

Jacobson radicals, abelian p -groups and the \oplus_c -topology

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ABSTRACT – In studying the Jacobson radical of the endomorphism ring of a separable abelian p -group, Sands (1984) identified the useful *Condition (C)*. Our central result is that the group G satisfies Condition (C) precisely when it is complete in its \oplus_c -topology, which uses the set of subgroups $X \leq G$ such that G/X is a direct sum of cyclics as a neighborhood base of 0. This equivalence is then used to compute the Jacobson radicals of the endomorphism rings of a variety of such groups, including those in the so-called *Keef class*. It is shown that Sands’ attempt to “complete” an arbitrary group with respect to Condition (C) is equivalent to D’Este’s (1980) attempt to show that the \oplus_c -topology is completable. Since Mader (1983) provided a counterexample to D’Este’s result, it follows that Sands’ result also fails.

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1. Introduction

In the following, all groups will be additively written abelian p -groups, where p is a fixed prime. Except when specifically noted, we will use the notation and terminology of [6]; we will use the notation \mathbb{Z}_{p^n} and \mathbb{Z}_{p^∞} instead of $\mathbb{Z}(p^n)$ and $\mathbb{Z}(p^\infty)$. By a Σ -cyclic group, we will mean one that is a direct sum of cyclics. We will denote the height function on a group G by $|\cdot|$, or perhaps $|\cdot|_G$ if there is some possibility of confusion. Restricting $|\cdot|$ to $G[p]$ turns it into a *valuated vector space* and we will assume some basic familiarity with the vocabulary of that subject (see [5]). We will be extensively discussing material from [23]; so in keeping with that work, we will write functions to the right of their arguments.

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Though an important way to study any algebraic structure is through an analysis of its endomorphisms, this is especially true for abelian p -groups. A classical result of Kaplansky [11] states that if G_1 and G_2 are such groups with endomorphism rings E_1 and E_2 , then any ring isomorphism $E_1 \rightarrow E_2$ is induced by a group isomorphism $G_1 \rightarrow G_2$. In particular, any automorphism of the endomorphism ring of a group is necessarily inner.

In analyzing any ring, it is natural to try to identify its Jacobson radical. In this note G will always denote some group, E will be the ring of all endomorphisms of G and J will be the Jacobson radical of E ; when we have occasion to consider some other group A , we will use the notation E_A and J_A .

Let $H \subseteq E$ be the ideal of all ϕ such that $|y| < |y\phi|$ for all $y \in G[p]$, called the *Pierce radical*. If G is separable (or more generally, fully transitive), then $J \subseteq H$, and if G is torsion-complete, Pierce [22] showed that $J = H$.

Next, let $M \subseteq E$ be the ideal of all ϕ such that $(p^n G[p])\phi = 0$ for some $n < \omega$. If we set $N = M \cap H$, we will always have $N \subseteq J$. If G is Σ -cyclic, Liebert [19] showed that $J = N$. This was later generalized to all totally projective groups by Hausen [8].

When G is separable, Sands [23] defined another ideal, $C \subseteq E$, as all ϕ such that whenever $\{g_n\}_{n < \omega}$ is a (p -adically) Cauchy sequence in $G[p]$, then $\{g_n\phi\}_{n < \omega}$ converges in $G[p]$. Note that any endomorphism ϕ of G extends uniquely to an endomorphism, $\bar{\phi}$, of the torsion-completion \bar{G} . So $\phi \in C$ if and only if $(\bar{G}[p])\bar{\phi} \subseteq G[p]$. From this characterization of C it follows easily that $M \subseteq C$.

Sands also defined the following condition on G :

CONDITION (C). Whenever $\{g_n\}_{n < \omega}$ is a sequence in $G[p]$ that is Cauchy, but not convergent (in the p -adic topology), then there is a Σ -cyclic group B and a homomorphism $\phi: G \rightarrow B$, such that $\{g_n\phi\}_{n < \omega}$ does not converge in B .

The following gives the main results of that paper:

THEOREM 1.1 ([23, Theorems 3 and 5]). *If G is a separable group, then $C \cap H \subseteq J \subseteq H$, and if G satisfies Condition (C), then $C \cap H = J$.*

In [23] it was asked whether $J = C \cap H$ for all separable groups, but in a very nice paper, Dugas [4] used a classical construction of Corner [2] to produce a counterexample.

Sands' Condition (C) has appeared in a number of other situations. For example, in [7, Proposition 4.1] it was shown that a group satisfying it may or may not have *minimal full inertia* (see that paper for the meaning of this term) and in [16] it was applied to the topic of *small pairs*.

In an apparently different direction, D'Este [3] defined the \oplus_c -topology on G to be the linear topology having $\mathcal{F}_G := \{X \leq G : G/X \text{ is } \Sigma\text{-cyclic}\}$ as a neighborhood

base of 0. The \oplus_c -completion of G , which we denote by \check{G} , is the inverse limit of the quotients G/X over all $X \in \mathcal{F}_C$. There is a natural homomorphism $G \rightarrow \check{G}$ whose kernel is $p^\omega G$; so when discussing this topology, it is customarily assumed that G is separable. It is clear that any large subgroup $L \subseteq G$ is in \mathcal{F}_G . So there is a natural embedding of \check{G} into the torsion-completion, \bar{G} .

Though in [7] and elsewhere, Condition (C) has been described as “technical”, the main purpose of this work is to show that this is not the case. In particular, connecting the above two notions, our central result (Theorem 2.2) states that a separable group G satisfies Condition (C) if and only if it is \oplus_c -complete.

If $\{A_i\}_{i \in I}$ is a collection of groups, then in [13] the torsion subgroup of the direct product, $\prod_{i \in I} A_i$, was referred to as their t -product, and was denoted by $\prod_{i \in I}^t A_i$. That work considered the smallest class of groups containing the cyclic groups that is closed under direct sums, summands and t -products over non-measurable index sets. Later, Fuchs [6, p. 341] referred to this as the *Keef class*, denoting it by K_p . In [13, Corollary 10] it was proved that every group in K_p is \oplus_c -complete, and in [12, Theorem 2] it was shown that two groups in K_p are isomorphic if and only if they have isometric socles. In particular, we can conclude that if G is in K_p , then $J = C \cap H$ (Corollary 2.4).

The expression $J = C \cap H$ is theoretically nice, and though the ideal H is straightforward, the ideal C may not be as simple. For some of the groups in K_p , it is shown how to describe J using decompositions of the group into direct sums and non-measurable t -products (Examples 2.6–2.9). These computations extend the aforementioned work of Liebert and Pierce. It is also shown how the Jacobson radical can be used to identify when $G \in K_p$ is either Σ -cyclic or torsion-complete (Theorem 2.13).

The \oplus_c -completion has other important applications besides those discussed here regarding Jacobson radicals. If $B = \bigoplus_{n < \omega} \mathbb{Z}_{p^{n+1}}$ is the standard Σ -cyclic group, then for a separable group G we can let $G^* = \text{Hom}(G, B)$. If $R = E_B$ is the endomorphism ring of B , then G^* and B will be (right) R -modules and we can let $G^{**} = \text{Hom}_R(G^*, B)$. There is clearly a homomorphism $G \rightarrow G^{**}$ and if G has non-measurable cardinality; then we can identify G^{**} with \check{G} (see, for example, [15, Proposition 2.2]). So the \oplus_c -completion (at least for groups of reasonable size) can be viewed as a natural embedding into a double-dual with respect to a fundamental Σ -cyclic group.

An important step in studying this duality was showing that when \bar{B} is torsion-complete and M is a reduced group such that $M/p^\omega M$ is Σ -cyclic, then $G := \text{Tor}(\bar{B}, M)$ is \oplus_c -complete ([15, Theorem 3.4]). Groups of this form were also used in [17]. We apply our central result to compute J for any such group (Proposition 3.1).

Not every group is \oplus_c -complete. For a separable group G , \check{G} will have two natural topologies: the completion topology and its own \oplus_c -topology. If these agree, then G is \oplus_c -completable. Mader [20] showed that G is \oplus_c -completable if and only if \check{G}/G is divisible. Suppose G is \oplus_c -completable and $\check{J} = J_{\check{G}}$. Since \check{G} is \oplus_c -complete, then, in some sense, we know \check{J} . What does this tell us about J itself? We show that if $\phi \in J$, then $\check{\phi} \in \check{J}$, and that if \check{G}/G has finite rank, then the converse holds (Theorem 3.2). We also provide an example that shows that this converse may fail if \check{G}/G has infinite rank (Example 3.4).

Mader [21] also showed that there are groups that are not \oplus_c -completable, contradicting several statements in [3]. In [23, Theorem 6] it was claimed that for a separable group G , even if G does not satisfy Condition (C), it naturally embeds in a group $\hat{G} \subseteq \bar{G}$ that does satisfy this property. We show that this result fails for almost precisely the same reason that D'Este's claim that every separable group is \oplus_c -completable fails (Theorem 3.6).

2. Our central result and the class K_p

In [13] an alternate description of the \oplus_c -completion was given, as opposed to its being an inverse limit. Suppose G is a separable group and $\{x_i\}_{i < \omega}$ is a Cauchy sequence in G in the p -adic topology on G . We say that $\{x_i\}_{i < \omega}$ is \oplus_c -Cauchy if for every $X \in \mathcal{F}_G$, $\{x_i + X\}_{i < \omega}$ converges in G/X (in its p -adic topology). This is applied in the following:

LEMMA 2.1 ([13, Lemma 2]). *If G is a separable group, then (the \oplus_c -completion) \check{G} can be identified with the subgroup of \bar{G} consisting of the p -adic limits of all \oplus_c -Cauchy sequences.*

We now state the central result of this note.

THEOREM 2.2. *A separable group G satisfies Condition (C) if and only if it is complete in its \oplus_c -topology.*

PROOF. Clearly, if $S_1 \subseteq S_2$ are separable groups and the sequence $\{x_n\}_{n < \omega} \subset S_1$ converges in S_1 , then it also converges in S_2 . This implies that in Condition (C), there is no loss of generality in assuming that each ϕ is surjective, or even a canonical epimorphism onto a factor group. In other words, Condition (C) is logically equivalent to the following:

(C') Given any Cauchy sequence $\{g_n\}_{n < \omega}$ in $G[p]$ which is not convergent in G , there exists a subgroup $X \in \mathcal{F}_G$ such that $\{g_n + X\}_{n < \omega}$ is not convergent in G/X .

Suppose first that G does not satisfy Condition (C); we need to show G is not \oplus_c -complete. Let $\{g_n\}_{n<\omega}$ be witness to the assumption that G does not satisfy condition (C'). In other words, $\{g_n\}_{n<\omega}$ is a (p -adically) Cauchy sequence in $G[p]$ that is not convergent in G , but, however, for all subgroups $X \in \mathcal{F}_G$, $\{g_n + X\}$ does converge in G/X in the p -adic topology on this quotient.

Since $\bar{G}[p]$ is complete (in the p -adic topology), there is a $y \in \bar{G}[p] \setminus G[p]$ such that $g_n \rightarrow y$. As $\{g_n\}_{n<\omega}$ is clearly a \oplus_c -Cauchy sequence, we can conclude that $y \in \check{G} \setminus G$, i.e., that G is not \oplus_c -complete.

Conversely, suppose G is not \oplus_c -complete; we need to show that Condition (C') also fails. We are assuming that $G \neq \check{G}$, and we can view \check{G} as a subgroup of \bar{G} . Let $z \in \check{G} \setminus G$; replacing z by $p^k z$ for some $k < \omega$, we may assume $z \in \check{G} \setminus G$, but $pz \in G$.

Since G is pure in \bar{G} , it follows that $pz = pw$ for some $w \in G$. Therefore, $y := z - w \in \check{G}[p] \setminus G$.

Let $\{g_n\}_{n<\omega}$ be a (p -adically) Cauchy sequence in $G[p]$ that converges to y ; in particular, it does not converge in G . On the other hand, since $y \in \check{G}$, for any $X \in \mathcal{F}_G$, $\{g_n + X\}_{n<\omega}$ does converge in G/X , by Lemma 2.1. This shows that G does not satisfy condition (C'). ■

So, using Theorem 2.2, we can restate the principal results of [23] as follows:

COROLLARY 2.3. *If G is a separable group, then $C \cap H \subseteq J \subseteq H$, and further, if G is \oplus_c -complete, then $C \cap H = J$.*

Since every group in K_p is known to be \oplus_c -complete, the next result follows immediately.

COROLLARY 2.4. *If G is in K_p , then $J = C \cap H$.*

Though this is a nice, general result, one can ask how it might be applied in specific cases. To that end, we briefly review how K_p can be obtained: Let $\mathcal{S}_0 = \mathcal{T}_0$ be the class of bounded groups (which are both Σ -cyclic and torsion-complete). Suppose now that α is some ordinal and we have defined \mathcal{S}_β and \mathcal{T}_β for all ordinals $\beta < \alpha$. Let $G \in \mathcal{S}_\alpha$ if $G \cong \bigoplus_{i \in I} A_i$, where for each $i \in I$, $A_i \in \mathcal{T}_\beta$ for some $\beta < \alpha$. Dually, we let $G \in \mathcal{T}_\alpha$ if $G \cong \prod_{i \in I} A_i$, where I is non-measurable and for each $i \in I$, $A_i \in \mathcal{S}_\beta$ for some $\beta < \alpha$. Letting \mathcal{R}_α be the union $\mathcal{S}_\alpha \cup \mathcal{T}_\alpha$, then \mathcal{R} , the union of the \mathcal{R}_α over all ordinals α , will be the smallest class containing the cyclic groups that is closed under direct sums and (non-measurable) t -products. Finally, K_p will be the groups isomorphic to a summand of some group in \mathcal{R} .

Note that \mathcal{S}_1 is the class of Σ -cyclic groups and \mathcal{T}_1 is the class of torsion-complete groups. Clearly, \mathcal{S}_2 will be the direct sums of torsion-complete groups and \mathcal{T}_2 will be the (non-measurable) t -products of Σ -cyclic groups.

To more clearly describe the Jacobson radicals of the endomorphism rings of the groups in some of these classes, thereby extending the results of Liebert (for Σ -cyclic groups) and Pierce (for torsion-complete groups), we introduce some convenient terminology: If G and A are separable groups, then a homomorphism $\phi: G \rightarrow A$ will be said to be *socle-bounded* if there is an $n < \omega$ such that $(p^n G[p])\phi = 0$. An endomorphism of G is in the ideal M if and only if it is socle-bounded, and N is the collection of all socle-bounded endomorphisms in H .

Any homomorphism $\phi: G \rightarrow A$ between separable groups will naturally extend to a homomorphism $\bar{\phi}: \bar{G} \rightarrow \bar{A}$; we will say that ϕ is *socle-extending* if $\bar{G}[p]\bar{\phi} \subseteq A[p]$. It is clear that if ϕ is socle-bounded, then it is socle-extending: Suppose $(p^n G[p])\phi = 0$ and $G = B \oplus C$, where B is a maximal p^n -bounded summand of G . It follows that $\bar{G}[p] = B[p] \oplus \bar{C}[p]$. Since $C[p] = p^n G[p]$ is dense in $\bar{C}[p]$, we have $\bar{C}[p]\bar{\phi} = 0$. Therefore, $\bar{G}[p]\bar{\phi} = B[p]\phi \subseteq A$.

If A is torsion-complete, then any $\phi: G \rightarrow A$ is socle-extending and if A is Σ -cyclic, then $\phi: G \rightarrow A$ is socle-extending if and only if it is socle-bounded. An endomorphism of G is in C if and only if it is socle-extending.

We also pause for a bit of notation: If $A = \bigoplus_{i \in I} A_i$, $I' \subseteq I$, and $A' = \bigoplus_{i \in I'} A_i$ then let $\sigma_{I'}: A \rightarrow A'$ be the usual projection. Similarly, if $A := \prod'_{i \in I} A_i$, $I' \subseteq I$, and $\prod'_{i \in I'} A_i$, then let $\pi_{I'}: A \rightarrow A'$ be the usual projection.

The following elementary observation is the key to our computations.

LEMMA 2.5. *Suppose G and A_i for $i \in I$ are separable groups.*

- (a) *A homomorphism $\phi: G \rightarrow \prod_I A_i$ is socle-extending if and only if for every $i \in I$, $\phi\pi_i$ is socle-extending.*
- (b) *A homomorphism $\phi: G \rightarrow \bigoplus_I A_i$ is socle-extending if and only if there is a finite subset $F \subseteq I$ with corresponding cofinite subset $Q = I \setminus F$, such that $\phi\sigma_Q$ is socle-bounded and $\phi\sigma_F$ is socle-extending.*

PROOF. Part (a) clearly holds. Part (b) is a variation on standard results on homomorphisms from direct products to direct sums (see, for example, [12, Main Lemma]). If ϕ is socle-extending, those results guarantee the existence of a cofinite subset Q such that $\phi\sigma_Q$ is socle-bounded; it is also clear that $\phi\sigma_F$ must be socle-extending. The converse is a consequence of the fact that any socle-bounded homomorphism is socle-extending. ■

Again, an endomorphism $\phi: G \rightarrow G$ is in Sands' ideal C if and only if it is socle-extending. Therefore, the following computations of the Jacobson radical J follow directly from the above discussion. (Each B is Σ -cyclic and each \bar{B} is torsion-complete.)

EXAMPLE 2.6. Suppose $G = \prod_{i \in I}^t B_i \in \mathcal{T}_2$ is a t -product of Σ -cyclic groups (where I is non-measurable). Then J is the collection of all $\phi \in H$ such that for all $i \in I$, there is an $n < \omega$ such that $(p^n G[p])\phi\pi_i = 0$. In other words, for all $i \in I$, $\phi\pi_i$ is socle-bounded.

EXAMPLE 2.7. Suppose $G = \bigoplus_{i \in I} \bar{B}_i \in \mathcal{S}_2$ is a direct sum of torsion-complete groups. Then J is the collection of all $\phi \in H$ such that for some $n < \omega$ and finite subset $F \subseteq I$, $(p^n G[p])\phi \subseteq \bigoplus_{i \in F} \bar{B}_i$. In other words, there is a cofinite subset $Q \subseteq I$ such that $\phi\sigma_Q$ is socle-bounded.

EXAMPLE 2.8. Suppose $G = \prod_{k \in K}^t \bigoplus_{i \in I_k} \bar{B}_{i,k} \in \mathcal{T}_3$ (K non-measurable). Then J is the collection of all $\phi \in H$ such that for all $k \in K$, there is an $n < \omega$ and a finite subset $F_k \subseteq I_k$ such that $(p^n G[p])\phi\pi_k \subseteq \bigoplus_{i \in F_k} \bar{B}_{i,k}$. In other words, for all $k \in K$, there is a cofinite subset $Q_k \subseteq I_k$ such that $\phi\pi_k\sigma_{Q_k}$ is socle-bounded.

EXAMPLE 2.9. Suppose $G = \bigoplus_{k \in K} \prod_{i \in I_k}^t B_{i,k} \in \mathcal{S}_3$ (each I_k non-measurable). Then J is the collection of all $\phi \in H$ such that there is a cofinite subset $Q \subseteq K$ such that $\phi\sigma_Q$ is socle-bounded and for every $k \in F = K \setminus Q$ and $i \in I_k$, $\phi\sigma_k\pi_i$ is socle-bounded.

Clearly, the pattern illustrated in the above examples could be extended indefinitely, giving explicit descriptions of J for G in an infinite number of such classes.

A homomorphism $\phi: G \rightarrow H$ is said to be *small* if for every $m < \omega$ there is an $n < \omega$ such that $(p^n G[p^m])\phi = 0$. Recall that G is *thick* if whenever B is Σ -cyclic, then every homomorphism $\phi: G \rightarrow B$ is small. This means that for thick groups, the \oplus_c -topology agrees with the large subgroup topology. It follows that a separable group G is thick if and only if $\check{G} = \bar{G}$. Dually, G is *thin* if whenever \bar{B} is torsion-complete, then every homomorphism $\phi: \bar{B} \rightarrow G$ is small.

Small homomorphisms are those whose kernels contain large subgroups. Since the elements of K_p are determined by their socles, we want to reinterpret these notions in terms of socle-bounded homomorphisms. By [14, Lemma 19], for given groups G and X , to show every homomorphism $G \rightarrow X$ is small, it suffices to show that for every $m < \omega$, every homomorphism $p^m G \rightarrow X$ is socle-bounded. We expand this idea in the next result, which, though not appearing in the literature in exactly this form, is nevertheless essentially well known.

LEMMA 2.10. *Suppose \mathcal{X} is a class of groups with the property that if $m < \omega$, then $X \in \mathcal{X}$ if and only if $p^m X \in \mathcal{X}$. For a group G , the following hold:*

- (a) *For all $X \in \mathcal{X}$, every homomorphism $G \rightarrow X$ is small if and only if for all $X \in \mathcal{X}$, every such homomorphism is socle-bounded.*
- (b) *For all $X \in \mathcal{X}$, every homomorphism $X \rightarrow G$ is small if and only if for all $X \in \mathcal{X}$, every such homomorphism is socle-bounded.*

PROOF. We will verify (a), the proof of (b) being analogous (and simpler). Since every small homomorphism is socle-bounded, the implication \Rightarrow is immediate. So suppose every such homomorphism is socle-bounded.

By [14, Lemma 19], we need to show that for every $X \in \mathcal{X}$ and $m < \omega$, every homomorphism $\phi: p^m G \rightarrow X$ is socle-bounded. It is elementary that there is a group X' such that $p^m X' = X$; and since $X \in \mathcal{X}$, we can conclude that $X' \in \mathcal{X}$, as well. It is also clear that $\phi: p^m G \rightarrow X = p^m X'$ must extend to a homomorphism $\phi': G \rightarrow X'$. By hypothesis, ϕ' is socle-bounded, which easily implies that ϕ is socle-bounded, completing the proof. ■

The last result allows us to interpret thin and thick groups in terms of socle-bounded homomorphisms. Again, it is a reformulation of essentially well-known results.

PROPOSITION 2.11. *In the following G and each A_i for $i \in I$ is a separable group.*

(a) *The following are equivalent:*

- (1a) *G is thick.*
- (2a) *Every homomorphism $G \rightarrow B$, where B is Σ -cyclic, is socle-bounded.*
- (3a) *For every homomorphism $\phi: G \rightarrow A$, where A is a direct sum $\bigoplus_{i \in I} A_i$, there is a cofinite subset $Q \subseteq I$ such that $\phi \sigma_Q$ is socle-bounded.*

(b) *The following are equivalent:*

- (1b) *G is thin.*
- (2b) *Every homomorphism $\bar{B} \rightarrow G$, where \bar{B} is torsion-complete, is socle-bounded.*
- (3b) *For every homomorphism $\phi: A \rightarrow G$, where A is a t -product $\prod_{i \in I}^t A_i$ (I non-measurable), there is a cofinite subset $Q \subseteq I$ such that if $\mu_Q: \prod_{i \in Q}^t A_i \rightarrow A$ is the obvious inclusion, then $\mu_Q \phi$ is socle-bounded.*

PROOF. The equivalence of (1a) and (2a) follows from Lemma 2.10 (a) (with \mathcal{X} as the Σ -cyclic groups). Likewise, the equivalence of (1b) and (2b) follows from Lemma 2.10 (b) (with \mathcal{X} as the torsion-complete groups).

We will show (3b) is equivalent to (1b) and (2b); the corresponding argument for (3a) is simpler, and in any case, will not be used in this note.

First, (3b) clearly implies (2b): if $B = \bigoplus_{i < \omega} B_i$, where each B_i is a direct sum of copies of $\mathbb{Z}_{p^{i+1}}$, we can just use $I = \omega$ and $A_i = B_i$ which easily gives (2b).

Conversely, showing (2b) implies (3b) is a variation on a standard argument regarding homomorphisms defined on products over a non-measurable index set; so we will be content with an outline. If $I = \omega$ and the conclusion failed, we could use the condition to construct a homomorphism $\bar{B} \rightarrow G$, where B is an unbounded Σ -cyclic group, that is not socle-bounded, contradicting (2b). The argument going from ω to an arbitrary non-measurable index set is then an exercise in Boolean algebras. ■

A valued vector space W with valuation $|\cdot|$ is *free* if it is isometric to a valued direct sum $\bigoplus_{n < \omega} W_n$, where $|x| = n$ whenever $0 \neq x \in W_n$; in other words, exactly when there is a Σ -cyclic group B whose socle is isometric to W . If V is a valued vector space, then a subspace $U \subseteq V$ is called *cofree* if there is a valued decomposition $V = W \oplus U$, where W is free. A useful fact is that if B is Σ -cyclic and $\phi: G \rightarrow B$ is a homomorphism with kernel K , then $U = K[p]$ is cofree in $V = G[p]$ (see, for example, [10, Lemma 1]).

LEMMA 2.12. *Suppose $A \in \mathcal{R}$ and G is a separable thin group. If $\phi: A \rightarrow G$ is a homomorphism, then there is a cofree subsocle $U \subseteq A[p]$ such that $U\phi = 0$.*

PROOF. Recall that for each ordinal α , $\mathcal{R}_\alpha = \mathcal{S}_\alpha \cup \mathcal{T}_\alpha$, where \mathcal{S}_α and \mathcal{T}_α are defined after Corollary 2.4 and $\mathcal{R} = \bigcup_\alpha \mathcal{R}_\alpha$. We induct on α , where $A \in \mathcal{R}_\alpha$. The result is trivial for $\alpha = 0$ since A would be bounded.

Case 1 – $A \in \mathcal{S}_\alpha$: It follows that $A = \bigoplus_{i \in I} A_i$, where for each $i \in I$, $A_i \in \mathcal{T}_\beta$ for some $\beta < \alpha$. So by induction, for each $i \in I$, there is a cofree $U_i \subseteq A_i[p]$ such that $U_i\phi = 0$. It follows that $U := \bigoplus_{i \in I} U_i$ is cofree in $A[p]$ and $U\phi = 0$, as desired.

Case 2 – $A \in \mathcal{T}_\alpha$: It follows that $A = \prod_{i \in I}^t A_i$, where for each $i \in I$, $A_i \in \mathcal{S}_\beta$ for some $\beta < \alpha$. By Proposition 2.11 (3b), for some cofinite subset $Q \subseteq I$ and $n < \omega$, if $Z = \prod_{i \in Q} p^n A_i[p]$, then $Z\phi = 0$. For ease of labeling, suppose $F := I \setminus Q = \{1, 2, \dots, k\}$. For each $i \in F$, restricting ϕ to A_i and using induction, there is a cofree subspace $U_i \subseteq A_i[p]$ such that $U_i\phi = 0$. If we let $U = U_1 \oplus \dots \oplus U_k \oplus Z$, then it is easy to check that U is cofree in $A[p]$ and $U\phi = 0$. ■

Though it is easy to inductively describe the elements of \mathcal{R} , less is known about their summands, i.e., the groups in K_p . Certainly, a summand of a group in \mathcal{S}_1 or \mathcal{T}_1 is of the same form. It is a classical result that a summand of a group in \mathcal{S}_2 remains in that class. It is a result of Lady [18] that if $A = \bigoplus_{i \in I} \bar{B}_i \in \mathcal{T}_2$, where each B_i is Σ -cyclic and I is *countable*, then any summand of A will have the same form. But even in the case of \mathcal{T}_2 , it is not known whether the class is *always* closed under summands.

We do have the following result, which describes the simplest examples of groups in K_p in several ways, including one using the Jacobson radical. We start with some additional terminology: A Σ -cyclic group B will be called *almost standard* if $B \cong \bigoplus_{n < \omega} \langle b_n \rangle$, where b_n has order p^{e_n} and $e_0 < e_1 < e_2 < \dots$; we also refer to \bar{B} as almost standard torsion-complete. Clearly, B is almost standard if and only if it is an unbounded summand of the standard Σ -cyclic group. The assignment $b_n \mapsto p^{e_{n+1}-e_n} b_{n+1}$ extends to an endomorphism $\rho: B \rightarrow B$ which we call a *right-shift* endomorphism; of course, there is also defined a right-shift endomorphism $\bar{\rho}: \bar{B} \rightarrow \bar{B}$. Note that $\rho \in H_B \setminus J_B$ and $\bar{\rho} \in J_{\bar{B}} \setminus N_{\bar{B}}$.

The next result again connects K_p with the classical computations of Jacobson radicals of endomorphism rings by Liebert and Pierce. It shows that, as observed by Pierce, not only is $J = H$ a *necessary* condition for an element of K_p to be torsion-complete, it is actually *sufficient*, as well. And dually, not only, as observed by Liebert, is $J = N$ a necessary condition for an element of K_p to be Σ -cyclic, it is also sufficient.

THEOREM 2.13. *Suppose G is in K_p .*

(a) *The following are equivalent:*

- (1a) *G is torsion-complete.*
- (2a) *$J = H$.*
- (3a) *G has no unbounded Σ -cyclic summand.*
- (4a) *G is thick.*

(b) *The following are equivalent:*

- (1b) *G is Σ -cyclic.*
- (2b) *$J = N$.*
- (3b) *G has no unbounded torsion-complete summand.*
- (4b) *G is thin.*

PROOF. Starting with (a), Pierce proved (1a) implies (2a). We prove (2a) implies (3a) by considering the contrapositive. So assume $G = B \oplus K$, where B is an unbounded direct sum of cyclics. Restricting to a summand, we may assume B is almost standard. Letting ρ be a right-shift endomorphism on B and 0 on K , it readily follows that $\rho \in H \setminus J$, so that (2a) fails, as desired.

We also prove (3a) implies (4a) by contrapositive. If G is not thick, then by Proposition 2.11 (2a) there is a non-socle-bounded homomorphism $\phi: G \rightarrow B$, where B is Σ -cyclic. If U is the kernel of ϕ intersected with $G[p]$, then there is a valuated decomposition $G[p] = W \oplus U$, where W is free and unbounded. Let A be Σ -cyclic such

that $A[p]$ is isometric to W . By [12, Theorem 1], the projection $G[p] \rightarrow W$ extends to a homomorphism $\gamma: G \rightarrow A$. Since for all $n < \omega$ we have $(p^n G[p])\phi = p^n A[p]$, it follows that K , the kernel of γ , is pure in G . As A is a pure projective, we can conclude that $G \cong A \oplus K$, so that (3a) fails, as desired.

Finally, since G is \oplus_c -complete, (4a) implies $G = \check{G} = \bar{G}$. Therefore, (4a) implies (1a), finishing the proof of (a).

Turning to (b), Liebert proved (1b) implies (2b). We show (2b) implies (3b) by considering the contrapositive. So assume $G = \bar{B} \oplus K$, where \bar{B} is unbounded and torsion-complete. Restricting to a summand, we may assume \bar{B} is almost standard. If we let $\bar{\rho}$ be a right-shift endomorphism on \bar{B} and 0 on K , it readily follows that $\bar{\rho} \in J \setminus N$ so that (2b) fails, as desired.

We also prove (3b) implies (4b) by contrapositive. If G is not thin, then by Proposition 2.11 (2b) there is a non-socle-bounded homomorphism $\phi: \bar{B} \rightarrow G$, where \bar{B} is torsion-complete. Restricting to a summand, we may assume \bar{B} is almost standard. Clearly, $\bar{B}[p]$ is compact in its p -adic topology, and this means that $V := (\bar{B}[p])\phi$ will still be compact and unbounded when we give it the p -adic topology from G . So $V \subseteq G[p]$ is complete in G 's p -adic topology. Therefore, there is an unbounded Σ -cyclic group A such that $\bar{A}[p]$ is isometric to V (that is, $|x|_{\bar{A}} = |x|_G$ for all $x \in V$). By [12, Theorem 1], the embedding $\bar{A}[p] \cong V \subseteq G[p]$ extends to a homomorphism $\gamma: \bar{A} \rightarrow G$. It follows that $\bar{A}\gamma$ will be a pure subgroup of G , and since it is torsion-complete, it is a summand of G , showing that (3b) fails, as desired.

Finally, suppose (4b) holds; we want to prove (1b) does, as well. Since $G \in K_p$ there is a group $A \in \mathcal{R}$ such that $A = G \oplus K$ for some group K . Let $\pi: A \rightarrow G$ be the usual projection. By Lemma 2.12 there is a cofree valuated summand $U \subseteq A[p]$ contained in $K[p]$. If $A[p] = W \oplus U$ with W free, it follows that $G[p]$ is isometric to a valuated summand of W , so that it is free. This, however, implies that G is Σ -cyclic, so that (1b) holds, as desired. ■

3. Other applications of the central result

Our next result pertains to a class of groups that had significant roles in both [15] and [17].

PROPOSITION 3.1. *Suppose \bar{B} is unbounded and torsion-complete, M is a reduced group such that $M/p^\omega M$ is Σ -cyclic and $G = \text{Tor}(\bar{B}, M)$. Then J is the collection of all $\phi \in H$ such that there is a finite subgroup $A \subseteq (p^\omega M)[p]$ and an $n < \omega$ with $(p^n G[p])\phi \subseteq \text{Tor}(\bar{B}, A) = \text{Tor}(\bar{B}[p], A)$.*

PROOF. By [15, Theorem 3.4], G is \oplus_c -complete, and hence by Corollary 2.3, $J = C \cap H$. Let K be a high subgroup of M (i.e., maximal with respect to $K \cap p^\omega M = 0$). It follows that K embeds in $M/p^\omega M$ and so is Σ -cyclic. It is a standard result that K is $p^{\omega+1}$ -pure in M . Therefore, there is a $p^{\omega+1}$ -pure sequence

$$0 \rightarrow \text{Tor}(\bar{B}, K) \rightarrow G \rightarrow \text{Tor}(\bar{B}, M/K) \rightarrow 0.$$

Since K is Σ -cyclic, so is the left-hand term in this sequence; so $W := \text{Tor}(\bar{B}, K)[p]$ is a free valued vector space.

Let $\{a_i\}_{i \in I}$ be any basis for the vector space $p^\omega M[p]$. So $(M/K)[p] \cong p^\omega M[p] \cong \bigoplus_{i \in I} \langle a_i \rangle$ and $M/K \cong \bigoplus_I \mathbb{Z}_{p^\infty}$, which shows that $\text{Tor}(\bar{B}, M/K) \cong \bigoplus_I \bar{B}$. So there are isometries

$$\bigoplus_I \bar{B}[p] \cong \text{Tor}(\bar{B}, M/K)[p] \cong \text{Tor}(\bar{B}, p^\omega M)[p] \subseteq G.$$

Therefore, there is a valued splitting

$$G[p] \cong \text{Tor}(\bar{B}, K)[p] \oplus \text{Tor}(\bar{B}, p^\omega M)[p] \cong W \oplus \left(\bigoplus_I \bar{B}[p] \right).$$

By definition, if $\phi \in E$, then $\phi \in C$ if and only if $(\bar{G}[p])\bar{\phi} \subseteq G[p]$. As usual, this happens precisely when there is an $n < \omega$ and a finite subset $F \subseteq I$ such that $(p^n G[p])\phi \subseteq \bigoplus_F \bar{B}[p]$. Setting $A = \bigoplus_{i \in F} \langle a_i \rangle$, the result clearly follows. ■

We now discuss whether there is an analogue of Sands' result when G is not necessarily \oplus_c -complete, but it is \oplus_c -completable.

THEOREM 3.2. *Suppose G is a separable group that is \oplus_c -completable. So every endomorphism $\phi: G \rightarrow G$ uniquely extends to an endomorphism $\check{\phi}: \check{G} \rightarrow \check{G}$. Let $\check{E} = E_{\check{G}}$ and $\check{J} = J_{\check{G}}$.*

(a) *If $\phi \in J$, then $\check{\phi} \in \check{J}$.*

(b) *If $\check{\phi} \in \check{J}$ and \check{G}/G has finite p -rank, then $\phi \in J$.*

PROOF. Since \check{G}/G is divisible, \check{G} is also pure in \bar{G} and $\bar{\check{G}} = \bar{G}$.

For (a), suppose $\phi \in J$; we want to verify $\check{\phi} \in \check{J}$. Defining \check{C} and \check{H} in the obvious way, we need to show $\check{\phi} \in \check{C}$ and $\check{\phi} \in \check{H}$. Since $\phi \in J \subseteq H$, it strictly increases heights in $G[p]$. Since $G[p]$ is dense in $\check{G}[p]$, it is easily argued that $\check{\phi} \in \check{H}$, i.e., it strictly increases heights in this larger socle. So we need to verify that $\check{\phi} \in \check{C}$.

Assume this fails, i.e., $(\bar{G}[p])\bar{\phi} \not\subseteq \bar{G}$. (The rest of the verification of (a) borrows from the proof of [23, Theorem 5]). If B is a basic subgroup of G , then there must

be a (p -adically) Cauchy sequence $\{x_n\}$ in $B[p]$ converging to some $y \in \bar{G}[p]$ such that $\{x_n\phi\} \subseteq G$ converges to $y\bar{\phi} \notin \check{G}$. Therefore, there is a subgroup $X \in \mathcal{F}_G$ and $\{x_n\phi + X\}$ does not converge in G/X .

Let $\alpha: B \oplus (G/X) \rightarrow G$ be $(b, g + X)\alpha = b$ and $\beta: G \rightarrow B \oplus (G/X)$ be $g\beta = (0, g + X)$. In the composition $\alpha\phi\beta$, $(x_n, 0) \mapsto (0, x_n\phi + X)$. If $\alpha\phi\beta$ were socle-bounded, it would follow that the sequence $x_n\phi + X$ is eventually constant, which is clearly not the case. So, since $B \oplus (G/X)$ is Σ -cyclic, we can conclude that $\alpha\phi\beta \notin J_{B \oplus (G/X)}$. Therefore, it follows from a result of Amitsur [1] that $\phi \notin J$, contrary to assumption.

Turning to (b), suppose that $\check{\phi} \in \check{J}$; we want to show $\phi \in J$. Let $\alpha \in E$ and consider $\xi := 1_G - \phi\alpha \in E$; we need to show ξ is an automorphism of G . Note that $\check{\xi}: 1_{\check{G}} - \check{\phi}\check{\alpha} \in \check{E}$, and since $\check{\phi} \in \check{J}$, $\check{\xi}$ is an automorphism of \check{G} . In particular, $\check{\xi}$, and therefore ξ , will be injective.

Since ξ induces an isomorphism $\check{G}/G \rightarrow \check{G}\check{\xi}/G\check{\xi} = \check{G}/G\xi$, $\check{G}/G\xi$ must be divisible. And since G is pure in \check{G} , $G/G\xi$ must be pure in $\check{G}/G\xi$, so that it is a summand. It follows that $\check{G}/G \cong \check{G}/G\xi \cong \check{G}/G \oplus G/G\xi$. Since \check{G}/G has finite rank, we can conclude that $G/G\xi = 0$, i.e., ξ is surjective. Therefore, ξ is an automorphism, as desired. ■

COROLLARY 3.3. *Let \bar{B} be an unbounded torsion-complete group and $G \subseteq \bar{B}$ be a pure dense subgroup such that \bar{B}/G has finite p -rank (i.e., is a direct sum of a finite number of copies of \mathbb{Z}_{p^∞}). Then $J = H$.*

PROOF. It is straightforward to check that G must be thick, so that $\check{G} = \bar{G}$ (for example, this follows as in [17, Proposition 2.9]). And since $\check{G}/G = \bar{G}/G$ is divisible, G is \oplus_c -completable. By Theorem 3.2, $\phi \in J$ if and only if $\check{\phi} \in \check{J}$ if and only if $\check{\phi} \in \check{H}$ if and only if $\phi \in H$. ■

Corollary 3.3 shows that outside of K_p , $J = H$ does not imply that G is torsion-complete (cf. Theorem 2.13). Similarly, outside of K_p , $J = N$ may not imply that G is Σ -cyclic: a group G is ω_1 -separable if every countable subgroup is contained in a Σ -cyclic summand; these are sometimes referred to as *sufficiently projective*. It was shown in [9] that $J = N$ for every ω_1 -separable group, but it is well known that there are such groups that fail to be Σ -cyclic.

The next example shows that the hypothesis that \check{G}/G has finite rank in Theorem 3.2 (b) is required. It is a simplified version of Dugas' construction in [4].

EXAMPLE 3.4. There is a thick group G (so, in particular, G is \oplus_c -completable with $\check{G} = \bar{G}$) with a $\phi \in E$ such that $\check{\phi} = \bar{\phi} \in \check{J} = \check{H}$, but $\phi \notin J$.

PROOF. Let B be the standard Σ -cyclic group, with \bar{B} the standard torsion-complete group. Let ϕ and $\bar{\phi}$ be right-shift endomorphisms on these groups. Let R be the p -adic completion of the polynomial ring $\mathbb{Z}[X]$, and for all $k(X) \in R$ and $y \in \bar{B}$, let $yk(X) = yk(\bar{\phi})$. So we can think of R as a complete separable subring of the endomorphism ring of \bar{B} .

It is easy to check that R satisfies Corner's condition (C) of [2, Theorem 2.1] (this is very different from Sands' Condition (C)). So by that result, there is a pure dense subgroup $G \subseteq \bar{B}$ containing B such that there is a decomposition $E = R \oplus E_s$, where E_s is the ideal of all small endomorphisms. By [16, Theorem 6.2], this G will be thick (as well as thin). Clearly $\check{\phi} = \bar{\phi} \in \check{H} = \check{J}$. On the other hand, there is a surjective ring homomorphism

$$E \rightarrow R \rightarrow R/pR \cong \mathbb{Z}_p[X].$$

If $\phi \in J$, it would follow that $1 + \phi$ is a unit of E . This would imply that $1 + X$ is a unit of $\mathbb{Z}_p[X]$, which is clearly not the case. Therefore, $\check{\phi} \in \check{J}$, but $\phi \notin J$, as desired. ■

Consider the following statement from [23]:

THEOREM 3.5 ([23, Theorem 6]). *Let G be a separable primary abelian group with basic subgroup B (so $B \subseteq G \subseteq \bar{B}$). Let*

$$\hat{G} = \{a \in \bar{B} : \text{for all } \alpha \in E_{\bar{B}}, G\alpha \subseteq B \text{ implies } a\alpha \in B\}.$$

Then \hat{G} is a separable primary abelian group satisfying Condition (C) (with $G \subseteq \hat{G} \subseteq \bar{B}$).

This result is incorrect; it is, in fact, a version of the error made by D'Este [3]. We can rewrite the above definition as follows:

$$\hat{G} = \{a \in \bar{B} : a\bar{\alpha} \in B \text{ whenever } \alpha \in \text{Hom}(G, B)\}.$$

THEOREM 3.6. *For a separable group G , we have $\check{G} = \hat{G}$.*

Before proving this result, recall Mader [21] showed that \check{G} may not be \oplus_c -complete, so, by Theorem 2.2, \hat{G} may not satisfy Condition (C), contradicting Theorem 3.5.

PROOF OF THEOREM 3.6. Clearly, if G is bounded, then $\hat{G} = G = \check{G}$ and the result is obvious. So we may assume that G and the basic subgroup B are unbounded.

Using the approach of Lemma 2.1, we can restate the result as follows:

For the Cauchy sequence $\{x_n\} \subseteq G$ with limit $a \in \bar{B}$, the following are equivalent:

(1) Whenever $X \in \mathcal{F}_G$, then $x_n + X$ converges in G/X , i.e., $\{x_n\}$ is \oplus_c -Cauchy.

(2) Whenever $\alpha: G \rightarrow B \subseteq G$ is a homomorphism, then $a\bar{\alpha} \in B$.

In the following argument, when we say that a sequence converges in some p -adic topology, we have to be clear as to the group (or subgroup) with respect to which our heights are being computed.

Suppose first that $\{x_n\}$ satisfies (1). Let $\alpha: G \rightarrow B$ be a homomorphism; we want to show $x_n\alpha$ converges in B (in its p -adic topology). If we let X be the kernel of α , then $G/X \cong G\alpha \subseteq B$ is Σ -cyclic. So, by (1), $x_n + X$ must converge in G/X to some $y + X$ (with $y \in G$) using the p -adic topology of $G/X \cong G\alpha$. This implies that $x_n\alpha$ must converge to $y\alpha \in B$ using the p -adic topology of B . Therefore, $a\bar{\alpha} = y\alpha \in B$, as desired.

Conversely, suppose $a \in \bar{B}$ satisfies (2) and $X \in \mathcal{F}_G$; we need to show that $x_n + X$ converges in the p -adic topology on G/X . Note that if $a \in \bar{B}[p^k]$, then we may assume $\{x_n\} \subseteq G[p^k]$. Fix this value of k .

If $G/X \cong S \oplus T$, where S is bounded and $\pi: G/X \rightarrow S$ is the usual projection, it is easy to see that $\{(x_n + X)\pi\} \subseteq S$ is actually finite. It follows that there is a decomposition $G/X = L \oplus M$, with $\{x_n + X\} \subseteq L$ and $L \cong \bigoplus_{n < \omega} L_n$, where each L_n is isomorphic to the direct sum of a finite number of copies of $\mathbb{Z}_{p^{n+1}}$; i.e., L is *semi-standard*. If $X' \subseteq G$ with $X'/X = M$, then $\{x_n + X\}$ converges in G/X if and only if $\{x_n + X'\}$ converges in $G/X' \cong L$. So, replacing X by X' , we may assume that the group G/X is semi-standard.

Suppose $G/X = \bigoplus_{m < \omega} \langle c_m \rangle$, where c_m has order p^{e_m} . Since G/X is semi-standard, after possibly reordering the c_m , we may assume $e_0 \leq e_1 \leq e_2 \leq \dots$. If G/X is bounded, it immediately follows that $\{x_n + X\}$ converges. So we may assume that the e_n increase without bound.

Since B is unbounded, there is clearly a decomposition $B = B_1 \oplus B_2$, where $B_1 = \bigoplus_{m < \omega} \langle b_m \rangle$, and if b_m has order p^{f_m} , then $e_m \leq f_m$ and $f_{m-1} \leq f_m$ (i.e., both the e_n and the f_n are increasing without bound and the f_n are at least as large as the e_n).

Mapping $c_m \mapsto c'_m := p^{f_m - e_m} b_m$ induces an injection $\gamma: G/X \rightarrow B$. If $\xi: G \rightarrow G/X$ is the canonical epimorphism, then $\alpha := \xi\gamma: G \rightarrow B$ also has kernel X . Let $A = G\alpha = (G/X)\gamma = \bigoplus_{m < \omega} \langle c'_m \rangle = \bigoplus_{m < \omega} \langle p^{f_m - e_m} b_m \rangle \subseteq B_1$.

Since

$$B/A \cong \left(\bigoplus_{m < \omega} \langle b_m \rangle / \langle c'_m \rangle \right) \oplus B_2$$

is separable, it follows that A is closed in B (using the p -adic topology of B or G). In addition, by (2), $a\bar{\alpha} \in B$, and since it is in the p -adic closure of A , $a\bar{\alpha}$ must, in fact, be in A .

Now, $\{x_n\alpha\}$ will converge to $a\bar{\alpha}$ using the p -adic topology of B or G , but what we need to show is that $\{x_n\alpha\}$ will converge to $a\bar{\alpha}$ using the p -adic topology of $A \cong G/X$.

Recall that $\{x_m\}$ and a are in $\bar{B}[p^k]$, so $\{x_m\alpha\}$ and $a\bar{\alpha}$ are in $A[p^k]$. We claim that $|-|_A$ and $|-|_G = |-|_B = |-|_{B_1}$ determine the same topology on $A[p^k]$. Fix some k' such that $k \leq e_{k'} \leq f_{k'}$. If $k' \leq j < \omega$, let

$$\begin{aligned} A_j &= A[p^k] \cap \left(\bigoplus_{j \leq m < \omega} \langle b_m \rangle \right) = \bigoplus_{j \leq m < \omega} \langle p^{f_{m-k}} b_m \rangle \\ &= A[p^k] \cap \left(\bigoplus_{j \leq m < \omega} \langle c'_m \rangle \right) = \bigoplus_{j \leq m < \omega} \langle p^{e_{m-k}} c'_m \rangle \\ &\cong \bigoplus_{j \leq m < \omega} \mathbb{Z}_{p^k}. \end{aligned}$$

It follows that when $k' \leq j < \omega$ we have

$$p^{f_j} B \cap A[p^k] \subseteq A_j \subseteq p^{f_{j-k}} B \cap A[p^k]$$

and

$$p^{e_j} A \cap A[p^k] \subseteq A_j \subseteq p^{e_{j-k}} A \cap A[p^k].$$

Therefore, $\{A_j\}_{k' \leq j < \omega}$ is a neighborhood base of 0 for either of the two relevant topologies on $A[p^k]$, so they must agree. And since $\{x_n\alpha\} \rightarrow a\bar{\alpha}$ using the p -adic topology of B or G , it follows that $\{x_n\alpha\} \rightarrow a\bar{\alpha}$ using the p -adic topology of $A \cong G/X$, as desired. ■

It appears that the error in the proof of [23, Theorem 6] was to assume that the topology on \hat{G} determined by $|-|_{\hat{G}}$ agrees with the topology on \hat{G} determined by $|-|_{\bar{G}}$. This, however, assumes that \hat{G} is pure in \bar{G} , i.e., that \hat{G}/G is divisible, i.e., that G is \oplus_c -completable.

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