Z. Anal. Anwend. (Online first)
DOI 10.4171/ZAA/1757



# A direct method of moving planes for logarithmic Schrödinger operator

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**Abstract.** In this paper, we study the radial symmetry and monotonicity of nonnegative solutions to nonlinear equations involving the logarithmic Schrödinger operator $(\mathcal{I}-\Delta)^{\log}$ corresponding to the logarithmic symbol $\log(1+|\xi|^2)$, which is a singular integral operator given by

$$(\mathcal{I}-\Delta)^{\log}u(x) = c_N \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^N}\kappa(|x-y|)dy,$$

where $c_N = \pi^{-\frac{N}{2}}$, $\kappa(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(r)$ and $\mathcal{K}_\nu$ is the modified Bessel function of the second kind with index $\nu$. The proof hinges on a direct method of moving planes for the logarithmic Schrödinger operator.

## 1. Introduction

The study of Schrödinger equations received a great deal of attention from researchers in the past decades because of its vast applications in several areas of mathematics and mathematical physics. In particular, Schrödinger equations arise in quantum field theory and in the Hartree–Fock theory (see [1,20,21,23]). Recently, there is a surge of interest to investigate integrodifferential operators of order close to zero and associated linear and nonlinear integrodifferential equations (see [5,6,16,18,19,22]). In particular, the logarithmic Laplacian and the logarithmic Schrödinger operator are two interesting examples of such a class of operators. The logarithmic Laplacian was first introduced by Chen and Weth in [5] as a limit of fractional Laplacian (see also [3,4] for the spectral properties of the logarithmic Laplacian). The logarithmic Schrödinger operator $(\mathcal{I}-\Delta)^{\log}$ (see [10]) and the logarithmic Laplacian $L_\Delta$ (see [2,5,11,12,24]) have the similar behavior locally concerning to the singularity of kernels but the logarithmic Schrödinger operator eliminates the integrability problem of the logarithmic Laplacian at infinity. To define the logarithmic Schrödinger operator, let us begin with the following observation:

$$\lim_{s\to 0^+} (\mathcal{I}-\Delta)^s u(x) = u(x) \quad \text{for } u \in C^2(\mathbb{R}^N), \tag{1.1}$$

*Mathematics Subject Classification 2020:* 35R11 (primary); 35B06, 35B50, 35B51, 35D30 (secondary).
*Keywords:* logarithmic Schrödinger operator, symmetry and monotonicity, the direct method of moving planes, logarithmic symbol, logarithmic Laplacian.

where, for $s \in (0,1)$, the operator $(\mathcal{I}-\Delta)^s$ stands for the relativistic Schrödinger operator, for sufficiently regular function $u : \mathbb{R}^N \to \mathbb{R}$, which can be represented via hypersingular integral (1.1) (see [8]),

$$(\mathcal{I}-\Delta)^s u(x) = u(x) + c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \varpi_s(|x-y|)dy, \tag{1.2}$$

where $c_{N,s} = \frac{\pi^{-\frac{N}{2}} 4^s}{\Gamma(-s)}$ is a normalization constant and the function $\varpi_s$ is given by

$$\varpi_s(r) = 2^{1-\frac{N+2s}{2}} r^{\frac{N+2s}{2}} \mathcal{K}_{\frac{N+2s}{2}}(r) = \int_0^{+\infty} t^{-1+\frac{N+2s}{2}} e^{-t-\frac{r^2}{4t}} dt. \tag{1.3}$$

Furthermore, if $u \in C^2(\mathbb{R}^N)$, then $(\mathcal{I}-\Delta)^s u(x)$ is well defined by (1.2) for every $x \in \mathbb{R}^N$. Here, the function $\mathcal{K}_\nu$ is the modified Bessel function of the second kind with index $\nu > 0$, and it is given by

$$\mathcal{K}_\nu(r) = \frac{(\frac{\pi}{2})^{\frac{1}{2}} r^\nu e^{-r}}{\Gamma(\frac{2\nu+1}{2})} \int_0^\infty \left(1+\frac{t}{2}\right)^{\nu-\frac{1}{2}} e^{-rt} t^{\nu-\frac{1}{2}} dt$$

for more properties of $\mathcal{K}_\nu$, see, e.g., [7, 9, 10, 14, 15] and references therein.

It is well known that $\mathcal{K}_\nu$ is a real and positive function satisfying

$$\mathcal{K}'_\nu(r) = -\frac{\nu}{r}\mathcal{K}_\nu(r) - \mathcal{K}_{\nu-1}(r) < 0 \tag{1.4}$$

for all $r > 0$, $\mathcal{K}_\nu = \mathcal{K}_{-\nu}$ for $\nu > 0$. Furthermore, for $\nu > 0$ (see [9, 15])

$$\mathcal{K}_\nu(r) \sim \begin{cases} \dfrac{\Gamma(\nu)}{2}\left(\dfrac{r}{2}\right)^{\nu}, & r \to 0, \\ \dfrac{\sqrt{\pi}}{\sqrt{2}} r^{-\frac{1}{2}} e^{-r}, & r \to \infty. \end{cases} \tag{1.5}$$

It follows from (1.1) that one may expect a Taylor expansion with respect to parameter $s$ of the operator $(\mathcal{I}-\Delta)^s$ near zero for $u \in C^2(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ as follows:

$$(\mathcal{I}-\Delta)^s u(x) = u(x) + s(\mathcal{I}-\Delta)^{\log} u(x) + o(s) \quad \text{as } s \to 0^+. \tag{1.6}$$

The logarithmic Schrödinger operator $(\mathcal{I}-\Delta)^{\log}$ appears as the first-order term in the above expansion.

In this paper, we study the integrodifferential operator $(\mathcal{I}-\Delta)^{\log}$ corresponding to the logarithmic symbol $\log(1+|\xi|^2)$, which is a singular integral operator given by

$$(\mathcal{I}-\Delta)^{\log} u(x) = c_N \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^N} \kappa(|x-y|) dy, \tag{1.7}$$

where $c_N = \pi^{-\frac{N}{2}} \Gamma(\frac{N}{2})$, P.V. stands for the Cauchy principal value of the integral, $\kappa(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(r)$ and $\mathcal{K}_\nu$ is the modified Bessel function of second kind with index $\nu$. One can also easily deduce from (1.4) that $\kappa'(r) < 0$ for $r > 0$. Using the expression (1.7), one can define $(\mathcal{I} - \Delta)^{\log}$ for a quite large class of functions $u$. To illustrate this, define the space $\mathcal{L}_0(\mathbb{R}^N)$ as the space of locally integrable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{\mathcal{L}_0(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \frac{|u(x)|e^{-|x|}}{(1+|x|)^{\frac{N+1}{2}}} dx < +\infty.$$

Then, it was proved by [10, Proposition 2.1] that for $u \in \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, which is also Dini continuous at some $x \in \mathbb{R}^N$, the quantity $[(\mathcal{I} - \Delta)^{\log} u](x)$ is well defined by the formula (1.7). Let us recall the definition of Dini continuity. Let $U$ be a measurable subset of $\mathbb{R}^N$ and let $u : U \to \mathbb{R}$ be a measurable function. The modulus of continuity $\Psi_{u,x,U} : (0, +\infty) \to [0, +\infty)$ of $u$ at a point $x \in U$ is defined by

$$\Psi_{u,x,U}(r) := \sup_{y \in U, |x-y| \le r} |u(x) - u(y)|.$$

We call the function $u$ Dini continuous at $x$ if

$$\int_0^1 \frac{\Psi_{u,x,U}(r)}{r} dr < \infty.$$

Using the generalized direct method of moving planes, in this note, we obtain the radial symmetry and monotonicity of nonnegative solutions for the nonlinear equations involving the logarithmic Schrödinger operator (see Theorem 1.1), namely, we consider the nonlinear Schrödinger equation

$$(\mathcal{I} - \Delta)^{\log} u(x) + m u(x) = u^p(x), \quad x \in \mathbb{R}^N, \tag{1.8}$$

with $m > 0$ and $u(x) \ge 0$ for all $x \in \mathbb{R}^N$.

The following results present symmetry and monotonicity properties of Schrödinger equation (1.8).

**Theorem 1.1.** *Let $u \in \mathcal{L}_0(\mathbb{R}^N)$ be a nonnegative Dini continuous solution of* (1.8) *with $m > 0$ and $1 < p < \infty$. If*

$$\lim_{|x| \to \infty} u(x) = a < \left(\frac{m}{p}\right)^{\frac{1}{p-1}}, \tag{1.9}$$

*then $u$ must be radially symmetric and monotone decreasing about some point in $\mathbb{R}^N$.*

**Remark 1.2.** The condition (1.9) in Theorem 1.1 is necessary for applying the method of moving planes using the decay at infinity principle (Theorem 2.3).

The paper is organized as follows: in Section 2, we prove some results for the logarithmic Schrödinger operator. By the direct method of moving planes, we obtain the symmetry and monotonicity of nonnegative solutions for the nonlinear equations involving logarithmic Schrödinger operator in Section 3.

## 2. Key ingredients for the method of moving planes

This section is devoted to developing basic and key results needed to apply the method of moving planes for establishing the proof of our main result in the next section. We first present some basic notation and nomenclatures which will be beneficial for the rest of the paper.

Choose an arbitrary direction, say, the $x_1$-direction. For arbitrary $\lambda \in \mathbb{R}$, let

$$T_\lambda = \left\{x \in \mathbb{R}^N \mid x_1 = \lambda\right\}$$

be the moving plane, and let

$$\Sigma_\lambda = \left\{x \in \mathbb{R}^N \mid x_1 < \lambda\right\}$$

be the region to the left of the plane $T_\lambda$, and let

$$x^\lambda = (2\lambda - x_1, x_2, \dots, x_N)$$

be the reflection of $x$ about the plane $T_\lambda$.

By denoting $u(x^\lambda) := u_\lambda(x)$, we define

$$\omega_\lambda(x) := u_\lambda(x) - u(x), \quad x \in \Sigma_\lambda,$$

to compare the values of $u(x)$ and $u_\lambda(x)$.

The following results on the strong maximum principle for the operator $(\mathcal{I} - \Delta)^{\log}$ can be deduced from [17, Theorem 1.1] (see also [13] and [10, Theorem 6.1]).

**Lemma 2.1** (Strong maximum principle). *Let $\Omega \subset \mathbb{R}^N$ be a domain, and let $u \in \mathcal{L}_0(\mathbb{R}^N)$ be a continuous function on $\overline{\Omega}$ satisfying*

$$\begin{cases} (\mathcal{I} - \Delta)^{\log} u(x) \geq 0 & \text{in } \Omega, \\ u(x) \geq 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.1}$$

*then $u > 0$ in $\Omega$ or $u = 0$ a.e. in $\mathbb{R}^N$.*

Now, we will prove the following maximum principles for the logarithmic Schrödinger operator.

**Theorem 2.2** (Maximum principle for antisymmetric functions). *Let $\Omega$ be a bounded domain in $\Sigma_\lambda$. Assume that $\omega_\lambda \in \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is Dini continuous on $\Omega$ and is lower semi-continuous on $\overline{\Omega}$. If*

$$\begin{cases} (\mathcal{I} - \Delta)^{\log} \omega_\lambda(x) \geq 0 & \text{in } \Omega, \\ \omega_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \backslash \Omega, \\ \omega_\lambda(x^\lambda) = -\omega(x) & \text{in } \Sigma_\lambda, \end{cases} \tag{2.2}$$

*then*

$$\omega_\lambda \geq 0 \quad in\ \Omega. \tag{2.3}$$

*Furthermore, if* $\omega_\lambda(x) = 0$ *at some point in* $\Omega$*, then we have*

$$\omega_\lambda = 0 \quad a.e.\ in\ \mathbb{R}^N. \tag{2.4}$$

*These conclusions hold for unbounded region* $\Omega$ *if we further assume that*

$$\liminf_{|x|\to\infty} \omega_\lambda(x) \geq 0.$$

*Proof.* If $\omega_\lambda$ is not nonnegative on $\Omega$, then the lower semi-continuity of $\omega_\lambda$ on $\bar{\Omega}$ implies that there exists a $x^o \in \bar{\Omega}$ such that

$$\omega_\lambda(x^o) := \min_{\bar{\Omega}} \omega_\lambda(x) < 0.$$

One can further deduce from (2.2) that $x^o$ is in the interior of $\Omega$. It follows that

$$\begin{aligned}(\mathcal{I} - \Delta)^{\log}\omega_\lambda(x^o) &= c_N \text{P.V.} \int_{\mathbb{R}^N} \frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y|^N} \kappa(|x^o - y|)dy \\ &= c_N \text{P.V.} \left( \int_{\Sigma_\lambda} \frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y|^N} \kappa(|x^o - y|)dy \right. \\ &\qquad \left. + \int_{\Sigma_\lambda} \frac{\omega_\lambda(x^o) - \omega_\lambda(y^\lambda)}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|)dy \right).\end{aligned} \tag{2.5}$$

Since $|x^o - y| \leq |x^o - y^\lambda|$, we have $\frac{1}{|x^o - y|} \geq \frac{1}{|x^o - y^\lambda|}$ and $\kappa(|x^o - y|) \geq \kappa(|x^o - y^\lambda|)$ as $\kappa$ is a decreasing function, and, therefore,

$$\frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y|^N} \kappa(|x^o - y|) \leq \frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|),$$

since $\omega_\lambda(x^o) - \omega_\lambda(y) \leq 0$.

Thus, we obtain from (2.5) that

$$\begin{aligned}(\mathcal{I} - \Delta)^{\log}\omega_\lambda(x^o) &\leq c_N \text{P.V.} \int_{\Sigma_\lambda} \left( \frac{\omega_\lambda(x^o) - \omega_\lambda(y)}{|x^o - y^\lambda|^N} + \frac{\omega_\lambda(x^o) + \omega_\lambda(y)}{|x^o - y^\lambda|^N} \right) \kappa(|x^o - y^\lambda|)dy \\ &= c_N \text{P.V.} \int_{\Sigma_\lambda} \frac{2\omega_\lambda(x^o)}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|)dy < 0,\end{aligned} \tag{2.6}$$

which contradicts (2.2). Therefore, our assumption is wrong, and, consequently, we have $\omega_\lambda(x) \geq 0$ in $\Omega$.

Now, we have proved that $\omega_\lambda(x) \geq 0$ in $\Omega$. If there is some point $\tilde{x} \in \Omega$ such that $\omega_\lambda(\tilde{x}) = 0$, then, from Lemma 2.1, we derive immediately $\omega_\lambda = 0$ a.e. in $\mathbb{R}^N$.

For unbounded domain $\Omega$, the condition

$$\liminf_{|x|\to\infty} \omega_\lambda(x) \geq 0$$

ensures that the negative minimum of $\omega_\lambda$ must be attained at some point $x^o$, then we can derive the same contradiction as above.

This completes the proof of Theorem 2.2. ■

The following decay at infinity will also be necessary for proving subsequent results.

**Theorem 2.3** (Decay at infinity). *Let $\Omega$ be an unbounded domain in $\Sigma_\lambda$. Suppose that a Dini continuous $\omega_\lambda \in \mathcal{L}_0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is a solution to*

$$\begin{cases} (\mathcal{I}-\Delta)^{\log}\omega_\lambda(x) + c(x)\omega_\lambda(x) \geq 0 & \text{in } \Omega, \\ \omega_\lambda(x) \geq 0 & \text{in } \Sigma_\lambda \backslash \Omega, \\ \omega_\lambda(x^\lambda) = -\omega(x) & \text{in } \Sigma_\lambda \end{cases} \tag{2.7}$$

*with the measurable function $c(x)$ such that*

$$\liminf_{|x|\to\infty} |x|^{\frac{1+N}{2}} c(x) \geq 0. \tag{2.8}$$

*Then, there exists a constant $R_o > 0$ such that if*

$$\omega_\lambda(x^o) = \min_{\Omega} \omega_\lambda(x) < 0, \tag{2.9}$$

*then*

$$|x^o| \leq R_o. \tag{2.10}$$

*Proof.* We prove the assertion by contradiction. Suppose that (2.10) is false, then by (2.7) and (2.9), we have

$$\omega_\lambda(x^o) = \min_{\Sigma_\lambda} \omega_\lambda(x) < 0.$$

After a direct calculation, we obtain

$$\begin{aligned}
&(\mathcal{I}-\Delta)^{\log}\omega_\lambda(x^o) \\
&= c_N \text{P.V.} \int_{\mathbb{R}^N} \frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y|^N}\kappa(|x^o-y|)dy \\
&= c_N \text{P.V.} \int_{\Sigma_\lambda} \left( \frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y|^N}\kappa(|x^o-y|) + \frac{\omega_\lambda(x^o)-\omega_\lambda(y^\lambda)}{|x^o-y^\lambda|^N}\kappa(|x^o-y^\lambda|) \right) dy \\
&\leq c_N \text{P.V.} \int_{\Sigma_\lambda} \left( \frac{\omega_\lambda(x^o)-\omega_\lambda(y)}{|x^o-y^\lambda|^N} + \frac{\omega_\lambda(x^o)+\omega_\lambda(y)}{|x^o-y^\lambda|^N} \right) \kappa(|x^o-y^\lambda|)dy \\
&= c_N \text{P.V.} \int_{\Sigma_\lambda} \frac{2\omega_\lambda(x^o)}{|x^o-y^\lambda|^N}\kappa(|x^o-y^\lambda|)dy < 0.
\end{aligned}$$

Now, we fix $\lambda$, and when $|x^o| \geq \lambda$, we have $B_{|x^o|}(\breve{x}) \subset \widetilde{\Sigma}_\lambda := \mathbb{R}^N \backslash \Sigma_\lambda$ with $\breve{x} = (3|x^o| + x_1^o, (x^o)')$. Then, for $y \in \widetilde{\Sigma}_\lambda$, if $|x^o| \geq \frac{R_\infty}{4}$ with sufficiently large $R_\infty$, we can deduce that $|x^o - y| \leq |x^o - \check{x}| + |\check{x} - y| \leq |x^o| + 3|x^o| = |4x^o|$ which together with the fact that $\kappa$ is a decreasing function implies that

$$\frac{\kappa(|x^o - y|)}{|x^o - y|} \geq \frac{\kappa(|4x^o|)}{|4x^o|}.$$

Thus, from (1.5) and $\kappa(r) = 2^{1-\frac{N}{2}} r^{\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(r)$, if $R_\infty$ is sufficiently large, we have

$$\begin{aligned}\int_{\Sigma_\lambda} \frac{1}{|x^o - y^\lambda|^N} \kappa(|x^o - y^\lambda|) dy &= \int_{\tilde{\Sigma}_\lambda} \frac{\kappa(|x^o - y|)}{|x^o - y|^N} dy \geq \int_{B_{|x^o|}(\check{x})} \frac{\kappa(|4x^o|)}{|4x^o|^N} dy \\ &\geq \int_{B_{|x^o|}(\check{x})} \frac{2^{1-\frac{N}{2}} \mathcal{K}_{\frac{N}{2}}(|4x^o|)}{|4x^o|^{\frac{N}{2}}} dy \\ &\geq \frac{c_\infty \omega_N}{2^{\frac{3N}{2}} |x^o|^{\frac{1+N}{2}} e^{4|x^o|}} := \frac{C}{|x^o|^{\frac{1+N}{2}} e^{4|x^o|}},\end{aligned} \tag{2.11}$$

where $C = c_\infty \omega_N 2^{-\frac{3N}{2}}$ is a positive constant.

It follows that

$$0 \leq (\mathcal{I} - \Delta)^{\log} \omega_\lambda(x^o) + c(x^o)\omega_\lambda(x^o) \leq \left( \frac{C}{|x^o|^{\frac{1+N}{2}} e^{4|x^o|}} + c(x^o) \right) \omega_\lambda(x^o),$$

or equivalently,

$$\frac{C}{|x^o|^{\frac{1+N}{2}} e^{4|x^o|}} + c(x^o) \leq 0.$$

Now, if $|x^o|$ is sufficiently large, this would contradict (2.8). Therefore, (2.10) holds.

This completes the proof of Theorem 2.3. ■

## 3. Proof of the main theorem

*Proof of Theorem 1.1.* Let $T_\lambda, \Sigma_\lambda, x^\lambda$, and $\omega_\lambda$ be defined as in the previous section. Then, at the points where $\omega_\lambda(x) < 0$, it is easy to verify that, for $\xi_\lambda(x) \in (u_\lambda(x), u(x))$, we have

$$(\mathcal{I} - \Delta)^{\log} \omega_\lambda(x) + m\omega_\lambda(x) = u_\lambda^p(x) - u^p(x) = p\xi_\lambda^{p-1}(x)\omega_\lambda(x) \geq pu^{p-1}(x)\omega_\lambda(x), \tag{3.1}$$

because $\omega_\lambda(x) < 0$ and $\xi_\lambda(x) < u(x)$.

*Step 1.* We will show that, for sufficiently negative $\lambda$,

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.2}$$

First, from the assumption (1.9), for each fixed $\lambda$, $\lim_{|x| \to \infty} \omega_\lambda(x) = 0$. In fact, by (1.9), we have $\lim_{|x| \to \infty} u(x) = a$, and $\lim_{|x| \to \infty} u_\lambda(x) = a$ implying that

$$\lim_{|x| \to \infty} \omega_\lambda(x) = 0.$$

Thus, if (3.2) is false, then the negative minimum of $\omega_\lambda$ can be obtained at some point, say, $x^o$ in $\Sigma_\lambda$, that is,

$$\omega_\lambda(x^o) = \min_{\Sigma_\lambda} \omega_\lambda(x) < 0.$$

Set $c(x) := m - pu^{p-1}(x)$ in (3.1), and then, the assumption (1.9) implies that $c \in L^\infty(\mathbb{R}^N)$ and

$$\lim_{|x|\to\infty} c(x) \geq 0.$$

Consequently, from Theorem 2.3, it follows that there exists $R_o > 0$ (independent of $\lambda$), such that

$$|x^o| \leq R_o. \tag{3.3}$$

Therefore, by choosing $\lambda < -R_o$ and, consequently, $|x^\lambda| > R_0$ for $x \in \Sigma_\lambda$, we obtain by (3.3) that

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.4}$$

*Step 2.* Step 1 provides a starting point, from which we can now move the plane $T_\lambda$ to the right as long as (3.2) holds to its limiting position. Define

$$\lambda_o := \sup\left\{\lambda \mid \omega_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \forall \mu \leq \lambda\right\}.$$

By (3.3), we know that $\lambda_o < \infty$.

Next, we will show via a contradiction argument that

$$\omega_{\lambda_o}(x) \equiv 0 \quad \forall\, x \in \Sigma_{\lambda_o}. \tag{3.5}$$

Suppose, on the contrary, that

$$\omega_{\lambda_o}(x) \geq 0 \quad \text{and} \quad \omega_{\lambda_o}(x) \not\equiv 0 \quad \text{in } \Sigma_{\lambda_o}, \tag{3.6}$$

then we must have

$$\omega_{\lambda_o}(x) > 0 \quad \forall\, x \in \Sigma_{\lambda_o}. \tag{3.7}$$

In fact, if (3.7) is violated, then there exists a point $\hat{x} \in \Sigma_{\lambda_o}$ such that

$$\omega_{\lambda_o}(\hat{x}) = \min_{\Sigma_{\lambda_o}} \omega_{\lambda_o}(x) = 0.$$

It means that $u_{\lambda_o}(\hat{x}) = u(\hat{x})$. Then, it follows from (3.1) that

$$(\mathcal{I} - \Delta)^{\log}\omega_{\lambda_o}(\hat{x}) = u^p_{\lambda_o}(\hat{x}) - u^p(\hat{x}) = u^p(\hat{x}) - u^p(\hat{x}) = 0.$$

Hence, Theorem 2.2 implies that $\omega_{\lambda_o}(\hat{x}) \equiv 0$ in $\Sigma_{\lambda_o}$, which contradicts (3.6). Thus, (3.7) holds.

Now, we will show that the plane $T_\lambda$ can be moved further right. More precisely, there exists an $\varepsilon > 0$ such that, for any $\lambda \in [\lambda_o, \lambda_o + \varepsilon)$, we have

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.8}$$

Once it is proved, this will contradict the definition of $\lambda_o$. Therefore, (3.5) must be valid.

Let us now prove (3.8). In fact, by (3.7), we have $\omega_{\lambda_o}(x) > 0$, $x \in \Sigma_{\lambda_o}$, which in turn implies that there is a constant $c_o > 0$ and $\delta > 0$ such that

$$\omega_{\lambda_o}(x) \geq c_o > 0, \quad x \in \overline{\Sigma_{\lambda_o-\delta} \cap B_{R_o}(0)}.$$

Since $\omega_\lambda$ is continuous with respect to $\lambda$, there exists an $\varepsilon > 0$ such that, for $\lambda \in [\lambda_o, \lambda_o + \varepsilon)$, we have

$$\omega_\lambda(x) \geq 0, \quad x \in \Sigma_{\lambda_o-\delta} \cap B_{R_o}(0). \tag{3.9}$$

Moreover, combining (3.3) with (3.9), we deduce that $w_\lambda(x) \geq 0$ on $\Sigma_{\lambda_0-\delta}$.

To proceed with the proof, we need the following small volume maximum principle (see [10, Theorem 6.1 (iii)] and [17, Theorem 1.3]).

**Lemma 3.1.** *Let $\Omega$ be an open set of $\mathbb{R}^N$. Consider the following problem on $\Omega$:*

$$\begin{cases} (\mathcal{I} - \Delta)^{\log} u(x) \geq c(x)u & \text{in } \Omega, \\ u \geq 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{3.10}$$

*with $c \in L^\infty(\mathbb{R}^N)$.*

*Then, there exists $\delta > 0$ such that for every open set $\Omega \subset \mathbb{R}^N$ with $|\Omega| \leq \delta$ and any solution $u \in \mathcal{V}_\omega(\Omega)$ of (3.10) in $\Omega$, where the space $\mathcal{V}_\omega(\Omega)$ is given in* [10, Section 6], *we have $u \geq 0$ in $\mathbb{R}^N$.*

Consequently, according to Lemma 3.1 (by taking $\Omega = (\Sigma_\lambda \setminus \Sigma_{\lambda_o-\delta}) \cap B_{R_o}(0)$), we obtain that (3.8) holds.

The arbitrariness of the $x_1$-direction leads to the radial symmetry of $u(x)$ about some point in $\mathbb{R}^N$, and the monotonicity is a consequence of the fact that (3.4) holds.

This completes the proof of Theorem 1.1. ∎

**Acknowledgments.** The authors thank the reviewers for several invaluable comments and suggestions that have led to an overall improvement in the quality of the manuscript.

**Funding.** RZ is supported by the Postdoctoral Fellowship Program of CPSF under Grant number GZC20232913. VK and MR are supported by the FWO Odysseus 1 Grant number G.0H94.18N: analysis and partial differential equations, the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021) and by FWO Senior Research Grant number G011522N. MR is also supported by EPSRC grant EP/R003025/2.

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Received 6 April 2023; revised 21 March 2024.

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