

# Profinite non-rigidity of arithmetic groups

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**Abstract.** We show that for a typical higher-rank arithmetic lattice  $\Gamma$ , there exist finite index subgroups  $\Gamma_1$  and  $\Gamma_2$  such that  $\Gamma_1 \not\cong \Gamma_2$  while  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ . But there are exceptions to that rule.

## 1. Introduction

Let  $\Gamma$  be a finitely generated residually finite group. We say that  $\Gamma$  is profinitely rigid if whenever  $\widehat{\Lambda} \cong \widehat{\Gamma}$  for some finitely generated residually finite group  $\Lambda$ , then  $\Lambda \cong \Gamma$ . Here,  $\widehat{\Gamma}$  (resp.  $\widehat{\Lambda}$ ) denotes the profinite completion of  $\Gamma$  (resp.  $\Lambda$ ).

Up until recently, the only profinite rigid groups were “small” (i.e., without non-abelian free subgroups). Recently, in a groundbreaking work, Bridson, McReynolds, Reid, and Spitler gave first examples of “big” groups which are profinitely rigid, among them are some fundamental groups of hyperbolic 3-manifolds [2] and some triangle groups [3].

Arithmetic subgroups of semisimple Lie groups need not be profinitely rigid [1, 7]. A well-known open problem asks:

For  $n \geq 2$ , is  $SL_n(\mathbb{Z})$  profinitely rigid?

While we will not answer this question, we will show that there are finite index subgroups of these groups (at least when  $n \geq 3$ ) which are not profinitely rigid. In fact, we will show a much more general result.

**Theorem 1.1** (Main theorem). *Let  $k$  be a number field and  $\mathbf{G}$  be a connected, simply connected, absolutely almost simple  $k$ -linear algebraic group of high  $\infty$ -rank such that  $\mathbf{G}(k)$  satisfies the congruence subgroup property and  $\Gamma \subseteq \mathbf{G}(k)$  an arithmetic subgroup:*

- (1) *Unless  $\mathbf{G}$  is of type  $G_2$ ,  $F_4$ , or  $E_8$  and  $k = \mathbb{Q}$ ,  $\Gamma$  has infinitely many pairs of finite index subgroups  $\Gamma_1$  and  $\Gamma_2$  which are not isomorphic but their profinite completions are.*
- (2) *The exceptional cases are truly exceptional and in these cases, there are no such pairs at all. In fact, if  $\Gamma_1, \Gamma_2 \subseteq \mathbf{G}(\mathbb{Q})$  are arithmetic subgroups with isomorphic profinite completions, then  $\Gamma_1$  and  $\Gamma_2$  are isomorphic.*

Note that we show that each such  $\Gamma$  has a finite index subgroup  $\Gamma_1$  which is not profinitely rigid by showing that  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$  for some  $\Gamma_2$  commensurable to it. This complements results of [1, 7, 8] which give examples of non-commensurable arithmetic groups which are profinitely isomorphic.

To illustrate our methods, let us now present them only for  $\Gamma = SL_4(\mathbb{Z})$  (Methods A and B) and  $\Gamma = SL_2(\mathbb{Z}[\sqrt{2}])$  (Method C).

**Method A:** Using the center of the simply connected form: Let  $2 \neq p, q$  be two different primes, and let  $\Lambda$  be the principal congruence subgroup corresponding to  $pq$ . Let  $\rho_p$  be the element of  $\widehat{SL}_4(\mathbb{Z}) \cong \prod_r SL_4(\mathbb{Z}_r)$  which is 1 at the places  $r \neq p$  and  $-1$  at the place  $p$ , similarly define  $\rho_q$ . Define  $\Gamma_1 := \langle \widehat{\Lambda}, \rho_p \rangle \cap SL_4(\mathbb{Z})$  and  $\Gamma_2 := \langle \widehat{\Lambda}, \rho_q \rangle \cap SL_4(\mathbb{Z})$ . Then  $\widehat{\Gamma}_1 \cong \widehat{\Lambda} \times \mathbb{Z}/2\mathbb{Z} \cong \widehat{\Gamma}_2$  but  $\Gamma_1$  and  $\Gamma_2$  cannot be isomorphic. For details see Theorem 3.1.

**Method B:** Using a non-trivial Dynkin automorphism: Let  $2, 3 \neq p, q$  be two different primes. Consider the following maximal parabolic subgroups of  $SL_4(\mathbb{Z}/p\mathbb{Z})$ :

$$P_{1,p} := \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \right\}, \quad P_{2,p} := \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & * & * & * \end{pmatrix} \right\}.$$

Similarly define the maximal parabolic subgroups  $P_{1,q}$  and  $P_{2,q}$  of  $SL_4(\mathbb{Z}/q\mathbb{Z})$ . Now, let  $\Gamma_1$  be the congruence subgroup corresponding to  $P_{1,p} \bmod p$ ,  $P_{1,q} \bmod q$  and is trivial mod 3, and  $\Gamma_2$  to be the congruence subgroup corresponding to  $P_{1,p} \bmod p$ ,  $P_{2,q} \bmod q$  and is trivial mod 3. Then the profinite completions of  $\Gamma_1$  and  $\Gamma_2$  are isomorphic via the automorphism of  $\widehat{SL}_4(\mathbb{Z})$  which is the non-trivial Dynkin automorphism at the place  $q$  and the identity elsewhere, but they themselves cannot be isomorphic. For details see Theorem 4.1.

**Method C:** Using the number field: Let  $2 \neq p, q$  be two different primes such that 2 is a square in  $\mathbb{Q}_p$  and  $\mathbb{Q}_q$ , hence  $p$  and  $q$  split completely in  $\mathbb{Q}[\sqrt{2}]$ . Set  $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$  and  $\mathfrak{p}_1, \mathfrak{p}_2 | p$ ,  $\mathfrak{q}_1, \mathfrak{q}_2 | q$  to be the primes lying over  $p$  and  $q$ , respectively. Let  $\Gamma_1$  be the principal congruence subgroup corresponding to  $\mathfrak{p}_1\mathfrak{q}_1$ , and  $\Gamma_2$  be the principal congruence subgroup corresponding to  $\mathfrak{p}_2\mathfrak{q}_1$ . As  $SL_2(\mathcal{O})$  has trivial congruence kernel,  $\widehat{SL}_2(\mathcal{O}) \cong \left( \prod_{l \neq p, q} SL_2(\mathcal{O}_l) \right) \times \prod_{i=1}^2 (SL_2(\mathcal{O}_{\mathfrak{p}_i}) \times SL_2(\mathcal{O}_{\mathfrak{q}_i}))$ , where for a prime  $l \neq p, q$ ,  $\mathcal{O}_l$  is the completion of  $\mathcal{O}$  with respect to the primes lying over  $l$ . Then the profinite completions,  $\widehat{\Gamma}_1$  and  $\widehat{\Gamma}_2$ , are isomorphic via the automorphism of  $\widehat{SL}_2(\mathbb{Z}[\sqrt{2}])$  which is the transposition of the places  $\mathfrak{p}_1$  and  $\mathfrak{p}_2$ . But  $\Gamma_1$  and  $\Gamma_2$  cannot be isomorphic. For details see Theorem 3.2.

The main theorem is proved by generalizing the above methods to more general arithmetic lattices. (In fact, only Methods A and C are really needed.)

The paper is organized as follows: After some preliminaries in Section 2, we will generalize Methods A and C in Section 3 deducing the first part of the main theorem.

In Section 4, we will elaborate on [Method B](#) and finally in Section 5 we will prove that the exceptional cases are true exceptions, concluding the main theorem. In Section 6, we will give a stronger and more general version of the main theorem, stating that it holds for  $S$ -arithmetic groups and not merely for arithmetic groups. Moreover, one can get any (finite) number of non-isomorphic subgroups with isomorphic profinite completions (not just pairs).

## 2. Preliminaries

Throughout we assume that  $k$  is a number field. The set of places of  $k$  is denoted by  $V(k)$ ; it is the union of the set of archimedean places  $V_\infty(k)$  and the set of finite places  $V_f(k)$ . The completion of  $k$  at  $v \in V(k)$  is denoted by  $k_v$ . Let  $\mathcal{O}_k$  denote the ring of integers of  $k$  and for a finite place  $v \in V_f(k)$ , denote by  $\mathcal{O}_{k,v}$  the ring of integers of  $k_v$ . The ring of finite adeles  $\mathbb{A}_k^f = \prod_{v \in V_f(k)}^* k_v := \{(x_v)_v \in \prod_{v \in V_f(k)} k_v : x_v \in \mathcal{O}_{k,v} \text{ for all but finitely many places}\}$  is the restricted product over all the finite completions of  $k$ . If  $k$  is clear from the context, we will omit the letter  $k$  from all the above.

Let  $\tilde{\mathbf{G}}$  be a connected, simply connected, absolutely almost simple  $k$ -linear algebraic group, with a fixed faithful  $k$ -representation  $\rho : \tilde{\mathbf{G}} \rightarrow GL(n_\rho)$ . A subgroup  $\Gamma \subseteq \tilde{\mathbf{G}}(k)$  is called arithmetic if it is commensurable with  $\tilde{\mathbf{G}}(\mathcal{O})$  (see [10, 13] for more details about arithmetic groups). We will usually write  $\mathbf{G}$  for the adjoint form of  $\tilde{\mathbf{G}}$  (which is the universal form), and by  $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$  the universal covering map, it is a central isogeny, and  $\mathbf{Z} := \ker \pi = \mathbf{Z}(\tilde{\mathbf{G}})$  is a finite group. The  $V_\infty(k)$ -rank of  $\tilde{\mathbf{G}}$  is  $\text{rank}_{V_\infty(k)} \tilde{\mathbf{G}} := \sum_{v \in V_\infty(k)} \text{rank}_{k_v} \tilde{\mathbf{G}}$ , where  $\text{rank}_{k_v} \tilde{\mathbf{G}}$  is the dimension of a maximal  $k_v$ -split torus;  $\tilde{\mathbf{G}}$  is said to have high  $\infty$ -rank if its  $V_\infty(k)$ -rank is  $\geq 2$ .

We will use Margulis' superrigidity in a rather delicate manner. The particular version we use is the following.

**Theorem 2.1** (Margulis' superrigidity). *Assume  $\tilde{\mathbf{G}}$  is of high  $\infty$ -rank, and let  $\Gamma_1, \Gamma_2 \subseteq \tilde{\mathbf{G}}(k)$  be arithmetic subgroups. Assume further that  $\Gamma_i \cap \mathbf{C}(k) = 1$ . If  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  is an isomorphism, then there exists a unique  $k$ -automorphism  $\Phi$  of  $\tilde{\mathbf{G}}$  and a unique automorphism  $\sigma$  of  $k$  such that  $\varphi(\gamma) = \Phi(\sigma^0(\gamma))$  for every  $\gamma \in \Gamma_1$ , where  $\sigma^0$  is the automorphism of  $\tilde{\mathbf{G}}(k)$  induced by  $\sigma$ .*

*Proof.* Identify  $\Gamma_1$  and  $\Gamma_2$  as arithmetic subgroups of the adjoint group  $\mathbf{G}(k)$  via the universal covering map  $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$ . Moving to the adjoint form enables us to use Margulis' superrigidity [9, Theorem VIII.3.6. (ii)] extending the isomorphism  $\varphi$  to an automorphism of  $\text{Res}_{k/\mathbb{Q}} \mathbf{G}$ . By the properties of the restriction of scalars functor, such an automorphism must be of the form  $\Phi \circ \sigma^0$  for a  $k$ -automorphism  $\Phi$  of  $\mathbf{G}$  and an automorphism  $\sigma$  of the field  $k$  [5, Proposition A.5.14].

The  $k$ -automorphism  $\Phi$  of  $\mathbf{G}$  is of the form  $C_g \circ \omega$  where  $C_g$  is conjugation by some  $g \in \mathbf{G}(\bar{k})$  (preserving  $\mathbf{G}(k)$ ) and  $\omega$  is a Dynkin automorphism. Both can be interpreted

as automorphisms of  $\tilde{\mathbf{G}}$  and these interpretations act the same way upon  $\Gamma_1$ . Hence, the assertion of the theorem. ■

If  $\sigma$  is an automorphism of  $k$ , it induces a permutation of the (finite) places of  $k$ , and thus an automorphism of the adelic group  $\tilde{\mathbf{G}}(\mathbb{A}_k^f)$  by permuting its factors according to  $\sigma$ , call this automorphism  $\sigma_{\mathbb{A}}^0$ . If  $\Phi$  is a  $k$ -automorphism of  $\tilde{\mathbf{G}}$ , it induces a unique  $k_v$ -automorphism  $\Phi_v : \tilde{\mathbf{G}}(k_v) \rightarrow \tilde{\mathbf{G}}(k_v)$  for every finite place  $v$  of  $k$  and the product  $(\Phi_v)_v : \prod_v \tilde{\mathbf{G}}(k_v) \rightarrow \prod_v \tilde{\mathbf{G}}(k_v)$  restricts to an automorphism  $\Phi_{\mathbb{A}} : \tilde{\mathbf{G}}(\mathbb{A}_k^f) \rightarrow \tilde{\mathbf{G}}(\mathbb{A}_k^f)$  [13, §5]. Clearly  $\Phi(\sigma^0(\gamma)) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$  for every  $\gamma \in \tilde{\mathbf{G}}(k)$  (we identify the group of rational points  $\tilde{\mathbf{G}}(k)$  with its diagonal embedding in the group of adelic points  $\tilde{\mathbf{G}}(\mathbb{A}_k^f)$ ) and is unique with this property. We thus get the following corollary.

**Corollary 2.2.** *Under the assumptions of the previous theorem. If  $\varphi : \Gamma_1 \rightarrow \Gamma_2$  is an isomorphism. Then there exist unique automorphisms  $\Phi_{\mathbb{A}}$  and  $\sigma_{\mathbb{A}}^0$  of the adelic group  $\tilde{\mathbf{G}}(\mathbb{A}_k^f)$ , such that  $\sigma_{\mathbb{A}}^0$  is induced from an automorphism of  $k$ , and  $\Phi_{\mathbb{A}}$  is induced from a  $k$ -automorphism of  $\tilde{\mathbf{G}}$  with  $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$  for every  $\gamma \in \Gamma_1$ .*

We will also need an adelic version of Margulis' superrigidity stated and proven by Kammeyer and Kionke [6, Theorem 3.2].

**Theorem 2.3.** *Let  $\mathbf{G}$  be a connected, absolutely almost simple  $\mathbb{Q}$ -linear algebraic group of high  $\infty$ -rank and  $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$  an arithmetic subgroup. If  $\varphi : \Gamma \rightarrow \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^f)$  is a homomorphism such that  $\overline{\varphi(\Gamma)}$  has non-empty interior, then there exist a homomorphism of adelic groups  $\eta : \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^f) \rightarrow \mathbf{G}(\mathbb{A}_{\mathbb{Q}}^f)$  and a group homomorphism  $\nu : \Gamma \rightarrow \mathbf{Z}(\mathbf{G})(\mathbb{A}_{\mathbb{Q}}^f)$  with finite image such that  $\varphi(\gamma) = \nu(\gamma)\eta(\gamma)$  for all  $\gamma \in \Gamma$ . Moreover,  $\eta$  and  $\nu$  are uniquely determined by this condition.*

## 2.1. Profinite groups and the congruence subgroup property

A family  $(\{G_i\}_{i \in I}, \{\phi_{i,j}\}_{i \geq j \in I})$  is an inverse system of finite groups over the directed set  $I$  if the  $G_i$ s are finite groups,  $\phi_{i,j} : G_i \rightarrow G_j$  are homomorphisms of groups whenever  $i \geq j$  such that  $\phi_{i,k} = \phi_{j,k}\phi_{i,j}$  for every  $i \geq j \geq k$  and  $\phi_{ii} = id_{G_i} \forall i$ . A group  $G$  is called profinite if it is the inverse limit of an inverse system of finite groups over some directed set. A profinite group is a compact, Hausdorff, totally disconnected topological group, and a map of profinite groups is a continuous homomorphism of groups.

**Example 2.4** (Profinite completion). Let  $\Gamma$  be a finitely generated group, and let  $\mathcal{N}$  be the set of finite index normal subgroups of  $\Gamma$ ; for  $M, N \in \mathcal{N}$ , declare that  $M \leq N$  whenever  $N \subseteq M$ ; it is a directed set. Consider the natural quotient homomorphisms  $\phi_{N,M} : \Gamma/N \rightarrow \Gamma/M$ , then the profinite group  $\hat{\Gamma} = \varprojlim_{N \in \mathcal{N}} \Gamma/N$  is called the *profinite completion* of  $\Gamma$ .

The profinite completion  $\hat{\Gamma}$  and the set  $\mathcal{C}(\Gamma)$  of isomorphism classes of the finite quotients of  $\Gamma$  hold the same information in the following manner.

**Theorem 2.5** ([16, Theorems 3.2.2 and 3.2.7]). *If  $\Gamma$  and  $\Lambda$  are two finitely generated residually finite groups, then  $\mathcal{C}(\Gamma) = \mathcal{C}(\Lambda)$  if and only if  $\widehat{\Gamma} \cong \widehat{\Lambda}$*

There is a natural map  $\iota : \Gamma \rightarrow \widehat{\Gamma}$  given by  $\gamma \mapsto (\gamma N)_N$ , and this map is injective if and only if  $\Gamma$  is residually finite; in this case, we identify  $\Gamma$  with its image  $\iota(\Gamma)$ . The pair  $(\widehat{\Gamma}, \iota)$  satisfies a universal property:  $\iota(\Gamma)$  is dense in  $\widehat{\Gamma}$ , and for every profinite group  $P$ , and every homomorphism  $\varphi : \Gamma \rightarrow P$ , there exists a unique homomorphism of profinite groups  $\widehat{\varphi} : \widehat{\Gamma} \rightarrow P$  such that  $\widehat{\varphi} \circ \iota = \varphi$ .

There is a strong connection between the finite index subgroups of  $\widehat{\Gamma}$  and those of  $\Gamma$ .

**Proposition 2.6** ([16, Proposition 3.2.2]). *Let  $\Gamma$  be a finitely generated residually finite group, then there is a one-to-one correspondence between the set  $\mathcal{X}$  of all finite index subgroups of  $\Gamma$  and the set  $\mathcal{Y}$  of all finite index subgroups of  $\widehat{\Gamma}$ , given by*

$$\begin{aligned} X &\mapsto \overline{X}, & X &\in \mathcal{X} \\ Y &\mapsto Y \cap \Gamma, & Y &\in \mathcal{Y}, \end{aligned}$$

where  $\overline{X}$  denote the closure of  $X$  in  $\widehat{\Gamma}$ . Moreover, this bijection preserves normality, index, and quotients.

Note that we have implemented in the above the assertion of [12] that every finite index subgroup of  $\widehat{\Gamma}$  is open.

**Example 2.7** (Congruence completion). Let  $\Gamma$  be an arithmetic subgroup of  $\widetilde{\mathbf{G}}(k)$ . Consider the set  $\mathcal{C}$  of all congruence subgroups, that is, subgroups that contain  $\Gamma[\mathcal{I}] := \Gamma \cap \ker(\phi_{\mathcal{I}} : \widetilde{\mathbf{G}}(\mathcal{O}_k) \rightarrow \widetilde{\mathbf{G}}(\mathcal{O}_k/\mathcal{I}))$  for some ideal  $\mathcal{I} \triangleleft \mathcal{O}_k$ , where  $\phi_{\mathcal{I}}$  is the reduction map mod  $\mathcal{I}$ . As in the profinite completion,  $\mathcal{C}$  is a directed set by the inverse of inclusion, and one can form the *congruence completion*  $\overline{\Gamma}$  of  $\Gamma$  with respect to this inverse system.

Thus, there is a surjective map  $\widehat{\Gamma} \rightarrow \overline{\Gamma}$  between the profinite completion and the congruence completion. Call  $C(\Gamma)$ , the kernel of this map, the *congruence kernel*. The group  $\Gamma$  is said to have the *congruence subgroup property* if the congruence kernel  $C(\Gamma)$  is a finite group. It is not difficult to see that the congruence subgroup property is actually a property of the ambient group  $\widetilde{\mathbf{G}}$  and the field  $k$ .

It was conjectured by Serre [17] that if  $\text{rank}_{V_{\infty}(k)}(\widetilde{\mathbf{G}}) \geq 2$  and  $\Gamma \subseteq \widetilde{\mathbf{G}}(k)$  is an arithmetic subgroup then  $C(\Gamma)$  is trivial or isomorphic to a subgroup of the roots of unity of  $k$  if  $k$  is totally imaginary. The conjecture has been proven in many instances, including for example, all the isotropic cases [15] and all anisotropic groups of type  $B_n, C_n, D_n$  (except for some triality forms of  $D_4$ ),  $E_7, E_8, F_4$ , and  $G_2$  ([13, Chapter 9] and [14]).

## 2.2. A number theoretic lemma

**Lemma 2.8.** *Let  $\widetilde{\mathbf{G}}$  be a connected, simply connected, absolutely almost simple  $k$ -linear algebraic group. There exist infinitely many finite places  $v \in V_f(k)$  such that  $\widetilde{\mathbf{G}}$  splits over  $k_v$ . Moreover, one can assume that for these places,  $\mathbf{Z}(\mathcal{O}_v) = \mathbf{Z}(\mathbb{C})$ .*

*Proof.* There exists a finite Galois field extension  $k'/k$  such that  $\tilde{\mathbf{G}}$  splits over  $k'$  and  $\mathbf{Z}(k') = \mathbf{Z}(\mathbb{C})$ . By Chebotarev's density theorem [11, Corollary 13.6], there exist infinitely many primes  $\mathfrak{p} \subseteq \mathcal{O}_k$  that split completely in  $k'$ . In particular, if such a prime lies under a prime  $\mathfrak{p}' \subseteq \mathcal{O}_{k'}$  then  $k_v \cong k'_{v'}$ , where  $v$  and  $v'$  are the places corresponding to the primes  $\mathfrak{p}$  and  $\mathfrak{p}'$ , respectively. Thus, for such a place  $v$ ,  $\tilde{\mathbf{G}}$  splits over  $k_v$ .

Moreover, as the center  $\mathbf{Z}(k')$  is finite, for all but finitely many places,  $\mathbf{Z}(k') = \mathbf{Z}(\mathcal{O}_{k',v'})$ . So there exist infinitely many places  $v \in V_f(k)$  with  $\mathbf{Z}(\mathcal{O}_{k,v}) = \mathbf{Z}(\mathbb{C})$  and  $\tilde{\mathbf{G}}$  splits over  $k_v$ . ■

### 3. First part of the main theorem: Existence

In this section we will prove the first part of Theorem 1.1, the existence part. It will follow from the two theorems below.

**Theorem 3.1.** *Let  $\Gamma \subseteq \tilde{\mathbf{G}}(k)$  be an arithmetic subgroup. Assume that  $\tilde{\mathbf{G}}$  has the congruence subgroup property and type different than  $E_8, F_4$ , or  $G_2$ . Then there exist two non-isomorphic finite index subgroups  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  with isomorphic profinite completions.*

*Proof* (Following Method A). Moving to a finite index subgroup, one can assume that there exists a finite set of primes  $S$  and a compact open subgroup  $\Lambda \subseteq \prod_{v \in S} \tilde{\mathbf{G}}(k_v)$  commensurable with  $\prod_{v \in S} \tilde{\mathbf{G}}(\mathcal{O}_v)$  such that

$$\hat{\Gamma} = \bar{\Gamma} \cong \Lambda \times \prod_{v \notin S} \tilde{\mathbf{G}}(\mathcal{O}_v). \quad (1)$$

Indeed, the congruence kernel  $C(\Gamma) \subseteq \hat{\Gamma}$  is finite, so one can find a finite index subgroup  $\Gamma'^P \subseteq \hat{\Gamma}$  of the above form; by Proposition 2.6, there exists a finite index subgroup  $\Gamma' \subseteq \Gamma$  such that  $\hat{\Gamma}' = \Gamma'^P$ . Let  $\pi : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$  be the canonical central isogeny to the adjoint form  $\mathbf{G}$ , by moving again to a finite index subgroup one can assume that  $\Gamma \cap C(k) = \{1\}$ , and  $\hat{\Gamma}$  is still of the same form as (1).

Let  $S_{\mathbb{Q}}$  be the set of all rational primes lying under some valuation in  $S$ , and  $S_{\text{full}} = \{v : v \mid p \text{ for some } p \in S_{\mathbb{Q}}\}$ . By Lemma 2.8, one can find two valuations  $\mathfrak{p}, \mathfrak{q} \notin S_{\text{full}}$  lying over different rational primes  $p$  and  $q$ , respectively, and such that  $\mathbf{Z}(\mathcal{O}_{\mathfrak{p}}) = \mathbf{Z}(\mathcal{O}_{\mathfrak{q}}) = \mathbf{Z}(\mathbb{C})$ , and thus also elements  $1 \neq \rho_{\mathfrak{p}} \in \mathbf{Z}(\mathcal{O}_{\mathfrak{p}})$  and  $1 \neq \rho_{\mathfrak{q}} \in \mathbf{Z}(\mathcal{O}_{\mathfrak{q}})$  of the same order. For every  $v \mid p, q$ , let  $\Delta_v$  be a finite index subgroup of  $\tilde{\mathbf{G}}(\mathcal{O}_v)$  with  $\mathbf{Z}(\mathcal{O}_v) \cap \Delta_v = \{1\}$ . Define  $\varepsilon_{\mathfrak{p}} = (\varepsilon_{\mathfrak{p},w})_w, \varepsilon_{\mathfrak{q}} = (\varepsilon_{\mathfrak{q},w})_w \in \hat{\Gamma}$ , where

$$\varepsilon_{\mathfrak{p},w} := \begin{cases} 1 & \text{if } w \neq \mathfrak{p} \\ \rho_{\mathfrak{p}} & \text{if } w = \mathfrak{p} \end{cases}, \quad \varepsilon_{\mathfrak{q},w} := \begin{cases} 1 & \text{if } w \neq \mathfrak{q} \\ \rho_{\mathfrak{q}} & \text{if } w = \mathfrak{q} \end{cases}.$$

Now, we define the following subgroups of  $\widehat{\Gamma}$ :

$$\begin{aligned}\Delta &= \Lambda \times \prod_{v \notin S, v \nmid p, v \nmid q} \widetilde{\mathbf{G}}(\mathcal{O}_v) \times \prod_{v|p \text{ or } v|q} \Delta_v \\ \Delta_1 &= \langle \Delta, \varepsilon_p \rangle \cong \Delta \times \langle \varepsilon_p \rangle \\ \Delta_2 &= \langle \Delta, \varepsilon_q \rangle \cong \Delta \times \langle \varepsilon_q \rangle.\end{aligned}$$

Clearly  $\Delta_1$  and  $\Delta_2$  are isomorphic finite index subgroups of  $\widehat{\Gamma}$ . By Proposition 2.6, there exist finite index subgroups  $\Gamma_i \subseteq \Gamma$  with  $\widehat{\Gamma}_i = \Delta_i$  for  $i = 1, 2$ . We will finish the proof by showing that  $\Gamma_1$  and  $\Gamma_2$  cannot be isomorphic.

Assume to the contrary that there exists an isomorphism  $\varphi : \Gamma_1 \rightarrow \Gamma_2$ . By Corollary 2.2, there exist unique adelic automorphisms  $\Phi_{\mathbb{A}}$  and  $\sigma_{\mathbb{A}}^0$  of  $\widetilde{\mathbf{G}}(\mathbb{A}_k^f)$  such that  $\sigma_{\mathbb{A}}^0$  is induced from an automorphism of  $k$  and  $\Phi_{\mathbb{A}}$  is induced from a  $k$ -automorphism of  $\widetilde{\mathbf{G}}$  such that  $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$  for every  $\gamma \in \Gamma_1$ , taking closures one deduces that  $(\widetilde{\Phi}_{\mathbb{A}} \circ \sigma_{\mathbb{A}}^0)(\widehat{\Gamma}_1) = \widehat{\Gamma}_2$ . On the other hand, the induced map between the profinite completions  $\widehat{\varphi} : \widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_2$  is unique with  $\widehat{\varphi}(\gamma) = \varphi(\gamma)$  for every  $\gamma \in \Gamma_1$ . Thus, it must be that  $\widehat{\varphi} \equiv (\widetilde{\Phi}_{\mathbb{A}} \circ \sigma_{\mathbb{A}}^0)|_{\widehat{\Gamma}_1}$ . In particular, it implies that the  $p$ th place of  $\widehat{\Gamma}_1$  is mapped isomorphically onto the  $\sigma(p)$ th place of  $\widehat{\Gamma}_2$ . This is a contradiction since  $\sigma(p) \nmid p$ , so the  $\sigma(p)$ th place of  $\widehat{\Gamma}_2$  is centerless, but the  $p$ th place of  $\widehat{\Gamma}_1$  has a non-trivial center. ■

**Theorem 3.2.** *Let  $\Gamma \subseteq \widetilde{\mathbf{G}}(k)$  be an arithmetic subgroup. Assume further that  $\widetilde{\mathbf{G}}(k)$  has the congruence subgroup property and that  $k$  is a number field of degree  $d \geq 2$  over  $\mathbb{Q}$ . Then there exist two non-isomorphic finite index subgroups  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  with isomorphic profinite completions.*

*Proof* (Following Method C). As before, by moving to a finite index subgroup we can assume that  $\Gamma \cap \mathbf{C}(k) = \{1\}$  and that  $\widehat{\Gamma} = \overline{\Gamma} = \Lambda \times \prod_{v \notin S} \widetilde{\mathbf{G}}(\mathcal{O}_v)$ , for some finite set of places  $S$  and  $\Lambda \subseteq \prod_{v \in S} \widetilde{\mathbf{G}}(k_v)$  commensurable with  $\prod_{v \in S} \widetilde{\mathbf{G}}(\mathcal{O}_v)$ . Let  $S_{\mathbb{Q}}$  and  $S_{\text{full}}$  be as before. By Chebotarev's density theorem, there exist infinitely many rational primes that split completely in  $k$  [11, Corollary 13.6]; pick two such different primes  $p, q \notin S_{\mathbb{Q}}$ . Say  $\mathfrak{p}_1, \dots, \mathfrak{p}_d | p$  and  $\mathfrak{q}_1, \dots, \mathfrak{q}_d | q$ . For  $i = 1, 2$ , let  $\Gamma_i := \Gamma(\mathfrak{p}_i \mathfrak{q}_1)$  be the principle congruence subgroups modulo  $\mathfrak{p}_i \mathfrak{q}_1$ , then obviously  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ .

By Corollary 2.2, if  $\Gamma_1$  and  $\Gamma_2$  are isomorphic. As above, there exist unique adelic automorphisms  $\Phi_{\mathbb{A}}$  and  $\sigma_{\mathbb{A}}^0$  of  $\widetilde{\mathbf{G}}(\mathbb{A}_k^f)$  such that  $\sigma_{\mathbb{A}}^0$  is induced from an automorphism of  $k$  and  $\Phi_{\mathbb{A}}$  is induced from a  $k$ -automorphism of  $\widetilde{\mathbf{G}}$  such that  $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$  for every  $\gamma \in \Gamma_1$ , taking closures one deduces that  $(\widetilde{\Phi}_{\mathbb{A}} \circ \sigma_{\mathbb{A}}^0)(\widehat{\Gamma}_1) = \widehat{\Gamma}_2$ . As  $\sigma_{\mathbb{A}}^0$  acts by permuting the places and  $\widetilde{\Phi}_{\mathbb{A}}$  acts place-wise, it must be that  $\sigma_{\mathbb{A}} : \mathfrak{q}_1 \mapsto \mathfrak{q}_1$ . We claim that such  $\sigma$  must be trivial, which will finish the proof.

Indeed, let  $N$  be the Galois closure of  $k$ ,  $G := \text{Gal}(N/\mathbb{Q})$ ,  $H := \text{Gal}(N/k)$ , let  $r$  be a prime lying over  $\mathfrak{p}_1$  and  $G_r$  be its decomposition group. The correspondence  $H \backslash G / G_r \rightarrow \{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$  given by  $H\sigma G_r \mapsto \sigma \mathfrak{p}_1$  is a one-to-one correspondence (of  $G$ -sets), hence  $G_r$  must be trivial (see [11, p. 55]). Thus,  $\text{Aut}(k)$  acts freely on the set of primes  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_d\}$ , as needed. ■

### 4. Another set of examples

The following theorem is not needed for the proof of the main theorem, but following [Method B](#), it gives many more examples of non-profinitely rigid arithmetic groups.

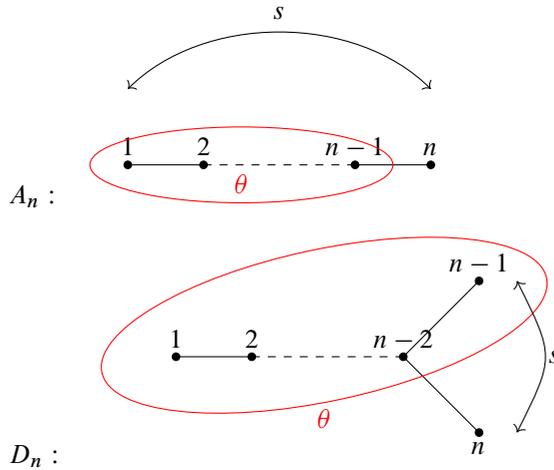
**Theorem 4.1.** *Let  $\Gamma \subseteq \tilde{\mathbf{G}}(k)$  be an arithmetic subgroup. Assume further that  $\tilde{\mathbf{G}}$  has (1) the congruence subgroup property; (2) type  $A_n (n \geq 2)$ ,  $D_n (n \geq 4)$ , or  $E_6$ ; and (3) there exists an archimedean place  $k_v$  such that  $\text{rank}_{k_v} \tilde{\mathbf{G}} \geq 2$ . Then there exist two non-isomorphic finite index subgroups  $\Gamma_1, \Gamma_2 \subseteq \Gamma$  with isomorphic profinite completions.*

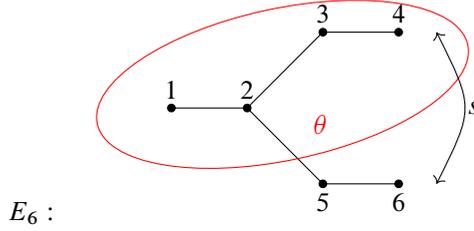
Note that  $A_n, D_n, E_6$  are exactly the types of Dynkin diagrams with a non-trivial symmetry.

*Proof* (Following [Method B](#)). As before, by moving to a finite index subgroup we can assume that  $\Gamma \cap \mathbf{C}(k) = \{1\}$  and that  $\hat{\Gamma} = \bar{\Gamma} = \Lambda \times \prod_{v \notin S} \tilde{\mathbf{G}}(\mathcal{O}_v)$ , for some finite set of places  $S$  and  $\Lambda \subseteq \prod_{v \in S} \tilde{\mathbf{G}}(k_v)$  commensurable with  $\prod_{v \in S} \tilde{\mathbf{G}}(\mathcal{O}_v)$ .

By Lemma 2.8, one can find two different primes  $p, q \notin S_{\text{full}}$  (where  $S_{\text{full}} \supseteq S$  is as in the previous section) lying over different rational primes, such that  $\tilde{\mathbf{G}}$  splits over both  $k_p$  and  $k_q$ . Fix root systems for  $\tilde{\mathbf{G}}(k_p)$  and  $\tilde{\mathbf{G}}(k_q)$ , and let  $\Sigma$  be the set of simple roots, identified for both groups. For  $v = p, q$ , the reduction maps  $\pi_v : \tilde{\mathbf{G}}(\mathcal{O}_v) \rightarrow \tilde{\mathbf{G}}(\mathcal{O}_v/v)$  are onto, and  $G_v := \tilde{\mathbf{G}}(\mathcal{O}_v/v)$  is the split universal Chevalley group of the same type as  $\tilde{\mathbf{G}}$  over the residue field. The root systems we fixed define corresponding root systems for  $G_v$ .

For each of the types involved, there exists a non-trivial symmetry  $s$  of the Dynkin diagram, pick a subset  $\theta \subseteq \Sigma$  which is non-invariant under  $s$ , for example,





The symmetry  $s$  of the Dynkin diagram induces an isomorphism  $\varphi_v$  of  $G_v$  [18, Corollary to Theorem 29]. For a subset  $R \subseteq \Sigma$  of simple roots, let  $P_{v,R}$  be the parabolic subgroup of  $G_v$  corresponding to  $R$ , then  $P_{v,\theta}$  and  $P_{v,s\theta}$  are non-conjugate in  $G_v$ , but isomorphic via  $\varphi_v$ .

Now, let  $\pi : \Gamma \rightarrow G_{\mathfrak{p}} \times G_{\mathfrak{q}}$  be the canonical reduction map modulo  $\mathfrak{p}\mathfrak{q}$ . Consider the following two congruence subgroups:

$$\begin{aligned}\Gamma_1 &:= \pi^{-1}(P_{\mathfrak{p},\theta} \times P_{\mathfrak{q},\theta}); \\ \Gamma_2 &:= \pi^{-1}(P_{\mathfrak{p},s\theta} \times P_{\mathfrak{q},\theta}).\end{aligned}$$

Then  $\widehat{\Gamma}_1$  and  $\widehat{\Gamma}_2$  are isomorphic via  $\Phi := (\Phi_v)_v$  where  $\Phi_v \equiv id$  for  $v \neq \mathfrak{p}$  and  $\Phi_{\mathfrak{p}}$  is the isomorphism of  $\widetilde{\mathbf{G}}(k_{\mathfrak{p}})$  induced by the non-trivial symmetry  $s$  of the Dynkin diagram. It remains to show that  $\Gamma_1$  and  $\Gamma_2$  cannot be isomorphic.

Assume to the contrary that there exists an isomorphism  $\varphi : \Gamma_1 \rightarrow \Gamma_2$ . By Corollary 2.2, there exist unique adelic automorphisms  $\Phi_{\mathbb{A}}$  and  $\sigma_{\mathbb{A}}^0$  of  $\widetilde{\mathbf{G}}(\mathbb{A}_k^f)$  such that  $\sigma_{\mathbb{A}}^0$  is induced from an automorphism  $\sigma$  of  $k$  and  $\Phi_{\mathbb{A}}$  is induced from a  $k$ -automorphism of  $\widetilde{\mathbf{G}}$  such that  $\varphi(\gamma) = \Phi_{\mathbb{A}}(\sigma_{\mathbb{A}}^0(\gamma))$  for every  $\gamma \in \Gamma_1$ . Moreover,  $\Phi_{\mathbb{A}}$  is of the form  $C_g \cdot \omega_{\mathbb{A}}$  where  $C_g$  is conjugation by some  $g \in \widetilde{\mathbf{G}}(\bar{k})$ ,  $\omega$  is an outer automorphism which comes from a symmetry of the Dynkin diagram, and  $\omega_{\mathbb{A}}$  just acts as  $\omega$  at each place. In particular,  $\pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\theta})$  is mapped onto  $g\pi_{\mathfrak{p}}^{-1}(P_{\sigma(\mathfrak{p}),\omega\theta})g^{-1}$  and  $\pi_{\mathfrak{q}}^{-1}(P_{\mathfrak{q},\theta})$  is mapped onto  $g\pi_{\mathfrak{q}}^{-1}(P_{\sigma(\mathfrak{q}),\omega\theta})g^{-1}$ . By our choice of  $\mathfrak{p}$  and  $\mathfrak{q}$  it must be that  $\sigma(\mathfrak{p}) = \mathfrak{p}$  and  $\sigma(\mathfrak{q}) = \mathfrak{q}$ .

We have that  $g^{-1}\pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\theta})g = \pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\omega\theta})$ , multiplying  $g$  from both sides by elements of  $\widetilde{\mathbf{G}}(k)$ , we can assume that conjugation by  $g$  preserves the root system that was fixed in the beginning of the proof. If  $\theta \neq \omega\theta$ , let  $\alpha$  be a simple root in  $\omega\theta \setminus \theta$ , then the action of  $g$  on the one-parameter unipotent subgroup  $U_{\alpha}$  must be as scalar multiplication by some  $f_{\alpha}$  with  $\text{val}_{\mathfrak{p}}(f_{\alpha}) = 1$ . Thus, the action on the opposite one-parameter unipotent subgroup  $U_{-\alpha}$  is given as scalar multiplication by  $1/f_{\alpha}$ , but then  $g^{-1}\pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\theta})g \not\subseteq \pi_{\mathfrak{p}}^{-1}(P_{\mathfrak{p},\omega\theta})$ . Hence,  $\theta$  must be equal to  $\omega\theta$ . The same argument implies that  $\omega\theta = s\theta$ , and so  $\theta = \omega\theta = s\theta$ . But  $\theta$  was chosen to be non-invariant under  $s$ , a contradiction. ■

## 5. Second part of the main theorem: The exceptional cases

**Lemma 5.1.** *Let  $\mathbf{G}$  be a connected, simply connected, and absolutely almost simple  $\mathbb{Q}$ -linear algebraic group of type  $E_8, F_4,$  or  $G_2$ . Then,  $\mathbf{G}$  splits over  $\mathbb{Q}_p$  for every  $p$ .*

*Proof.* The  $\mathbb{Q}_p$  forms of the group  $\mathbf{G}$  are classified by the first Galois cohomology set  $H^1(\mathbb{Q}_p, \text{Aut}(\mathbf{G}))$ . For the groups considered, the universal and the adjoint forms coincide; moreover, there are no symmetries for their Dynkin diagram, hence  $\text{Aut}(\mathbf{G}) \cong \mathbf{G}$ . As the field  $\mathbb{Q}_p$  is local and non-archimedean, by [13, Theorem 6.4], the Galois cohomology group  $H^1(\mathbb{Q}_p, \mathbf{G})$  is trivial. Thus, there is only one (up to an isomorphism)  $\mathbb{Q}_p$  form for  $\mathbf{G}$ ; in particular, this form must be the split form. ■

**Remark.** If  $\mathbf{G}$  is a  $k$ -split simple  $k$ -linear algebraic group, then the automorphism group,  $\text{Aut}(\mathbf{G}(k))$ , of  $\mathbf{G}(k)$  is completely known. Precisely, following the notations of [18, Theorem 30], each automorphism can be written as the product of an inner, a diagonal, a graph, and a field automorphism. We have used graph automorphisms for [Method B](#) and field automorphisms for [Method C](#). The group of diagonal automorphisms (modulo the inner ones) has a connection with the center of the universal form [18, Exercise following Theorem 30], which was used for [Method A](#). Thus, restricting ourselves to the exceptional cases, where  $\mathbf{G}$  has type  $E_8, F_4,$  or  $G_2$  and  $k = \mathbb{Q}$  (and also for  $\mathbb{Q}_p$ ), all automorphisms are inner.

**Theorem 5.2.** *Let  $\mathbf{G}$  be a connected, simply connected, and absolutely almost simple high  $\infty$ -rank  $\mathbb{Q}$ -linear algebraic group of type  $E_8, F_4,$  or  $G_2$ . If  $\Gamma_1, \Gamma_2 \subseteq \mathbf{G}(\mathbb{Q})$  are two arithmetic subgroups with isomorphic profinite completions, then  $\Gamma_1$  and  $\Gamma_2$  are isomorphic.*

*Proof.* Let  $\Gamma_1, \Gamma_2 \subseteq \mathbf{G}(\mathbb{Q})$  be two arithmetic subgroups. As noted in the preliminaries, the congruence kernel is trivial for these groups, so one can write  $\overline{\Gamma}_i = \widehat{\Gamma}_i = \Lambda_i \times \prod_{p \notin S} \mathbf{G}(\mathbb{Z}_p)$  where  $S$  is a finite set and  $\Lambda_i$  are commensurable with  $\prod_{p \in S} \mathbf{G}(\mathbb{Z}_p)$ . Assume that  $\Phi : \overline{\Gamma}_1 \cong \widehat{\Gamma}_1 \xrightarrow{\sim} \widehat{\Gamma}_2 \cong \overline{\Gamma}_2 \subseteq \mathbf{G}(\mathbb{A})$  is an isomorphism between the profinite completions of the two. By adelic superrigidity 2.3, there exists a unique homomorphism of adelic groups

$$\tilde{\Phi} : \mathbf{G}(\mathbb{A}^f) \rightarrow \mathbf{G}(\mathbb{A}^f)$$

such that  $\tilde{\Phi}|_{\Gamma_1} \equiv \Phi \circ \iota|_{\Gamma_1}$ , as  $\mathbf{G}(\mathbb{A}^f)$  is centerless by our assumption on the type of  $\mathbf{G}$ . Moreover, using the uniqueness of the map,  $\tilde{\Phi}$  must be an isomorphism, and  $\tilde{\Phi}|_{\overline{\Gamma}_1} \equiv \Phi$ .

Consider the homomorphisms  $\tilde{\Phi}_{p,q} : \mathbf{G}(\mathbb{Q}_p) \rightarrow \mathbf{G}(\mathbb{Q}_q)$  which are the composite

$$\mathbf{G}(\mathbb{Q}_p) \xrightarrow{\iota_p} \mathbf{G}(\mathbb{A}^f) \xrightarrow{\tilde{\Phi}} \mathbf{G}(\mathbb{A}^f) \xrightarrow{\pi_q} \mathbf{G}(\mathbb{Q}_q)$$

of the inclusion in the  $p$ th place,  $\tilde{\Phi}$ , and the projection onto the  $q$ th place. This is a continuous homomorphism between a  $p$ -adic group and a  $q$ -adic group, so if  $p \neq q$  it must

be a locally constant map. So its image is a normal countable subgroup of  $\mathbf{G}(\mathbb{Q}_q)$ ; in particular, it is not of finite index, and hence must be trivial [13, Proposition 3.17]. Thus,  $\tilde{\Phi} = (\tilde{\Phi}_{p,p})_p$  is given by an isomorphism at each place, and  $\tilde{\Phi}_{p,p}$  must be conjugation by some  $y'_p \in \mathbf{G}(\mathbb{Q}_p)$  (see the remark above).

We truncate  $\tilde{\Phi}$  in the following manner:

$$\text{write } y = (y_p)_p, \quad \text{where } y_p = \begin{cases} y_p = y'_p & p \in S \\ y_p = 1 & p \notin S \end{cases}.$$

By our choice of  $S$ , conjugation by  $y$  is again an isomorphism between  $\overline{\Gamma}_1$  and  $\overline{\Gamma}_2$ . By the strong approximation theorem [13, Theorem 7.12], there exists some  $g \in \mathbf{G}(\mathbb{Q})$  with  $g \in y\overline{\Gamma}_1$ . Thus, conjugation by  $g \in \mathbf{G}(\mathbb{Q})$  is an isomorphism between  $\overline{\Gamma}_1$  and  $\overline{\Gamma}_2$ . We have that  $\Gamma_i = \overline{\Gamma}_i \cap \mathbf{G}(\mathbb{Q})$ , which imply that  $\Gamma_1$  and  $\Gamma_2$  can be conjugated by  $g$ , as needed. ■

## 6. Final remarks

It is possible to generalize our methods even further. For example, using [Method A](#), we can find finite index subgroups of  $\Gamma := SL_2(\mathbb{Z}[1/p])$  which are not profinitely rigid. Explicitly (for  $p \neq 2, 3, 5$ ), the following finite index subgroups of  $SL_2(\mathbb{Z}[1/p])$  are non-isomorphic, but their profinite completions are

$$\Gamma_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[1/p]) : \begin{array}{l} b, c \equiv 0 \pmod{3, 5} \\ a, d \equiv \pm 1 \pmod{3} \\ a, d \equiv 1 \pmod{5} \end{array} \right\};$$

$$\Gamma_2 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[1/p]) : \begin{array}{l} b, c \equiv 0 \pmod{3, 5} \\ a, d \equiv 1 \pmod{3} \\ a, d \equiv \pm 1 \pmod{5} \end{array} \right\}.$$

Indeed, just as in Section 3,  $\widehat{\Gamma}_1 \cong \widehat{\Gamma(15)} \times \mathbb{Z}/2\mathbb{Z} \cong \widehat{\Gamma}_2$ , where  $\Gamma(15)$  is the principle congruence subgroup of  $\Gamma$  of level 15.

Let us stress out that it is still unknown whether or not  $SL_2(\mathbb{Z}[1/p])$  itself is profinitely rigid, and in fact, there are some reasons to believe it is profinitely rigid (see, for example, [4, §4]). On the other hand, increasing slightly the dimension, it has been shown that  $SL_4(\mathbb{Z}[1/p])$  is not profinitely rigid [4].

We would like to state a stronger version of Theorem 1.1 which includes the above example. First, we need some further notations. Let  $S \subseteq V(k)$  be a finite set of places containing all the archimedean places. The ring of  $S$ -integers of the number field  $k$  is

$$\mathcal{O}_{k,S} := \{x \in k : v(x) \geq 0 \forall v \notin S\}.$$

Let  $\tilde{\mathbf{G}}$  be a connected, simply connected, absolutely almost simple  $k$ -linear algebraic group with a fixed faithful  $k$ -representation  $\rho : \tilde{\mathbf{G}} \rightarrow GL(n_\rho)$ . A subgroup  $\Gamma \subseteq \tilde{\mathbf{G}}(k)$

is called an  $S$ -arithmetic subgroup if it is commensurable with  $\tilde{\mathbf{G}}(\mathcal{O}_S)$ . As in Section 2, there is a map from the profinite completion to the congruence completion, denote its kernel by  $C(\Gamma, S)$ . The group  $\Gamma$  is said to have the congruence subgroup property (with respect to  $S$ ) if  $C(\Gamma, S)$  is a finite group. Again, this is actually a property of the ambient group  $\tilde{\mathbf{G}}$ , the field  $k$ , and the set  $S$ . The proofs given throughout the paper carry over to establish.

**Theorem 6.1.** *Let  $n$  be a positive integer,  $k$  a number field,  $S$  a finite set of places of  $k$  containing all the archimedean places, and  $\tilde{\mathbf{G}}$  a connected, simply connected, and absolutely almost simple  $k$ -linear algebraic group such that  $\sum_{v \in S} \text{rank}_{k_v} \tilde{\mathbf{G}} \geq 2$  and such that  $\tilde{\mathbf{G}}(k)$  satisfies the congruence subgroup property (with respect to  $S$ ). Let  $\Gamma \subseteq \tilde{\mathbf{G}}(k)$  be an  $S$ -arithmetic subgroup. Then, unless  $\tilde{\mathbf{G}}$  has type  $G_2$ ,  $F_4$ , or  $E_8$  and  $k = \mathbb{Q}$ ,  $\Gamma$  has infinitely many sequences of pairwise non-isomorphic finite index subgroups  $\Gamma_1, \dots, \Gamma_n$  with isomorphic profinite completions.*

As in the main theorem, the exceptional cases are indeed exceptional. Moreover, in these cases, if  $\Gamma_1$  is an  $S_1$ -arithmetic subgroup and  $\Gamma_2$  is an  $S_2$ -arithmetic subgroup with  $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$  then  $S_1 = S_2$  and  $\Gamma_1 \cong \Gamma_2$ .

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