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Norm inflation for solutions of semi-linear one-dimensional Klein–Gordon equations

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Abstract. In space dimension larger than or equal to 2, the nonlinear Klein–Gordon equation with small, smooth, decaying initial data has global-in-time solutions. This no longer holds true in one space dimension, where examples of blowing-up solutions are known. On the other hand, it has been proved that if the nonlinearity satisfies a convenient compatibility condition, the "null condition", one recovers global existence and that the solutions satisfy the same dispersive bounds as linear solutions. The goal of this paper is to show that, in the case of cubic semi-linear nonlinearities, this null condition is optimal, in the sense that, when it does not hold, one may construct small, smooth, decaying initial data giving rise to solutions that display inflation of their L^∞ and L^2-norms in finite time.

Introduction

It is well known that quasi-linear Klein–Gordon equations with smooth, small, decaying initial data have global-in-time solutions, in space dimension larger than or equal to 3, as has been proved independently by Klainerman [21] and Shatah [28]. The same holds true in two space dimensions, according to Simon–Taflin [29] and Ozawa–Tsutaya–Tsutsumi [27]. On the other hand, in one space dimension, finite-time blow-up may occur. Examples of nonlinearities for which this happens have been obtained by Yordanov [32] and Keel–Tao [18]. In [10], we introduced for a general quasi-linear nonlinearity a "null condition", expressed explicitly in terms of the quadratic and cubic parts of the nonlinearity, and we conjectured that, under that null condition, small data that are smooth and have some decay at infinity should give rise to global solutions. We showed in [11, 12] that this conjecture holds true for C_0^∞ initial data. We refer to Lindblad–Soffer [23–25] for nonlinearities depending only on u, to Hayashi–Naumkin [17] and to Stingo [31] for more general data, and to the bibliography of [11] for references about the state of the art at the time of publication of that paper.

The goal of the present paper is to show that, in the case of cubic semi-linear nonlinearities, i.e. for the equation

$$(\partial_t^2 - \partial_x^2 + 1)u = P(u, \partial_t u, \partial_x u), \tag{1}$$

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where P is a homogeneous polynomial of degree three, our null condition is optimal, in that sense that if it is not satisfied, one can construct solutions, with small and decaying initial data, that *do not* enjoy the same dispersive bounds as the ones that hold true for linear solutions (or nonlinear global solutions when the null condition is satisfied). More precisely, the null condition was obtained in [10] extracting from the PDE an ODE which has global solutions for small data if and only if the null condition holds. When this is the case, the asymptotics of the solution of this ODE give the asymptotic behavior of the global solution of the PDE. When the null condition is *not* satisfied, this ODE blows up at some finite time, depending on the parameter $y = \frac{x}{t} \in]-1, 1[$. The minimal blow-up time for y describing $]-1, 1[$ is of the form e^{S_*/ε^2} for some $S_* > 0$ (when blow-up occurs in the future), $\varepsilon \ll 1$ being the size of the initial condition. In [10] it was shown that the solution exists and has L^∞-norm at time t which is $O(\frac{\varepsilon}{\sqrt{t}})$ for $t < e^{A/\varepsilon^2}$, for any constant $A < S_*$. The main result of this paper (see Theorem 1.2.1 below) asserts that one may construct initial data so that for $t = T(\varepsilon)$ close enough to e^{S_*/ε^2}, one has inflation of norms in the sense that

$$\sqrt{T(\varepsilon)}(\|u(T(\varepsilon))\|_{L^\infty} + \|\partial_t u(T(\varepsilon))\|_{L^\infty}) \geq cT(\varepsilon)^{\frac{1}{2}-c} \sim e^{\frac{c'}{\varepsilon^2}}, \quad \varepsilon \to 0$$

for positive constants c, c'. In other words, the solution is still small at time $T(\varepsilon)$, but exponentially large when compared to the size of linear solutions. Of course, this norm inflation result does not mean that the solution does blow up, but we explain in the remarks that follow the statement of Theorem 1.2.1 that this is the best we may expect, if we want to single out a property of the solution that follows *only* from the violation of the null condition, and that is in contrast with the kinds of estimates that hold true under the null condition.

The proof of the main theorem relies on the construction of an approximate blowing-up solution, that was inspired for us by the papers of Cazenave–Martel–Zhao [6] and Cazenave–Han–Martel [5]. In these references, the authors construct blowing-up solutions for Schrödinger equations of the form

$$(i\partial_t - \partial_x^2)u = \alpha|u|^2 u, \quad \alpha \in \mathbb{C} - \mathbb{R}. \tag{2}$$

(In fact, their result is not limited to one space dimension nor to cubic nonlinearities.) They first look for an approximate solution given in terms of a profile that satisfies some ODE and blows up at time $t = 1$. Next they write the equation satisfied by the difference between this approximate solution and the exact one. They prove that this equation has a global backwards solution with zero initial condition at (or close to) the blow-up time. The sum of this solution and of the approximate one brings thus an exact solution to (2) that blows up at time $t = 1$. See also Liu–Zhang [26] and for blowing-up solutions of Schrödinger equations with small data, the preprint by Kita [19].

Our general strategy is the same, except that we have to cope with some difficulties inherent to the Klein–Gordon equation. To describe the strategy, let us write equation (1) as a first-order system on $(u_+, \bar{u}_+)$, where u_+ is a new complex-valued unknown deduced from u, with first equation

$$\begin{aligned}\big(D_t - \sqrt{1+D_x^2}\big)u_+ &= M^{(1)}(u_+,u_+,u_+) + M^{(2)}(u_+,u_+,\bar{u}_+)\\ &\quad + M^{(3)}(u_+,\bar{u}_+,\bar{u}_+) + M^{(4)}(\bar{u}_+,\bar{u}_+,\bar{u}_+),\end{aligned} \tag{3}$$

$M^{(j)}$ being nonlocal expression of their arguments, homogeneous of degree 3. The difference from (2) comes from $M^{(1)}$,$M^{(3)}$, $M^{(4)}$ which are not invariant under $u_+ \to zu_+$ for $z \in U(1)$. On the other hand, these terms are "noncharacteristic" ones, since when computed on a linear solution, they oscillate along a noncharacteristic phase for the linear part of (3). Our proof thus has two steps, as in [5, 6]. First, we construct an approximate solution starting from small initial data $(\varepsilon f_0, \varepsilon g_0)$ with f_0, g_0 in $\mathcal{S}(\mathbb{R})$. If the null condition is not satisfied, choosing f_0, g_0 conveniently, we have constructed in [10] an approximate solution defined on some interval $[1, e^{S_*/\varepsilon^2}[$ as

$$\begin{aligned}u^1_{\text{app}}(t,x) = 2\,\text{Re}\Big[&\frac{\varepsilon}{\sqrt{t}}a_{1,1}\Big(\varepsilon^2\log t, \frac{x}{t}\Big)e^{i\sqrt{t^2-x^2}}\\ &+ \frac{\varepsilon^3}{t^{\frac{3}{2}}}a_{3,3}\Big(\varepsilon^2\log t, \frac{x}{t},\varepsilon\Big)e^{3i\sqrt{t^2-x^2}} + O(\varepsilon t^{-\frac{5}{2}})\Big],\end{aligned} \tag{4}$$

where $a_{1,1}(s,y)$, $a_{3,3}(s,y,\varepsilon)$ are functions supported for $|y| \le 1$, smooth in (s,y) for $s < S_*$. As a consequence of the violation of the null condition, one may construct $a_{1,1}(s,y)$, $a_{3,3}(s,y,\varepsilon)$ that blow up if $s \to S_*-$, so that (4) provides a useful approximate solution only for $t < e^{A/\varepsilon^2}$ with $A < S_*$. If one wants to study what happens for t close to e^{S_*/ε^2}, one has to construct a more accurate approximate solution, gluing (4) for say $t < e^{3S_*/4\varepsilon^2}$ to another approximate solution, defined on $e^{S_*/2\varepsilon^2} < t < e^{S_*/\varepsilon^2}$, given by an ansatz of the form

$$\begin{aligned}u^2_{\text{app}}(t,x) = 2\,\text{Re}\Bigg[&\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{N} \varepsilon^{2-\ell}t^{-\frac{\ell}{2}}e^{i\sqrt{t^2-x^2}}a_{\ell,1}\Big(\varepsilon^2\log t, \frac{x}{t},\varepsilon\Big)\\ &+ \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N}\sum_{\substack{3\le q\le \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{iq\sqrt{t^2-x^2}}a_{\ell,q}\Big(\varepsilon^2\log t, \frac{x}{t},\varepsilon\Big)\Bigg],\end{aligned} \tag{5}$$

where $a_{\ell,q}(s,y,\varepsilon)$ are functions that blow up at $s = S_*$ like $(S_* - s)^{-\frac{\ell}{2}-0}$. If $s = \varepsilon^2\log t$ is close to S_*, the $a_{\ell,q}$ terms in the two sums in (5) are thus larger and larger, so that (5) cannot provide an approximate solution. But we may exploit the dispersive decay factor $t^{-\frac{\ell}{2}}$ and limit ourselves to times $t < T(\varepsilon)$, where $T(\varepsilon)$ is such that $T(\varepsilon)^{-1}(S_* - \varepsilon^2\log T(\varepsilon))^{-1} \ll 1$. Under this restriction, (5) provides a function satisfying (1) up to a small remainder. Moreover, $T(\varepsilon)$ is close enough to e^{S_*/ε^2} so that $u^2_{\text{app}}(T(\varepsilon),x)\sqrt{T(\varepsilon)}$ will be large (actually of size e^{c'/ε^2}) in L^∞.

The second step of the proof is to look for an exact solution $u(t,x) = u_{\text{app}}(t,x) + r(t,x)$, where u_{app} is the approximate solution obtained gluing together u^1_{app} and u^2_{app} above, and r a remainder that will be zero as well as its time derivative at $t = T(\varepsilon)$.

Then r solves the backwards equation with force term deduced from (1) replacing u by $u_{\mathrm{app}} + r$. One has to show that if the approximate solution has been constructed in an accurate enough way, the remainder r exists down to time $t \sim 1$ and that at this initial time, it perturbs the initial condition $(\varepsilon f_0, \varepsilon g_0)$ used to construct the approximate solution only at order $o(\varepsilon)$. The general strategy employed to prove such properties is to use the methods that are useful in the study of global existence (normal forms, energy estimates for the action of $x \pm t\frac{x}{\langle x\rangle}$ on the solution of the reduced system obtained by normal forms for the remainder). A difference from problems of global existence is that the equation satisfied by the remainder contains linear terms (coming from the linearization of the approximate solution). The coefficients of these linear terms being expressions containing the approximate solution, they are relatively large close to $T(\varepsilon)$, and thus cannot be treated as perturbations. In order to overcome this difficulty, we use an idea of Cazenave–Han–Martel [5]: we remark that in a Grönwall inequality, the growth of the amplifying factor coming from this large coefficient is more than compensated by the fact that the source term against which it is integrated – which comes from the error in the equation applied to the approximate solution – may be made as small as we want.

The plan of the paper is as follows: In Section 1 we recall the definition of the null condition and state the main theorem. Section 2 is devoted to the construction of the approximate solution. In Section 3 we study the remainder given by the difference between the exact and the approximate solutions. We express it as a solution of a (2×2)-system with source term, and obtain energy estimates for the Sobolev norm of the remainder and for the L^2-norm of the action of $L_+ = x + t\frac{D_x}{\langle D_x\rangle}$ on it. Finally, in Section 4 we conclude the proof using a bootstrap argument and a Klainerman–Sobolev estimate to control L^∞-norms. The appendix is devoted to some technical results used in the proof.

To conclude this introduction, let us give some references to other works concerning the construction of blowing-up solutions for nonlinear *wave* equations instead of Klein–Gordon ones. In the quasi-linear case, recall that in three space dimensions, the null condition was introduced by Christodoulou [9] and by Klainerman [20, 22], who proved that global existence with small decaying initial data holds true under that assumption. In two space dimensions, Alinhac [3] defined the (more complicated) corresponding version of the null condition and also proved global existence when it holds.

When the null condition is *not* satisfied, the study of blowing-up solutions and of their asymptotic behavior was undertaken by Alinhac in a series of papers [1, 2, 4]. For more recent references on that and further results, we refer to the book by Speck [30] and especially its preface and introduction. We notice also that the situation considered in all these papers is quite different from the one we encounter in the present work, as in these quasi-linear models, the singularities that form are of shock type, i.e. the quantities that blow up are *second*-order derivatives, while in our setting, the function itself (or its time derivative) will display norm inflation. For the construction of blowing-up solutions for semi-linear wave equations with a nonlinearity depending only on the function itself, and not on its derivatives, we refer to the papers by Cazenave–Martel–Zhao [7, 8] and their bibliographies.

1. Statement of the main theorem

1.1. Semi-linear Klein–Gordon equation and the null condition

We consider the cubic semi-linear Klein–Gordon equation in one space dimension

$$(\partial_t^2 - \partial_x^2 + 1)u = P(u, \partial_t u, \partial_x u), \tag{1.1.1}$$

where P is a polynomial homogeneous of degree 3, with real coefficients, which we write in the form

$$P(u, \partial_t u, \partial_x u) = \sum_{k=0}^{3} P_k(u; \partial_t u, \partial_x u), \tag{1.1.2}$$

where $P_k(T; Z_1, Z_2)$ is homogeneous of degree k in (Z_1, Z_2) and $3 - k$ in T, with real coefficients. We define, for $y \in \,]-1, 1[$,

$$\omega_0(y) = \frac{1}{\sqrt{1-y^2}}, \quad \omega_1(y) = -\frac{y}{\sqrt{1-y^2}}, \tag{1.1.3}$$

and we set

$$\begin{aligned} p_k(\omega_0(y), \omega_1(y)) &= P_k(1; \omega_0(y), \omega_1(y)), \\ \phi(y) &= (p_1 + 3p_3)(\omega_0(y), \omega_1(y)), \\ \psi(y) &= -(3p_0 + p_2)(\omega_0(y), \omega_1(y)). \end{aligned} \tag{1.1.4}$$

We recall the following definition from [10]:

Definition 1.1.1. One says that the nonlinearity in (1.1.1) satisfies the null condition if $\phi \equiv 0$.

Assume that the null condition is satisfied and take in (1.1.1) initial conditions of the form $u(1, x) = \varepsilon f(x)$, $\partial_t u(1, x) = \varepsilon g(x)$ with $f, g \in C_0^\infty(\mathbb{R})$. Then, it was proved in [11, 12] (see also Stingo [31]), including in the case of quasi-linear equations with quadratic and cubic nonlinearities (for which one has to modify the expression of ϕ in (1.1.4)), that, if the null condition is satisfied, for $\varepsilon > 0$ small enough, the solution to (1.1.1) is globally defined for $t \geq 1$ and satisfies L^∞ bounds of the form $\|\partial_x^k u(t, \cdot)\|_{L^\infty} = O(\varepsilon t^{-\frac{1}{2}})$ when t goes to $+\infty$. The solution thus decays like a solution of the linear Klein–Gordon equation in one space dimension. Of course, a similar statement holds when t goes to $-\infty$. On the other hand, it was also proved that scattering does not hold (one has only modified scattering).

We are interested here in the case when the null condition is *not* satisfied, and we want to construct initial data that generate inflation of the norms of the solution in finite time, i.e. we want to show for instance that the L^∞-norm will *not* satisfy the dispersive bounds that hold true under the null condition. Consequently, in order to ensure that the null condition does not hold, we assume

$$\sup_{y \in]-1,1[} \phi(y) > 0. \tag{1.1.5}$$

This will allow us to construct solutions that display norm inflation at some positive time. If in (1.1.5) ϕ was replaced by $-\phi$, we would in the same way get inflation of the norms at some negative time.

1.2. Main theorem and norm inflation

Let f_0, g_0 be two real-valued functions in $\mathcal{S}(\mathbb{R})$. We associate to them a quantity that will appear in the expression (2.1.9) of the modulus of the solution to the ODE (2.1.8) below, and that will control the blowing-up time. Namely, we set

$$\Gamma(y) = \frac{1}{8\pi}(1-y^2)^{-1}\left|\hat{f}_0(\omega_1(y)) - i\sqrt{1-y^2}\hat{g}_0(\omega_1(y))\right|^2, \tag{1.2.1}$$

which is a smooth function on $]-1,1[$ that, extended by zero outside this interval, gives a smooth function on $\mathbb{R}$. (This function was introduced in [10, formula (1.18)], but the expression given there is correct only if f_0, g_0 satisfy some evenness or oddness conditions. In general, the correct expression is (1.2.1).) By (1.1.5), we may choose f_0, g_0 in $\mathcal{S}(\mathbb{R})$ such that $\sup_{y\in]-1,1[}(\Gamma(y)\phi(y))$ is positive, and we define $S_* > 0$ by

$$\frac{1}{S_*} = \sup_{y\in]-1,1[}(\Gamma(y)\phi(y)). \tag{1.2.2}$$

As $\Gamma(y)$ vanishes at infinite order at $y = \pm 1$, and ϕ grows at most polynomially at these points, the supremum is reached at some points in $]-1,1[$. We shall assume

$$\begin{aligned}&y \to \Gamma(y)\phi(y) \text{ reaches its maximum at a unique point } y_0 \in\,]-1,1[\,,\\ &\text{and, moreover, there is } \kappa_0 \in \mathbb{N}^* \text{ such that } \partial_y^\alpha(\Gamma(y)\phi(y))|_{y=y_0} = 0\\ &\text{for } \alpha = 0,\dots,2\kappa_0 - 1 \text{ and } \partial_y^{2\kappa_0}(\Gamma(y)\phi(y))|_{y=y_0} < 0.\end{aligned} \tag{1.2.3}$$

Of course, one may always choose functions $\hat{f}_0$, $\hat{g}_0$ in $\mathcal{S}(\mathbb{R})$ such that (1.2.3) holds, because of (1.1.5).

Let $\gamma > 0$, $\delta' > 0$ be fixed positive numbers. For $\varepsilon > 0$ small, define

$$\varepsilon' = \varepsilon^{-\frac{2+\gamma+2\delta'}{1+2\delta'}}\exp\Big(-\frac{S_*}{\varepsilon^2(1+2\delta')}\Big) \ll 1. \tag{1.2.4}$$

Let $u(\varepsilon')$ be the unique small solution satisfying $u(0) = 0$ of the equation

$$u = \varepsilon' \exp\Big(\frac{u}{1+2\delta'}\Big)$$

so that $u(\varepsilon') = \varepsilon' + O(\varepsilon'^2)$, $\varepsilon' \to 0$. We define

$$T(\varepsilon) = \exp\Big(\frac{S_*}{\varepsilon^2} - u(\varepsilon')\Big) = e^{\frac{S_*}{\varepsilon^2}}(1 - \varepsilon' + O(\varepsilon'^2)), \quad \varepsilon' \to 0. \tag{1.2.5}$$

Our main theorem is the following one:

Theorem 1.2.1. *Let $f_0, g_0 \in \mathcal{S}(\mathbb{R})$ be given such that assumption* (1.2.3) *holds. Let $c > 0$, $\theta > 0$ be given small numbers. Let $s_0 \in \mathbb{N}$ be a large enough integer. There is $\delta'_0 > 0$ and for any $\delta' \in]0, \delta'_0]$, any $\gamma \geq 2(\delta' + 2)$, there are $\varepsilon_0 > 0$, $C > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[$, there are functions $x \to (f(x,\varepsilon), g(x,\varepsilon))$ in $H^{s_0+1}(\mathbb{R}) \times H^{s_0}(\mathbb{R})$, small in the sense that*

$$\begin{aligned} \|f(\cdot,\varepsilon)\|_{H^{s_0+1}} + \|g(\cdot,\varepsilon)\|_{H^{s_0}} &\leq C\varepsilon^{1-\theta}, \\ \|xf(\cdot,\varepsilon)\|_{H^1} + \|xg(\cdot,\varepsilon)\|_{L^2} &\leq C\varepsilon^{1-\theta}, \end{aligned} \tag{1.2.6}$$

so that the unique solution u of (1.1.1) *with initial data*

$$u(1,x) = \varepsilon(f_0(x) + f(x,\varepsilon)), \quad \partial_t u(1,x) = \varepsilon(g_0(x) + g(x,\varepsilon)) \tag{1.2.7}$$

is defined for $t \in [1, T(\varepsilon)]$ and satisfies

$$\begin{aligned} \|u(T(\varepsilon),\cdot)\|_{L^\infty} + \|\partial_t u(T(\varepsilon),\cdot)\|_{L^\infty} &= \frac{\varepsilon}{\sqrt{T(\varepsilon)}} I(\varepsilon), \\ \|u(T(\varepsilon),\cdot)\|_{L^2} + \|\partial_t u(T(\varepsilon),\cdot)\|_{L^2} &= \varepsilon J(\varepsilon), \end{aligned} \tag{1.2.8}$$

where

$$I(\varepsilon) \geq cT(\varepsilon)^{\frac{1}{2}-c}, \quad J(\varepsilon) \geq cT(\varepsilon)^{\frac{1}{2}-\frac{1}{4\kappa_0}-c}. \tag{1.2.9}$$

Remarks. Let us make the following comments:

- By (1.2.5), $T(\varepsilon)$ is exponentially large when $\varepsilon \to 0+$. Then (1.2.8) and the first inequality (1.2.9) show that one has inflation of the estimate of the L^∞-norm of the solution by a factor $I(\varepsilon)$ in comparison with the $O(\varepsilon/\sqrt{T(\varepsilon)})$ bound that holds when the null condition is satisfied. In the same way, if $\kappa_0 \geq 2$, (1.2.8) and the lower bound for $J(\varepsilon)$ in (1.2.9) imply inflation of the L^2-norms in comparison with the $O(T(\varepsilon)^\alpha)$ ($\alpha > 0$ arbitrary) bound that holds under the null condition.
- The solution u will be written as the sum of an approximate solution and of a remainder. The lower bounds (1.2.8) are those of this approximate solution (constructed from f_0, g_0) at time $T(\varepsilon)$.
- The exact solution will be given by the sum of the approximate solution and of an error obtained solving a backwards Klein–Gordon equation with zero data at $t = T(\varepsilon)$ and source term determined by the approximate solution. This error generates in the initial conditions (1.2.7) the $O(\varepsilon^{2-\theta})$ perturbation of $(\varepsilon f_0, \varepsilon g_0)$.
- As mentioned in the introduction, our method of proof is inspired by the construction of blowing-up solutions for nonlinear Schrödinger equations by Cazenave–Martel–Zhao [6] and Cazenave–Han–Martel [5]. For Klein–Gordon equations that do not satisfy the null condition, in general we cannot expect to get blowing-up solutions, but only norm inflation. Actually, the null condition provides a global existence criterion only in the framework of *small data*: in order to uncover it, one has to make some reductions (through normal forms) in order to eliminate some noncharacteristic contributions to the nonlinearity. These reductions bring new terms in the nonlinearity, vanishing at order five at the origin. As long as data are small, these quintic

corrections are negligible, but they could play a prominent role for larger solutions. As a toy example, consider the ODE $\dot{y} = \frac{1}{2}y^3$, with data $y(0) = \varepsilon$, whose solution $y(t) = \frac{\varepsilon}{\sqrt{1-t\varepsilon^2}}$ blows up at time $t = \frac{1}{\varepsilon^2}$. The perturbed equation $\dot{y} = \frac{1}{2}y^3(1 - y^2)$ with the same initial condition has solutions that are globally defined for $t \geq 0$. If we set

$$a(\varepsilon) = \Big(1 - \varepsilon^2 \log \frac{\varepsilon^2}{1-\varepsilon^2}\Big)^{-\frac{1}{2}}$$

the solution satisfies

$$y(t)\Big(1 - y(t)^2 \log \frac{y(t)^2}{1-y(t)^2}\Big)^{-\frac{1}{2}} = \frac{\varepsilon a(\varepsilon)}{(1 - t\varepsilon^2 a(\varepsilon)^2)^{\frac{1}{2}}}. \tag{1.2.10}$$

At time $t_\varepsilon = \varepsilon^{-2}a(\varepsilon)^{-2}(1 - \varepsilon^{2-2\delta})$ with $\delta > 0$ small, we deduce from (1.2.10) that $y(t_\varepsilon)$ will be of size essentially ε^δ, much larger than the size ε of the initial data (though still small). This is the same phenomenon as the one that happens in the theorem.

2. Construction of approximate solution

2.1. Construction for moderate time

Our first goal is to construct an approximate solution for equation (1.1.1) with initial condition

$$u(1, \varepsilon) = \varepsilon f_0(x), \quad \partial_t u(1, \varepsilon) = \varepsilon g_0(x) \tag{2.1.1}$$

where (f_0, g_0) are functions in $\mathcal{S}(\mathbb{R})$ chosen so that (1.2.2) and (1.2.3) hold true. In this subsection, we construct the solution up to time $e^{3S_*/4\varepsilon^2}$, essentially following [10]. We introduce the notation

$$p(\xi) = \sqrt{1+\xi^2}, \quad L_\pm = x \pm tp'(D_x).$$

We first take as an approximate solution over an interval $[1, \varepsilon^{-1+\theta}]$, where $\theta > 0$ is small, the solution u_0 of the linear equation

$$\begin{aligned} &(\partial_t^2 - \partial_x^2 + 1)u_0 = 0, \\ &u_0(1, \cdot) = \varepsilon f_0, \quad \partial_t u_0(1, \cdot) = \varepsilon g_0. \end{aligned} \tag{2.1.2}$$

Proposition 2.1.1. *Set*

$$r_0(t, x) = (\partial_t^2 - \partial_x^2 + 1)u_0 - P(u_0, \partial_t u_0, \partial_x u_0). \tag{2.1.3}$$

Then for any $s_0 \in \mathbb{N}$, $\theta > 0, c > 0$, there is $C > 0$ such that

$$\begin{aligned} &\int_1^{c\varepsilon^{-1+\theta}} \|r_0(\tau, \cdot)\|_{H^{s_0}}\, d\tau \leq C\varepsilon^{3-0}, \\ &\int_1^{c\varepsilon^{-1+\theta}} \|L_\pm r_0(\tau, \cdot)\|_{H^1}\, d\tau \leq C\varepsilon^{2+\theta}, \end{aligned} \tag{2.1.4}$$

where ε^{3-0} *means* $\varepsilon^{3-\kappa}$ *for any* $\kappa > 0$. *Moreover, if for* $|y| < 1$, *we denote*

$$\varphi(y) = \sqrt{1-y^2},$$

then for $t \geq 1$, *we may write* u_0 *in the form*

$$u_0(t,x) = 2\,\mathrm{Re}\Big[\frac{\varepsilon}{\sqrt{t}} e^{it\varphi(x/t)}\Big(a_1^0\Big(\frac{x}{t}\Big) + \frac{1}{t} b_1^0\Big(\frac{x}{t}\Big) + \frac{1}{t^2} c_1^0\Big(t, \frac{x}{t}\Big)\Big)\Big] + \varepsilon e(t,x), \quad (2.1.5)$$

where $a_1^0(y)$, $b_1^0(y)$, *(resp.* $c_1^0(t,y)$*) are smooth functions on* $\mathbb{R}$ *(resp.* $[1,+\infty[\times\mathbb{R}$*), supported for* $|y| \leq 1$, *with*

$$a_1^0(y) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi}}(1-y^2)^{-\frac{3}{4}}\big[\hat{f}_0(\omega_1(y)) - i\sqrt{1-y^2}\,\hat{g}_0(\omega_1(y))\big] \quad (2.1.6)$$

for $|y| < 1$, *with* c_1^0 *satisfying for any* α, β, N *in* $\mathbb{N}$,

$$|\partial_t^\alpha \partial_y^\beta c_1^0(t,y)| \leq C_{\alpha,\beta,N} t^{-\alpha}(1-|y|)^N, \quad (2.1.7)$$

and where $e(t,x)$ *is a real-valued function in* $\mathcal{S}([1,+\infty]\times\mathbb{R})$.

Proof. Expansion (2.1.5) is given in [10, Proposition 2.1.1]. To get estimates (2.1.4), we just notice that $r_0(t,x) = -P(u_0, \partial_t u_0, \partial_x u_0)$ may be written according to (2.1.5) as the sum of an element of $\mathcal{S}([1,+\infty[\times\mathbb{R})$ that is $O(\varepsilon^3)$ in that space, which trivially satisfies (2.1.4), and of expressions of the form

$$\frac{\varepsilon^3}{t^{\frac{3}{2}}} e^{iqt\varphi(x/t)} c\Big(t, \frac{x}{t}\Big)$$

for some function c of the same form as c_1^0 in (2.1.7) and some q in $\mathbb{Z}$. Such terms satisfy (2.1.4). This concludes the proof. ∎

Our next step is to construct the approximate solution for t up to $e^{3S_*/4\varepsilon^2}$. We first introduce the solution $s \to a_{1,1}(s,y)$ of the differential equation

$$\begin{aligned} \omega_0(y)\partial_s a_{1,1}(s,y) &= \frac{1}{2}(\phi(y) + i\psi(y))|a_{1,1}(s,y)|^2 a_{1,1}(s,y), \\ a_{1,1}(0,y) &= a_1^0(y), \end{aligned} \quad (2.1.8)$$

where $a_1^0(y) \in C_0^\infty(\mathbb{R})$ with support in $[-1,1]$ is defined in (2.1.6) and where $\omega_0(y)$, $\phi(y)$, $\psi(y)$ have been introduced in (1.1.3), (1.1.4). It follows from (2.1.8) that

$$\partial_s |a_{1,1}(s,y)|^2 = \phi(y)\omega_0(y)^{-1}|a_{1,1}(s,y)|^4$$

when $|y| < 1$, so that

$$|a_{1,1}(s,y)|^2 = \frac{|a_1^0(y)|^2}{1 - |a_1^0(y)|^2\phi(y)\sqrt{1-y^2}\,s} = \frac{\Gamma(y)\omega_0(y)}{1-\Gamma(y)\phi(y)s} \quad (2.1.9)$$

using definition (2.1.6) of a_1^0 and notation (1.2.1). By (1.2.2), $a_{1,1}$ is thus defined for $s \in [1, S_*[$ and plugging (2.1.9) into (2.1.8), we get the explicit expression

$$a_{1,1}(s,y) = a_1^0(y)(1-\Gamma(y)\phi(y)s)^{-\frac{1}{2}} \exp\Big[-\frac{i}{2}\frac{\psi(y)}{\phi(y)}\log(1-\Gamma(y)\phi(y)s)\Big]. \quad (2.1.10)$$

In particular, $a_{1,1}$ is a smooth function of $(s,y) \in [0, S_*[\times]-1,1[$ that extended by zero for $|y| \geq 1$ is smooth on $[0, S_*[\times \mathbb{R}$, since a_1^0 and Γ are C^∞ on $\mathbb{R}$, supported in $[-1,1]$.

We shall construct an approximate solution of (1.1.1) defined for $t \in [c\varepsilon^{-1+\theta}, e^{3S_*/4\varepsilon^2}]$ that will match with u_0 defined in Proposition 2.1.1.

Proposition 2.1.2. *There are, in addition to function $a_{1,1}$ introduced in (2.1.10), smooth functions $(s,y) \to a_{3,3}^1(s,y)$ (resp. $(s,y,\varepsilon) \to a_{5,3}^1(s,y,\varepsilon)$, $(s,y,\varepsilon) \to a_{5,5}^1(s,y,\varepsilon)$) defined on $[0, \frac{3S_*}{4}] \times \mathbb{R}$ (resp. on $[0, \frac{3S_*}{4}] \times \mathbb{R} \times [0,1]$), supported for $|y| \leq 1$, such that if we define for $c\varepsilon^{-1+\theta} \leq t \leq e^{3S_*/4\varepsilon^2}$,*

$$\begin{aligned} u_{\mathrm{app}}^1(t,x) = 2\,\mathrm{Re}\Big[&\frac{\varepsilon}{\sqrt{t}} a_{1,1}\Big(\varepsilon^2 \log t, \frac{x}{t}\Big) e^{it\varphi(x/t)} \\ &+ \frac{\varepsilon^3}{t^{\frac{3}{2}}} a_{3,3}^1\Big(\varepsilon^2 \log t, \frac{x}{t}\Big) e^{3it\varphi(x/t)} \\ &+ \frac{\varepsilon}{t^{\frac{5}{2}}} a_{5,3}^1\Big(\varepsilon^2 \log t, \frac{x}{t}, \varepsilon\Big) e^{3it\varphi(x/t)} \\ &+ \frac{\varepsilon^5}{t^{\frac{5}{2}}} a_{5,5}^1\Big(\varepsilon^2 \log t, \frac{x}{t}, \varepsilon\Big) e^{5it\varphi(x/t)}\Big] \end{aligned} \quad (2.1.11)$$

the following holds true: the remainder

$$r_{\mathrm{app}}^1(t,x) = (\partial_t^2 - \partial_x^2 + 1)u_{\mathrm{app}}^1 - P(u_{\mathrm{app}}^1, \partial_t u_{\mathrm{app}}^1, \partial_x u_{\mathrm{app}}^1) \quad (2.1.12)$$

may be written as

$$r_{\mathrm{app}}^1(t,x) = 2\,\mathrm{Re}\Big[\varepsilon t^{-\frac{5}{2}} e^{it\varphi(x/t)} \tilde{c}_{5,1}\Big(\varepsilon^2 \log t, \frac{x}{t}, \frac{1}{t}, \varepsilon\Big)\Big] + F_{\mathrm{app}}^1, \quad (2.1.13)$$

where $\tilde{c}_{5,1}(s,y,h,\varepsilon)$ is continuous on $[0, \frac{3S_*}{4}] \times \mathbb{R} \times]0,1] \times [0,1]$, with uniform estimates for the function and all its ∂_s, ∂_y, $h\partial_h$-derivatives, supported for $|y| \leq 1$, and F_{app}^1 satisfies for any $s \in \mathbb{N}$ estimates

$$\begin{aligned} \int_{c\varepsilon^{-1+\theta}}^{\exp(3S_*/4\varepsilon^2)} \|F_{\mathrm{app}}^1(t,\cdot)\|_{H^s}\,dt &\leq C\varepsilon^{2-\theta}, \\ \int_{c\varepsilon^{-1+\theta}}^{\exp(3S_*/4\varepsilon^2)} \|L_\pm F_{\mathrm{app}}^1(t,\cdot)\|_{H^1}\,dt &\leq C\varepsilon^{2-\theta}. \end{aligned} \quad (2.1.14)$$

Before starting the proof of the proposition, we introduce some notation. We shall denote by $\mathscr{P}$ the ring of continuous functions $(y,h,\varepsilon) \to \omega(y,h,\varepsilon)$ defined on the domain $]-1,1[\times]0,1] \times [0,1]$, such that for any α, α' in $\mathbb{N}^2$, there is $K_{\alpha,\alpha'}$ in $\mathbb{N}$ so that the function $(1-y^2)^{K_{\alpha,\alpha'}}\partial_y^\alpha(h\partial_h)^{\alpha'}\omega$ is uniformly bounded. Then the space of functions of (y,h,ε) defined and continuous on $\mathbb{R}\times]0,1]\times[0,1]$, bounded as well as their ∂_y, $h\partial_h$ derivatives on that domain, and supported for $|y|\le 1$, is a $\mathcal{P}$-module. If $(s,y)\to a(s,y)$ is a smooth function on $[0,S_*[\times\mathbb{R}$, supported for $|y|\le 1$, and if q,ℓ are (odd) integers with $1\le|q|\le\ell$, we notice that

$$\begin{aligned}(\partial_t^2-\partial_x^2+1)&\Big[e^{itq\varphi(x/t)}t^{-\frac{\ell}{2}}a\Big(\varepsilon^2\log t,\frac{x}{t}\Big)\Big]\\ &=(1-q^2)t^{-\frac{\ell}{2}}e^{itq\varphi(y)}a(s,y)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\ &\quad+2iqt^{-\frac{\ell}{2}-1}e^{itq\varphi(y)}\omega_0(y)\Big[\varepsilon^2\partial_s a-\frac{1}{2}(\ell-1)a\Big](s,y)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\ &\quad+t^{-\frac{\ell}{2}-2}e^{itq\varphi(y)}R_2(a)(s,y,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}},\end{aligned}\tag{2.1.15}$$

where ω_0 was defined in (1.1.3) and $R_2(a)$ belongs to the $\mathcal{P}$-module generated by $\partial_s^\alpha\partial_y^{\alpha'}a(s,y)$, $\alpha+\alpha'\le 2$.

Let us first compute the linear part in expression (2.1.12) of r^1_{app}.

Lemma 2.1.3. *There is a smooth function $(s,y,\varepsilon)\to b_{5,3}(s,y,\varepsilon)$ (resp. a smooth function $(s,y,\varepsilon)\to b_{5,1}(s,y,\varepsilon)$), defined on $[0,\frac{3S_*}{4}]\times\mathbb{R}\times[0,1]$, supported for $|y|\le 1$, which is fully determined by $a^1_{3,3}$ (resp. $a_{1,1}$) in (2.1.11), and there are continuous functions $(s,y,h,\varepsilon)\to b_{7,q}(s,y,h,\varepsilon)$, defined on $[0,\frac{3S_*}{4}]\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $|y|\le 1$, bounded as well as their ∂_s, ∂_y, $(h\partial_h)$-derivatives, for $q=3,5,7$, fully determined by $a_{1,1}$, $a^1_{3,3}$, $a^1_{5,3}$, $a^1_{5,5}$ such that the following equality holds true:*

$$\begin{aligned}(\partial_t^2-\partial_x^2+1)u^1_{\text{app}}=2\,\mathrm{Re}\Big[&2i\omega_0(y)\frac{\varepsilon^3}{t^{\frac{3}{2}}}e^{it\varphi(y)}\partial_s a(s,y)\\ &-8\frac{\varepsilon^3}{t^{\frac{3}{2}}}e^{3it\varphi(y)}a^1_{3,3}(s,y)\\ &+\frac{\varepsilon}{t^{\frac{5}{2}}}e^{it\varphi(y)}b_{5,1}(s,y,\varepsilon)\\ &-\frac{\varepsilon}{t^{\frac{5}{2}}}e^{3it\varphi(y)}[8a^1_{5,3}(s,y,\varepsilon)-b_{5,3}(s,y,\varepsilon)]\\ &-24\frac{\varepsilon^5}{t^{\frac{5}{2}}}e^{5it\varphi(y)}a^1_{5,5}(s,y,\varepsilon)\\ &+\frac{\varepsilon}{t^{\frac{7}{2}}}\sum_{q=1}^{5}e^{itq\varphi(y)}b^1_{7,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}.\end{aligned}\tag{2.1.16}$$

Proof. We apply (2.1.15) to each term in definition (2.1.11) of u^1_{app}. The $a_{1,1}$ term in (2.1.11) brings the first term on the right-hand side of (2.1.16) and the third one. If we apply (2.1.15) to the $t^{-\frac{3}{2}}\varepsilon^3e^{3i\varphi(y)}a^1_{3,3}(s,y)$ term in (2.1.11), we get the second term on the right-hand side of (2.1.16), the $b_{5,3}$ term (which depends only on $a^1_{3,3}$) and contributions to the last sum. In the same way, applying (2.1.15) to the $\varepsilon t^{-\frac{5}{2}} a^1_{5,3} e^{3i\varphi}$ term in (2.1.11), we get the $a^1_{5,3}$ term in (2.1.16) and contributions to the last sum. Finally, the last term of (2.1.11) brings the $a^1_{5,5}$ term in (2.1.16) and contributions to the last sum. ■

Next we compute the nonlinear part in definition (2.1.13) of r^1_{app}.

Lemma 2.1.4. *There are continuous functions $(s, y) \to c_{3,q}(s, y)$, $q = 1, 3$ (resp. $(s, y, \varepsilon) \to c_{5,q}(s, y, \varepsilon)$, $1 \leq q \leq 5$, q odd, resp. $(s, y, h, \varepsilon) \to c_{7,q}(s, y, h, \varepsilon)$, $1 \leq q \leq 15$, q odd) defined on $[0, \frac{3S_*}{4}] \times \mathbb{R}$ (resp. $[0, \frac{3S_*}{4}] \times \mathbb{R} \times [0, 1]$, resp. $[0, \frac{3S_*}{4}] \times \mathbb{R} \times]0, 1] \times [0, 1]$), supported for $|y| \leq 1$, with all their ∂_s, ∂_y, $h\partial_h$-derivatives bounded, such that $P(u^1_{\mathrm{app}}, \partial_t u^1_{\mathrm{app}}, \partial_x u^1_{\mathrm{app}})$ may be written in the form*

$$
\begin{aligned}
2\,\mathrm{Re}\Big[& \varepsilon^3 t^{-\frac{3}{2}} \sum_{q=1,3} e^{itq\varphi(y)} c_{3,q}(s, y) \\
& + \varepsilon^3 t^{-\frac{5}{2}} \sum_{q=1,3} e^{itq\varphi(y)} c_{5,q}\big(s, y, \varepsilon\big) + \varepsilon^5 t^{-\frac{5}{2}} e^{it5\varphi(y)} c_{5,5}(s, y, \varepsilon) \\
& + \varepsilon^3 t^{-\frac{7}{2}} \sum_{\substack{q \text{ odd} \\ 1\leq q\leq 15}} e^{itq\varphi(y)} c_{7,q}\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2 \log t \\ y=x/t}}.
\end{aligned} \tag{2.1.17}
$$

Moreover, $c_{3,1}$ is given by

$$
c_{3,1}(s, y) = i(\phi(y) + i\psi(y))|a_{1,1}(s, y)|^2 a_{1,1}(s, y) \tag{2.1.18}
$$

with ϕ, ψ defined in (1.1.4) *and*

$$
\begin{aligned}
& c_{3,3} \text{ depends only on } a_{1,1}, \\
& c_{5,q},\ q = 1, 3, 5 \text{ depends only on } a_{1,1},\ a^1_{3,3}, \\
& c_{7,q},\ q \text{ odd},\ 1 \leq q \leq 15, \text{ depends only on } a_{1,1},\ a^1_{3,3},\ a^1_{5,q'},\ q' = 3, 5.
\end{aligned} \tag{2.1.19}
$$

Proof. For the proof, we introduce the notation

$$
\begin{aligned}
U_1(t, x) &= \frac{\varepsilon}{\sqrt{t}} e^{it\varphi(x/t)} a_{1,1}\Big(\varepsilon^2 \log t, \frac{x}{t}\Big), \\
U_3(t, x) &= \frac{\varepsilon^3}{t^{\frac{3}{2}}} e^{3it\varphi(x/t)} a^1_{3,3}\Big(\varepsilon^2 \log t, \frac{x}{t}\Big).
\end{aligned}
$$

Then, by (2.1.11), we may write

$$
P(u^1_{\mathrm{app}}, \partial_t u^1_{\mathrm{app}}, \partial_x u^1_{\mathrm{app}}) - P\big(2\,\mathrm{Re}(U_1 + U_3, \partial_t(U_1 + U_3), \partial_x(U_1 + U_3))\big) \tag{2.1.20}
$$

as a linear combination of expressions of the form

$$
\varepsilon^p t^{-\frac{\ell}{2}} e^{iqt\varphi(x/t)} c\Big(\varepsilon^2 \log t, \frac{x}{t}, \varepsilon\Big)
$$

with $p \geq 3$, $\ell \geq 7$, $1 \leq |q| \leq 15$, q odd, and $c(s, y, \varepsilon)$ smooth on $[0, \frac{3S_*}{4}] \times \mathbb{R} \times [0, 1]$, supported for $|y| \leq 1$, i.e. (2.1.20) contributes to the last term in (2.1.17). We are thus reduced to the study of $P(\mathcal{U}_1 + \mathcal{U}_3)$, where

$$\mathcal{U}_j = (U_j + \bar{U}_j, \partial_t(U_j + \bar{U}_j), \partial_x(U_j + \bar{U}_j)).$$

By Taylor expansion,

$$P(\mathcal{U}_1 + \mathcal{U}_3) \sim P(\mathcal{U}_1) + DP(\mathcal{U}_1) \cdot \mathcal{U}_3$$

modulo terms that contribute again to the last term in (2.1.17). The last term $DP(\mathcal{U}_1) \cdot \mathcal{U}_3$ may be written as contributions to the $t^{-\frac{5}{2}}$-expression in (2.1.17) with coefficients $c_{5,q}$ satisfying (2.1.19) and as contributions to the last term in (2.1.17). It remains to study

$$P(\mathcal{U}_1) = P(U_1 + \bar{U}_1, \partial_t(U_1 + \bar{U}_1), \partial_x(U_1 + \bar{U}_1)). \tag{2.1.21}$$

When computing $\partial_t U_1$, $\partial_x U_1$, if the derivative does not fall on the exponential, we get an extra t^{-1} factor, so that (2.1.21) may be written as new contributions to the last two sums in (2.1.17) and as the expression

$$\frac{\varepsilon^3}{t^{\frac{3}{2}}} P\big(e^{it\varphi(y)} a_{1,1}(s, y)\Omega(y) + e^{-it\varphi(y)} \bar{a}_{1,1}(s, y)\overline{\Omega(y)}\big), \tag{2.1.22}$$

with the notation

$$\Omega(y) = (1, i\omega_0(y), i\omega_1(y)). \tag{2.1.23}$$

Then (2.1.22) provides the $t^{-\frac{3}{2}}$ term in (2.1.17), and to prove (2.1.18) we have to explicitly compute the $e^{it\varphi(y)}$ term in (2.1.22), which gives

$$\begin{aligned} c_{3,1}(s, y) &= DP(a_{1,1}(s, y)\Omega(y)) \cdot \bar{a}_{11}(s, y)\overline{\Omega(y)} \\ &= |a_{1,1}(s, y)|^2 a_{1,1}(s, y) DP(\Omega(y)) \cdot \overline{\Omega(y)} \end{aligned} \tag{2.1.24}$$

since P is homogeneous of degree 3. The explicit expression of $c_{3,1}$ given by (2.1.18) follows from the next lemma. ■

Lemma 2.1.5. *Let P be the cubic polynomial given by* (1.1.2) *and let Ω be given by* (2.1.23)*. Then*

$$DP(\Omega(y)) \cdot \overline{\Omega(y)} = i(\phi(y) + i\psi(y)) \tag{2.1.25}$$

with ϕ, ψ defined in (1.1.4)*. Moreover,*

$$D^2P(\Omega(y)) \cdot (\overline{\Omega(y)}, \Omega(y)) = 2i(\phi(y) + i\psi(y)). \tag{2.1.26}$$

Proof. Since P is homogeneous of order 3, $DP(X)X = 3P(X)$, whence the equality $D^2P(X)(Y, X) = 2DP(X) \cdot Y$, so that (2.1.25) implies (2.1.26). Let us show (2.1.25). Writing as (T, Z_1, Z_2) the variables of P, we have by (2.1.23),

$$\begin{aligned} DP(\Omega) \cdot \bar{\Omega} &= \frac{\partial P}{\partial T}(1, i\omega_0, i\omega_1) - i\omega_0 \frac{\partial P}{\partial Z_1}(1, i\omega_0, i\omega_1) - i\omega_1 \frac{\partial P}{\partial Z_2}(1, i\omega_0, i\omega_1) \\ &= (T\partial_T - Z_1\partial_{Z_1} - Z_2\partial_{Z_2})P(1, i\omega_0, i\omega_1). \end{aligned} \tag{2.1.27}$$

Write the decomposition (1.1.2) as $P(T, Z_1, Z_2) = \sum_{k=0}^{3} P_k(T; Z_1, Z_2)$, where P_k is homogeneous of degree k in (Z_1, Z_2) and $3 - k$ in T. We get that (2.1.27) is given by

$$\begin{aligned}(3P_0 - P_2)(1, i\omega_0, i\omega_1) &+ (P_1 - 3P_3)(1, i\omega_0, i\omega_1)\\ &= (3P_0 + P_2)(1, \omega_0, \omega_1) + i(P_1 + 3P_3)(1, \omega_0, \omega_1).\end{aligned}$$

Going back to definition (1.1.4) of ϕ, ψ we obtain (2.1.25). ■

Proof of Proposition 2.1.2. By (2.1.12), r^1_{app} is the difference of (2.1.16) and (2.1.17). We first choose $a_{1,1}$ as the solution to equation (2.1.8). By (2.1.18) this implies that the first term on the right-hand side of (2.1.16) cancels out the $c_{3,1}$ term in (2.1.17). By (2.1.19), the $c_{3,3}$ term in (2.1.17) is now determined, and we may eliminate it from (2.1.17), choosing $a^1_{3,3}(s, y) = -\frac{1}{8} c_{3,3}(s, y)$ in (2.1.16). By (2.1.19), the $c_{5,q}$ are now determined, as are $b_{5,1}$ and $b_{5,3}$ in (2.1.16), according to the statement of Lemma 2.1.3. The $b_{5,1}$ contribution to (2.1.16) and the $c_{5,1}$ contribution in (2.1.17) will form part of the $\tilde{c}_{5,1}$ term in (2.1.13). On the other hand, if we set

$$\begin{aligned}a^1_{5,3} &= -\frac{1}{8}(c_{5,3}\varepsilon^2 - b_{5,3}),\\ a^1_{5,5} &= -\frac{1}{24} c_{5,5},\end{aligned}$$

we cancel out the $t^{-\frac{5}{2}}$ terms in (2.1.16) and (2.1.17).

We are thus left with only the $t^{-\frac{7}{2}}$ contributions coming from (2.1.16), (2.1.17), which are all of the form

$$\varepsilon e^{itq\varphi(y)} t^{-\frac{7}{2}} c\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big|_{\substack{s=\varepsilon^2 \log t\\ y=x/t}} \tag{2.1.28}$$

for continuous functions on $[0, \frac{3Y_*}{4}] \times \mathbb{R} \times]0, 1] \times [0, 1]$, supported for $|y| \leq 1$, bounded as well as their ∂_s, ∂_y, $(h\partial_h)$-derivatives. The Sobolev norms of (2.1.28) integrated for $t \geq c\varepsilon^{-1+\theta}$ are thus $O(\varepsilon^{3-2\theta})$, which is better than the first inequality (2.1.14). If we make $L_\pm$ act on (2.1.28) before computing the H^1-norm, we lose an extra power of t and get instead, after integration, an $O(\varepsilon^{2-\theta})$ bound that brings the second estimate (2.1.14). This concludes the proof. ■

Next we glue together the function u_0, solution to (2.1.2), which is an approximate solution of (2.1.1) for small times according to Proposition 2.1.1, and the function u^1_{app} defined by (2.1.11), which is also an approximate solution for intermediate times.

Proposition 2.1.6. *Let χ_0 in $C^\infty(\mathbb{R})$ be equal to* 1 *close to* 0*. Define for* $1 \leq t \leq e^{3S_*/4\varepsilon^2}$,

$$u^{\text{M}}_{\text{app}}(t, x) = \chi_0(\varepsilon^{1-\theta}(t-1))u_0(t, x) + (1 - \chi_0)(\varepsilon^{1-\theta}(t-1))u^1_{\text{app}}(t, x) \tag{2.1.29}$$

and

$$r^{\text{M}}_{\text{app}}(t, x) = (\partial_t^2 - \partial_x^2 + 1)u^{\text{M}}_{\text{app}} - P(u^{\text{M}}_{\text{app}}, \partial_t u^{\text{M}}_{\text{app}}, \partial_x u^{\text{M}}_{\text{app}}). \tag{2.1.30}$$

One may write

$$r_{\mathrm{app}}^{\mathrm{M}}(t,x) = 2\,\mathrm{Re}\left[\frac{\varepsilon}{t^{\frac{5}{2}}} e^{it\varphi(y)}\chi_1(\varepsilon^{1-\theta}t)c_{5,1}\Big(s,y,\frac{1}{t},\varepsilon\Big)\right]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} + F_{\mathrm{app}}^{\mathrm{M}}(t,x) \tag{2.1.31}$$

where $\chi_1 \in C^\infty(\mathbb{R})$ is equal to 0 *close to* 0 *and equal to* 1 *outside a neighborhood of* 0*, $c_{5,1}(s,y,h,\varepsilon)$ is a continuous function, bounded as well as its ∂_s, ∂_y, $(h\partial_h)$-derivatives on $[0,\frac{3S_*}{4}]\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $|y|\le 1$, and where $F_{\mathrm{app}}^{\mathrm{M}}$ satisfies*

$$\int_1^{\exp(3S_*/4\varepsilon^2)} \|F_{\mathrm{app}}^{\mathrm{M}}(t,\cdot)\|_{H^s}\,dt \le C_s\varepsilon^{2-\theta} \tag{2.1.32}$$

for any $s\in\mathbb{N}$ and

$$\int_1^{\exp(3S_*/4\varepsilon^2)} \|L_\pm F_{\mathrm{app}}^{\mathrm{M}}(t,\cdot)\|_{H^1}\,dt \le C\varepsilon^{2-\theta}. \tag{2.1.33}$$

Proof. We decompose $r_{\mathrm{app}}^{\mathrm{M}}$ using notation (2.1.3), (2.1.12) as

$$r_{\mathrm{app}}^{\mathrm{M}} = r_{\mathrm{app,L}}^{\mathrm{M}} + r_{\mathrm{app,NL}}^{\mathrm{M}} + \chi_0(\varepsilon^{1-\theta}(t-1))r_0 + (1-\chi_0)(\varepsilon^{1-\theta}(t-1))r_{\mathrm{app}}^1 \tag{2.1.34}$$

with

$$\begin{aligned} r_{\mathrm{app,L}}^{\mathrm{M}} &= (\partial_t^2-\partial_x^2+1)u_{\mathrm{app}}^{\mathrm{M}} - \chi_0(\varepsilon^{1-\theta}(t-1))(\partial_t^2-\partial_x^2+1)u_0\\ &\quad - (1-\chi_0)(\varepsilon^{1-\theta}(t-1))(\partial_t^2-\partial_x^2+1)u_{\mathrm{app}}^1 \end{aligned} \tag{2.1.35}$$

and

$$\begin{aligned} r_{\mathrm{app,NL}}^{\mathrm{M}} &= -P(u_{\mathrm{app}}^{\mathrm{M}},\partial_t u_{\mathrm{app}}^{\mathrm{M}},\partial_x u_{\mathrm{app}}^{\mathrm{M}}) + \chi_0(\varepsilon^{1-\theta}(t-1))P(u_0,\partial_t u_0,\partial_x u_0)\\ &\quad + (1-\chi_0)(\varepsilon^{1-\theta}(t-1))P(u_{\mathrm{app}}^1,\partial_t u_{\mathrm{app}}^1,\partial_x u_{\mathrm{app}}^1). \end{aligned} \tag{2.1.36}$$

Let us study (2.1.35) and (2.1.36) successively.

- *Study of* (2.1.35). By definition (2.1.29) of $u_{\mathrm{app}}^{\mathrm{M}}$, we may write (2.1.35) as

$$2\varepsilon^{1-\theta}\chi_0'(\varepsilon^{1-\theta}(t-1))(\partial_t u_0 - \partial_t u_{\mathrm{app}}^1) + \varepsilon^{2-2\theta}\chi_0''(\varepsilon^{1-\theta}(t-1))(u_0-u_{\mathrm{app}}^1). \tag{2.1.37}$$

By (2.1.5) and (2.1.11), we have

$$\begin{aligned} u_0(t,x)-u_{\mathrm{app}}^1(t,x) = 2\,\mathrm{Re}\Bigg[&\frac{\varepsilon}{\sqrt{t}}e^{it\varphi(y)}(a_1^0(y)-a_{1,1}(s,y))\\ &+\frac{\varepsilon}{t^{\frac{3}{2}}}e^{it\varphi(y)}c_{3,1}\Big(y,\frac{1}{t}\Big)\\ &+\frac{\varepsilon^3}{t^{\frac{3}{2}}}e^{3it\varphi(y)}c_{3,3}(s,y)\\ &+\frac{\varepsilon}{t^{\frac{5}{2}}}e^{3it\varphi(y)}c_{5,3}(s,y,\varepsilon)\\ &+\frac{\varepsilon}{t^{\frac{5}{2}}}e^{5it\varphi(y)}c_{5,5}(s,y,\varepsilon)\Bigg]\Bigg|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\ &+\varepsilon e(t,x), \end{aligned} \tag{2.1.38}$$

where the functions $c_{\ell,q}(s,y,h,\varepsilon)$ are continuous functions of their arguments $s \in [0, \frac{3S_*}{4}]$, $y \in \mathbb{R}$, $h \in]0,1]$, $\varepsilon \in [0,1]$, supported for $|y| \leq 1$, bounded as well as their ∂_s, ∂_y, $(h\partial_h)$-derivatives on their domain of definition, and where $e(t,x)$ is in $\mathcal{S}([1,+\infty[\times\mathbb{R})$.

Denote by $\tilde{\chi}_0$ some function in $C_0^\infty(]0,+\infty[)$. If we take a time derivative of the second term on the right-hand side of (2.1.38) and multiply it by $\varepsilon^{1-\theta}\tilde{\chi}_0(\varepsilon^{1-\theta}(t-1)) = t^{-1}t\varepsilon^{1-\theta}\tilde{\chi}_0(\varepsilon^{1-\theta}(t-1))$, we get an expression of the form of the first term on the right-hand side of (2.1.31). This shows that the contribution of the $c_{3,1}$ term in (2.1.38) to (2.1.37) has such a form.

We need thus to prove that all other terms in (2.1.38) give, when plugged into (2.1.37), contributions to $F^{\mathrm{M}}_{\mathrm{app}}$ in (2.1.31). By (2.1.8), $a_{1,1}(0,y) = a_1^0(y)$, so that the product of the first term on the right-hand side of (2.1.38) by $\varepsilon^{1-\theta}\tilde{\chi}_0(\varepsilon^{1-\theta}(t-1))$ is bounded in modulus by

$$\frac{\varepsilon}{t^{\frac{5}{2}}}\varepsilon^{1-\theta}t|\tilde{\chi}_0(\varepsilon^{1-\theta}(t-1))|\varepsilon^2 t\log t\,\mathbb{1}_{|x/t|\leq 1}. \tag{2.1.39}$$

Similar or better estimates hold if we take ∂_t or ∂_x-derivatives, so that the contribution of the first term on the right-hand side of (2.1.38) to (2.1.37) satisfies, as well as its derivatives, bound (2.1.39). As the $L^2(dx)$-norm of (2.1.39) is $O(\varepsilon^{2+\theta-0}t^{-2}\mathbb{1}_{t\sim\varepsilon^{-1+\theta}})$, we see that a bound of the form (2.1.32) holds. If we make $L_\pm$ act on the corresponding term before computing an H^1-norm, we get a bound in $O(\varepsilon^{2+\theta-0}t^{-1}\mathbb{1}_{t\sim\varepsilon^{-1+\theta}})$ which implies that (2.1.33) holds as well.

We are thus reduced to showing that the third to the last terms on the right-hand side of (2.1.38) also give contributions satisfying (2.1.32), (2.1.33) when plugged into (2.1.37). This is evident for the last term. The other ones bring to (2.1.37) expressions of the form

$$\frac{\varepsilon^a}{t^{\frac{\ell}{2}+1}}\tilde{\chi}(\varepsilon^{1-\theta}(t-1))e^{iqt\varphi(y)}c(s,y,\varepsilon)|_{\substack{s=\varepsilon^2\log t,\\ y=x/t}} \tag{2.1.40}$$

with either $a=3$, $\ell=3$ or $a=1$, $\ell\geq 5$. The L^2-norm of (2.1.40) and its derivatives is $O(\varepsilon^a t^{-\frac{\ell+1}{2}}\mathbb{1}_{t\sim\varepsilon^{-1+\theta}})$, whose integral largely satisfies (2.1.32). To obtain (2.1.33), one has to make $L_\pm$ act on (2.1.40), which makes one factor t appear, so that the H^1-norm is $O(\varepsilon^a t^{-\frac{\ell-1}{2}}\mathbb{1}_{t\sim\varepsilon^{-1+\theta}})$. Because of the conditions on a, ℓ, one gets an $O(\varepsilon^{2-\theta})$ bound as in (2.1.33). This concludes the estimate of (2.1.35).

• *Estimate of* (2.1.36). From definition (2.1.29) of $u^{\mathrm{M}}_{\mathrm{app}}$, we may write (2.1.36) as the sum of expressions

$$\begin{aligned}P(u^{\mathrm{M}}_{\mathrm{app}}, \chi_0\partial_t u_0 + (1-\chi_0)\partial_t u^1_{\mathrm{app}}, \partial_x u^{\mathrm{M}}_{\mathrm{app}})\\ - P\big(u^{\mathrm{M}}_{\mathrm{app}}, \partial_t(\chi_0 u_0 + (1-\chi_0)u^1_{\mathrm{app}}), \partial_x u^{\mathrm{M}}_{\mathrm{app}}\big)\end{aligned} \tag{2.1.41}$$

and

$$\begin{aligned}\chi_0 P(u_0,\partial_t u_0,\partial_x u_0) + (1-\chi_0)P(u^1_{\mathrm{app}},\partial_t u^1_{\mathrm{app}},\partial_x u^1_{\mathrm{app}})\\ - P(\chi_0 u_0 + (1-\chi_0)u^1_{\mathrm{app}}, \chi_0\partial_t u_0 + (1-\chi_0)\partial_t u^1_{\mathrm{app}},\\ \chi_0\partial_x u_0 + (1-\chi_0)\partial_x u^1_{\mathrm{app}}).\end{aligned} \tag{2.1.42}$$

Difference (2.1.41) may be bounded pointwise by

$$C\varepsilon^{1-\theta}|\chi_0'(\varepsilon^{1-\theta}(t-1))|\,|u_0 - u^1_{\mathrm{app}}|\Big(\sum_{\alpha+\beta\le 1}(|\partial_t^\alpha\partial_x^\beta u^1_{\mathrm{app}}| + |\partial_t^\alpha\partial_x^\beta u_0|)\Big)^2. \tag{2.1.43}$$

The difference $u_0 - u^1_{\mathrm{app}}$ is given by (2.1.38), so that its modulus is bounded from above on the support of $\chi_0'(\varepsilon^{1-\theta}(t-1))$ by $C\varepsilon t^{-\frac{3}{2}}$ (using that the first term in (2.1.38) is $O(\frac{\varepsilon}{\sqrt{t}}s)$ with $s = \varepsilon^2\log t = O(\frac{1}{t})$ if $t \sim \varepsilon^{-1+\theta}$). In addition, u^1_{app}, u_0 are $O(\varepsilon t^{-\frac{1}{2}})$, as well as their derivatives. Then (2.1.43) is bounded from above by

$$C\varepsilon^{4-\theta}|\chi'(\varepsilon^{1-\theta}(t-1))|t^{-\frac{5}{2}}(\mathbb{1}_{|x|\le t} + O(\langle x\rangle^{-N})). \tag{2.1.44}$$

A similar bound holds for the derivatives of (2.1.41), so that (2.1.32) is largely satisfied. To get (2.1.33), one has to bound the L^2-norm of (2.1.44) multiplied by t, so that the conclusion follows as well.

It remains to study (2.1.42), which may be written as

$$\chi_0(1-\chi_0)M(u_0, \partial_t u_0, \partial_x u_0, u^1_{\mathrm{app}}, \partial_t u^1_{\mathrm{app}}, \partial_x u^1_{\mathrm{app}}) \tag{2.1.45}$$

for some cubic expression M. Since by (2.1.5), (2.1.11), u_0, u^1_{app} and their derivatives are $O(\varepsilon t^{-\frac{1}{2}}\mathbb{1}_{|x|\le t}) + O(\varepsilon t^{-N}\langle x\rangle^{-N})$, we get that the Sobolev norm of (2.1.45) is of magnitude $O(\varepsilon^3 t^{-1}\mathbb{1}_{t\sim\varepsilon^{-1+\theta}})$, which brings an estimate of the form (2.1.32). In the same way, the integrand in (2.1.33) is $O(\varepsilon^3\mathbb{1}_{t\sim\varepsilon^{-1+\theta}})$, which gives an $O(\varepsilon^{2+\theta})$ bound for the integral.

This concludes the proof, since we have shown that (2.1.35) may be written as a contribution to the $c_{5,1}$ term in (2.1.31) and as a remainder that may be integrated to $F^{\mathrm{M}}_{\mathrm{app}}$, since (2.1.36) is also of the form $F^{\mathrm{M}}_{\mathrm{app}}$ and since the remaining terms $\chi_0 r_0 + (1-\chi_0)r^1_{\mathrm{app}}$ in (2.1.34) are of the form of the right-hand side of (2.1.31) by Proposition 2.1.1 and (2.1.13). ∎

2.2. Construction for large time

Our next goal is to extend the approximate solution that has been constructed up to time $e^{3S_*/4\varepsilon^2}$ in order to almost reach the blow-up time e^{S_*/ε^2}. At this time, the main part of the profile (2.1.11) blows up and we introduce notation for spaces describing the solution close to the blow-up time.

Definition 2.2.1. Let $m \in \mathbb{R}$, y_0 be a point in $]-1,1[$ and $\kappa_0 \in \mathbb{N}^*$. We denote by Σ^m the space of continuous functions $(s,y,h,\varepsilon) \to a(s,y,h,\varepsilon)$ defined on $[0,S_*[\,\times\mathbb{R}\times\,]0,1]\times[0,1]$, with values in $\mathbb{C}$, smooth in (s,y,h), supported for $|y|\le 1$, that satisfy for any integers α,β,ζ,N, any (s,y,h,ε) in the domain of definition, estimates

$$|\partial_s^\alpha\partial_y^\beta(h\partial_h)^\zeta a(s,y,h,\varepsilon)| \le C_{\alpha,\beta,\zeta,N}(S_* - s + |y-y_0|^{2\kappa_0})^{m-\alpha-\frac{\beta}{2\kappa_0}}(1-|y|)^N. \tag{2.2.1}$$

In particular, Σ^m is a $\mathcal{P}$-module (for $\mathcal{P}$ defined after Proposition 2.1.2). Moreover, $\partial_s^\alpha\partial_y^\beta a$ belongs to $\Sigma^{m-\alpha-\frac{\beta}{2\kappa_0}} \subset \Sigma^{m-\alpha-\beta}$. When a does not depend on one of the variables h or ε, we remove it from the notation.

Example. Consider the function $a_{1,1}(s,y)$ defined in (2.1.10) with a_1^0 smooth on $\mathbb{R}$, supported for $|y| \leq 1$. Then $a_{1,1}$ is smooth on $[0,S_*] \times \mathbb{R} - \{(S_*, y_0)\}$ because of (1.2.2), (1.2.3). Moreover, for y close to y_0, (1.2.3) implies that $\Gamma(y)\phi(y) = \frac{1}{S_*} - (y-y_0)^{2\kappa_0}\Theta(y)$ for some smooth positive function Θ, so that we get estimates of the form (2.2.1) with $m = -\frac{1}{2}$, i.e. $a_{1,1}$ belongs to $\Sigma^{-\frac{1}{2}}$.

Our goal is to prove the following proposition:

Proposition 2.2.2. *Let $\delta > 0$ be a small number, $N \in \mathbb{N}$. One may construct for all odd integers q, ℓ satisfying $1 \leq q \leq \ell \leq N$ elements $a_{\ell,q}$ of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ with $a_{1,1}$ given by* (2.1.10) *so that, if we define for $t \in [e^{\frac{S_*}{2\varepsilon^2}}, e^{\frac{S_*}{\varepsilon^2}}[$, $x \in \mathbb{R}$,*

$$u^2_{\mathrm{app}}(t,x) = 2\,\mathrm{Re}\Big[\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{N} \varepsilon^{2-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} a_{\ell,1}(s,y,\varepsilon) + \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N} \sum_{\substack{3\leq q\leq \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{itq\varphi(y)} a_{\ell,q}(s,y,\varepsilon)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \tag{2.2.2}$$

then

$$r^2_{\mathrm{app}} = (\partial_t^2 - \partial_x^2 + 1)u^2_{\mathrm{app}} - P(u^2_{\mathrm{app}}, \partial_t u^2_{\mathrm{app}}, \partial_x u^2_{\mathrm{app}}) \tag{2.2.3}$$

may be written as the sum of the noncharacteristic expression

$$2\,\mathrm{Re}\Big[\sum_{\substack{\ell=N+2\\ \ell \text{ odd}}}^{3N} \sum_{\substack{3\leq q\leq \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{itq\varphi(y)} d_{\ell,q}(s,y,\varepsilon)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \tag{2.2.4}$$

with, for $3 \leq q \leq \ell$,

$$d_{\ell,q} \in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)},$$

and of a characteristic expression

$$2\,\mathrm{Re} \sum_{\substack{\ell=N+4\\ \ell \text{ odd}}}^{3N} \varepsilon^{6-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} d_{\ell,1}(s,y,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \tag{2.2.5}$$

with

$$d_{\ell,1} \in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}.$$

Before proving the proposition, we establish several lemmas.

Lemma 2.2.3. *Assume we are given N an odd integer and for any odd integers ℓ, q satisfying $1 \leq q \leq \ell \leq N$ continuous functions $(s,y,\varepsilon) \to a_{\ell,q}(s,y,\varepsilon)$ on $[0,S_*[\times \mathbb{R} \times [0,1]$, smooth in (s,y), supported for $|y| \leq 1$. Let $\mathcal{P}_0$ be the ring of functions $(y,\varepsilon) \to \gamma(y,\varepsilon)$ continuous on $]-1,1[\times [0,1]$, that are smooth in y and have at most algebraic* *growth, as well as their ∂_y-derivatives when $y^2 \to 1$ (uniformly in ε). For each q, ℓ as above, denote by $\mathcal{C}_{\ell,q}$ the $\mathcal{P}_0$-module generated by all cubic expressions of the form*

$$\prod_{j=1}^{3} \partial_s^{\alpha_j} \partial_y^{\beta_j} a_{\ell_j,q_j}(s,y,\varepsilon), \tag{2.2.6}$$

where $\ell_j \in \mathbb{N}$, $q_j \in \mathbb{Z}$ are odd, $\alpha_j, \beta_j \in \mathbb{N}$, $a_{\ell_j,-q_j} = \bar{a}_{\ell_j,q_j}$, and where the following inequalities hold true:

$$\sum_{j=1}^{3} (\ell_j + 2\alpha_j + 2\beta_j) \le \ell, \quad q = |q_1 + q_2 + q_3|. \tag{2.2.7}$$

Introduce

$$U(t,x) = \sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{N} \sum_{\substack{1\le q\le \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{itq\varphi(y)} a_{\ell,q}(s,y,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{2.2.8}$$

Then we may write

$$\begin{aligned} P(2\operatorname{Re}(U,\partial_t U,\partial_x U)) = 2\operatorname{Re}\Bigg[&\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{3N} \sum_{\substack{3\le q\le \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{itq\varphi(y)} c_{\ell,q}(s,y,\varepsilon) \\ &+ \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{3N} \varepsilon^{6-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} c_{\ell,1}(s,y,\varepsilon)\Bigg]\Bigg|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \end{aligned} \tag{2.2.9}$$

where $c_{\ell,q}$ belongs to $\mathcal{C}_{\ell,q}$. Moreover, for $3 \le q \le \ell$,

$$c_{\ell,q} \text{ depends only on } a_{\ell',q'}, \quad 1 \le q' \le \ell' \le \ell - 2. \tag{2.2.10}$$

In addition, $c_{3,1}$ is given explicitly by

$$c_{3,1}(s,y) = i(\phi(y) + i\psi(y))|a_{1,1}|^2 a_{1,1}(s,y) \tag{2.2.11}$$

and for $\ell \ge 5$, one may decompose

$$c_{\ell,1}(s,y,\varepsilon) = c'_{\ell,1}(s,y,\varepsilon) + c''_{\ell,1}(s,y,\varepsilon), \tag{2.2.12}$$

where $c'_{\ell,1}$ is given explicitly by

$$c'_{\ell,1}(s,y,\varepsilon) = 2i(\phi(y) + i\psi(y))\Big(|a_{1,1}|^2 a_{\ell-2,1} + \frac{1}{2} a_{1,1}^2 \bar{a}_{\ell-2,1}\Big)(s,y,\varepsilon) \tag{2.2.13}$$

and

$$\begin{aligned} c''_{\ell,1} &\text{ depends only on } a_{\ell',q'} \text{ for } 1 \le q' \le \ell' \le \ell - 4 \\ &\text{or on } a_{\ell-2,q'},\ 3 \le q' \le \ell - 2, \text{ and } a_{1,1}. \end{aligned} \tag{2.2.14}$$

Proof. We notice first that (2.2.7) implies that

$$\sum_{j=1}^{3}(2|q_j|-\ell_j) \geq 2q-\ell \tag{2.2.15}$$

and that for terms (2.2.6) that are characteristic, i.e. such that $q=|q_1+q_2+q_3|=1$, we have $|q_1|+|q_2|+|q_3|-q \geq 2$, so that

$$\sum_{j=1}^{3}(2|q_j|-\ell_j) \geq 2|q|+4-\ell \geq 6-\ell. \tag{2.2.16}$$

Let us compute (2.2.9). From (2.2.8), and the expressions that may be obtained for $\partial_t U$, $\partial_x U$ from that formula, we see that the $t^{-\frac{\ell}{2}}$ terms in (2.2.9) are given by the product of $e^{\pm iqt\varphi(x/t)}$ ($q \in \mathbb{N}$, q odd), of an element of $\mathcal{C}_{\ell,q}$ and of a power of ε of the form

$$\varepsilon^{\sum_{j=1}^{3}(2|q_j|-\ell_j)+a} \tag{2.2.17}$$

for some $a \geq 0$. In the noncharacteristic case $q \neq 1$, it follows from (2.2.15) that (2.2.17) is $O(\varepsilon^{2q-\ell})$ and in the characteristic case, (2.2.17) will be $O(\varepsilon^{6-\ell})$ by (2.2.16). We thus obtain the structure indicated in (2.2.9). Let us check properties (2.2.10) to (2.2.14).

Since in (2.2.7) all ℓ_j are larger than or equal to 1, and $c_{\ell,q}$ is given by a cubic expression of the form (2.2.6), (2.2.10) holds necessarily.

Let us consider now specifically the characteristic terms $c_{\ell,1}$ in (2.2.9) with $\ell \geq 5$. These terms are given by (2.2.6) with indices satisfying (2.2.7). In this property, either one ℓ_j is equal to $\ell-2$ and then the others are equal to 1 and $\alpha_j=\beta_j=0$ for all j, or all ℓ_j are smaller than or equal to $\ell-4$ (recall that they are odd). This last case corresponds to contributions $c''_{\ell,1}$ satisfying the first alternative in (2.2.14). On the other hand, if one ℓ_j is equal to $\ell-2$, say $\ell_3=\ell-2$, then $\ell_1=\ell_2=1$. If the q_3 associated to ℓ_3 satisfies $|q_3| \geq 3$, we get a contribution to $c''_{\ell,1}$ corresponding to the second alternative in (2.2.14). We are thus left with terms of the form (2.2.6) with

$$\alpha_j=\beta_j=0, \quad \ell_3=\ell-2, \quad |q_3|=1, \quad \ell_1=\ell_2=1, \quad |q_1+q_2+q_3|=1. \tag{2.2.18}$$

These terms give $c'_{\ell,1}$ in (2.2.12) and have to be computed explicitly. Notice also that in the case $\ell=3$, $c_{3,1}$ is itself of that form. Moreover, we have also $|q_j| \leq \ell_j = 1$, $j=1,2$ so that we see that we have to compute those terms of (2.2.9) that oscillate on the phases $\pm t\varphi(y)$ and that come from the contribution to U given by

$$U'(t,x)=\sum_{\ell'=1}^{N}\varepsilon^{2-\ell'}t^{-\frac{\ell'}{2}}e^{it\varphi(y)}a_{\ell',1}(s,y)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{2.2.19}$$

Denote by $U'_{\ell'}$ the general term of that sum and set

$$\mathcal{U}'_{\ell'}=(U'_{\ell'}+\bar{U}'_{\ell'}, \partial_t(U'_{\ell'}+\bar{U}'_{\ell'}), \partial_x(U'_{\ell'}+\bar{U}'_{\ell'})). \tag{2.2.20}$$

We thus have to compute the contribution to (2.2.9) given by those terms in

$$P\Big(\sum_{\substack{\ell'=1\\ \ell' \text{ odd}}}^{N} (\mathcal{U}'_{\ell'} + \bar{\mathcal{U}}'_{\ell'})\Big) \tag{2.2.21}$$

that oscillate along the phases $e^{\pm it\varphi(x/t)}$ and that come from the terms (2.2.20), where ∂_t and ∂_x act on the oscillatory factors coming from (2.2.19) (since in (2.2.18), $\alpha_j = \beta_j = 0$ for any j). Using the notation $\Omega(y) = (1, i\omega_0(y), i\omega_1(y))$ we see that we may reduce (2.2.21) to the expression

$$P\bigg(2\,\mathrm{Re}\bigg[\Omega(y)e^{it\varphi(y)} \sum_{\substack{\ell'=1\\ \ell' \text{ odd}}}^{N} \varepsilon^{2-\ell'} t^{-\frac{\ell'}{2}} a_{\ell',1}(s,y)\bigg]\bigg)\bigg|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{2.2.22}$$

By (2.2.18) we are only interested in contributions to (2.2.22) that are at least quadratic in $a_{1,1}, \bar{a}_{1,1}$, i.e. we may reduce (2.2.22) to

$$\begin{aligned}
&\varepsilon^3 P(2\,\mathrm{Re}[\Omega(y)e^{it\varphi(y)}t^{-\frac{1}{2}}a_{1,1}(s,y)])\\
&\quad+\sum_{\substack{\ell'=1\\ \ell' \text{ odd}}}^{N} \varepsilon^2 DP(2\,\mathrm{Re}[\Omega(y)e^{it\varphi(y)}t^{-\frac{1}{2}}a_{1,1}(s,y)])\\
&\qquad\qquad \cdot (2\,\mathrm{Re}[\Omega(y)e^{it\varphi(y)}t^{-\frac{\ell'}{2}}a_{\ell',1}(s,y)\varepsilon^{2-\ell'}])|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}.
\end{aligned} \tag{2.2.23}$$

The first term in (2.2.23) has been already computed in (2.1.22), (2.1.24) and brings $c_{3,1}$ given by (2.2.11). The term in $t^{-\frac{\ell}{2}}e^{it\varphi(y)}$ coming from the sum in (2.2.23) is obtained when $\ell' = \ell - 2$ and is equal to the $e^{it\varphi(y)}$ term in

$$\varepsilon^{6-\ell}t^{-\frac{\ell}{2}} DP(\Omega e^{it\varphi}a_{1,1} + e^{-it\varphi}\bar{\Omega}\bar{a}_{1,1})\cdot(\Omega e^{it\varphi}a_{\ell-2,1} + e^{-it\varphi}\bar{\Omega}\bar{a}_{\ell-2,1}).$$

Taylor expanding this expression, we see that we have to consider

$$\varepsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi}[a_{1,1}^2\bar{a}_{\ell-2,1}DP(\Omega)\cdot\bar{\Omega} + |a_{1,1}|^2 a_{\ell-2,1}D^2P(\Omega)\cdot(\bar{\Omega},\Omega)].$$

By (2.1.25), (2.1.26), this gives (2.2.13) and concludes the proof of the lemma. ∎

We apply the preceding lemma to compute $P(u^2_{\mathrm{app}}, \partial_t u^2_{\mathrm{app}}, \partial_x u^2_{\mathrm{app}})$.

Corollary 2.2.4. *Assume that u^2_{app} is given by (2.2.2). Then*

$$\begin{aligned}
&P(u^2_{\mathrm{app}}, \partial_t u^2_{\mathrm{app}}, \partial_x u^2_{\mathrm{app}})\\
&\quad= 2\,\mathrm{Re}\bigg[\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{3N} \varepsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}c_{\ell,1}(s,y,\varepsilon)\\
&\qquad\qquad+\sum_{\substack{\ell=3\\ q \text{ odd}}}^{3N}\sum_{\substack{q=3\\ \ell \text{ odd}}}^{\ell} \varepsilon^{2q-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}c_{\ell,q}(s,y,\varepsilon)\bigg]\bigg|_{\substack{s=\varepsilon^2\log t\\ y=x/t}},
\end{aligned} \tag{2.2.24}$$

where $c_{\ell,q}$ *is an element of* $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)+2\delta}$ *which, for* $q \geq 3$*, depends only on* $a_{\ell',q'}$*,* $1 \leq q' \leq \ell' \leq \ell - 2$*, where* $c_{3,1}$ *is given by* (2.2.11) *and for* $\ell \geq 5$*,* $c_{\ell,1}$ *may be decomposed into the form* (2.2.12) *with* (2.2.13) *and* (2.2.14) *holding true.*

Proof. The left-hand side of (2.2.24) is (2.2.9), so that we obtain expression (2.2.24). The coefficients $c_{\ell,q}$ belong to $\mathscr{C}_{\ell,q}$, i.e. are given (up to $\mathscr{P}_0$-multiplicative factors) by expressions of the form (2.2.6) with indices satisfying (2.2.7). Since a_{ℓ_j,q_j} belongs to $\Sigma^{-\frac{\ell_j}{2}-(\ell_j-1)\delta}$, it follows from the definition of these classes and from (2.2.7) that $c_{\ell,q}$ is in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)+2\delta}$. The other assertions of the corollary follow from (2.2.10) to (2.2.14). ■

In order to prove Proposition 2.2.2, we also need the following result.

Lemma 2.2.5. *Let* $y \to \Theta(y)$ *be a complex-valued smooth function defined on* $]-1,1[$*, with at most algebraic growth when* $|y| \to 1-$*, as well as its derivatives. Let* $a(s,y)$ *be an element of* $\Sigma^{-\frac{1}{2}}$*. Assume that there is an open neighborhood* V *of* y_0 *in* $]-1,1[$ *and* $c > 0$ *such that for any* y *in* V*, any* $s \in [0, S_*[$*,*

$$|\mathrm{Re}\,\Theta(y)| \geq c, \tag{2.2.25}$$

$$|a(s,y)| \geq c(S_* - s + |y-y_0|^{2\kappa_0})^{-\frac{1}{2}}, \tag{2.2.26}$$

and that a *solves the ODE* $\partial_s a(s,y) = \Theta(y)|a(s,y)|^2 a(s,y)$*. Let* ℓ *be an odd integer,* $\ell \geq 5$*, and let* r *be an element of* $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$*. Let* $(s,y) \to b(s,y)$ *be the solution of*

$$\begin{aligned} &\partial_s b(s,y) = \Theta(y)(2|a(s,y)|^2 b(s,y) + a(s,y)^2\overline{b(s,y)}) + r(s,y),\\ &b(0,y) = 0. \end{aligned} \tag{2.2.27}$$

Then b *belongs to* $\Sigma^{-\frac{\ell-2}{2}-\delta(\ell-3)}$*.*

Proof. We notice first that if y stays outside V, then by definition of $\Sigma^{-\frac{1}{2}}$, the coefficients on the right-hand side of (2.2.27) are smooth functions on $[0, S_*] \times (\mathbb{R} - V)$, so that the same holds true for the solution b, which is moreover supported for $|y| \leq 1$.

We may thus assume that y stays close to y_0, so that (2.2.25), (2.2.26) hold true. We introduce $B(s,y) = \left[\begin{smallmatrix} b(s,y) \\ \overline{b(s,y)} \end{smallmatrix}\right]$ that solves the system

$$\partial_s B(s,y) = M(s,y)B(s,y) + R(s,y), \tag{2.2.28}$$

with

$$R(s,y) = \begin{bmatrix} r(s,y) \\ \overline{r(s,y)} \end{bmatrix}, \quad M(s,y) = \begin{bmatrix} 2\Theta|a|^2 & \Theta a^2 \\ \bar{\Theta}\bar{a}^2 & 2\bar{\Theta}|a|^2 \end{bmatrix}.$$

Define the two functions

$$\Phi_1(s,y) = \begin{bmatrix} ia(s,y) \\ -i\overline{a(s,y)} \end{bmatrix}, \quad \Phi_2(s,y) = \begin{bmatrix} \partial_s a(s,y) \\ \overline{\partial_s a(s,y)} \end{bmatrix} = |a(s,y)|^2 \begin{bmatrix} \Theta a \\ \bar{\Theta}\bar{a} \end{bmatrix}.$$

Then Φ_j solves the homogeneous equation $\partial_s \Phi_j = M(s,y)\Phi_j$ and the wronskian $w(s,y)$ of $\Phi_1(s,y)$, $\Phi_2(s,y)$ is equal to $2i \operatorname{Re}\Theta(y)|a(s,y)|^4$, so satisfies for $y \in V$, $|w(s,y)| \geq cA(s,y)^{-4}$ according to (2.2.25), (2.2.26) if we set

$$A(s,y) = (S_* - s + |y-y_0|^{2\kappa_0})^{\frac{1}{2}}.$$

The fact that $a \in \Sigma^{-\frac{1}{2}}$ and that (2.2.25), (2.2.26) hold for y close to y_0 imply that the wronskian matrix $W(s,y)$ and its inverse $W(s',y)$ satisfy

$$W(s,y) = \begin{bmatrix} O(A(s,y)^{-1}) & O(A(s,y)^{-3}) \\ O(A(s,y)^{-1}) & O(A(s,y)^{-3}) \end{bmatrix},$$
$$W(s',y)^{-1} = \begin{bmatrix} O(A(s',y)) & O(A(s',y)) \\ O(A(s',y)^3) & O(A(s',y)^3) \end{bmatrix}.$$

Since $s \to A(s,y)$ is decreasing, we conclude that for $0 \leq s' \leq s$,

$$W(s,y)W(s',y)^{-1} = O\Big(\Big(\frac{A(s',y)}{A(s,y)}\Big)^3\Big). \tag{2.2.29}$$

Writing the solution to (2.2.28) with zero initial condition at $s=0$ in the form

$$\int_0^s W(s,y)W(s',y)^{-1}R(s',y)\,ds' \tag{2.2.30}$$

and using that $r \in \Sigma^{-\frac{\ell}{2}-\delta(\ell-1)+2\delta}$, we get from (2.2.29), (2.2.30),

$$|B(s,y)| \leq C\int_0^s A(s',y)^{3-\ell-2\delta(\ell-1)+4\delta}\,ds' A(s,y)^{-3}.$$

Since $\ell \geq 5$ and $\delta > 0$, this is $O(A(s,y)^{2-\ell-2\delta(\ell-3)})$, i.e. B satisfies (2.2.1) with $\alpha = \beta = 0$, $m = -\frac{\ell-2}{2} - \delta(\ell-3)$ for y close to y_0. If we take ∂_y or ∂_s derivatives in (2.2.30), we get in the same way the estimates (2.2.1) for positive α or β. This concludes the proof. ■

Proof of Proposition 2.2.2. We shall compute first the action of $\partial_t^2 - \partial_x^2 + 1$ on u^2_{app} given by (2.2.2) and use the fact that the last term in the expression (2.2.3) of r^2_{app} was computed in Corollary 2.2.4. We shall then construct the $a_{\ell,q}$ recursively in order to reduce r^2_{app} to an expression of the form (2.2.4).

• *Linear term in* (2.2.3). We apply (2.1.15) to the general term of the sums in (2.2.2). We get on the one hand the characteristic contribution

$$2\operatorname{Re}\Big[2i\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N+2} \varepsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}\omega_0(y)\partial_s a_{\ell-2,1}(s,y,\varepsilon)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}$$
$$+\,2\operatorname{Re}\Big[\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+4} \varepsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}R_2(a_{\ell-4,1})(s,y,\varepsilon)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \tag{2.2.31}$$

where $R_2(a_{\ell-4,1})$ belongs to the $\mathcal{P}$-module generated by $\partial_s^\alpha \partial_y^\beta a_{\ell-4,1}$ for $\alpha+\beta \leq 2$, so that

$$R_2(a_{\ell-4,1}) \in \Sigma^{-\frac{\ell}{2}-\delta(\ell-5)}$$

by the definition of this class. On the other hand, the second sum in (2.2.2) provides to the linear term in (2.2.3) the noncharacteristic contribution

$$\begin{aligned}
&2\operatorname{Re}\Big[\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N}\sum_{\substack{3\leq q\leq \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}}(1-q^2)e^{itq\varphi(y)}a_{\ell,q}(s,y,\varepsilon)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\
&\quad+2\operatorname{Re}\Big[\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+2}\sum_{\substack{3\leq q\leq \ell-2\\ q \text{ odd}}} \varepsilon^{2q+2-\ell} t^{-\frac{\ell}{2}}e^{itq\varphi(y)}R_1(a_{\ell-2,q})\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\
&\quad+2\operatorname{Re}\Big[\sum_{\substack{\ell=7\\ \ell \text{ odd}}}^{N+4}\sum_{\substack{3\leq q\leq \ell-4\\ q \text{ odd}}} \varepsilon^{2q+4-\ell} t^{-\frac{\ell}{2}}e^{itq\varphi(y)}R_2(a_{\ell-4,q})\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}},
\end{aligned} \tag{2.2.32}$$

where $R_1(a_{\ell-2,q})$ (resp. $R_2(a_{\ell-4,q})$) is in the $\mathcal{P}$-module generated by $\partial_s^\alpha \partial_y^\beta a_{\ell-2,q}$ (resp. $\partial_s^\alpha \partial_y^\beta a_{\ell-4,q}$) for $\alpha+\beta\leq 1$ (resp. $\alpha+\beta\leq 2$). Thus, $R_1(a_{\ell-2,q})$ (resp. $R_2(a_{\ell-4,q})$) is in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$ (resp. $\Sigma^{-\frac{\ell}{2}-\delta(\ell-5)}$). It follows from (2.2.31) and (2.2.32) that

$$\begin{aligned}
&(\partial_t^2-\partial_x^2+1)u^2_{\text{app}}\\
&\quad=2\operatorname{Re}\Big[2i\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N+2}\varepsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}\omega_0(y)\partial_s a_{\ell-2,1}(s,y,\varepsilon)\\
&\qquad+\sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N}\sum_{\substack{3\leq q\leq \ell\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}}(1-q^2)e^{itq\varphi(y)}a_{\ell,q}(s,y,\varepsilon)\\
&\qquad+\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+4}\varepsilon^{6-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}b_{\ell,1}(s,y,\varepsilon)\\
&\qquad+\sum_{\substack{\ell=5\\ \ell \text{ odd}}}^{N+4}\sum_{\substack{3\leq q\leq \ell-2\\ q \text{ odd}}} \varepsilon^{2q-\ell} t^{-\frac{\ell}{2}}e^{itq\varphi(y)}b_{\ell,q}(s,y,\varepsilon)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}},
\end{aligned} \tag{2.2.33}$$

where for $5\leq \ell \leq N+4$,

$$\begin{aligned}
&b_{\ell,1}\in \Sigma^{-\frac{\ell}{2}-\delta(\ell-5)} \text{ and depends only on } a_{\ell-4,1},\\
&b_{\ell,q}\in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)} \text{ and depends only on } a_{\ell',q},\ \ell'\leq \min(\ell-2,N) \text{ when } q\geq 3.
\end{aligned} \tag{2.2.34}$$

- *Nonlinear term in* (2.2.3). This term is given by formula (2.2.24).

• *Determination of the* $a_{\ell,q}$. To prove Proposition 2.2.2, we have to choose the $a_{\ell,q}$ recursively in order to eliminate most terms in the difference (2.2.2) between (2.2.33) and (2.2.24), to be left only with terms of the form (2.2.4) or (2.2.5). We determine first the characteristic coefficient $a_{1,1}$. Equating the $t^{-\frac{3}{2}}e^{it\varphi(y)}$ terms in (2.2.24) and (2.2.33), and using expression (2.2.11) for $c_{3,1}$, we obtain

$$\omega_0(y)\partial_s a_{1,1} = \frac{1}{2}(\phi(y) + i\psi(y))|a_{1,1}(s,y)|^2 a_{1,1}(s,y).$$

If we take for $a_{1,1}$ the function (2.1.10), this equality is satisfied by (2.1.8), and the explicit formula (2.1.10) shows that $a_{1,1}$ belongs to $\Sigma^{-\frac{1}{2}}$.

Next we determine the coefficient $a_{3,3}$, equating the $t^{-\frac{3}{2}}e^{3it\varphi(y)}$ coefficients in (2.2.24) and (2.2.33). We get $-8a_{3,3} = c_{3,3}$, where $c_{3,3}$ is determined by $a_{1,1}$ according to Corollary 2.2.4 and belongs to $\Sigma^{-\frac{3}{2}} \subset \Sigma^{-\frac{3}{2}-2\delta}$.

Assume by induction that we have determined for some $\ell \geq 5$,

$$a_{\ell',q'}, 1 \leq q' \leq \ell' \leq \ell - 4 \quad \text{and} \quad a_{\ell-2,q'}, 3 \leq q' \leq \ell - 2. \tag{2.2.35}$$

Let us determine $a_{\ell-2,1}$. Equating the $t^{-\frac{\ell}{2}}e^{it\varphi(y)}$ terms in (2.2.24) and (2.2.33), we get, also using expressions (2.2.12) to (2.2.14),

$$\partial_s a_{\ell-2,1}(s,y) = \Theta(y)[2|a_{1,1}|^2 a_{\ell-2,1} + a_{1,1}^2 \bar{a}_{\ell-2,1}](s,y) + r_{\ell-2,1}(s,y), \tag{2.2.36}$$

where $\Theta(y) = \frac{1}{2}\omega_0(y)^{-1}(\phi(y) + i\psi(y))$ and

$$r_{\ell-2}(s,y) = -\frac{i}{2\omega_0(y)}(c''_{\ell,1}(s,y) - b_{\ell,1}(s,y)).$$

By Corollary 2.2.4, and decompositions (2.2.12)–(2.2.14), $c''_{\ell,1}$ is in the space $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$ and depends only on $a_{\ell',q'}$ for $1 \leq q' \leq \ell' \leq \ell - 4$ and on $a_{\ell-2,q'}$ for $3 \leq q' \leq \ell - 2$. These coefficients are determined by assumption (2.2.35). Moreover, by (2.2.34), $b_{\ell,1}$ belongs to $\Sigma^{-\frac{\ell}{2}-\delta(\ell-5)}$ and depends only on coefficients already determined. It follows that $r_{\ell-2,1}$ is known and belongs to $\Sigma^{-\frac{\ell}{2}-\delta(\ell-3)}$. If we supplement (2.2.36) by the initial condition $a_{\ell-2,1}(0,y) = 0$, we may thus apply Lemma 2.2.5 with $a \equiv a_{1,1}$ to conclude that $a_{\ell-2,1}$ belongs to $\Sigma^{-\frac{\ell-2}{2}-\delta(\ell-3)}$ as wanted in the statement of the proposition, if we check that assumptions (2.2.25), (2.2.26) hold. The first one, which is equivalent to $\frac{1}{2}\omega_0(y_0)^{-1}\phi(y_0) \neq 0$, follows from conditions (1.2.2), (1.2.3). The second one is implied by the explicit expression (2.1.10) of $a_{1,1}$ and the fact that by (2.1.6), (1.2.1), and (1.2.2), $a_1^0(y_0)$ does not vanish.

We have thus determined $a_{\ell',q'}$ for $1 \leq q' \leq \ell' \leq \ell - 2$. To obtain (2.2.35) with ℓ replaced by $\ell + 2$, we are left with finding $a_{\ell,q}$ for $3 \leq q \leq \ell$. Equating terms in $t^{-\frac{\ell}{2}}e^{itq\varphi(y)}$ in (2.2.24) and (2.2.33), we obtain an equation

$$(1 - q^2)a_{\ell,q} = c_{\ell,q} - b_{\ell,q} \in \Sigma^{-\frac{\ell}{2}-\delta(\ell-3)} \subset \Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}, \tag{2.2.37}$$

where $c_{\ell,q}, b_{\ell,q}$ depend only on $a_{\ell',q'}$ with $1 \leq q' \leq \ell - 2$ by (2.2.34) and Corollary 2.2.4. Consequently, the right-hand side of (2.2.37) is already determined, so that we have defined $a_{\ell-2,q}$ for $3 \leq q \leq \ell$. We have finally recovered (2.2.35) at rank $\ell + 2$.

Consequently, we have eliminated all characteristic terms in (2.2.33) that are $O(t^{-\frac{\ell}{2}})$ for $\ell \leq N+2$ and all noncharacteristic terms that are $O(t^{-\frac{\ell}{2}})$ for $\ell \leq N$. We are thus left with the terms in the third (resp. fourth) sum in (2.2.33) corresponding to $\ell = N+4$ (resp. $\ell = N+2$ or $N+4$) and with the terms in the first (resp. second) sum in (2.2.24) corresponding to $N+4 \leq \ell \leq 3N$ (resp. $N+2 \leq \ell \leq 3N$). These terms contribute to (2.2.4) and (2.2.5). This concludes the proof. ∎

We construct now an approximate solution to equation (1.1.1) defined for $t \in [1, e^{\frac{S_*}{\varepsilon^2}}[$, gluing together the approximate solution for moderate times $u^{\mathrm{M}}_{\mathrm{app}}$ that was defined in Proposition 2.1.6 and the approximate solution u^2_{app} of Proposition 2.2.2.

Corollary 2.2.6. *Let $\tilde{\chi}_0$ be in $C_0^\infty([0, \frac{3S_*}{4}[)$ and be equal to 1 on $[0, \frac{S_*}{2}]$. Define for $t \in [1, e^{\frac{S_*}{\varepsilon^2}}[$,*

$$u_{\mathrm{app}}(t,x) = \tilde{\chi}_0(\varepsilon^2 \log t) u^{\mathrm{M}}_{\mathrm{app}}(t,x) + (1 - \tilde{\chi}_0)(\varepsilon^2 \log t) u^2_{\mathrm{app}}(t,x). \tag{2.2.38}$$

Then

$$r_{\mathrm{app}}(t,x) = (\partial_t^2 - \partial_x^2 + 1) u_{\mathrm{app}} - P(u_{\mathrm{app}}, \partial_t u_{\mathrm{app}}, \partial_x u_{\mathrm{app}}) \tag{2.2.39}$$

may be written as the sum

$$\begin{aligned} &2\,\mathrm{Re}\Big[\frac{\varepsilon}{t^{\frac{5}{2}}} e^{it\varphi(y)} \chi_1(\varepsilon^{1-\theta} t) c_{5,1}\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2 \log t\\ y = x/t}} \\ &\quad + (1 - \tilde{\chi}_0)(\varepsilon^2 \log t) r^2_{\mathrm{app}}(t,x) + F_{\mathrm{app}}(t,x), \end{aligned} \tag{2.2.40}$$

where χ_1 is smooth, equal to 0 close to 0 and to 1 outside a neighborhood of 0, where $c_{5,1}(s, y, h, \varepsilon)$ is a continuous function on $[0, +\infty[\times \mathbb{R} \times]0,1] \times [0,1]$, supported for $s \leq \frac{3S_*}{4}$ and $|y| \leq 1$, bounded as well as all its ∂_s, ∂_y, $h\partial_h$ derivatives on that domain, where r^2_{app} given by (2.2.3) is the sum of (2.2.4) and (2.2.5) and where F_{app} is compactly supported for $t \leq e^{3S_*/4\varepsilon^2}$ and satisfies

$$\begin{aligned} \int_1^{\exp(S_*/\varepsilon^2)} \|F_{\mathrm{app}}(\tau,\cdot)\|_{H^s}\, d\tau &\leq C\varepsilon^{2-\theta}, \\ \int_1^{\exp(S_*/\varepsilon^2)} \|L_\pm F_{\mathrm{app}}(\tau,\cdot)\|_{H^1}\, d\tau &\leq C\varepsilon^{2-\theta}. \end{aligned} \tag{2.2.41}$$

Proof. By the definition of u_{app} and (2.1.30), (2.2.3), we may write

$$\begin{aligned} r_{\mathrm{app}}(t,x) &= \tilde{\chi}_0(\varepsilon^2 \log t) r^{\mathrm{M}}_{\mathrm{app}}(t,x) + (1 - \tilde{\chi}_0)(\varepsilon^2 \log t) r^2_{\mathrm{app}}(t,x) \\ &\quad + 2\frac{\varepsilon^2}{t} \tilde{\chi}_0'(\varepsilon^2 \log t) \partial_t (u^{\mathrm{M}}_{\mathrm{app}} - u^2_{\mathrm{app}})(t,x) \end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{\varepsilon^4}{t^2}\tilde{\chi}_0''(\varepsilon^2\log t)-\frac{\varepsilon^2}{t^2}\tilde{\chi}_0'(\varepsilon^2\log t)\Big)(u_{\mathrm{app}}^{\mathrm{M}}-u_{\mathrm{app}}^2)(t,x)\\
&-P(u_{\mathrm{app}},\partial_t u_{\mathrm{app}},\partial_x u_{\mathrm{app}})+\tilde{\chi}_0(\varepsilon^2\log t)P(u_{\mathrm{app}}^{\mathrm{M}},\partial_t u_{\mathrm{app}}^{\mathrm{M}},\partial_x u_{\mathrm{app}}^{\mathrm{M}})\\
&+(1-\tilde{\chi}_0)(\varepsilon^2\log t)P(u_{\mathrm{app}}^2,\partial_t u_{\mathrm{app}}^2,\partial_x u_{\mathrm{app}}^2)\\
={}&\mathrm{I}+\cdots+\mathrm{VII}.
\end{aligned}
$$

We examine terms I to VII successively.

• *Contribution of term* I. By Proposition 2.1.6, we get contributions to the first term in (2.2.40) and to F_{app}.

• *Contribution of term* II. This is the second term in (2.2.40).

• *Contributions of terms* III + IV. On the support of $\tilde{\chi}_0'(\varepsilon^2\log t)$, $u_{\mathrm{app}}^{\mathrm{M}}$ coincides with u_{app}^1 by (2.1.29), so that we have to estimate $u_{\mathrm{app}}^1-u_{\mathrm{app}}^2$ and its time derivative. This difference may be computed from (2.1.11) and (2.2.2). The $t^{-\frac{1}{2}}$ terms cancel out. We are thus reduced to the following terms:

– Characteristic terms in $O(t^{-\frac{3}{2}})$ coming from the $t^{-\frac{3}{2}}a_{3,1}$ term in (2.2.2): This provides a contribution to the first term in (2.2.40).

– Noncharacteristic terms in $O(t^{-\frac{3}{2}})$ coming from the $a_{3,3}$ term in (2.2.2) and the $a_{3,3}^1$ term in (2.1.11): When plugged into III + IV, these terms give contributions

$$
\mathrm{Re}\Big[\frac{\varepsilon^5}{t^{\frac{5}{2}}}\underline{\chi}_0(s)e^{3it\varphi(y)}\tilde{a}_{3,3}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \tag{2.2.42}
$$

where $\underline{\chi}_0\in C_0^\infty(]0,+\infty[)$ and where $\tilde{a}_{3,3}(s,y,h,\varepsilon)$ is a continuous function on the product $[0,+\infty[\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $s\le\frac{3S_*}{4}$ and $|y|\le 1$, bounded as well as all its ∂_s, ∂_y, $h\partial_h$ derivatives on that domain. The Sobolev norm of (2.2.42) is $O(\varepsilon^5t^{-2})$, so that the first estimate (2.2.41) largely holds. If we make $L_\pm$ act on (2.2.42) and bound the H^1-norm, we get an $O(\varepsilon^5t^{-1})$ estimate. Integrating for $1\le t\le e^{3S_*/4\varepsilon^2}$ gives an $O(\varepsilon^3)$ bound, better than the right-hand side of the second inequality (2.2.41). Thus (2.2.42) may be included in F_{app} in (2.2.40).

– Characteristic or noncharacteristic terms coming from (2.1.11) or (2.2.2) that are $O(t^{-\frac{5}{2}})$, i.e. terms in $a_{5,3}^1$, $a_{5,5}^1$ in (2.1.11) and $a_{\ell,q}$, $\ell\ge 5$ in (2.2.2): The contributions of all such terms to III + IV may be written in the form

$$
\mathrm{Re}\Big[\frac{\varepsilon^{6-\ell}}{t^{\frac{\ell}{2}}}\underline{\chi}_0(s)e^{iqt\varphi(y)}\tilde{a}_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \tag{2.2.43}
$$

with $\ell\ge 7$ and $\tilde{a}_{\ell,q}$ satisfying the same estimates as $\tilde{a}_{3,3}$ above. Then the Sobolev norm of (2.2.43) or its H^1-norm after action of $L_\pm$, integrated for t in the support of $\underline{\chi}_0(\varepsilon^2\log t)$ is $O(e^{-c/\varepsilon^2})$, so that (2.2.41) is largely verified and these terms may be included inside F_{app}.

- *Contributions of* V + VI + VII. We write this contribution as the sum of

$$P(u_{\text{app}}, \tilde{\chi}_0 \partial_t u^{\text{M}}_{\text{app}} + (1-\tilde{\chi}_0)\partial_t u^2_{\text{app}}, \partial_x u_{\text{app}}) \\ - P\big(u_{\text{app}}, \partial_t(\tilde{\chi}_0 u^{\text{M}}_{\text{app}} + (1-\tilde{\chi}_0)u^2_{\text{app}}), \partial_x u_{\text{app}}\big) \tag{2.2.44}$$

and of

$$\tilde{\chi}_0 P(u^{\text{M}}_{\text{app}}, \partial_t u^{\text{M}}_{\text{app}}, \partial_x u^{\text{M}}_{\text{app}}) + (1-\tilde{\chi}_0) P(u^2_{\text{app}}, \partial_t u^2_{\text{app}}, \partial_x u^2_{\text{app}}) \\ - P(\tilde{\chi}_0 u^{\text{M}}_{\text{app}} + (1-\tilde{\chi}_0)u^2_{\text{app}}, \tilde{\chi}_0\partial_t u^{\text{M}}_{\text{app}} + (1-\tilde{\chi}_0)\partial_t u^2_{\text{app}}, \\ \tilde{\chi}_0 \partial_x u^{\text{M}}_{\text{app}} + (1-\tilde{\chi}_0)\partial_x u^2_{\text{app}}). \tag{2.2.45}$$

Consider first (2.2.44). This expression may be bounded pointwise by

$$\frac{\varepsilon^2}{t} |\tilde{\chi}_0'(\varepsilon^2 \log t)|\, |u^{\text{M}}_{\text{app}} - u^2_{\text{app}}| \Big(\sum_{\alpha+\beta\le 1} |\partial_t^\alpha \partial_x^\beta u^{\text{M}}_{\text{app}}| + |\partial_t^\alpha \partial_x^\beta u^2_{\text{app}}| \Big)^2. \tag{2.2.46}$$

We have seen in the study of III + IV that the $t^{-\frac{1}{2}}$ terms cancel out in $u^{\text{M}}_{\text{app}} - u^2_{\text{app}}$, so that this difference is $O(\varepsilon^{-a} t^{-\frac{3}{2}})$ for some a. The squared factor in (2.2.45) is moreover $O(\varepsilon^2/t)$, so that we may get for (2.2.46) a bound in $O(\varepsilon^{-a} t^{-\frac{7}{2}} \mathbb{1}_{|x|\le t})$. The same holds for derivatives of (2.2.44) so that, computing its H^s-norm or the H^1-norm of the action of $L_\pm$ on it, we shall obtain, as at the end of the study of III + IV, that the time integral of these quantities is $O(e^{-c/\varepsilon^2})$. Thus (2.2.44) largely satisfies (2.2.41).

Finally, consider (2.2.45) which may be written as $-\psi(\tilde{\chi}_0(\varepsilon^2 \log t))$ with

$$\psi(\mu) = P(\mu u^{\text{M}}_{\text{app}} + (1-\mu)u^2_{\text{app}}, \mu\partial_t u^{\text{M}}_{\text{app}} + (1-\mu)\partial_t u^2_{\text{app}}, \mu\partial_x u^{\text{M}}_{\text{app}} + (1-\mu)\partial_x u^2_{\text{app}}) \\ - \mu P(u^{\text{M}}_{\text{app}}, \partial_t u^{\text{M}}_{\text{app}}, \partial_x u^{\text{M}}_{\text{app}}) - (1-\mu) P(u^2_{\text{app}}, \partial_t u^2_{\text{app}}, \partial_x u^2_{\text{app}}).$$

As $\psi(1) = \psi(0) = 0$, we have

$$|\psi(\tilde{\chi}_0(\varepsilon^2 \log t))| \le (1-\tilde{\chi}_0)\tilde{\chi}_0 \sup_{\mu\in[0,1]} |\psi''(\mu)| \\ \le C(1-\tilde{\chi}_0)\tilde{\chi}_0 [|u^{\text{M}}_{\text{app}} - u^2_{\text{app}}| + |\partial_t(u^{\text{M}}_{\text{app}} - u^2_{\text{app}})| + |\partial_x(u^{\text{M}}_{\text{app}} - u^2_{\text{app}})|]^2 \\ \times \sum_{\alpha+\beta\le 1} (|\partial_t^\alpha \partial_x^\beta u^{\text{M}}_{\text{app}}| + |\partial_t^\alpha \partial_x^\beta u^2_{\text{app}}|). \tag{2.2.47}$$

We have seen in the study of (2.2.44) that $u^{\text{M}}_{\text{app}} - u^2_{\text{app}}$ is $O(\varepsilon^{-a} t^{-\frac{3}{2}})$, as well as its derivatives, on the support of $(1-\tilde{\chi}_0)\tilde{\chi}_0(\varepsilon^2 \log t)$. It follows that again (2.2.47) is $O(\varepsilon^{-a} t^{-\frac{7}{2}})$ and supported for $|x| \le t$. As the same bound holds for derivatives of (2.2.45), we conclude that this term satisfies (2.2.41) as well. This concludes the proof. ∎

3. Reduction to a system and normal form

In this section we shall reduce equation (1.1.1) to a first-order system. We shall then look for the solution as the sum of an approximate solution deduced from u_{app} constructed in

Section 2 and of a remainder. Finally, in Section 3.2, we shall perform a normal form procedure in order to eliminate part of the cubic nonlinearity.

3.1. Reduction to a system

Let us introduce some notation that will be used in the rest of the paper. We shall denote by $M_0(\xi_1,\dots,\xi_n)$ a smooth positive function on $\mathbb{R}^n$, valued in $\mathbb{R}_+^*$, such that $M_0(\xi_1,\dots,\xi_n)$ is equivalent to $1+\max_2(|\xi_1|,\dots,|\xi_n|)$, where $\max_2$ stands for the second largest among $|\xi_1|,\dots,|\xi_n|$. For instance, we may take

$$M_0(\xi_1,\dots,\xi_n) = \Bigg(\sum_{\substack{\alpha=(\alpha_1,\dots,\alpha_n)\\ |\alpha|=n\\ \max(\alpha_j)\le n-1}} (\xi^\alpha)^2+1\Bigg)^{\frac{1}{2}}\Bigg(1+\sum_{j=1}^n \xi_j^2\Bigg)^{-\frac{n-1}{2}}. \tag{3.1.1}$$

Definition 3.1.1. Let $n\in\mathbb{N}^*$, $\nu\in\mathbb{R}$, $\kappa\in\mathbb{R}_+$, $\beta\in\mathbb{R}_+$. We denote by $S_{\kappa,\beta}(M_0^\nu,n)$ the space of smooth functions on $\mathbb{R}\times\mathbb{R}^n$, $(x,\xi_1,\dots,\xi_n)\to m(x,\xi_1,\dots,\xi_n)$, with values in $\mathbb{C}$, satisfying for any $\alpha_0\in\mathbb{N}$, $\alpha\in\mathbb{N}^*$, $N\in\mathbb{N}$ estimates

$$|\partial_x^{\alpha_0}\partial_\xi^\alpha m(x,\xi_1,\dots,\xi_n)| \le C_{\alpha_0,\alpha,N} M_0(\xi)^{\nu+\kappa(\alpha_0+|\alpha|)}(1+\beta h^\beta|\xi|)^{-N}. \tag{3.1.2}$$

Remark. Most of the time we shall only need the special case $\beta=0$, so that the last factor on the right-hand side of (3.1.2) disappears. If m is in $S_{\kappa,0}(M_0^\nu,n)$ and $\chi\in C_0^\infty(\mathbb{R}^n)$, then $m(x,\xi)\chi(h^\beta\xi)$ is in $S_{\kappa,\beta}(M_0^\nu,n)$ for $\beta>0$.

If m is in $S_{\kappa,\beta}(M_0^\nu,n)$ and if $u_1,\dots,u_n$ are in $\mathcal{S}(\mathbb{R}^n)$, we set

$$\begin{aligned}\mathrm{Op}(m)(u_1,\dots,u_n) = \frac{1}{(2\pi)^n}\int & e^{ix(\xi_1+\dots+\xi_n)} m(x,\xi_1,\dots,\xi_n)\\ &\times\prod_{j=1}^n \hat{u}_j(\xi_j)\,d\xi_1\cdots d\xi_n.\end{aligned} \tag{3.1.3}$$

In Appendix A.2 we observe that (3.1.3) remains meaningful when u_j belongs to Sobolev spaces of high enough order, so that we may use (3.1.3) for the solution to our problem.

Let $u\to u(t,x)$ be defined on $[1,T[\times\mathbb{R}$ for some $T\in]1,e^{S_*/\varepsilon^2}[$ with values in $\mathbb{R}$, which is in $C^0([1,T[,H^s(\mathbb{R}))\cap C^1([1,T[,H^{s-1}(\mathbb{R}))$ for some large enough s, solving equation (1.1.1). We define, with the notation $p(D_x)=\sqrt{1+D_x^2}$,

$$u_\pm = (D_t \pm p(D_x))u \tag{3.1.4}$$

so that

$$u_- = -\bar{u}_+,\quad u=\frac{1}{2}p(D_x)^{-1}(u_+-u_-),\quad \partial_t u = \frac{i}{2}(u_++u_-). \tag{3.1.5}$$

If $I=(i_1,i_2,i_3)$ is an element of $\{-,+\}^3$, we set $u_I=(u_{i_1},u_{i_2},u_{i_3})$. If we express u and its derivatives from (3.1.5) in (1.1.2), we may write

$$P(u,\partial_t u,\partial_x u) = -\sum_{I\in\{-,+\}^3}\mathrm{Op}(m_I)(u_I) \tag{3.1.6}$$

for some m_I in $S_{0,0}(1,3)$ (with constant coefficients). Consequently, equation (1.1.1) is equivalent to

$$(D_t - p(D_x))u_+ = \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I)(u_I). \tag{3.1.7}$$

Of course, by conjugation, using (3.1.5), we have

$$(D_t + p(D_x))u_- = \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I^-)(u_{\bar I}), \tag{3.1.8}$$

where $\bar I = -I$ and

$$m_I^-(x,\xi_1,\dots,\xi_n) = (-1)^n\overline{m(x,-\xi_1,\dots,-\xi_n)}.$$

Let us construct an approximate solution u_+^{app} of equation (3.1.7) from the approximate solution u_{app} of Corollary 2.2.6. We shall do that when the time t stays smaller than the time $T(\varepsilon)$ defined in (1.2.5). We shall use the following inequality, with $\delta > 0$ introduced in Proposition 2.2.2 and $\delta' > 0$, $\gamma > 0$ to be chosen:

$$\begin{gathered}\text{there is } \varepsilon_0 \in]0,1] \text{ such that if } 0<\varepsilon<\varepsilon_0 \text{ and } t\in[e^{S_*/2\varepsilon^2}, T(\varepsilon)[,\\ \text{then } t^{-\frac12}(S_* - \varepsilon^2\log t)^{-\frac12-\delta} < \varepsilon^{\frac{\gamma}{2}-\delta'}(S_*-\varepsilon^2\log t)^{\delta'-\delta}.\end{gathered} \tag{3.1.9}$$

Actually, this inequality is trivial if $S_* - \varepsilon^2\log t \geq \varepsilon^2$ since, as $t \geq e^{S_*/2\varepsilon^2}$, the factor $t^{-\frac12}$ in the left-hand side is then exponentially decaying, so that (3.1.9) holds for small enough ε. If $u = \frac{S_*-\varepsilon^2\log t}{\varepsilon^2} \leq 1$, then inequality (3.1.9) is equivalent to

$$ue^{-\frac{u}{1+2\delta'}} > \varepsilon^{-\frac{2+\gamma+2\delta'}{1+2\delta'}} e^{-\frac{S_*}{\varepsilon^2(1+2\delta')}}, \tag{3.1.10}$$

whose right-hand side is the quantity ε' introduced in (1.2.4). Since $u \to ue^{-\frac{u}{1+2\delta'}}$ is strictly increasing on $[0,1]$ if $\delta' > 0$, inequality (3.1.10) is equivalent to $u > u(\varepsilon')$ where $u(\varepsilon')$ was defined before (1.2.5). But by the definitions of u and of $T(\varepsilon)$ in (1.2.5), this means $t < T(\varepsilon)$. In the sequel, the parameters δ, δ', γ will be chosen positive, with δ and δ' small, satisfying the inequalities

$$\delta' > \delta, \quad \gamma \geq 2(\delta' + 2). \tag{3.1.11}$$

We notice for further reference that (3.1.9) implies that $t^{-1}(S_* - \varepsilon^2\log t)^{-1} = O(1)$, so that, when $t \in [1, T(\varepsilon)[$, definition (2.2.1) of classes Σ^m shows that

$$a \in \Sigma^m \Rightarrow \partial_t^\alpha\partial_x^\beta\Big[a\Big(\varepsilon^2\log t, \frac{x}{t}, \frac{1}{t}, \varepsilon\Big)\Big] = b\Big(\varepsilon^2\log t, \frac{x}{t}, \frac{1}{t}, \varepsilon\Big) \tag{3.1.12}$$

for some b in Σ^m. We define, from the approximate solution u_{app} of Corollary 2.2.6,

$$\tilde u_+^{\mathrm{app}} = (D_t + p(D_x))u_{\mathrm{app}}, \quad \tilde u_-^{\mathrm{app}} = -\overline{\tilde u_+^{\mathrm{app}}} \tag{3.1.13}$$

and $\tilde{u}_I^{\mathrm{app}} = (\tilde{u}_{i_1}^{\mathrm{app}}, \tilde{u}_{i_2}^{\mathrm{app}}, \tilde{u}_{i_3}^{\mathrm{app}})$ if $I = (i_1, i_2, i_3)$. Then, by (2.2.38), (2.2.39), (2.2.40), and (3.1.6),

$$\begin{aligned}(D_t - p(D_x))\tilde{u}_+^{\mathrm{app}} &- \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I)(\tilde{u}_I^{\mathrm{app}})\\ &= -\Big(\frac{\varepsilon}{t^{\frac{5}{2}}} e^{it\varphi(y)}\chi_1(\varepsilon^{1-\theta}t)c_{5,1}\Big(s,y,\frac{1}{t},\varepsilon\Big)\\ &\quad + \frac{\varepsilon}{t^{\frac{5}{2}}} e^{-it\varphi(y)}\chi_1(\varepsilon^{1-\theta}t)c_{5,-1}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\ &\quad - F_{\mathrm{app}}(t,x) - (1-\tilde{\chi}_0)(\varepsilon^2\log t)r^2_{\mathrm{app}}(t,x),\end{aligned} \tag{3.1.14}$$

where $c_{5,-1} = \bar{c}_{5,1}$ is supported for $s \le \frac{3S_*}{4}$, $|y| \le 1$, and $F_{\mathrm{app}}(t,x)$ for $t \le e^{3S_*/4\varepsilon^2}$, and satisfies (2.2.41). On the right-hand side of (3.1.14), we have an $e^{it\varphi(x/t)}$ term that is characteristic for $D_t - p(D_x)$ and an $e^{-it\varphi(x/t)}$ term that is noncharacteristic for the same operator. We start by eliminating the noncharacteristic term, introducing a modification u_+^{app} of $\tilde{u}_+^{\mathrm{app}}$. We first define this function and study its structure.

Lemma 3.1.2. *Define*

$$u_+^{\mathrm{app}}(t,x) = \tilde{u}_+^{\mathrm{app}}(t,x) - \frac{\varepsilon}{2t^{\frac{5}{2}}} e^{-it\varphi(y)}\chi_1(\varepsilon^{1-\theta}t)\sqrt{1-y^2}\,c_{5,-1}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{3.1.15}$$

Then we may write, for $1 \le t \le T(\varepsilon)$,

$$\begin{aligned}u_+^{\mathrm{app}}(t,x) &= \chi_0(\varepsilon^{1-\theta}(t-1))u_{0,+}(t,x)\\ &\quad + e^{it\varphi(y)}(1-\chi_0)(\varepsilon^{1-\theta}(t-1))\frac{\varepsilon}{\sqrt{t}}a^+_{1,1}(s,y)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\ &\quad + \sum_{\substack{\ell=3\\ \ell\ \mathrm{odd}}}^{N+1}\sum_{\substack{1\le|q|\le\ell\\ q\ \mathrm{odd}}} e^{itq\varphi(y)}t^{-\frac{\ell}{2}}a^+_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)e_{\ell,q}(t,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}\\ &\quad + \varepsilon r(t,x),\end{aligned} \tag{3.1.16}$$

where χ_0 *was introduced in Proposition* 2.1.6, *where we denoted*

$$u_{0,+}(t,x) = (D_t + p(D_x))u_0, \tag{3.1.17}$$

where $a^+_{1,1}(s,y) = 2(1-y^2)^{-\frac{1}{2}}a_{1,1}(s,y)$, $a_{1,1}$ *being defined in* (2.1.10), *where the coefficients* $a^+_{\ell,q}(s,y,h,\varepsilon)$ *are elements of* $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ *and where* $e_{\ell,q}(t,\varepsilon)$ *satisfy, for any* ζ,

$$\begin{aligned}(t\partial_t)^\zeta e_{\ell,q}(t,\varepsilon) &= O(\varepsilon) && \text{if } t \le e^{3S_*/4\varepsilon^2},\\ (t\partial_t)^\zeta e_{\ell,q}(t,\varepsilon) &= O(\varepsilon^{2|q|-\ell}) && \text{if } t \ge e^{S_*/2\varepsilon^2},\ q \ne -1,\\ (t\partial_t)^\zeta e_{\ell,-1}(t,\varepsilon) &= O(\varepsilon^{4-\ell}) && \text{if } t \ge e^{S_*/2\varepsilon^2},\end{aligned} \tag{3.1.18}$$

and where $r(t,x)$ is a smooth function satisfying, for all α, β, N,

$$|\partial_t^\alpha \partial_x^\beta r(t,x)| \le C(t+|x|)^{-N}. \tag{3.1.19}$$

Moreover, we may write $u_{0,+}(t,x)$ in the form

$$u_{0,+}(t,x) = \frac{\varepsilon}{\sqrt{t}} e^{it\varphi(x/t)} a_1^+\Big(\frac{x}{t},\frac{1}{t}\Big) + \frac{\varepsilon}{t^{\frac{3}{2}}} e^{-it\varphi(x/t)} a_1^-\Big(\frac{x}{t},\frac{1}{t}\Big) + \varepsilon r_0(t,x), \tag{3.1.20}$$

where $a_1^\pm(y,h)$ are continuous on $\mathbb{R}\times]0,1]$, bounded as well as their ∂_y and $h\partial_h$-derivatives on that domain, supported for $|y|\le 1$, and where r_0 satisfies (3.1.19).

Proof. Consider first an element $a_{\ell,q}$ of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$, with $1\le |q|\le \ell$. We may apply Corollary A.1.4 of the appendix to compute $p(D_x)[e^{iqt\varphi(x/t)}a_{\ell,q}(\varepsilon^2\log t, \frac{x}{t}, \frac{1}{t}, \varepsilon)]$ since the assumption $t(S_*-s)^{\frac{1}{2\kappa_0}} \ge c$ of the appendix is satisfied: this is trivial for $s\le \frac{S_*}{2}$ and holds for $s=\varepsilon^2\log t \ge \frac{S_*}{2}$ and $t<T(\varepsilon)$ by (3.1.9) (for ε small enough). By this corollary and estimates (2.2.1), we have

$$\begin{aligned}
(D_t + p(D_x))\Big[e^{iqt\varphi(y)} a_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \\
= \Bigg(\frac{q+\sqrt{1+(q^2-1)y^2}}{\sqrt{1-y^2}}\Bigg) e^{itq\varphi(y)} a_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \\
+ \frac{1}{t} e^{itq\varphi(y)} a^1_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} + r(t,x),
\end{aligned} \tag{3.1.21}$$

where $a^1_{\ell,q}$ is in $\Sigma^{-\frac{\ell}{2}-1-\delta(\ell-1)}$ and r satisfies (3.1.19).

First we compute $\tilde{u}_+^{\rm app}$ in (3.1.15) from its definition (3.1.13) making $(D_t+p(D_x))$ act on definition (2.2.38) of $u_{\rm app}$. Using expression (2.1.29) of $u^{\rm M}_{\rm app}$, we get

$$\begin{aligned}
\tilde{u}_+^{\rm app} &= \chi_0(\varepsilon^{1-\theta}(t-1))(D_t+p(D_x))u_0 \\
&\quad + (1-\chi_0)(\varepsilon^{1-\theta}(t-1))\tilde{\chi}_0(\varepsilon^2\log t)(D_t+p(D_x))u^1_{\rm app} \\
&\quad + (1-\tilde{\chi}_0)(\varepsilon^2\log t)(D_t+p(D_x))u^2_{\rm app} \\
&\quad - i\varepsilon^{1-\theta}\chi_0'(\varepsilon^{1-\theta}(t-1))u_0(t,x) \\
&\quad + i\big(\varepsilon^{1-\theta}\chi_0'(\varepsilon^{1-\theta}(t-1)) - \varepsilon^2 t^{-1}\tilde{\chi}_0'(\varepsilon^2\log t)\big)u^1_{\rm app}(t,x) \\
&\quad + i\varepsilon^2 t^{-1}\tilde{\chi}_0'(\varepsilon^2\log t)u^2_{\rm app}(t,x) \\
&= \mathrm{I} + \cdots + \mathrm{VI}.
\end{aligned} \tag{3.1.22}$$

We study terms I to VI above successively, in order to obtain expressions (3.1.16) from (3.1.15).

- *Term* I. This provides the first term on the right-hand side of (3.1.16) by (3.1.17).

• *Term* II. Recall that u^1_{app} is given by (2.1.11). We may apply (3.1.21) to all terms in that sum. Since each of these terms is supported for $s = \varepsilon^2 \log t \leq \frac{3S_*}{4}$ and since the first term on the right-hand side of (3.1.21) vanishes if $q = -1$, we shall get

$$\begin{aligned} \mathrm{II} &= (1-\chi_0)(\varepsilon^{1-\theta}(t-1))\tilde{\chi}_0(s)\frac{\varepsilon}{\sqrt{t}}e^{it\varphi(y)}a^+_{1,1}(s,y,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \\ &+ \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{5} \sum_{\substack{1\leq|q|\leq\ell\\ q \text{ odd}}} (1-\chi_0)(\varepsilon^{1-\theta}(t-1))\frac{\varepsilon}{t^{\frac{\ell}{2}}}e^{itq\varphi(y)}a^{+,1}_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \\ &+ \varepsilon r(t,x), \end{aligned} \tag{3.1.23}$$

with $a^{+,1}_{1,1}(s,y)$ defined in the statement of the lemma, and where the $a^{+,1}_{\ell,q}$, $\ell = 3,5$ are elements of $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ and are supported for $0 \leq s \leq \frac{3S_*}{4}$ and $|y| \leq 1$. Thus, (3.1.23) provides a contribution to the last sum in (3.1.16) and to εr. Notice that the last term in (3.1.15) may also be written as a contribution to the sum in (3.1.16) with $\ell = 5$, $q = -1$.

• *Term* III. We make $D_t + p(D_x)$ act on the sums (2.2.2). Consider first the contributions coming from the second sum,

$$(D_t + p(D_x))[\varepsilon^{2|q|-\ell}t^{-\frac{\ell}{2}}e^{itq\varphi(y)}a_{\ell,q}(s,y,\varepsilon)]|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{3.1.24}$$

According to (3.1.21), we get a first term which is of the form of the (ℓ,q) term in the sum (3.1.16) with $3 \leq |q| \leq \ell$. The second term on the right-hand side of (3.1.21) also brings to (3.1.24) a contribution in the form of the (ℓ,q) term in (3.1.16): actually, we may write it as

$$t^{-\frac{\ell}{2}}e^{itq\varphi(y)}(S_*-s)a^1_{\ell,q}\Big(s,y,\frac{1}{t},\varepsilon\Big)e_{\ell,q}(t,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}},$$

with $e_{\ell,q}(t,\varepsilon) = \varepsilon^{2|q|-\ell}t^{-1}(S_*-\varepsilon^2\log t)^{-1}\underline{\chi}_1(\varepsilon^2\log t)$ for some function $\underline{\chi}_1$ supported for $s \geq \frac{S_*}{2}$. Then property (3.1.9) shows that for $t < T(\varepsilon)$, $e_{\ell,q}$ satisfies the second inequality (3.1.18) when $3 \leq |q| \leq \ell$.

We consider next the contributions coming from the first sum in (2.2.2). We have to study

$$(D_t + p(D_x))[\varepsilon^{2-\ell}t^{-\frac{\ell}{2}}e^{it\varphi(y)}a_{\ell,1}(s,y,\varepsilon)]|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \tag{3.1.25}$$

$$(D_t + p(D_x))[\varepsilon^{2-\ell}t^{-\frac{\ell}{2}}e^{-it\varphi(y)}a_{\ell,-1}(s,y,\varepsilon)]|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}, \tag{3.1.26}$$

with $a_{\ell,-1} = \overline{a_{\ell,1}}$. We apply (3.1.21) to (3.1.25). We get a first term that may be written as

$$\varepsilon^{2-\ell}t^{-\frac{\ell}{2}}2(1-y^2)^{-\frac{1}{2}}e^{it\varphi(y)}a_{\ell,1}(s,y,\varepsilon)|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{3.1.27}$$

For $\ell = 1$, this brings the second term on the right-hand side of (3.1.16), when we combine it with the first term on the right-hand side of II in (3.1.23), since we defined $a^+_{1,1}(s, y, \varepsilon) = 2(1 - y^2)^{-\frac{1}{2}} a_{1,1}(s, y, \varepsilon)$. Terms (3.1.27) with $\ell \geq 3$ contribute to the last sum in (3.1.16) with $q = 1$. On the other hand, the second term on the right-hand side of (3.1.21) applied to (3.1.25) is of the form

$$\varepsilon^{2-\ell} t^{-\frac{\ell+2}{2}} e^{it\varphi(y)} a^1_{\ell,1}\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big|_{\substack{s=\varepsilon^2 \log t\\ y=x/t}}, \tag{3.1.28}$$

with $a^1_{\ell,1}$ in $\Sigma^{-\frac{\ell+2}{2}-\delta(\ell-1)} \subset \Sigma^{-\frac{\ell+2}{2}-\delta(\ell+2-1)}$. For any $\ell \geq 1$, we may incorporate that term to the sum in (3.1.16) with coefficients $e_{\ell+2,1}$ satisfying (3.1.18) with $q = 1$. Notice also that the remainder in (3.1.21) may be incorporated in the one in (3.1.16), in spite of the negative powers of ε that may appear, since term III is supported for $t \geq e^{S_*/\varepsilon^2}$, so that the rapid decay in (3.1.19) also brings smallness in ε.

We still have to cope with (3.1.26). Because the oscillatory term is $e^{-it\varphi(x/t)}$, when we apply (3.1.21) with $q = -1$, the first term disappears, and we are left only with a term of the form (3.1.28) with $e^{it\varphi}$ replaced by $e^{-it\varphi}$. Such a term may be rewritten as

$$\varepsilon^{4-\ell} e^{-it\varphi} t^{-\frac{\ell}{2}} a^+_{\ell,-1}\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big|_{\substack{s=\varepsilon^2 \log t\\ y=x/t}}$$

for some $a^+_{\ell,-1}$ in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$ and $\ell \geq 3$, i.e. brings a contribution to the sum in (3.1.16) with $q = -1$ and a coefficient $e_{\ell,-1}$ which is $O(\varepsilon^{4-\ell})$ as in the last equality (3.1.18) and not just $O(\varepsilon^{2-\ell})$.

This concludes the treatment of term III in (3.1.22).

- *Term* IV. If we use expansion (2.1.5) of u_0 and again (3.1.21), we see that this term may be rewritten as a contribution to the $t^{-\frac{3}{2}} e^{\pm it\varphi}$ term in the sum (3.1.16), with a coefficient $e_{3,1}$ satisfying the first bound (3.1.18) and to the remainder εr.

- *Term* V. Using (2.1.11), we see in the same way that this term may be written as a contribution to the sum in (3.1.16), with coefficients $e_{\ell,q}$ satisfying the first bound (3.1.18).

- *Term* VI. We use (2.2.2), which implies that VI may be written as a contribution to the last sum in (3.1.16) with coefficients satisfying the second or third equality in (3.1.18). This concludes the proof of equality (3.1.16).

To obtain (3.1.20), we notice that we may apply Corollary A.1.4 in the special case when functions $a(s, y, h, \varepsilon)$ of that corollary are replaced by smooth functions of the sole variable y supported for $|y| \leq 1$ and use (3.1.21) again in that context. Using expansion (2.1.5) of u_0, we thus get (3.1.20). ∎

Next we shall check that the function u^{app}_+ defined in (3.1.15) will provide an approximate solution for the nonlinear equation given by the left-hand side of (3.1.14).

Proposition 3.1.3. *Let N_0 be an integer. Then if we define the approximate solution u_+^{app} by (3.1.15), (3.1.16), with N large enough relative to N_0, u_+^{app} solves an equation*

$$(D_t - p(D_x))u_+^{\mathrm{app}} - \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I)(u_I^{\mathrm{app}}) = -(F + r_{\mathrm{app}}), \tag{3.1.29}$$

where $u_I^{\mathrm{app}} = (u_{i_1}^{\mathrm{app}}, u_{i_2}^{\mathrm{app}}, u_{i_3}^{\mathrm{app}})$, and where the source term is given from a function $F(t,x)$ supported for $1 \le t \le e^{3S_*/4\varepsilon^2}$, that satisfies for any s_0 in $\mathbb{R}$,

$$\begin{aligned} \int_1^{+\infty} \|F(t,\cdot)\|_{H^{s_0}}\, dt &\le C_{s_0}\varepsilon^{2-\theta}, \\ \int_1^{+\infty} \|L_+F(t,\cdot)\|_{H^1}\, dt &\le C\varepsilon^{2-\theta}, \end{aligned} \tag{3.1.30}$$

and from a function $(t,x) \to r_{\mathrm{app}}(t,x)$ supported for $t \ge e^{S_*/2\varepsilon^2}$ that satisfies for any $t < T(\varepsilon)$, any s_0,

$$\begin{aligned} \|r_{\mathrm{app}}(t,\cdot)\|_{H^{s_0}} &\le C_{s_0} t^{-2}\varepsilon^{N_0}(S_* - \varepsilon^2\log t)^{N_0}, \\ \|L_+r_{\mathrm{app}}(t,\cdot)\|_{H^1} &\le C t^{-1}\varepsilon^{N_0}(S_* - \varepsilon^2\log t)^{N_0}. \end{aligned} \tag{3.1.31}$$

Proof. To compute the left-hand side of (3.1.29), we use definition (3.1.15) of u_+^{app}, (3.1.14), and the fact that we may apply Corollary A.1.4 with $\psi = -\varphi$ in order to compute the action of $D_t - p(D_x)$ on the last term in (3.1.15). (Notice that the assumption $t(S_* - s)^{1/2\kappa_0} \ge c$ holds on the support of that function.) By (A.1.23), the action of that operator on this last term is equal to

$$\frac{\varepsilon}{t^{\frac{5}{2}}} e^{-it\varphi(y)} \chi_1(\varepsilon^{1-\theta}t) c_{5,-1}\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t \\ y=x/t}}, \tag{3.1.32}$$

modulo a term of the same form where $t^{-\frac{5}{2}}$ is replaced by $t^{-\frac{7}{2}}$ and $c_{5,-1}$ by a function $c_{7,-1}$ satisfying the same conditions, and modulo a remainder satisfying (A.1.24) and supported for $s \le \frac{3S_*}{4}$. Since (3.1.32) compensates the second term on the right-hand side of (3.1.14), we get, more precisely,

$$\begin{aligned} (D_t - p(D_x))u_+^{\mathrm{app}} &- \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I)(\tilde{u}_I^{\mathrm{app}}) \\ = -\Big[&\frac{\varepsilon}{t^{\frac{5}{2}}} e^{it\varphi(y)} \chi_1(\varepsilon^{1-\theta}t) c_{5,1}\Big(s, y, \frac{1}{t}, \varepsilon\Big) \\ &+ \frac{\varepsilon}{t^{\frac{7}{2}}} e^{-it\varphi(y)} \chi_1(\varepsilon^{1-\theta}t) c_{7,-1}\Big(s, y, \frac{1}{t}, \varepsilon\Big) \\ &+ \chi_1(\varepsilon^{1-\theta}t)\varepsilon r\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big]\Big|_{\substack{s=\varepsilon^2\log t \\ y=x/t}} \\ &- (1 - \tilde{\chi}_0)(\varepsilon^2\log t) r_{\mathrm{app}}^2 - F_{\mathrm{app}} \\ = \mathrm{I} + \cdots + \mathrm{V}, \end{aligned} \tag{3.1.33}$$

where $c_{7,-1}(s,y,h,\varepsilon)$ is continuous on $[0,+\infty[\times\mathbb{R}\times]0,1]\times[0,1]$, supported for $s\leq\frac{3S_*}{4}$ and $|y|\leq 1$, bounded as well as all its ∂_s, ∂_y, $h\partial_h$-derivatives, where χ_1 is a new function supported inside a neighborhood of 0 (which may vary from line to line), and where r satisfies (A.1.24) and is supported for $s\leq\frac{3S_*}{4}$. If in the cubic term on the left-hand side of (3.1.33), we replace $\tilde{u}_+^{\mathrm{app}}$ by u_+^{app} using (3.1.15), we generate on the right-hand side a perturbation

$$\sum_{I\in\{-,+\}^3}(\mathrm{Op}(m_I)(u_I^{\mathrm{app}}+\Delta_I^{\mathrm{app}})-\mathrm{Op}(m_I)(u_I^{\mathrm{app}})), \tag{3.1.34}$$

where

$$\Delta_+^{\mathrm{app}}=\frac{\varepsilon}{2t^{\frac{5}{2}}}e^{-it\varphi(y)}\chi_1(\varepsilon^{1-\theta}t)\sqrt{1-y^2}c_{5,-1}\Big(s,y,\frac{1}{t},\varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}}. \tag{3.1.35}$$

It remains to show that terms I to V in (3.1.33) and (3.1.35) may be written as contributions to $F+r_{\mathrm{app}}$ on the right-hand side of (3.1.29). We start with the terms supported for $s\leq\frac{3S_*}{4}$, i.e. I to III and V in (3.1.33) and (3.1.34).

• *Term* I *in* (3.1.33). Since $c_{5,1}$ is bounded, as well as its ∂_s and ∂_y derivatives, and supported for $|y|\leq 1$, the Sobolev norm of I is $O(\varepsilon t^{-2})$, so that its integral for $t\geq\varepsilon^{-1+\theta}$ is $O(\varepsilon^{2-\theta})$ as in (3.1.30). If we make L_+ act on I and use (A.1.27) with $q=1$ (and a symbol a supported for $s\leq\frac{3S_*}{4}$), we obtain the same estimate for the H^1-norm of L_+I integrated for $t\geq\varepsilon^{-1+\theta}$, so that the second inequality (3.1.30) holds as well.

• *Term* II *in* (3.1.33). The reasoning is the same, except that we use (A.1.27) with $q=-1$, so that the first term on the right-hand side of this equality remains. We thus get an $O(|x|)=O(t)$ factor, which is compensated by the fact that $c_{7,1}$ is $O(t^{-\frac{7}{2}})$ instead of $O(t^{-\frac{5}{2}})$.

• *Term* III *in* (3.1.33). By (A.1.24), this term is rapidly decaying in t and $|\frac{x}{t}|$, so that the bounds (3.1.30) are trivial when integrating for $t\geq\varepsilon^{-1+\theta}$.

• *Term* V *in* (3.1.33). This term is F_{app} coming from (3.1.14) which by (2.2.41) satisfies (3.1.30).

• *Term* (3.1.34). Note that Δ_+^{app} in (3.1.35) is supported for $s\leq\frac{3S_*}{4}$, as the same holds for $c_{5,1}$. We have to study terms of the form

$$\begin{gathered}\mathrm{Op}(m_I)(u_{i_1}^{\mathrm{app}},u_{i_2}^{\mathrm{app}},\Delta_{i_3}^{\mathrm{app}}),\quad \mathrm{Op}(m_I)(u_{i_1}^{\mathrm{app}},\Delta_{i_2}^{\mathrm{app}},\Delta_{i_3}^{\mathrm{app}}),\\ \mathrm{Op}(m_I)(\Delta_{i_1}^{\mathrm{app}},\Delta_{i_2}^{\mathrm{app}},\Delta_{i_3}^{\mathrm{app}}),\end{gathered} \tag{3.1.36}$$

with $I=(i_1,i_2,i_3)\in\{-,+\}^3$, $\Delta_-^{\mathrm{app}}=-\overline{\Delta_+^{\mathrm{app}}}$, and m_I in $S_{0,0}(1,3)$. By inequality (A.2.2), there is $\rho_0\in\mathbb{R}^+$ such that for any $s_0\in\mathbb{N}$, the H^{s_0}-norm of any term in (3.1.36) is bounded from above by

$$\begin{aligned}C(&\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|\Delta_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})^2\|\Delta_+^{\mathrm{app}}\|_{H^{s_0}}\\ &+(\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}+\|\Delta_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|\Delta_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\|u_+^{\mathrm{app}}\|_{H^{s_0}}.\end{aligned} \tag{3.1.37}$$

Notice that for t in the support of (3.1.16), i.e. $\varepsilon^{-1+\theta} \leq t \leq e^{3S_*/4\varepsilon^2}$, we have

$$\|u_+^{\mathrm{app}}(t,\cdot)\|_{H^{s_0}} = O(\varepsilon), \qquad \|u_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}} = O\Big(\frac{\varepsilon}{\sqrt{t}}\Big), \tag{3.1.38}$$

$$\|\Delta_+^{\mathrm{app}}(t,\cdot)\|_{H^{s_0}} = O(\varepsilon t^{-2}), \quad \|\Delta_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}} = O(\varepsilon t^{-\frac{5}{2}}). \tag{3.1.39}$$

Actually, u_+^{app} is given by (3.1.16), with the $e_{\ell,q}$ bounded by the first inequality (3.1.18) by our assumption on t, and with s in (3.1.16) smaller than $\frac{3S_*}{4}$, so that the functions $a_{\ell,q}^+(s,y,h,\varepsilon)$ are uniformly bounded. Then (3.1.38) follows. On the other hand, (3.1.39) follows from (3.1.35).

If we plug these estimates into (3.1.37), we get a bound in $O(\varepsilon^3 t^{-3})$, whose time integral largely satisfies the first inequality (3.1.30). If we make L_+ act on (3.1.34) before computing the L^2-norm, we get an $O(\varepsilon^3 t^{-2})$ estimate that is still sufficient to obtain (3.1.30).

• *Term* IV *in* (3.1.33). Term IV is supported for $t \geq e^{S_*/2\varepsilon^2}$ and is expressed in terms of r^2_{app} coming from (3.1.14), i.e. from (2.2.40), and is given by (2.2.3), i.e. by the sum of (2.2.4) and (2.2.5). The general term in these sums is bounded from above, if $t < T(\varepsilon)$, by

$$\begin{aligned} &C\varepsilon^{6-\ell} t^{-\frac{\ell}{2}} (S_* - \varepsilon^2 \log t)^{-\ell(\frac{1}{2}+\delta)} \mathbb{1}_{|x|\leq t} \\ &\quad \leq C t^{-\frac{5}{2}} \varepsilon^{6-\ell} (S_* - \varepsilon^2 \log t)^{-5(\frac{1}{2}+\delta)} \\ &\qquad \times [\varepsilon^{\frac{\gamma}{2}-\delta'} (S_* - \varepsilon^2 \log t)^{\delta'-\delta}]^{\ell-5} \mathbb{1}_{|x|\leq t}, \end{aligned} \tag{3.1.40}$$

where we have used (3.1.9) and that $\ell \geq N+2 \geq 5$. As we assumed that (3.1.11) holds, if N is so large that

$$\begin{aligned} (\delta'-\delta)(N-3) &\geq N_0 + 5\Big(\frac{1}{2}+\delta\Big), \\ \Big(\frac{\gamma}{2} - \delta' - 1\Big)(N-3) &\geq N_0 - 1, \end{aligned}$$

we get a bound in $\varepsilon^{N_0} t^{-\frac{5}{2}} (S_* - \varepsilon^2 \log t)^{N_0}$.

If we take ∂_x derivatives of the general sum in (2.2.4), (2.2.5), we may use (3.1.12) to see that we still get expressions of the same type so that (3.1.40) will still hold true. This implies that for any s_0 in $\mathbb{N}$,

$$\|(1-\tilde{\chi}_0)(\varepsilon^2 \log t) r^2_{\mathrm{app}}(t,\cdot)\|_{H^{s_0}} \leq C t^{-2} \varepsilon^{N_0} (S_* - \varepsilon^2 \log t)^{N_0},$$

i.e. the first estimate (3.1.31) holds. The second one holds in the same way, since the action of L_+ makes one lose at most $O(t)$. This concludes the proof of the proposition. ∎

To finish this subsection, we introduce the equation satisfied by the difference $v_+ = u_+ - u_+^{\mathrm{app}}$ between the solution of (3.1.7) and the approximate solution u_+^{app} of Lemma 3.1.2.

Proposition 3.1.4. *The function $v_+ = u_+ - u_+^{\mathrm{app}}$ satisfies, with symbols $m_I^{(j)}$ in $S_{0,0}(1,3)$,*

$$\begin{aligned}(D_t - p(D_x))v_+ = &\sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3})\\ &+ \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\mathrm{app}})\\ &+ \sum_{I\in\{-,+\}^3} \mathrm{Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\mathrm{app}}, u_{i_3}^{\mathrm{app}})\\ &+ F + r_{\mathrm{app}},\end{aligned} \tag{3.1.41}$$

where F is supported for $t \le e^{3S_*/4\varepsilon^2}$, r_{app} is supported for $t \ge e^{S_*/2\varepsilon^2}$, and F (resp. r_{app}) satisfies (3.1.30) *(resp.* (3.1.31)*).*

Proof. One has just to consider the difference between (3.1.7) and (3.1.29). ∎

3.2. Normal forms

Identifying $\{-,+\}$ to $\{-1,1\}$, we shall respectively denote by

$$\mathcal{I}_{\mathrm{c}} = \big\{I = (i_1, i_2, i_3) \in \{-,+\}^3; \textstyle\sum_{\ell=1}^3 i_\ell = 1\big\}$$

and

$$\mathcal{I}_{\mathrm{nc}} = \big\{I = (i_1, i_2, i_3) \in \{-,+\}^3; \textstyle\sum_{\ell=1}^3 i_\ell \neq 1\big\}$$

the sets of characteristic and noncharacteristic indices. We shall eliminate by normal forms all noncharacteristic terms on the right-hand side of (3.1.41). We recall that normal forms for Klein–Gordon equations were introduced by Shatah [28] and for further results on these methods, we refer to the review paper of Germain [16] and references therein.

Consider $I = (i_1, i_2, i_3) \in \mathcal{I}_{\mathrm{nc}}$. Up to permutations, we have thus either $(i_1, i_2, i_3) = (1,1,1)$, or $(i_1, i_2, i_3) = (1,-1,-1)$, or $(i_1, i_2, i_3) = (-1,-1,-1)$. We set

$$\begin{aligned}D_I(\xi_1,\xi_2,\xi_3) = i_1\sqrt{1+\xi_1^2} + i_2\sqrt{1+\xi_2^2} + i_3\sqrt{1+\xi_3^2}\\ - \sqrt{1+(\xi_1+\xi_2+\xi_3)^2}.\end{aligned} \tag{3.2.1}$$

Since $i_2 = i_3$, we may write with some $c > 0$,

$$\begin{aligned}|D_I(\xi_1,\xi_2,\xi_3)| &\ge \sqrt{1+\xi_2^2} + \sqrt{1+\xi_3^2} - |\xi_2+\xi_3|\\ &\ge c(1+\min(|\xi_2|,|\xi_3|))^{-1} \ge cM_0(\xi_1,\xi_2,\xi_3)^{-1}\end{aligned}$$

if $M_0(\xi_1,\xi_2,\xi_3)$ is defined by (3.1.1) and so is equivalent to the second largest among $1+|\xi_1|, 1+|\xi_2|, 1+|\xi_3|$. This implies that for any noncharacteristic index I, $D_I(\xi_1,\xi_2,\xi_3)^{-1}$ belongs to the class $S_{1,0}(M_0,3)$ of Definition 3.1.1. Consider the symbols $m_I^{(\ell)}$ on the right-hand side of (3.1.41) and define when $I \in \mathcal{I}_{\mathrm{nc}}$,

$$\widehat{m}_I^{(\ell)}(\xi_1,\xi_2,\xi_3) = m_I^{(\ell)}(\xi_1,\xi_2,\xi_3)D_I(\xi_1,\xi_2,\xi_3)^{-1} \in S_{1,0}(M_0,3). \tag{3.2.2}$$

We shall prove the following proposition:

Proposition 3.2.1. *Define from the solution v_+ of (3.1.41),*

$$w_+ = v_+ - \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\text{nc}}}} [\text{Op}(\widehat{m}_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3}) + \text{Op}(\widehat{m}_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\text{app}}) + \text{Op}(\widehat{m}_I^{(3)})(v_{i_1}, u_{i_2}^{\text{app}}, u_{i_3}^{\text{app}})]. \tag{3.2.3}$$

Then w_+ solves, for $t < T(\varepsilon)$, an equation of the form

$$(D_t - p(D_x))w_+ = \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\text{c}}}} [\text{Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3}) + \text{Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\text{app}}) + \text{Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\text{app}}, u_{i_3}^{\text{app}})] + \mathcal{R}, \tag{3.2.4}$$

where $\mathcal{R}$ is the sum of terms of the following form:

- *a contribution $F(t,x)$, supported for $t \le e^{3S_*/4\varepsilon^2}$, satisfying (3.1.30);*
- *a term $r_{\text{app}}(t,x)$, supported for $e^{S_*/2\varepsilon^2} \le t$, satisfying (3.1.31);*
- *"quintic" terms of the form*

$$\begin{aligned} &\text{Op}(\widetilde{m})(v_{J_1}, u_{J_2}^{\text{app}}), && |J_1| + |J_2| = 5,\ |J_1| \ge 1,\\ &\text{Op}(\widetilde{m})(v_{J_1}, u_{J_2}^{\text{app}}, F_{i_3}), && |J_1| + |J_2| = 2,\\ &\text{Op}(\widetilde{m})(v_{J_1}, u_{J_2}^{\text{app}}, r_{\text{app},i_3}), && |J_1| + |J_2| = 2, \end{aligned} \tag{3.2.5}$$

for different symbols $\widetilde{m}$ belonging to $S_{1,0}(M_0^\nu, 5)$ (resp. $S_{1,0}(M_0^\nu, 3)$) for the first line (resp. the second and third lines) for some $\nu \in \mathbb{N}$, where we denoted $F_+ = F$, $F_- = -\overline{F}$, $r_{\text{app},+} = r_{\text{app}}$, $r_{\text{app},-} = -\overline{r_{\text{app}}}$.

Proof. We make $D_t - p(D_x)$ act on (3.2.3). We get, using (3.2.2) and (3.2.1),

$$\begin{aligned}(D_t - p(D_x))w_+ = (D_t - p(D_x))v_+ &- \sum_{\substack{I=(i_1,i_2,i_3)\\ I\in\mathcal{I}_{\text{nc}}}} [\text{Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3}) + \text{Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\text{app}})\\ &+ \text{Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\text{app}}, u_{i_3}^{\text{app}})] + \mathcal{R}',\end{aligned} \tag{3.2.6}$$

where $\mathcal{R}'$ is the sum of expressions of the following form, up to permutation of factors:

$$\text{Op}(m_I^{(\ell)})((D_t - i_1 p(D_x))v_{i_1}, v_{i_2}, v_{i_3}), \tag{3.2.7}$$

$$\text{Op}(m_I^{(\ell)})((D_t - i_1 p(D_x))v_{i_1}, v_{i_2}, u_{i_3}^{\text{app}}), \tag{3.2.8}$$

$$\text{Op}(m_I^{(\ell)})((D_t - i_1 p(D_x))v_{i_1}, u_{i_2}^{\text{app}}, u_{i_3}^{\text{app}}), \tag{3.2.9}$$

$$\text{Op}(m_I^{(\ell)})(v_{i_1}, v_{i_2}, (D_t - i_3 p(D_x))u_{i_3}^{\text{app}}), \tag{3.2.10}$$

$$\text{Op}(m_I^{(\ell)})(v_{i_1}, u_{i_2}^{\text{app}}, (D_t - i_3 p(D_x))u_{i_3}^{\text{app}}), \tag{3.2.11}$$

where $I = (i_1, i_2, i_3)$ is in $\mathcal{I}_{\text{nc}}$.

In (3.2.7), we replace $(D_t - i_1 p(D_x))v_{i_1}$ by the right-hand side of (3.1.41) if $i_1 = 1$, and by the opposite of the conjugate of this right-hand side if $i_1 = -1$. Using Lemma A.2.1 we get expressions of the form (3.2.5). The same conclusion holds for (3.2.8), (3.2.9). Using (3.1.29) instead of (3.1.41), we see in the same way that (3.2.10), (3.2.11) may be written as contributions to (3.2.5). Thus $\mathcal{R}'$ contributes to $\mathcal{R}$ in (3.2.4). Finally, if on the right-hand side of (3.2.6), we replace $(D_t - p(D_x))v_+$ by its expression coming from (3.1.41), the noncharacteristic contributions cancel each other, and we are left only with the characteristic ones, as on the right-hand side of (3.2.4), and the contribution $F + r_{\mathrm{app}}$ to $\mathcal{R}$. This concludes the proof. ∎

To prepare the energy estimates of next section, we notice that getting bounds on w_+ or v_+ will be essentially equivalent, up to small errors.

Lemma 3.2.2. *There is $\rho_0 \in \mathbb{N}$ such that for any s_0 in $\mathbb{N}$,*

$$\|w_+ - v_+\|_{H^{s_0}} \le C[\|v_+\|^2_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|^2_{W^{\rho_0+s_0,\infty}}]\|v_+\|_{H^{s_0}}, \tag{3.2.12}$$

$$\begin{aligned}\|L_+(w_+ - v_+)\|_{L^2} \le{}& C\|v_+\|^2_{W^{\rho_0,\infty}}[\|L_+v_+\|_{L^2} + t\|v_+\|_{L^2}]\\ &+ C[\|L_+u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}]\\ &\times(\|v_+\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v_+\|_{L^2}.\end{aligned} \tag{3.2.13}$$

Proof. To get (3.2.12), we express $w_+ - v_+$ from (3.2.3). We apply (A.2.2) to the first term in the sum on the right-hand side. To treat the two remaining ones, we use (A.2.7) with $\ell = 2$ or $\ell = 1$ respectively. We obtain a bound by the right-hand side of (3.2.12).

Let us prove (3.2.13). We may write for any functions f_1, f_2, f_3 and any symbol m in $S_{1,0}(M_0, 3)$,

$$\begin{aligned}L_+\mathrm{Op}(m)(f_1, f_2, f_3) ={}& \mathrm{Op}(\tilde{m})(f_1, f_2, f_3) + \mathrm{Op}(m)(f_1, f_2, xf_3)\\ &+ tp'(D_x)\mathrm{Op}(m)(f_1, f_2, f_3)\end{aligned}$$

for some $\tilde{m}$ in $S_{1,0}(M_0^2, 3)$. Then writing $xf_3 = (x + i_3tp'(D_x))f_3 - i_3tp'(D_x)f_3$, we obtain

$$\begin{aligned}L_+\mathrm{Op}(m)(f_1, f_2, f_3) ={}& \mathrm{Op}(m)(f_1, f_2, L_{i_3}f_3) + \mathrm{Op}(\tilde{m})(f_1, f_2, f_3)\\ &- i_3t\mathrm{Op}(m)(f_1, f_2, p'(D_x)f_3)\\ &+ tp'(D_x)(f_1, f_2, f_3).\end{aligned} \tag{3.2.14}$$

We write $L_+(w_+ - v_+)$ from (3.2.3) on which we make L_+ act. We apply equality (3.2.14) with $(f_1, f_2, f_3) = (v_{i_1}, v_{i_2}, v_{i_3})$ to the $\mathrm{Op}(\widehat{m}_I^{(1)})$ term in (3.2.3). By (A.2.3) applied with $j = 3$, we get that the L^2-norm of the action of L_+ on the first term in the sum (3.2.3) is estimated from

$$(\|L_+v_+\|_{L^2} + t\|v_+\|_{L^2})\|v_+\|^2_{W^{\rho_0,\infty}}. \tag{3.2.15}$$

In the same way, applying (3.2.14) to $(f_1, f_2, f_3) = (v_{i_1}, v_{i_2}, u_{i_3}^{\mathrm{app}})$, and using (A.2.3) with $j = 1$, we estimate the L^2-norm of the action of L_+ on the $\mathrm{Op}(\widehat{m}_I^{(2)})$ term in (3.2.3) by

$$(\|L_+ u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v_+\|_{W^{\rho_0,\infty}}\|v_+\|_{L^2}. \tag{3.2.16}$$

Finally, doing the same for the $\mathrm{Op}(\widehat{m}_I^{(3)})$ term, we get a bound in

$$(\|L_+ u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}\|v_+\|_{L^2}.$$

Together with (3.2.15) and (3.2.16), this gives (3.2.13). ∎

4. Construction of the solution and proof of the main theorem

Recall that we want to construct a solution u to equation (1.1.1) that displays inflation of its norms and that we have rewritten that equation as a first-order system (3.1.7)–(3.1.8). We look next for the solution $(u_+, u_- = -\bar{u}_+)$ of that system in the form $u_+ = u_+^{\mathrm{app}} + v_+$, where u_+^{app} is the approximate solution defined in (3.1.15), which solves (3.1.29), and which blows up at time e^{S_*/ε^2}, and where v_+ is the perturbation introduced in Proposition 3.1.4, which solves equation (3.1.41). We shall construct v_+ solving that equation backwards, starting at time $T(\varepsilon)$ introduced in (1.2.5), with initial condition $v_+|_{t=T(\varepsilon)} = 0$. In order to show that v_+ exists up to time $t = 1$, and remains under control down to that time, we shall prove in this section a priori estimates for $\|v_+(t,\cdot)\|_{H^{s_0}}$ for s_0 large enough and $\|L_+ v_+(t,\cdot)\|_{L^2}$. In order to do so, we shall exploit the fact that $\|v_+\|_{W^{\rho_0,\infty}}$ remains small, so that Lemma 3.2.2 will imply that the H^s (resp. L^2) norm of v_+ (resp. $L_+ v_+$) is equivalent to the H^s (resp. L^2) norm of w_+ (resp. $L_+ w_+$), where w_+ solves equation (3.2.4), in which the explicit cubic terms on the right-hand side are all *characteristic*. In the following subsections, we shall successively prove estimates for $\|v_+(t,\cdot)\|_{H^{s_0}}$, $\|L_+ v_+(t,\cdot)\|_{L^2}$, and then perform the bootstrap argument that gives the proof of the main theorem.

4.1. Sobolev estimates

In the estimates of this subsection and the following ones, it will be important to track the dependence of some constants on others. We shall fix indices of smoothness ρ_0, s_0 (which will be taken large enough), as well as the parameters δ, δ', γ that satisfy (3.1.11). A universal constant will be a constant that depends possibly on these parameters, but on no other quantity. Next we shall have constants like N (the order at which we construct the approximate solutions (3.1.16)) or N_0 in (3.1.31), as well as the constants A_0, A_1, B that we introduce below in the estimates of v_+. It will be important to track how other constants depend on them. Because of that, when we introduce a constant like $K(A_0, A_1, B, \dots)$, we mean that K depends only on the quantities explicitly mentioned in the argument.

Proposition 4.1.1. *Let $\rho_0 \in \mathbb{N}$ be fixed such that the estimates of Proposition A.2.2 of the appendix hold. Let $s_0 \in \mathbb{N}$ be given. There is an integer $N_{0,\min} > 0$ such that for any $N_0 \geq N_{0,\min}$, the following holds:*

The choice of ρ_0, s_0, N_0 determines the constants on the right-hand side of (3.1.30), (3.1.31). *For any couple of constants (A_0, B) with A_0 large enough relative to N_0, there is $\varepsilon_0 \in]0, 1]$ such that, for any $\varepsilon \in]0, \varepsilon_0]$, the following bootstrap holds: Denote by v_+ the backwards solution of* (3.1.41)*, with initial condition $v_+(T(\varepsilon), \cdot) = 0$, and source term $F + r_{\text{app}}$ (with F, r_{app} satisfying* (3.1.30), (3.1.31)*). Assume that this solution is defined on an interval $[T, T(\varepsilon)]$ for some $T \in [1, T(\varepsilon)[$ and that the following a priori estimates hold true for any $t \in [T, T(\varepsilon)]$:*

$$\|v_+(t, \cdot)\|_{W^{\rho_0,\infty}} \leq \frac{B}{\sqrt{t}}\varepsilon^{2-\theta}. \tag{4.1.1}$$

Then, for any $t \in [T, T(\varepsilon)]$, one has

$$\|v_+(t, \cdot)\|_{H^{s_0}} \leq \frac{A_0}{2}\varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_0}. \tag{4.1.2}$$

Before starting the proof, we introduce notation for the cubic terms on the right-hand side of (3.1.41), namely

$$\begin{aligned}\mathscr{F}_3(v_+, u_+^{\text{app}}) = &\sum_{I\in\{-,+\}^3} \text{Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3})\\ &+ \sum_{I\in\{-,+\}^3} \text{Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\text{app}})\\ &+ \sum_{I\in\{-,+\}^3} \text{Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\text{app}}, u_{i_3}^{\text{app}}).\end{aligned} \tag{4.1.3}$$

As $m_I^{(j)}$ is in $S_{0,0}(1, 3)$, independent of x, we may apply (A.2.2) to the first sum in (4.1.3) and (A.2.7) to the second and third ones, with $\ell = 2$ and $\ell = 1$ respectively. We get for any $s_0 \in \mathbb{N}$,

$$\|\mathscr{F}_3(v_+, u_+^{\text{app}})\|_{H^{s_0}} \leq C(\|v_+\|^2_{W^{\rho_0,\infty}} + \|u_+^{\text{app}}\|^2_{W^{\rho_0+s_0,\infty}})\|v_+\|_{H^{s_0}}. \tag{4.1.4}$$

To prove Proposition 4.1.1, we shall need a bound for $\|u_+^{\text{app}}\|_{W^{\rho_0+s_0,\infty}}$ on the right-hand side of (4.1.4).

Lemma 4.1.2. *For any $\rho > 0$, there are $C_0(\rho)$, $\theta' > 0$, and for any $N \in \mathbb{N}$, there is a constant $K(N)$ such that the approximate solution u_+^{app} given by* (3.1.16) *with that value of N satisfies*

$$\begin{aligned}\|u_+^{\text{app}}\|_{W^{\rho,\infty}} \leq\; &C_0(\rho)\frac{\varepsilon}{\sqrt{t}}(S_* - \varepsilon^2 \log t)^{-\frac{1}{2}}\\ &+ K(N)\frac{\varepsilon^{1+\theta'}}{\sqrt{t}}(S_* - \varepsilon^2 \log t)^{-\frac{1}{2}} + K(N)\frac{\varepsilon}{\sqrt{t}}.\end{aligned} \tag{4.1.5}$$

Remark. We shall use (4.1.5) to estimate $\|u_+^{\mathrm{app}}\|_{W^{\rho_0+s_0,\infty}}$ on the right-hand side of (4.1.4), so that the first multiplicative constant on the right-hand side $C_0(\rho_0+s_0)=C_0$ will be a universal constant with the terminology introduced at the beginning of this section. In particular, it is independent of N. The two other constants in (4.1.5) do depend on N, but they are either multiplied by a small factor $\varepsilon^{\theta'}$ or are not affected by the large factor $(S_*-\varepsilon^2\log t)^{-\frac{1}{2}}$.

Proof of Lemma 4.1.2. We bound the $W^{\rho,\infty}$-norm of each term on the right-hand side of (3.1.16).

- By (3.1.17) and (3.1.20), the first term on the right-hand side of (3.1.16) has $W^{\rho,\infty}$-norm bounded by $C(\rho)\frac{\varepsilon}{\sqrt{t}}$ for some constant $C(\rho)$ depending only on ρ.
- In the second term on the right-hand side of (3.1.16), $a_{1,1}^+(s,y)$ is equal to the quantity $2(1-y^2)^{-\frac{1}{2}}a_{1,1}(s,y)$, where $a_{1,1}$ is the element of $\Sigma^{-\frac{1}{2}}$ given explicitly by (2.1.10). It depends only on the initial data of (2.1.8). By definition (2.2.1) of class $\Sigma^{-\frac{1}{2}}$,

$$\partial_x^\ell\Big(a_{1,1}^+\Big(\varepsilon^2\log t,\frac{x}{t}\Big)\Big)=t^{-\ell}b_\ell\Big(\varepsilon^2\log t,\frac{x}{t}\Big) \tag{4.1.6}$$

  for some $b_\ell\in\Sigma^{-\frac{1}{2}-\ell}$. Since by (3.1.9), $t^{-\ell}(S_*-\varepsilon^2\log t)^{-\ell}=O(1)$, the $W^{\rho,\infty}$-norm of the second term on the right-hand side of (3.1.16) is bounded by $C_0(\rho)\frac{\varepsilon}{\sqrt{t}}(S_*-\varepsilon^2\log t)^{-\frac{1}{2}}$ for some constant $C_0(\rho)$ depending only on ρ.
- Consider next the $W^{\rho,\infty}$-norm of each term in the last sum in (3.1.16). Since as above in (4.1.6), any ∂_x derivative may be written as an expression of the same form as the general term in that sum, it is enough to bound the L^∞-norm, which is smaller than

$$C_\ell t^{-\frac{\ell}{2}}(S_*-\varepsilon^2\log t)^{-\frac{\ell}{2}-\delta(\ell-1)}e_\ell(t,\varepsilon) \tag{4.1.7}$$

  with a factor $e_\ell(t,\varepsilon)$ that satisfies, according to (3.1.18),

$$\begin{aligned} e_\ell(t,\varepsilon)&=O(\varepsilon) && \text{if } t\le e^{3S_*/4\varepsilon^2},\\ e_\ell(t,\varepsilon)&=O(\varepsilon^{2-\ell}) && \text{if } t\ge e^{S_*/2\varepsilon^2}. \end{aligned} \tag{4.1.8}$$

  Using (3.1.9), we estimate (4.1.7) when $t\ge e^{S_*/2\varepsilon^2}$ by

$$C_\ell t^{-\frac{1}{2}}(S_*-\varepsilon^2\log t)^{-\frac{1}{2}}e_\ell(t,\varepsilon)\varepsilon^{(\ell-1)(\frac{\gamma}{2}-\delta')}. \tag{4.1.9}$$

  Using (4.1.8), (3.1.11), and the fact that $\ell\ge 3$, we get that (4.1.9) is estimated by the second term on the right-hand side of (4.1.5) with $\theta'\ge 2$. The sum of all these terms for $3\le\ell\le N+1$, ℓ odd, is thus also controlled by this quantity.

For $t\le e^{3S_*/4\varepsilon^2}$ (4.1.8) implies that the sum of expressions (4.1.7) is smaller than the last term in (4.1.5). This concludes the proof of the lemma. ■

Next we show the following lemma:

Lemma 4.1.3. *Assume the a priori inequality* (4.1.1) *and that the source term in* (3.1.41) *satisfies* (3.1.30), (3.1.31). *Let* N_0 *be given in* $\mathbb{N}$. *Then there are a universal constant* C_0 *and a constant* $K(N_0)$ *depending on* N_0, *a constant* $K(B)$ *depending on* B *in* (4.1.1), *such that*

$$\begin{aligned}\|v_+(t,\cdot)\|_{H^s_0} \leq &\int_t^{T(\varepsilon)} [\|F(\tau,\cdot)\|_{H^{s_0}} + \|r_{\mathrm{app}}(\tau,\cdot)\|_{H^{s_0}}]\,d\tau\\ &+ K(N_0)\varepsilon^2 \int_t^{T(\varepsilon)} [\varepsilon^{2\theta'}(S_* - \varepsilon^2\log\tau)^{-1} + 1]\|v_+(\tau,\cdot)\|_{H^{s_0}}\frac{d\tau}{\tau}\\ &+ C_0\varepsilon^2\int_t^{T(\varepsilon)} (S_* - \varepsilon^2\log\tau)^{-1}\|v_+(\tau,\cdot)\|_{H^{s_0}}\frac{d\tau}{\tau}\\ &+ K(B)\varepsilon^{4-2\theta}\int_t^{T(\varepsilon)} \|v_+(\tau,\cdot)\|_{H^{s_0}}\frac{d\tau}{\tau}\end{aligned} \tag{4.1.10}$$

for any $t \in [T, T(\varepsilon)]$.

Proof. We write the backwards energy inequality for the solution to (3.1.41) with zero initial condition at $t = T(\varepsilon)$ using notation (4.1.3). We obtain

$$\begin{aligned}\|v_+(t,\cdot)\|_{H^{s_0}} \leq &\int_t^{T(\varepsilon)} \|\mathscr{F}_3(v_+, u_+^{\mathrm{app}})(\tau,\cdot)\|_{H^{s_0}}\,d\tau\\ &+ \int_t^{T(\varepsilon)} \|F(\tau,\cdot)\|_{H^{s_0}}\,d\tau + \int_t^{T(\varepsilon)} \|r_{\mathrm{app}}(\tau,\cdot)\|_{H^{s_0}}\,d\tau.\end{aligned}$$

Into the first integral on the right-hand side, we plug (4.1.4). By estimate (4.1.1), the $\|v_+\|^2_{W^{\rho_0,\infty}}$ term on the right-hand side of (4.1.4) brings the last term in (4.1.10). To study the contribution of the $\|u_+^{\mathrm{app}}\|^2_{W^{\rho_0+s_0,\infty}}$ term of (4.1.4), we apply (4.1.5) with $\rho = \rho_0 + s_0$ for an N taken large enough relative to N_0 so that Proposition 3.1.3 holds. We obtain thus from the right-hand side of (4.1.5) the second and third terms on the right-hand side of (4.1.10), with a constant K that depends on N, and thus on N_0. This concludes the proof. ■

Proof of Proposition 4.1.1. We make the change of variable $t = e^{\frac{s}{\varepsilon^2}}$ with $s \in [0, S(\varepsilon)]$, $S(\varepsilon) = \varepsilon^2 \log T(\varepsilon)$, and rewrite (4.1.10) as

$$f(s) \leq \int_s^{S(\varepsilon)} g(\sigma)\,d\sigma + \int_s^{S(\varepsilon)} \psi(\sigma) f(\sigma)\,d\sigma, \tag{4.1.11}$$

where

$$\begin{aligned}f(s) &= \|v_+(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{H^{s_0}},\\ g(\sigma) &= e^{\frac{\sigma}{\varepsilon^2}}\varepsilon^{-2}[\|F(e^{\frac{\sigma}{\varepsilon^2}},\cdot)\|_{H^{s_0}} + \|r(e^{\frac{\sigma}{\varepsilon^2}},\cdot)\|_{H^{s_0}}],\\ \psi(\sigma) &= K(N_0)(\varepsilon^{2\theta'}(S_* - \sigma)^{-1} + 1) + C_0(S_* - \sigma)^{-1} + K(B)\varepsilon^{2-2\theta}.\end{aligned} \tag{4.1.12}$$

Denote

$$\Phi(s) = -\int_s^{S(\varepsilon)} \psi(\sigma)\, d\sigma \tag{4.1.13}$$

so that (4.1.11) implies by the Grönwall inequality,

$$f(s) \le \int_s^{S(\varepsilon)} e^{\Phi(s')-\Phi(s)} g(s')\, ds'. \tag{4.1.14}$$

Assume that ε is small enough so that in the expression of $\psi(\sigma)$ in (4.1.12), $K(N_0)\varepsilon^{2\theta'} + K(B)\varepsilon^{2-2\theta} \le 1$. Then (4.1.13) implies that for $T \le s \le s' \le T(\varepsilon) < S_*$,

$$e^{\Phi(s')-\Phi(s)} \le e^{(K(N_0)+1)S_*}\Big(\frac{S_*-s}{S_*-s'}\Big)^{C_0+1}.$$

We thus get from (4.1.14),

$$\begin{aligned}\|v_+(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{H^{s_0}} \le K(N_0)\int_s^{S(\varepsilon)} \Big(\frac{S_*-s}{S_*-s'}\Big)^{C_0+1} e^{\frac{s'}{\varepsilon^2}}\varepsilon^{-2} \\ \times [\|F(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{H^{s_0}} + \|r_{\mathrm{app}}(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{H^{s_0}}]\, ds'\end{aligned} \tag{4.1.15}$$

for a new constant $K(N_0)$. Next we use (3.1.30), (3.1.31) to estimate the right-hand side. If $s \ge \frac{3S_*}{4}$, then the $F(e^{\frac{s}{\varepsilon^2}},\cdot)$-contribution on the right-hand side of (4.1.15) vanishes, so that by (3.1.31), we get the estimate

$$C_{s_0}K(N_0)\int_s^{S(\varepsilon)} (S_*-s)^{C_0+1}(S_*-s')^{-C_0-1+N_0} e^{-\frac{s'}{\varepsilon^2}}\varepsilon^{N_0-2}\, ds'. \tag{4.1.16}$$

If $N_0 > C_0$, which may be imposed since C_0 is a universal constant, we get a bound in $K(N_0)e^{-\frac{s}{\varepsilon^2}}(S_*-s)^{N_0+1}\varepsilon^{N_0-2}$ (for a new $K(N_0)$) that largely implies an estimate of the form (4.1.2), if we assume $N_0 \ge 4$ and $\varepsilon < \varepsilon_0$ small enough.

If on the other hand in (4.1.15), $s < \frac{3S_*}{4}$, the integral on the right-hand side of (4.1.15) for $s \in [\frac{3S_*}{4}, S(\varepsilon)]$ is estimated as above and the remaining one by

$$\begin{aligned}K(N_0)\int_s^{3S_*/4} (S_*-s)^{C_0+1}(S_*-s')^{-C_0-1} e^{\frac{s'}{\varepsilon^2}}\varepsilon^{-2}\|r_{\mathrm{app}}(e^{\frac{s'}{\varepsilon^2}},\cdot)\|_{H^{s_0}}\, ds' \\ + 4^{C_0+1}K(N_0)\int_1^{e^{3S_*/4}} \|F(\tau,\cdot)\|_{H^{s_0}}\, d\tau.\end{aligned} \tag{4.1.17}$$

The first term may be bounded again by (4.1.16) and then by $K(N_0)\varepsilon^{2-\theta}$ if N_0 is large enough. By (3.1.30), the last contribution to (4.1.17) is also in $K(N_0)\varepsilon^{2-\theta}$ for a new constant depending on N_0. If the constant A_0 is chosen large enough in the function of N_0, we may ensure that (4.1.2) holds. This concludes the proof. ∎

4.2. Estimates for the action of L_+

We want to prove estimates for the L^2-norm of L_+v_+ analogous to those of Proposition 4.1.1 in the case of Sobolev norms. To do so, we shall have to use the auxiliary unknown w_+ of Proposition 3.2.1.

Proposition 4.2.1. *Assume given large enough integers ρ_0, s_0. Assume also given a large enough integer N_1 and an integer N_0 satisfying $N_0 \geq N_1 + 1 + 2\delta$. For any constant $A_0 > 0$ (which depends on the preceding ones), there is $A_1 > 0$ and, for any constant $B > 0$ (which may depend on A_0, A_1), there is $\varepsilon_0 \in]0,1]$, such that the following holds true for any $\varepsilon \in]0, \varepsilon_0]$:*

Let $T \in]0, T(\varepsilon)[$ and let v_+ be a solution of equation (3.1.41) defined on $[T, T(\varepsilon)]$, with the initial condition $v_+(T(\varepsilon), \cdot) = 0$, such that v_+ satisfies, for any $t \in [T, T(\varepsilon)]$, the following estimates:

$$\begin{aligned} \|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq \frac{B}{\sqrt{t}}\varepsilon^{2-\theta}, \\ \|v_+(t,\cdot)\|_{H^{s_0}} &\leq A_0\varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_0}. \end{aligned} \tag{4.2.1}$$

Then for any $t \in [T, T(\varepsilon)]$, we have the estimate

$$\|L_+v_+(t,\cdot)\|_{L^2} \leq \frac{A_1}{2}\varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_1}. \tag{4.2.2}$$

To prove the proposition, we first need an estimate for $\|L_+u_+^{\text{app}}\|_{W^{\rho,\infty}}$ for any ρ.

Lemma 4.2.2. *For any $\rho > 0$, any $N \in \mathbb{N}^*$, there is a constant $K(\rho, N)$ such that if u_+^{app} is defined by (3.1.16), one has for any $t \leq T(\varepsilon)$ the bound*

$$\|L_+u_+^{\text{app}}(t,\cdot)\|_{W^{\rho,\infty}} \leq K(N,\rho)\frac{\varepsilon}{\sqrt{t}}(S_* - \varepsilon \log t)^{-\frac{3}{2}-2\delta}. \tag{4.2.3}$$

Proof. We make L_+ act on (3.1.16). We get a first term

$$\chi_0(\varepsilon^{1-\theta}(t-1))L_+u_{0,+}. \tag{4.2.4}$$

If we apply Corollary A.1.5 with $q = 1$ or $q = -1$ to expression (3.1.20) of $u_{0,+}$, we conclude that the $W^{\rho,\infty}$-norm of (4.2.4) is $O(\varepsilon/\sqrt{t})$. Applying Corollary A.1.5 again with $q = 1$ to the second term on the right-hand side of (3.1.16), we get that the action of L_+ on it gives an expression

$$e^{it\varphi(y)}(1-\chi_0)(\varepsilon^{1-\theta}(t-1))\frac{\varepsilon}{\sqrt{t}}\tilde{a}_{1,1}^+\Big(s, y, \frac{1}{t}, \varepsilon\Big) + \tilde{r}\Big(s, y, \frac{1}{t}, \varepsilon\Big)\Big|_{\substack{s=\varepsilon^2\log t\\ y=x/t}} \tag{4.2.5}$$

with $\tilde{a}_{1,1}^+$ in $\Sigma^{-\frac{1}{2}-\frac{1}{2\kappa_0}} \subset \Sigma^{-\frac{3}{2}}$ and $\tilde{r}$ with all its ∂_s, ∂_y, $h\partial_h$-derivatives smaller than $h^N\langle y\rangle^{-N}$ for any N. Then the $W^{\rho,\infty}$-norm of (4.2.5) is $O(\frac{\varepsilon}{\sqrt{t}}(S_* - \varepsilon^2 \log t)^{-\frac{3}{2}})$ since the action of each ∂_x-derivative makes one lose at most $1 + t^{-1}(S_* - \varepsilon^2 \log t)^{-1}$, which is $O(1)$ (using (3.1.9) when t satisfies $e^{S_*/2\varepsilon^2} < t < T(\varepsilon)$).

We consider next the action of L_+ on the last sum in (3.1.16). We have on the one hand the characteristic terms corresponding to $q = 1$, $\ell \geq 3$. We apply Corollary A.1.5 with $q = 1$ to see that the action of L_+ on these terms is given by a sum for $3 \leq \ell \leq N + 1$ of expressions

$$e^{it\varphi(y)} t^{-\frac{\ell}{2}} a^{+,2}_{\ell,1}\Big(s, y, \frac{1}{t}, \varepsilon\Big) e_{\ell,1}(t, \varepsilon)|_{\substack{s=\varepsilon^2 \log t\\ y=x/t}} \tag{4.2.6}$$

with $a^{+,2}_{\ell,1}$ in $\Sigma^{-\frac{\ell}{2}-1-\delta(\ell-1)}$ and $e_{\ell,1}(t, \varepsilon)$ given by (3.1.18), modulo a remainder $\varepsilon\tilde{r}$, with $\tilde{r}$ as in (4.2.5), so that it will trivially satisfy a bound of the form (4.2.3). In (4.2.6), $e_{\ell,1}(t, \varepsilon) = O(\varepsilon)$ if $t \leq e^{S_*/2\varepsilon^2}$, so that in this case bounds (4.2.3) hold immediately. If $t > e^{S_*/2\varepsilon^2}$, we bound the modulus of (4.2.6) by

$$t^{-\frac{\ell}{2}}(S_* - \varepsilon^2 \log t)^{-\frac{\ell}{2}-1-\delta(\ell-1)} \varepsilon^{2-\ell} = t^{-\frac{1}{2}}(t^{-\frac{1}{2}}(S_* - s)^{-\frac{1}{2}-\delta})^{\ell-1} \varepsilon^{2-\ell}(S_* - s)^{-\frac{3}{2}}|_{s=\varepsilon^2 \log t}.$$

By (3.1.9) this is $O(\frac{\varepsilon}{\sqrt{t}}(S_* - \varepsilon^2 \log t)^{-\frac{3}{2}})$ since $(\ell - 1)(\frac{\gamma}{2} - \delta') + 2 - \ell \geq 1$ by (3.1.11). Since the same estimates hold for ∂_x-derivatives of (4.2.6), we get for the $W^{\rho,\infty}$-norm of the action of L_+ on the characteristic terms in the sum in (3.1.16) a bound by the right-hand side of (4.2.3).

We still have to study the noncharacteristic terms in that sum, i.e. those for which $q \neq 1$. By Corollary A.1.5 and (3.1.9), we get that the action of L_+ on these terms gives

$$e^{itq\varphi(y)} t^{-\frac{\ell}{2}+1} a^{+,2}_{\ell,q}\Big(s, y, \frac{1}{t}, \varepsilon\Big) e_{\ell,q}(t, \varepsilon)|_{\substack{s=\varepsilon^2 \log t\\ y=x/t}} \tag{4.2.7}$$

with $a^{+,2}_{\ell,q}$ in $\Sigma^{-\frac{\ell}{2}-\delta(\ell-1)}$, $\ell \geq 3$, modulo again a remainder that is again like $\varepsilon\tilde{r}$ in (4.2.5). The modulus of (4.2.7) is bounded by

$$\begin{aligned} &t^{-\frac{\ell}{2}+1}(S_* - \varepsilon^2 \log t)^{-\frac{\ell}{2}-\delta(\ell-1)}|e_{\ell,q}(t, \varepsilon)| \\ &\quad = t^{-\frac{1}{2}}(t^{-\frac{1}{2}}(S_* - s)^{-\frac{1}{2}-\delta})^{\ell-3}(S_* - s)^{-\frac{3}{2}-2\delta}|e_{\ell,q}(t, \varepsilon)||_{s=\varepsilon^2 \log t}. \end{aligned} \tag{4.2.8}$$

When $t \geq e^{S_*/2\varepsilon^2}$, we use (3.1.9) to estimate (4.2.8) from

$$t^{-\frac{1}{2}} \varepsilon^{(\ell-3)(\frac{\gamma}{2}-\delta')}|e_{\ell,q}(t, \varepsilon)|(S_* - \varepsilon^2 \log t)^{-\frac{3}{2}-2\delta}. \tag{4.2.9}$$

If $\ell = 3$, $q = -1$, the last inequality (3.1.18) gives a bound of the form (4.2.3). If $\ell = 3$, $|q| = 3$, the second estimate (3.1.18) shows that $e_{\ell,q}(t, \varepsilon) = O(\varepsilon^3)$, so that we obtain again the wanted bound. If $\ell \geq 5$, using that $e_{\ell,q}(t, \varepsilon) = O(\varepsilon^{2-\ell})$ and (3.1.11), we obtain that (4.2.9) is controlled by the right-hand side of (4.2.3). When $t \leq e^{S_*/2\varepsilon^2}$, the first estimate (3.1.18) shows that the bound of (4.2.8) by (4.2.3) holds trivially. Finally, since similar bounds are satisfied by ∂_x-derivatives, we get that the noncharacteristic terms in the sum in (3.1.16) are controlled as in (4.2.3). This concludes the proof. ∎

We prove next a lemma relating estimates for L_+v_+ and L_+w_+.

Lemma 4.2.3. *Let v_+ be a function defined on some interval $[T, T(\varepsilon)]$ satisfying, for any $t \in [T, T(\varepsilon)]$, estimates*

$$\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq \frac{B}{\sqrt{t}}\varepsilon^{2-\theta} \tag{4.2.10}$$

for some constant B. There is $\varepsilon_0 > 0$, depending on B, such that if $\varepsilon \in]0, \varepsilon_0[$ and (4.2.10) holds, then w_+ defined by (3.2.3) from v_+ and u_+^{app} given by (3.1.15) satisfies

$$\begin{aligned}\|L_+v_+(t,\cdot)\|_{L^2} &\leq 2\|L_+w_+(t,\cdot)\|_{L^2}\\ &\quad + K(N, B\varepsilon^{1-\theta})\varepsilon^2(S_* - \varepsilon^2\log t)^{-1-2\delta}\|v_+(t,\cdot)\|_{L^2}\end{aligned} \tag{4.2.11}$$

(with N equal to the order at which u_+^{app} has been constructed in (3.2.14)).

Proof. By (4.1.5) and (4.2.3) we have

$$\begin{aligned}&\|L_+u_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}} + t\|u_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}}\\ &\quad\leq \varepsilon K(N)\sqrt{t}(S_* - \varepsilon^2\log t)^{-\frac{1}{2}-2\delta}[1 + t^{-1}(S_* - \varepsilon^2\log t)^{-1}].\end{aligned} \tag{4.2.12}$$

On the other hand, still by (4.1.5) and the a priori estimate of $\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}}$ in (4.2.10), we have

$$\|u_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}} + \|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq \frac{\varepsilon K(N, B\varepsilon^{1-\theta})}{\sqrt{t}}(S_* - \varepsilon^2\log t)^{-\frac{1}{2}}. \tag{4.2.13}$$

Using (3.1.9), we see that the product of (4.2.12) by (4.2.13) is smaller than

$$\varepsilon^2K(N, B\varepsilon^{1-\theta})(S_* - \varepsilon^2\log t)^{-1-2\delta}.$$

Plugging this into (3.2.13), and also using the a priori estimate (4.2.10) of $\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}}$, we get

$$\begin{aligned}\|L_+(w_+ - v_+)(t,\cdot)\|_{L^2} &\leq K(B\varepsilon^{1-\theta})\frac{\varepsilon^2}{t}\|L_+v_+(t,\cdot)\|_{L^2}\\ &\quad + K(B\varepsilon^{1-\theta})\varepsilon^2\|v_+(t,\cdot)\|_{L^2}\\ &\quad + \varepsilon^2K(N, B\varepsilon^{1-\theta})(S_* - \varepsilon^2\log t)^{-1-2\delta}\|v_+(t,\cdot)\|_{L^2},\end{aligned}$$

which implies (4.2.11). This concludes the proof. ∎

We prove next an energy inequality for $\|L_+w_+(t,\cdot)\|_{L^2}$.

Lemma 4.2.4. *Assume that for t in some interval $[T, T(\varepsilon)]$, the following a priori estimate holds true:*

$$\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq \frac{B}{\sqrt{t}}\varepsilon^{2-\theta}. \tag{4.2.14}$$

Then for t in the same interval, one has an inequality

$$
\begin{aligned}
&\|(D_t - p(D_x))L_+ w_+(t,\cdot)\|_{L^2} \\
&\quad \le \frac{\varepsilon^2}{t}[(S_* - \varepsilon^2 \log t)^{-1}(C_0 + K(N,B)\varepsilon^{\theta''}) + K(N)]\|L_+ v_+(t,\cdot)\|_{L^2} \\
&\quad + \frac{\varepsilon^2}{t}(S_* - \varepsilon^2 \log t)^{-2-2\delta}(K(N) + K(N,B)\varepsilon^{\theta''})\|v_+(t,\cdot)\|_{H^{s_0}} \\
&\quad + \mathcal{R}_L(t) + \mathcal{R}_H(t),
\end{aligned} \tag{4.2.15}
$$

where C_0 is a universal constant, $\theta'' > 0$, and $\mathcal{R}_H$, $\mathcal{R}_L$ satisfy

$$
\begin{aligned}
\int_1^{+\infty} \|\mathcal{R}_L(t)\|_{L^2}\,dt &\le (K(N) + K(N,B)\varepsilon^{\theta''})\varepsilon^{2-\theta}, \\
\|\mathcal{R}_H(t)\|_{L^2} &\le K(N,B)t^{-1}\varepsilon^{N_0}(S_* - \varepsilon^2 \log t)^{N_0-1},
\end{aligned} \tag{4.2.16}
$$

where N_0 is the integer introduced in Proposition 3.1.3, and $\mathcal{R}_L$ is supported for $t \le e^{3S_*/4\varepsilon^2}$.

Proof. We make L_+ act on equation (3.2.4) to get

$$
\begin{aligned}
&(D_t - p(D_x))L_+ w_+ \\
&\quad = \sum_{\substack{I=(i_1,i_2,i_3) \\ I \in \mathcal{I}_c}} [L_+ \mathrm{Op}(m_I^{(1)})(v_{i_1}, v_{i_2}, v_{i_3}) + L_+ \mathrm{Op}(m_I^{(2)})(v_{i_1}, v_{i_2}, u_{i_3}^{\mathrm{app}}) \\
&\qquad + L_+ \mathrm{Op}(m_I^{(3)})(v_{i_1}, u_{i_2}^{\mathrm{app}}, u_{i_3}^{\mathrm{app}})] + L_+ \mathcal{R} \\
&\quad = \mathrm{I} + \cdots + \mathrm{IV}.
\end{aligned} \tag{4.2.17}
$$

We estimate the L^2-norm of the terms on the right-hand side. Since the index I is characteristic, we may use Proposition A.3.1 in order to estimate $\mathrm{I} + \mathrm{II} + \mathrm{III}$.

- *Estimates of* I, II, III. By (A.3.1), we get

$$
\begin{aligned}
\|\mathrm{I}\|_{L^2} &\le C\|v_+\|^2_{W^{\rho_0,\infty}}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \\
&\le CB^2\frac{\varepsilon^{4-2\theta}}{t}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}})
\end{aligned} \tag{4.2.18}
$$

by (4.2.14). We estimate II using (A.3.2). We get

$$
\begin{aligned}
\|\mathrm{II}\|_{L^2} \le\ & 2C\|v_+\|_{W^{\rho_0,\infty}}\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}) \\
& + C\|v_+\|_{W^{\rho_0,\infty}}(\|L_+ u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v_+\|_{L^2}.
\end{aligned}
$$

Using (4.2.14), bound (4.1.5) of u_+^{app} which implies

$$
\|u_+^{\mathrm{app}}(t,\cdot)\|_{W^{\rho_0,\infty}} \le K(N)\frac{\varepsilon}{\sqrt{t}}(S_* - \varepsilon^2 \log t)^{-\frac{1}{2}}, \tag{4.2.19}
$$

and (4.2.3), we get

$$\begin{aligned}\|\mathrm{II}\|_{L^2} \leq K(N,B)\frac{\varepsilon^{3-\theta}}{t}(S_* - \varepsilon^2\log t)^{-\frac{1}{2}}\|L_+v_+(t,\cdot)\|_{L^2}\\ + K(N,B)\frac{\varepsilon^{3-\theta}}{t}(S_* - \varepsilon^2\log t)^{-\frac{3}{2}-2\delta}\|v_+(t,\cdot)\|_{H^{s_0}}.\end{aligned} \tag{4.2.20}$$

To estimate III, we use (A.3.3). We obtain

$$\begin{aligned}\|\mathrm{III}\|_{L^2} \leq 2C\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}(\|Lu_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})\|v\|_{L^2}\\ + C\|u_+^{\mathrm{app}}\|^2_{W^{\rho_0,\infty}}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}}).\end{aligned}$$

Using (4.1.5) to estimate $\|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}}$ and (4.2.3), we obtain a bound

$$\begin{aligned}\|\mathrm{III}\|_{L^2} \leq C_0\frac{\varepsilon^2}{t}(S_* - \varepsilon^2\log t)^{-1}(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}})\\ + K(N)\frac{\varepsilon^2}{t}(\varepsilon^{2\theta'}(S_* - \varepsilon^2\log t)^{-1} + 1)(\|Lv_+\|_{L^2} + \|v_+\|_{H^{s_0}})\\ + K(N)\frac{\varepsilon^2}{t}(S_* - \varepsilon^2\log t)^{-2-2\delta}\|v_+(t,\cdot)\|_{L^2}.\end{aligned} \tag{4.2.21}$$

Summing (4.2.18), (4.2.20), and (4.2.21), we deduce that

$$\begin{aligned}&\|\mathrm{I}+\mathrm{II}+\mathrm{III}\|_{L^2}\\ &\leq \frac{\varepsilon^2}{t}[C_0(S_* - \varepsilon^2\log t)^{-1} + K(N,B)\varepsilon^{\theta''}(S_* - \varepsilon^2\log t)^{-1} + K(N)]\|L_+v_+\|_{L^2}\\ &+ (K(N) + K(N,B)\varepsilon^{\theta''})\frac{\varepsilon^2}{t}(S_* - \varepsilon^2\log t)^{-2-2\delta}\|v_+(t,\cdot)\|_{H^{s_0}}\end{aligned} \tag{4.2.22}$$

for some $\theta'' > 0$, which is controlled by the right-hand side of (4.2.15).

• *Estimate of* IV. We estimate now the L^2-norm of the last term $L_+\mathcal{R}$ in (4.2.17), where $\mathcal{R}$ is the last term in (3.2.4) and has the structure described in the statement of Proposition 3.2.1. The contribution L_+F to $L_+\mathcal{R}$ satisfies (3.1.30) and is supported for $t \leq e^{3S_*/4\varepsilon^2}$, so may be incorporated into $\mathcal{R}_L$ in (4.2.15), with $\mathcal{R}_L$ satisfying (4.2.16). The contribution L_+r_{app} to $L_+\mathcal{R}$ is supported for $t \geq e^{S_*/2\varepsilon^2}$ and satisfies (3.1.31), so that we may incorporate it into $\mathcal{R}_H$ in (4.2.15), with $\mathcal{R}_H$ satisfying (4.2.16).

We are left with studying the quintic terms obtained by making L_+ act on (3.2.5). We consider first the action of L_+ on the first term in (3.2.5). Since $|J_1| \geq 1$, the first argument in $\mathrm{Op}(\widetilde{m})(\cdots)$ is equal to $v_\pm$. When we make $L_+ = x + tp'(D_x)$ act on it, we argue as in (3.2.14), and rewrite the resulting expression as a sum of terms of the following forms:

$$\begin{aligned}&\mathrm{Op}(m)(L_\pm v_\pm, v_{J_1'}, u_{J_2'}^{\mathrm{app}}),\\ &\mathrm{Op}(m)(v_\pm, v_{J_1'}, u_{J_2'}^{\mathrm{app}}),\\ &tp'(D_x)\mathrm{Op}(m)(v_\pm, v_{J_1'}, u_{J_2'}^{\mathrm{app}}),\\ &\pm t\mathrm{Op}(m)(p'(D_x)v_\pm, v_{J_1'}, u_{J_2'}^{\mathrm{app}}),\end{aligned}$$

where $|J_1'| + |J_2'| = 4$ and m is a new symbol in the class $S_{1,0}(M_0^\nu, 3)$ for some ν, with constant coefficients. We apply (A.2.3) with $j = 1$ to all these expressions. We get an estimate of their L^2-norms by

$$(\|v_+\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})^4(\|L_+v_+\|_{L^2} + t\|v_+\|_{L^2}).$$

By (4.2.19) and the a priori assumption (4.2.14), we get a bound

$$K(N,B)\frac{\varepsilon^4}{t}(S_* - \varepsilon^2\log t)^{-2}[t^{-1}\|L_+v_+\|_{L^2} + \|v_+\|_{L^2}].$$

Again using that, by (3.1.9), $t^{-1}(S_* - \varepsilon^2\log t)^{-1} = O(1)$, we get finally the upper bound

$$\begin{aligned} &K(N,B)\frac{\varepsilon^4}{t}(S_* - \varepsilon^2\log t)^{-1}\|L_+v_+\|_{L^2} \\ &\quad + K(N,B)\frac{\varepsilon^4}{t}(S_* - \varepsilon^2\log t)^{-2}\|v_+\|_{L^2}, \end{aligned} \tag{4.2.23}$$

which is better than the right-hand side of (4.2.15). Finally, we have to estimate the L^2-norm of the action of L_+ on the last two terms in (3.2.5). Arguing again as in (3.2.14), we have to study

$$\begin{aligned} &\mathrm{Op}(m)(v_{J_1}, u_{J_2}^{\mathrm{app}}, L_{i_3}G_{i_3}), \\ &\mathrm{Op}(m)(v_{J_1}, u_{J_2}^{\mathrm{app}}, G_{i_3}), \\ &tp'(D_x)\mathrm{Op}(m)(v_{J_1}, u_{J_2}^{\mathrm{app}}, G_{i_3}), \\ &t\mathrm{Op}(m)(v_{J_1}, u_{J_2}^{\mathrm{app}}, p'(D_x)G_{i_3}), \end{aligned}$$

for symbols m in $S_{1,0}(M_0^\nu, 3)$ with constant coefficients, $|J_1| + |J_2| = 2$, $G_+ = F$ or r_{app}, $G_- = -\overline{F}$ or $-\overline{r_{\mathrm{app}}}$. Using (A.2.3) with $j = 3$, we bound the L^2-norm of all these terms by

$$(\|v_+\|_{W^{\rho_0,\infty}} + \|u_+^{\mathrm{app}}\|_{W^{\rho_0,\infty}})^2(\|L_+G_+\|_{L^2} + t\|G_+\|_{L^2}). \tag{4.2.24}$$

When $G_+ = F$, since this term is supported for $t \le e^{3S_*/4\varepsilon^2}$, it follows from (4.2.19) and (4.2.14) that this is bounded by

$$K(B,N)\varepsilon^2(\|L_+F(t,\cdot)\|_{L^2} + \|F(t,\cdot)\|_{L^2}).$$

By (3.1.30), the integral in t of that quantity is $O(K(B,N)\varepsilon^{4-\theta})$, so may be incorporated into $\mathcal{R}_L$, satisfying (4.2.16).

When $G_+ = r_{\mathrm{app}}$, we use (4.2.19) and (4.2.14) again to bound (4.2.24) by

$$\begin{aligned} &\Big(K(N)\frac{\varepsilon^2}{t}(S_* - \varepsilon^2\log t)^{-1} + K(B)\frac{\varepsilon^2}{t}\Big)\|L_+r_{\mathrm{app}}(t,\cdot)\|_{L^2} \\ &\quad + K(N,B)\varepsilon^2(S_* - \varepsilon^2\log t)^{-1}\|r_{\mathrm{app}}(t,\cdot)\|_{L^2}. \end{aligned}$$

If we plug (3.1.31) into this inequality, we largely get an estimate in

$$\frac{\varepsilon^2}{t} K(N,B)\varepsilon^{N_0}(S_* - \varepsilon^2 \log t)^{N_0-1},$$

so that we obtain a contribution to $\mathcal{R}_H$ satisfying (4.2.16). Combining this with (4.2.22) and (4.2.23) we get (4.2.15). ∎

Proof of Proposition 4.2.1. We assume a priori inequalities (4.2.1). For $\varepsilon_0 > 0$ small enough, if $\varepsilon < \varepsilon_0$, inequality (4.2.11) holds. Plugging this inequality into the right-hand side of (4.2.15), and also assuming ε_0 small enough so that $K(N,B)\varepsilon^{\theta''} \leq 1$ and $B\varepsilon^{1-\theta} \leq 1$, we get

$$\begin{aligned}
&\|(D_t - p(D_x))L_+ w_+(t,\cdot)\|_{L^2} \\
&\quad \leq \frac{\varepsilon^2}{t} 2[(C_0+1)(S_* - \varepsilon^2 \log t)^{-1} + K(N)]\|L_+ w_+(t,\cdot)\|_{L^2} \\
&\qquad + \frac{\varepsilon^2}{t} K(N)(S_* - \varepsilon^2 \log t)^{-2-2\delta}\|v_+(t,\cdot)\|_{H^{s_0}} + \mathcal{R}_L(t) + \mathcal{R}_H(t). \qquad (4.2.25)
\end{aligned}$$

On the right-hand side of (4.2.25), we plug in the second a priori estimate (4.2.1) and we write the energy inequality associated to (4.2.25), starting from time $t = T(\varepsilon)$ at which $L_+ w_+$ vanishes. We get for $T \leq t \leq T(\varepsilon)$, using also (4.2.16),

$$\begin{aligned}
\|L_+ w_+(t,\cdot)\|_{L^2} \leq &\int_t^{T(\varepsilon)} 2[(C_0+1)(S_* - \varepsilon^2 \log \tau)^{-1} + K(N)]\|L_+ w_+(\tau,\cdot)\|_{L^2}\varepsilon^2 \frac{d\tau}{\tau} \\
&+ K(N,A_0)\varepsilon^{2-\theta} \int_t^{T(\varepsilon)} (S_* - \varepsilon^2 \log \tau)^{N_0-2-2\delta}\varepsilon^2 \frac{d\tau}{\tau} \\
&+ \int_t^{T(\varepsilon)} \|\mathcal{R}_L(\tau)\|_{L^2}\, d\tau \\
&+ K(N,B)\varepsilon^{N_0-2} \int_t^{T(\varepsilon)} (S_* - \varepsilon^2 \log \tau)^{N_0-1}\varepsilon^2 \frac{d\tau}{\tau}. \qquad (4.2.26)
\end{aligned}$$

We set $t = e^{\frac{s}{\varepsilon^2}}$, $\tau = e^{\frac{s'}{\varepsilon^2}}$, $S(\varepsilon) = \varepsilon^2 \log T(\varepsilon)$, $f(s) = \|L_+ w_+(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{L^2}$, and

$$g(s) = \varepsilon^{2-\theta}(K(N,A_0) + \varepsilon^\theta K(N,B))(S_* - s)^{N_1-1} + \|\mathcal{R}_L(e^{\frac{s}{\varepsilon^2}})\|_{L^2}\varepsilon^{-2}e^{\frac{s}{\varepsilon^2}},$$

with $N_1 \leq N_0 - 1 - 2\delta$ and N_1 large enough so that $N_0 \geq 4$. We may thus rewrite (4.2.26) in the form

$$f(s) \leq \int_s^{S(\varepsilon)} \psi(s') f(s')\, ds' + \int_s^{S(\varepsilon)} g(s')\, ds'$$

with

$$\psi(s) = 2(C_0+1)(S_* - s)^{-1} + 2K(N).$$

We may apply estimate (4.1.14), with notation (4.1.13). We obtain, with new constants,

$$\begin{aligned}\|L_+ w_+(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{L^2} \\ \leq \int_s^{S(\varepsilon)} \Big(\frac{S_*-s}{S_*-s'}\Big)^{2(C_0+1)} [\varepsilon^{2-\theta}(K(N,A_0)+\varepsilon^{\theta}K(N,B))(S_*-s')^{N_1-1} \\ + K(N)\|\mathcal{R}_L(e^{\frac{s'}{\varepsilon^2}},\cdot)\|_{L^2}\varepsilon^{-2}e^{\frac{s'}{\varepsilon^2}}]\,ds'.\end{aligned}$$

Since C_0 is a universal constant, we may take N_1 large enough so that $N_1 - 2(C_0+1) > 0$. Moreover, as $\mathcal{R}_L(e^{\frac{s'}{\varepsilon^2}},\cdot)$ is supported for $s' \leq \frac{3S_*}{4}$, $(S_* - s')^{-1}$ stays bounded on the support of that function. Using (4.2.16), we get finally

$$\|L_+ w_+(e^{\frac{s}{\varepsilon^2}},\cdot)\|_{L^2} \leq \varepsilon^{2-\theta}(K(N,A_0)+\varepsilon^{\theta''}K(N,B))(S_*-s)^{N_1},$$

for some constants depending on N, A_0, B and a new $\theta'' > 0$. By (4.2.11) and the second a priori inequality in (4.2.1), we get

$$\|L_+ v_+(t,\cdot)\|_{L^2} \leq \varepsilon^{2-\theta}(K(N,A_0)+\varepsilon^{\theta''}K(N,B))(S_*-\varepsilon^2\log t)^{N_1}$$

for new constants $K(N,A_0)$, $K(N,B)$, using again that $N_0 \geq N_1 + 1 + 2\delta$. We take A_1 large enough so that $K(N,A_0) \leq \frac{A_1}{4}$ and $\varepsilon < \varepsilon_0$ small enough so that $\varepsilon^{\theta''}K(N,B) \leq \frac{A_1}{4}$ in order to obtain (4.2.2). ∎

4.3. Proof of the main theorem

We shall deduce the proof of Theorem 1.2.1 from the preceding subsections. Let us recall how the constants are chosen:

- First one fixes $\theta > 0$ small and δ, δ', γ satisfying (3.1.11), with δ, δ' small. One also fixes $\rho_0 \in \mathbb{N}$ large enough so that the estimates in Proposition A.2.2 hold true (for a fixed large enough ν) and ρ_0 larger than $\tilde{\rho}_0$ in Proposition A.3.1. This ρ_0 is universal and does not depend on any of the constants that we shall introduce in the forthcoming points. It determines the constant C_0 in Lemma 4.1.2.
- Next one chooses $s_0 \in \mathbb{N}$ large enough, such that Proposition A.3.1 holds true and s_0 large enough relative to ρ_0 so that Proposition A.4.1 holds true.
- One takes N_1 large enough as in Proposition 4.2.1. Once N_1 has been chosen, we take N_0 so that Propositions 4.2.1 and 4.1.1 hold true. Once N_1 and N_0 have been fixed, the order N at which one has to construct the approximate solution so that u_+^{app} in Proposition 3.1.3 satisfies (3.1.29)–(3.1.31) is also determined.
- Once N_0 is determined, the constant A_0 is taken large enough in Proposition 4.1.1.
- Once A_0 is fixed, the constant A_1 is determined by Proposition 4.2.1. Next we choose B large enough relative to A_0, A_1 as in (4.3.4) below.
- Finally, ε is taken in $]0,\varepsilon_0]$ for some ε_0 small enough relative to all preceding constants.

Proof of Theorem 1.2.1. To construct the solution u of the theorem, one considers the solution $(u_+, u_- = -\bar{u}_+)$ of the equivalent system (3.1.7), (3.1.8): one looks for u_+ in the form $u_+ = u_+^{\text{app}} + v_+$, where u_+^{app} is defined in (3.1.15) and v_+ satisfies equation (3.1.41). One then wants to solve this equation for v_+ backwards from $t = T(\varepsilon)$, with zero initial data at $t = T(\varepsilon)$, and prove that the solution exists down to time $t = 1$. By local existence theory, there is $T_0 < T(\varepsilon)$ such that the solution exists on $[T_0, T(\varepsilon)]$ and we denote by $T \geq 1$ the infimum of the $\widetilde{T} \geq 1$ such that the solution exists on $[\widetilde{T}, T(\varepsilon)]$ and satisfies for all $t \in [\widetilde{T}, T(\varepsilon)]$ a priori estimates

$$
\begin{aligned}
\|v_+(t,\cdot)\|_{H^{s_0}} &\leq A_0 \varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_0},\\
\|L_+ v_+(t,\cdot)\|_{L^2} &\leq A_1 \varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_1},\\
\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq B \frac{\varepsilon^{2-\theta}}{\sqrt{t}},
\end{aligned}
\tag{4.3.1}
$$

where the parameters $s_0, \rho_0, N_0, N_1, A_0, A_1, B$ are chosen as explained at the beginning of this subsection. If we apply Proposition 4.1.1, we get that it implies that for t in the same interval,

$$
\|v_+(t,\cdot)\|_{H^{s_0}} \leq \frac{A_0}{2} \varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_0} \tag{4.3.2}
$$

if $\varepsilon < \varepsilon_0$ small enough. Then, applying Proposition 4.2.1, we get for $\varepsilon < \varepsilon_0$,

$$
\|L_+ v_+(t,\cdot)\|_{L^2} \leq \frac{A_1}{2} \varepsilon^{2-\theta}(S_* - \varepsilon^2 \log t)^{N_1}. \tag{4.3.3}
$$

By (A.4.4), we deduce from the first two inequalities (4.3.1),

$$
\|v_+(t,\cdot)\|_{W^{\rho_0,\infty}} \leq C \frac{\varepsilon^{2-\theta}}{\sqrt{t}} \left(A_1 + \sqrt{A_0}\sqrt{A_0 + A_1}\right) \leq \frac{B}{2\sqrt{t}} \varepsilon^{2-\theta} \tag{4.3.4}
$$

if B is chosen large enough relative to A_0, A_1.

By the bootstrap (4.3.2), (4.3.3), (4.3.4), we get that the solution v_+ exists on the interval $[1, T(\varepsilon)]$ and satisfies (4.3.1) at any t in that interval. Writing these estimates at $t = 1$, we get from (4.3.1),

$$
\begin{aligned}
\|u_+(1,\cdot) - u_+^{\text{app}}(1,\cdot)\|_{H^{s_0}} &= O(\varepsilon^{2-\theta}),\\
\|x(u_+(1,\cdot) - u_+^{\text{app}}(1,\cdot))\|_{L^2} &= O(\varepsilon^{2-\theta}).
\end{aligned}
$$

By (3.1.15) and recalling that χ_1 vanishes close to zero, we get for small enough ε,

$$
\begin{aligned}
\|u_+(1,\cdot) - \tilde{u}_+^{\text{app}}(1,\cdot)\|_{H^{s_0}} &= O(\varepsilon^{2-\theta}),\\
\|x(u_+(1,\cdot) - \tilde{u}_+^{\text{app}}(1,\cdot))\|_{L^2} &= O(\varepsilon^{2-\theta}).
\end{aligned}
$$

The definition (3.1.4) (resp. (3.1.13)) of u_+ (resp. $\tilde{u}_+^{\text{app}}$) from u (resp. u_{app}) and the fact that u, u_{app} are real-valued functions imply

$$
\begin{aligned}
\|u(1,\cdot) - u_{\text{app}}(1,\cdot)\|_{H^{s_0+1}} + \|D_t u(1,\cdot) - D_t u_{\text{app}}(1,\cdot)\|_{H^{s_0}} &= O(\varepsilon^{2-\theta}),\\
\|x(u(1,\cdot) - u_{\text{app}}(1,\cdot))\|_{H^1} + \|x(D_t u(1,\cdot) - D_t u_{\text{app}}(1,\cdot))\|_{L^2} &= O(\varepsilon^{2-\theta}).
\end{aligned}
\tag{4.3.5}
$$

By (2.2.38), (2.1.29), and (2.1.2), $(u_{\mathrm{app}}(1,\cdot), \partial_t u_{\mathrm{app}}(1,\cdot))$ are the initial conditions $(\varepsilon f_0, \varepsilon g_0)$ chosen in the statement of the theorem so that (1.2.3) holds. Thus (4.3.5) shows that the initial conditions of our solution u have structure (1.2.7), with perturbation $(f(x,\varepsilon), g(x,\varepsilon))$ satisfying (1.2.6).

It remains to prove (1.2.8). At time $t = T(\varepsilon)$, the value of u (resp. $\partial_t u$) is given by $u_{\mathrm{app}}(T(\varepsilon),\cdot)$ (resp. $\partial_t u_{\mathrm{app}}(T(\varepsilon),\cdot)$). By (2.2.38), these quantities are equal to $u^2_{\mathrm{app}}(T(\varepsilon),\cdot)$ (resp. $\partial_t u^2_{\mathrm{app}}(T(\varepsilon),\cdot)$) with u^2_{app} given by (2.2.2). All contributions corresponding to $\ell \geq 3$ in (2.2.2), as well as their derivatives, have modulus bounded from above by

$$\begin{aligned}&\varepsilon^{2-\ell} T(\varepsilon)^{-\frac{1}{2}} (S_* - \varepsilon^2 \log T(\varepsilon) + |y - y_0|^{2\kappa_0})^{-\frac{1}{2}} \\ &\quad \times \big(T(\varepsilon)^{-\frac{1}{2}} (S_* - \varepsilon^2 \log T(\varepsilon))^{-\frac{1}{2}-\delta}\big)^{\ell-1}. \end{aligned} \tag{4.3.6}$$

Since $\ell \geq 3$, (3.1.9) implies a bound in

$$T(\varepsilon)^{-\frac{1}{2}} (S_* - \varepsilon^2 \log T(\varepsilon))^{-\frac{1}{2}} \big(\varepsilon^{\frac{\gamma}{2}-\delta'} (S_* - \varepsilon^2 \log T(\varepsilon))^{\delta'-\delta}\big)^{\ell-1} \varepsilon^{2-\ell}. \tag{4.3.7}$$

By (1.2.5), (1.2.4), $S_* - \varepsilon^2 \log T(\varepsilon) = \varepsilon^2 u(\varepsilon')$ is exponentially small in $e^{-\frac{c}{\varepsilon^2}}$, so that since $\delta' > \delta$ (4.3.7), and thus all terms with $\ell \geq 3$ in (2.2.2) computed at $t = T(\varepsilon)$, are negligible relative to $\varepsilon T(\varepsilon)^{-\frac{1}{2}} (S_* - \varepsilon^2 \log T(\varepsilon))^{-\frac{1}{2}}$. On the other hand, by (2.1.10) and (1.2.3), (1.2.2), the coefficient of $\frac{\varepsilon}{\sqrt{t}} e^{it\varphi(x/t)}$ in (2.2.2), computed at $t = T(\varepsilon)$, $\frac{x}{t} = y_0$ satisfies

$$|a_{1,1}(\varepsilon^2 \log T(\varepsilon), y_0)| = |a_1^0(y_0)| \Big(1 - \frac{\varepsilon^2 \log T(\varepsilon)}{S_*}\Big)^{-\frac{1}{2}}$$

since $\Gamma(y_0)\phi(y_0) = S_*^{-1}$ by (1.2.2), (1.2.3). Moreover, since $|a_1^0(y_0)| = (1 - y_0^2)^{-\frac{1}{4}} \times \Gamma(y_0)^{\frac{1}{2}}$ by (2.1.6), (1.2.1), with $\Gamma(y_0) \neq 0$ by (1.2.2), we get that all terms with $\ell \geq 3$ in (2.2.2) at time $T(\varepsilon)$ are $o(\varepsilon T(\varepsilon)^{-\frac{1}{2}} |a_{1,1}(\varepsilon^2 \log T(\varepsilon), y_0)|)$. We conclude that the main contribution to (2.2.2) at time $t = T(\varepsilon)$ and $x = y_0 T(\varepsilon)$ is

$$2\,\mathrm{Re}[\varepsilon T(\varepsilon)^{-\frac{1}{2}} e^{iT(\varepsilon)\varphi(y_0)} a_{1,1}(\varepsilon^2 \log T(\varepsilon), y_0)]$$

and its time derivative is

$$2\,\mathrm{Re}[\varepsilon T(\varepsilon)^{-\frac{1}{2}} i\omega(y_0) e^{iT(\varepsilon)\varphi(y_0)} a_{1,1}(\varepsilon^2 \log T(\varepsilon), y_0)].$$

Thus,

$$\begin{aligned}&|u^2_{\mathrm{app}}(T(\varepsilon), y_0 T(\varepsilon))| + |\partial_t u^2_{\mathrm{app}}(T(\varepsilon), y_0 T(\varepsilon))| \\ &\quad \sim \varepsilon T(\varepsilon)^{-\frac{1}{2}} |a_{1,1}(\varepsilon^2 \log T(\varepsilon), y_0)| \\ &\quad \sim \varepsilon T(\varepsilon)^{-\frac{1}{2}} (S_* - \varepsilon^2 \log T(\varepsilon))^{-\frac{1}{2}} \\ &\quad \sim T(\varepsilon)^{-\frac{1}{2}} u(\varepsilon')^{-\frac{1}{2}} \sim T(\varepsilon)^{-\frac{1}{2}} \varepsilon'^{-\frac{1}{2}} \end{aligned} \tag{4.3.8}$$

by (1.2.5). If $c > 0$ is given and if δ' in (1.2.4) is taken small enough with respect to c, one has $\varepsilon' \leq e^{-\frac{S_*}{\varepsilon^2}(1-2c)} \sim T(\varepsilon)^{-1+2c}$. Thus (4.3.8) is bounded from below by $T(\varepsilon)^{-c}$ which gives the first equality (1.2.8).

To get the second one, we proceed in the same way, except that we have to estimate from below

$$\frac{\varepsilon}{\sqrt{T(\varepsilon)}}\Big\|a_{1,1}\Big(\varepsilon^2\log T(\varepsilon),\frac{x}{T(\varepsilon)}\Big)\Big\|_{L^2(dx)} = \varepsilon\|a_{1,1}(\varepsilon^2\log T(\varepsilon),y)\|_{L^2(dy)}.$$

By (1.2.3), the example following Definition 2.2.1 and the expression (2.1.10) of $a_{1,1}$, one has

$$|a_{1,1}(s,y)| \sim (S_* - s + |y-y_0|^{2\kappa_0})^{-\frac{1}{2}},$$

so that if $\kappa_0 > 0$,

$$\|a_{1,1}(\varepsilon^2\log T(\varepsilon),y)\|_{L^2(dy)} \sim (S_* - \varepsilon^2\log T(\varepsilon))^{-\frac{1}{2}+\frac{1}{4\kappa_0}}.$$

We have seen above that $(S_* - \varepsilon^2\log T(\varepsilon))^{-1} \geq T(\varepsilon)^{1-2c}\varepsilon^{-2}$. Then the second inequality (1.2.8) for the $a_{1,1}$ term in (2.2.2) with the lower bound (1.2.9) follows from that, up to changing the definition of c. Since the contributions to (2.2.2) indexed by $\ell \geq 3$ are bounded pointwise by (4.3.6) and thus by $T(\varepsilon)^{-\frac{1}{2}}|a_{1,1}(\varepsilon^2\log T(\varepsilon),\frac{x}{T(\varepsilon)})|e^{-\frac{c}{\varepsilon^2}}$ for some $c > 0$, as seen after (4.3.7), they are negligible perturbations, so that (1.2.8), (1.2.9) hold for $u^2_{\mathrm{app}}(T(\varepsilon),\cdot)$. This concludes the proof. ∎

A. Calculus and estimates for pseudo-differential and multilinear operators

A.1. Pseudo-differential operators

In this subsection we prove several results on pseudo-differential operators used in the bulk of the proof.

Definition A.1.1. Let $p(x,\xi)$ be a smooth function on $\mathbb{R}\times\mathbb{R}$, satisfying for some $\mu\in\mathbb{R}$ and all α,β in $\mathbb{N}$,

$$|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{\mu-|\beta|}. \tag{A.1.1}$$

Then if $u\in\mathcal{S}(\mathbb{R})$, we set

$$p(x,D_x)u = \frac{1}{2\pi}\int e^{ix\xi}p(x,\xi)\hat{u}(\xi)\,d\xi, \tag{A.1.2}$$

and if $h\in\,]0,1]$ is a semi-classical parameter, we set

$$\begin{aligned} p(x,hD_x)v &= \frac{1}{2\pi}\int e^{ix\xi}p(x,h\xi)\hat{v}(\xi)\,d\xi \\ &= \frac{1}{2\pi h}\int e^{i\frac{(x-y)\xi}{h}}p(x,\xi)v(y)\,dy\,d\xi, \end{aligned}$$

where the last integral is an oscillatory one. This is related to (A.1.2) by the conjugation formula

$$\Theta_h^{-1}\circ p(x,hD_x)\circ\Theta_h = p(hx,D_x) \tag{A.1.3}$$

if we define

$$(\Theta_h u)(x) = \frac{1}{\sqrt{h}} u\Big(\frac{x}{h}\Big). \tag{A.1.4}$$

We want to study the action of $p(D_x)$ on oscillating expressions of the form used to construct an approximate solution in Section 2. First we define the following:

Definition A.1.2. Let $x_0 \in]-1,1[$, $\kappa_0 \in \mathbb{N}$. For $m \in \mathbb{R}$, we denote by $\widetilde{\Sigma}^m$ the space of continuous functions

$$\begin{gathered}(x,\lambda,h,\varepsilon) \to \sigma(x,\lambda,h,\varepsilon),\\ \mathbb{R}\times[1,+\infty[\times]0,1]\times[0,1] \to \mathbb{C},\end{gathered}$$

smooth in (x,λ,h), supported for $|x|\le 1$, that satisfy for any α, β, ζ, N in $\mathbb{N}$, any $(x,\lambda,h,\varepsilon)$ in $[-1,1]\times[1,+\infty[\times]0,1]\times[0,1]$,

$$\begin{aligned}&|\partial_x^\alpha\partial_\lambda^\beta(h\partial_h)^\zeta\sigma(x,\lambda,h,\varepsilon)|\\ &\qquad\le C_{\alpha,\beta,\zeta}\lambda^{-m+\alpha-\beta}(1+\lambda|x-x_0|)^{m-\alpha-2\kappa_0\beta}(1-|x|)^N.\end{aligned} \tag{A.1.5}$$

Let $\psi\colon]-1,1[\to\mathbb{R}$ be a smooth function such that for some $A\in\mathbb{R}_+$ and any $\alpha\in\mathbb{N}$,

$$|\partial_x^\alpha\psi(x)| \le C_\alpha(1-|x|)^{-A-|\alpha|} \quad \text{for all } x\in]-1,1[. \tag{A.1.6}$$

Finally, let $\xi\to p(\xi)$ be a symbol independent of x, satisfying (A.1.1).

Proposition A.1.3. *Let σ be in $\widetilde{\Sigma}^m$. Then for any $(x,\lambda,h,\varepsilon)$ satisfying $\lambda h\le 1$, we have*

$$\begin{aligned}p(hD_x)[e^{\frac{i}{h}\psi(x)}\sigma(x,\lambda,h,\varepsilon)] &= p(d\psi(x))\sigma(x,\lambda,h,\varepsilon)e^{\frac{i}{h}\psi(x)}\\ &\quad + h\sigma_1(x,\lambda,h,\varepsilon)e^{\frac{i}{h}\psi(x)} + \underline{r}(x,\lambda,h,\varepsilon),\end{aligned} \tag{A.1.7}$$

where $\sigma_1\in\widetilde{\Sigma}^{m-1}$ and where $\underline{r}$ is a continuous function on $\mathbb{R}\times[1,+\infty[\times]0,1]\times[0,1]$, smooth in (x,λ,h), satisfying for any $\alpha,\beta,\zeta,N\in\mathbb{N}$,

$$|\partial_x^\alpha\partial_\lambda^\beta(h\partial_h)^\zeta\underline{r}(x,\lambda,h,\varepsilon)| \le Ch^N(1+|x|)^{-N}. \tag{A.1.8}$$

Proof. The left-hand side of (A.1.7) is

$$\frac{1}{2\pi h}\int e^{\frac{i}{h}[(x-y)\xi+\psi(y)]}p(\xi)\sigma(y,\lambda,h,\varepsilon)\,dy\,d\xi. \tag{A.1.9}$$

Let $(x,y)\to\theta(x,y)$ be a smooth function on $\mathbb{R}\times]-1,1[$, supported for $|x-y|\ll 1-|y|$, such that for any α, β,

$$|\partial_x^\alpha\partial_y^\beta\theta(x,y)| \le C(1-|y|)^{-\alpha-\beta}. \tag{A.1.10}$$

Assume also that $\theta(x,y)=1$ if $|x-y|\le c(1-|y|)$ for some small $c>0$. If we insert the cut-off $1-\theta$ under the integral, and make N' integrations by parts in ξ, we get an integrand bounded by

$$Ch^{N'}\lambda^{-m}(1+\lambda|y-x_0|)^m(1-|y|)^{N-N'}\langle\xi\rangle^{\mu-N'}(1+|x-y|)^{-N'},$$

by (A.1.1), (A.1.5). If we make $h\partial_h$ act on (A.1.9), we also get a similar bound, with a different N', using (A.1.6) as well. We thus see that (A.1.9) with the cut-off $1-\theta$ under the integral brings a contribution to $\underline{r}$ in (A.1.7), using that by assumption $\lambda = O(1/h)$ in order to control any positive power of λ, like those coming from ∂_x-derivatives. We are thus reduced to

$$\frac{1}{2\pi h}\int e^{\frac{i}{h}[(x-y)\xi+\psi(y)]}\theta(x,y)p(\xi)\sigma(y,\lambda,h,\varepsilon)\,dy\,d\xi. \tag{A.1.11}$$

Define

$$\psi_1(x,y) = \int_0^1 \psi'(\tau y + (1-\tau)x)\,d\tau.$$

As on the support of θ, $1-|x| \sim 1-|y|$, we see using (A.1.6) that for $\theta(x,y) \neq 0$,

$$|\partial_x^\alpha \partial_y^\beta \psi_1(x,y)| \leq C_{\alpha,\beta}(1-|y|)^{-A-1-\alpha-\beta} \tag{A.1.12}$$

and $\psi(y) = \psi(x) - \psi_1(x,y)(x-y)$, so that (A.1.11) may be written

$$\frac{1}{2\pi h}e^{i\frac{\psi(x)}{h}}\int e^{\frac{i}{h}(x-y)\eta}\theta(x,y)p(\eta+\psi_1(x,y))\sigma(y,\lambda,h,\varepsilon)\,dy\,d\eta. \tag{A.1.13}$$

Inside this integral, we decompose

$$p(\eta+\psi_1(x,y)) = p(\psi_1(x,y)) + \eta q(x,y,\eta), \tag{A.1.14}$$

where $q(x,y,\eta) = \int_0^1 p'(\psi_1(x,y)+\tau\eta)\,d\tau$ satisfies, according to (A.1.1), (A.1.12) and for (x,y) staying in the support of θ, bounds of the form

$$|\partial_x^\alpha \partial_y^\beta \partial_\eta^\gamma (\eta\partial_\eta)^\zeta q(x,y,\eta)| \leq C\langle\eta\rangle^{\max(\mu-1,0)}(1-|y|)^{-K_{\alpha,\beta,\gamma,\zeta}} \tag{A.1.15}$$

for some positive exponents $K_{\alpha,\beta,\gamma,\zeta}$. We substitute (A.1.14) into (A.1.13). The first term on the right-hand side of (A.1.14) gives the first term on the right-hand side of (A.1.7). Consider next the term in (A.1.13) coming from the last term in (A.1.14), i.e. the product of $e^{i\frac{\psi(x)}{h}}$ and

$$-\frac{i}{2\pi}\int e^{\frac{i}{h}(x-y)\eta}\tilde{q}(x,y,\eta,\lambda,h,\varepsilon)\,dy\,d\eta, \tag{A.1.16}$$

with

$$\tilde{q}(x,y,\eta,\lambda,h,\varepsilon) = \partial_y[\theta(x,y)q(x,y,\eta)\sigma(y,\lambda,h,\varepsilon)].$$

It follows from (A.1.5), (A.1.10), (A.1.15) that

$$\begin{aligned}&|\partial_y^{\alpha'}\partial_\eta^\gamma(\eta\partial_\eta)^{\gamma'}(h\partial_h)^\zeta\tilde{q}(x,y,\eta,\lambda,h,\varepsilon)|\\ &\quad\leq C\lambda^{-m+1+\alpha'}(1+\lambda|y-x_0|)^{m-1}(1-|y|)^N\langle\eta\rangle^{\max(\mu-1,0)}\end{aligned}$$

for any $\alpha',\gamma,\gamma',\zeta,N$. Inside integral (A.1.16) we perform integrations by parts using the operator $\langle\frac{\eta}{\lambda h}\rangle^{-2}(1-\frac{\eta}{\lambda^2 h}D_y)$. We shall obtain a new expression

$$\int e^{\frac{i}{h}(x-y)\eta}\tilde{q}_1(x,y,\eta,\lambda,h,\varepsilon)\,dy\,d\eta, \tag{A.1.17}$$

where $\tilde{q}_1$ satisfies since $\lambda h \leq 1$ and since $1-|x| \sim 1-|y|$ on the support of $\theta(x,y)$,

$$|\partial_y^{\alpha'}\partial_\eta^{\gamma}(\eta\partial_\eta)^{\gamma'}(h\partial_h)^{\zeta}\tilde{q}_1(x,y,\eta,\lambda,h,\varepsilon)|$$
$$\leq C\lambda^{-m+1+\alpha'}(\lambda h)^{-\gamma}\Big\langle\frac{\eta}{\lambda h}\Big\rangle^{-N_0}(1+\lambda|y-x_0|)^{m-1}(1-|x|)^N\langle\eta\rangle^{\max(\mu-1,0)} \quad \text{(A.1.18)}$$

for an arbitrary large N_0. We perform next integrations by parts in (A.1.17) using

$$\langle\lambda(x-y)\rangle^{-2}(1+\lambda^2 h(x-y)\cdot D_\eta).$$

It follows from (A.1.18) that the modulus of (A.1.17) is bounded by

$$C\lambda^{-m+1}\int\Big\langle\frac{\eta}{\lambda h}\Big\rangle^{-N_0}\langle\lambda(x-y)\rangle^{-N_0}(1+\lambda|y-x_0|)^{m-1}$$
$$\times\langle\eta\rangle^{\max(\mu-1,0)}\,dy\,d\eta(1-|x|)^N. \quad \text{(A.1.19)}$$

Since $\lambda h \leq 1$, the modulus of (A.1.19) is $O(h\lambda^{-m+1}(1+\lambda|x-x_0|)^{m-1}(1-|x|)^N)$.

If we make a ∂_x-derivative act on the integral in (A.1.13), one ∂_y-integration by parts together with (A.1.12) and estimates (A.1.5), (A.1.15) shows that we get the same estimates as in (A.1.15)–(A.1.19), with m replaced by $m-1$. In the same way, a ∂_λ-derivative acting on the integral gives rise to an extra factor $\lambda^{-1}(1+\lambda|y-x_0|)^{-2\kappa_0}$, which induces in the estimates of (A.1.19) a corresponding factor $\lambda^{-1}(1+\lambda|x-x_0|)^{-2\kappa_0}$.

Finally, an $h\partial_h$-derivative acting on the exponential in (A.1.13) may be traded off against an $\eta\partial_\eta$-derivative, so that by integration by parts, the final expression (A.1.19) still has the same estimates. We thus see that (A.1.13) with $p(\eta+\psi_1(x,y))$ replaced by $\eta q(x,y,\eta)$ may be written as the second term on the right-hand side of (A.1.7). This concludes the proof. ■

We shall translate Proposition A.1.3 on the class of symbols Σ^m introduced in Definition 2.2.1. Notice that if a belongs to Σ^m and if, for $s \in [0, S_*[$, we set

$$\lambda = (S_*-s)^{-\frac{1}{2\kappa_0}}, \quad s = S_* - \lambda^{-2\kappa_0}, \quad \text{(A.1.20)}$$

then for $a(s,y,h,\varepsilon)$ in Σ^m, the function

$$\sigma(y,\lambda,h,\varepsilon) = a(S_*-\lambda^{-2\kappa_0}, y, h, \varepsilon) \quad \text{(A.1.21)}$$

satisfies, by (2.2.1),

$$|\sigma(y,\lambda,h,\varepsilon)| \leq C\lambda^{-2\kappa_0 m}(1+\lambda|y-x_0|)^{2\kappa_0 m}(1-|y|)^N$$

for any N, i.e. bound (A.1.5) with $\alpha=\beta=\zeta=0$ and m replaced by $\tilde{m} = 2\kappa_0 m$. If we take a ∂_y-derivative of σ, we get in the same way estimate (A.1.5) with $\tilde{m} = 2\kappa_0 m$ and $\alpha = 1$. One checks similarly that ∂_λ, $h\partial_h$ derivatives acting on (A.1.21) give rise to similar bounds for (A.1.5). In other words, with definition (A.1.21) of σ in terms of a, we have the equivalence

$$a \in \Sigma^m \Leftrightarrow \sigma \in \tilde{\Sigma}^{\tilde{m}} \quad \text{with } \tilde{m} = 2\kappa_0 m. \quad \text{(A.1.22)}$$

Corollary A.1.4. *Let $p(\xi)$ be a function independent of x and satisfying (A.1.1). Let $(s,y,h,\varepsilon)\to a(s,y,h,\varepsilon)$ be an element of Σ^m defined as in Definition 2.2.1 for some m in $\mathbb{R}$. Let ψ be a real phase function defined on $]-1,1[$ satisfying (A.1.6). Then if $t(S_*-s)^{\frac{1}{2\kappa_0}}\geq c>0$, we have*

$$p(D_x)\Big[e^{it\psi(\frac{x}{t})}a\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)\Big]=e^{it\psi(\frac{x}{t})}p\Big(\psi'\Big(\frac{x}{t}\Big)\Big)a\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)+\frac{1}{t}e^{it\psi(\frac{x}{t})}a_1\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)+r\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big),\quad\text{(A.1.23)}$$

where $a_1\in\Sigma^{m-\frac{1}{2\kappa_0}}\subset\Sigma^{m-1}$ and r satisfies

$$|\partial_s^\alpha\partial_y^\beta(h\partial_h)^\zeta r(s,y,h,\varepsilon)|\leq C_N h^N(1+|y|)^{-N}\quad\text{(A.1.24)}$$

for all α,β,ζ,N.

Proof. If we set $h=\frac{1}{t}$, we have according to (A.1.4),

$$a\Big(s,\frac{x}{t},h,\varepsilon\Big)=\sqrt{t}(\Theta_h^{-1}a)(s,x,h,\varepsilon),\quad\text{(A.1.25)}$$

so that the left-hand side of (A.1.23) may be written according to (A.1.3) as

$$\frac{1}{\sqrt{h}}\Theta_h^{-1}\big[p(hD_x)[e^{i\frac{\psi(x)}{h}}a(s,x,h,\varepsilon)]\big].\quad\text{(A.1.26)}$$

We notice that if λ is defined by (A.1.20), the assumption $\lambda h\leq c$ of Proposition A.1.3 is equivalent to the condition $t(S_*-s)^{\frac{1}{2\kappa_0}}\geq c^{-1}$ that we impose in the corollary. If we apply Proposition A.1.3 to the symbol σ defined from a by (A.1.21), we deduce that (A.1.26) is equal to

$$\frac{1}{\sqrt{h}}\Theta_h^{-1}[p(d\psi(x))e^{i\frac{\psi(x)}{h}}a(s,x,h,\varepsilon)+he^{i\frac{\psi(x)}{h}}\sigma_1(x,\lambda,h,\varepsilon)+\underline{r}(x,\lambda,h,\varepsilon)]$$

for some element $\sigma_1\in\widetilde{\Sigma}^{\widetilde{m}-1}$ with $\widetilde{m}=2\kappa_0 m$ by (A.1.22). We denote by $a_1\in\Sigma^{m-\frac{1}{2\kappa_0}}$ the symbol associated to σ_1 by (A.1.22), so that by (A.1.25), we obtain (A.1.23) with $r(s,x,h,\varepsilon)=\underline{r}(x,\lambda,h,\varepsilon)$ that satisfies (A.1.24) by (A.1.8), (A.1.20) and the fact that $\lambda(s)h\leq 1$. This concludes the proof. ∎

Corollary A.1.5. *Denote $\varphi(x)=\sqrt{1-x^2}$ for $|x|<1$, and set $p(\xi)=\sqrt{1+\xi^2}$. Let m be an element of Σ^m and q be in $\mathbb{Z}$. We have*

$$\begin{aligned}(x+tp'(D_x))\Big[e^{itq\varphi(x/t)}a\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)\Big]&\\=x\Big(1-\frac{q}{\sqrt{1+(q^2-1)(x/t)^2}}\Big)e^{itq\varphi(x/t)}a\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)&\\+e^{itq\varphi(x/t)}a_1\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)+r\Big(s,\frac{x}{t},\frac{1}{t},\varepsilon\Big)&\quad\text{(A.1.27)}\end{aligned}$$

for some $a_1\in\Sigma^{m-\frac{1}{2\kappa_0}}\subset\Sigma^{m-1}$ and r satisfying (A.1.24) (with $h=\frac{1}{t}$).

Proof. We just apply (A.1.23), noticing that $y + p'(q\varphi'(y)) = y(1 - \frac{q}{\sqrt{1+(q^2-1)y^2}})$ by the definitions of p, φ. ∎

A.2. Properties of multilinear operators

We gather here some properties of multilinear operators that we use in the bulk of the paper. Some of them follow from the appendices in [14].

Lemma A.2.1. *Let $m_1 \in S_{1,0}(M^{\nu_1}, p)$, $m_2 \in S_{1,0}(M^{\nu_2}, q)$ for some $p, q \in \mathbb{N}^*$, some $\nu_1, \nu_2 \in \mathbb{N}$, with the notation introduced in Definition 3.1.1. Assume moreover that m_1, m_2 have constant coefficients. Then there is m in $S_{1,0}(M_0^{\nu_1+\nu_2}, p+q-1)$ such that*

$$\operatorname{Op}(m_1)(u_1, \dots, u_{p-1}, \operatorname{Op}(m_2)(u_p, \dots, u_{p+q-1})) = \operatorname{Op}(m)(u_1, \dots, u_{p+q-1}) \quad \text{(A.2.1)}$$

for any functions $u_1, \dots, u_{p+q-1}$.

Proof. Equality (A.2.1) follows from (3.1.3) setting

$$m(\xi_1, \dots, \xi_{p+q-1}) = m_1(\xi_1, \dots, \xi_{p-1}, \xi_p + \dots + \xi_{p+q-1}) m_2(\xi_p, \dots, \xi_{p+q-1}).$$

The conclusion follows from

$$\begin{aligned} &M_0(\xi_1, \dots, \xi_{p-1}, \xi_p + \dots + \xi_{p+q-1})^{\nu_1} M_0(\xi_p, \dots, \xi_{p+q-1})^{\nu_2} \\ &\quad \le C M_0(\xi_1, \dots, \xi_{p+q-1})^{\nu_1+\nu_2} \end{aligned}$$

since $M_0(\xi_1, \dots, \xi_n)$ is equivalent to the second largest among $|\xi_1| + 1, \dots, |\xi_n| + 1$. ∎

We recall some results about boundedness properties of operators associated to symbols in the class $S_{\kappa,0}(M_0^\nu, p)$ from [14]. Recall that we defined

$$\|u\|_{W^{\rho,\infty}} = \|\langle D_x\rangle^\rho w\|_{L^\infty}.$$

Then, by [14, Proposition D.1.1] (applied with $h = 1$ and to symbols independent of x, y, with the notation of that reference), we have the following proposition:

Proposition A.2.2. *Let $n \in \mathbb{N}^*$, $\kappa \in \mathbb{N}$, $\nu \ge 0$. There is $\rho_0 \in \mathbb{N}$ such that for any $m \in S_{\kappa,0}(M_0^\nu, n)$, independent of x, the following estimates hold for any $s \in \mathbb{N}$, any $v_1, \dots, v_n$:*

$$\|\operatorname{Op}(m)(v_1, \dots, v_n)\|_{H^s} \le C_s \sum_{j=1}^n \Big(\prod_{\ell \ne j} \|v_\ell\|_{W^{\rho_0,\infty}}\Big) \|v_j\|_{H^s}, \quad \text{(A.2.2)}$$

and moreover, for any fixed j in $\{1, \dots, n\}$,

$$\|\operatorname{Op}(m)(v_1, \dots, v_n)\|_{L^2} \le C \Big(\prod_{\ell \ne j} \|v_\ell\|_{W^{\rho_0,\infty}}\Big) \|v_j\|_{L^2}, \quad \text{(A.2.3)}$$

$$\|\operatorname{Op}(m)(v_1, \dots, v_n)\|_{H^s} \le C_s \Big(\prod_{\ell \ne j} \|v_\ell\|_{W^{\rho_0+s,\infty}}\Big) \|v_j\|_{H^s}. \quad \text{(A.2.4)}$$

If one assumes in addition that m is supported for $|\xi_1| + \cdots + |\xi_{n-1}| \leq C(1 + |\xi_n|)$ *for some constant* C*, one gets instead of* (A.2.2)*,*

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \leq C_s \Big(\prod_{\ell=1}^{n-1} \|v_\ell\|_{W^{\rho_0,\infty}} \Big) \|v_n\|_{H^s}, \tag{A.2.5}$$

and for any $j < n$*,*

$$\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \leq C_s \|v_j\|_{L^2} \Big(\prod_{\substack{\ell \neq j \\ \ell \neq n}} \|v_\ell\|_{W^{\rho_0,\infty}} \Big) \|v_n\|_{W^{\rho_0+s,\infty}}. \tag{A.2.6}$$

Without the support condition on m, we get instead, for any $1 \leq \ell \leq n-1$*,*

$$\begin{aligned} &\|\mathrm{Op}(m)(v_1, \dots, v_n)\|_{H^s} \\ &\quad \leq C_s \Big[\sum_{j=1}^{\ell} \Big(\prod_{\ell' \neq j} \|v_{\ell'}\|_{W^{\rho_0,\infty}} \Big) \|v_j\|_{H^s} \\ &\qquad + \sum_{j=\ell+1}^{n} \sum_{j'=1}^{\ell} \|v_j\|_{W^{\rho_0+s,\infty}} \|v_{j'}\|_{L^2} \prod_{\substack{\ell' \neq j,j' \\ 1 \leq \ell' \leq n}} \|v_{\ell'}\|_{W^{\rho_0,\infty}} \Big]. \end{aligned} \tag{A.2.7}$$

Finally, inequality (A.2.3) *holds also for* x*-dependent symbols in* $S_{\kappa,\beta}(M_0^\nu, n)$ *for any* $\kappa \geq 0$, $\beta \geq 0$.

Proof. Estimates (A.2.2) and (A.2.3) are inequalities (D.6) and (D.7) of [14, Proposition D.1.1]. Inequality (A.2.4) follows from (A.2.3) if we make s ∂_x-derivatives act on $\mathrm{Op}(m)(v_1, \dots, v_n)$ and use the Leibniz rule. In addition, (A.2.3) holds for general symbols in $S_{\kappa,\beta}(M_0^\nu, n)$ by [14, Proposition D.1.1 (iii)]. Estimate (A.2.5) is just [14, inequality (D.5)]. Let us prove (A.2.6) when $j = 1$ for instance. Using the support property of m, we may write for any $\alpha \in \mathbb{N}$, $\alpha \leq s$,

$$\partial_x^\alpha \mathrm{Op}(m)(v_1, \dots, v_n) = \mathrm{Op}(\widetilde{m})(v_1, \dots, v_{n-1}, \langle D_x \rangle^s v_n)$$

for another symbol $\widetilde{m}$ in $S_{\kappa,0}(M_0^\nu, n)$. Applying (A.2.3) we get (A.2.6).

To prove (A.2.7), we decompose

$$m(\xi_1, \dots, \xi_n) = \sum_{j=1}^{n} m_j(\xi_1, \dots, \xi_n),$$

where m_j is in $S_{\kappa,0}(M_0^\nu, n)$ and supported for $|\xi_1| + \cdots + \widehat{|\xi_j|} + \cdots + |\xi_n| \leq C(1 + |\xi_j|)$. For $1 \leq j \leq \ell$, we apply (A.2.5) with n replaced by j to bound $\|\mathrm{Op}(m_j)(v_1, \dots, v_n)\|_{H^s}$ by the first sum on the right-hand side of (A.2.7). For $\ell + 1 \leq j \leq n$ we bound the Sobolev norm $\|\mathrm{Op}(m_j)(v_1, \dots, v_n)\|_{H^s}$ using (A.2.6) with (j, n) replaced by (j', j). This concludes the proof. ∎

A.3. Action of L_+ on characteristic cubic expressions

Consider m an element of $S_{1,0}(M_0, 3)$ with constant coefficients, with the notation introduced in Definition 3.1.1. Let $I = (i_1, i_2, i_3)$ be a characteristic index, i.e. an element of $\{-1, 1\}^3$ with $i_1 + i_2 + i_3 = 1$. The goal of this subsection is to obtain L^2 estimates for the action of L_+ on a characteristic cubic term.

Proposition A.3.1. *There are integers $\tilde{\rho}_0$, $\tilde{s}_0$ in $\mathbb{N}$ such that for any functions w_1, w_2, w_3 the following estimate holds true:*

$$\|L_+ \mathrm{Op}(m)(w_1, w_2, w_3)\|_{L^2} \leq C \sum_{\ell=1}^{3} (\|L_{i_\ell} w_\ell\|_{L^2} + \|w_\ell\|_{H^{\tilde{s}_0}}) \prod_{\substack{1 \leq j \leq 3 \\ j \neq \ell}} \|w_j\|_{W^{\tilde{\rho}_0,\infty}}. \tag{A.3.1}$$

In addition, one has also the bounds

$$\begin{aligned} &\|L_+ \mathrm{Op}(m)(w_1, w_2, w_3)\|_{L^2} \\ &\quad \leq C(\|L_{i_1} w_1\|_{L^2} + \|w_1\|_{H^{\tilde{s}_0}}) \|w_2\|_{W^{\tilde{\rho}_0,\infty}} \|w_3\|_{W^{\tilde{\rho}_0,\infty}} \\ &\qquad + C \|w_1\|_{W^{\tilde{\rho}_0,\infty}} (\|L_{i_2} w_2\|_{L^2} + \|w_2\|_{H^{\tilde{s}_0}}) \|w_3\|_{W^{\tilde{\rho}_0,\infty}} \\ &\qquad + C \|w_1\|_{L^2} \|w_2\|_{W^{\tilde{\rho}_0,\infty}} (\|L_{i_3} w_3\|_{W^{\tilde{\rho}_0,\infty}} + \|w_3\|_{W^{\tilde{\rho}_0,\infty}}) \end{aligned} \tag{A.3.2}$$

and

$$\begin{aligned} &\|L_+ \mathrm{Op}(m)(w_1, w_2, w_3)\|_{L^2} \\ &\quad \leq C(\|L_{i_1} w_1\|_{L^2} + \|w_1\|_{H^{\tilde{s}_0}}) \|w_2\|_{W^{\tilde{\rho}_0,\infty}} \|w_3\|_{W^{\tilde{\rho}_0,\infty}} \\ &\qquad + C \|w_1\|_{L^2} \|w_2\|_{W^{\tilde{\rho}_0,\infty}} (\|L_{i_3} w_3\|_{W^{\tilde{\rho}_0,\infty}} + \|w_3\|_{W^{\tilde{\rho}_0,\infty}}) \\ &\qquad + C \|w_1\|_{L^2} (\|L_{i_2} w_2\|_{W^{\tilde{\rho}_0,\infty}} + \|w_2\|_{W^{\tilde{\rho}_0,\infty}}) \|w_3\|_{W^{\tilde{\rho}_0,\infty}}. \end{aligned} \tag{A.3.3}$$

Moreover, estimates similar to (A.3.2), (A.3.3) *hold if one makes any permutation of* $(1, 2, 3)$ *on the right-hand side.*

To prove the proposition, we shall apply some results from [14]. In order to do so, we reduce ourselves to the framework of the appendices of that reference, using the rescaling (A.1.4). Set $h = \frac{1}{t}$ and

$$v_j = (\Theta_h w_j)(x) = \frac{1}{\sqrt{h}} w_j\Big(\frac{x}{h}\Big). \tag{A.3.4}$$

Then if we set

$$\|v\|_{H^s_h} = \|\langle h D_x \rangle^s v\|_{L^2}, \quad \|v\|_{W^{\rho,\infty}_h} = \|\langle h D_x \rangle^\rho\|_{L^\infty},$$

one has

$$\|v_j\|_{H^s_h} = \|w_j\|_{H^s}, \quad \|v_j\|_{W^{\rho,\infty}_h} = h^{-\frac{1}{2}} \|w_j\|_{W^{\rho,\infty}}. \tag{A.3.5}$$

Define

$$\mathrm{Op}_h(m)(v_1,v_2,v_3) = \frac{1}{(2\pi)^3}\int e^{ix(\xi_1+\xi_2+\xi_3)} m(h\xi_1,h\xi_2,h\xi_3)\prod_{j=1}^{3}\hat{v}_j(\xi_j)\,d\xi_1\,d\xi_2\,d\xi_3.$$

Then

$$\Theta_h^{-1}\mathrm{Op}_h(m)(\Theta_h w_1,\Theta_h w_2,\Theta_h w_3) = h^{-1}\mathrm{Op}(m)(w_1,w_2,w_3).$$

Moreover, by (A.1.3), if we set

$$\mathcal{L}_\pm = \frac{1}{h}\mathrm{Op}_h(x\pm p'(\xi)) = \frac{1}{h}(x\pm p'(hD_x)), \tag{A.3.6}$$

we get

$$\Theta_h^{-1}\circ\mathcal{L}_\pm\circ\Theta_h w = L_\pm w. \tag{A.3.7}$$

It follows from (A.3.4), (A.3.5)–(A.3.7), that inequality (A.3.1) is equivalent to

$$\|\mathcal{L}_+\mathrm{Op}_h(m)(v_1,v_2,v_3)\|_{L^2} \le C\sum_{\ell=1}^{3}(\|\mathcal{L}_{i_\ell}v_\ell\|_{L^2}+\|v_\ell\|_{H_h^{\tilde{s}_0}})\prod_{\substack{1\le j\le 3\\ j\neq\ell}}\|v_\ell\|_{W_h^{\tilde{\rho}_0,\infty}}. \tag{A.3.8}$$

In the same way, (A.3.2) is equivalent to

$$\begin{aligned}\|\mathcal{L}_+\mathrm{Op}_h(m)(v_1,v_2,v_3)\|_{L^2} &\\ \le C\big[&(\|\mathcal{L}_{i_1}v_1\|_{L^2}+\|v_1\|_{H_h^{\tilde{s}_0}})\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\\ &+\|v_1\|_{W_h^{\tilde{\rho}_0,\infty}}(\|\mathcal{L}_{i_2}v_2\|_{L^2}+\|v_2\|_{H_h^{\tilde{s}_0}})\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\\ &+\|v_1\|_{L^2}\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}(\|\mathcal{L}_{i_3}v_3\|_{W_h^{\tilde{\rho}_0,\infty}}+\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}})\big],\end{aligned} \tag{A.3.9}$$

and (A.3.3) is equivalent to

$$\begin{aligned}\|\mathcal{L}_+\mathrm{Op}_h(m)(v_1,v_2,v_3)\|_{L^2} &\\ \le C\big[&(\|\mathcal{L}_{i_1}v_1\|_{L^2}+\|v_1\|_{H_h^{\tilde{s}_0}})\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\\ &+\|v_1\|_{L^2}\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}(\|\mathcal{L}_{i_3}v_3\|_{W_h^{\tilde{\rho}_0,\infty}}+\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}})\\ &+\|v_1\|_{L^2}(\|\mathcal{L}_{i_2}v_2\|_{W_h^{\tilde{\rho}_0,\infty}}+\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}})\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\big].\end{aligned} \tag{A.3.10}$$

Moreover, estimates of Proposition A.2.2 hold (uniformly in $h\in\,]0,1]$) if everywhere we replace $\mathrm{Op}(m)$ by $\mathrm{Op}_h(m)$, $\|\cdot\|_{H^s}$ by $\|\cdot\|_{H_h^s}$, and $\|\cdot\|_{W^{\rho,\infty}}$ by $\|\cdot\|_{W_h^{\rho,\infty}}$.

Proof of Proposition A.3.1. Let us decompose

$$m(\xi_1,\xi_2,\xi_3) = m^L(\xi_1,\xi_2,\xi_3)+m^H(\xi_1,\xi_2,\xi_3),$$

$$m^H(\xi_1,\xi_2,\xi_3) = \sum_{j=1}^{3} m_j^H(\xi_1,\xi_2,\xi_3),$$

where for some $\beta > 0$ small, m^L is supported for $|\xi_1| + |\xi_2| + |\xi_3| \leq Ch^{-\beta}$, while m_j^H is supported for $|\xi_\ell| \leq C|\xi_j|$, $\ell \neq j$ and $|\xi_j| \geq ch^{-\beta}$, each of these symbols being in $S_{1,0}(M_0, 3)$.

- *Contribution of m^H to* (A.3.8)–(A.3.10). Write

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m_1^H)(v_1, v_2, v_3) = {}& \mathrm{Op}_h(\tilde{m}_1^H)(v_1, v_2, v_3) + \mathrm{Op}_h(m_1^H)(\mathcal{L}_{i_1}v_1, v_2, v_3)\\ &+ h^{-1}p'(hD_x)\mathrm{Op}_h(m_1^H)(v_1, v_2, v_3)\\ &- i_1h^{-1}\mathrm{Op}_h(m_1^H)(p'(hD_x)v_1, v_2, v_3),\end{aligned} \tag{A.3.11}$$

where $\tilde{m}_1^H = i\frac{\partial m_1^H}{\partial \xi_1} \in S_{1,0}(M_0^2, 3)$. In the arguments of each term on the right-hand side, we may replace v_1 by $\mathrm{Op}_h((1-\chi_0)(h^\beta\xi_1))v_1$ for $\chi_0 \in C_0^\infty(\mathbb{R})$, equal to 1 close to 0, with small enough support, by the support property of m_1^H. We estimate then the L^2-norm of (A.3.11) using the version of (A.2.3) for $\mathrm{Op}_h(m)$. We obtain

$$\begin{aligned}\|\mathcal{L}_+\mathrm{Op}_h(m_1^H)(v_1, v_2, v_3)\|_{L^2} \leq {}& C\big(h^{-1}\|\mathrm{Op}_h((1-\chi_0)(h^\beta\xi))v_1\|_{L^2} + \|\mathcal{L}_{i_1}v_1\|_{L^2}\big)\\ &\times \|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}\end{aligned} \tag{A.3.12}$$

if $\tilde{\rho}_0$ is taken large enough. Moreover, in the first factor on the right-hand side, we may bound

$$h^{-1}\|\mathrm{Op}_h((1-\chi_0)(h^\beta\xi))v_1\|_{L^2} \leq Ch^{-1+\beta\tilde{s}_0}\|v_1\|_{H_h^{\tilde{s}_0}} \leq C\|v_1\|_{H_h^{\tilde{s}_0}} \tag{A.3.13}$$

if $\tilde{s}_0$ is chosen large enough so that $\tilde{s}_0\beta \geq 1$. Thus the left-hand side of (A.3.12) is bounded from above by the first term on the right-hand side of (A.3.8). By symmetry, we thus get that (A.3.8) for m replaced by m^H holds.

Let us prove (A.3.9) for m^H. By (A.3.11) to (A.3.13), the contribution of m_1^H to the left-hand side of (A.3.9) is estimated by the first term on the right-hand side of this inequality. In the same way, the contribution of m_2^H is bounded by the second term on the right-hand side. For m_3^H, instead of (A.3.11) write

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m_3^H)(v_1, v_2, v_3) = {}& \mathrm{Op}_h(\tilde{m}_3^H)(v_1, v_2, v_3) + \mathrm{Op}_h(m_3^H)(v_1, v_2, \mathcal{L}_{i_3}v_3)\\ &+ h^{-1}p'(hD_x)\mathrm{Op}_h(m_3^H)(v_1, v_2, v_3)\\ &- i_3h^{-1}\mathrm{Op}_h(m_3^H)(v_1, v_2, p'(hD_x)v_3).\end{aligned} \tag{A.3.14}$$

We use next (A.2.3) with $j = 1$. The L^2-norm of (A.3.14) is bounded from above by

$$C\|v_1\|_{L^2}\|v_2\|_{W_h^{\tilde{\rho}_0,\infty}}(\|\mathcal{L}_{i_3}v_3\|_{W_h^{\tilde{\rho}_1,\infty}} + h^{-1}\|\mathrm{Op}_h(1-\chi(h^\beta\xi))v_3\|_{W_h^{\tilde{\rho}_1,\infty}}) \tag{A.3.15}$$

for some large enough $\tilde{\rho}_1$. If $\tilde{\rho}_0$ is such that $(\tilde{\rho}_0 - \tilde{\rho}_1)\beta > 1$, we may bound the last term by $\|v_3\|_{W_h^{\tilde{\rho}_0,\infty}}$, using that operators of negative order are bounded on L^∞-spaces. This gives an estimate of (A.3.15) by the last term on the right-hand side of (A.3.9).

Finally, let us prove (A.3.10) for m^H. The contributions of m_1^H, m_3^H are treated as in the study of (A.3.8) and (A.3.9) above. For m_2^H, we write (A.3.14) for m_2^H instead of m_3^H with indices 2 and 3 interchanged on the right-hand side. This gives an estimate for the m_2^H-contribution to the left-hand side of (A.3.10) by the third term on the right-hand side.

- *Contribution of* m^L *to* (A.3.8)–(A.3.10). Since m^L is supported for $|\xi_1|+|\xi_2|+|\xi_3| \leq Ch^{-\beta}$ by construction, m_L satisfies estimate (3.1.2) with $\beta > 0$, $\nu = 1$, $\kappa = 1$, i.e. belongs to the class $S_{1,\beta}(M_0, 3)$. This allows us to apply [14, Proposition F.2.1] which asserts that a Leibniz rule holds, in that sense that if (i_1, i_2, i_3) is characteristic,

$$\begin{aligned}\mathcal{L}_+\mathrm{Op}_h(m^L)(v_1, v_2, v_3) = {} & \mathrm{Op}_h(m_1^L)(\mathcal{L}_{i_1}v_1, v_2, v_3)\\ & + \mathrm{Op}_h(m_2^L)(v_1, \mathcal{L}_{i_2}v_2, v_3)\\ & + \mathrm{Op}_h(m_3^L)(v_1, v_2, \mathcal{L}_{i_2}v_3)\\ & + \mathrm{Op}_h(r)(v_1, v_2, v_3), \end{aligned} \tag{A.3.16}$$

where m_j^L, $j = 1, 2, 3$ and r are elements of the class $S_{1,\beta}(M_0^\nu, 3)$ for some $\nu \in \mathbb{N}$. Actually, in [14], there is also a weight $\prod_{j=1}^3 \langle \xi_j \rangle^{-1}$ on the right-hand side of the inequalities (3.1.2) that define the symbols, but that does not play any role in the proofs. In [14, Proposition F.2.1] there is also an extra term on the right-hand side of (A.3.16), of the form $h^{-1}\mathrm{Op}_h(r')(v_1, v_2, v_3)$ for some r'. Such a term does not appear here because our symbols are constant coefficients and in particular do not depend on the y-variable in [14, Proposition F.2.1]: see the last three lines in [14, Proposition B.2.1].

To obtain (A.3.8) for m^L, we now just have to use estimate (A.2.3) for each term on the right-hand side of (A.3.16), putting the L^2-norm on the factor in $\mathcal{L}_{i_j}v_j$ for the first three terms on the right-hand side.

One obtains (A.3.9) for m^L in the same way, except that we treat the $\mathrm{Op}_h(m_3^L)$ term on the right-hand side of (A.3.16), putting the L^2-norms on the factor v_1 in estimate (A.2.3). Finally, to get (A.3.10) for m^L, we argue in the same way, controlling the L^2-norms of the $\mathrm{Op}_h(m_2^L)$ and $\mathrm{Op}_h(m_3^L)$ terms using (A.2.3), where we put the L^2-norm on the v_1 term on the right-hand side.

This concludes the proof. ∎

A.4. Klainerman–Sobolev estimates

In this subsection we prove a Klainerman–Sobolev estimate for the one-dimensional Klein–Gordon equation. This estimate is not new and may be found implicitly in a weaker form in [13, 31] for instance. We first introduce some notation.

If $\delta \in [0, 1]$, let us introduce $\widetilde{S}_\delta(1)$, the space of smooth functions $(x, \xi) \to a(x, \xi, h)$ from $\mathbb{R}^2$ to $\mathbb{C}$, depending also on a parameter $h \in]0, 1]$, such that for any α, β in $\mathbb{N}$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_\alpha h^{-\delta(\alpha+\beta)}.$$

For $u \in \mathcal{S}(\mathbb{R})$, define the semi-classical Weyl quantization of a acting on u by

$$\mathrm{Op}_h^{\mathrm{W}}(a)u = \frac{1}{2\pi h}\int e^{i(x-y)\frac{\xi}{h}} a\Big(\frac{x+y}{2}, \xi, h\Big) u(y)\, dy\, d\xi.$$

If $b(x,\xi,y,\eta)$ is a smooth function, define

$$\sigma(D_x, D_\xi, D_y, D_\eta) b(x,\xi,y,\eta) = (D_\xi D_y - D_x D_\eta) b(x,\xi,y,\eta)$$

and recall that if $\delta \in [0, \frac{1}{2}[$, if a_1, a_2 are in $\widetilde{S}_\delta(1)$, there is a symbol $a_1 \#_h a_2$ in $\widetilde{S}_\delta(1)$, such that for any N,

$$a_1 \#_h a_2 - \sum_{k=0}^{N} \frac{1}{k!}\Big(\frac{ih}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\Big)^k (a_1(x,\xi) a_2(y,\eta))|_{x=y, \xi=\eta} \tag{A.4.1}$$

is in $h^{(N+1)(1-2\delta)}\widetilde{S}_\delta(1)$ and

$$\mathrm{Op}_h^{\mathrm{W}}(a_1) \circ \mathrm{Op}_h^{\mathrm{W}}(a_2) = \mathrm{Op}_h^{\mathrm{W}}(a_1 \#_h a_2). \tag{A.4.2}$$

In particular, if a_1 and a_2 have disjoint supports, $\mathrm{Op}_h^{\mathrm{W}}(a_1) \circ \mathrm{Op}_h^{\mathrm{W}}(a_2)$ may be written for any N in $\mathbb{N}$ as $h^N \mathrm{Op}_h^{\mathrm{W}}(r)$ with r in $\widetilde{S}_\delta(1)$, since $\delta < \frac{1}{2}$.

Recall also that if $a \in \widetilde{S}_\delta(1)$, $\mathrm{Op}_h^{\mathrm{W}}(a)$ is bounded on H_h^s with uniform estimates

$$\|\mathrm{Op}_h^{\mathrm{W}}(a)u\|_{H_h^s} \le C\|u\|_{H_h^s}. \tag{A.4.3}$$

All the results above may be found, for instance, in the book by Dimassi–Sjöstrand [15, Chapter 7].

Our goal is to prove the following proposition:

Proposition A.4.1. *Let $\rho_0 \in \mathbb{N}$. There is $s_0 \in \mathbb{N}$ such that for any function w, one has the bound*

$$\|w\|_{W^{\rho_0,\infty}} \le \frac{C}{\sqrt{t}}((\|L_+ w\|_{L^2} + \|w\|_{H^{s_0}})^{\frac{1}{2}} \|w\|_{H^{s_0}}^{\frac{1}{2}} + \|L_+ w\|_{L^2}), \tag{A.4.4}$$

where $L_+ = x + tp'(D_x)$ with $p(\xi) = \sqrt{1+\xi^2}$.

We define from w a function v by (A.3.4) and using notation (A.3.6) and (A.3.5), we see that (A.4.4) is equivalent to

$$\|v\|_{W_h^{\rho_0,\infty}} \le C((\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{H_h^{s_0}})^{\frac{1}{2}} \|v\|_{H_h^{s_0}}^{\frac{1}{2}} + \|\mathcal{L}_+ v\|_{L^2}). \tag{A.4.5}$$

Let us notice that one may further reduce to proving that there is some large enough $\tilde{s}_0$ such that the following estimate holds:

$$\|\langle hD_x\rangle^{-3} v\|_{L^\infty} \le C((\|\mathcal{L}_+ v\|_{L^2} + \|v\|_{H_h^{\tilde{s}_0}})^{\frac{1}{2}} \|v\|_{H_h^{\tilde{s}_0}}^{\frac{1}{2}} + \|\mathcal{L}_+ v\|_{H_h^{-\tilde{s}_0}}). \tag{A.4.6}$$

Actually, if (A.4.6) is proved, we may apply it to $v_k = \mathrm{Op}_h^{\mathrm{W}}(\chi(2^{-k}\xi))v$ for some $\chi \in C_0^\infty(\mathbb{R}^*)$ and $k \in \mathbb{N}$. We have then

$$\begin{aligned}\|v_k\|_{W_h^{\rho_0,\infty}} &\leq C2^{k(\rho_0+3)}\|\langle hD_x\rangle^{-3}v_k\|_{L^\infty}\\ &\leq C2^{k(\rho_0+3)}((\|\mathcal{L}_+v_k\|_{L^2}+\|v_k\|_{H_h^{\tilde{s}_0}})^{\frac{1}{2}}\|v_k\|^{\frac{1}{2}}_{H_h^{\tilde{s}_0}}+\|\mathcal{L}_+v_k\|_{H_h^{-\tilde{s}_0}})\\ &\leq C2^{k(\rho_0+3+\frac{\tilde{s}_0}{2}-\frac{s_0}{2})}(\|\mathcal{L}_+v\|_{L^2}+\|v\|_{H_h^{s_0}})^{\frac{1}{2}}\|v\|^{\frac{1}{2}}_{H_h^{s_0}}\\ &\quad + C2^{k(\rho_0+3-\tilde{s}_0)}(\|\mathcal{L}_+v\|_{L^2}+\|v\|_{L^2}),\end{aligned}$$

from which (A.4.5) follows by summation of a Littlewood–Paley decomposition if $s_0 > \tilde{s}_0 + 2(\rho_0+3)$ and $\tilde{s}_0 > \rho_0 + 3$.

In the rest of this subsection, we shall prove (A.4.6). Before starting the proof, we make some reductions.

Lemma A.4.2. *Let $\gamma, \chi \in C_0^\infty(\mathbb{R})$ be equal to 1 close to 0, with small enough support. Let $M \in \mathbb{N}$. There is $\beta > 0$ and a family of smooth functions $x \to \theta_h(x)$, depending on a parameter $h \in]0,1]$, with for any $\alpha \in \mathbb{N}$, $\partial_x^\alpha\theta_h(x) = O(h^{-2\beta\alpha})$, θ_h being supported in $[-1+ch^{2\beta}, 1-ch^{2\beta}]$ for some $c > 0$, such that for any function v,*

$$\begin{aligned}&\big\|\langle hD_x\rangle^{-2}v - \mathrm{Op}_h^{\mathrm{W}}\big(\gamma\big((x+p'(\xi))\langle\xi\rangle^2\big)\chi(h^\beta\xi)\theta_h(x)\langle\xi\rangle^{-2}\big)v\big\|_{L^\infty}\\ &\quad\leq C(\|v\|_{H_h^{-1+\frac{1}{2\beta}}}+\|\mathcal{L}_+v\|_{H_h^{-2M+1}}).\end{aligned}\tag{A.4.7}$$

Proof. By semi-classical Sobolev embedding, one has for any $\varepsilon > 0$,

$$\big\|\langle hD_x\rangle^{-2}\big(v-\mathrm{Op}_h^{\mathrm{W}}(\chi(h^\beta\xi))v\big)\big\|_{L^\infty} \leq Ch^{-\frac{1}{2}+\beta(s+\frac{3}{2}-\varepsilon)}\|v\|_{H_h^s}\tag{A.4.8}$$

if $s > -\frac{3}{2}+\varepsilon$, so that we have an upper bound by the right-hand side of (A.4.7). We shall study next the L^∞-norm of $\mathrm{Op}_h^{\mathrm{W}}(a(x,\xi))v$ if

$$a(x,\xi) = \chi(h^\beta\xi)\langle\xi\rangle^{-2}(1-\gamma)\big((x+p'(\xi))\langle\xi\rangle^2\big) = a_1(x,\xi)(x+p'(\xi)),$$

where $a_1 = \chi(h^\beta\xi)\gamma_1((x+p'(\xi))\langle\xi\rangle^2)$ with $\gamma_1(z) = \frac{1-\gamma(z)}{z}$. Then a and a_1 belong to $\widetilde{S}_\delta(1)$ with $\delta = 2\beta < \frac{1}{2}$ for small enough $\beta > 0$. We use (A.4.2), (A.4.1) to write with some r in $\widetilde{S}_\delta(1)$,

$$\begin{aligned}\mathrm{Op}_h^{\mathrm{W}}(a)v &= \mathrm{Op}_h^{\mathrm{W}}(a_1)\circ\mathrm{Op}_h^{\mathrm{W}}(x+p'(\xi))v + h^{1-2\delta}\mathrm{Op}_h^{\mathrm{W}}(r)\\ &= h\mathrm{Op}_h^{\mathrm{W}}(a_1)\mathcal{L}_+v + h^{1-2\delta}\mathrm{Op}_h^{\mathrm{W}}(r).\end{aligned}\tag{A.4.9}$$

In the right-hand side write

$$\mathrm{Op}_h^{\mathrm{W}}(a_1)\mathcal{L}_+v = \mathrm{Op}_h^{\mathrm{W}}(a_1)\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{2M})(\langle hD_x\rangle^{-2M}\mathcal{L}_+v)\tag{A.4.10}$$

and use that since M is an integer, we have an exact composition formula (A.4.2),

$$\begin{aligned}&\mathrm{Op}_h^{\mathrm{W}}(a_1)\circ\mathrm{Op}_h^{\mathrm{W}}(\langle\xi\rangle^{2M})\\&\quad=\sum_{k=0}^{2M}\frac{1}{k!}\Big(i\frac{h}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\Big)^k(a_1(x,\xi)\langle\eta\rangle^{2M})|_{x=y,\xi=\eta}.\end{aligned}\tag{A.4.11}$$

Now, since $|\xi|=O(h^{-\beta})$ on the support of a_1, we get that (A.4.11) is of the form $h^{-2M\delta}\mathrm{Op}_h^{\mathrm{W}}(a_2)$ with some $a_2\in\widetilde{S}_\delta(1)$. Applying the semi-classical Sobolev inequality again, we deduce from (A.4.10), (A.4.11),

$$\|\mathrm{Op}_h^{\mathrm{W}}(a_1)\mathcal{L}_+v\|_{L^\infty}\le Ch^{-\frac12-2M\delta}\|\mathcal{L}_+v\|_{H^{-2M+\frac12+\varepsilon}}.$$

Plugging this into (A.4.9), we get

$$\|\mathrm{Op}_h^{\mathrm{W}}(a)v\|_{L^\infty}\le Ch^{\frac12-2M\delta}\|\mathcal{L}_+v\|_{H^{-2M+\frac12+\varepsilon}}+Ch^{\frac12-2\delta}\|\mathrm{Op}_h^{\mathrm{W}}(r)v\|_{H^{\frac12+\varepsilon}}.\tag{A.4.12}$$

If $\delta=2\beta$ is small enough relative to $1/M$, this implies that (A.4.12) is bounded by the right-hand side of (A.4.7). Taking into account (A.4.8), we thus see that it remains to consider

$$\mathrm{Op}_h^{\mathrm{W}}\big(\chi(h^\beta\xi)\gamma\big((x+p'(\xi))\langle\xi\rangle^2\big)(1-\theta_h)(x)\langle\xi\rangle^{-2}\big)v.\tag{A.4.13}$$

We shall be done if we prove that, if $\mathrm{Supp}\,\gamma$ has been taken small enough, we may choose θ_h such that it is equal to 1 on the support of $\chi(h^\beta\xi)\gamma((x+p'(\xi))\langle\xi\rangle^2)$ so that (A.4.13) vanishes identically. This follows from the fact that, if $\mathrm{Supp}\,\gamma$ is small enough and $\gamma((x+p'(\xi))\langle\xi\rangle^2)\neq 0$, then when $\xi\to+\infty$ (resp. $\xi\to-\infty$), $x+1$ (resp. $x-1$) stays in an interval $[c_1/\xi^2,c_2/\xi^2]$ (resp. $[-c_2/\xi^2,-c_1/\xi^2]$) for some $0<c_1<c_2$. If, in addition, $|\xi|=O(h^{-\beta})$, this implies that x belongs to the interval $[-1+ch^{2\beta},1-ch^{2\beta}]$ for some $c>0$, which allows one to construct the wanted θ_h. This concludes the proof. ∎

Proof of Proposition A.4.1. With the notation of Lemma A.4.2 we set

$$\tilde v=\mathrm{Op}_h^{\mathrm{W}}\big(\gamma\big((x+p'(\xi))\langle\xi\rangle^2\big)\theta_h(x)\chi(h^\beta\xi)\langle\xi\rangle^{-3}\big)v\tag{A.4.14}$$

so that, by that lemma, inequality (A.4.6), which implies Proposition A.4.1, will follow if we prove

$$\|\tilde v\|_{L^\infty}\le C(\|v\|_{H_h^{\tilde s_0}}+\|\mathcal{L}_+v\|_{L^2})^{\frac12}\|v\|_{H_h^{\tilde s_0}}^{\frac12}.\tag{A.4.15}$$

Take $\tilde\theta_h\in C_0^\infty(]-1,1[)$ equal to 1 on the support of θ_h, satisfying $\partial_x^\alpha\tilde\theta_h=O(h^{-\delta\alpha})$ (with $\delta=2\beta$) for any α. Since the symbol of the operator defining $\tilde v$ in (A.4.14) is in $\widetilde{S}_\delta(1)$, it follows from (A.4.2) and the remark following it, that $(1-\tilde\theta_h)\tilde v=h^N\mathrm{Op}_h^{\mathrm{W}}(r)v$ for some symbol r in $\widetilde{S}_\delta(1)$ and any N. Then, using the semi-classical Sobolev estimate and (A.4.3) again, we see that $\|(1-\tilde\theta_h)\tilde v\|_{L^\infty}$ is estimated by the right-hand side of (A.4.15). We are thus left with studying $\tilde\theta_h\tilde v$. If $\varphi(x)=\sqrt{1-x^2}$ for $x\in\,]-1,1[$, write

$$\begin{aligned}\|\tilde\theta_h\tilde v\|_{L^\infty}=\|e^{-i\frac{\varphi}{h}}\tilde\theta_h\tilde v\|_{L^\infty}&\le Ch^{-\frac12}\|hD_x(e^{-i\frac{\varphi}{h}}\tilde\theta_h\tilde v)\|_{L^2}^{\frac12}\|e^{-i\frac{\varphi}{h}}\tilde\theta_h\tilde v\|_{L^2}^{\frac12}\\ &\le Ch^{-\frac12}\big(\|\tilde\theta_h(x)(hD_x-d\varphi(x))\tilde v\|_{L^2}^{\frac12}\\ &\quad+\|(hD_x\tilde\theta_h)\tilde v\|_{L^2}^{\frac12}\big)\|\tilde v\|_{L^2}^{\frac12}.\end{aligned}\tag{A.4.16}$$

Note that $(hD_x\tilde\theta_h)\tilde v=-ih^{1-\delta}\tilde\theta_h^1(x)\tilde v$ for a function $\tilde\theta_h^1$ again satisfying $\partial_x^\alpha\tilde\theta_h^1=O(h^{-\delta\alpha})$, whose support does not intersect the support of the symbol defining $\tilde v$ in (A.4.14). Using (A.4.2) again, we conclude that $\|(hD_x\tilde\theta_h)\tilde v\|_{L^2}\le c_Nh^N\|v\|_{L^2}$, so that to show that the right-hand side of (A.4.16) is bounded by the right-hand side of (A.4.15), it is enough to prove that

$$h^{-\frac12}\|\tilde\theta_h(x)(hD_x-d\varphi(x))\tilde v\|_{L^2}^{\frac12}\|\tilde v\|_{L^2}^{\frac12}\le C(\|v\|_{H_h^{\tilde s_0}}+\|\mathcal{L}_+v\|_{L^2})^{\frac12}\|v\|_{H_h^{\tilde s_0}}^{\frac12}.\tag{A.4.17}$$

Notice that in (A.4.17), $h^\beta\tilde\theta_h(x)\,d\varphi(x)$ is an element of $\widetilde S_\delta(1)$ and that

$$\tilde\theta_h(x)hD_x=\mathrm{Op}_h^{\mathrm W}(\tilde\theta_h\xi)+i\frac h2\tilde\theta_h'(x).\tag{A.4.18}$$

Again, the last contribution in (A.4.18) will bring a trivial term to estimate in (A.4.17), so that we are reduced to the study of

$$h^{-\frac12}\big\|\mathrm{Op}_h^{\mathrm W}\big(\tilde\theta_h(x)(\xi-d\varphi(x))\big)\tilde v\big\|_{L^2}^{\frac12}\|\tilde v\|_{L^2}^{\frac12}.\tag{A.4.19}$$

If we express $\tilde v$ from (A.4.14) and use (A.4.2), (A.4.1) at order $N=1$, we obtain

$$\mathrm{Op}_h^{\mathrm W}\big(\tilde\theta_h(x)(\xi-d\varphi(x))\big)\tilde v=\mathrm{Op}_h^{\mathrm W}(a_0(x,\xi)+ha_1(x,\xi)+h^{2(1-2\delta)-\beta}r)v,\tag{A.4.20}$$

where $r\in\widetilde S_\delta(1)$ is the remainder in (A.4.1) (the extra power $h^{-\beta}$ coming from the fact that $\tilde\theta_h\,d\varphi$ is not in $\widetilde S_\delta(1)$, but only in $h^{-\beta}\widetilde S_\delta(1)$), and where a_0,a_1 are the first two terms in expansion (A.4.1) and are given explicitly by

$$\begin{aligned}a_0&=\theta_h(x)(\xi-d\varphi(x))\gamma\big((x+p'(\xi))\langle\xi\rangle^2\big)\chi(h^\beta\xi)\langle\xi\rangle^{-3},\\ a_1&=-\frac i2\{(\xi-d\varphi(x)),\theta_h(x)\gamma\big((x+p'(\xi))\langle\xi\rangle^2\big)\chi(h^\beta\xi)\langle\xi\rangle^{-3}\}.\end{aligned}\tag{A.4.21}$$

If δ,β are small enough, the r term in (A.4.20) brings to (A.4.19) a contribution bounded by the right-hand side of (A.4.17). We thus have to study a_0,a_1. We use [13, Lemma 1.8] to rewrite a_0,a_1. According to [13, (1.28), (1.29)] with $\kappa=2$, we may write

$$a_0(x,\xi)=(x+p'(\xi))b_0(x,\xi),$$

where b_0 is supported for $|\xi|\le h^{-\beta}$, $1-x^2\ge c\langle\xi\rangle^{-2}$ and satisfies estimates of the form $|\partial_x^\alpha\partial_\xi^{\alpha'}b_0(x,\xi)|\le C\langle\xi\rangle^{2\alpha-\alpha'}$. Actually, as already seen in the proof of Lemma A.4.2,

$\theta_h \equiv 1$ on the support of $\gamma((x + p'(\xi))\langle\xi\rangle^2)\chi(h^\beta\xi)$, so that this factor θ_h may be omitted in definition (A.4.21) of a_0. It follows then that b_0 is in $\widetilde{S}_\delta(1)$ with $\delta = 2\beta$.

We may thus apply (A.4.2), (A.4.1) to write

$$\begin{aligned}\operatorname{Op}_h^{\mathrm{W}}(a_0)v &= \operatorname{Op}_h^{\mathrm{W}}\big(b_0(x,\xi)(x + p'(\xi))\big)v \\ &= \operatorname{Op}_h^{\mathrm{W}}(b_0)\operatorname{Op}_h^{\mathrm{W}}(x + p'(\xi))v - \frac{h}{2i}\operatorname{Op}_h^{\mathrm{W}}\Big(\frac{\partial b_0}{\partial\xi} - p''(\xi)\frac{\partial b_0}{\partial x}\Big)v \\ &\quad + h^{2-4\delta}\operatorname{Op}_h^{\mathrm{W}}(r)v, \end{aligned}\tag{A.4.22}$$

for some r in $\widetilde{S}_\delta(1)$. In the above expression, $\frac{\partial b_0}{\partial\xi} - p''(\xi)\frac{\partial b_0}{\partial x}$ is in $\widetilde{S}_\delta(1)$ since $p''(\xi) = O(\langle\xi\rangle^{-3})$. Applying (A.4.3) to the three terms on the right-hand side of (A.4.22), we get that for $\beta > 0$ small enough,

$$\begin{aligned}\|\operatorname{Op}_h^{\mathrm{W}}(a_0)v\|_{L^2} &\le C\big(\|\operatorname{Op}_h^{\mathrm{W}}(x + p'(\xi))v\|_{L^2} + h\|v\|_{L^2}\big) \\ &\le Ch(\|\mathcal{L}_+v\|_{L^2} + \|v\|_{L^2}).\end{aligned}\tag{A.4.23}$$

Consider next a_1 given by (A.4.21) (where $\theta_h(x)$ may be removed). As on the support of a_1, $\partial_x^\alpha(d\varphi(x)) = O(\langle\xi\rangle^{1+2\alpha})$, it follows that a_1 is in $\widetilde{S}_\delta(1)$, so that the second term on the right-hand side of (A.4.20) satisfies

$$h\|\operatorname{Op}_h^{\mathrm{W}}(a_1)v\|_{L^2} \le Ch\|v\|_{L^2}.\tag{A.4.24}$$

Plugging (A.4.20), (A.4.23), and (A.4.24) into (A.4.19), we get that this expression is bounded from above by the right-hand side of (A.4.17). This concludes the proof. ∎

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