B \to \mathcal{E}_1, \sigma_2 \colon B \to \mathcal{E}_2$ be rational sections of two elliptic schemes such that at least one of them is non-torsion. Suppose that there exists an isogeny $\phi \colon \mathcal{E}_1 \to \mathcal{E}_2$. Let us consider the abelian scheme $\pi \colon \mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$ endowed with the (non-torsion) section $\sigma = (\sigma_1, \sigma_2)$. We have the following situation:*

(1) *If $\phi \circ \sigma_1, \sigma_2$ are linearly dependent over $\mathbb{Z}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^2$.*

(2) *If $\phi \circ \sigma_1, \sigma_2$ are linearly independent over $\mathbb{Z}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^4$.*

THEOREM (Theorem 4.11). *Let $\sigma_i \colon B \to \mathcal{E}_i, i = 1, 2,$ be rational sections of two non-isogenous elliptic schemes and suppose they are not both torsion sections. Let us consider the abelian scheme $\pi \colon \mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$ endowed with the (non-torsion) section $\sigma = (\sigma_1, \sigma_2)$. We have the following situation:*

(1) *If one of σ_1 and σ_2 is a torsion section, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\{0\}$ or to $\mathbb{Z}^2$.*

(2) *If neither* σ_1 *nor* σ_2 *is a torsion section, the cover* $B_\sigma \to B^*$ *has infinite degree and its Galois group is isomorphic to* $\mathbb{Z}^4$.

These results determine the relative monodromy of abelian logarithms with respect to periods in the cases of fibered products of elliptic schemes.

2. Abelian schemes and logarithms of sections

Let $\pi\colon \mathcal{A} \to B$ be a complex abelian scheme of relative dimension g; here B is a quasi-projective smooth curve, $\mathcal{A}$ is a quasi-projective variety and $\pi\colon \mathcal{A} \to B$ is a proper surjective morphism all of whose fibers $\mathcal{A}_b := \pi^{-1}(b)$ are abelian varieties of constant dimension g. We always suppose that the abelian scheme $\pi\colon \mathcal{A} \to B$ has no fixed part and that there exists a section $\sigma_0\colon B \to \mathcal{A}$ which marks the origin in each fiber.

Over every point $b \in B$, we have an abelian exponential map $\mathrm{Lie}(\mathcal{A}_b) \to \mathcal{A}_b$, whose kernel is the period lattice. Since $\mathrm{Lie}(\mathcal{A}_b)$ is a complex vector space of dimension g, the period lattice can be seen as a lattice in $\mathbb{C}^g$ for each $b \in B$. The family of Lie algebras $\mathrm{Lie}(\mathcal{A}) \to B$ defines a vector bundle over B and we have the exponential map

$$\exp\colon \mathrm{Lie}(\mathcal{A}) \to \mathcal{A}.$$

Any fiber $\mathcal{A}_b$ is analytically isomorphic to a complex torus $\mathbb{C}^g/\Lambda_b$, where Λ_b is a lattice of (maximal) rank $2g$. On suitable open subsets $U \subset B$ in the complex topology, we can find holomorphic functions $\omega_{U,1}, \dots, \omega_{U,2g}\colon U \to \mathbb{C}^g$ such that $\omega_{U,1}(b), \dots, \omega_{U,2g}(b)$ is a basis of Λ_b for each $b \in U$. Moreover, we may assume that U is simply connected and that B is covered by such sets.

Observe that by restricting the map exp to $U \times \mathbb{C}^g$, we obtain the covering map

$$U \times \mathbb{C}^g \to \mathcal{A}_{|U},$$

where we denote $\pi^{-1}(U)$ by $\mathcal{A}_{|U}$.

Definition 2.1. Let $\sigma\colon B \to \mathcal{A}$ be a section of the abelian scheme and let $U \subset B$ be an open set as above. A *logarithm of* σ *in* U is a lifting of $\sigma_{|U}$ to $U \times \mathbb{C}^g$.

In other words, we have the following commutative diagram:

$$\begin{tikzcd}
 & U \times \mathbb{C}^g \arrow[d] \\
B \supset U \arrow[ru, dashed, "\xi"] \arrow[r, "\sigma_{|U}"'] & \pi^{-1}(U) \subset \mathcal{A}.
\end{tikzcd}$$

Observe that ξ is of the form $\xi = (\mathrm{id}, \tilde{\sigma})$; the holomorphic map $\tilde{\sigma}\colon U \to \mathbb{C}^g$ will be called a *logarithm of* σ *in* U. By definition, saying that $\xi(b)$ is a logarithm of $\sigma(b)$ means that $\exp_b \circ\, \tilde{\sigma}(b) = \sigma(b)$ on U. In what follows, we will denote a logarithm of a section σ by $\log_\sigma$ (instead of $\tilde{\sigma}$).

2.1 – *Monodromy representations*

Given an abelian scheme $\mathcal{A} \to B$, let us consider the fundamental group $G := \pi_1(B, b)$, where $b \in B$ is a fixed base point. Given $h \in G$, we can consider the analytic continuation of the periods along any loop in B belonging to the homotopy class h: to be more precise, if h is the homotopy class of a loop $\alpha\colon [0,1] \to B$ and ω is a period, when we need it we will denote by $c_h(\omega)$ the analytic continuation of ω in $\alpha(1)$ along α. This procedure induces a change of basis of the lattice Λ_b. In other words, the monodromy action of G on periods induces a homomorphism to $\mathrm{GL}_{2g}(\mathbb{Z})$; since the action preserves the orientation of the basis, the image of the said homomorphism is contained in $\mathrm{SL}_{2g}(\mathbb{Z})$, so that we obtain a representation $\rho\colon G \to \mathrm{SL}_{2g}(\mathbb{Z})$ which describes the monodromy of periods. If $\omega = u_1\omega_1 + \cdots + u_{2g}\omega_{2g}$ is a period, the monodromy action on the $\mathbb{Z}$-module generated by periods is given by

$$h \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_{2g} \end{pmatrix} = \rho(h) \begin{pmatrix} u_1 \\ \vdots \\ u_{2g} \end{pmatrix}.$$

Moreover, given a section $\sigma\colon B \to \mathcal{A}$, observe that two branches of a logarithm over $b \in B$ have to differ by an element of Λ_b: thus, for fixed $h \in G$ we have that $\log_\sigma$ transforms in the following way:

$$\log_\sigma \mapsto \log_\sigma + (u_1, \ldots, u_{2g}) \cdot \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix},$$

where $u_1, \ldots, u_{2g} \in \mathbb{Z}$. Observe that the monodromy group of the logarithm, as a function defined locally on the B^* considered in the introduction, is a subgroup of $\mathbb{Z}^{2g}$.

Remark 2.2. Let us consider a torsion section $\sigma\colon B \to \mathcal{A}$. The Betti map of such a section (see [2] for a definition) is constant and a logarithm is a rational constant combination of periods. In other words, we have

$$\log_\sigma = q_1\omega_1 + \cdots + q_{2g}\omega_{2g},$$

where $q_1, \dots, q_{2g} \in \mathbb{Q}$. Therefore, a loop which leaves unchanged periods via analytic continuation, also leaves the logarithm of such a section unchanged. In other words, the cover $B_\sigma \to B^*$ defined in the introduction is trivial in this case. This is why we only consider non-torsion sections in what follows.

In the particular case of an elliptic scheme $\pi\colon \mathcal{E} \to B$, if we look at the simultaneous monodromy action of $G := \pi_1(B)$ on periods and logarithm of a section $\sigma\colon B \to \mathcal{E}$, we provide a representation

$$\theta_\sigma\colon G \to \mathrm{SL}_3(\mathbb{Z}),$$

where every matrix $\theta_\sigma(g)$ is of the form

$$\theta_\sigma(g) = \begin{pmatrix} T_g & w_g \\ 0 & 1 \end{pmatrix},$$

where $T_g = \rho(g) \in \mathrm{SL}_2(\mathbb{Z})$ is a matrix acting on periods (so it does not depend on σ), and $w_g = (u_g, v_g)^\intercal \in \mathbb{Z}^2$ is a vector which corresponds to the following monodromy action on logarithm:

$$\log_\sigma \xrightarrow{g} \log_\sigma + u_g \omega_1 + v_g \omega_2.$$

Theorem 1.1, which determines the relative monodromy group of the logarithm with respect to periods, can be restated in terms of representations as follows:

THEOREM 2.3. *Given a non-torsion (rational) section $\sigma\colon B \to \mathcal{E}$, the kernel of the homomorphism $\theta_\sigma(G) \to \mathrm{SL}_2(\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$, which is equivalent to saying*

$$\theta_\sigma(\ker \rho) \cong \mathbb{Z}^2.$$

REMARK 2.4. Observe that the conclusion of Theorem 2.3 is stronger than knowing the kernel of the homomorphism $\theta_\sigma(G)^{\mathrm{Zar}} \to \mathrm{SL}_2$, obtained by taking the Zariski closure of the group $\theta_\sigma(G)$, as the following example shows. Define H to be the subgroup of $\mathrm{SL}_3(\mathbb{Z})$ generated by the matrices

$$A := \begin{pmatrix} 1 & 2 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{pmatrix} =: \begin{pmatrix} A_0 & v \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & w_1 \\ 2 & 1 & w_2 \\ 0 & 0 & 1 \end{pmatrix} =: \begin{pmatrix} B_0 & w \\ 0 & 1 \end{pmatrix},$$

where A_0, B_0 are the standard unipotent generators of Γ_2 and v, w are a basis for $\mathbb{Z}^2$. It can be shown that the Zariski closure of H is the full semidirect product of SL_2 by $\mathbb{G}_a^2$, whereas the kernel of the natural map to SL_2 is trivial; the details can be found in [3].

3. Monodromy of abelian logarithms: Invariance under isogeny

Let $\pi\colon \mathcal{A} \to B$ be a complex abelian scheme of relative dimension g and let us consider a section $\sigma\colon B \to \mathcal{A}$. We will call $B^* \to B$ the minimal (unramified) cover of B on which a basis for the period lattice can be globally defined; moreover, we set $B_\sigma \to B^*$ to be the minimal cover of B^* on which we can define the logarithm of σ. The tower of covers is represented in the diagram

$$B_\sigma \to B^* \to B.$$

Our aim is to prove something similar to Theorem 2.3 for double elliptic schemes (i.e. fibered products of elliptic schemes). Let us start by formulating the following conjecture (which is beyond our aims since it concerns abelian schemes of arbitrary relative dimension) and by proving it is invariant under isogeny:

Conjecture 3.1. *Let $\pi\colon \mathcal{A} \to B$ be an abelian scheme of relative dimension g which has no fixed part. If the image of $\sigma\colon B \to \mathcal{A}$ is not contained in any proper group-subscheme, then the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^{2g}$.*

Observe that the hypothesis that the image of $\sigma\colon B \to \mathcal{A}$ is not contained in any proper group-subscheme is necessary, as shown by the following example.

Example 3.2. Let $\pi_{\mathcal{E}}\colon \mathcal{E} \to B$ be an elliptic scheme with zero-section denoted by σ_0 and let $\sigma\colon B \to \mathcal{E}$ be a non-torsion section. By Theorem 2.3, we know that the Galois group of $B_\sigma \to B^*$ is isomorphic to $\mathbb{Z}^2$. We can consider the fibered product $\pi_{\mathcal{A}}\colon \mathcal{A} := \mathcal{E} \times_B \mathcal{E} \to B$, which gives rise to an abelian scheme of relative dimension 2. If we denote by $\mathcal{E}_b := \pi_{\mathcal{E}}^{-1}(b)$ the fiber of the elliptic scheme over a point b, then the fiber $\mathcal{A}_b$ is given by the product $\mathcal{E}_b \times \mathcal{E}_b$. The morphism

$$\tilde{\sigma} := (\sigma_0, \sigma)\colon B \to \mathcal{A}, \qquad b \mapsto (0_b, \sigma(b))$$

is a section of the abelian scheme, whose image is contained in the proper group-subscheme $\sigma_0(B) \times_B \mathcal{E} \to B$ of $\mathcal{A} \to B$. Note that the cover $B^* \to B$ is the same for the two schemes $\mathcal{A} \to B$ and $\mathcal{E} \to B$. Moreover, the cover $B_{\tilde{\sigma}} \to B^*$ is the same as the cover $B_\sigma \to B^*$, i.e. the Galois group of $B_{\tilde{\sigma}} \to B^*$ is isomorphic to $\mathbb{Z}^2$, thus in this case it is not as large as possible.

More generally, similar examples can be obtained by considering a section $\tilde{\sigma} = (\sigma_1, \sigma_2)$ where σ_1, σ_2 are linearly dependent sections of the elliptic scheme $\mathcal{E} \to B$. The case of the abelian scheme $\mathcal{E} \times_B \mathcal{E} \to B$ is fully covered in Section 4.1, where we prove the conjecture for the product of isogenous elliptic schemes.

Proof of "invariance under isogeny"

Consider two abelian schemes $\pi\colon \mathcal{A} \to B$, $\pi'\colon \mathcal{A}' \to B$. Recall the following definition:

Definition 3.3. A morphism $f\colon \mathcal{A} \to \mathcal{A}'$ of group schemes over a scheme B is said to be an *isogeny* if f is surjective and if its kernel $\ker f$ is a flat finite group B-scheme.

We start by showing that the above conjecture is isogeny invariant; in other words, if $\mathcal{A}$ and $\mathcal{A}'$ are isogenous, then the conjecture for $\mathcal{A}'$ implies the conjecture for $\mathcal{A}$. In order to prove this, let consider a non-torsion section $\sigma\colon B \to \mathcal{A}$ and suppose that the theorem is true for $\mathcal{A}' \to B$.

Lemma 3.4. *If $f\colon \mathcal{A} \to \mathcal{A}'$ is an isogeny of abelian schemes, then the map $b \mapsto \#\ker(f_{|\mathcal{A}_b})$ is constant.*

Proof. Let us consider the B-scheme $\ker f \xrightarrow{\pi_{|\ker f}} B$ and recall that the fiber of $\pi_{|\ker f}$ over a point $b \in B$ is given by

$$(\ker f)_b = \ker(f_{|\mathcal{A}_b}) = \ker f \times_B \operatorname{Spec} \mathbb{C}(b).$$

By definition of isogeny, the restriction

$$\pi_{|\ker f}\colon \ker f \to B$$

is a flat finite morphism. Then the map

$$B \to \mathbb{N}, \quad b \mapsto \dim_{\mathbb{C}(b)}((\pi_* \mathcal{O}_{\ker f})_b \otimes_{\mathcal{O}_{B,b}} \mathbb{C}(b))$$

is locally constant. Since B is connected, then this function is constant, say

$$\dim_{\mathbb{C}(b)}((\pi_* \mathcal{O}_{\ker f})_b \otimes_{\mathcal{O}_{B,b}} \mathbb{C}(b)) = q \quad \text{for each } b \in B,$$

where $q \in \mathbb{N}$. Then $(\pi_* \mathcal{O}_{\ker f})_b \otimes_{\mathcal{O}_{B,b}} \mathbb{C}(b)$ is isomorphic to $\mathbb{C}(b)^q$ as a vector space. Since the fiber $(\ker f)_b$ is an algebraic group (in characteristic zero), hence it is reduced, then $\ker(f_{|\mathcal{A}_b})$ is a disjoint union of q points. ■

Let consider the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "f"] \arrow[dr, "\pi"] & & \mathcal{A}' \arrow[dl, "\pi'"'] \\
& B \arrow[ul, bend left, "\sigma"] \arrow[ur, dashed, bend right, "\sigma':=f\circ\sigma"'] &
\end{tikzcd}$$

where given σ as above we define $\sigma' := f \circ \sigma$. Obviously, this last is a section of $\mathcal{A}' \to B$ since

$$\pi' \circ \sigma' = \pi' \circ f \circ \sigma = \pi \circ \sigma = \mathrm{id}_B.$$

Proposition 3.5. *If σ is a non-torsion section of $\mathcal{A} \to B$, then σ' is a non-torsion section of $\mathcal{A}' \to B$.*

Proof. As we have just observed, σ' is a section of $\mathcal{A}' \to B$. We prove the equivalent statement "σ' torsion $\Rightarrow \sigma$ torsion". So suppose $k\sigma' = 0$ for some k. Since $f_{|\mathcal{A}_b}$ is a morphism, we have

$$f(k\sigma(b)) = k(f \circ \sigma(b)) = k\sigma'(b) = 0$$

for each b. This means $k\sigma(b) \in \ker(f_{|\mathcal{A}_b})$ for each b. By Lemma 3.4, $\ker(f_{|\mathcal{A}_b})$ is a finite group of fixed order q for each $b \in B$. Therefore $qk\sigma(b) = 0$ for each b. This means $(qk)\sigma = 0$; in other words, σ is torsion. ■

Theorem 3.6 (Invariance under isogeny). *Let $\mathcal{A} \to B$, $\mathcal{A}' \to B$ be two abelian schemes of relative dimension g and $f\colon \mathcal{A} \to \mathcal{A}'$ an isogeny. If Conjecture 3.1 holds for $\mathcal{A}' \to B$, then it holds for $\mathcal{A} \to B$.*

Proof. Let $\sigma\colon B \to \mathcal{A}$ be a non-torsion section and define, as above, $\sigma' := f \circ \sigma$. We have the following two towers of coverings:

$$B_\sigma \to B_1^* \to B,$$
$$B_{\sigma'} \to B_2^* \to B,$$

which correspond to the relative monodromy problems for $\log_\sigma$, $\log_{\sigma'}$, respectively. Since $\mathcal{A}$ and $\mathcal{A}'$ are isogenous, the periods of $\mathcal{A}$ are related to those of $\mathcal{A}'$ through a matrix in $\mathrm{GL}_{2g}(\mathbb{Q})$ (this matrix does not depend on $b \in B$). In order to prove this, let us consider the fibers $\mathcal{A}_b$, $\mathcal{A}'_b$ and let us denote by $\omega_i, \omega'_i \in \mathbb{C}^g$ (as row vectors), for $i = 1, \dots, 2g$, the corresponding periods. The isogeny f induces an isogeny on the fibers, i.e. $f_b\colon \mathcal{A}_b \to \mathcal{A}'_b$. So there exists $M(= M_b) \in \mathrm{GL}_g(\mathbb{C})$ such that

$$\begin{aligned}\omega_1 \cdot M &= a_{1,1}\omega'_1 + \cdots + a_{1,2g}\omega'_{2g},\\ &\vdots\\ \omega_{2g} \cdot M &= a_{2g,1}\omega'_1 + \cdots + a_{2g,2g}\omega'_{2g},\end{aligned}$$

where $a_{i,j} \in \mathbb{Z}$ for each i, j. Thus we obtain the relation

$$\begin{pmatrix} \omega_1 M \\ \vdots \\ \omega_{2g} M \end{pmatrix} = \zeta \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix}, \tag{3.1}$$

where we denote by ζ the matrix $(a_{i,j})_{i,j=1,\dots,2g} \in \mathrm{GL}_{2g}(\mathbb{Q})$.

Remark 3.7. Let us consider an isogeny $f\colon A_1 \to A_2$ between complex abelian varieties of dimension g. Then we have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^g \arrow[r, "\varphi"] \arrow[d] & \mathbb{C}^g \arrow[d] \\
\mathbb{C}^g/\Lambda_1 \arrow[r, "f"] & \mathbb{C}^g/\Lambda_2,
\end{tikzcd}$$

where φ is an isomorphism obtained by covering theory. The isomorphism φ can be expressed by right multiplication by a matrix $M \in \mathrm{GL}_g(\mathbb{C})$. Let us prove that M is uniquely determined by f. In fact, if two matrices M, N induce the same isogeny, we have

$$z \cdot M \equiv z \cdot N \pmod{\Lambda_2} \quad \text{for all } z \in \mathbb{C}^g.$$

Hence the map $z \mapsto z \cdot (M - N)$ sends $\mathbb{C}^g$ to Λ_2. Since Λ_2 is discrete, the map must be constant. This implies $M = N$.

Remark 3.8. By the previous remark, the matrix $M = M_b$ considered above is uniquely determined by f_b; moreover, the function $b \mapsto M_b$ is a holomorphic function on B. In fact, we can consider the diagram

$$\begin{tikzcd}
U \times \mathbb{C}^g \arrow[r, "\varphi"] \arrow[d, "\exp"] \arrow[dr, dashed, "\psi"] & U \times \mathbb{C}^g \arrow[d, "\exp'"] \\
\mathcal{A}_{|U} \arrow[r, "f"] & \mathcal{A}'_{|U},
\end{tikzcd}$$

where U is a simply connected open set, $\psi := f \circ \exp$ and φ is a lift of ψ (it exists because $U \times \mathbb{C}^g$ is simply connected). We necessarily have

$$\varphi_{|\{b\}\times\mathbb{C}^g} = [M_b],$$

where $[M_b]$ is right multiplication by M_b. Since φ is holomorphic, so is $b \mapsto M_b$. Moreover, since M_b is uniquely determined by f_b, then the function $b \mapsto M_b$ cannot have non-trivial monodromy along loops. So it is well defined on the whole of B.

Now let us return to equation (3.1). In particular, it means that the monodromy action of $\pi_1(B)$ on periods of $\mathcal{A}$ is determined by the monodromy action on the periods of $\mathcal{A}'$. To be more precise, let us consider a period ω' with coordinate vector $(u'_1, \dots, u'_{2g})$ with respect to the basis ω'_i, i.e.

$$\omega' := (u'_1, \dots, u'_{2g}) \cdot \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix}.$$

By (3.1), we obtain

$$\omega' = (u'_1, \dots, u'_{2g})\zeta^{-1} \begin{pmatrix} \omega_1 M \\ \vdots \\ \omega_{2g} M \end{pmatrix}.$$

Let us look at the monodromy action on the bases ω'_i and ω_i; denote by ρ, ρ' the monodromy representations of $\mathcal{A}$, $\mathcal{A}'$ respectively. Since M_b varies holomorphically and without monodromy with respect to $b \in B$ and ζ does not depend on $b \in B$, we obtain

$$h \cdot \omega' = h \cdot (u'_1, \dots, u'_{2g}) \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix} = (u'_1, \dots, u'_{2g})\rho'(h)^{\mathsf{T}} \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix},$$

$$h \cdot \omega' = h \cdot (u'_1, \dots, u'_{2g})\zeta^{-1} \begin{pmatrix} \omega_1 M \\ \vdots \\ \omega_{2g} M \end{pmatrix} = (u'_1, \dots, u'_{2g})\zeta^{-1}\rho(h)^{\mathsf{T}} \begin{pmatrix} \omega_1 M \\ \vdots \\ \omega_{2g} M \end{pmatrix}.$$

Combining the previous relations, we obtain

$$\begin{aligned} (u'_1, \dots, u'_{2g})\rho'(h)^{\mathsf{T}} \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix} &= (u'_1, \dots, u'_{2g})\zeta^{-1}\rho(h)^{\mathsf{T}} \begin{pmatrix} \omega_1 M \\ \vdots \\ \omega_{2g} M \end{pmatrix} \\ &= (u'_1, \dots, u'_{2g})\zeta^{-1}\rho(h)^{\mathsf{T}}\zeta \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix}, \end{aligned}$$

for all $u'_1, \dots, u'_{2g} \in \mathbb{Z}$. In other terms, the two representations are conjugated between them, i.e. $\rho'(h) = \zeta^{\mathsf{T}}\rho(h)(\zeta^{\mathsf{T}})^{-1}$. Therefore, the periods of $\mathcal{A}$ are defined over a cover $B^* \to B$ if and only if the periods of $\mathcal{A}'$ are. In other words, we have $B^*_1 = B^*_2$.

Now let us study the logarithms of the two abelian schemes. Let $U \subset B$ be a simply connected open set and consider $\log_\sigma$, $\log_{\sigma'}$:

$$\begin{tikzcd}
& U \times \mathbb{C}^g \arrow[r, "\varphi"] \arrow[d, "\exp"] & U \times \mathbb{C}^g \arrow[d, "\exp'"] & \\
B \supset U \arrow[ur, dashed, "{(\mathrm{id},\log_\sigma)}"] \arrow[r, "\sigma"'] & \mathcal{A}_{|U} \arrow[r, "f"] & \mathcal{A}'_{|U} & U \subset B. \arrow[l, "\sigma'"] \arrow[ul, dashed, "{(\mathrm{id},\log_{\sigma'})}"']
\end{tikzcd}$$

As stated above, the periods of $\mathcal{A}$ are related to those of $\mathcal{A}'$ through a matrix ζ in $\mathrm{GL}_{2g}(\mathbb{Q})$; we continue to use the above notation. The induced isogeny f_b is given by right multiplication with a matrix $M_b \in \mathrm{GL}_g(\mathbb{C})$; in other words, we have the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^g \arrow[r, "\cdot M_b"] \arrow[d] & \mathbb{C}^g \arrow[d] \\
\mathcal{A}_b \arrow[r, "f_{|\mathcal{A}_b}"] & \mathcal{A}'_b.
\end{tikzcd}$$

This means that we can choose $\log_{\sigma'}(b) = \log_\sigma(b) \cdot M_b$, for a point $b \in B$. For fixed $h \in G$ let us recall that $\log_\sigma$ transforms in the following way:

$$\log_\sigma \mapsto \log_\sigma + (u_1, \ldots, u_{2g}) \cdot \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix}.$$

Moreover, as remarked above, $M_b = M$ varies holomorphically with respect to b and there is no monodromy action on it. Therefore, for σ' we have

$$\begin{aligned}
\log_{\sigma'} = \log_\sigma \cdot M_b \mapsto & \left(\log_\sigma + (u_1, \ldots, u_{2g}) \cdot \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_{2g} \end{pmatrix} \right) \cdot M_b \\
& = \log_\sigma \cdot M + (u_1, \ldots, u_{2g}) \cdot \begin{pmatrix} \omega_1 M \\ \vdots \\ \omega_{2g} M \end{pmatrix} \\
& = \log_{\sigma'} + (u_1, \ldots, u_{2g}) \zeta \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_{2g} \end{pmatrix}.
\end{aligned}$$

Thus we obtain that the monodromies of logarithms are related in the following way:

$$\begin{pmatrix} u'_1 \\ \vdots \\ u'_{2g} \end{pmatrix} = \zeta^\intercal \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_{2g} \end{pmatrix},$$

where u_i and u'_i describe the monodromies of $\log_\sigma$, $\log_{\sigma'}$, respectively. It follows that $B_\sigma = B_{\sigma'}$; this means that if Conjecture 3.1 holds for $\mathcal{A}' \to B$, then it holds for $\mathcal{A} \to B$. ■

4. Monodromy of double elliptic logarithms

Now we will analyze *double elliptic schemes* over the same base, usually a curve.

Such a scheme may be seen as a fiber product of elliptic schemes, or a pair of elliptic curves defined over a function field of the (same) curve.

To settle things in precise terms, let us suppose we are given two non-isotrivial elliptic schemes $\mathcal{E}_i \to B$, for $i = 1, 2$, over the same base B, supposed to be an affine (ramified) cover of $S := \mathbb{P}_1 - \{0, 1, \infty\}$. By taking a cover of B if necessary, we may assume that these elliptic schemes are pullbacks of the Legendre scheme (i.e. the elliptic scheme $\mathcal{L} \to S$ defined by the equation $y^2z = x(x-z)(x-\lambda z)$, $\lambda \in S$). Each of these elliptic schemes has associated periods and monodromy action of $\pi_1(B)$ on the corresponding periods: this action yields subgroups G_1, G_2 of $\Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z})$, both of finite index. In other words, we have the corresponding monodromy representations:

$$\rho_{\mathcal{E}_1}\colon \pi_1(B) \to G_1 \subset \Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z}),$$
$$\rho_{\mathcal{E}_2}\colon \pi_1(B) \to G_2 \subset \Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z}).$$

This setting is equivalent to considering the abelian scheme $\mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$. Hence, putting together what we said about $\mathcal{E}_1$, $\mathcal{E}_2$, we have a representation

$$\rho = (\rho_{\mathcal{E}_1}, \rho_{\mathcal{E}_2})\colon \pi_1(B) \to G_1 \times G_2 \subset \Gamma_2 \times \Gamma_2 \subset \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z}),$$

where we identify $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ as a subgroup of $\mathrm{SL}_4(\mathbb{Z})$. Observe that ρ is exactly the monodromy representation associated with $\mathcal{A}$, so we denote it by $\rho_{\mathcal{A}}$. Thus, we have

$$\rho_{\mathcal{A}}(g) = \begin{pmatrix} \rho_{\mathcal{E}_1}(g) & 0 \\ 0 & \rho_{\mathcal{E}_2}(g) \end{pmatrix}.$$

If $\mathcal{E}_1 \to B$, $\mathcal{E}_2 \to B$ are isogenous elliptic schemes (we may assume the isogeny to be defined over $\mathbb{C}(B)$), then the periods of $\mathcal{E}_1$ are related to those of $\mathcal{E}_2$ through a matrix in $\mathrm{GL}_2(\mathbb{Q})$. This reflects in the fact that there exists a constant matrix $\zeta \in \mathrm{GL}_2(\mathbb{Q})$ such that $\rho_{\mathcal{E}_2} = \zeta^{-1}\rho_{\mathcal{E}_1}\zeta$. Thus, in particular, the image of $\rho_{\mathcal{A}}$ is a graph, and the same holds for its Zariski closure in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$, so we may express this by considering it to be "small". The following theorem, which we only state, establishes a converse assertion, namely whether a "small" image necessarily implies the existence of an isogeny. A detailed proof of it is carried out by Lang in a different setting in [5] with methods which take the Galois action directly into account; instead, an alternative treatment based entirely on the Galois action as induced by the monodromy action is given by Corvaja and Zannier [3].

THEOREM 4.1 (Isogeny theorem). *Let $\mathcal{E}_1$, $\mathcal{E}_2$ be elliptic schemes over B, as above, and consider the monodromy representations as above. Then either the Zariski closure of the image of ρ is the whole of $\mathrm{SL}_2 \times \mathrm{SL}_2$, or $\mathcal{E}_1$, $\mathcal{E}_2$ are isogenous (over a cover of B) and there exists $\sigma \in \mathrm{GL}_2(\mathbb{Q})$ such that $\rho_2(g) = \sigma^{-1}\rho_1(g)\sigma$ for all $g \in \pi_1(B)$.*

Also, for a large enough prime number p, either the image of ρ in $\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$ *is dense in the whole group or we fall into the same conclusion.*

Now let us consider two sections σ_1 and σ_2 of $\mathcal{E}_1 \to B$ and $\mathcal{E}_2 \to B$, respectively. This setting is equivalent to considering the abelian scheme $\mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$ with a section σ, whose components are σ_1, σ_2. Any loop γ whose homotopy class g is in $\pi_1(B, b_0)$ gives rise to a matrix $\rho_{\mathcal{A}}(g) \in \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ which describes the monodromy of periods and to a column vector $w_g \in \mathbb{Z}^4$ which describes the monodromy of logarithm. Thus we have a representation of the fundamental group $\pi_1(B)$ in $\mathrm{SL}_5(\mathbb{Z})$, given by

$$\begin{aligned} \theta_\sigma \colon \pi_1(B) &\to \mathrm{SL}_5(\mathbb{Z}), \\ g &\mapsto \begin{pmatrix} \rho_{\mathcal{A}}(g) & w_g \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where

$$\rho_{\mathcal{A}}(g) = \begin{pmatrix} \rho_{\mathcal{E}_1}(g) & 0 \\ 0 & \rho_{\mathcal{E}_2}(g) \end{pmatrix}, \quad w_g = \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ v_{1,g} \\ v_{2,g} \end{pmatrix}.$$

Observe that the logarithms $\log_{\sigma_1}, \log_{\sigma_2}$ transform in the following way:

$$\begin{aligned} \log_{\sigma_1} &\overset{g}{\mapsto} \log_{\sigma_1} + u_{1,g}\omega_{1,\mathcal{E}_1} + u_{2,g}\omega_{2,\mathcal{E}_1}, \\ \log_{\sigma_2} &\overset{g}{\mapsto} \log_{\sigma_2} + v_{1,g}\omega_{1,\mathcal{E}_2} + v_{2,g}\omega_{2,\mathcal{E}_2}, \end{aligned}$$

where $\omega_{1,\mathcal{E}_i}, \omega_{2,\mathcal{E}_i}$ denote the periods of $\mathcal{E}_i \to B$. Moreover, we have

$$w_{gh} = w_g + \rho_{\mathcal{A}}(g) \cdot w_h = \begin{pmatrix} \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} + \rho_{\mathcal{E}_1}(g) \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} \\ \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} + \rho_{\mathcal{E}_2}(g) \cdot \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix} \end{pmatrix}.$$

Finally, let us recall the following notation for the elliptic schemes $\mathcal{E}_i \to B$, $i = 1, 2$:

$$\begin{aligned} \theta_{\sigma_i} \colon \pi_1(B) &\to \mathrm{SL}_3(\mathbb{Z}), \\ g &\mapsto \begin{pmatrix} \rho_{\mathcal{E}_i}(g) & w_{i,g} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

where

$$w_{1,g} := \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix}, \quad w_{2,g} := \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix}.$$

Before dealing with the main theorems, we prove a lemma which will be very useful in what follows.

LEMMA 4.2. *Let* $H \trianglelefteq G := \pi_1(B)$ *be a normal subgroup of* $\pi_1(B)$. *If* $H \subset \ker \rho_{\mathcal{E}_1}$ *(resp.* $H \subset \ker \rho_{\mathcal{E}_2}$*), then* $\theta_{\sigma_1}(H)$ *(resp.* $\theta_{\sigma_2}(H)$*) is isomorphic to either* $\{0\}$ *or* $\mathbb{Z}^2$.

PROOF. We will give the proof only for the case $H \subset \ker \rho_{\mathcal{E}_1}$, since the other case is analogous. Since we are going to work only with the scheme $\mathcal{E}_1 \to B$, let us denote the periods $\omega_{1,\mathcal{E}_1}, \omega_{2,\mathcal{E}_1}$ simply by ω_1, ω_2 in this lemma.

First, since $H \subset \ker \rho_{\mathcal{E}_1}$ observe that $\theta_{\sigma_1}(H)$ is a subgroup of $\mathbb{Z}^2$; so it is isomorphic to either $\{0\}$ or $\mathbb{Z}$ or $\mathbb{Z}^2$. We want to prove that the case $\mathbb{Z}$ is excluded. Suppose by contradiction that $\theta_{\sigma_1}(H)$ is infinite cyclic: this means that for every $h \in H$, the logarithm $\log_{\sigma_1}$ of σ_1 is transformed by h as

$$\log_{\sigma_1} \overset{h}{\mapsto} \log_{\sigma_1} + \chi(h)\omega_{\sigma_1},$$

for a fixed non-zero period ω_{σ_1} and a homomorphism $\chi\colon H \to \mathbb{Z}$. In particular, let us choose h such that $\chi(h) = 1$. Recall that, for $g \in G = \pi_1(B)$, the logarithm $\log_{\sigma_1}$ will be sent by g to a new determination of the form

$$\log_{\sigma_1} + \omega_g,$$

where ω_g is a period. Recall that the monodromy group $G_1 = \rho_{\mathcal{E}_1}(G)$ is Zariski dense in $\mathrm{SL}_2(\mathbb{Z})$: in fact, up to a finite base change, an arbitrary non-isotrivial elliptic scheme can always be supposed to be obtained as a pullback of the Legendre elliptic scheme, which has a Zariski-dense monodromy group (for more details see [3]). Therefore, the group $G_1 = \rho_{\mathcal{E}_1}(G)$ acts irreducibly on the lattice of periods, since it is Zariski dense in $\mathrm{SL}_2(\mathbb{Z})$. Then there exists $g \in G$ such that ω_{σ_1} is not an eigenvector of $\rho_{\mathcal{E}_1}(g)$. Since $H \trianglelefteq G$, we have $h' = g^{-1}hg \in H$, where g, h are the ones just considered. Let us calculate the action of the element $h' = g^{-1}hg$. Recalling the notation introduced in Section 2.1, given $h \in \pi_1(B)$ and a period ω, we denote by $c_h(\omega)$ the analytic continuation of ω along any loop whose homotopy class is h. Thus, we have

$$\log_{\sigma_1} \xmapsto{g} \log_{\sigma_1} + \omega_g \xmapsto{h} \log_{\sigma_1} + \omega_g + \omega_{\sigma_1} \xmapsto{g^{-1}} \log_{\sigma_1} + c_{g^{-1}}(\omega_{\sigma_1}).$$

Since $c_{g^{-1}}(\omega_{\sigma_1}) = \rho_{\mathcal{E}_1}(g^{-1})\omega_{\sigma_1}$, we obtain $\rho_{\mathcal{E}_1}(g^{-1})\omega_{\sigma_1} = \chi(h')\omega_{\sigma_1}$, but this is a contradiction since ω_{σ_1} is not an eigenvector of $\rho_{\mathcal{E}_1}(g)$ (nor of $\rho_{\mathcal{E}_1}(g^{-1})$). This concludes the proof. ∎

4.1 – *Case* 1*: Product of isogenous elliptic schemes*

In this section we will formulate a result on the monodromy of the logarithm of a section σ in the case in which $\mathcal{E}_1$, $\mathcal{E}_2$ are isogenous; we use the above notation $\mathcal{A} = \mathcal{E}_1 \times_B \mathcal{E}_2$. In this case the monodromy representations are conjugate (see Theorem 4.1), so we have

$$\ker \rho_{\mathcal{A}} = \ker \rho_{\mathcal{E}_1} = \ker \rho_{\mathcal{E}_2}.$$

Moreover, in what follows we will make the following identifications: if $g \in \ker \rho_{\mathcal{A}}$ we identify $\theta_{\sigma_i}(g) \equiv w_{i,g} \in \mathbb{Z}^2$ and $\theta_\sigma(g) \equiv w_g \in \mathbb{Z}^4$. Moreover, we define

$$H_1 := \theta_{\sigma_2}(\ker \theta_{\sigma_1}), \quad H_2 := \theta_{\sigma_1}(\ker \theta_{\sigma_2}).$$

Now we are ready for the results of this section.

Lemma 4.3. *The groups H_1, H_2 are isomorphic to either $\{0\}$ or $\mathbb{Z}^2$.*

Proof. This follows by Lemma 4.2, since $\ker \theta_{\sigma_1}$ and $\ker \theta_{\sigma_2}$ are normal subgroups of $\pi_1(B)$ and $\ker \rho_{\mathcal{E}_1} = \ker \rho_{\mathcal{E}_2}$. ■

Proposition 4.4. *Suppose that at least one of σ_1 and σ_2 is non-torsion. The group $\theta_\sigma(\ker \rho_{\mathcal{A}})$ is isomorphic to either $\mathbb{Z}^2$ or $\mathbb{Z}^4$.*

Proof. We prove the theorem supposing that σ_1 is non-torsion.

Recall that by Theorem 2.3, we have $\theta_{\sigma_1}(\ker \rho_{\mathcal{E}_1}) \cong \mathbb{Z}^2$. Since $\ker \rho_{\mathcal{A}} = \ker \rho_{\mathcal{E}_1}$, then we have $2 \leq \operatorname{rank} \theta_\sigma(\ker \rho_{\mathcal{A}}) \leq 4$. By the previous lemma, we only have two possibilities for H_1, i.e. $H_1 \cong \{0\}$ or $H_1 \cong \mathbb{Z}^2$.

Case 1: $H_1 \cong \{0\}$. The condition $H_1 \cong \{0\}$ means that for each $g \in \ker \rho_{\mathcal{A}}$ if $u_{1,g} = u_{2,g} = 0$, then $v_{1,g} = v_{2,g} = 0$, where the notation is the same as above. Let us prove that $\operatorname{rank} \theta_\sigma(\ker \rho_{\mathcal{A}}) = 2$ by proving that any three elements of the form $w_g, w_h, w_k \in \theta_\sigma(\ker \rho_{\mathcal{A}})$ are linearly dependent on $\mathbb{Z}$.

Since $\theta_{\sigma_1}(\ker \rho_{\mathcal{E}_1}) \cong \mathbb{Z}^2$, given any three elements $g, h, k \in \ker \rho_{\mathcal{A}} \subset \ker \rho_{\mathcal{E}_1}$, there always exists $n_g, n_h, n_k \in \mathbb{Z}$, not all zero, such that

$$n_g \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} + n_h \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} + n_k \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we have

$$\theta_\sigma(k^{n_k} h^{n_h} g^{n_g}) = \begin{pmatrix} & 0 \\ & 0 \\ \mathrm{id}_4 & n_g v_{1,g} + n_h v_{1,h} + n_k v_{1,k} \\ & n_g v_{2,g} + n_h v_{2,h} + n_k v_{2,k} \\ 0 & 1 \end{pmatrix}.$$

For what we observed at the beginning of this proof, $H_1 \cong \{0\}$ implies that

$$n_g \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} + n_h \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix} + n_k \begin{pmatrix} v_{1,k} \\ v_{2,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, any three elements of $\mathbb{Z}^4$ of the form w_g, w_h, w_k are linearly dependent on $\mathbb{Z}$, so $\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2$.

Case 2: $H_1 \cong \mathbb{Z}^2$. Observe that this condition says that σ_2 is a non-torsion section too.

Since $H_1 \cong \mathbb{Z}^2$, we can consider a $\mathbb{Z}$-basis for it and the following corresponding elements of $\theta_\sigma(\ker \rho_{\mathcal{A}})$:

$$w_3 = \begin{pmatrix} 0 \\ 0 \\ v_{1,k} \\ v_{2,k} \end{pmatrix}, \quad w_4 = \begin{pmatrix} 0 \\ 0 \\ v_{1,l} \\ v_{2,l} \end{pmatrix}, \quad \text{where } k, l \in \ker \theta_{\sigma_1} \subset \ker \rho_{\mathcal{A}}.$$

Since $\theta_{\sigma_1}(\ker \rho_{\mathcal{E}_1})$ also has rank 2, let us choose a $\mathbb{Z}$-basis $\left(\begin{smallmatrix} u_{1,g} \\ u_{2,g} \end{smallmatrix}\right), \left(\begin{smallmatrix} u_{1,h} \\ u_{2,h} \end{smallmatrix}\right)$ of it, where $g, h \in \ker \rho_{\mathcal{A}}$, and consider the corresponding elements of $\theta_\sigma(\ker \rho_{\mathcal{A}})$:

$$z_1 = \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ v_{1,g} \\ v_{2,g} \end{pmatrix}, \quad z_2 = \begin{pmatrix} u_{1,h} \\ u_{2,h} \\ v_{1,h} \\ v_{2,h} \end{pmatrix}.$$

With an appropriate linear combination of z_1, z_2, w_3, w_4 we obtain that

$$w_1 = \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} u_{1,h} \\ u_{2,h} \\ 0 \\ 0 \end{pmatrix}$$

are elements of $\theta_\sigma(\ker \rho_{\mathcal{A}})$. Moreover, w_1, w_2, w_3, w_4 are linearly independent over $\mathbb{Z}$. Thus $\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^4$. ■

4.1.1. Main theorem. Recall that we are considering an abelian scheme $\mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2$, where we assume that $\mathcal{E}_1$ and $\mathcal{E}_2$ are isogenous, i.e. there exists an isogeny $\phi\colon \mathcal{E}_1 \to \mathcal{E}_2$. This isogeny induces an isogeny $\phi_{\mathcal{A}} := (\phi, \mathrm{id}_{\mathcal{E}_2})\colon \mathcal{E}_1 \times_B \mathcal{E}_2 \to \mathcal{E}_2 \times_B \mathcal{E}_2$. Since our theorem is invariant under isogeny, we can just study the case $\mathcal{A} := \mathcal{E} \times_B \mathcal{E}$, where $\mathcal{E} \to B$ is an elliptic scheme.

Theorem 4.5. *Let $\sigma_1, \sigma_2 \colon B \to \mathcal{E}$ be rational sections of an elliptic scheme and suppose that at least one of the sections σ_1, σ_2 is non-torsion. Let us consider the abelian scheme $\pi \colon \mathcal{A} := \mathcal{E} \times_B \mathcal{E} \to B$ endowed with the (non-torsion) section $\sigma = (\sigma_1, \sigma_2)$. We have the following situation:*

1. *If σ_1, σ_2 are linearly dependent over $\mathbb{Z}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^2$.*
2. *If σ_1, σ_2 are linearly independent over $\mathbb{Z}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^4$.*

Proof. Let us prove the two cases separately.

(1) We are supposing that σ_1, σ_2 are linearly dependent over $\mathbb{Z}$. So there exist $n_1, n_2 \in \mathbb{Z}$ such that

$$n_1\sigma_1 + n_2\sigma_2 = 0.$$

Now let us consider the corresponding elliptic logarithms $\log_{\sigma_1}$, $\log_{\sigma_2}$. On some domain $U \subset B$ on which they are well defined, the linear dependence relation between the sections induces the relation

$$n_1 \log_{\sigma_1} + n_2 \log_{\sigma_2} = \omega,$$

where $\omega(b) \in \Lambda_b$ is a period for each $b \in U$. By Theorem 2.3 we know that $\theta_{\sigma_1}(\ker \rho_{\mathcal{E}}) \cong \mathbb{Z}^2$. So let us fix a loop α in B whose homotopy class is $g \in \ker \rho_{\mathcal{E}}$ and also denote

$$w_{1,g} = \theta_{\sigma_1}(g) = \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix}, \quad w_{2,g} = \theta_{\sigma_2}(g) = \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} \in \mathbb{Z}^2.$$

Now we analytically continue the relation $n_1 \log_{\sigma_1} + n_2 \log_{\sigma_2} = \omega$ along α, by considering that ω remains unchanged since $g \in \ker \rho_{\mathcal{E}}$. So we obtain

$$n_1 \log_{\sigma_1} + n_1 u_{1,g}\omega_1 + n_1 u_{2,g}\omega_2 + n_2 \log_{\sigma_2} + n_2 v_{1,g}\omega_1 + n_2 v_{2,g}\omega_2 = \omega.$$

Therefore we have

$$(n_1 u_{1,g} + n_2 v_{1,g})\omega_1 + (n_1 u_{2,g} + n_2 v_{2,g})\omega_2 = 0,$$

which is the same as writing

$$n_1 \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} + n_2 \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By the arbitrariness of g we have $n_1 w_{1,g} + n_2 w_{2,g} = 0$ for all $g \in \ker \rho_{\mathcal{A}}$. In other words, we have

$$w_{2,g} = -\frac{n_1}{n_2} w_{1,g}.$$

Then the map

$$\begin{pmatrix} w_{1,g} \\ w_{2,g} \end{pmatrix} \mapsto w_{1,g}$$

is an isomorphism, so

$$\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2$$

and the first part is proved.

(2) Now let σ_1, σ_2 be linearly independent sections and let us suppose by contradiction that $\theta_\sigma(\ker \rho_{\mathcal{A}})$ is not isomorphic to $\mathbb{Z}^4$. Let us introduce the following notation:

$$K := \theta_\sigma(\ker \rho_{\mathcal{A}}) = \left\{ \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ v_{1,g} \\ v_{2,g} \end{pmatrix} : g \in \ker \rho_{\mathcal{A}} \right\},$$

$$K_1 := \theta_{\sigma_1}(\ker \rho_{\mathcal{A}}) = \left\{ \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} : g \in \ker \rho_{\mathcal{A}} \right\},$$

$$K_2 := \theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) = \left\{ \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} : g \in \ker \rho_{\mathcal{A}} \right\}.$$

By Proposition 4.4, we have that $K \cong \mathbb{Z}^2$. Moreover, by Theorem 2.3, since σ_1, σ_2 are linearly independent hence in particular non-torsion, we also have $K_1 \cong \mathbb{Z}^2$ and $K_2 \cong \mathbb{Z}^2$.

Claim 1. *There exists a matrix* $M \in \mathrm{GL}_2(\mathbb{Q})$ *such that*

$$M \cdot \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} = \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix}$$

for all $g \in \ker \rho_{\mathcal{A}}$.

Proof of Claim 1. Let us define the projections $p_1 \colon K \to K_1$ and $p_2 \colon K \to K_2$ as

$$p_1 \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ v_{1,g} \\ v_{2,g} \end{pmatrix} = \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix}, \quad p_2 \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ v_{1,g} \\ v_{2,g} \end{pmatrix} = \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix}.$$

Obviously, the projections p_1 and p_2 are surjective. Thus, in our hypothesis, both the projections p_1, p_2 have to be injective. In conclusion, p_1 and p_2 are isomorphisms. Therefore, we can define the isomorphism $\varphi := p_2 \circ p_1^{-1} \colon K_1 \to K_2$ which maps

$$\begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} \mapsto \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix}.$$

Since φ is an isomorphism between two full-rank subgroups of $\mathbb{Z}^2$, it induces an automorphism of $\mathbb{Q}^2$.Then, there exists a matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}),$$

such that

$$\begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} = M \cdot \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} = \begin{pmatrix} \alpha u_{1,g} + \beta u_{2,g} \\ \gamma u_{1,g} + \delta u_{2,g} \end{pmatrix} \quad \text{for all } g \in \ker \rho_{\mathcal{A}}. \qquad \blacksquare$$

Claim 2. *The matrix M is of the form*

$$M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where $\alpha \in \mathbb{Q}$.

Proof of Claim 2. Now let us choose an element $h \in \ker \rho_{\mathcal{A}}$ and an element $g \in \pi_1(B)$. Let us use the notation

$$\theta_{\sigma_1}(h) = \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix}, \quad \theta_{\sigma_2}(h) = \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix},$$

and let us consider the periods

$$\omega_{h,\sigma_1} := u_{1,h}\omega_1 + u_{2,h}\omega_2, \quad \omega_{h,\sigma_2} := v_{1,h}\omega_1 + v_{2,h}\omega_2.$$

Moreover, we will indicate with ω_{g,σ_1}, ω_{g,σ_2} the variations of $\log_{\sigma_1}$, $\log_{\sigma_2}$ along g, respectively.

Finally, let us consider the element $h' := ghg^{-1} \in \ker \rho_{\mathcal{A}}$ and use analogous notation to above, i.e.

$$\omega_{h',\sigma_1} := u_{1,h'}\omega_1 + u_{2,h'}\omega_2, \quad \omega_{h',\sigma_2} := v_{1,h'}\omega_1 + v_{2,h'}\omega_2.$$

Recalling the notation introduced in Section 2.1, given $h \in \pi_1(B)$ and a period ω, we denote by $c_h(\omega)$ the analytic continuation of ω along any loop whose homotopy class is h. If we look at the action of h' on the determination of $\log_{\sigma_1}$ we obtain

$$\begin{aligned}\log_{\sigma_1} \xrightarrow{g^{-1}} \log_{\sigma_1} - c_{g^{-1}}(\omega_{g,\sigma_1}) &\xrightarrow{h} \log_{\sigma_1} - c_{g^{-1}}(\omega_{g,\sigma_1}) + \omega_{h,\sigma_1} \\ &\xrightarrow{g} \log_{\sigma_1} + c_g(\omega_{h,\sigma_1}).\end{aligned}$$

In the same way, if we look at the action of h' on the determination of $\log_{\sigma_2}$ we obtain

$$\log_{\sigma_2} \xrightarrow{h'} \log_{\sigma_2} + c_g(\omega_{h,\sigma_2}).$$

Equivalently, the following equations hold:

$$\omega_{h',\sigma_1} = c_g(\omega_{h,\sigma_1}), \quad \omega_{h',\sigma_2} = c_g(\omega_{h,\sigma_2}).$$

In terms of coordinates, this means

$$\begin{pmatrix} u_{1,h'} \\ u_{2,h'} \end{pmatrix} = \rho_{\mathcal{E}}(g) \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix}, \quad \begin{pmatrix} v_{1,h'} \\ v_{2,h'} \end{pmatrix} = \rho_{\mathcal{E}}(g) \cdot \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix}. \tag{4.1}$$

Moreover, since $h, h' \in \ker \rho_{\mathcal{A}}$, by Claim 1 we have that

$$M \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} = \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix}, \quad M \cdot \begin{pmatrix} u_{1,h'} \\ u_{2,h'} \end{pmatrix} = \begin{pmatrix} v_{1,h'} \\ v_{2,h'} \end{pmatrix}. \tag{4.2}$$

Now we are ready to put it all together: by (4.1) and (4.2) we obtain

$$(M\rho_{\mathcal{E}}(g) - \rho_{\mathcal{E}}(g)M) \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} = 0$$

for all $g \in \pi_1(B)$ and $h \in \ker \rho_{\mathcal{A}}$. Observe that the relation does not depend on $u_{1,g}, u_{2,g}$. By the arbitrariness of h, we obtain

$$M\rho_{\mathcal{E}}(g) - \rho_{\mathcal{E}}(g)M = 0$$

for every $g \in \pi_1(B)$. Since $\rho_{\mathcal{E}}(\pi_1(B))$ is Zariski dense in $\mathrm{SL}_2(\mathbb{Z})$, this last relation has to be true for every matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ in place of $\rho_{\mathcal{E}}(g)$. In other words, we have just proved that M commutes with $\mathrm{SL}_2(\mathbb{Z})$. Therefore, by the Schur lemma, the matrix M is a scalar matrix, i.e. it has the form

$$M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

where $\alpha \in \mathbb{Q}$, say $\alpha := \frac{m}{n}$. ■

Claims 1 and 2 mean that $v_{1,h} = \alpha u_{1,h}$, $v_{2,h} = \alpha u_{2,h}$ for all $h \in \ker \rho_{\mathcal{A}}$. In other words, the logarithm of σ_2 has the following variation under the action of each $h \in \ker \rho_{\mathcal{A}}$:

$$\log_{\sigma_2} \mapsto \log_{\sigma_2} + \alpha u_{1,h}\omega_1 + \alpha u_{2,h}\omega_2.$$

Now let us consider the sections $m \cdot \sigma_1$ and $n\sigma_2$. Observe that for each $h \in \ker \rho_{\mathcal{A}}$ we have

$$\begin{aligned} \log_{m\sigma_1} &= m \log_{\sigma_1} \mapsto m \log_{\sigma_1} + m u_{1,h}\omega_1 + m u_{2,h}\omega_2, \\ \log_{n\sigma_2} &= n \log_{\sigma_2} \mapsto n \log_{\sigma_2} + m u_{1,h}\omega_1 + m u_{2,h}\omega_2. \end{aligned}$$

Therefore, we can define the section $\tilde{\sigma} := m\sigma_1 - n\sigma_2$ of $\mathcal{E} \to B$. Observe that for each $h \in \ker \rho_{\mathcal{A}} = \ker \rho_{\mathcal{E}}$ the action of h on $\log_{\tilde{\sigma}}$ is trivial. This means that $\theta_{\tilde{\sigma}}(\ker \rho_{\mathcal{E}})$ is trivial. By Theorem 2.3, it follows that $\tilde{\sigma}$ is a torsion section, i.e. we have

$$k\tilde{\sigma} = 0,$$

and this means that σ_1 and σ_2 are linearly dependent over $\mathbb{Z}$; this contradiction concludes the proof. ■

REMARK 4.6. Let us consider the abelian scheme $\mathcal{A} := \mathcal{E} \times_B \mathcal{E} \to B$. It is well known that a pair $(P_1, P_2) \in \mathcal{A}_b$ is contained in a proper group-subscheme of $\mathcal{A} \to B$ if and only if there exist $n_1, n_2 \in \mathbb{Z}$ such that $n_1 P_1 + n_2 P_2 = 0$ (see for example [8, Lemma 1] or [6, Section 3.3, Lemma 2], remembering that the generic fiber of a non-isotrivial elliptic scheme has no complex multiplication). In other words, saying that the image of a section $\sigma = (\sigma_1, \sigma_2)$ is not contained in a proper group-subscheme is equivalent to saying that σ_1, σ_2 are linearly independent over $\mathbb{Z}$. Thus, Theorem 4.5 proves Conjecture 3.1 for the case $\mathcal{A} = \mathcal{E} \times_B \mathcal{E}$.

By Proposition 3.6, we deduce the theorem in the case of the product of two isogenous elliptic schemes, which reads as follows:

THEOREM 4.7. *Let $\sigma_1 \colon B \to \mathcal{E}_1$, $\sigma_2 \colon B \to \mathcal{E}_2$ be rational sections of two elliptic schemes such that at least one of them is non-torsion. Suppose that there exists an isogeny $\phi \colon \mathcal{E}_1 \to \mathcal{E}_2$. Let us consider the abelian scheme $\pi \colon \mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$ endowed with the (non-torsion) section $\sigma = (\sigma_1, \sigma_2)$. We have the following situation:*

1. *If $\phi \circ \sigma_1$, σ_2 are linearly dependent over $\mathbb{Z}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^2$.*
2. *If $\phi \circ \sigma_1$, σ_2 are linearly independent over $\mathbb{Z}$, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^4$.*

Example 4.8. Let us consider the following two algebraic sections of the Legendre scheme:

$$\sigma_1(\lambda) = \big(2, \sqrt{2(2-\lambda)}\big), \quad \sigma_2(\lambda) = \big(\lambda+1, \sqrt{\lambda(\lambda+1)}\big).$$

The base B on which the two sections become well defined may be taken as the (ramified) cover of $\mathbb{P}_1 - \{0, 1, \infty\}$ defined by taking the square roots of $2-\lambda$ and of $\lambda(\lambda+1)$. This cover has degree 4 and is ramified above $\lambda = 2$ and $\lambda = -1$. Let us define the elliptic scheme $\mathcal{E} \to B$ obtained extending the Legendre scheme by base change to B and consider the abelian family $\mathcal{A} := \mathcal{E} \times_B \mathcal{E} \to B$ (observe that the abelian family $\mathcal{A}$ is obtained as the fiber square of the Legendre scheme, extended by base change to B). The above two sections give a rational section $\sigma\colon B \to \mathcal{A}$, whose components we continue to denote by σ_1, σ_2.

Note that none of the sections is identically torsion: in fact, it is known that every torsion section can be defined over a base which is an unramified cover of $\mathbb{P}_1 - \{0, 1, \infty\}$, whereas to define σ_1 (resp. σ_2) we need the base B to be ramified (at least) above the point 2 (resp. -1) of $\mathbb{P}_1 - \{0, 1, \infty\}$. Moreover, the fact that the minimal ramification necessary to define σ_1, σ_2 is different for the two sections implies that they are linearly independent over $\mathbb{Z}$. To prove this assertion, let us look at the monodromy action on a possible relation $n_1\sigma_1 + n_2\sigma_2 = 0$ and observe that for these sections a different choice of the square root would merely change sign to the section. Thus, if we look at the monodromy action induced by a “small loop” turning around 2 in $\mathbb{P}_1 - \{0, 1, \infty\}$ on the dependence relation, we change sign to σ_1 but leave unchanged σ_2 and so obtain $n_1 = 0$. Analogously, we obtain $n_2 = 0$.

Theorem 4.7 yields that the relative monodromy group of the logarithm of σ with respect to periods is isomorphic to $\mathbb{Z}^4$. Moreover, we are able to construct an explicit loop which leaves periods unchanged but not logarithms. In [7] we constructed such a loop Γ for the logarithm of σ_1. This loop Γ is one of the loops we are looking for: in fact it obviously also works for the logarithm of σ with respect to the periods of the abelian scheme, since the periods of $\mathcal{A} \to B$ are determined by the periods of $\mathcal{E} \to B$.

4.2 – *Case 2: Product of non-isogenous elliptic schemes*

In this last section, we now formulate a result on the monodromy of the logarithm of a section $\sigma\colon B \to \mathcal{A} = \mathcal{E}_1 \times_B \mathcal{E}_2$, in the case in which $\mathcal{E}_1$, $\mathcal{E}_2$ are not isogenous.

In what follows we will make the following identifications: if $g \in \ker \rho_{\mathcal{E}_i}$ we identify $\theta_{\sigma_i}(g) \equiv w_{i,g} \in \mathbb{Z}^2$; if $g \in \ker \rho_{\mathcal{A}}$ we identify $\theta_\sigma(g) \equiv w_g \in \mathbb{Z}^4$. Moreover, we denote by $\omega_{1,\mathcal{E}_i}, \omega_{2,\mathcal{E}_i}$ the periods of $\mathcal{E}_i \to B$.

Recall that unlike the case in which $\mathcal{E}_1$, $\mathcal{E}_2$ are isogenous schemes, in this case the representations $\rho_{\mathcal{E}_1}$ and $\rho_{\mathcal{E}_2}$ are not conjugate. Rather, the image $\rho_{\mathcal{A}}(\pi_1(B))$ is Zariski dense in $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ (see Theorem 4.1).

Now we are ready for the results of this section. First of all, suppose that both σ_1 and σ_2 are non-torsion and let us take a look at the difference from the case where the two schemes are isogenous. In both cases, we can use Theorem 2.3 for σ_1, σ_2 and obtain that

$$\theta_{\sigma_1}(\ker \rho_{\mathcal{E}_1}) \cong \theta_{\sigma_2}(\ker \rho_{\mathcal{E}_2}) \cong \mathbb{Z}^2.$$

The problem is that when the schemes are isogenous, we have

$$\ker \rho_{\mathcal{E}_1} = \ker \rho_{\mathcal{E}_2} = \ker \rho_{\mathcal{A}}$$

(since the monodromy groups of periods are conjugate). Thus we deduce immediately that $\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0$.

Instead, when the two schemes are not isogenous, the group $\ker \rho_{\mathcal{A}} = \ker \rho_{\mathcal{E}_1} \cap \ker \rho_{\mathcal{E}_2}$ can be smaller than $\ker \rho_{\mathcal{E}_1}$ and $\ker \rho_{\mathcal{E}_2}$. Therefore, this time Theorem 2.3 does not allow us to conclude directly that $\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0$. However, the conclusion is still true and we prove it (and a little more) in the next proposition.

Proposition 4.9. *If σ_1, σ_2 are both non-torsion, then $\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0$. Moreover, at least one of the two groups $\theta_{\sigma_1}(\ker \rho_{\mathcal{A}})$ and $\theta_{\sigma_2}(\ker \rho_{\mathcal{A}})$ is isomorphic to $\mathbb{Z}^2$.*

Proof. First of all, observe that if $\ker \rho_{\mathcal{E}_1} \subseteq \ker \rho_{\mathcal{E}_2}$, then we have $\ker \rho_{\mathcal{A}} = \ker \rho_{\mathcal{E}_1}$. Thus, the conclusion follows directly by Theorem 2.3 applied to $\mathcal{E}_1 \to B$: in fact, we obtain

$$\theta_{\sigma_1}(\ker \rho_{\mathcal{A}}) = \theta_{\sigma_1}(\ker \rho_{\mathcal{E}_1}) \cong \mathbb{Z}^2,$$

which in particular implies $\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0$.

Similarly, if $\ker \rho_{\mathcal{E}_2} \subseteq \ker \rho_{\mathcal{E}_1}$, by Theorem 2.3 applied to $\mathcal{E}_2 \to B$ we obtain

$$\theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) = \theta_{\sigma_2}(\ker \rho_{\mathcal{E}_2}) \cong \mathbb{Z}^2, \quad \theta_\sigma \ker \rho_{\mathcal{A}} \neq 0.$$

In particular, if we have $\ker \rho_{\mathcal{E}_1} = \ker \rho_{\mathcal{E}_2}$, we obtain

$$\theta_{\sigma_1}(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2, \quad \theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2, \quad \theta_\sigma \ker \rho_{\mathcal{A}} \neq 0.$$

Thus, for the rest of this proof we suppose we are not in one of the previous cases. In other words, we suppose that the two sets $\ker \rho_{\mathcal{E}_1} \setminus \ker \rho_{\mathcal{E}_2}$ and $\ker \rho_{\mathcal{E}_2} \setminus \ker \rho_{\mathcal{E}_1}$ are not empty.

Let us consider $g_1 \in \ker \rho_{\mathcal{E}_1}$ and $g_2 \in \ker \rho_{\mathcal{E}_2}$. We want to prove that the commutator $g_1 g_2 g_1^{-1} g_2^{-1}$ is an element of $\ker \rho_{\mathcal{A}}$. To this end, by looking at the representation θ_σ, we obtain the following two matrices:

$$\theta_\sigma(g_1) = \begin{pmatrix} I_2 & 0 & \begin{matrix} u_{1,g_1} \\ u_{2,g_1} \end{matrix} \\ 0 & \rho_{\mathcal{E}_2}(g_1) & \begin{matrix} v_{1,g_1} \\ v_{2,g_1} \end{matrix} \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta_\sigma(g_2) = \begin{pmatrix} \rho_{\mathcal{E}_1}(g_2) & 0 & \begin{matrix} u_{1,g_2} \\ u_{2,g_2} \end{matrix} \\ 0 & I_2 & \begin{matrix} v_{1,g_2} \\ v_{2,g_2} \end{matrix} \\ 0 & 0 & 1 \end{pmatrix}.$$

If we compute $\theta_\sigma(g_1 g_2 g_1^{-1} g_2^{-1})$ we obtain

$$\theta_\sigma(g_1 g_2 g_1^{-1} g_2^{-1}) = \begin{pmatrix} I_2 & 0 & \begin{pmatrix} u_{1,g_1} \\ u_{2,g_1} \end{pmatrix} - \rho_{\mathcal{E}_1}(g_2) \begin{pmatrix} u_{1,g_1} \\ u_{2,g_1} \end{pmatrix} \\ 0 & I_2 & \rho_{\mathcal{E}_2}(g_1) \begin{pmatrix} v_{1,g_2} \\ v_{2,g_2} \end{pmatrix} - \begin{pmatrix} v_{1,g_2} \\ v_{2,g_2} \end{pmatrix} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

In particular, this proves that $g_1 g_2 g_1^{-1} g_2^{-1} \in \ker \rho_{\mathcal{A}}$ for each $g_1 \in \ker \rho_{\mathcal{E}_1}$, $g_2 \in \ker \rho_{\mathcal{E}_2}$. Observe that we have not proved that $\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0$ yet: in fact, a priori, the first four components of the last column of the matrix in (4.3) could be zero for the chosen g_1, g_2.

For fixed $g_2 \in \ker \rho_{\mathcal{E}_2} \setminus \ker \rho_{\mathcal{E}_1}$, by Theorem 2.3 applied to $\mathcal{E}_1 \to B$, we can choose $g_1 \in \ker \rho_{\mathcal{E}_1}$ in such a way that

$$\begin{pmatrix} u_{1,g_1} \\ u_{2,g_1} \end{pmatrix} - \rho_{\mathcal{E}_1}(g_2) \begin{pmatrix} u_{1,g_1} \\ u_{2,g_1} \end{pmatrix} \neq 0.$$

Similarly, for fixed $g_1 \in \ker \rho_{\mathcal{E}_1} \setminus \ker \rho_{\mathcal{E}_2}$, by Theorem 2.3 applied to $\mathcal{E}_2 \to B$, we can choose $g_2 \in \ker \rho_{\mathcal{E}_2}$ such that

$$\rho_{\mathcal{E}_2}(g_1) \begin{pmatrix} v_{1,g_2} \\ v_{2,g_2} \end{pmatrix} - \begin{pmatrix} v_{1,g_2} \\ v_{2,g_2} \end{pmatrix} \neq 0.$$

Since $g_1 g_2 g_1^{-1} g_2^{-1} \in \ker \rho_{\mathcal{A}}$ for each $g_1 \in \ker \rho_{\mathcal{E}_1}$, $g_2 \in \ker \rho_{\mathcal{E}_2}$, we conclude that

$$\theta_\sigma(\ker \rho_{\mathcal{A}}),\ \theta_{\sigma_1}(\ker \rho_{\mathcal{A}}),\ \theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) \neq 0.$$

In particular, by Lemma 4.2 we have

$$\theta_{\sigma_1}(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2, \quad \theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2. \qquad \blacksquare$$

Let us define

$$H_1 := \theta_{\sigma_2}(\ker\theta_{\sigma_1} \cap \ker\rho_{\mathcal{A}}), \quad H_2 := \theta_{\sigma_1}(\ker\theta_{\sigma_2} \cap \ker\rho_{\mathcal{A}}).$$

Since $\ker\theta_{\sigma_1} \cap \ker\rho_{\mathcal{A}}$ is a normal subgroup of G which is contained in $\ker\rho_{\mathcal{E}_2}$, by Lemma 4.2 we have that H_1 is isomorphic either to $\{0\}$ or to $\mathbb{Z}^2$. The same is true for H_2.

PROPOSITION 4.10. *If both σ_1, σ_2 are non-torsion, then the group $\theta_\sigma(\ker\rho_{\mathcal{A}})$ is isomorphic to either $\mathbb{Z}^2$ or $\mathbb{Z}^4$.*

PROOF. First of all, we are going to prove a quick claim.
CLAIM 3. $\theta_\sigma(\ker\rho_{\mathcal{A}}) \neq \mathbb{Z}$.

PROOF OF CLAIM 3. Let us suppose rank $\theta_\sigma(\ker\rho_{\mathcal{A}}) < 2$. Observe that

$$\operatorname{rank}\theta_{\sigma_1}(\ker\rho_{\mathcal{A}}) \leq \operatorname{rank}\theta_\sigma(\ker\rho_{\mathcal{A}}),$$
$$\operatorname{rank}\theta_{\sigma_2}(\ker\rho_{\mathcal{A}}) \leq \operatorname{rank}\theta_\sigma(\ker\rho_{\mathcal{A}}).$$

Thus, by Lemma 4.2, this implies that

$$\theta_{\sigma_1}(\ker\rho_{\mathcal{A}}) \cong \{0\} \quad \text{and} \quad \theta_{\sigma_2}(\ker\rho_{\mathcal{A}}) \cong \{0\},$$

which is in contradiction with Proposition 4.9. ■

FINAL PROOF. Since both the sections are non-torsion, by Proposition 4.9 and by Claim 3, we have that

$$2 \leq \operatorname{rank}\theta_\sigma(\ker\rho_{\mathcal{A}}) \leq 4.$$

Moreover, by Proposition 4.9 we also have $\theta_{\sigma_1}(\ker_{\mathcal{A}}) \cong \mathbb{Z}^2$ or $\theta_{\sigma_2}(\ker_{\mathcal{A}}) \cong \mathbb{Z}^2$. Without loss of generality, let us suppose

$$\theta_{\sigma_1}(\ker_{\mathcal{A}}) \cong \mathbb{Z}^2$$

for the rest of this proof: if we have the other case we just need to consider H_2 in place of H_1 in the following lines for the proof to work.

Let us consider the group H_1. By Lemma 4.2, we only have two possibilities for H_1, i.e. $H_1 \cong \{0\}$ or $H_1 \cong \mathbb{Z}^2$.

Case 1: $H_1 \cong \{0\}$. The condition $H_1 \cong \{0\}$ means that for each $g \in \ker\rho_{\mathcal{A}}$ if $u_{1,g} = u_{2,g} = 0$, then $v_{1,g} = v_{2,g} = 0$, where the notation is the same as above. Let us prove that rank $\theta_\sigma(\ker\rho_{\mathcal{A}}) = 2$ by proving that any three elements of the form $w_g, w_h, w_k \in \theta_\sigma(\ker\rho_{\mathcal{A}})$ are linearly dependent on $\mathbb{Z}$.

Since $\theta_{\sigma_1}(\ker\rho_{\mathcal{E}_1}) \cong \mathbb{Z}^2$, given any three elements $g, h, k \in \ker\rho_{\mathcal{A}} \subset \ker\rho_{\mathcal{E}_1}$, there always exists $n_g, n_h, n_k \in \mathbb{Z}$, not all zero, such that

$$n_g \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} + n_h \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} + n_k \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we have

$$\theta_\sigma(k^{n_k} h^{n_h} g^{n_g}) = \begin{pmatrix} & 0 \\ & 0 \\ \mathrm{I}_4 & n_g v_{1,g} + n_h v_{1,h} + n_k v_{1,k} \\ & n_g v_{2,g} + n_h v_{2,h} + n_k v_{2,k} \\ 0 & 1 \end{pmatrix}.$$

The condition $H_1 \cong \{0\}$ implies that

$$n_g \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} + n_h \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix} + n_k \begin{pmatrix} v_{1,k} \\ v_{2,k} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore, any three elements of $\mathbb{Z}^4$ of the form w_g, w_h, w_k are linearly dependent on $\mathbb{Z}$, so $\theta_\sigma(\ker\rho_{\mathcal{A}}) \cong \mathbb{Z}^2$.

Case 2: $H_1 \cong \mathbb{Z}^2$. Since $H_1 \cong \mathbb{Z}^2$ we can consider a $\mathbb{Z}$-basis for it and the following corresponding elements of $\theta_\sigma(\ker\rho_{\mathcal{A}})$:

$$w_3 = \begin{pmatrix} 0 \\ 0 \\ v_{1,k} \\ v_{2,k} \end{pmatrix}, \quad w_4 = \begin{pmatrix} 0 \\ 0 \\ v_{1,l} \\ v_{2,l} \end{pmatrix}, \quad \text{where } k, l \in \ker\theta_{\sigma_1} \cap \ker\rho_{\mathcal{A}} \subset \ker\rho_{\mathcal{A}}.$$

Recall that $\theta_{\sigma_1}(\ker\rho_{\mathcal{A}}) \cong \mathbb{Z}^2$ by hypothesis. Therefore, let us choose a $\mathbb{Z}$-basis of it, say $\left(\begin{smallmatrix} u_{1,g} \\ u_{2,g} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} u_{1,h} \\ u_{2,h} \end{smallmatrix}\right)$ where $g, h \in \ker\rho_{\mathcal{A}}$, and consider the corresponding elements of $\theta_\sigma(\ker\rho_{\mathcal{A}})$:

$$z_1 = \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ v_{1,g} \\ v_{2,g} \end{pmatrix}, \quad z_2 = \begin{pmatrix} u_{1,h} \\ u_{2,h} \\ v_{1,h} \\ v_{2,h} \end{pmatrix}.$$

With an appropriate linear combination of z_1, z_2, w_3, w_4 we obtain that

$$w_1 = \begin{pmatrix} u_{1,g} \\ u_{2,g} \\ 0 \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} u_{1,h} \\ u_{2,h} \\ 0 \\ 0 \end{pmatrix}$$

are elements of $\theta_\sigma(\ker \rho_{\mathcal{A}})$. Moreover, w_1, w_2, w_3, w_4 are linearly independent over $\mathbb{Z}$. Thus $\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^4$. ■

4.2.1. Main theorem. Now recall that we are considering an abelian scheme $\mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are not isogenous.

THEOREM 4.11. *Let $\sigma_i \colon B \to \mathcal{E}_i, i = 1, 2$, be rational sections of two non-isogenous elliptic schemes and suppose they are not both torsion sections. Let us consider the abelian scheme $\pi \colon \mathcal{A} := \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$ endowed with the (non-torsion) section $\sigma = (\sigma_1, \sigma_2)$. We have the following situation:*

(1) *If one of σ_1 and σ_2 is a torsion section, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\{0\}$ or to $\mathbb{Z}^2$.*

(2) *If neither σ_1 nor σ_2 is a torsion section, the cover $B_\sigma \to B^*$ has infinite degree and its Galois group is isomorphic to $\mathbb{Z}^4$.*

PROOF. Let us prove the two cases separately.

(1) Suppose that one of σ_1 and σ_2 is a torsion section, say for example σ_1. We suppose that $\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0$ and prove that $\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2$. Since σ_1 is torsion, we have $\theta_{\sigma_1}(\ker \rho_{\mathcal{E}_1}) = 0$ (see Remark 2.2); in particular, this implies that $\theta_{\sigma_1}(\ker \rho_{\mathcal{A}}) = 0$. The two conditions

$$\theta_\sigma(\ker \rho_{\mathcal{A}}) \neq 0 \quad \text{and} \quad \theta_{\sigma_1}(\ker \rho_{\mathcal{A}}) = 0$$

imply that

$$\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) \quad \text{and} \quad \theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) \neq 0.$$

By Lemma 4.2, we have

$$\theta_{\sigma_2}(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2.$$

Thus, it follows that $\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2$ and the first part is proved.

(2) Now let σ_1, σ_2 both be non-torsion sections and let us suppose by contradiction that $\theta_\sigma(\ker \rho_{\mathcal{A}})$ is not isomorphic to $\mathbb{Z}^4$. By Proposition 4.10, we can only have the case

$$\theta_\sigma(\ker \rho_{\mathcal{A}}) \cong \mathbb{Z}^2.$$

With the same calculations detailed in the case of the product of isogenous curves, we obtain that there exists a matrix $M \in \mathrm{GL}_2(\mathbb{Q})$ such that

$$M \cdot \begin{pmatrix} u_{1,g} \\ u_{2,g} \end{pmatrix} = \begin{pmatrix} v_{1,g} \\ v_{2,g} \end{pmatrix} \tag{4.4}$$

for all $g \in \ker \rho_{\mathcal{A}}$.

Claim 4. *The matrix M is the zero matrix, i.e. $M = 0$.*

Proof of Claim 4. Now let us choose an element $h \in \ker \rho_{\mathcal{A}}$ and an element $g \in \pi_1(B)$. Let us use the notation

$$\theta_{\sigma_1}(h) = \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix}, \quad \theta_{\sigma_2}(h) = \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix},$$

and let us consider the periods

$$\omega_{h,\sigma_1} := u_{1,h}\omega_{1,\mathcal{E}_1} + u_{2,h}\omega_{2,\mathcal{E}_1}, \quad \omega_{h,\sigma_2} := v_{1,h}\omega_{1,\mathcal{E}_2} + v_{2,h}\omega_{2,\mathcal{E}_2}.$$

Moreover, we will indicate with $\omega_{g,\sigma_1}, \omega_{g,\sigma_2}$ the variation of $\log_{\sigma_1}, \log_{\sigma_2}$ along g, respectively.

Finally, let us consider the element $h' := ghg^{-1} \in \ker \rho_{\mathcal{A}}$ and use analogous notation to above, i.e.

$$\omega_{h',\sigma_1} := u_{1,h'}\omega_{1,\mathcal{E}_1} + u_{2,h'}\omega_{2,\mathcal{E}_1}, \quad \omega_{h',\sigma_2} := v_{1,h'}\omega_{1,\mathcal{E}_2} + v_{2,h'}\omega_{2,\mathcal{E}_2}.$$

If we look at the action of h' on the determination of $\log_{\sigma_1}$ we obtain

$$\begin{aligned} \log_{\sigma_1} \xrightarrow{g^{-1}} \log_{\sigma_1} - \rho_{\mathcal{E}_1}(g^{-1})(\omega_{g,\sigma_1}) &\xrightarrow{h} \log_{\sigma_1} - \rho_{\mathcal{E}_1}(g^{-1})(\omega_{g,\sigma_1}) + \omega_{h,\sigma_1} \\ &\xrightarrow{g} \log_{\sigma_1} + \rho_{\mathcal{E}_1}(g)(\omega_{h,\sigma_1}). \end{aligned}$$

In the same way, if we look at the action of h' on the determination of $\log_{\sigma_2}$ we obtain

$$\log_{\sigma_2} \xrightarrow{h'} \log_{\sigma_2} + \rho_{\mathcal{E}_2}(g)(\omega_{h,\sigma_2}).$$

Equivalently, the following equations hold:

$$\omega_{h',\sigma_1} = \rho_{\mathcal{E}_1}(g)(\omega_{h,\sigma_1}), \quad \omega_{h',\sigma_2} = \rho_{\mathcal{E}_2}(g)(\omega_{h,\sigma_2}).$$

In terms of coordinates, this means

$$\begin{pmatrix} u_{1,h'} \\ u_{2,h'} \end{pmatrix} = \rho_{\mathcal{E}_1}(g) \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix}, \quad \begin{pmatrix} v_{1,h'} \\ v_{2,h'} \end{pmatrix} = \rho_{\mathcal{E}_2}(g) \cdot \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix}. \tag{4.5}$$

Moreover, since $h, h' \in \ker \rho_{\mathcal{A}}$, by (4.4) we have

$$M \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} = \begin{pmatrix} v_{1,h} \\ v_{2,h} \end{pmatrix}, \quad M \cdot \begin{pmatrix} u_{1,h'} \\ u_{2,h'} \end{pmatrix} = \begin{pmatrix} v_{1,h'} \\ v_{2,h'} \end{pmatrix}. \tag{4.6}$$

Now we are ready to put it all together: by (4.5) and (4.6) we obtain

$$(M\rho_{\mathcal{E}_1}(g) - \rho_{\mathcal{E}_2}(g)M) \cdot \begin{pmatrix} u_{1,h} \\ u_{2,h} \end{pmatrix} = 0 \tag{4.7}$$

for all $g \in \pi_1(B)$ and $h \in \ker \rho_{\mathcal{A}}$. Observe that the relation does not depend on $u_{1,g}, u_{2,g}$. By the arbitrariness of h, equation (4.7) reads

$$M\rho_{\mathcal{E}_1}(g) = \rho_{\mathcal{E}_2}(g)M$$

for every $g \in \pi_1(B)$. Since the two representations $\rho_{\mathcal{E}_1}, \rho_{\mathcal{E}_2}$ are not equivalent, by the Schur lemma we deduce that $M = 0$. ■

Since $M \in \mathrm{GL}_2(\mathbb{Q})$, this is a contradiction, concluding the proof. ■

Remark 4.12. Observe that if E_1, E_2 are complex elliptic curves which are not isogenous, then every proper connected algebraic subgroup of $E_1 \times E_2$ is of one of the shapes $0 \times E_2$ or $E_1 \times 0$. In other words, if $\mathcal{A} = \mathcal{E}_1 \times_B \mathcal{E}_2 \to B$ is a product of two non-isogenous elliptic schemes, then saying that the image of a section $\sigma = (\sigma_1, \sigma_2)$ is not contained in a proper group-subscheme is equivalent to saying that neither σ_1 nor σ_2 is a torsion section. Thus, Theorem 4.11 proves Conjecture 3.1 in the case of a fibered product of non-isogenous elliptic schemes.

Example 4.13. Let us consider the line B in S^2 (where $S = \mathbb{P}_1 - \{0, 1, \infty\}$) defined by $x + y = 2$, which is isomorphic under the first projection to $\mathbb{P}_1 - \{0, 1, 2, \infty\}$. Let us consider the scheme over B whose fiber over the point $(\lambda, \mu) \in B$ is the product $\mathcal{L}_\lambda \times \mathcal{L}_\mu$ of the corresponding Legendre curves; denote it by $\mathcal{A} \to B$. This is a product of two non-isogenous elliptic schemes, since the curve $\mathcal{L}_\lambda$ is not isogenous generically to $\mathcal{L}_{2-\lambda}$: in fact, if two elliptic schemes are isogenous then their j-invariants must have the same poles in B; but in this case, the schemes corresponding to $\mathcal{L}_\lambda$, $\mathcal{L}_\mu$ have a different set of bad reduction. We may consider the section $\sigma \colon B \to \mathcal{A}$ given by

$$\mathbb{P}_1 - \{0, 1, 2, \infty\} \ni \lambda \mapsto \big((2, \sqrt{2-\lambda}), (2, \sqrt{2\lambda})\big),$$

whose components are non-torsion sections. Theorem 4.11 yields that the relative monodromy group of the logarithm of σ with respect to periods is isomorphic to $\mathbb{Z}^4$.

Can we say something about an explicit loop which leaves periods unchanged but not logarithms? We have such a loop Γ_1 (resp. Γ_2) for the logarithm of σ_1 (resp. σ_2). Unlike Example 4.8, this time the loops Γ_1, Γ_2 do not work for the logarithm of σ, since the periods of the two factors of $\mathcal{A} \to B$ are not the same. Anyway, we can obtain such a loop as explained in the proof of Proposition 4.9, i.e. taking the commutator of suitable loops which work for $\log_{\sigma_1}$ and $\log_{\sigma_2}$ (these last can be found by looking at the construction in [7]).

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References

[1] Y. André, Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compos. Math.* **82** (1992), no. 1, 1–24. Zbl 0770.14003 MR 1154159

[2] Y. André – P. Corvaja – U. Zannier, The Betti map associated to a section of an abelian scheme. *Invent. Math.* **222** (2020), no. 1, 161–202. Zbl 1460.11083 MR 4145789

[3] P. Corvaja – U. Zannier, Poncelet games and elliptic schemes. 2021, unpublished preprint.

[4] P. Corvaja – U. Zannier, Unramified sections of the Legendre scheme and modular forms. *J. Geom. Phys.* **166** (2021), article no. 104266. Zbl 1497.11147 MR 4260019

[5] S. Lang, *Elliptic functions*. 2nd edn., Grad. Texts in Math. 112, Springer, New York, 1987. Zbl 0615.14018 MR 0890960

[6] D. W. Masser – G. Wüstholz, Fields of large transcendence degree generated by values of elliptic functions. *Invent. Math.* **72** (1983), no. 3, 407–464. Zbl 0516.10027 MR 0704399

[7] F. Tropeano Monodromy of elliptic logarithms: Some topological methods and effective results. 2024, arXiv:2402.07741v1.

[8] E. Viada, The intersection of a curve with algebraic subgroups in a product of elliptic curves. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **2** (2003), no. 1, 47–75. Zbl 1170.11314 MR 1990974

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