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Lack of local controllability for a water-tank system when the time is not large enough

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Abstract. We consider the small-time local-controllability property of a water tank modeled by one-dimensional Saint-Venant equations, where the control is the acceleration of the tank. It is known from the work of Dubois et al. that the linearized system is not controllable. Moreover, concerning the linearized system, they showed that a traveling time T_* is necessary to bring the tank from one position to another for which the water is still at the beginning and at the end. Concerning the nonlinear system, Coron showed that local controllability around equilibrium states holds for a time large enough. In this paper, we show that for local controllability of the nonlinear system around the equilibrium states, the necessary time is at least $2T_*$ even for the tank being still at the beginning and at the end. The key point of the proof is a coercivity property for the quadratic approximation of the water-tank system.

1. Introduction

1.1. Statement of the main result

We consider a water tank with a length $L > 0$ in the time interval $(0, T)$ modeled by the following one-dimensional Saint-Venant system (see Figure 1):

$$\begin{cases} \partial_t H + \partial_x(vH) = 0 & \text{for } (t,x) \in (0,T) \times (0,L), \\ \partial_t v + \partial_x\Big(gH + \dfrac{v^2}{2}\Big) = -u(t) & \text{for } (t,x) \in (0,T) \times (0,L), \\ v(t,0) = v(t,L) = 0 & \text{for } t \in (0,T), \end{cases} \tag{1.1}$$

and

$$\ddot{D}(t) = u(t) \quad \text{for } t \in (0,T). \tag{1.2}$$

Here H denotes the height of the water, v is the horizontal velocity field of the water, u is the acceleration that is imposed on the tank, D is the position of the tank, and g is the gravity. Given $H_{\text{eq}} > 0$, one can easily check that $(H_{\text{eq}}, 0)$ is a solution of (1.1) and thus is an equilibrium of (1.1).

The well-posedness of system (1.1) will be discussed in Proposition 4.1. In this article, we are interested in *local controllability* of this system, in the following sense:

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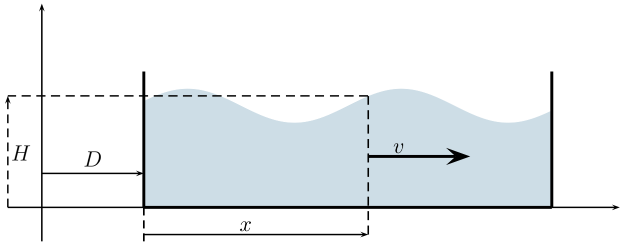

Figure 1. Water-tank problem.

Definition 1.1 (Local controllability of the water tank). Let $T > 0$, $H_{\text{eq}} > 0$. The water-tank system (1.1)–(1.2) is locally controllable around $(H, v) = (H_{\text{eq}}, 0)$ in time T if for every $\varepsilon > 0$, there exists $\eta > 0$ such that for every $H_0, H_1 \in C^1([0, L])$, with $\partial_x H_i(0) = \partial_x H_i(L)$, for every $v_0, v_1 \in C^1([0, L])$ with $v_i(0) = v_i(L) = 0$, and for every D_0, D_1, $s_0, s_1 \in \mathbb{R}$, if

$$\|H_0 - H_{\text{eq}}\|_{C^1} + \|v_0\|_{C^1} < \eta, \qquad \|H_1 - H_{\text{eq}}\|_{C^1} + \|v_1\|_{C^1} < \eta,$$
$$|D_0 + s_0 T - D_1| + |s_0 - s_1| < \eta, \quad \int_{[0,L]} H_0 = \int_{[0,L]} H_1 = L H_{\text{eq}},$$

there exists $u \in C^0([0, T])$ such that $u(0) = -\partial_x H_0(0)$, $\|u\|_{C^0} < \varepsilon$ and such that the solution $(H, v, D, \dot{D})$ of the water-tank system (1.1)–(1.2) with initial conditions $(H, v)(0, \cdot) = (H_0, v_0)$ and $(D, \dot{D})(0) = (D_0, s_0)$ is such that $(H, v)(T, \cdot) = (H_1, v_1)$ and $(D, \dot{D})(T) = (D_1, s_1)$.

When T is too small, we prove that system (1.1)–(1.2) is not locally controllable around $(H, v) = (H_{\text{eq}}, 0)$. In fact, we exhibit a family of simple trajectories that are impossible:

Theorem 1.2. *Let $L > 0$, $g > 0$, and $H_{\text{eq}} > 0$. Set*

$$T_* := \frac{L}{\sqrt{H_{\text{eq}} g}}. \tag{1.3}$$

Let $T \in (T_*, 2T_*)$. There exists $\eta > 0$ such that for every $u \in C^0([0, T])$ with $u(0) = 0$ and $\|u\|_{C^0([0,T])} < \eta$, if the solution $(H, v) \in (C^1([0, T] \times [0, L]))^2$ of the water-tank system (1.1) *with the initial data*

$$H(0, \cdot) = H_{\text{eq}} \quad \text{and} \quad v(0, \cdot) = 0, \tag{1.4}$$

satisfies

$$H(T, \cdot) = H_{\text{eq}} \quad \text{and} \quad v(T, \cdot) = 0, \tag{1.5}$$

and the solution D of (1.2) *satisfies*

$$\dot{D}(T) = \dot{D}(0) = 0, \tag{1.6}$$

then

$$u = 0 \text{ in } (0, T).$$

Conditions (1.4) and (1.5) read "u steers the water-tank system from $(H_{\mathrm{eq}}, 0)$ to $(H_{\mathrm{eq}}, 0)$ at time T", while condition (1.6) reads "the water tank ends with the same speed as it started with". As a consequence of Theorem 1.2, the water-tank system is not locally controllable around $(H, v) = (H_{\mathrm{eq}}, 0)$ and $(D, \dot{D}) = (0, 0)$ for time smaller than $2T_*$ (with controls small in $C^0([0, T])$).

Remark 1.3. Let us comment on Theorem 1.2:

(1) The regularity required for the control u, namely C^0, might be somehow unexpected. Standard well-posedness theorems for classical solutions of hyperbolic systems [11, §3.4] and [27, Chapter 4] would assume the source term u is (small) in C^1. The specific form of the source term ($u(t)$ instead of $u(t, x)$) is used for this point.

(2) The time T_* is the time needed for waves of the linearized equation to travel from one end of the tank to the other end, as observed in [22].

(3) The water-tank system (1.1) is a hyperbolic system. As such, there is a finite speed of propagation, and it is no surprise that local controllability fails in small time (see Remark 4.2). The interest of this theorem is that local controllability fails even for times larger than what the finite speed of propagation would suggest.

(4) Does a similar theorem hold for system (1.1) (without $(D, \dot{D})$ as part of the state)? This is an open problem, but an essential part of our method, the so-called quadratic drift, irremediably breaks down. We discuss this in Remark 3.10.

The controllability of the water-tank system was initially considered by Dubois, Petit and Rouchon [22], where the linearized system was considered. In particular, they proved for the linearized system that, given $T > T_*$, there exists a control that steers an equilibrium $(H_{\mathrm{eq}}, 0)$ back to itself while moving the water tank.

Concerning the nonlinear system, local controllability was investigated by Coron [12] using the *return method*. More precisely, Coron proved that local controllability around equilibrium states $(H_{\mathrm{eq}}, 0)$ for (H, v) starting with $(\dot{D}(0), D(0))$ near (s_0, D_0) and ending with $(\dot{D}(T), D(T))$ near $(s_0, D_0 + Ts_0)$ for a time T large enough. In particular, local controllability around $(H_{\mathrm{eq}}, 0)$ (for (H, v)) and $(0, 0)$ (for $(\dot{D}, D)$) holds for a large enough time.

Theorem 1.2 reveals new properties for local controllability of the nonlinear water-tank problem. First, Theorem 1.2 reveals that for $T_* < T < 2T_*$, contrary to the linearized system, one cannot steer an equilibrium $H(0, x) = H_{\mathrm{eq}}$, $v(0, x) = 0$ back to itself if the water tank ends with the same speed as it started with (except for the trivial trajectory where $u = 0$). Thus, Theorem 1.2 also points out that local controllability around $(H_{\mathrm{eq}}, 0)$ (for (H, v)) and $(0, 0)$ (for $(\dot{D}, D)$) proven by Coron [12] cannot hold in time less than $2T_*$.

The optimal time for the boundary controllability of hyperbolic systems has been studied extensively; see [19–21, 24–26], where the controls are on one side. This is different from the water-tank problem, which can be seen as a boundary control problem (see [22, §2] or equations (4.1), (4.2)), *where the control is the same on both sides*. This rigid structure on the control yields new phenomena and obstructions that require new ingredients to describe.

1.2. The main ideas of the proof and the organization of the paper

Using standard scaling arguments (see for instance [12, Section 2]), namely setting

$$\begin{aligned} H^*(t,x) &:= \frac{1}{H_{\mathrm{eq}}} H\left(\frac{L}{\sqrt{H_{\mathrm{eq}} g}} t, Lx\right), \\ v^*(t,x) &:= \frac{1}{\sqrt{H_{\mathrm{eq}} g}} v\left(\frac{L}{\sqrt{H_{\mathrm{eq}} g}} t, Lx\right), \end{aligned}$$

we may assume that $L = 1$, $g = 1$, and $H_{\mathrm{eq}} = 1$ and this will be assumed from now on. Note that in this case, T_* defined in Theorem 1.2 is $T_* = 1$.

The proof of Theorem 1.2 has its root in the *power series expansion method*; see, e.g., [16] and [14, Chapter 8]: since the linearized system does not give enough information to conclude about the local controllability of (1.1), we consider the second-order approximation. Indeed, the linearized system of (1.1) around the equilibrium $(1, 0)$ is[1]

$$\begin{cases} \partial_t h_1 + \partial_x v_1 = 0 & \text{for } (t,x) \in (0,T) \times (0,1), \\ \partial_t v_1 + \partial_x h_1 = -u(t) & \text{for } (t,x) \in (0,T) \times (0,1), \\ v_1(t,0) = v_1(t,1) = 0 & \text{for } t \in (0,T). \end{cases} \tag{1.7}$$

Simple computations prove that if $h_1(0,x) = 0$ and $v_1(0,x) = 0$, then $h_1(t, 1-x) = -h_1(t,x)$ and $v_1(t, 1-x) = v_1(t,x)$ whatever u is. Thus, the linearized system is not controllable. As usual, the second-order approximation system is given as

$$\begin{cases} \partial_t h_2 + \partial_x v_2 = -\partial_x (h_1 v_1) & \text{for } (t,x) \in (0,T) \times (0,1), \\ \partial_t v_2 + \partial_x h_2 = -\partial_x \left(\dfrac{v_1^2}{2}\right) & \text{for } (t,x) \in (0,T) \times (0,1), \\ v_2(t,0) = v_2(t,L) = 0 & \text{for } t \in (0,T). \end{cases} \tag{1.8}$$

The main idea is to prove that if a control steers the linearized system from 0 to 0, this second order always lies in some half-space, at least when $T < 2T_*$. More precisely, for $T_* < T < 2T_*$, we prove that for well-chosen functions ϕ, ψ, there exists $c > 0$

[1] We use lowercase h_1 instead of H_1, because h_1 is not an approximation of H but of $H - 1$.

such that for every control u that steers the linearized system from 0 to 0 and such that $\int_0^T u(s)\,\mathrm{d}s = 0$, with

$$U(t) := \int_0^t u(s)\,\mathrm{d}s, \tag{1.9}$$

we have the coercivity estimate

$$(h_2(T,\cdot),\phi) + (v_2(T,\cdot),\psi) \geq c\|U\|_{L^2}^2. \tag{1.10}$$

This means that the quadratic approximation of the water-tank system cannot be steered into the open half-space $\{(h,v) \in (L^2(0,1))^2,\ (h,\phi) + (v,\psi) < 0\}$. The rest of the proof consists in estimating the difference between the quadratic approximation and the nonlinear system in an appropriate way to deal with controls small in C^0.

We will use this notation U for the primitive of u that vanishes at zero throughout the article. The paper is organized as follows:

(1) in Section 2 we characterize the controls that steer the linearized system from 0 to 0;

(2) in Section 3 we analyze the second-order term and prove that it satisfies a "conditional H^{-1}-coercivity" property;

(3) in Section 4 we study the nonlinear system, and in particular we prove that the error between the nonlinear solution and the second-order approximation cannot counter the positivity of the second-order term.

1.3. Bibliographical comments

Our proof relies on the positivity of a scalar product of the quadratic approximation of the water-tank system (1.1). This kind of phenomenon was at the heart of several "lack of small-time local controllability" results for systems modeled by partial differential equations. Concerning examples in a finite-dimensional system, we refer to Beauchard and Marbach [5], and the references therein.

The quadratic obstructions for small-time local controllability were previously observed for the Schrödinger equation with bilinear control [7, 9, 13], the viscous Burgers equation [28], nonlinear heat equations [6], and a KdV system [18] where the speed of the propagation is infinite. All these results share the same core idea: the scalar product of the second-order approximation with appropriate test functions enjoys a coercivity property. Let us give a little detail for each of these cases.

For the Schrödinger equation with bilinear control, the existing results rely heavily on explicit computation using the eigenfunctions and eigenvalues of the operator $-\partial_x^2$. Note that in Coron's result [13] as well as Beauchard and Morancey's result [7], the equivalent of our coercivity estimate (1.10) also has $\|U\|_{L^2}^2$ on the right-hand side, leading to a lack of small-time local controllability with controls small in L^∞-norm. Bournissou [9] also has a similar coercivity estimate, with the nth iterated integral of the control instead of U, where n depends on the structure of the potential. This leads to a lack of small-time local controllability with controls small either in $W^{-1,\infty}$ (when $n = 1$) or H^{2n-3} (when $n \geq 2$).

Marbach [28] considered a viscous Burgers equation with control $u(t)$ as a source term. The main difficulty is the fact that the kernel of the quadratic approximation does not seem to be explicitly computable in a usable form. To tackle the problem, he rescaled the equation in time to transform the "small-time" aspect of the problem into a small-viscosity problem. This allowed him to compute an asymptotic expansion of the kernel of the quadratic approximation of the viscous Burgers equation in low-viscosity limit. Using this, Marbach succeeded in disproving the small-time local controllability with controls small in L^2-norm. A striking feature of [28] is the fact that the coercivity is associated with a noninteger Sobolev norm, which is in contrast with the finite-dimensional case; see [5].

Beauchard and Marbach [6] considered a class of nonlinear heat equation. They exhibit a range of phenomena. For instance, for some nonlinearities, they prove a coercivity estimate with the H^{-s}-norm of the control for some $s > 0$ that depends on the nonlinearity and that can be fractional. Also, for other nonlinearities, the quadratic term can actually *help* recover the small-time local controllability. This is the first example in which the quadratic term gives the local-controllability result.

Concerning the KdV equations [18], we proved that the KdV equation with Dirichlet boundary conditions and Neumann boundary control on the right is not small-time locally controllable with controls small in H^1 for some critical lengths, introduced previously by Rosier [31]. This fact is surprising when compared with known results on internal controls for the corresponding KdV system for which the small-time result holds (see, e.g., [30]). One of the main difficulties was to characterize the controls that steer the linearized equation from 0 to 0. The analysis is based on a complete characterization of controls which bring 0 to 0 for the linearized system that involves the Paley–Wiener theorem. The equivalent of the coercivity estimate (1.10) has the $H^{-2/3}$-norm of the control on the right-hand side.

The result of this paper compares to the previous ones in the following aspects:

- The control is internal, as was the case for the bilinear Schrödinger equation and the viscous Burgers equation, and unlike the KdV equation (where the control was at the boundary).
- Even if the computations are lengthy, we are able to compute the kernel of the second-order approximation in a very simple closed-form expression, which was more or less the case for the bilinear Schrödinger equation, but was not the case for the viscous Burgers equation and the KdV equation, where only an asymptotic expansion of the kernel was computed in closed form.
- We are able to disprove the small-time local controllability with controls small in C^0, which is a natural space for the known well-posedness results for C^1 solutions. This is different from some bilinear Schrödinger equations, some nonlinear heat equations, and the KdV equation, where the existing results require the control to be quite regular.

It is worth noting that less regular controls can change the situation. This is done for the Schrödinger equation by Bournissou [10], where the cubic terms surprisingly help recover local controllability even in the case where the quadratic term gives the obstruction if regular controls are used.

It is worth noting that the asymptotic stabilization of the water-tank control system (1.1) is a challenging open problem. Since the linearized control system is not asymptotically stabilizable, the usual techniques relying on this asymptotic stabilizability cannot be used. A possibility might be to use the phantom tracking method introduced in [2, 15]. A first step in this direction is the construction in [17] of feedback laws which asymptotically stabilize the linearized control system around the equilibrium ($H(x) = H_{\mathrm{eq}}(1+\gamma(L-2x))$, $v(x) = 0$, $u = -2\gamma$) if γ is a constant such that $|\gamma|$ is small enough but not 0.

Finally, we note that even with infinite speed of propagation in the linear setting, there might not be small-time controllability when there is a concentration of eigenfunctions [3,4,23] or when there is condensation of eigenvalues or eigenfunctions [8] (see also references therein).

2. Preliminary properties of the linearized system

As explained in Section 1.2, without loss of generality, we may assume that $g = 1$ and $L = H_{\mathrm{eq}} = 1$. Then the linearization of system (1.1) around the equilibrium $(H_{\mathrm{eq}}, 0) = (1, 0)$ is given by system (1.7).

This system can be rewritten as $\partial_t F + \mathcal{A} F = U(t)$ with $F = (h_1, v_1) \in (L^2)^2$, $U(t) = (0, -u(t))$ and $\mathcal{A}$ is the unbounded operator on $H = (L^2)^2$ with domain $D(\mathcal{A}) := H^1 \times H_0^1$ and defined by $\mathcal{A}(h, v) = (\partial_x v, \partial_x h)$. One can prove this system is well posed thanks, e.g., to the Lumer–Phillips theorem [29, Theorem 4.3].

2.1. Periodic change of variables

From now on, we denote

$$\mathbb{T} := \mathbb{R}/2\mathbb{Z}.$$

It is convenient to introduce the following periodic change of variables.

Definition 2.1. Given $F = (h, v) \in (L^2(0,1))^2$, define $\mathcal{C} F \in L^2(\mathbb{T})$ by

$$\mathcal{C} F(x) = \begin{cases} h(x) + v(x) & \text{for } 0 < x < 1, \\ h(-x) - v(-x) & \text{for } -1 < x < 0. \end{cases}$$

This change of variables transforms the linearized water-tank system into a transport equation with periodic boundary conditions:

Proposition 2.2. *Let $(H, v) \in (C^1([0,T] \times [0,1]))^2$ such that $v(t,0) = v(t,1) = 0$ and denote*

$$\zeta(t,\cdot) = \mathcal{C}(H(t,\cdot), v(t,\cdot)) \quad \textit{for } t \in [0,T].$$

Then

- ζ *is continuous in* $[0,T]\times\mathbb{T}$ *and is* C^1 *in* $[0,T]\times(\mathbb{T}\setminus\{0,1\})$*;*
- *if in addition* $W\in L^\infty([0,T]\times[0,1])^2$ *and*

$$\partial_t(H,v)(t,x)+\mathcal{A}(H,v)(t,x)=W(t,x)\quad \textit{for } (t,x)\in[0,T]\times[0,1],$$

then

$$\partial_t\zeta(t,x)+\partial_x\zeta(t,x)=\mathcal{C}\,W(t,x)\quad \textit{for every } t\geq 0 \textit{ and } x\in\mathbb{T}\setminus\{0,1\}. \tag{2.1}$$

Proof. The fact that $\zeta=\mathcal{C}(H,v)$ is C^1 in $[0,T]\times(\mathbb{T}\setminus\{0,1\})$ is a direct consequence of the definition of $\mathcal{C}$. The continuity at $x=0$ and $x=1$ results from the boundary conditions $v(t,0)=v(t,1)=0$.

The second point results from elementary computations. ■

Remark 2.3. We can check that $\mathcal{C}$ is an isometry (up to a factor 2) from $L^2(0,1)^2$ to $L^2(\mathbb{T})$, and that if $F=(H,v)\in C^1([0,1])^2$ with $v(0)=v(1)=0$, then $\|\mathcal{C}\,F\|_{W^{1,\infty}}\leq 2\|F\|_{C^1}$.

Using the characteristic method, one can obtain the following formula for the solution of (2.1):

$$\zeta(t,x)=\int_0^t w(s,x+s-t)\,\mathrm{d}s. \tag{2.2}$$

The linearized system (1.7) with zero initial conditions can be rewritten in the $\zeta_1(t,x)=\mathcal{C}(h_1,v_1)(t,x)$ variable as

$$(\partial_t+\partial_x)\zeta_1(t,x)=u(t)\theta(x),\quad \zeta_1(u,0,\cdot)=0, \tag{2.3}$$

where θ is a "square wave" function that is 2-periodic defined by

$$\theta(x)=\begin{cases}1 & \text{on } (-1,0),\\ -1 & \text{on } (0,1).\end{cases}$$

By the characteristic formula (2.2), we have

$$\zeta_1(t,x)=\int_0^t u(s)\theta(x+s-t)\,\mathrm{d}s.$$

Remark 2.4. We remark that $\theta(x+1)=-\theta(x)$, thus $\zeta_1(t,x+1)=-\zeta_1(t,x)$.

We will sometimes emphasize that $\zeta_1(t,x)$ depends on u by denoting it by $\zeta_1(u,t,x)$. This will be useful later on when we are considering several controls in Sections 4.2 and 4.3. But as long as there is no ambiguity in the control, we refrain from doing so. We will use similar notation for every quantity that depends on the control.

We will use the notation U defined in equation (1.9). Another useful formula for ζ_1 is the following:

Lemma 2.5. *Let $u \in L^2(0,T)$, extended by 0 for $t < 0$. Then, for $0 < x < 1$ and $t > 0$,*[2]

$$\zeta_1(t,x) = -U(t) + 2\sum_{k=0}^{+\infty}(-1)^k U(t-x-k).$$

Proof. If we define $\tilde{\zeta}_1$ as the right-hand side of this formula, we see that $\tilde{\zeta}_1(t,1) = -\tilde{\zeta}_1(t,0)$, so that the 1-antiperiodic extension of $\tilde{\zeta}_1$ is continuous in $(t,x) \in [0,T]\times\mathbb{T}$. Moreover, we see that for $0 < x < 1$ and $t > 0$,

$$(\partial_t + \partial_x)\tilde{\zeta}_1(t,x) = -u(t).$$

Thus, if we still denote by $\tilde{\zeta}_1$ the 1-antiperiodic extension of $\tilde{\zeta}_1$, we have $(\partial_t + \partial_x)\tilde{\zeta}_1 = u(t)\theta(x)$. Thus, $\tilde{\zeta}_1 = \zeta_1$. ∎

Let us finally give some estimates for ζ_1. In what follows, for $T > 0$, we use the notation $L^2_t L^2_x$ and $L^\infty_t L^2_x$ as shorthand for $L^2(0,T;L^2(\mathbb{T}))$ and $L^\infty(0,T;L^2(\mathbb{T}))$.

Proposition 2.6. *Let $T > 0$. The solution ζ of $(\partial_t + \partial_x)\zeta = w$ satisfies for some C independent of w,*

$$\|\zeta\|_{L^2_t L^2_x} \le C\|w\|_{L^2_t L^2_x}. \tag{2.4}$$

Moreover, in the case that the right-hand side is $w(t,x) = u(t)\theta(x)$, the solution ζ_1 satisfies, for some C independent of u,

$$\|\zeta_1\|_{L^2_t L^2_x} \le C\|U\|_{L^2}.$$

Proof. The first inequality is standard, and is proved with the characteristic formula (equation (2.2)) and the Cauchy–Schwarz inequality:

$$\begin{aligned}
\|\zeta\|^2_{L^2_t L^2_x} &= \int_{[0,T]^3\times\mathbb{T}} \mathbb{1}_{s_1,s_2\le t}\, w(s_1, x+s_1-t)w(s_2, x+s_2-t)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\,\mathrm{d}x \\
&\le \int_{[0,T]^3\times\mathbb{T}} w(s_1, x+s_1-t)^2\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_{[0,T]^3\times\mathbb{T}} w(s_1, x')^2\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t\,\mathrm{d}x',
\end{aligned}$$

where we also used the change of variables $x' = x + s_1 - t$. This implies the claimed estimate (2.4).

The second estimate is a direct consequence of Lemma 2.5. ∎

[2] Note that with this extension of u, we have for $t \le 0$, $U(t) = 0$, so that there are only a finite number of nonzero terms in the sum.

2.2. Control of the linearized system

We next discuss control properties for the linearized system. We give a controllability result when the target is 1-antiperiodic and we characterize the controls that steer 0 to 0.

Lemma 2.7. *Let $T > 1$. For any $\zeta_T \in H^1(\mathbb{T})$ that is 1-anti-periodic (i.e., $\zeta_T(x+1) = -\zeta_T(x)$), there exists a control $u \in L^2(0,T)$ such that $\int_0^T u(t)\,\mathrm{d}t = 0$, the solution ζ of the linear equation (2.3) with initial condition 0 satisfies $\zeta(T,\cdot) = \zeta_T$, and $\|U\|_{L^2(0,T)} \leq C\|\zeta_T\|_{L^2(\mathbb{T})}$ for some C independent of ζ_T.*

Proof. We construct the control using the so-called *flatness method*. The main point, inspired by Dubois, Petit, and Rouchon [22, Section 3.4], is that if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is in $H^1(\mathbb{R})$, then the function $\zeta\colon \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ defined by

$$\zeta(t,x) = \begin{cases} 2\varphi\Big(t - x + \dfrac{1}{2}\Big) - \varphi\Big(t + \dfrac{1}{2}\Big) - \varphi\Big(t - \dfrac{1}{2}\Big) & \text{if } 0 < x < 1, \\ -2\varphi\Big(t - x - \dfrac{1}{2}\Big) + \varphi\Big(t + \dfrac{1}{2}\Big) + \varphi\Big(t - \dfrac{1}{2}\Big) & \text{if } -1 < x < 0, \end{cases}$$

satisfies $(\partial_t + \partial_x)\zeta(t,x) = u(t)\theta(x)$ with $u(t) := \varphi'(t+1/2) + \varphi'(t-1/2)$. We aim to construct a function φ such that the trajectory associated to this formula goes from 0 at time 0 to ζ_T at time T.

To construct φ, for $T - 1/2 < x < T + 1/2$, we set $\varphi(x) := \zeta_T(T - x + 1/2)/2$, and we extend this as a function in $H^1(\mathbb{R})$, which is still denoted by φ, such that $\varphi = 0$ in $(-\infty, 1/2]$ (this is possible because $T > 1$). This extension can be done so that $\zeta_T \in L^2(\mathbb{T}) \mapsto \varphi \in L^2(\mathbb{R})$ is linear and continuous.

The first condition ensures that for $0 < x < 1$, the corresponding trajectory ζ satisfies $\zeta(T,x) = \zeta_T(x)$. Since ζ_T is 1-antiperiodic, we also have $\zeta(T,x) = \zeta_T(x)$ for $-1 < x < 0$. The fact that φ is zero on $[-1/2, 1/2]$ ensures that $\zeta(0,\cdot) = 0$.

For the last points, recall that $u(t) = \varphi'(t+1/2) + \varphi'(t-1/2)$ and $\varphi = 0$ on $(-\infty, 1/2]$, hence

$$U(t) = \int_0^t u(s)\,\mathrm{d}s = \varphi(t+1/2) + \varphi(t-1/2).$$

Thus,

$$\int_0^T u(t)\,\mathrm{d}t = \zeta_T(1)/2 + \zeta_T(0)/2 = 0$$

because ζ_T is assumed to be 1-antiperiodic. We also deduce that

$$\|U\|_{L^2(0,T)} \leq 2\|\varphi\|_{L^2(-1/2,T+1/2)} \leq C\|\zeta_T\|_{L^2}. \qquad \blacksquare$$

We now study the controls that steer 0 to 0. We only prove that the following condition is necessary, which is all we need, but we could also prove that it is sufficient.

Proposition 2.8. *Let $T \in (1,2)$ and let $u \in L^2(0,T)$ such that the solution ζ_1 of $(\partial_t + \partial_x)\zeta_1(t,x) = u(t)\theta(x)$, $\zeta_1(0,\cdot) = 0$ satisfies $\zeta_1(T,\cdot) = 0$. Then*

$$u(t) = 0 \ \text{ for } t \in (T-1, 1) \quad \text{and} \quad u(t+1) = u(t) \ \text{ for } t \in (0, T-1).$$

Proof. We use the formula for ζ_1 given by Lemma 2.5. Since $1 < T < 2$, $U(T - x - k)$ is zero whenever $k \geq 2$ and $0 < x < 1$. Hence, for $0 < x < 1$,

$$\zeta_1(T, x) = -U(T) + 2U(T - x) - 2U(T - x - 1).$$

Since $\zeta_1(T, x) = 0$, by differentiating in x, we get that for $0 < x < 1$,

$$u(T - x) = u(T - x - 1).$$

If $0 < t < T - 1$, we choose $x = T - t - 1$. This proves that $u(t + 1) = u(t)$ as claimed. If $T - 1 < t < 1$, we choose $x = T - t$, which gives $u(t) = u(t - 1)$. But $u(t - 1) = 0$ (we extended u by 0 on $(-\infty, 0)$), which proves that $u(t) = 0$. ∎

3. Second-order approximation for the nonlinear system

3.1. Periodic change of variables

In this section, we deal with the second-order approximation system given by (1.8). We rewrite it in the $\zeta_2 = \mathcal{C}(h_2, v_2)$ variables, which is done thanks to the following computation:

Lemma 3.1. *Let Q be the quadratic form on $\mathbb{R}^2$ defined by $Q(a, b) := (3a^2 - 2ab - b^2)/8$. Let $\zeta = \mathcal{C}(h, v)$ for some $(h, v) \in (C^1([0, 1]))^2$ with $v(0) = v(1) = 0$. Set*

$$w(x) := \mathcal{C}(-\partial_x(hv), -\partial_x(v^2/2)) \quad \text{and} \quad r(x) := Q(\zeta(x), \zeta(-x)).$$

Then

$$w(x) = -\partial_x r(x).$$

In the case that $\zeta(x) = \zeta_1(t, x)$, we will denote accordingly $w(x)$ by $w_1(t, x)$ and $r(x)$ by $r_1(t, x)$.

Proof. First, using the definition of $\mathcal{C}$,

$$w(x) = \begin{cases} -\partial_x(hv + v^2/2)(x) & \text{for } 0 < x < 1, \\ -\partial_x(hv - v^2/2)(-x) & \text{for } -1 < x < 0. \end{cases}$$

Inverting the relation $\zeta = \mathcal{C}(h, v)$, we get for $0 < x < 1$,

$$h(x) = \frac{1}{2}(\zeta(x) + \zeta(-x)),$$
$$v(x) = \frac{1}{2}(\zeta(x) - \zeta(-x)).$$

So,

$$hv(x) = \frac{1}{4}(\zeta^2(x) - \zeta^2(-x)),$$
$$\frac{1}{2}v^2(x) = \frac{1}{8}(\zeta(x) - \zeta(-x))^2,$$

thus,

$$hv(x) + \frac{1}{2}v^2(x) = \frac{1}{8}(3\zeta^2(x) - 2\zeta(x)\zeta(-x) - \zeta^2(-x)),$$
$$hv(x) - \frac{1}{2}v^2(x) = \frac{1}{8}(\zeta^2(x) + 2\zeta(x)\zeta(-x) - 3\zeta^2(-x)).$$

Finally, according to the definition of r, we rewrite this as

$$hv(x) + \frac{1}{2}v^2(x) = r(x),$$
$$hv(x) - \frac{1}{2}v^2(x) = -r(-x).$$

In both cases $0 < x < 1$ and $-1 < x < 0$, we find

$$w(x) = -\partial_x r(x).$$

Moreover, since $v(0) = v(1) = 0$, $\zeta = \mathscr{C}(h, v)$ is continuous on $\mathbb{T}$, and so is r. Thus, we conclude that in the sense of distributions,

$$w = -\partial_x r. \qquad \blacksquare$$

If we set

$$\zeta_2 := \mathscr{C}(h_2, v_2),$$

according to Proposition 2.2 and Lemma 3.1,

$$(\partial_t + \partial_x)\zeta_2(t, x) = -\mathscr{C}(\partial_x(h_1 v_1), \partial_x(v_1^2/2)) = w_1(t, x) = -\partial_x r_1(t, x). \tag{3.1}$$

Remark 3.2. We recall that $\zeta_1(t, x + 1) = -\zeta_1(t, x)$, so that $r_1(t, x + 1) = r_1(t, x)$. So, w_1 as well as ζ_2 is 1-periodic in x.

3.2. Kernel for ζ_2

In this section, we express ζ_2 (or more precisely scalar products of ζ_2) via a kernel that we compute explicitly. For $a, b \in \mathbb{R}$, we denote

$$a \vee b := \max\{a, b\} \quad \text{and} \quad a \wedge b := \min\{a, b\}.$$

We begin with the following lemma:

Lemma 3.3. *Let ϕ be a* 1*-periodic C^1 function. Let $u \in L^2(0,T)$ and let $\zeta_2(\cdot,\cdot)$ be the second-order correction for the water-tank system, i.e., the solution of $(\partial_t + \partial_x)\zeta_2(t,x) = w_1(t,x)$, $\zeta_2(0,\cdot) = 0$ (where w_1 was defined in Lemma 3.1). There exists a symmetric function $K_t(\cdot,\cdot) \in L^2((0,T)^2)$ (depending on ϕ) such that*

$$(\zeta_2(t,\cdot),\phi)_{L^2(\mathbb{T})} = \int_{[0,t]^2} K_t(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2.$$

This function is given by the following explicit formula, where q is the bilinear symmetric form on $\mathbb{R}^2$ associated to the quadratic form Q defined in Lemma 3.1, i.e., $q(a,b,a',b') = (3aa' - ab' - a'b - bb')/8$, and where $\Omega = \Omega(s_1,s_2) = \{(t_1,t_2) \in \mathbb{R}^2 \colon 2(s_1 \vee s_2 - t) < t_1 + t_2 < 0,\ 0 < t_1 - t_2 < 2\}$:

$$\begin{aligned} K_t(s_1,s_2) := \int_\Omega &\phi'(t_1 + t - s_1 \vee s_2) \\ &\times q\big(\theta(t_1 - |s_2 - s_1|), \theta(t_2 - |s_2 - s_1|), \theta(t_1), \theta(t_2)\big)\,\mathrm{d}t_1\,\mathrm{d}t_2. \end{aligned} \tag{3.2}$$

Proof. This is a mostly straightforward computation using the characteristics formula. Since ζ_2 satisfies the equation $(\partial_t + \partial_x)\zeta_2(t,x) = w_1(t,x)$ with $\zeta_2(0,\cdot) = 0$, then we have according to the characteristics formula,

$$\zeta_2(t,x) = \int_0^t w_1(s, s + x - t)\,\mathrm{d}s.$$

Since $w_1(s,x) = -\partial_x r_1(s,x)$, integrating by parts in x, and keeping in mind that everything is 2-periodic in x, we have

$$\begin{aligned} (\zeta_2(t,\cdot),\phi)_{L^2(\mathbb{T})} &= -\int_{\mathbb{T}\times[0,t]} \phi(x)\partial_x r_1(s, x + s - t)\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_{\mathbb{T}\times[0,t]} \phi'(x) r_1(s, x + s - t)\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Since the integrand is in fact 1-periodic (r_1 is according to Remark 3.2, and we assumed that ϕ is 1-periodic), we rewrite this as

$$(\zeta_2(t,\cdot),\phi)_{L^2(\mathbb{T})} = 2\int_{[0,1]\times[0,t]} \phi'(x) r_1(s, x + s - t)\,\mathrm{d}x\,\mathrm{d}s. \tag{3.3}$$

Recall that if Q is a quadratic form on $\mathbb{C}^d$ and q is its associated bilinear form, Fubini's theorem implies that for any compact subset X of $\mathbb{R}^n$ and $f\colon X \to \mathbb{C}^d$ measurable bounded, we have $Q(\int_X f(s)\,\mathrm{d}s) = \int_{X^2} q(f(s_1), f(s_2))\,\mathrm{d}s_1\,\mathrm{d}s_2$. Then, using the fact that $r_1(s,x) = Q(\zeta_1(s,x), \zeta_1(s,-x))$ and $\zeta_1(s,x) = \int_0^s u(s')\theta(x + s' - s)\,\mathrm{d}s'$, we get

$$r_1(s,x) = \int_{[0,s]^2} q\big(u(s_1)\theta(x+s_1-s), u(s_1)\theta(-x+s_1-s), u(s_2)\theta(x+s_2-s), u(s_2)\theta(-x+s_2-s)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2$$
$$= \int_{[0,s]^2} u(s_1)u(s_2)q\big(\theta(x+s_1-s), \theta(-x+s_1-s), \theta(x+s_2-s), \theta(-x+s_2-s)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2.$$

Plugging this into equation (3.3), we get that the formula

$$(\zeta_2(t,\cdot),\phi) = \int_{[0,t]^2} K_t(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

holds with

$$K_t(s_1,s_2) = 2\int_{[0,1]\times[0,t]} \mathbb{1}_{s_1,s_2\le s}\phi'(x)q\big(\theta(x+s_1-t), \theta(-x+s_1-2s+t), \theta(x+s_2-t), \theta(-x+s_2-2s+t)\big)\,\mathrm{d}x\,\mathrm{d}s.$$

We see from this expression and the symmetry of q that $K_t(s_1,s_2) = K_t(s_2,s_1)$. Since the integrand is 1-periodic in x, the change of variables $x' = x+s-t$ gives

$$K_t(s_1,s_2) = 2\int_{[0,1]\times[s_1\vee s_2,t]} \phi'(x-s+t)q\big(\theta(x+s_1-s), \theta(-x+s_1-s), \theta(x+s_2-s), \theta(-x+s_2-s)\big)\,\mathrm{d}x\,\mathrm{d}s.$$

Since $K_t(s_1,s_2) = K_t(s_2,s_1)$, to simplify the notation, we may assume that $s_2 = s_1\vee s_2$ and $s_1 = s_1\wedge s_2$. Then the change of variables $t_1 = x+s_2-s$, $t_2 = -x+s_2-s$, which satisfies $\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\,\mathrm{d}t_1\,\mathrm{d}t_2$ and $x-s+t = t_1-s_2+t$ proves

$$K_t(s_1,s_2) = \int_{\Omega} \phi'(t_1-s_2+t)q\big(\theta(t_1+s_1-s_2), \theta(t_2+s_1-s_2), \theta(t_1), \theta(t_2)\big)\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

where Ω is the image of $[0,1]\times[s_2,t]$ by the change of variables. Since $x = (t_1-t_2)/2$ and $s_2 - s = (t_1+t_2)/2$, $\Omega = \{0 < t_1 - t_2 < 2,\ 2(s_2-t) < t_1+t_2 < 0\}$. By possibly swapping s_1 and s_2, we assumed that $s_2 = s_1\vee s_2$ and $s_1 = s_1\wedge s_2$, hence this is the claimed formula. ■

The following proposition, with the symmetry of K_t proved in the previous lemma, allows us to compute K_t in closed form when $0 < t < 2$.

Proposition 3.4. *Let ϕ be a C^1 1-periodic function and $t \in (0,2)$. Let K_t be the kernel defined in Lemma 3.3. For every $0 < s_1, s_2 < t$ such that $1 < s_2 - s_1$, we have $K_t(s_1,s_2) =$*

$-K_t(s_1+1, s_2)$. *Moreover, for* $0 < s_1 < s_2 < t$ *and* $s_2 - s_1 < 1$, *we have*

$$2K_t(s_1,s_2) = \begin{cases} \int_{-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s \\ \quad + 2(t-s_2)\phi(t-s_2) \\ \quad - 4(t-s_2)\phi(t-s_1) & \text{if } 2t-1 < s_1+s_2 < 2t, \\ \int_{s_2-s_1}^{2-2t+s_2+s_1} \phi(s+t-s_2)\,\mathrm{d}s \\ \quad + (-1+4t-3s_2-s_1)\phi(t-s_2) \\ \quad - (1+2t-3s_2+s_1)\phi(t-s_1) & \text{if } 2t-2 < s_1+s_2 < 2t-1, \\ \int_{2-2t+2s_2}^{0} \phi(s+t-s_2)\,\mathrm{d}s \\ \quad + (1+2t-2s_2)\phi(t-s_2) \\ \quad - (-1+4t-4s_2)\phi(t-s_1) & \text{if } 2t-3 < s_1+s_2 < 2t-2, \\ \int_{s_2-s_1}^{4-2t+s_2+s_1} \phi(s+t-s_2)\,\mathrm{d}s \\ \quad + (-2+4t-3s_2-s_1)\phi(t-s_2) \\ \quad - (2+2t-3s_2+s_1)\phi(t-s_1) & \text{if } 2t-4 < s_1+s_2 < 2t-3. \end{cases} \tag{3.4}$$

Proof. If $0 < s_1 < s_2 - 1 < s_2 < t$, we see that $s_1 \vee s_2 = (s_1+1) \vee s_2$ and that the integration set in formula (3.2) is the same for $K_t(s_1,s_2)$ and $K_t(1+s_1,s_2)$. Then, using the fact that $\theta(x+1) = -\theta(x)$ and the bilinearity of q, we get that $K_t(s_1+1,s_2) = -K_t(s_1,s_2)$. Thus, we only need to compute $K_t(s_1,s_2)$ when $0 < s_2 - s_1 < 1$.

Since $\theta(x) \in \{-1,1\}$ the term $q(\theta(t_1-s_2+s_1), \theta(t_2-s_2+s_1), \theta(t_1), \theta(t_2))$ only takes a finite number of values. To simplify notation, we set $\sigma = |s_2-s_1|$, $\tau = t - s_2$ and

$$\alpha_\sigma(t_1,t_2) = q\big(\theta(t_1-\sigma), \theta(t_2-\sigma), \theta(t_1), \theta(t_2)\big).$$

The proof then consists in identifying which values α_σ takes and on which subsets of Ω. Then we integrate $\int \phi'(t_1+\tau)$ on these sets and sum everything with the right coefficient.

We remark that if a, b, a', b' are equal to ± 1, then $q(a,b,a',b')$ is equal to 0 or $\pm 1/2$. Indeed, $q(1,1,1,1) = 0$, $q(1,-1,1,1) = 1/2$, $q(1,-1,1,-1) = 1/2$ and we get the other values using the bilinearity and the symmetry of q.

Remark that α_σ can only change value when t_1 or t_2 crosses the value k or $\sigma + k$ for some $k \in \mathbb{Z}$. We represent this in Figure 2.

We remark that the set where $\alpha_\sigma = 1/2$ is the intersection of three rectangles and Ω:

$$\underbrace{(\Omega \cap [-2+\sigma,-1] \times [-3+\sigma,-2])}_{\Omega_{11}} \cup \underbrace{(\Omega \cap [-1+\sigma,0] \times [-2,-1+\sigma])}_{\Omega_{12}}$$
$$\cup \underbrace{(\Omega \cap [\sigma,1] \times [-1,0])}_{\Omega_{13}},$$

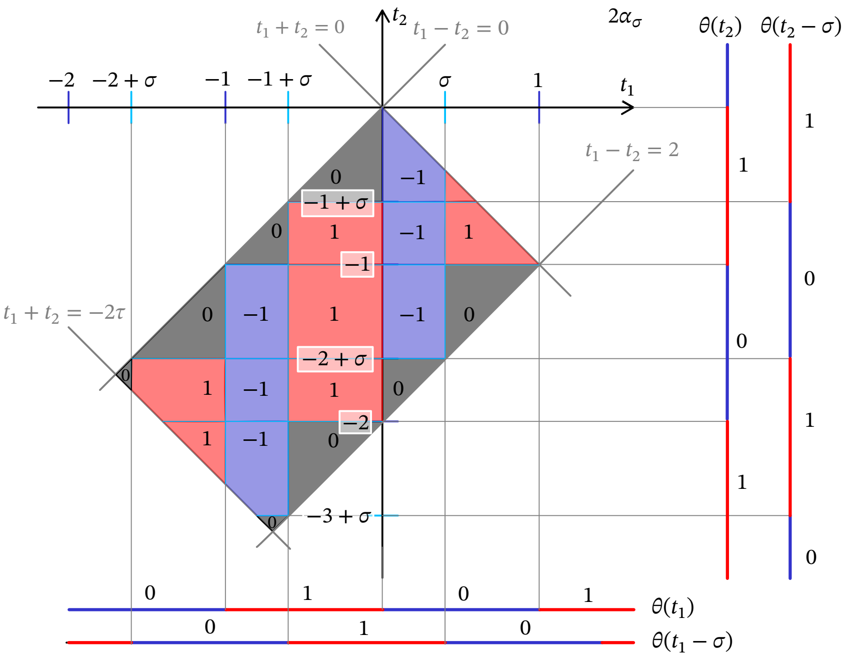

Figure 2. In light blue we show the potential thresholds for t_1 and t_2 where α_σ might change value. On the right we show the values of $\theta(t_2 - \sigma)$ and $\theta(t_2)$, and at the bottom, the values of $\theta(t_1 - \sigma)$ and $\theta(t_1)$. The diagonally placed rectangle is Ω. Inside Ω, we write the value of $2\alpha_\sigma(t_1, t_2)$.

while the set where $\alpha_\sigma = -1/2$ is the intersection of two rectangles and Ω:

$$\underbrace{(\Omega \cap [-1, -1+\sigma] \times [-3+\sigma, -1])}_{\Omega_{-11}} \cup \underbrace{(\Omega \cap [0, \sigma] \times [-2+\sigma, 0])}_{\Omega_{-12}}.$$

In other words, with the notation above,

$$K_t(s_1, s_2) = \frac{1}{2} \int_{\Omega_{11} \cup \Omega_{12} \cup \Omega_{13}} \phi'(t_1 + \tau)\, \mathrm{d}t_1\, \mathrm{d}t_2 - \frac{1}{2} \int_{\Omega_{-11} \cup \Omega_{-12}} \phi'(t_1 + \tau)\, \mathrm{d}t_1\, \mathrm{d}t_2.$$

Using Green's theorem, we get

$$K_t(s_1, s_2) = \frac{1}{2} \sum_{i=1}^{3} \oint_{\partial \Omega_{1i}} \phi(t_1 + \tau)\, \mathrm{d}t_2 - \frac{1}{2} \sum_{i=1}^{2} \oint_{\partial \Omega_{-1i}} \phi(t_1 + \tau)\, \mathrm{d}t_2.$$

The only thing left to do is to identify the different cases where the $\Omega_{i,j}$ are empty, triangles, some other 4-polygon or 5-polygon and compute each of these integrals.

We detail one case, and give the results for the others with just a figure as explanation.

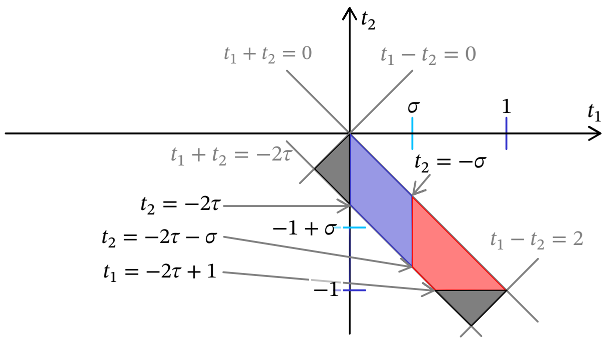

Figure 3. The equivalent of Figure 2 when $2t - 1 < s_1 + s_2$.

Step 1*: Case* $2t - 1 < s_1 + s_2 < 2t$ *(Figure 3).* In this case, the domains $\Omega_{i,j}$ look like that of Figure 3.

$$\begin{aligned}
2K_t(s_1, s_2) &= \underbrace{\int_0^\sigma \phi(s+\tau)\,\mathrm{d}s - \int_0^\sigma \phi(s+\tau)\,\mathrm{d}s}_{\text{“Diagonal” part of } \int_{\partial\Omega_{-1,2}}} + \underbrace{2\tau\phi(\tau) - 2\tau\phi(\sigma+\tau)}_{\text{“Vertical” part of } \int_{\partial\Omega_{-1,2}}} \\
&\quad \underbrace{- \int_\sigma^{-2\tau+1} \phi(s+\tau)\,\mathrm{d}s + \int_\sigma^1 \phi(s+\tau)\,\mathrm{d}s}_{\text{“Diagonal” part of } \int_{\partial\Omega_{13}}} \underbrace{- 2\tau\phi(\sigma+\tau)}_{\text{“Vertical” part of } \int_{\partial\Omega_{13}}} \\
&= \int_{-2\tau+1}^1 \phi(s+\tau)\,\mathrm{d}s + 2\tau\phi(\tau) - 4\tau\phi(\sigma+\tau) \\
&= \int_{-2t+2s_2}^0 \phi(s+t-s_2)\,\mathrm{d}s + 2(t-s_2)\phi(t-s_2) - 4(t-s_2)\phi(t-s_1).
\end{aligned}$$

Step 2*: Case* $s_1 + s_2 < 2t - 1 < 2s_1 + 1$ *(Figure 4).* We have

$$\begin{aligned}
2K_t(s_1, s_2) &= \int_{s_2-s_1}^{2-2t+s_2+s_1} \phi(s - s_2 + t)\,\mathrm{d}s \\
&\quad + (4t - 1 - 3s_2 - s_1)\phi(t-s_2) - (1 + 2t - 3s_2 + s_1)\phi(t-s_1).
\end{aligned}$$

Step 3*: Case* $2s_1 < 2t - 2 < s_1 + s_2$ *(Figure 5).* We have

$$\begin{aligned}
2K_t(s_1, s_2) &= -\int_{2-2t+s_2+s_1}^{s_2-s_1} \phi(s - s_2 + t)\,\mathrm{d}s \\
&\quad + (4t - 1 - 3s_2 - s_1)\phi(t-s_2) - (1 + 2t - 3s_2 + s_1)\phi(t-s_1).
\end{aligned}$$

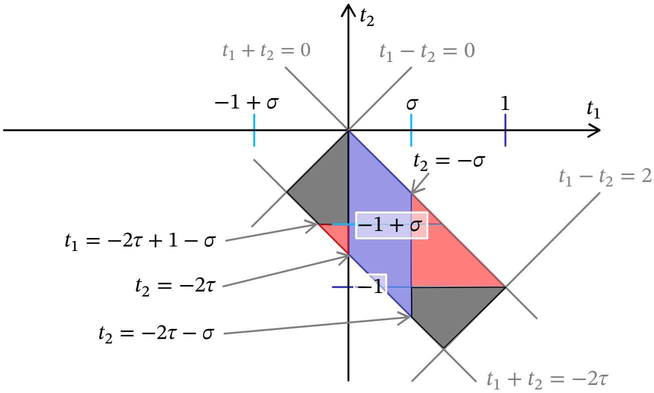

Figure 4. The equivalent of Figure 2 when $s_1 + s_2 < 2t - 1 < 2s_1 + 1$.

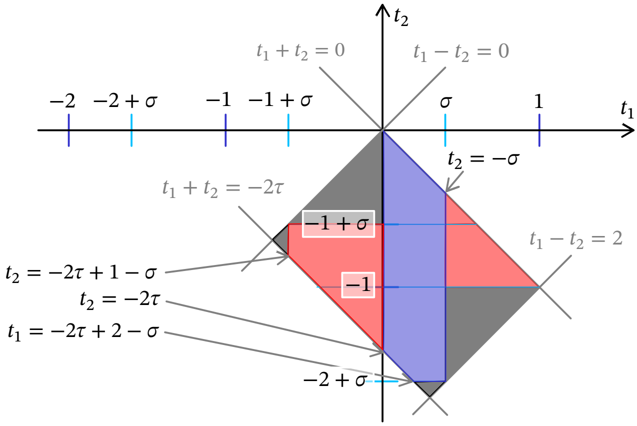

Figure 5. The equivalent of Figure 2 when $2s_1 < 2t - 2 < s_1 + s_2$.

Step 4: Case $s_1 + s_2 < 2t - 2 < 2s_2$ (Figure 6). We have

$$2K_t(s_1, s_2) = -\int_0^{2-2t+2s_2} \phi(s + t - s_2)\, \mathrm{d}s + (1 + 2t - 2s_2)\phi(t - s_2) - (-1 + 4t - 4s_2)\phi(t - s_1).$$

Step 5: Case $2s_2 - 1 < 2t - 3 < s_1 + s_2$ (Figure 7). We have

$$2K_t(s_1, s_2) = \int_{2-2t+2s_2}^{0} \phi(s - s_2 + t)\, \mathrm{d}s + (1 + 2t - 2s_2)\phi(t - s_2) - (-1 + 4t - 4s_2)\phi(t - s_1).$$

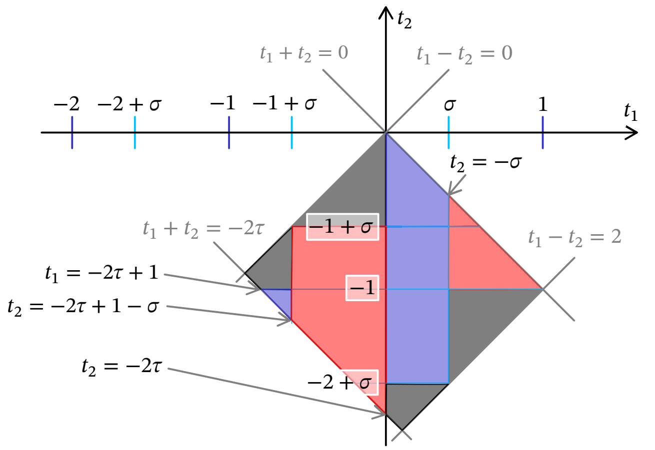

Figure 6. The equivalent of Figure 2 when $s_1+s_2<2t-2<2s_2$.

Step 6*: Case* $s_1+s_2<2t-3<2s_1+1$ *(Figure* 8*).* We have

$$2K_t(s_1,s_2)=\int_{s_2-s_1}^{4-2t+s_2+s_1}\phi(s+t-s_2)\,\mathrm{d}s \\ +(-2+4t-3s_2-s_1)\phi(t-s_2)-(2+2t-3s_2+s_1)\phi(t-s_1).$$

Step 7*: Case* $2s_1<2t-4<s_1+s_2$ *(Figure* 9*).* We have

$$2K_t(s_1,s_2)=\int_{s_2-s_1}^{4-2t+s_2+s_1}\phi(s+t-s_2)\,\mathrm{d}s \\ +(-2+4t-3s_2-s_1)\phi(t-s_2)-(2+2t-3s_2+s_1)\phi(t-s_1). \quad \blacksquare$$

When the control u steers the linearized equation (1.7) from 0 to 0, we can prove that this kernel acts as another, simpler one.

Proposition 3.5. *Let* $T\in(1,2)$*. Let* $\phi\in C^1(\mathbb{T})$ *be* 1*-periodic. We define the* reduced kernel $K_T^{\mathrm{red}}\colon[0,T-1]^2\to\mathbb{R}$ *by*

$$K_T^{\mathrm{red}}(s_1,s_2):=\frac{3}{2}(1-|s_2-s_1|)(\phi(T-s_1\vee s_2)-\phi(T-s_1\wedge s_2)).$$

Let $u\in L^2(0,T)$ *steer the linearized equation* (2.3) *from* 0 *to* 0 *(i.e.,* $\zeta_1(T,\cdot)=0$*). Let* $\zeta_2(\cdot,\cdot)$ *be the second-order correction for the water-tank system, i.e., the solution of* (3.1)*. Then*

$$(\zeta_2(T,\cdot),\phi)_{L^2(\mathbb{T})}=\int_{[0,T-1]^2}K_T^{\mathrm{red}}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2. \tag{3.5}$$

The two important points about this formula are that the expression of the reduced kernel is simpler, and that we integrate on $[0,T-1]^2$ instead of $[0,T]^2$.

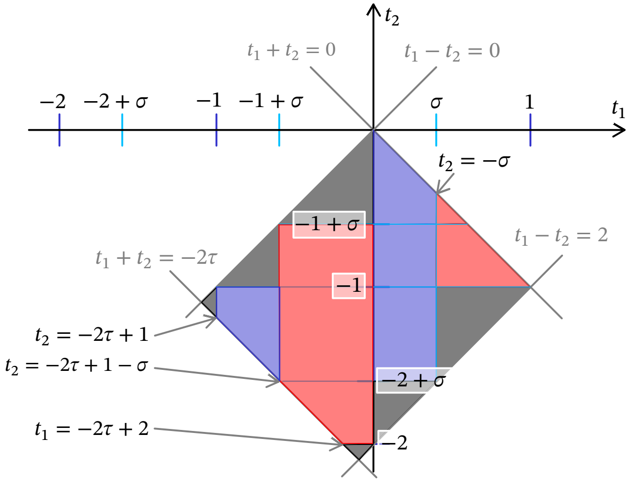

Figure 7. The equivalent of Figure 2 when $2s_2 - 1 < 2t - 3 < s_1 + s_2$.

Proof. Step 1*: Expression of* K_T^{red} *as a function of* K_T. According to Proposition 2.8, we have for every $T - 1 < s < 1$, $u(s) = 0$ and for every $0 < s < T - 1$, $u(s + 1) = u(s)$. Thus, according to Proposition 3.4 we have

$$\begin{aligned}(\zeta_2(T,\cdot),\phi) &= \int_{[0,T]^2} K_T(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\\ &= \int_{[0,T-1]^2} (K_T(s_1,s_2) + K_T(1+s_1,s_2) + K_T(s_1,1+s_2)\\ &\qquad + K_T(1+s_1,1+s_2))u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2.\end{aligned}$$

Thus, equation (3.5) holds with

$$K_T^{\mathrm{red}}(s_1,s_2) = K_T(s_1,s_2) + K_T(1+s_1,s_2) + K_T(s_1,1+s_2) + K_T(1+s_1,1+s_2).$$

Since K_T (and also K_T^{red}) are symmetric in s_1, s_2, we may assume that $s_1 \le s_2$. Then, with $s_2' := 1 + s_2$ and $s_1' := s_1$, we have $s_1' + 1 \le s_2'$; thus, according to Proposition 3.4, we have $K_T(s_1', s_2') = -K_T(1 + s_1', s_2')$. Thus, $K_T(s_1, 1 + s_2) + K_T(1 + s_1, 1 + s_2) = 0$ and $K_T^{\mathrm{red}}(s_1, s_2) = K_T(s_1, s_2) + K_T(1 + s_1, s_2)$.

We end the computation by using the formula for K_T from Proposition 3.4. We have $0 < s_1 \le s_2 < T - 1$ and $1 < T < 2$. So $2T - 4 < 0 < s_1 + s_2 < 2T - 2$. We consider two cases: $2T - 3 < s_1 + s_2 < 2T - 2$ and $2T - 4 < s_1 + s_2 < 2T - 3$.

Step 2*: Case* $2T - 3 < s_1 + s_2 < 2T - 2$. To compute $K_T(s_1, s_2)$, we use the third case of expression (3.4) of K_T. To compute $K_T(1 + s_1, s_2)$, we remark that with $s_1' := s_2$ and

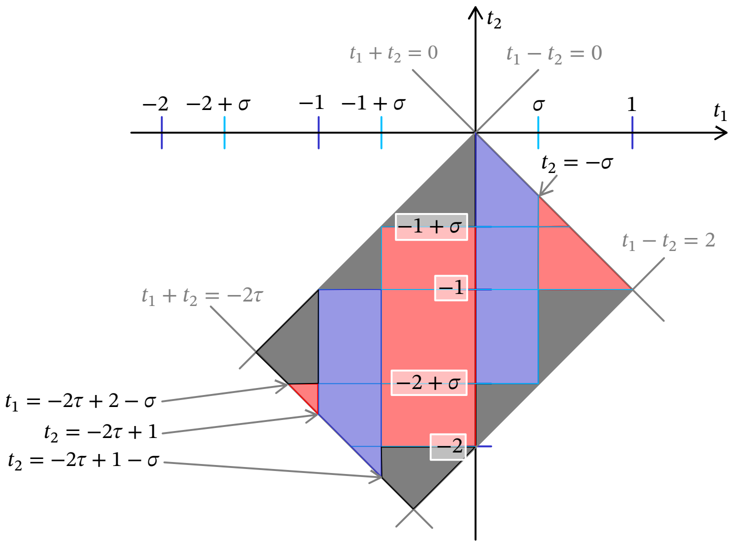

Figure 8. The equivalent of Figure 2 when $s_1 + s_2 < 2t - 3 < 2s_1 + 1$.

$s_2' := 1 + s_1$, we have $s_1' < s_2'$ and $2T - 2 < s_1' + s_2' < 2T - 1$. Thus, $K_T(1 + s_1, s_2) = K_T(s_1', s_2')$ is computed with the second case of expression (3.4) of K_T. We get

$$
\begin{aligned}
2K_T^{\mathrm{red}}(s_1, s_2) &= 2K_T(s_1, s_2) + 2K_T(s_1', s_2') \\
&= \int_{2-2T+2s_2}^{0} \phi(s + T - s_2)\,\mathrm{d}s + (1 + 2T - 2s_2)\phi(T - s_2) \\
&\quad - (-1 + 4T - 4s_2)\phi(T - s_1) + \int_{s_2'-s_1'}^{2-2T+s_2'+s_1'} \phi(s + T - s_2')\,\mathrm{d}s \\
&\quad + (-1 + 4T - 3s_2' - s_1')\phi(T - s_2') - (1 + 2T - 3s_2' + s_1')\phi(T - s_1') \\
&= \int_{2-2T+2s_2}^{0} \phi(s + T - s_2)\,\mathrm{d}s + (1 + 2T - 2s_2)\phi(T - s_2) \\
&\quad - (-1 + 4T - 4s_2)\phi(T - s_1) + \int_{1+s_1-s_2}^{3-2T+s_1+s_2} \phi(s + T - s_1)\,\mathrm{d}s \\
&\quad + (-4 + 4T - 3s_1 - s_2)\phi(T - s_1) - (-2 + 2T - 3s_1 + s_2)\phi(T - s_2) \\
&= \int_{2-2T+2s_2}^{0} \phi(s + T - s_2)\,\mathrm{d}s + \int_{1+s_1-s_2}^{3-2T+s_1+s_2} \phi(s + T - s_1)\,\mathrm{d}s \\
&\quad + (3 - 3s_2 + 3s_1)\phi(T - s_2) - (3 - 3s_2 + 3s_1)\phi(T - s_1).
\end{aligned}
$$

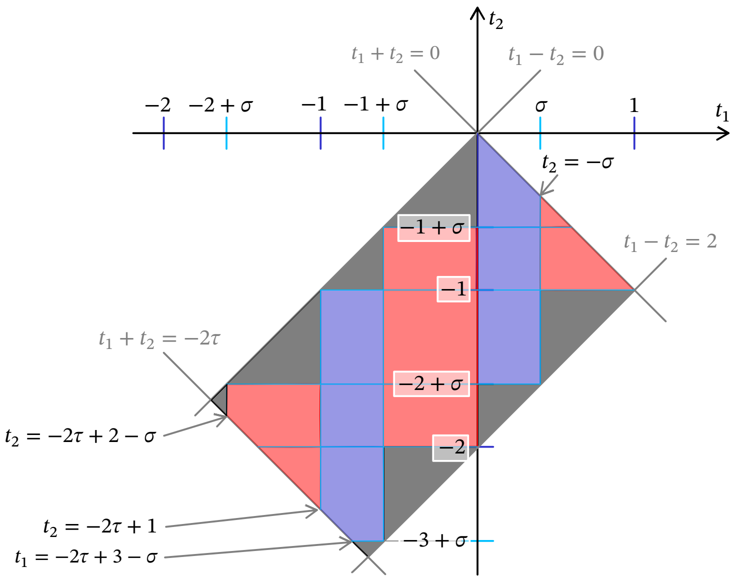

Figure 9. The equivalent of Figure 2 when $2s_1 < 2t - 4 < s_1 + s_2$.

In the second integral, we make the change of variable $s' = s + s_2 - s_1$:

$$\begin{aligned} 2K_T^{\text{red}}(s_1, s_2) &= \int_{2-2T+2s_2}^{0} \phi(s + T - s_2)\,\mathrm{d}s + \int_{1}^{3-2T+2s_2} \phi(s + T - s_2)\,\mathrm{d}s \\ &\quad + 3(1 - s_2 + s_1)(\phi(T - s_2) - \phi(T - s_1)) \\ &= 3(1 - s_2 + s_1)(\phi(T - s_2) - \phi(T - s_1)), \end{aligned}$$

where we used the 1-periodicity of ϕ to cancel the two integrals. Since we swapped s_1 and s_2 to have $s_1 = s_1 \vee s_2$ and $s_2 = s_1 \wedge s_2$, this is indeed the claimed formula.

Step 3: Case $2T - 4 < s_1 + s_2 < 2T - 3$. This case is treated in the same way, the only difference being that $K_T(s_1, s_2)$ is computed using the fourth case of expression (3.4) of K_T, and $K_T(1 + s_1, s_2)$ is computed using the third case of the same expression. We get the same formula. ∎

3.3. Coercivity of the kernel

In this section we use the expression for $(\zeta_2(T, \cdot), \phi)$ given in Proposition 3.5 to prove that when $1 < T < 2$, $|\zeta_2(T, \cdot)|$ is lower-bounded by $\|U\|_{L^2}^2$, where, as before, $U(t) = \int_0^t u(s)\,\mathrm{d}s$. To do that, we first have to choose the right function ϕ.

Definition 3.6. Let $1 < T < 2$ and let ϕ be a C^∞ 1-periodic function such that $\phi(s) = s$ in $[1, T]$.

We will discuss how we identified this test function as the right one in Remark 3.9.

Proposition 3.7. *If* $1 < T < 2$ *and* $u \in L^2(0,T)$ *steers the solution of the linearized equation* (2.3) *from* 0 *to* 0 *(i.e.,* $\zeta_1(T,\cdot) = 0$*) and if* $\int_0^T u(t)\,\mathrm{d}t = 0$*, then,*

$$(\zeta_2(T,\cdot),\phi)_{L^2} \geq 3(2-T)\|U\|^2_{L^2(0,T-1)},$$

where ϕ *is a function given in Definition* 3.6.

This proposition uses the following computation:

Lemma 3.8. *Let* $I = (a,b)$ *with* $a < b$*, and let* $K \in H^1(I^2) \cap H^2(I^2 \setminus \{s_1 = s_2\})$*. Let* $R \in L^2(I^2)$ *such that for* $s_1 \neq s_2$*,* $R(s_1,s_2) = \partial_{s_1,s_2}K(s_1,s_2)$*, let* $w(s) := \partial_{s_1}K(s, s+0) - \partial_{s_1}K(s,s-0)$*, and let* $g(s) := \partial_{s_1}K(s,b) + \partial_{s_2}K(b,s)$*. Then, for every* $u \in L^2(a,b)$*, with* $U(t) := \int_a^t u(s)\,\mathrm{d}s$*, we have*

$$\begin{aligned}\int_{I^2} K(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 = &\int_I w(s)|U(s)|^2\,\mathrm{d}s \\ &+ \int_{I^2} R(s_1,s_2)U(s_1)U(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &- U(b)\int_I g(s)U(s)\,\mathrm{d}s + K(b,b)U(b)^2.\end{aligned}$$

Proof. First, integrate by parts in s_1:

$$\begin{aligned}\int_{I^2} K(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 = &-\int_{I^2}\partial_{s_1}K(s_1,s_2)U(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &+ \int_I K(b,s_2)U(b)u(s_2)\,\mathrm{d}s_2.\end{aligned} \tag{3.6}$$

Now we split the first integral into two parts: $s_2 < s_1$ and $s_1 < s_2$. With

$$\mathcal{I} = \int_{I^2}\partial_{s_1}K(s_1,s_2)U(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

one has

$$\begin{aligned}\mathcal{I} &= \int_I\left(\int_a^{s_1}\partial_{s_1}K(s_1,s_2)u(s_2)\,\mathrm{d}s_2 + \int_{s_1}^b \partial_{s_1}K(s_1,s_2)u(s_2)\,\mathrm{d}s_2\right)U(s_1)\,\mathrm{d}s_1 \\ &= \int_I\Bigg(-\int_a^{s_1}\partial_{s_1,s_2}K(s_1,s_2)U(s_2)\,\mathrm{d}s_2 + \partial_{s_1}K(s_1,s_1-0)U(s_1) \\ &\qquad -\int_{s_1}^b \partial_{s_1,s_2}K(s_1,s_2)U(s_2)\,\mathrm{d}s_2\partial_{s_1}K(s_1,b)U(b) \\ &\qquad -\partial_{s_1}K(s_1,s_1+0)U(s_1)\Bigg)U(s_1)\,\mathrm{d}s_1 \\ &= -\int_{I^2}R(s_1,s_2)U(s_1)U(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 - \int_I w(s)U(s)^2\,\mathrm{d}s \\ &\quad + U(b)\int_I \partial_{s_1}K(s_1,b)U(s_1)\,\mathrm{d}s_1.\end{aligned}$$

Moreover,

$$\int_I K(b,s_2)u(s_2)\,\mathrm{d}s_2 = -\int_I \partial_{s_2} K(b,s_2)U(s_2)\,\mathrm{d}s_2 + K(b,b)U(b).$$

Plugging these two formulas into equation (3.6) proves the lemma. ∎

Proof of Proposition 3.7. We first simplify the expression of K_T^{red} given by Proposition 3.5. For $0 < s_1, s_2 < T-1$, we have $1 < T - s_1 \vee s_2 \le T - s_1 \wedge s_2 < T$; thus, according to the definition of ϕ, we have for $0 < s_1, s_2 < T-1$,

$$\begin{aligned}
K_T^{\mathrm{red}}(s_1,s_2) &= \frac{3}{2}(1-|s_2-s_1|)((T-s_1\vee s_2)-(T-s_1\wedge s_2))\\
&= -\frac{3}{2}(1-|s_2-s_1|)|s_2-s_1|\\
&= \frac{3}{2}(-|s_2-s_1|+(s_2-s_1)^2).
\end{aligned}$$

Thus, according to Proposition 3.5, if u is as in the statement of Proposition 3.7,

$$\begin{aligned}
(\zeta_2(u,T,\cdot),\phi) = &-\frac{3}{2}\int_{[0,T-1]^2} |s_2-s_1|u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2\\
&+\frac{3}{2}\int_{[0,T-1]^2}(s_2-s_1)^2u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2.
\end{aligned} \tag{3.7}$$

With the notation of Lemma 3.8 with $K = K_T^{\mathrm{red}}$, we have

$$R(s_1,s_2) = -3, \quad w(s) = 3.$$

Moreover, since $\int_0^T u(t)\,\mathrm{d}t = 0$, according to Proposition 2.8, we have

$$\int_0^T u(t)\,\mathrm{d}t = 2\int_0^{T-1} u(t)\,\mathrm{d}t = 0.$$

Hence the boundary term $U(T-1)$ is zero. Plugging the formula of Lemma 3.8 into expression (3.7), we get

$$(\zeta_2(u,T,\cdot),\phi) = 3\|U\|^2_{L^2(0,T-1)} - 3\Big(\int_0^{T-1} U(s)\,\mathrm{d}s\Big)^2.$$

According to the Cauchy–Schwarz inequality, we have

$$\Big|\int_0^{T-1} U(s)\,\mathrm{d}s\Big| \le \sqrt{T-1}\|U\|_{L^2(0,T-1)}.$$

Thus

$$(\zeta_2(u,T,\cdot),\phi) \ge 3(2-T)\|U\|^2_{L^2(0,T-1)}.$$ ∎

Remark 3.9. How did we choose the ϕ of Definition 3.6? It turns out that if ϕ is monotone on $[1,T]$, the assertion

$$\int_0^{T-1} u(t)\,\mathrm{d}t = 0 \Rightarrow \int_{[0,T-1]^2} K_T^{\mathrm{red}}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \geq c\|U\|^2_{L^2(0,T-1)}$$

is equivalent to the condition $\int_1^T \phi'(s)\,\mathrm{d}s \int_1^T (\phi'(s))^{-1}\,\mathrm{d}s < (3-T)^2$ (we sketch the proof of this fact in Appendix A). Hence, the smaller the left-hand side of this condition, the larger the time for which we will be able to disprove local controllability. With some calculus of variations, we can see that if ϕ minimizes the left-hand side, then ϕ' is constant on $[1,T]$, hence our choice of ϕ.

Remark 3.10. The hypothesis that $\int_0^T u(t)\,\mathrm{d}t = 0$ in Proposition 3.7 is essential. Indeed, K_T^{red} is continuous and $K_T^{\mathrm{red}}(s,s) = 0$. Hence, if we choose a sequence $(u_n)_{n\in\mathbb{N}}$ that approximates δ_{t_0} nicely enough for some fixed $t_0 \in (0, T-1)$ (for instance $u_n(t) = n\varphi((t-t_0)/n)$ where $\varphi \in C_c(0,T-1)$, $\varphi \geq 0$, $\int_0^{T-1}\varphi(x)\,\mathrm{d}x = 1$), we get

$$\int_{[0,T-1]^2} K_T^{\mathrm{red}}(s_1,s_2)u_n(s_1)u_n(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2 \xrightarrow[n\to+\infty]{} K_T^{\mathrm{red}}(t_0,t_0) = 0.$$

Moreover, we have

$$U_n(t) := \int_0^t u_n(s)\,\mathrm{d}s \xrightarrow[n\to+\infty]{} \begin{cases} 0 & \text{if } t < t_0, \\ 1 & \text{if } t > t_0, \end{cases}$$

hence $\|U_n\|_{L^2(0,T-1)} \xrightarrow[n\to+\infty]{} \sqrt{T-t_0} > 0$. This proves that the quadratic map

$$u \mapsto \int_{[0,T-1]^2} K_T^{\mathrm{red}}(s_1,s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$$

has no "H^{-1}-coercivity".

4. Nonlinear equation

Proposition 3.7 shows that if the time T is smaller than 2 and if u steers the linearized equation (2.3) from 0 to 0, then $\|\zeta_2(u,T,\cdot)\|_{L^2} \geq c\|U\|^2_{L^2(0,T)}$. As in the previous section, we fix $T \in (1,2)$. Our aim now is to prove that the solution of the *nonlinear* equation also has this property, as long as $\|u\|_{C^0}$ is small enough. As a consequence, one cannot move the water tank in time T with a control small in C^0-norm, and that finishes the proof of Theorem 1.2.

To this end, we use the fact that if $\|u\|$ is small enough, the solution of the nonlinear equation is well approximated by $(h_1,v_1)+(h_2,v_2)$, where (h_1,v_1) solves the linearized system (1.7) and (h_2,v_2) solves the second-order system (1.8).

4.1. Well-posedness of the water-tank system

In this section we state several basic results on the nonlinear system related to the water-tank system (1.1). We begin with the well-posedness of the water-tank system, where, as in the rest of the article, $g = 1$ and $L = 1$ and $U(t) = \int_0^t u(s)\,\mathrm{d}s$.

Proposition 4.1. *Let $T > 0$. There exists $\varepsilon > 0$ such that for $(H_0, v_0) \in (C^1([0,1]))^2$ that satisfies*

$$\|u\|_{C^0([0,T])} + \|(H_0, v_0) - (1,0)\|_{C^1([0,1])} < \varepsilon,$$

as well as the compatibility conditions

$$\partial_x H_0(0) = \partial_x H_0(1) = -u(0),$$

there exists a unique solution $(H_{\mathrm{nl}}, v_{\mathrm{nl}}) \in (C^1([0,T] \times [0,1]))^2$ of the water-tank system (1.1) with $H_{\mathrm{nl}}(0,x) = H_0(x)$ and $v_{\mathrm{nl}}(0,x) = v_0(x)$. Moreover,

$$\|(H_{\mathrm{nl}}, v_{\mathrm{nl}}) - (1,0)\|_{C^1([0,T]\times[0,1])} \le C(\|u\|_{C^0([0,T])} + \|(H_0, v_0) - (1,0)\|_{C^1([0,1])}),$$

for some positive constant C depending only on T.

Proof. In this proof, we omit the index nl and write just (H, v) for $(H_{\mathrm{nl}}, v_{\mathrm{nl}})$.

Standard results for the classical solutions of hyperbolic systems (see for instance [11, §3.4] and [27, Chapter 4]) assume that all coefficients are at least C^1, but here we assume that u is only C^0. In order to achieve that, we note that if (H, v) solves the water-tank system (1.1), then with V defined by $v(t,x) = V(t,x) - U(t)$, the water-tank system becomes

$$\begin{cases} \partial_t H + \partial_x((V-U)H) = 0, \\ \partial_t V + \partial_x\Big(H + \dfrac{(V-U)^2}{2}\Big) = 0, \\ V(t,0) = V(t,1) = U(t), \end{cases}$$

where all the coefficients are now C^1. This system can be written in the form

$$\partial_t \begin{pmatrix} H \\ V \end{pmatrix} + \begin{pmatrix} V-U & H \\ 1 & V-U \end{pmatrix} \partial_x \begin{pmatrix} H \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } [0,T] \times [0,1] \tag{4.1}$$

and

$$V(t,0) = V(t,1) = U(t) \quad \text{in } [0,T]. \tag{4.2}$$

System (4.1) and (4.2) is in the standard form of the quasilinear hyperbolic system for U with $\|U\|_{C^1(0,T)}$ small; see, e.g., [32, Theorem 2.1]. The proof of the well-posedness and the estimate are based on fixed point arguments (see also [27, Chapter 4] and [20, proof of Lemma 2.2]). For the convenience of the reader, we sketch the ideas of the proof.

Set

$$(H^{(0)}, V^{(0)})(t,x) = (H_0(x), V_0(x)) \quad \text{in } [0,T] \times [0,1],$$

and define $(H^{(n)}, V^{(n)})$ in $[0, T] \times [0, L]$ for $n \geq 1$ by

$$\partial_t \begin{pmatrix} H^{(n)} \\ V^{(n)} \end{pmatrix} + \begin{pmatrix} V^{(n-1)} - U & H^{(n-1)} \\ 1 & V^{(n-1)} - U \end{pmatrix} \partial_x \begin{pmatrix} H^{(n)} \\ V^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } [0, T] \times [0, 1], \tag{4.3}$$

with the corresponding boundary conditions. Let us assume that

$$\|(H^{(n-1)} - 1, V^{(n-1)}, U)\|_{C^1([0,T]\times[0,1])} \leq C_1, \tag{4.4}$$

for some small positive constant C_1. Then, by considering (4.3) as a *linear* hyperbolic system where the matrix

$$\begin{pmatrix} V^{(n-1)} - U & H^{(n-1)} \\ 1 & V^{(n-1)} - U \end{pmatrix}$$

is a given function of t and x and satisfies (4.4), and by applying the characteristic method again, one gets the existence of a positive constant C_2 (independent of $\varepsilon > 0$ small enough and n) such that

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^0([0,T]\times[0,1])} \leq C_2(\|U\|_{C^0([0,T])} + \|(H_0, V_0) - (1, 0)\|_{C^0([0,1])}).$$

By taking the derivative of (4.3) with respect to t, we also obtain [3]

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^1([0,T]\times[0,1])} \leq C_2(\|u\|_{C^0([0,T])} + \|(H_0, V_0) - (1, 0)\|_{C^1([0,1])}).$$

We derive that

$$\|(H^{(n)} - 1, V^{(n)})\|_{C^1([0,T]\times[0,1])} \leq C\varepsilon.$$

In order to prove the uniform equicontinuity of $(\nabla_{t,x} H^{(n)}, \nabla_{t,x} V^{(n)})$ in $[0, T] \times [0, 1]$, we estimate their modulus of continuity defined by

$$\rho_n(r) := \sup_{\substack{|(t,x)-(s,y)|<r \\ (t,x),(s,y)\in[0,T]\times[0,1]}} |(\nabla_{t,x} H^{(n)}(t, x) - \nabla_{s,y} H^{(n)}(s, y), \nabla_{t,x} V^{(n)}(t, x) - \nabla_{s,y} V^{(n)}(s, y))|.$$

Using the characteristics method again, we can prove that there exists a positive constant γ depending only on T such that for ε sufficiently small,

$$\rho_n(r) \leq C\Big(\sup_{\substack{|t-s|<\gamma r \\ t,s\in[0,T]}} |u(t) - u(s)| + \sup_{\substack{|x-y|<\gamma r \\ x,y\in[0,1]}} |(H_0', V_0')(x) - (H_0', V_0')(y)| \Big). \tag{4.5}$$

[3] Rigorous arguments can start by first approximating (H^{n-1}, V^{n-1}) and U by smooth solutions and then passing to the limit. The meaning of the broad solutions, see, e.g., [11], can also be used to give the result. The details are omitted.

Using Ascoli's theorem, one can conclude from (4.5) that $(H^{(n)}, V^{(n)})$ converges to (H, V) in $C^1([0, T] \times [0, 1])$ up to a subsequence.

It thus suffices to prove the uniqueness of (H, V) to conclude the proof. Let (H, V) and $(\widehat{H}, \widehat{V})$ be two solutions in $C^1([0, T] \times [0, 1])$ such that their norms in $C^1([0, T] \times [0, 1])$ are bounded by $C\varepsilon$ for some positive constant C independent of ε. Consider the system solved by $(\widehat{H} - H, \widehat{V} - V)$: one can check that

$$\|(\widehat{H} - H, \widehat{V} - V)\|_{C^0([0,T]\times[0,1])} \le C\varepsilon \|(\widehat{H} - H, \widehat{V} - V)\|_{C^0([0,T]\times[0,1])}. \tag{4.6}$$

This implies $(\widehat{H}, \widehat{V}) = (H, V)$ for ε sufficiently small. The uniqueness is established.

The proof is complete. ■

Remark 4.2. We do not need this for the proofs below, but it is worth noting that standard methods using the propagation along characteristics can be used to prove the lack of local controllability around equilibrium states in time $T < T_*$. Let us sketch it. Consider the characteristic speeds $\lambda_\pm$ and Riemann invariants $R_\pm$, which are given by[4]

$$\lambda_\pm = v \pm \sqrt{H},$$
$$R_\pm = v \pm 2\sqrt{H} + U.$$

We have

$$\begin{cases} (\partial_t + \lambda_\pm \partial_x) R_\pm = 0, \\ R_+(t, 0) + R_-(t, 0) = R_+(t, 1) + R_-(t, 1) = 2U(t). \end{cases}$$

Consider also the characteristics, i.e., the solutions $x_\pm$ of the Cauchy problem

$$\begin{cases} \partial_t x_\pm(t, t_0, x_{t_0}) = \lambda_\pm(t, x_\pm(t, t_0, x_{t_0})), \\ x_\pm(t_0, t_0, x_{t_0}) = x_{t_0}. \end{cases}$$

Then, differentiating in t and using the equation for $R_\pm$, we get that $R_\pm(t, x_\pm(t, t_0, 0))$ does not depend on t (as long as $x_\pm(t, t_0, 0)$ is defined, i.e., stays inside $[0, 1]$). Hence

$$R_+(t, x_+(t, t_0, 0)) = R_+(t_0, 0) \quad \text{and} \quad R_-(t_0, 0) = R_-(0, x_-(0, t_0, 0)).$$

Hence, if $R_\pm(0, \cdot) = R_\pm(T, \cdot) = 0$, $0 < t_0 < T$, and if $x_+(T, t_0, 0)$ and $x_-(0, t_0, 0)$ are defined,

$$2U(t_0) = R_+(t_0, 0) + R_-(t_0, 0) = R_+(T, x_+(T, t_0, 0)) + R_-(0, x_-(0, t_0, 0)) = 0.$$

The characteristic speed depends on the solution, and thus on the control, but if the control is small, the characteristic speeds are $\lambda_\pm(t, x) = \pm 1 + O(\|u\|_{C^0})$, which implies that $x_\pm(t, t_0, 0) = \pm(t - t_0) + O(\|u\|_{C^0})$. Hence, the computations outlined above are valid if $T < 1 - O(\|u\|_{C^0})$.

[4]The Riemann invariants as defined in [1, Section 1.4] do not have the $+U$ term. But in our case, it is convenient to add it.

4.2. Error estimates

In this section, $(H_{\rm nl}, v_{\rm nl}) = (1 + h_{\rm nl}, v_{\rm nl})$ is the solution of the water-tank system (1.1) with control u. We will often conflate this solution and $\zeta_{\rm nl} := \mathscr{C}(h_{\rm nl}, v_{\rm nl})$. The same will be done for the solution (h_1, v_1) of the linearized system (1.7) and $\zeta_1 := \mathscr{C}(h_1, v_1)$ (solution of (2.3)), as well as the solution (h_2, v_2) of (1.8) and $\zeta_2(u) := \mathscr{C}(h_2, v_2)$. If anything, this will make the notation lighter.

We will also set $w_{\rm nl} := -\mathscr{C}(\partial_x(h_{\rm nl}v_{\rm nl}), \partial_x(v_{\rm nl}^2/2))$, so that $\zeta_{\rm nl}$ satisfies $(\partial_t + \partial_x)\zeta_{\rm nl}(t, x) = w_{\rm nl}(t, x) + u(t)\theta(x)$. We also denote the right-hand side of equation (3.1) satisfied by ζ_2 by $w_1(t, x)$, i.e., $w_1 = -\mathscr{C}(\partial_x(h_1 v_1), \partial_x(v_1^2/2))$. Finally, we set $\delta_1 := \zeta_{\rm nl} - \zeta_1$ and $\delta_2 := \zeta_{\rm nl} - \zeta_1 - \zeta_2$.

We also keep the notation $U(t) = \int_0^t u(s)\,{\rm d}s$ of the previous sections.

In this subsection we prove estimates on the following error terms:

- in Lemma 4.4, an estimate on $\delta_2 = \zeta_{\rm nl} - \zeta_1 - \zeta_2$,
- in Lemma 4.5, we bound $\zeta_2(\tilde{u}, T, \cdot) - \zeta_2(u, T, \cdot)$.

The aim is to prove that these terms cannot counter the positivity of the term $3(2 - T)\|U\|_{L^2}^2$ that appears in Proposition 3.7.

From now on, $\|u\|_{C^0}$ means $\|u\|_{C^0([0,T])}$. We start with an estimate for the nonlinear equation, which is a consequence of the nonlinear well-posedness (Proposition 4.1) and the linear estimates (Proposition 2.6):

Corollary 4.3. *Let $T > 0$. There exist $\eta > 0$ and $C > 0$ such that for every $u \in C^0([0, T])$ with $u(0) = 0$ and $\|u\|_{C^0} < \delta$, there exists a unique solution $(H_{\rm nl}, v_{\rm nl}) \in (C^1([0, T] \times [0, 1]))^2$ of the water-tank system (1.1) with $(H_{\rm nl}, v_{\rm nl})(0, \cdot) = (1, 0)$. Moreover, with the notation $\zeta_{\rm nl}$ defined at the beginning of this section, we have for some C independent of u,*

$$\|\zeta_{\rm nl}\|_{L_t^2 L_x^2} \le C\|U\|_{L^2(0,T)}.$$

Proof. The existence and uniqueness are a consequence of the well-posedness (Proposition 4.1). Let us now prove the inequality. We write $\zeta_{\rm nl} = \zeta_1 + \delta_1$.

We have $(\partial_t + \partial_x)\zeta_1(u, t, x) = u(t)\theta(x)$ and $(\partial_t + \partial_x)\delta_1(u, t, x) = w_{\rm nl}(u, t, x)$. Hence, according to Proposition 2.6, we have $\|\zeta_1\|_{L_t^2 L_x^2} \le C\|U\|_{L^2}$ and $\|\delta_1\|_{L_t^2 L_x^2} \le C\|w_{\rm nl}\|_{L^2}^2$. Since $w_{\rm nl}$ can be written as $-\partial_x r_{\rm nl}$, where $r_{\rm nl}(t, x)$ is a quadratic form of $\zeta_{\rm nl}(t, x)$ and $\zeta_{\rm nl}(t, -x)$ (Lemma 3.1), we have $\|w_{\rm nl}\|_{L_t^2 L_x^2} \le C\|\partial_x\zeta_{\rm nl}\|_{L^\infty}\|\zeta_{\rm nl}\|_{L_t^2 L_x^2}$. Thus,

$$\|\zeta_{\rm nl}\|_{L_t^2 L_x^2} \le C(\|U\|_{L^2} + \|\partial_x\zeta_{\rm nl}\|_{L^\infty}\|\zeta_{\rm nl}\|_{L_t^2 L_x^2}).$$

Finally, since $\|\partial_x\zeta_{\rm nl}\|_{L^\infty} \le \|\mathscr{C}(h_{\rm nl}, v_{\rm nl})\|_{W^{1,\infty}} \le 2\|(h_{\rm nl}, v_{\rm nl})\|_{C^1}$ (see Remark 2.3), we have according to the well-posedness estimate of Proposition 4.1, $\|\partial_x\zeta_{\rm nl}\|_{L^\infty} \le C\|u\|_{C^0} \le C\eta$. Thus,

$$\|\zeta_{\rm nl}\|_{L_t^2 L_x^2} \le C\|U\|_{L^2} + C\eta\|\zeta_{\rm nl}\|_{L_t^2 L_x^2},$$

which implies for η small enough,

$$\|\zeta_{\rm nl}\|_{L_t^2 L_x^2} \le \frac{C}{1 - C\eta}\|U\|_{L^2}. \qquad \blacksquare$$

Next we prove the approximation property:

Lemma 4.4. *Let* $\phi \in C^1(\mathbb{T})$*. Let* $T > 0$ *and* $u \in C^0(0,T)$ *with* $u(0) = 0$ *and* $\|u\|_{C^0} < \eta$*. Then, with the notation above, for some* $C > 0$ *independent of* u*,*

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C\|U\|_{L^2(0,T)}\|u\|_{C^0},$$
$$|(\delta_2(T,\cdot),\phi)| \leq C\|U\|_{L^2(0,T)}^2\|u\|_{C^0}.$$

Proof. Step 1*: Estimate of* δ_1 *in* L^2*-norm.* We have $(\partial_t + \partial_x)\delta_1 = w_{\mathrm{nl}}$; thus, using Duhamel's formula,

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C\|w_{\mathrm{nl}}\|_{L_t^1 L_x^2}.$$

Since $w_{\mathrm{nl}} = -\mathscr{C}(\partial_x(h_{\mathrm{nl}}v_{\mathrm{nl}}), \partial_x(v_{\mathrm{nl}}^2/2))$, we can use Lemma 3.1 to write $w_{\mathrm{nl}} = -\partial_x r_{\mathrm{nl}}$ with $r_{\mathrm{nl}}(t,x) = Q(\zeta_{\mathrm{nl}}(t,x), \zeta_{\mathrm{nl}}(t,-x))$. Thus,

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C\|\partial_x r_{\mathrm{nl}}\|_{L_t^2 L_x^2}.$$

Since Q is a quadratic form (see Lemma 3.1), $\partial_x r_{\mathrm{nl}}$ is the sum of products of ζ_{nl} and $\partial_x\zeta_{\mathrm{nl}}$ evaluated at (t,x) or $(t,-x)$. Thus, we get

$$\begin{aligned}\|\delta_1\|_{L_t^\infty L_x^2} &\leq C\|\zeta_{\mathrm{nl}}\|_{L_t^2 L_x^2}\|\partial_x\zeta_{\mathrm{nl}}\|_{L_t^\infty L_x^\infty}\\ &\leq C\|\zeta_{\mathrm{nl}}\|_{L_t^2 L_x^2}\|(h_{\mathrm{nl}}, v_{\mathrm{nl}})\|_{C^1([0,T]\times[0,1])},\end{aligned}$$

where we used that the change of variables $\mathscr{C}$ is such that for $(h,v) \in C^1([0,1])$ with $v(0) = v(1) = 0$, then $\|\mathscr{C}(h,v)\|_{W^{1,\infty}} \leq 2\|(h,v)\|_{C^1}$ (Remark 2.3). Finally, using the well-posedness estimates of Proposition 4.1 and Corollary 4.3, we get

$$\|\delta_1\|_{L_t^\infty L_x^2} \leq C\|U\|_{L^2}\|u\|_{C^0}.$$

Step 2*: Estimation on* (δ_2,ϕ)*.* The function δ_2 is a solution of $(\partial_t + \partial_x)\delta_2 = w_{\mathrm{nl}} - w_1$. Thus, using the characteristics formula (equation (2.2)),

$$\begin{aligned}(\delta_2(T,\cdot),\phi) &= \int_{x\in\mathbb{T}} \delta_2(T,x)\phi(x)\,\mathrm{d}x\\ &= \int_{[0,T]\times\mathbb{T}} (w_{\mathrm{nl}} - w_1)(s, x+s-T)\phi(x)\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

We can use Lemma 3.1 to write $w_{\mathrm{nl}} = -\partial_x r_{\mathrm{nl}}$, with $r_{\mathrm{nl}}(t,x) = Q(\zeta_{\mathrm{nl}}(t,x), \zeta_{\mathrm{nl}}(t,-x))$ and similarly for w_1. Thus, integrating by parts,

$$(\delta_2(T,\cdot),\phi) = \int_{[0,T]\times\mathbb{T}} (r_{\mathrm{nl}} - r_1)(s, x+s-T)\partial_x\phi(x)\,\mathrm{d}s\,\mathrm{d}x.$$

Thus,

$$|(\delta_2(T,\cdot),\phi)| \leq \|r_{\mathrm{nl}} - r_1\|_{L_t^1 L_x^1}\|\phi(x)\|_{C^1}.$$

We recall that $r_{\rm nl}(t,x) = Q(\zeta_{\rm nl}(t,x), \zeta_{\rm nl}(t,-x))$, where Q is a quadratic form, and similarly for r_1. Thus, writing $aa' - bb' = ((a-b)(a'+b') + (a'-b')(a+b))/2$, we get

$$\begin{aligned}|(\delta_2(T,\cdot),\phi)| \le C\big(&\|(\zeta_1-\zeta_{\rm nl})(t,x)(\zeta_1(t,x)+\zeta_{\rm nl}(t,x))\|_{L^1_tL^1_x}\\ &+\|(\zeta_1-\zeta_{\rm nl})(t,-x)(\zeta_1(t,x)+\zeta_{\rm nl}(t,x))\|_{L^1_tL^1_x}\\ &+\|(\zeta_1-\zeta_{\rm nl})(t,x)(\zeta_1(t,-x)+\zeta_{\rm nl}(t,-x))\|_{L^1_tL^1_x}\\ &+\|(\zeta_1-\zeta_{\rm nl})(t,-x)(\zeta_1(t,-x)+\zeta_{\rm nl}(t,-x))\|_{L^1_tL^1_x}\big)\\ \le{}& C\|\zeta_1-\zeta_{\rm nl}\|_{L^2_tL^2_x}(\|\zeta_1\|_{L^2_tL^2_x}+\|\zeta_{\rm nl}\|_{L^2_tL^2_x}).\end{aligned}$$

Finally, using the estimate on δ_1 we obtained in the first step, the regularity estimate on ζ_1 of Proposition 2.6, and the estimate on $\zeta_{\rm nl}$ of Corollary 4.3,

$$|(\delta_2(T,\cdot),\phi)| \le C\|U\|_{L^2}\|u\|_{C^0}\|U\|_{L^2}. \qquad \blacksquare$$

In the following proposition, we make it explicit that ζ_2 depends on u by denoting it by $\zeta_2(u,t,x)$, and similarly for other quantities that depend on u, because we need to estimate $\zeta_2(u,\cdot,\cdot) - \zeta_2(\tilde u,\cdot,\cdot)$.

Lemma 4.5. *Let $\phi \in C^1(\mathbb{T})$, $T > 0$ and $u, \tilde u \in L^2$. With the notation of Lemma 4.4, and with $U(t) := \int_0^t u(s)\,\mathrm{d}s$ and $\widetilde U(t) := \int_0^t \tilde u(s)\,\mathrm{d}s$, for some $C > 0$ independent of u, $\tilde u$,*

$$|(\zeta_2(u,T,\cdot) - \zeta_2(\tilde u, T\cdot), \phi)| \le C\|U - \widetilde U\|_{L^2(0,T)}(\|U\|_{L^2(0,T)} + \|\widetilde U\|_{L^2(0,T)}).$$

Proof. We use the same notation w_1 and r_1 as Lemma 3.1. The function $\zeta_2(u) - \zeta_2(\tilde u)$ satisfies

$$(\partial_t + \partial_x)(\zeta_2(u) - \zeta_2(\tilde u)) = w_1(u) - w_1(\tilde u).$$

Thus, according to the characteristics formula,

$$(\zeta_2(u,T,\cdot) - \zeta_2(\tilde u,T\cdot),\phi) = \int_{[0,T]\times\mathbb{T}} (w_1(u,s,x+s-T) - w_1(\tilde u,s,x+s-T))\phi\,\mathrm{d}s\,\mathrm{d}x.$$

Since $w_1(u) = -\partial_x r_1(u)$, we integrate by parts to get

$$(\zeta_2(u,T,\cdot) - \zeta_2(\tilde u,T\cdot),\phi) = \int_{[0,T]\times\mathbb{T}} (r_1(u,s,x+s-T) - r_1(\tilde u,s,x+s-T))\partial_x\phi\,\mathrm{d}s\,\mathrm{d}x.$$

Thus,

$$|(\zeta_2(u,T,\cdot) - \zeta_2(\tilde u,T\cdot),\phi)| \le C\|r_1(u) - r_1(\tilde u)\|_{L^1_tL^1_x}.$$

Recall that $r_1(u,t,x)$ is a linear combination of quadratic terms involving $\zeta_1(u,t,x)$ and $\zeta_1(u,t,-x)$ (see Lemma 3.1). Thus, writing $aa' - bb' = ((a-b)(a'+b') + (a'-b')(a+b))/2$, and using Hölder's inequality, we get

$$|(\zeta_2(u,T,\cdot) - \zeta_2(\tilde u,T\cdot),\phi)| \le C\|\zeta_1(u-\tilde u)\|_{L^2_tL^2_x}(\|\zeta_1(u)\|_{L^2_tL^2_x} + \|\zeta_1(\tilde u)\|_{L^2_tL^2_x}).$$

Finally, the regularity estimate for the linear equation (Proposition 2.6) proves the theorem. ■

4.3. Quadratic drift

We prove in this section a "quadratic drift" (in the words of Beauchard and Marbach [5, §5.4]). Theorem 1.2 follows easily from this result. We keep the notation ζ_{nl}, ζ_1, δ_1, etc. defined at the start of the previous subsection.

Lemma 4.6. *Let* $\Pi\colon \zeta \in L^2(\mathbb{T}) \mapsto (\zeta - \zeta(\cdot + 1))/2$*, which is the orthogonal projection on the reachable space for the linearized equation (Remark 2.4 and Lemma 2.7). Let* $T \in (1,2)$*. There exist* $\phi \in C^\infty(\mathbb{T})$*,* $c = c_T > 0$*, and* $\eta > 0$ *such that for every* $u \in C^0([0,T])$ *with* $u(0) = 0$ *and* $\|u\|_{C^0} < \eta$*, if* $\Pi\zeta_{\mathrm{nl}}(u,T,\cdot) = 0$ *and* $\int_0^T u(t)\,\mathrm{d}t = 0$*,*

$$(\phi, \zeta_{\mathrm{nl}}(u,T,\cdot))_{L^2(\mathbb{T})} \geq c\|U\|^2_{L^2(0,T)}.$$

Proof. Let $T \in (1,2)$. Let $\eta > 0$ be such that Lemma 4.4 holds. Reducing η if necessary, we may assume that $\eta < 1$. Let $u \in C^0(0,T)$ with $u(0) = 0$ and $\|u\|_{C^0} < \eta$ such that $\Pi\zeta_{\mathrm{nl}}(u,T,\cdot) = 0$.

Step 1*: There exists a control* $\tilde{u}$ *close to* u *that steers the linearized equation from* 0 *to* 0*.* We are looking for a control $\tilde{u}$ close to u such that $\zeta_1(\tilde{u},T,\cdot) = 0$. We look for $\tilde{u}$ with the form $\tilde{u} = u + v$. The condition $\zeta_1(u+v,T,\cdot) = 0$ is equivalent to $\zeta_1(v,T,\cdot) = -\zeta_1(u,T,\cdot)$. Since $\Pi\zeta_{\mathrm{nl}}(u,T,\cdot) = 0$ by hypothesis and since $\Pi\zeta_1(u,T,\cdot) = \zeta_1(u,T,\cdot)$ (Remark 2.4), we rewrite this as

$$\zeta_1(v,T,\cdot) = \Pi\delta_1(u,T,\cdot).$$

According to Lemma 2.7, such a control v exists, and we can also choose it such that $\int_0^T v(t)\,\mathrm{d}t = 0$ and such that $\mathcal{V}(t) := \int_0^t v(s)\,\mathrm{d}s$ satisfies $\|\mathcal{V}\|_{L^2(0,T)} \leq C\|\Pi\delta_1(u,T,\cdot)\|_{L^2} \leq C\|\delta_1(u,T,\cdot)\|_{L^2}$. According to the estimate on δ_1 of Lemma 4.4, this control is such that

$$\|\mathcal{V}\|_{L^2(0,T)} \leq C\|u\|_{C^0}\|U\|_{L^2(0,T)}. \tag{4.7}$$

Step 2*: Estimating the difference* $(\zeta_{\mathrm{nl}}(u,T,\cdot),\phi) - (\zeta_2(\tilde{u},T,\cdot),\phi)$*.* Since $\zeta_1(u,T,\cdot)$ is 1-antiperiodic (Remark 2.4), and since ϕ is 1-periodic, $(\zeta_1(u,T,\cdot),\phi) = 0$. Thus using the triangle inequality,

$$\begin{aligned}|(\zeta_{\mathrm{nl}}(u,T,\cdot) - \zeta_2(\tilde{u},T,\cdot),\phi)| &= |(\zeta_{\mathrm{nl}}(u,T,\cdot) - \zeta_1(u,T,\cdot) - \zeta_2(\tilde{u},T,\cdot),\phi)| \\ &\leq |(\zeta_{\mathrm{nl}}(u,T,\cdot) - \zeta_1(u,T,\cdot) - \zeta_2(u,T,\cdot),\phi)| \\ &\quad + |(\zeta_2(u,T,\cdot) - \zeta_2(\tilde{u},T,\cdot),\phi)|.\end{aligned}$$

The first term on the right-hand side is $|(\delta_2(u,T,\cdot),\phi)|$, and according to Lemma 4.4, we have $|(\delta_2(u,T,\cdot),\phi)| \leq C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}$. According to Lemma 4.5, the second term is bounded by $C\|\mathcal{V}\|_{L^2(0,T)}(\|U\|_{L^2(0,T)} + \|\mathcal{V}\|_{L^2(0,T)})$. Thus,

$$\begin{aligned}|(\zeta_{\mathrm{nl}}(u,T,\cdot) - \zeta_2(\tilde{u},T,\cdot),\phi)| &\leq C\|U\|^2_{L^2(0,T)}\|u\|_{C^0} \\ &\quad + C\|\mathcal{V}\|_{L^2(0,T)}(\|\mathcal{V}\|_{L^2(0,T)} + \|U\|_{L^2(0,T)}).\end{aligned}$$

Now, plugging in the estimate for $\|\mathcal{V}\|_{L^2(0,T)}$ (equation (4.7)), we get

$$\begin{aligned}|(\zeta_{\mathrm{nl}}(u,T,\cdot)-\zeta_2(\tilde u,T,\cdot),\phi)| &\le C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}\\ &\quad + C\|U\|_{L^2(0,T)}\|u\|_{C^0}(\|U\|_{L^2(0,T)}\|u\|_{C^0}+\|U\|_{L^2(0,T)})\\ &= C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}+C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}(\|u\|_{C^0}+1).\end{aligned}$$

Since we assumed that $\|u\|_{C^0}<\eta<1$, we have

$$|(\zeta_{\mathrm{nl}}(u,T,\cdot)-\zeta_2(\tilde u,T,\cdot),\phi)| \le C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}. \tag{4.8}$$

Step 3*: Using the coercivity of the kernel.* According to estimate (4.8) from the previous step and the inverse triangle inequality, we have

$$\begin{aligned}(\zeta_{\mathrm{nl}}(u,T,\cdot),\phi) &\ge (\zeta_2(\tilde u,T,\cdot),\phi)-|(\zeta_{\mathrm{nl}}(u,T,\cdot)-\zeta_2(\tilde u,T,\cdot),\phi)|\\ &\ge (\zeta_2(\tilde u,T,\cdot),\phi)-C\|U\|^2_{L^2(0,T)}\|u\|_{C^0}.\end{aligned}$$

Recall that $\zeta_1(\tilde u,T,\cdot)=0$, and $\int_0^T\tilde u(t)\,\mathrm{d}T=0$. Hence we can plug in the coercivity estimate of Proposition 3.7, which gives us

$$(\zeta_{\mathrm{nl}}(u,T,\cdot),\phi)\ge 3(2-T)\|\widetilde U\|^2_{L^2(0,T-1)}-C\|u\|_{C^0}\|U\|^2_{L^2(0,T)}, \tag{4.9}$$

where $\widetilde U(s)=\int_0^s\tilde u(s')\,\mathrm{d}s'$.

Step 4*: Conclusion.* We claim that $\|\widetilde U\|^2_{L^2(0,T)}=2\|\widetilde U\|^2_{L^2(0,T-1)}$. Indeed, recall that

$$\zeta_1(\tilde u,T,\cdot)=0\quad\text{and}\quad\int_0^T\tilde u(s)\,\mathrm{d}s=0.$$

Hence, according to the characterization of the linear controls that steer 0 to 0 (Proposition 2.8), $\tilde u(t+1)=\tilde u(t)$ for $0<t<T-1$ and $\tilde u(t)=0$ for $T-1<t<1$. Thus, $\int_0^{T-1}\tilde u(s)\,\mathrm{d}s=\frac12\int_0^T\tilde u(s)\,\mathrm{d}s=0$ and

$$\widetilde U(t)=\int_0^t\tilde u(s)\,\mathrm{d}s=\begin{cases}\widetilde U(t) & \text{for } 0<t<T-1,\\ \displaystyle\int_0^{T-1}\tilde u(s)\,\mathrm{d}s+\int_{T-1}^t 0\,\mathrm{d}s=0 & \text{for } T-1<t<1,\\ \displaystyle\int_0^{T-1}\tilde u(s)\,\mathrm{d}s+\int_{T-1}^1 0\,\mathrm{d}s & \\ \displaystyle\quad+\int_1^T\tilde u(s-1)\,\mathrm{d}s=\widetilde U(t-1) & \text{for } 1<t<T.\end{cases}$$

This proves the claim that $\|\widetilde U\|^2_{L^2(0,T)}=2\|\widetilde U\|^2_{L^2(0,T-1)}$. Plugging this into equation (4.9), we get

$$(\zeta_{\mathrm{nl}}(u,T,\cdot),\phi)\ge\frac32(2-T)\|\widetilde U\|^2_{L^2(0,T)}-C\|u\|_{C^0}\|U\|^2_{L^2(0,T)}. \tag{4.10}$$

We now bound the term $\|\widetilde{U}\|^2_{L^2(0,T)}$ from below. We have

$$2(U, \mathcal{V})_{L^2(0,T)} \leq \frac{1}{2}\|U\|^2_{L^2(0,T)} + 2\|\mathcal{V}\|^2_{L^2(0,T)}.$$

Since, $\tilde{u} = u + v$, this implies

$$\begin{aligned}\|\widetilde{U}\|^2_{L^2(0,T)} &= \|U\|^2_{L^2(0,T)} - 2(U, \mathcal{V})_{L^2(0,T)} + \|\mathcal{V}\|^2_{L^2(0,T)} \\ &\geq \frac{1}{2}\|U\|^2_{L^2(0,T)} - \|\mathcal{V}\|^2_{L^2(0,T)}.\end{aligned}$$

Using the bound on $\|\mathcal{V}\|_{L^2(0,T)}$ (equation 4.7), and reducing η if necessary, we get

$$\begin{aligned}\|\widetilde{U}\|^2_{L^2(0,T)} &\geq \frac{1}{2}\|U\|^2_{L^2(0,T)} - C\|u\|^2_{C^0}\|U\|^2_{L^2(0,T)} \\ &\geq \frac{1}{4}\|U\|^2_{L^2(0,T)}.\end{aligned}$$

Finally, plugging this into equation (4.10), we get

$$\begin{aligned}(\zeta_{\text{nl}}(u, T, \cdot), \phi) &\geq \frac{3}{8}(2 - T)\|U\|^2_{L^2(0,T)} - C\|u\|_{C^0}\|U\|^2_{L^2(0,T)} \\ &\geq \left(\frac{3}{8}(2 - T) - C\eta\right)\|U\|^2_{L^2(0,T)}.\end{aligned}$$

Reducing η if necessary, this proves the claimed estimate. ∎

A. On the positivity of a class of quadratic forms

In this appendix, we sketch the proof of the following proposition.

Proposition A.1. *Let $I = [a, b]$ with $a < b$, let $\phi\colon I \to \mathbb{R}$ be C^1 and such that $\phi' \geq c > 0$ and ϕ' nonconstant, and let $\varepsilon \in \mathbb{R}^*$. Set $K(s_1, s_2) := (1 + \varepsilon|s_2 - s_1|)(\phi(s_1 \wedge s_2) - \phi(s_1 \vee s_2))$, and denote by Q_K the associated quadratic form, i.e., $Q_K(u) := \int_{I^2} K(s_1, s_2)u(s_1)u(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2$. The following assertions are equivalent:*

1. *there exists $c > 0$ such that for every $u \in L^2(I)$ with $\int_a^b u(t)\,\mathrm{d}t = 0$, $Q_K(u) > c\|U\|^2_{L^2(I)}$, where $U(t) := \int_a^t u(s)\,\mathrm{d}s$;*
2. $\int_I \phi'(s)\,\mathrm{d}s \int_I \frac{\mathrm{d}s}{\phi'(s)} < (b - a + 2\varepsilon^{-1})^2$.

On the other hand, if $\int_I \phi'(s)\,\mathrm{d}s \int_I (\phi'(s))^{-1}\,\mathrm{d}s > (b - a + 2\varepsilon^{-1})^2$, there exist $u_1, u_2 \in L^2(I)$ with $\int_I u_1(s)\,\mathrm{d}s = \int_I u_2(s)\,\mathrm{d}s = 0$ such that $Q_K(u_1) > 0$ and $Q_K(u_2) < 0$.

The hypothesis that ϕ' is not constant is useful to avoid some degeneracy several times in the proof, but the result still holds if ϕ' is constant by perturbing ϕ.

We first start by recasting the quadratic form in a more manageable way for us. This is done thanks to the following lemma.

Lemma A.2. *Define Q_K as in Proposition A.1. Then, for every $u \in L^2(I)$ with $\int_0^T u(t)\,\mathrm{d}t = 0$,*

$$Q_K(u) = 2\int_I \phi'(s)(U(s))^2\,\mathrm{d}s + 2\varepsilon \int_I \phi'(s)U(s)\,\mathrm{d}s \int_I U(s)\,\mathrm{d}s,$$

where $U(t) := \int_a^t u(s)\,\mathrm{d}s$. We will denote the right-hand side of the expression as $\widetilde{Q}_K(U)$, which makes sense for each $U \in L^2(I)$. With this notation, $Q_K(u) = \widetilde{Q}_K(U)$.

This formula actually holds without the assumptions $\phi'(s) \geq c > 0$ and $\varepsilon = 0$, with the same proof.

Sketch of the proof. With K as in Proposition A.1 and w, R as in Lemma 3.8, straightforward computations show that $w(s) = 2\phi'(s)$ and $R(s_1, s_2) = \varepsilon(\phi'(s_1) + \phi'(s_2))$. The terms $g(s)$ and $K(b, b)$ do not matter since $U(b) = 0$. ∎

The expression of this lemma suggests that we work in the weighted space $L^2_{\phi'} := L^2(I, \phi'(s)\,\mathrm{d}s)$. This is where the hypothesis $\phi'(s) > 0$ is useful: to make sense of this space. We will denote by $\|\cdot\|_{\phi'}$ the norm in $L^2_{\phi'}$ and by $(\cdot,\cdot)_{\phi'}$ the scalar product. The main consequence of working in this space is that on a space of codimension 2, $Q_K(u) = 2\|U\|^2_{\phi'}$.

Lemma A.3. *Let $\widetilde{Q}_K$ be as in Lemma A.2. Let S be the symmetric operator (for the $L^2_{\phi'}$ scalar product) associated with $\widetilde{Q}_K$. Let $E := \{U \in L^2_{\phi'},\ \int_I U(s)\,\mathrm{d}s = \int_I \phi'(s)U(s)\,\mathrm{d}s = 0\}$ and $F := \mathrm{Span}(1, (\phi')^{-1})$. Then,*

- *E is the orthogonal of F (for the $L^2_{\phi'}$ scalar product);*
- *E and F are stable by S;*
- *the restriction of S on E is $S_{|E} = 2I$.*

Sketch of the proof. The orthogonality of E and F results from simple computations. Since E is of codimension 2, $E + F = L^2_{\phi'}$. For the other two points, let us denote by $M(U)$ the constant function equal to $\int_I U(s)\,\mathrm{d}s$ and M^* the adjoint of this operator M for the $L^2_{\phi'}$-scalar product. Straightforward computations show that

$$S(U) = 2U + \varepsilon(M(U) + M^*(U)) = 2U + \varepsilon\left(\int_I U(s)\,\mathrm{d}s + \frac{1}{\phi'}\int_I \phi'(s)U(s)\,\mathrm{d}s\right).$$

With this expression of S, the last two points are immediate. ∎

With these lemmas, we can prove Proposition A.1.

Sketch of the proof of Proposition A.1. The main idea is that according to Lemma A.3, the only possible counterexamples to the coercivity inequality $Q_K(U) \geq c\|U\|^2_{\phi'}$ are in F; thus we are left to study whether a 2×2 matrix is positive.

Let us first compute the matrix of the restriction of $\widetilde{Q}_K$ to F in the basis $(1, (\phi')^{-1})$. Here, we use the fact that ϕ' is not constant; otherwise, the family $(1, (\phi')^{-1})$ would not be linearly independent. For simplicity, write $U_1 := 1$, $U_2 := (\phi')^{-1}$, and $M(U)$ the constant function equal to $\int_I U(s)\,\mathrm{d}s$. Then

$$\begin{aligned} A &:= \mathrm{Matrix}_{(U_1,U_2)}(\widetilde{Q}_K)_{|F} \\ &= \begin{pmatrix} 2|U_1|^2 + 2\varepsilon(M(U_1),U_1) & 2(U_1,U_2) + \varepsilon(M(U_1),U_2) + \varepsilon(U_1,M(U_2)) \\ 2(U_1,U_2) + \varepsilon(M(U_1),U_2) + \varepsilon(U_1,M(U_2)) & 2|U_2|^2 + 2\varepsilon(M(U_2),U_2) \end{pmatrix}, \end{aligned}$$

where all the norms and scalar products are taken in $L^2_{\phi'}$. Finally, if we set $\alpha := \int_I \phi'(s)\,\mathrm{d}s$ and $\beta := \int_I (\phi'(s))^{-1}\,\mathrm{d}s$, some straightforward (again) computations prove that this matrix is

$$A = \begin{pmatrix} 2\alpha(1+\varepsilon(b-a)) & 2(b-a) + \varepsilon(b-a)^2 + \varepsilon\alpha\beta \\ 2(b-a) + \varepsilon(b-a)^2 + \varepsilon\alpha\beta & 2\beta(1+\varepsilon(b-a)) \end{pmatrix}.$$

To study the positivity of $\widetilde{Q}_K$, we compute the trace and determinant of A. Straightforward computations show that

$$\mathrm{Tr}(A) = 2(\alpha+\beta)(1+\varepsilon(b-a)), \tag{A.1}$$

$$\det(A) = -\varepsilon^2(\alpha\beta - (b-a)^2)(\alpha\beta - (b-a-2\varepsilon^{-1})^2). \tag{A.2}$$

Finally, let us note that thanks to the Cauchy–Schwarz inequality, $(b-a)^2 < \alpha\beta$, where the inequality is strict because we assumed that ϕ' is not constant.

Step 1*:* (1) $\Rightarrow$ (2). If assertion (1) holds, $\widetilde{Q}_K$ is positive definite, thus the matrix A is positive definite. Hence, $\det(A) > 0$. Since $(b-a)^2 < \alpha\beta$, according to expression (A.2) of $\det(A)$, we have $\alpha\beta < (b-a+2\varepsilon^{-1})^2$, which is exactly assertion (2).

Step 2*:* (2) $\Rightarrow$ (1). If assertion (2) holds, according to expression (A.2) of $\det(A)$ and the fact $(b-a)^2 < \alpha\beta$, we have $\det(A) > 0$. Moreover, since $(b-a)^2 < \alpha\beta < (b-a+2\varepsilon^{-1})^2$, we have $b-a < |b-a+2\varepsilon^{-1}|$, i.e., $b-a < b-a+2\varepsilon^{-1}$ or $b-a < -(b-a) - 2\varepsilon^{-1}$. In both cases, we get $1+\varepsilon(b-a) > 0$. Hence, according to expression (A.1) of $\mathrm{Tr}(A)$, we have $\mathrm{Tr}(A) > 0$. Thus, A is positive definite. Finally, according to Lemma A.3, we deduce that for each $U \in L^2$, $\widetilde{Q}_K(U) > c\|U\|^2_{\phi'}$. Since $\phi' \geq c > 0$, the $L^2_{\phi'}$ and L^2 norm are equivalent, hence assertion (1) holds.

Step 3*: Last assertion.* If $\alpha\beta > (b-a+2\varepsilon^{-1})^2$, according to expression (A.2) of $\det(A)$ and the fact $(b-a)^2 < \alpha\beta$, we have $\det(A) < 0$, hence A has a positive and a negative eigenvalue, and so does $\widetilde{Q}_K$. Hence, we can find $\widetilde{U}_1, \widetilde{U}_2 \in L^2(I)$ such that $\widetilde{Q}_K(\widetilde{U}_1) > 0$ and $\widetilde{Q}_K(\widetilde{U}_2) < 0$. By approximating $\widetilde{U}_i$ in the L^2-norm by some $U_i \in H^1_0(I)$, we find $U_1, U_2 \in H^1_0(I)$ such that $\widetilde{Q}_K(U_1) > 0$ and $\widetilde{Q}_K(U_2) < 0$. Since $Q_K(U') = \widetilde{Q}_K(U)$, this proves the proposition. ■

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