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L^2-cohomology and quasi-isometries on the ends of unbounded geometry

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Abstract – The paper explores the minimal and maximal L^2-cohomology of oriented Riemannian manifolds, focusing on both the reduced and the unreduced versions. The main result is the proof of the invariance of the L^2-cohomology groups under uniform homotopy equivalences that are quasi-isometric on the unbounded ends. A *uniform map* is a uniformly continuous map such that the diameter of the preimage of a subset is bounded in terms of the diameter of the subset itself. Moreover, a map f between two Riemannian manifolds (X, g) and (Y, h) is *quasi-isometric on the unbounded ends* if $X = M \cup E_X$, where M is the interior of a manifold of bounded geometry with boundary, E_X is an open subset of X and the restriction of f to E_X is a quasi-isometry. Finally, some consequences are shown: the main ones are the definition of a mapping cone for L^2-cohomology and the invariance of the L^2-signature.

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Introduction

In this paper, a generalization of the author's previous work in [14] is presented. In that study, the author examined uniform maps $f : (M, g) \to (N, h)$ between Riemannian manifolds of bounded geometry and introduced a bounded operator $T_f : \mathcal{L}^2(N, h) \to \mathcal{L}^2(M, g)$ related to f between the spaces of square-integrable forms. A *uniform* map is defined as a uniformly continuous map where, for every compact subset A, the diameter of its preimage is bounded in terms of the diameter of A. A *manifold of*

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bounded geometry is a Riemannian manifold whose curvature is uniformly bounded and the injectivity radius is bounded from below. The operator T_f in [14] induces a linear operator between the reduced and unreduced L^2-cohomology groups, replacing the pullback operator which is not well defined between the $\mathcal{L}^2$-spaces. Consequently, the invariance of reduced and the unreduced L^2-cohomology groups under uniform homotopy equivalences is proved.

The aim of this paper is to prove a similar result for the minimal and maximal L^2-cohomologies of possibly not complete Riemannian manifolds. This result will hold for both the reduced and unreduced versions. In order to reach our goal we need some additional assumptions on the homotopy equivalence. Let us briefly introduce these assumptions.

Given a Riemannian manifold (X, g), it can be decomposed as $X = M \cup E_X$, where E_X is an open subset of X and M is the interior of a manifold with boundary of bounded geometry (this notion is introduced in Schick [12]). We will say that M is an *open subset of bounded geometry* and E_X will be the *unbounded ends* of X. Such a manifold $(X = M \cup E_X, g)$, decomposed in this manner, is termed a manifold of bounded geometry with unbounded ends.

In this paper we investigate uniform maps $f : (X, g) \to (Y, h)$ between oriented manifolds which are *quasi-isometric on unbounded ends*, i.e. given two manifolds of bounded geometry with unbounded ends $(X = M \cup E_X, g)$ and $(Y = N \cup E_Y, h)$, then $f(E_X) \subseteq E_Y$ and $f_{|E_X}$ is a quasi-isometry.

In particular, in Theorem 4.6, we prove that if $f : (X, g) \to (Y, h)$ is a uniform homotopy equivalence which is quasi-isometric on the unbounded ends. Then

$$\begin{cases} H^k_{2,\max}(X, g) \cong H^k_{2,\max}(Y, h), \\ \bar{H}^k_{2,\max}(X, g) \cong \bar{H}^k_{2,\max}(Y, h), \\ H^k_{2,\min}(X, g) \cong H^k_{2,\min}(Y, h), \\ \bar{H}^k_{2,\min}(X, g) \cong \bar{H}^k_{2,\min}(Y, h), \end{cases}$$

where $H^k_{2,\min\backslash\max}(X, g)$ and $H^k_{2,\min\backslash\max}(Y, h)$ are the minimal and maximal k-th group of L^2-cohomology and $\bar{H}^k_{2,\min\backslash\max}(X, g)$ and $\bar{H}^k_{2,\min\backslash\max}(Y, h)$ are their reduced versions. Finally, some consequences are shown.

The paper is structured as follows: Section 1 introduces the notions of uniform maps quasi-isometric on the unbounded ends. Section 2 defines the minimal and maximal L^2-cohomology of a Riemannian manifold. In Section 3 we recall the definitions and the key properties of three necessary components for proving Theorem 4.6. These components are the following:

- The *Radon–Nikodym–Lipschitz maps*. These are maps between Riemannian manifolds such that their pullback induces a well-defined $\mathcal{L}^2$-bounded operator.
- The *generalized Sasaki metrics* on vector bundles. These are some Riemannian metrics induced by a connection on the vector bundle, a bundle metric and a Riemannian metric on the base space,
- The *Mathai–Quillen–Thom forms*, which constitute a specific family of Thom form on a vector bundle.

In Section 4 we introduce the new version of the operator T_f and in Theorem 4.6 the main result is proved.

Finally, in Section 5 we explore three consequences of the existence of T_f:

- we define a mapping cone for minimal and maximal L^2-cohomology,
- we demonstrate the invariance of the L^2-signature for complete $4k$-dimensional manifolds under uniform homotopy equivalences that preserve orientation,
- we prove a result similar to Lott [9, Proposition 5 (1)], accompanied by an illustrative example.

1. Quasi-isometries on the unbounded ends and open subsets of bounded geometry

In this section we establish the geometric framework of the manuscript. In particular, we introduce the concepts of *open subset of bounded geometry* of a Riemannian manifold and of *uniform homotopy quasi-isometric on the unbounded ends*.

1.1 – *Manifold of bounded geometry with some unbounded ends*

The following definition is Schick [12, Definition 2.2].

Definition 1.1. Let $(\bar{M}, g)$ be an oriented Riemannian manifold with boundary $\partial\bar{M}$ (possibly empty). Fix on $\partial\bar{M}$ the Riemannian metric induced by $\bar{M}$. Denote by l the second fundamental form of $\partial\bar{M}$, by R the curvature tensor of $\bar{M}$ and by $\bar{\nabla}$ the Levi-Civita connection on $\partial\bar{M}$. Then we say that $(\bar{M}, g)$ is a *manifold with boundary and of bounded geometry* if the following hold:

(1) There exists a number $r_c > 0$ such that

$$\begin{aligned} K\colon \partial\bar{M} \times [0, r_c) &\longrightarrow \bar{M}, \\ (x, t) &\longrightarrow \exp_x(t\nu_x) \end{aligned}$$

is a diffeomorphism with its image (ν_x is the unit inward normal vector). Given r in $[0, r_c)$ we denote by $\mathcal{N}(r)$ the set $K(\partial \bar{M} \times [0, r))$.The set $\mathcal{N}(r)$ is the *normal collar* of length r of $\partial \bar{M}$.

(2) The injectivity radius $\mathrm{inj}_{\partial \bar{M}}$ of $\partial \bar{M}$ is positive.

(3) There is an $r_i > 0$ such that if $r \leq r_i$, then, for each p in $\bar{M} \setminus \mathcal{N}(r)$, the exponential map is a diffeomorphism on $B_{0_p}(r) \subset T_p\bar{M}$.

(4) For every $k \in \mathbb{N}$ there is a constant C_k so that $|\nabla^i R| \leq C_i$ and $|\bar{\nabla}^i l| \leq C_i$ for each $i = 0, \dots, k$.

Remark 1. The first point of Definition 1.1 provides some coordinates on $\mathcal{N}(r_c)$. Indeed, given a constant r_1 which is smaller than the injectivity radius of $\partial \bar{M}$, it is possible to define the chart

$$\begin{aligned} k_{x'}\colon B_{r_1}(0) \times [0, r_c] &\longrightarrow \mathcal{N}(r_c), \\ (v, t) &\longrightarrow \exp_{\exp'_x(v)}(t\nu), \end{aligned}$$

where x' is a point in $\partial \bar{M}$ and $B_{r_1}(0)$ is the euclidean ball of the same dimension as $\partial \bar{M}$ centered at 0 with radius r_1. We call these coordinates *collar coordinates*. On the other hand, given a point p in $\bar{M} \setminus \mathcal{N}(s)$ we term *normal coordinates* or *Gaussian coordinates* the coordinates induced by the exponential map of $\bar{M}$ on a ball of radius s around p.

Remark 2. By [12, Theorem 2.5], we know that on a manifold with boundary of bounded geometry, the metric components g_{ij} of the metric g with respect to some collar or Gaussian coordinates satisfy the following inequalities: for each α in $\mathbb{N}$, there is a number C_α such that

$$\left| \frac{\partial^\alpha g_{ij}}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}} \right| \leq C_\alpha \quad \text{and} \quad \left| \frac{\partial^\alpha g^{ij}}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}} \right| \leq C_\alpha,$$

where $\alpha = \sum_{i=1}^m \alpha_i$.

Definition 1.2. Given a Riemannian manifold (X, g), an *open subset of bounded geometry of* (X, g) is an open subset $M \subset X$ such that its closure $\bar{M}$ in (X, g), endowed with the Riemannian metric induced by g, is a manifold with boundary of bounded geometry.

The *normal collar* of length r of M is the set $\mathcal{N}_X(r) := \mathcal{N}(r) \cap M$, where $\mathcal{N}(r)$ is the normal collar of length r of the manifold with boundary of bounded geometry $\bar{M}$.

Definition 1.3. A *manifold of bounded geometry with some (possibly) unbounded ends* is an oriented Riemannian manifold (X, g) such that $X = M \cup E_X$ and

- M is an open subset of bounded geometry,
- E_X is an open subset of X,
- $\mathcal{N}_X(r_X) \subset M \cap E_X$ for some constant $r_X > 0$.

The open subset E_X is called the *unbounded ends of X*.

LEMMA 1.1. *For each Riemannian manifold (X, g) there are two open subsets M and E_X of X which make (X, g) a manifold of bounded geometry with unbounded ends.*

PROOF. Fix a point p on X and let $M = B_\delta(p)$, where δ is smaller than the injectivity radius in p. Let E_X be the complement of $B_{\frac{\delta}{2}}(p)$ in X. It easily follows that M and E_X satisfy all the conditions in Definition 1.3. ∎

Notice that the open subset of bounded geometry M of a manifold of bounded geometry with unbounded ends $X = M \cup E_X$ is not necessarily compact.

EXAMPLE 1.1. Let $X = \mathbb{R}^2 \times S^1$ and let $g = dx^2 + dy^2 + x^2\, d\theta^2$, where $d\theta^2$ is the euclidean Riemannian metric on $S^1 \subset \mathbb{R}^2$ and $dx^2 + dy^2$ is the euclidean metric on $\mathbb{R}^2$. Then, given two positive numbers a, b such that $a < b$, if we fix $M = (-b, b) \times \mathbb{R} \times S^1$ and $E_X = [(-\infty, -a) \cup (a, +\infty)] \times \mathbb{R} \times S^1$, it is easy to check that $(X, g) = (M \cup E_X, g)$ is a manifold of bounded geometry with two unbounded ends.

EXAMPLE 1.2. Each manifold of bounded geometry (X, g) is a manifold of bounded geometry with unbounded ends. In this case, $M = X$, $\partial M = \emptyset$ and so also $\mathcal{N}(r) = E_X = \emptyset$.

REMARK 3. Let $(X = M \cup E_X, g)$ be a manifold of bounded geometry with unbounded ends. Let p be a point in $\mathcal{N}_X(r_X)$. Since $\mathcal{N}_X(r_X) \subset \mathcal{N}(r_X)$, the normal collar of $\partial \bar{M}$, it is possible to identify p as a point (x_0, t_0) of $\partial M \times (0, r_X)$. Indeed,

$$p = K(x_0, t_0),$$

where K is the map in the first point of Definition 1.1.

LEMMA 1.2. *Let $(X = M \cup E_X, g)$ be a manifold of bounded geometry with unbounded ends. Let $p = (x_0, t_0)$ be a point in $\mathcal{N}_X(r_X)$. Then the distance between ∂M and p satisfies*

$$d_g(\partial M, p) = t_0.$$

PROOF. Fix some normal coordinates (U, x^i) centered at x_0 in ∂M and let $(U \times (0, r_c), x^i, t)$ be the collar coordinates on $\pi^{-1}(U) \subset M$. Notice that, thanks to [12, Proposition 2.8], we know the Gram matrix of g with respect to x^i and t has the form

$$g(x,t) = \begin{bmatrix} g_{ij}(x,t) & 0 \\ 0 & 1 \end{bmatrix}.$$

It immediately follows that the curve $\sigma\colon (0,t_0] \to \mathcal{N}_X(r_X) \subset M$ defined as $\sigma(t) = (0,t)$ is length minimizing. ■

1.2 – *Uniform maps quasi-isometric on the unbounded ends*

In this subsection we introduce the maps under study in this manuscript. Fix two Riemannian manifolds (X,g) and (Y,h).

Definition 1.4. A map $f\colon (X,g) \to (Y,h)$ is *uniformly continuous* if for each $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for each x_1, x_2 in X,

$$d_X(x_1,x_2) \leq \delta(\epsilon) \implies d_Y(f(x_1), f(x_2)) \leq \epsilon.$$

Moreover, f is *uniformly (metrically) proper*, if for each $R \geq 0$ there is a number $S(R) > 0$ such that for each subset A of (Y, d_Y),

$$\text{diam}(A) \leq R \implies \text{diam}(f^{-1}(A)) \leq S(R).$$

We say that a map $f\colon (X,g) \to (Y,h)$ is a *uniform map* if it is uniformly continuous and uniformly proper.

Definition 1.5. Two maps f_0 and $f_1\colon (X,d_X) \to (Y,d_Y)$ are *uniformly homotopic* if they are homotopic with a uniformly continuous homotopy $H\colon (X \times [0,1], g + dt) \to (Y,h)$. We will denote it by

$$f_1 \sim f_2.$$

Definition 1.6. A map $f\colon (X,g) \to (Y,h)$ is a *uniform homotopy equivalence* if f is uniformly continuous and there is map s such that

- s is a homotopy inverse of f,
- s is uniformly continuous,
- $f \circ s$ is uniformly homotopic to id_N and $s \circ f$ is uniformly homotopic to id_M.

To define the class of maps we are interested in, we first need to introduce the notion of *quasi-isometry*.

Definition 1.7. Let (X, g) and (Y, h) be two Riemannian manifolds. Then a *quasi-isometry* is a local diffeomorphism $f : (X, g) \to (Y, h)$ such that f^*h and g are *quasi-isometric* metric, i.e. there is a constant $K \geq 1$ such that

$$K^{-1}g \leq f^*h \leq Kg.$$

Definition 1.8. Let $(X = M \cup E_X, g)$ and $(Y = N \cup E_Y, h)$ be two manifolds of bounded geometry with unbounded ends. A map $f : (X, g) \to (Y, h)$ is *quasi-isometric on the unbounded ends* if

- $f(E_X) \subseteq E_Y$, $f(M) \subset N$, $f(\partial M) \subseteq \partial N$,
- $f_{|E_X} : E_X \to E_Y$ is a quasi-isometry.

A *uniform homotopy equivalence isometric on the ends* is a uniform homotopy equivalence which is isometric on the unbounded ends. Finally, f is a *uniform homotopy equivalence quasi-isometric on the ends* if $f_{|E_X}$ is a quasi-isometry.

Remark 4. Fix a uniform homotopy equivalence isometric on the ends $f : (X, g) \to (Y, h)$. Let $r_0 := \min\{r_X, r_Y\}$. Then $f(\mathcal{N}_X(r_0)) \subseteq \mathcal{N}_Y(r_0)$. Moreover, if we choose some collar coordinates $\{x^i, t\}$ on $\mathcal{N}_X(r_0)$ and $\{y^j, s\}$ on $\mathcal{N}_Y(r_0)$, then f has the form

$$f(x^i, t) = (F(x^i, t), t).$$

This is an immediate consequence of Lemma 1.2.

Our next step is to introduce a family of maps equivalent to the C^k_b-maps defined for manifolds of bounded geometry by Eldering [5].

Definition 1.9. Let $f : (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a quasi-isometry on the ends. Then f is a $C^k_{b,\mathrm{loc}}$*-map* if there is an $r_f \leq r_X$ and there are two constants δ_X and δ_Y such that, for each point p in $M \setminus \mathcal{N}_X(r_f)$, the functions $\widetilde{F}_p : B_{\delta_X}(0_p) \subset T_pX \to B_{\delta_Y}(0_{f(p)}) \subset T_{f(p)}Y$ defined as

$$\widetilde{F}_x := \exp^{-1}_{f(x)} \circ f \circ \exp_x$$

have uniformly bounded C^k-norms as maps between euclidean spaces.

Example 1.3. Let $(X = M \cup E_X, g)$ be a manifold of bounded geometry with unbounded ends. Fix $r < r_X$ and let $\{y^i, U\}$ be some normal or collar coordinates on M, which is an open subset of bounded geometry. Then for each i the functions $\{y^i\}$ defined on U are $C^k_{b,\mathrm{loc}}$-maps for each k in $\mathbb{N}$. This fact is a direct consequence of [12, Proposition 3.3].

Proposition 1.3. *Let* $(X = M \cup E_X, g)$ *and* $(Y = N \cup E_Y, h)$ *be two manifolds of bounded geometry with unbounded ends. Fix a uniform map quasi-isometric on the ends* $f\colon (X, g) \to (Y, h)$.

Then, for each ϵ *small enough, there is a map* $f_\epsilon\colon (X, g) \to (Y, h)$ *such that*

(1) $d_Y(f_\epsilon(p), f(p)) \leq \epsilon$ *and* $f_\epsilon \sim f$,

(2) *for each* k *in* $\mathbb{N}$, *the map* f *is a* $C^k_{b,\mathrm{loc}}$*-map,*

(3) *letting* $B_\epsilon(E_Y)$ *be an* ϵ*-neighborhood of* E_Y, *then* $f_\epsilon\colon (X = M \cup E_X, g) \to (Y = N \cup B_\epsilon(E_Y), h)$ *is a smooth uniform map quasi-isometric on the ends.*

Proof. Points (1) and (2) can be proved following a strategy very similar to the proof of [14, Proposition 1.3]. In this proposition, given two constants $\delta_1 < \delta_2$, a cover of balls $\{B_{\delta_2}(x_i)\}$ of a manifold of bounded geometry is fixed. This cover has the property that $\{B_{\delta_1}(x_i)\}$ is again a cover and for each x the ball $B_{\delta_2}(x)$ intersects at most R balls $B_{\delta_2}(x_i)$. Since f is a uniform map, there exists a sufficiently small constant σ_2 such that we can ensure $f(B_{\delta_2}(x_i)) \subseteq B_\epsilon(f(p))$. Therefore f can be modified recursively on each ball of the cover. We mean that, given $F_0 := f$, in each step, F_i is replaced by a function F_{i+1} which is defined as

$$F_{i+1}(p) := \begin{cases} \exp_{f(x_i)} \circ G_{i,\nu(\epsilon)} \circ \exp_{x_i} & \text{if } p \in B_{\delta_2}(x_i), \\ F_i(p) & \text{otherwise,} \end{cases}$$

where

$$G_{i,\nu(\epsilon)}\colon B_{\delta_2}(0) \subset T_{x_i}M \cong \mathbb{R}^m \longrightarrow B_{\sigma_2}(0) \subset T_{f(x_i)}N \subset \mathbb{R}^n$$

is a C^k_b-approximation of $g_i = \exp^{-1}_{f(x_i)} \circ F_i \circ \exp_{x_i}$ (see Eldering [5, Lemma 2.34] for a detailed definition of $G_{i,\nu(\epsilon)}$). This function $G_{i,\nu(\epsilon)}$ is C^k_b and its norm is bounded in terms of ϵ and k. Then, for each x, the map f_ϵ was defined as

$$f_\epsilon(x) = \lim_{i \to +\infty} F_i(p).$$

The idea is to replace the cover $\{B_{\delta_2}(x_i)\}$ with a cover of $M \setminus \mathcal{N}_X(r_f)$ for some $r_f < r_X$. We can find a suitable cover in [12, Proposition 3.2]: in this proposition the author gives a cover of a manifold of bounded geometry with boundary. By removing subsets which intersect the boundary, we obtain a constant r and a cover of balls $\{B_r(x_i)\}$ of $M \setminus \mathcal{N}_X(\frac{r_X}{2})$. Importantly, $\{B_{\frac{r}{2}}(x_i)\}$ is also a cover and if p is a point of $M \setminus \mathcal{N}_X(\frac{r_X}{2})$, then there are at most R balls such that $B_r(x) \cap B_r(x_i) \neq \emptyset$.

If we define f_ϵ as in [14, Proposition 1.7], we obtain a map which satisfies points (1) and (2). The proof is the same as given in [14, Proposition 1.7].

The remaining task is to prove point (3). To establish point (3), we need to demonstrate that f_ϵ is a diffeomorphism on E_X and that $f^*_\epsilon(h)$ is quasi-isometric to g on each point p of E_X. Fix some normal coordinates $\{x_i\}$ around p and let $\{y_j\}$ be normal coordinates around $f(p)$.

Notice that, if we denote the Jacobian matrices of f and f_ϵ in p by $J_f(p)$ and $J_{f_\epsilon}(p)$ respectively, then the (i,j)-component

$$|(J_f - J_{f_\epsilon})_{i,j}(p)| < C \cdot \epsilon,$$

where C is a constant which does not depend on p. This is because, in normal coordinates, f_ϵ approximates f as a C^1-function ([5, Lemma 2.34], [14, Lemma 1.2 (3)]). Since the norm of J_f is bounded from below by K^{-1}, then if ϵ is small enough, J_{f_ϵ} is also invertible and so f_ϵ is a diffeomorphism in p. Moreover, if we denote the Gram matrices in p of f^*h and f_ϵ^* by $F^*H(p)$ and $F_\epsilon^*H(p)$, then for each (i,j) we have

$$|(F^*H - F_\epsilon^*H)_{i,j}| \leq D \cdot \epsilon,$$

where D does not depend on p. Indeed, if p is in $E_X \setminus M$, then the Gram matrices are equal; if p is in M then the bound D is a consequence of the boundedness of the metric h in normal coordinates around $f(p)$ and of the C^1-approximation of f_ϵ to f. Then it is an easy exercise to conclude that $f_\epsilon^*(h)$ is quasi-isometric to g in p. ■

Lemma 1.4. *Let $f\colon X \to (Y,h)$ be a map between two manifolds. Let g_1 and g_2 be two quasi-isometric Riemannian metrics on X. Then $f\colon (X,g_1) \to (Y,h)$ is a uniform map if and only if $f\colon (X,g_2) \to (Y,h)$ is a uniform map.*

Proof. Let γ be a differentiable curve on X. Fix $i = 1,2$ and denote by $L_i(\gamma)$ the length of γ with respect to the metric g_i. Since g_1 and g_2 are quasi-isometric, we obtain that there is a constant K, which does not depend on γ, such that

$$K^{-1} \cdot L_1(\gamma) \leq L_2(\gamma) \leq K \cdot L_1(\gamma).$$

So this implies that for each couple of points a and b in X we obtain

$$K^{-1} \cdot d_1(a,b) \leq d_2(a,b) \leq K \cdot d_1(a,b),$$

where d_i is the distance induced by g_i. The claim immediately follows. ■

Proposition 1.5. *Let $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a smooth uniform map which is quasi-isometric on the unbounded ends. Assume that f is a $C^3_{b,\mathrm{loc}}$-map. Then there is a metric $\tilde{g}$ on X such that*

(1) *g and $\tilde{g}$ are quasi-isometric,*

(2) *$M \subset (X, \tilde{g})$ is an open subset of bounded geometry,*

(3) *there are* $\widetilde{E}_X \subset E_X$ *and* $\widetilde{E}_Y \subset E_Y$ *such that* $f\colon (X = M \cup \widetilde{E}_X, \tilde{g}) \to (Y = N \cup \widetilde{E}_Y, h)$ *is a uniform map which is isometric on the unbounded ends.*

Proof. Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\phi \cong 1$ on $[1, +\infty)$, $\phi \cong 0$ on $(-\infty, 0]$ and $\phi(x) \in [0, 1]$ otherwise. Fix $r_0 := \min\{r_X, r_Y\}$, where r_X and r_Y are two constants such that $\mathcal{N}_X(r_X) \subset M \cap E_X$ and $\mathcal{N}_Y(r_Y) \subset N \cap E_Y$. Let $r_f < r_X$ be the constant such that f is a C^3_b-map on $M \setminus \mathcal{N}_X(r_f)$.

Denote by $d_g(\partial M, x)$ the distance of a point x in X from ∂M with respect to the Riemannian metric g. Then we can define

$$\chi(x) := \begin{cases} \phi\Big(\dfrac{1}{r_x - r_f}(d_g(\partial M, x) - r_f)\Big) & \text{if } x \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the function χ is $C^k_{b,\mathrm{loc}}$ for each k in $\mathbb{N}$. This fact is a consequence of Lemma 1.2 and [12, Proposition 3.3]. Then we define the metric $\tilde{g}$ on $x \in X$ as

$$\tilde{g}_x := \chi(x) g_x + (1 - \chi(x)) f^* h_x.$$

Observe that $\tilde{g}$ and g are quasi-isometric. Indeed, since there is a K such that

$$K^{-1} \cdot g \leq f^* h \leq K \cdot g,$$

then

$$\frac{1}{1+K} g_x \leq \tilde{g} \leq (1 + K) g_x.$$

Our next step is to prove that $(\overline{M}, \tilde{g})$ is a manifold with boundary of bounded geometry.

Let us verify the conditions in Definition 1.1:

- Conditions (1) and (2) are satisfied. Let $\mathcal{N}_X(r_f)$ be the r_f-normal collar of M with respect to g and let $\mathcal{N}_Y(r_f)$ be the r-normal collar of N. Since f is a quasi-isometry on the unbounded ends, there is a δ_0 such that $f^{-1}(\mathcal{N}_Y(\delta_0)) \subset \mathcal{N}_X(r)$. Notice that $\chi \cong 0$ on $\mathcal{N}_X(r)$. This implies that the δ_0-neighborhood of $\partial\overline{M}$ (with respect to $\tilde{g}$) is isometric to the δ_0-neighborhood of $\partial\overline{N}$. Then, since $\overline{N}$ is a manifold with boundary of bounded geometry, the first two conditions of Definition 1.1 are satisfied.
- Condition (3) is also satisfied. Given that the δ_0-neighborhood of ∂M is isometric to the δ_0-neighborhood of ∂N, we now need to ensure that the injectivity radius on M is bounded from below.

  Let p be a point on M, let $\{e_i\}$ be an orthonormal basis of $T_p X$ and fix some normal coordinates $\{(U, x^i)\}$ (with respect to g) referred to as p and $\{e_i\}$. We obtain the coordinates $\{x^i, \mu^j\}$ on TX, where $\{\mu^j\}$ are the components of $\frac{\partial}{\partial x^i}$.

It is important to note that in these coordinates, the components of the Gram matrix $\tilde{g}_{ij}$ and its derivatives are uniformly bounded. This is because f and χ have uniformly bounded derivatives of each order with respect to the coordinates $\{x^i\}$. Also, the components of the inverse of $\tilde{g}_{ij}$ are uniformly bounded due to the determinant of $\tilde{g}_{ij}$ being bounded from below by the minimum of the union of the spectra of g and f^*h. Lastly, the derivative of the components of the inverse of $\tilde{g}_{ij}$ are uniformly bounded (see [12, Lemma 2.17]).

Notice that a $\tilde{g}$-geodesic has to satisfy the equations

$$\begin{cases} \dot{\mu}^k = -\widetilde{\Gamma}^k_{ij}\mu^i\mu^j, \\ \dot{x}^i = \mu^i, \end{cases}$$

where $\widetilde{\Gamma}^k_{ij}$ are the components of the Levi-Civita connection of $\tilde{g}$. Observe also that $\widetilde{\Gamma}^k_{ij}$ and their derivatives are uniformly bounded. Assume that C is the bound on $|\widetilde{\Gamma}^k_{ij}|$ and that L is the bound on the derivatives of $\widetilde{\Gamma}^k_{ij}$. This means that if we fix $s = \min\{\frac{1}{C \cdot \mathrm{inj}_M}, \frac{1}{L}\}$ on the ball[1] $B_s(0_p) \subset T_pM$, the exponential map $\exp'_p$ with respect to $\tilde{g}$ is well defined. This is a direct consequence of Picard–Lindelöf theorem.

We know that the Jacobian matrix $J_{\exp'_p}(0_p)$ is the identity. Furthermore, it is established that the derivatives of the entries of $J_{\exp'_p}$ are uniformly bounded by a constant P: this follows directly from [12, Lemma 3.4]. So this means that there is a constant D such that for each v_p in $B_s(0_p)$

$$\|J_{\exp'_p}(v_p) - J_{\exp'_p}(0_p)\| = \|J_{\exp'_p}(v_p) - \mathrm{Id}\| \leq D \cdot \|v_p\|.$$

Then, if $R = \min\{\frac{1}{2D}, s\}$ on $B_R(0_p)$ the exponential map $\exp'_p$ is invertible. Considering that g and $\tilde{g}$ are quasi-isometric, it follows that $B_R(0_p)$ contains a ball of radius $\frac{R}{K+1}$ with respect to the metric $\tilde{g}$. This implies that the injectivity radius of $\tilde{g}$ on $M \setminus \mathcal{N}_X(\delta_0)$ is bounded from below by $\frac{R}{K+1}$.

- Condition (4) is satisfied. In particular, the boundedness of the derivatives of the second fundamental form of ∂M follows because $\bar{M}$ is isometric to $\bar{N}$ near ∂M. On the other hand, the boundedness of the covariant derivatives of the Riemann tensor of $\tilde{g}$ follows from two factors. Firstly, the norms induced by g and $\tilde{g}$ on each fiber F_x of a tensor multi-product of TX and T^*X are equivalent to some constants which do not depend on x. Secondly, χ and the components of f^*h have uniformly bounded derivatives in normal coordinates with respect to g.

(1) The radius is given with respect to g.

Finally, we can conclude the proof by noting that $f:(X,\tilde{g}) \to (N,h)$ is a uniform map over M (Lemma 1.4) and it is an isometry on $\widetilde{E}_X := (X \setminus M) \cup \mathcal{N}(\delta_0)$, where $\mathcal{N}(\delta_0)$ is the δ_0-neighborhood of ∂M in M with respect to $\tilde{g}$. So the last point holds true if we define $\widetilde{E}_Y := f(\widetilde{E}_X)$ ■

2. Minimal and maximal domains

In this section we introduce the L^2-cohomology groups of a Riemannian manifold, we introduce a regularizing operator for L^2-forms and we conclude by proving some technical lemmas.

2.1 – *The space of square-integrable differential forms*

Let (X,g) be an oriented Riemannian manifold and let $\Omega_c^k(X)$ be the space of complex differential forms with compact support. The Riemannian metric g induces for every $k \in \mathbb{N}$ a scalar product on $\Omega_c^k(X)$,

$$\langle \alpha, \beta \rangle_{\mathcal{L}^2(X,g)} := \int_X \alpha \wedge \star \bar{\beta},$$

where $\star$ is the Hodge star operator induced by g. This scalar product induces a norm on $\Omega_c^k(X)$. We denote this norm by $|\cdot|_{\mathcal{L}^2\Omega^k(X,g)}$, and $\mathcal{L}^2\Omega^k(X,g)$ will be the Hilbert space given by the closure of $\Omega_c^*(X)$ with respect to this norm. Finally, we also define the Hilbert space $\mathcal{L}^2(X,g)$ as

$$\mathcal{L}^2(X,g) := \bigoplus_{k\in\mathbb{N}} \mathcal{L}^2\Omega^k(X,g).$$

The norm of $\mathcal{L}^2(X,g)$ is denoted by $|\cdot|_{\mathcal{L}^2(X,g)}$. Moreover, for a fixed Riemannian manifold (Y,g), a linear operator $A\colon \Omega_c^*(X) \to \mathcal{L}^2(Y,h)$, which is bounded with respect to $|\cdot|_{\mathcal{L}^2(Y,h)}$ and $|\cdot|_{\mathcal{L}^2(X,g)}$, is said to be an $\mathcal{L}^2$-bounded operator. Finally, an element in $\mathcal{L}^2(X,g)$ is an $\mathcal{L}^2$-form.

Definition 2.1. Let (X,g) be an oriented Riemannian manifold whose dimension is m. The *chirality operator* is defined as the operator $\tau_X\colon \mathcal{L}^2(X,g) \to \mathcal{L}^2(X,g)$ such that, for each α in $\Omega_c^*(X)$,

$$\tau_X(\alpha) := i^{\frac{m}{2}} \star \alpha$$

if m is even and

$$\tau_X(\alpha) := i^{\frac{m+1}{2}} \star \alpha$$

if m is odd.

REMARK 5. The chirality operator is an $\mathcal{L}^2$-bounded operator. In particular, with respect to $|\cdot|_{\mathcal{L}^2(X,g)}$, it is an isometric involution. Moreover, τ_X is also self-adjoint.

DEFINITION 2.2. Let (X,g) and (Y,h) be two Riemannian manifolds. Fix an operator $A\colon \mathrm{dom}(A) \subset \mathcal{L}^2(X) \to \mathcal{L}^2(Y)$, and let $A^*\colon \mathrm{dom}(A^*) \subset \mathcal{L}^2(Y) \to \mathcal{L}^2(X)$ be its adjoint operator. Then we denote by $A^\dagger$ the operator defined as

$$A^\dagger := \tau_X \circ A^* \circ \tau_Y.$$

2.2 – *A regularizing operator*

In this subsection we discuss C^k-forms over a manifold X. These are sections of class C^k of the bundle $\Lambda^*(X)$. If $k > 1$, then the differential of a C^k-form is defined locally in the usual way and it is a C^{k-1}-form.

Let (X,g) be a Riemannian manifold and fix $\epsilon > 0$. Gol'dshtein and Troyanov [7] studied the de Rham regularizing operator R_ϵ^X. In their paper they proved that, given $k > 0$ in $\mathbb{N}$, then for each compactly supported C^k-form ω,

- $R_\epsilon^X \omega$ is in $\Omega_c^*(X)$,
- $\lim_{\epsilon \to 0} |R_\epsilon^X \omega - \omega|_{\mathcal{L}^2(X,g)} = 0$,
- $R_\epsilon^X d\omega = dR_\epsilon^X \omega$.

This operator will be useful in proving the technical lemmas at the end of this section. For this reason it is important to recall the definition of R_ϵ^X.

The operator R_ϵ^X is defined as follows: let n be the dimension of X and fix a mollifier $\rho\colon \mathbb{R}^n \to \mathbb{R}$. Then let $h\colon B_1(0) \subset \mathbb{R}^n \to \mathbb{R}^n$ be a radial diffeomorphism such that $h(x) = x$ if $\|x\| \leq \frac{1}{3}$ and

$$h(x) = \frac{1}{\|x\|} \exp\Big(\frac{1}{1 - \|x\|^2}\Big) \cdot x \quad \text{if } \|x\| \geq \frac{2}{3}.$$

Then define the submersion $s\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ as

$$s(x,v) := \begin{cases} h^{-1}(h(x) + v) & \text{if } \|x\| \leq 1, \\ x & \text{otherwise.} \end{cases}$$

Let U be a bounded convex domain of $\mathbb{R}^n$ which contains the ball $B_1(0)$. Denote by dt^2 the euclidean Riemannian metric on U. Then, for each $\epsilon > 0$, the *local* regularizing operator $R_\epsilon\colon \mathcal{L}^2(U, dt^2) \to \mathcal{L}^2(U, dt^2)$ is defined as

$$R_\epsilon \omega := \int_{\mathbb{R}^n} s^*\omega \wedge \rho_\epsilon(v) dv^1 \wedge \cdots \wedge dv^n,$$

where $\rho_\epsilon(v) := \rho(\frac{v}{\epsilon})$. To obtain a global regularizing operator we need to fix a constant J and consider a countable atlas $\{(V_i, \phi_i)\}$ of M such that for each i there are at most K charts (V_j, ϕ_j) such that $V_i \cap V_j \neq \emptyset$. Assume that $B_1(0) \subseteq \phi_i(V_i) \subset \mathbb{R}^n$ for each i and that $\{\phi_i^{-1}(B_1(0))\}$ is again a cover of M. Then we define for each k in $\mathbb{N}$,

$$R_{\epsilon,k} := R_{\epsilon,V_0} \circ \cdots \circ R_{\epsilon,V_k},$$

where $R_{\epsilon,V_i} := \phi_i^* \circ R_\epsilon \circ [\phi_i^{-1}]^*$ on V_i and it is the identity outside V_i. Finally, the operator R_ϵ^X is defined as $R_\epsilon^X := \lim_{k\to+\infty} R_{\epsilon,k}$.

2.3 – *Reduced and unreduced cohomologies*

Let (X, g) be a Riemannian manifold. The exterior derivative operator $d : \Omega_c^*(X) \to \Omega_c^*(X)$ can be seen as an unbounded operator with respect to the $\mathcal{L}^2$-norm on (X, g). So we can define some different closures $\bar{d} : \mathrm{dom}(\bar{d}) \subset \mathcal{L}^2(X, g) \to \mathcal{L}^2(X, g)$ of $(d, \Omega_c^*(X))$. In this manuscript we will concentrate on the maximal and the minimal closures of d.

Definition 2.3. The *minimal domain* of the exterior derivative on (X, g) is defined as the subset $\mathrm{dom}(d_{\min})$ of $\mathcal{L}^2(X, g)$ given by the form α such that there is a sequence of $\{\omega_k\} \subset \Omega_c^*(X)$ such that $\lim_{k\to+\infty} |\omega_k - \alpha|_{\mathcal{L}^2(X,g)} = 0$ and the sequence of $\{d\omega_k\}$ converges in $\mathcal{L}^2(X, g)$ with respect to the $\mathcal{L}^2$-norm.

Then we can define $d_{\min}(\alpha) := \lim_{k\to+\infty} d\omega_k$.

Definition 2.4. The *maximal domain* of the exterior derivative on (X, g) is defined as the subset $\mathrm{dom}(d_{\max})$ of $\mathcal{L}^2(X, g)$ given by the k-forms α such that there exists an $\mathcal{L}^2$-form η_α such that for each β in $\Omega_c^*(X)$,

$$\int_X \alpha \wedge d\beta = (-1)^{k+1} \int_X \eta_\alpha \wedge \beta.$$

Then we can define $d_{\max}(\alpha) := \eta_\alpha$.

Remark 6. Notice that, given a Riemannian manifold (X, g), then $d_{\max} = d_{\min}^\dagger$.

A well-known result about the minimal and the maximal closures of d is the following. Let $(\bar{d}, \mathrm{dom}(\bar{d}))$ be a closed extension of d on $\mathcal{L}^2(X, g)$. Then

$$(\mathrm{dom}(d_{\min}), d_{\min}) \subseteq (\bar{d}, \mathrm{dom}(\bar{d})) \subseteq (\mathrm{dom}(d_{\max}), d_{\max}).$$

Moreover, Gaffney [6] proved that if (X, g) is a complete Riemannian manifold, then $\mathrm{dom}(d_{\min}) = \mathrm{dom}(d_{\max})$ and so there is just one closed extension of d.

One of the most important properties of $d_{\min}$ and $d_{\max}$ is that

$$d_{\min\backslash\max}(\mathrm{dom}(d_{\min\backslash\max})) \subseteq \mathrm{dom}(d_{\min\backslash\max})$$

and, in particular, $d^2_{\min\backslash\max} := d_{\min\backslash\max} \circ d_{\min\backslash\max} = 0$. So it is possible to define the cohomology groups of L^2-cohomology as follows:

DEFINITION 2.5. The *k-th group of minimal L^2-cohomology* is the group defined as

$$H^k_{2,\min}(X, g) := \frac{\ker(d^k_{\min})}{\mathrm{im}(d^{k-1}_{\min})}.$$

Moreover, the *k-th group of reduced minimal L^2-cohomology* is given by

$$\bar{H}^k_{2,\min}(X, g) := \frac{\ker(d^k_{\min})}{\overline{\mathrm{im}(d^{k-1}_{\min})}},$$

where $d^k_{\min}$ is an operator defined on $\mathrm{dom}(d_{\min}) \cap \mathcal{L}^2\Omega^k(X, g)$.

DEFINITION 2.6. The *k-th group of maximal L^2-cohomology* is the group defined as

$$H^k_{2,\max}(X, g) := \frac{\ker(d^k_{\max})}{\mathrm{im}(d^{k-1}_{\max})}.$$

Moreover, the *k-th group of reduced maximal L^2-cohomology* is given by

$$\bar{H}^k_{2,\max}(X, g) := \frac{\ker(d^k_{\max})}{\overline{\mathrm{im}(d^{k-1}_{\max})}},$$

where $d^k_{\max}$ is the operator $d^k_{\max}$ defined on $\mathrm{dom}(d_{\max}) \cap \mathcal{L}^2\Omega^k(X, g)$.

In general, for each $k \in \mathbb{N}$, the groups $H^k_{2,\min}(X, g)$, $\bar{H}^k_{2,\min}(X, g)$, $H^k_{2,\max}(X, g)$ and $\bar{H}^k_{2,\max}(X, g)$ can be different.

On the other hand, we know that if (X, g) is a complete Riemannian manifold then $d_{\min} = d_{\max}$. In this case, we obtain

$$H^k_{2,\min}(X, g) = H^k_{2,\max}(X, g) \quad \text{and} \quad \bar{H}^k_{2,\min}(X, g) = \bar{H}^k_{2,\max}(X, g).$$

REMARK 7. If two Riemannian manifolds (X, g) and (Y, h) are quasi-isometric manifolds, then their L^2-cohomology groups (resp. minimal or maximal, reduced or unreduced) are isomorphic.

Finally, we conclude this section with the following lemmas.

Lemma 2.1. *Let (X, g) be a Riemannian manifold and let $\alpha \in \mathrm{dom}(d_{\max})$. Then, for each $k \geq 1$ and for each β in $C_c^k(\Lambda^* T^* X)$, we have*

$$\int_X \alpha \wedge d\beta = (-1)^k \int_X d_{\max}\alpha \wedge \beta.$$

This is a classical result; see for example [4, Remark 1 of Section 8.2]. However, here we give an explicit proof.

Proof of Lemma 2.1. Let us denote by R_ϵ^X the de Rham regularizing operator. We can assume, without loss of generality, that there is a j such that $\mathrm{supp}(\beta)$ is contained in $\phi_j^{-1}(B_1(0)) \subseteq V_j$. Here, $\{(V_i, \phi_i)\}$ is the atlas used to define R_ϵ^X. Observe that $\alpha \wedge R_{\frac{1}{n}}^X \beta$ converges pointwise to $\alpha \wedge \beta$.

Notice that on V_i, α has the form $\alpha(x) = \alpha_I(x)\, dx^I$ such that

$$\int_{V_i} g^{IJ}(x)\alpha_I(x)\alpha_J(x)\, dx^1 \wedge \cdots \wedge dx^n$$

is bounded. Here, the capital letter I is an ordered multi-index, i.e. $I = (i_1, \ldots, i_k)$ where $i_1 < i_2 < \cdots < i_k$. Moreover, we also denote by $N - I$ the ordered multi-index such that $(I, N - I) = (\sigma(1), \ldots, \sigma(n))$, where σ is a permutation. So the integral of $\alpha \wedge d\beta$ has the form $\int_{\phi_i^{-1}(B_1(0))} \alpha_I \cdot (d\beta)_{N-I}\, dx^1 \wedge \cdots \wedge dx^n$. Notice that $|\alpha_I \cdot (d\beta)_{N-I}| \leq C_\beta \cdot |\alpha_I|$. This means that if $|\alpha_I|$ is in $L^1(\phi_j^{-1}(B_1(0)))$ then we could apply the dominated convergence theorem to the sequence $\alpha_I \cdot \lim_{k\to+\infty}(dR_{\frac{1}{k}}^X \beta)_{N-I}$.

Observe that $|\alpha_I|$ is in $L^2(\phi_j^{-1}(B_1(0)))$. Indeed,

$$\begin{aligned}\int_{\phi_j^{-1}(B_1(0))} |\alpha_I|^2\, dx^1 \wedge \cdots \wedge dx^n &\leq \frac{1}{m_j} \int_{\phi_j^{-1}(B_1(0))} m_j |\alpha_I|^2\, dx^1 \wedge \cdots \wedge dx^n \\ &\leq \frac{1}{m_j} \int_{\phi_j^{-1}(B_1(0))} g^{IJ} \alpha_I \alpha_J\, dx^1 \wedge \cdots \wedge dx^n \\ &\leq C_j,\end{aligned}$$

where m_i is the infimum of the eigenvalues of g on the closure of V_i. So, since $\phi_j^{-1}(B_1(0))$ is compact, then $|\alpha_I|$ is also an L^1 function. This means that, thanks to the dominated convergence theorem,

$$\begin{aligned}\int_{V_i} \alpha \wedge d\beta &= \int_{V_i} \alpha_I \cdot (d\beta)_{N-I} dx^1 \wedge \cdots \wedge dx^n \\ &= \int_{V_i} \alpha_I \cdot \lim_{s\to+\infty} (dR_{\frac{1}{s}}^X \beta)_{N-I}\, dx^1 \wedge \cdots \wedge dx^n\end{aligned}$$

$$= \int_{V_i} \lim_{s\to+\infty} \alpha_I \cdot (dR^X_{\frac{1}{s}}\beta)_{N-I}\, dx^1 \wedge \cdots \wedge dx^n$$
$$= \lim_{s\to+\infty} \int_{V_i} \alpha \wedge dR^X_{\frac{1}{s}}\beta.$$

Observe that $dR^X_{\frac{1}{s}}\beta$ is a compactly supported smooth form: this means that

$$\int_{V_i} \alpha \wedge d\beta = \lim_{s\to+\infty} (-1)^k \int_{V_i} d\alpha \wedge R^X_{\frac{1}{s}}\beta.$$

So, by applying the dominated convergence theorem again, we conclude. ■

LEMMA 2.2. *Let (X, g) and (Y, h) be two Riemannian manifolds. Denote by $d_{\min,X}$ and $d_{\max,X}$ the minimal and the maximal extensions of the exterior derivative operator on X. Similarly, $d_{\min,Y}$ and $d_{\max,Y}$ are the minimal and the maximal extensions of the exterior derivative operator on Y. Fix an $\mathcal{L}^2$-bounded operator $A\colon \Omega^*_c(Y) \to C^k_c(\Lambda^*(T^*X))$ for some $k > 1$. Suppose that $A^\dagger\colon \Omega^*_c(X) \to C^k_c(\Lambda^*(T^*Y))$. Finally, assume that, for each smooth form ω,*

$$dA\omega \pm Ad\omega = B\omega,$$

where $B\colon \Omega^*_c(X) \to C^k_c(\Lambda^* T^* Y)$ is an $\mathcal{L}^2$-bounded operator. Then $A(\mathrm{dom}(d_{\min,Y})) \subseteq \mathrm{dom}(d_{\min,X})$ and $A(\mathrm{dom}(d_{\max,Y})) \subseteq \mathrm{dom}(d_{\max,X})$.

PROOF. Let α be a k-form of $\mathrm{dom}(d_{\max,Y})$: there is an $\mathcal{L}^2$-form $d\alpha$ such that, for each β in $\Omega^*_c(Y)$,

$$\int_Y d\alpha \wedge \beta = (-1)^k \int_Y \alpha \wedge d\beta.$$

Let us concentrate on $A\alpha$. Let γ be a form in $\Omega^*_c(X)$. We obtain

$$\begin{aligned}
\int_X A\alpha \wedge d\gamma &= \langle A\alpha, \tau_X d\gamma\rangle_{\mathcal{L}^2(X,g)} \\
&= \langle \alpha, A^* \tau_X d\gamma\rangle_{\mathcal{L}^2(Y,h)} \\
&= \langle \alpha, \tau_Y A^\dagger \tau_X \tau_X d\gamma\rangle_{\mathcal{L}^2(Y,h)} \\
&= \langle \alpha, \tau_Y A^\dagger d\gamma\rangle_{\mathcal{L}^2(Y,h)} \\
&= (-1)^k \langle \alpha, \tau_Y A^\dagger d^\dagger \gamma\rangle_{\mathcal{L}^2(Y,h)} \\
&= (-1)^k \langle \alpha, \tau_Y (d^\dagger [\mp A]^\dagger + B^\dagger)\gamma\rangle_{\mathcal{L}^2(Y,h)} \\
&= \langle \alpha, \tau_Y (d [\mp A]^\dagger + B^\dagger)\gamma\rangle_{\mathcal{L}^2(Y,h)} \\
&= \langle d_{\max,X}\alpha, \tau_Y [\mp A]^\dagger \gamma\rangle_{\mathcal{L}^2(X,g)} + \langle B\alpha, \tau_Y \gamma\rangle_{\mathcal{L}^2(X,g)} \\
&= \langle (\mp A d_{\max,X} + B)\alpha, \gamma\rangle_{\mathcal{L}^2(X,g)}.
\end{aligned}$$

So this means that $A(\mathrm{dom}(d_{\max,Y})) \subseteq \mathrm{dom}(d_{\max,X})$ and $d_{\max,X}A = \mp Ad_{\max,Y} + B$.

Let β be an element in $\mathrm{dom}(d_{\min,Y})$. Then there is a sequence $\{\beta_n\}$ such that $\beta = \lim_{n\to+\infty}\beta_n$ and $d\beta = \lim_{n\to+\infty} d\beta_n$ with respect to the $\mathcal{L}^2$-norm. Since A is an $\mathcal{L}^2$-bounded operator, we obtain

$$A\beta = \lim_{n\to+\infty} A\beta_n.$$

Fix $n \in \mathbb{N}$ and let $K(n)$ be a number such that

$$|A\beta - A\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \leq \frac{1}{2n}$$

and

$$|Ad\beta - Ad\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \leq \frac{1}{2n}.$$

Then let $\epsilon(n)$ be small enough that

$$|A\beta_{K(n)} - R^X_{\epsilon(n)}A\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \leq \frac{1}{2n}$$

and

$$|Ad\beta_{K(n)} - R^X_{\epsilon(n)}Ad\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \leq \frac{1}{2n}.$$

So let $\{\gamma_n\}$ be the sequence defined as $R^X_{\epsilon(n)}A\beta_{K(n)}$ for each n. This is a sequence of compactly supported smooth forms which converges, with respect to the $\mathcal{L}^2$-norm, to $A\beta$. We just have to show that $dR^X_{\epsilon(n)}A\beta_{K(n)}$ converges to an element of $\mathcal{L}^2(X, g)$.

Observe that

$$\begin{aligned} dR^X_{\epsilon(n)}A\beta_{K(n)} &= R^X_{\epsilon(n)}d_{\max,X}A\beta_{K(n)} \\ &= R^X_{\epsilon(n)}(\mp Ad_{\max,X} + B)\beta_{K(n)} \\ &= R^X_{\epsilon(n)}(\mp Ad + B)\beta_{K(n)}. \end{aligned}$$

This means that

$$\begin{aligned} &|Ad\beta - R^X_{\epsilon(n)}(\mp Ad + B)\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \\ &\quad \leq |Ad\beta - Ad\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \\ &\qquad + |Ad\beta_{K(n)} - R^X_{\epsilon(n)}(\mp Ad + B)\beta_{K(n)}|_{\mathcal{L}^2(X,g)} \\ &\quad \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}. \end{aligned}$$

So we just proved that $A(\mathrm{dom}(d_{\min,Y})) \subseteq \mathrm{dom}(d_{\min,X})$ and $d_{\min,X}A = \mp Ad_{\min,Y} + B$. ∎

3. R.–N.–Lipschitz maps and vector bundles

This section is devoted to recalling three necessary components for proving the main result in Theorem 4.6. These notions are also studied in [14]. In particular, we need

- some assumptions which allow a map between Riemannian manifolds to induce an $\mathcal{L}^2$-bounded pullback,
- a specific Riemannian metric and a family of Thom forms on a vector bundle.

In next subsection, we address the first bullet.

3.1 – *Radon–Nikodym–Lipschitz maps*

Let (M, ν) and (N, μ) be two measured spaces and let $f\colon (M, \nu) \to (N, \mu)$ be a function such that the pushforward measure $f_\star(\nu)$ is absolutely continuous with respect to μ.

Definition 3.1. Let (N, μ) be σ-finite. Then the *fiber volume* of f is the Radon–Nikodym derivative

$$\mathrm{Vol}_{f,\nu,\mu} := \frac{\partial f_\star \nu}{\partial \mu}.$$

When M and N are Riemannian manifolds (X, g) and (Y, h) and their measures ν and μ are those induced by the metrics g and h, then we denote the fiber volume of a map $f\colon (X, g) \to (Y, h)$ by Vol_f.

Let (M, d_M, μ_M) and (N, d_N, μ_N) be two measured and metric spaces.

Definition 3.2. A map $f\colon (M, d_M, \mu_M) \to (N, d_N, \mu_N)$ is *Radon–Nikodym–Lipschitz* or *R.–N.–Lipschitz* if

- f is Lipschitz,
- f has a well-defined and bounded fiber volume.

Remark 8. As shown in [14, Remark 12], there is an equivalent definition of Radon–Nikodym–Lipschitz map. A map $f\colon (M, d_M, \nu_M) \to (N, d_N, \mu_N)$ is a *R.–N.–Lipschitz map* if it is Lipschitz and there is a constant C such that, for each measurable set $A \subseteq N$,

$$\mu_M(f^{-1}(A)) \leq C\mu_N(A).$$

So this implies that a composition of R.–N.–Lipschitz maps is an R.–N.–Lipschitz map.

The following Proposition 3.1 about R.–N.–Lipschitz maps is proved in [14, Proposition 2.4].

PROPOSITION 3.1. *Let (M, g) and (N, h) be Riemannian manifolds and let $f: (M, g) \to (N, h)$ be an R.–N.–Lipschitz map. Then f^*, which is the pullback along f, is an $\mathcal{L}^2$-bounded operator.*

In general, a Lipschitz map is not R.–N.–Lipschitz: for example, if (M, g) and (N, h) are two Riemannian manifolds such that $\dim(M) < \dim(N)$ then there is no Lipschitz embedding $i: (M, g) \to (N, h)$ which is R.–N.–Lipschitz.

However, we are interested in the fiber volume of a submersion and, when $f: (M, g) \to (N, h)$ is a Lipschitz submersion between oriented Riemannian manifolds, we know how to compute its fiber volume. In order to explain how to compute it, we recall the notion of a quotient between two differential forms.

DEFINITION 3.3. Let us consider a differentiable manifold X. Given two differential forms $\alpha \in \Omega^k(X)$, $\beta \in \Omega^n(X)$, we define *a quotient between α and β*, denoted by $\frac{\alpha}{\beta}$, as a (possibly not continuous) section of $\Lambda^{k-n}(X)$ such that for all p in M,

$$\alpha(p) = \beta(p) \wedge \frac{\alpha}{\beta}(p).$$

In general, given two smooth differential forms, we do not know whether there is a quotient between them. Moreover, if a quotient between α and β exists, it may be not unique. However, given a submersion between Riemannian manifolds $\pi: (X, g) \to (Y, h)$ and denoting by Vol_X and Vol_Y the volume forms on (X, g) and (Y, h), then it is possible to define a quotient between Vol_X and $\pi^*\mathrm{Vol}_Y$ and the pullback of this quotient on a fiber of π is a smooth form which does not depend on the choice of the quotient (see [14, p. 15]). So this means that if we denote the embedding of the fiber of q in X by $i_q: F_q \to X$, then

$$\int_{F_q} i_q^*\Big(\frac{\mathrm{Vol}_X}{\pi^*\mathrm{Vol}_Y}\Big)$$

is a well-defined real number for each q in N.

PROPOSITION 3.2. *Let (X, g) and (Y, h) be two oriented, Riemannian manifolds. Denote by Vol_X (resp. Vol_Y) the volume form on (X, g) (resp. (Y, h)). Let $\pi: (X, g) \to (Y, h)$ be a submersion. Then, if q is on $\mathrm{im}(\pi)$,*

$$\mathrm{Vol}_\pi(q) = \int_F \frac{\mathrm{Vol}_X}{\pi^*\mathrm{Vol}_Y}(q),$$

and $\mathrm{Vol}_\pi(q) = 0$ otherwise.

REMARK 9. If the submersion $f: X \to Y$ is a diffeomorphism between oriented manifolds which preserves the orientations, then the integration along the fibers of f is the pullback $(f^{-1})^*$. This means that the fiber volume of f is given by $|(f^{-1})^* \frac{\mathrm{Vol}_X}{f^*(\mathrm{Vol}_Y)}|$.

We also recall [14, Proposition 2.6]: this proposition allows us to compute the fiber volume of the composition of two submersions.

Proposition 3.3. *Let $f\colon (X,g) \to (Y,h)$ and $g\colon (Y,h) \to (W,l)$ be two submersions between oriented Riemannian manifolds. Denote by* Vol_X, Vol_Y *and* Vol_W *the volume forms on X, Y and W induced by their Riemannian metrics. Then*

$$\mathrm{Vol}_{g\circ f}(q) = \int_{g^{-1}(q)} \left(\int_{f^{-1}g^{-1}(q)} \frac{\mathrm{Vol}_X}{f^*(\mathrm{Vol}_Y)} \right) \frac{\mathrm{Vol}_Y}{g^*\mathrm{Vol}_W}.$$

Finally, we conclude this subsection by showing the relation between R.–N.–Lipschitz maps and the quasi-isometries.

Proposition 3.4. *Let $f\colon (X,g) \to (Y,h)$ be a quasi-isometry. Then f is an R.–N.–Lipschitz map.*

Proof. Notice that f and its inverse f^{-1} are Lipschitz maps: this directly follows from the definition of quasi-isometry. So we just have to show that f has a bounded fiber volume. This can be proved by showing that id: $(X,g) \to (X, f^*h)$ has bounded fiber volume. Let x be a point of X. Thanks to a simultaneous diagonalization argument, we can find a basis $\{e_1, \dots, e_n\}$ of T_xX such that the Gram matrices of f^*h and g are both diagonals. Let K be the Lipschitz constant of f^{-1}. Observe that $g(e_i,e_i) \leq Kf^*h(e_i,e_i)$ for each $i = 1, \dots, n$. So this means that

$$\det(g_{jl}(x)) = \prod_{i=1}^{n} g(e_i,e_i) \leq K^n \prod_{i=1}^{n} f^*h(e_i,e_i) = K^n \det(f^*h_{jl}(x)).$$

Notice that the fiber volume of id on a point $y = f(x)$ is given by

$$\sqrt{\frac{\det(g_{jl}(x))}{\det(f^*h_{jl}(x))}} \leq K^{\frac{n}{2}}.$$

This concludes the proof. ∎

3.2 – *Generalized Sasaki metrics*

In this subsection a Riemannian metric on a vector bundle is introduced. This metric is a generalization of the metric defined by Sasaki in [11] for the tangent bundle of a Riemannian manifold. The definition of this generalized Sasaki metric can be found in Boucetta and Essoufi [3, p. 2].

Let (N,h) be a Riemannian manifold and let $\pi_E\colon E \to N$ be a vector bundle. Fix a bundle metric $H_E \in \Gamma(E^* \otimes E^*)$ and a linear connection ∇_E which preserves H_E.

Fix $\{s_\alpha\}$, a local frame of E. Given a system of local coordinates $\{x^i\}$ over $U \subseteq N$, we obtain the system of coordinates $\{x^i, \mu^\alpha\}$ on $\pi_E^{-1}(U)$, where the μ^α are the components with respect to $\{s_\alpha\}$. The map $K\colon TE \to E$ is then defined as

$$K\Big(b^i \frac{\partial}{\partial x^i}\Big|_{(x_0,\mu_0)} + z^\alpha \frac{\partial}{\partial \mu^\alpha}\Big|_{(x_0,\mu_0)}\Big) := (z^\alpha + b^i \mu^j \Gamma^\alpha_{ij}(x_0))s_\alpha(x_0),$$

where the Γ^l_{ij} are the Christoffel symbols of ∇_E. The *Christoffel symbols* of a connection ∇_E are some coefficients locally implicitly defined by the equality

$$\nabla^E_{\frac{\partial}{\partial x^j}} s_\eta(x) := \Gamma^\gamma_{\eta j}(x)s_\gamma(x).$$

Definition 3.4. The *Sasaki metric* on E is the Riemannian metric h^E defined for all A, B in $T_{(p,v_p)}E$ as

$$h^E(A,B) := h(d\pi_{E,v_p}(A), d\pi_{E,v_p}(B)) + H_E(K(A), K(B)).$$

Remark 10. Fix the system of coordinates $\{x^i, \mu^j\}$ on E. The Gram matrix of h^E is

$$\begin{cases} h^E_{ij}(x,\mu) = h_{ij}(x) + H_{\alpha\gamma}(x)\Gamma^\alpha_{\beta i}(x)\Gamma^\gamma_{\eta j}(x)\mu^\beta\mu^\eta, \\ h^E_{i\sigma}(x,\mu) = H_{\sigma\alpha}(x)\Gamma^\alpha_{\beta i}(x)\mu^\beta, \\ h^E_{\sigma\tau}(x,\mu) = H_{\sigma,\tau}(x), \end{cases}$$

where $i, j = 1, \dots, n$ and $\sigma, \tau = n+1, \dots, n+m$. Consider a point $x_0 = (x_0^1, \dots, x_0^n)$ in N. If all the Christoffel symbols of ∇_E in x_0 are zero, then, in local coordinates, the matrix of h^E at a point (x_0, μ) is

$$\begin{bmatrix} h_{i,j}(x_0) & 0 \\ 0 & H_{\sigma,\tau}(x_0) \end{bmatrix}.$$

Moreover, with respect to the coordinates (x^i, μ^σ), the inverse of the Gram matrix is given by

$$\begin{cases} h^{ij}_E(x,\mu) = h^{ij}(x), \\ h^{i\sigma}_E = -\Gamma^\sigma_{\beta j}(x)h^{ij}(x)\mu^\beta, \\ h^{\sigma\tau}_E = H^{\sigma\tau}(x) + h^{ij}(x)\Gamma^\sigma_{\beta i}(x)\Gamma^\tau_{\eta j}(x)\mu^\beta\mu^\eta, \end{cases}$$

where $H^{\sigma\tau}$ and $h^{ij}(x)$ are the components of the inverse matrices of h_{ij} and $H_{\sigma,\tau}$.

Example 3.1. Let (X, g) be a Riemannian manifold and let $E = TX$. Choose as h_E the metric g itself and fix $\nabla_E = \nabla^{\mathrm{LC}}_g$, the Levi-Civita connection of g. We denote by g_S the Sasaki metric induced by g and ∇^{LC}_g.

Example 3.2. Let $f\colon (X,g)\to(Y,h)$ be a smooth map. Let $\pi\colon f^*TY\to X$ be the pullback bundle of TY. Then the Riemannian metric h can be seen as a bundle metric on TY and so we obtain a bundle metric f^*h on f^*TY. Fix the connection $f^*\nabla_h^{\mathrm{LC}}$ on f^*TY which is the pullback of the Levi-Civita connection on (Y,h). Let us denote by $g_{S,f}$ the Sasaki metric induced by $f^*\nabla_h^{\mathrm{LC}}$, f^*h and g.

Remark 11. Let (Y,h) be a Riemannian manifold. Then $(\mathrm{id}^*TY, g_{S,\mathrm{id}})$ and (TY, g_S) are isometric Riemannian manifolds, and the isometry is the map ID: $(\mathrm{id}^*TY, g_{S,\mathrm{id}})\to(TY,g_S)$ which is the bundle map induced by the identity. For this reason, we will identify $(\mathrm{id}^*TY, g_{S,\mathrm{id}})$ and (TY,g_S) both as bundles and as Riemannian manifolds.

Definition 3.5. Let (X,g) be a Riemannian manifold and fix a vector bundle $\pi\colon E\to X$. Let H_E and ∇_E be a bundle metric and a connection on X. Then, given a constant δ, we denote by E^δ the following disk-bundle:

$$E^\delta=\left\{e_p\in E\;\middle|\; h^E(e_p,e_p)<\delta\right\}.$$

Let $f\colon (X,g)\to(Y,h)$ be a smooth map between two Riemannian manifolds. When $E=f^*TY$, $\nabla_E=f^*\nabla_h^{\mathrm{LC}}$ and H_E is the pullback bundle metric of h, then instead of E^δ we use the notation $f^*T^\delta Y$.

Remark 12. Let $f\colon (X,g)\to(Y,h)$ be a smooth map. Fix a bundle E on Y and let ∇^E be a connection and h^E be a bundle metric on E. Then the induced bundle map

$$\begin{aligned} F\colon (f^*T^\delta Y, g_{S,f}) &\longrightarrow (T^\delta Y, g_S),\\ (p, w_{f(p)}) &\longrightarrow w_{f(p)} \end{aligned}$$

is a smooth Lipschitz map.

Remark 13. Notice that if $\pi\colon (X,g)\to(Y,h)$ is a submersion, then $(\pi^*)^\dagger=\pi_\star$, which is the integration along the fibers of π. This is a consequence of the projection formula (see [2] or [13]).

Proposition 3.5. *Let $\pi\colon E\to X$ be a vector bundle on a Riemannian manifold (X,g). On E, fix a metric bundle H_E and a connection ∇_E which preserves H_E. Let h^E be the generalized Sasaki metric on E induced by g, H_E and ∇_E. Assume that at each point p of X there is a system of normal coordinates $\{x^i\}$ centered in p and a frame $\{s_\alpha\}$ such that*

- *$\{s_\alpha\}$ is orthonormal in p,*
- *all the Christoffel symbols of ∇_E with respect to $\{x^i\}$ and $\{s_\alpha\}$ vanish in p.*

Then $\pi\colon (E^\delta, h^E) \to (X, g)$ is an R.–N.–Lipschitz map for each $\delta > 0$. This means that π^* is an $\mathcal{L}^2$-bounded operator. Moreover, as a consequence of Remark 13, integration along the fibers $\pi_\star = (\pi^*)^\dagger$ is also an $\mathcal{L}^2$-bounded operator.

Proof. The submersion $\pi\colon E^\delta \to X$ is a Riemannian submersion so, in particular, it is Lipschitz. Moreover, for fixed a point p on M, if $\mathrm{Vol}_X(p) = e_1 \wedge \cdots \wedge e_n$, then $\mathrm{Vol}_E(q) = \pi^*(e_1)(q) \wedge \cdots \wedge \pi^*(e_n)(q) \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$, where q is in the fiber of π and $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ is a volume form of the fiber of q. Then the fiber volume of π is given by

$$\begin{aligned}\int_F \frac{\mathrm{Vol}_E}{\pi^*\mathrm{Vol}_X} &= \int_{B_\delta(0_p)} \frac{\pi^*(e_1)(q) \wedge \cdots \wedge \pi^*(e_n)(q) \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_n}{\pi^*(e_1)(q) \wedge \cdots \wedge \pi^*(e_n)(q)} \\ &= \int_{B_\delta(0_p)} \varepsilon_1 \wedge \cdots \wedge \varepsilon_n = \mu_{\mathrm{eucl}}(B_\delta(0)).\end{aligned}$$ ∎

Remark 14. The metrics of Examples 3.1 and 3.2 satisfy the assumptions of Proposition 3.5.

3.3 – *Mathai–Quillen–Thom forms*

The last component that we need in order to prove the main result of this paper is the Mathai–Quillen–Thom forms of a vector bundle. This is a family of Thom forms, so we will start by introducing the notion of a Thom form.

Definition 3.6. Let $\pi\colon E \to M$ be a vector bundle. A smooth form ω in $\Omega^*_{cv}(E)$ is a *Thom form* if it is closed and its integral along the fibers of π is equal to the constant function 1.

Given a Thom form ω of f^*TY such that $\mathrm{supp}(\omega)$ is contained in a $\delta_0 < \delta$ neighborhood of the null section, let us denote by $e_\omega\colon \Omega^*(f^*(T^\delta Y)) \to \Omega^*(f^*(T^\delta Y))$, the operator defined for every smooth form α as

$$e_\omega(\alpha) := \alpha \wedge \omega.$$

Denote by $|\cdot|_{(p,w_{f(p)})}$ the norm on $\Lambda^*_{(p,w_{f(p)})}(f^*TY)$ induced by the Riemannian metric $g_{S,f}$. If there is a constant C such that

$$|\omega|_{(p,w_{f(p)})} < C,$$

then the operator e_ω is an $\mathcal{L}^2$-bounded operator (see [14, Proposition 4.4]). In this case, $e_\omega^\dagger = r_\omega$ where $r_\omega(\beta) := \omega \wedge \beta$.

In this manuscript, we use the Thom forms introduced by Mathai and Quillen [10]. In their work, the authors defined a Thom form for a vector bundle $\pi\colon E \to (M, g)$ induced by the Riemannian metric g, a metric bundle h_E and a connection ∇_E. In particular, for each $\delta > 0$ they found a Thom form whose support is contained on E^δ. So we will consider, on the tangent bundle of a Riemannian manifold (Y, h), the Mathai–Quillen–Thom ω_Y form induced by h and ∇_h^{LC}. On the other hand, given a map $f\colon X \to Y$, we fix on f^*TY the pullback $F^*\omega_Y$, where $F\colon f^*(TY) \to TY$ is the bundle map induced by f.

As shown in [14, Section 4.2.], if h has the same bounds as a manifold of bounded geometry and if f is a smooth Lipschitz map, then there is a constant C such that for each $(p, w_{f(p)})$ in f^*TY the norm $|F^*\omega_Y|_{(p,w_{f(p)})} \leq C$. In this case, the operator $\Omega_c^*(TY) \to \Omega_c^*(TY)$ is an $\mathcal{L}^2$-bounded operator (see [14, Propositions 4.3 and 4.4]).

4. Stability of cohomologies

In this section, given a uniform map $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ which is isometric on the ends, we define an operator $T_f\colon \mathcal{L}^2(Y, h) \to \mathcal{L}^2(X, g)$. Moreover, we also prove that if f is a homotopy equivalence, then T_f induces the required isomorphisms in L^2-cohomology.

In this section we always assume that f is a smooth $C^k_{b,\mathrm{loc}}$-map for each k in $\mathbb{N}$. If f is not a $C^k_{b,\mathrm{loc}}$-map, then we replace it with an ϵ-approximation f_ϵ provided by Proposition 1.3. Moreover, we also assume that f is an isometry on the unbounded ends: we know, thanks to Proposition 1.5, that the metric on the domain can be replaced by another metric which is quasi-isometric to the first one and which makes f an isometry on the ends.

4.1 – *A submersion related to a uniform map*

Fix $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$, a smooth $C^k_{b,\mathrm{loc}}$-map which is isometric on unbounded ends. Let r_0 be a constant such that the restriction of f to $\mathcal{N}_X(r_0)$ is an isometry. Fix $\delta = \frac{1}{4}r_0$ and denote by $\pi\colon f^*T^\delta Y \to X$ the fiber bundle given by the vectors on $f^*T^\delta Y$ whose norm is less than or equal to δ. Consider the map

$$\begin{aligned} p_f\colon \mathrm{dom}(p_f) \subset f^*T^\delta Y &\longrightarrow Y, \\ (p, v_{f(p)}) &\longrightarrow \exp_{f(p)}(v_{f(p)}). \end{aligned}$$

Notice that outside $\pi^{-1}(M) \subseteq f^*T^\delta Y$ the map p_f could be not defined on all $f^*T^\delta Y$ since Y is not complete. However, it is defined on a neighborhood of the 0-section.

Thanks to [14, Lemma 3.3] we know that, on its domain, p_f is a smooth submersion, such that $p_f(0_{f(p)}) = f(p)$. Let p be a point in X and fix a system of normal coordinates $\{x^i\}$ around p and let $\{y^j\}$ be a system of normal coordinates around $f(p)$ in Y. Let $\{x^i, \mu^j\}$ be the system of coordinates on $f^*T^\delta Y$, where μ^j are the coordinates referred to the frame $\{\frac{\partial}{\partial y^j}\}$. Since f is a $C^k_{b,\text{loc}}$-map, thanks to [14, Lemma 3.3] we know that p_f has uniformly bounded derivatives with respect to $\{x^i, \mu^j\}$. This implies that on

$$A_f := \big\{(p, w_{f(p)}) \in f^*T^\delta Y \text{ such that } f(p) \subset N \\ \text{and } |w_{f(p)}|_h \leq d(f(p), \partial N)\big\}, \tag{4.1}$$

the map p_f is Lipschitz. This can be proved as a consequence of Schick [12, Lemma 3.4].

Let $\psi\colon \mathbb{R} \to \mathbb{R}$ be the map defined as

$$\psi(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t \geq \delta, \\ P(\frac{t}{\delta}) & \text{if } t \in (0, \delta), \end{cases}$$

where $P(s) := -20s^7 + 70s^6 - 84s^5 + 35s^4$. Thanks to Eldering [5, Lemma 2.34], we can fix $\epsilon < \frac{\delta}{3}$ and $\nu \leq \frac{r_0}{8}$ and find a map $\psi_\epsilon\colon \mathbb{R} \to \mathbb{R}$ such that

- $\psi = \psi_\epsilon$ on $\mathbb{R} \setminus (\delta - \nu, \delta + \nu)$,
- $d(\psi_\epsilon(t), \psi(t)) \leq \epsilon$ for each t,
- ψ is a smooth function on $\mathbb{R} \setminus \{0\}$,
- ψ_ϵ is a map of class C^3 on $\mathbb{R}$.

Then we can define the map $\phi\colon Y \to \mathbb{R}$ as

$$\phi(q) := \begin{cases} 0 & \text{if } q \in Y \setminus N, \\ 1 & \text{if } q \in N \setminus \mathcal{N}_Y(r_0), \\ \psi_\epsilon(d(q, \partial N)) & \text{if } q \in \mathcal{N}_Y(r_0). \end{cases}$$

By some easy computations, it is possible to notice that ϕ is a Lipschitz function in $C^3(Y)$ and it is smooth outside ∂N. Indeed, because of Remark 4, if we fix some collar coordinates (y, t) on $\mathcal{N}_Y(r_0)$, then ϕ only depends on t.

Let $\tilde{p}_f\colon f^*T^\delta Y \to Y$ be the map defined as

$$\tilde{p}_f(v_{f(p)}) := p_f(\phi(f(p)) \cdot v_{f(p)}).$$

Observe that $\tilde{p}_f$ is defined on all $f^*T^\delta Y$ and it is a submersion. Indeed, on M, $\tilde{p}_f$ is a submersion because $\phi \neq 0$ and p_f is a submersion; on $X \setminus M$, $\tilde{p}_f$ is a submersion since f is an isometry.

Proposition 4.1. *Let $(X = M \cup E_X, g)$ and $(Y = N \cup E_Y, h)$ be two manifolds with unbounded ends. Let $f\colon (X,g) \to (N,h)$ be a uniform map isometric on the unbounded ends. Assume that $f_{|\mathcal{N}_X(r_0)}$ is an isometry and fix $\delta = \frac{r_0}{4}$. Denote by $g_{S,f}$ the generalized Sasaki metric induced by g, f^*h and $f^*\nabla_h^{\mathrm{LC}}$ on f^*TN.*

Then the map $\tilde{p}_f\colon (f^*T^\delta, g_{S,f}) \to (Y,h)$ is an R.–N.–Lipschitz map.

Proof. Observe that $\tilde{p}_f$ is a Lipschitz map since

- p_f is Lipschitz on the subset A_f defined in equality (4.1),
- f is an isometry on the unbounded ends,
- ϕ is Lipschitz.

It remains to prove the boundedness of the fiber volume. Observe that if $q \in Y \setminus N$, then $p_f^{-1}(q)$ is the fiber of $f^*T^\delta N$ over $p_0 = f^{-1}(q)$. Then, since f is an isometry on the ends, the fiber volume of q in $Y \setminus N$ is equal to the volume of a ball of radius δ in $\mathbb{R}^n$.

Let q be a point in N such that $B_\delta(q) \cap \mathcal{N}_Y(\delta) = \emptyset$. Observe that for each p_0 in X, since $\phi \leq 1$,

$$d(f(p_0), \tilde{p}_f(w_{f(p_0)})) \leq \delta \cdot f^*\phi(p_0) \leq \delta.$$

This means that

$$\tilde{p}_f^{-1}(q) \subset \pi^{-1} f^{-1}(B_\delta(q)).$$

Notice that, since $B_\delta(q) \cap \mathcal{N}_Y(r_0) = \emptyset$, we obtain that $f^*\phi$ on $f^{-1}(B_\delta(q))$ is equal to 1.

This means that $\tilde{p}_f$, on the fiber of q, is equal to the submersion p_f defined in [14]. So the fiber volume of q can be computed as in [14, Corollary 4.2] and it is bounded by a constant which only depends on the curvatures of M and N.

Finally, we have to calculate the fiber volume of a point q in N such that $B_\delta(q) \cap \mathcal{N}_Y(\delta) \neq \emptyset$. Since $\delta = \frac{r_0}{4}$ we obtain that $B_\delta(q) \cap N \subset \mathcal{N}_Y(r_0)$.

Since $\tilde{p}_f = f \circ \pi$ outside M and since $f_{|E_X}$ is an isometry, then $\tilde{p}_f^{-1}(q)$ is contained in $\pi^{-1}(\mathcal{N}_X(r_0))$. Recall that, since M is an open subset of bounded geometry, we can identify each point p in $\pi^{-1}(\mathcal{N}_X(r_0))$ as a couple (x,t) in $\partial M \times (0, r_0)$. Let us define the map

$$\begin{aligned} \tilde{t}_f\colon \pi^{-1}(\mathcal{N}_X(r_0)) &\longrightarrow M \times N, \\ w_{f(x,t)} &\longrightarrow (x, t, \tilde{p}_f(w_{f(x,t)})). \end{aligned}$$

Observe that $\tilde{p}_f = \mathrm{pr}_N \circ \tilde{t}_f$ where $\mathrm{pr}_N\colon M \times N \to N$ is the projection on the second component.

Thanks to Proposition 3.3, we obtain

$$\mathrm{Vol}_{\tilde{p}_f}(q) = \int_X \mathrm{Vol}_{\tilde{t}_f}(p,q)\, d\mu_X.$$

Let us focus on $\mathrm{Vol}_{\tilde{t}_f}$. Fix some normal coordinates $\{V, y^i\}$ around q and some collar coordinates $\{U, x^j, t\}$ around $p_0 = (x_0, t_0) = f^{-1}(q)$. Assume that $f(U) \subseteq V$. Then on $\pi^{-1}(U) \subset f^*T^\delta Y$ we have the fibered coordinates $\{x^i, t, \mu^j\}$, where $\{\mu^j\}$ are the coordinates relative to the frame $\{\frac{\partial}{\partial y^i}\}$. Observe that

$$\tilde{t}_f(x_0, t_0, \mu^j) = (x_0, t_0, \psi_\epsilon(t_0) \cdot \mu^j).$$

Moreover, since the Christoffel symbols of the pullback connection $f^*\nabla_h^{\mathrm{LC}}$ vanish in (x_0, t_0), the volume form of $f^*T^\delta Y$ in (x_0, t_0, μ^j) is given by

$$\mathrm{Vol}_{f^*T^\delta Y}(x_0, t_0, \mu^j) = \det(g_{ij}(x_0, t_0))\, dx^I \wedge dt \wedge d\mu^J.$$

Observe that $\det(g_{ij}(x_0, t_0))$ is uniformly bounded since M is an open subset of bounded geometry. Let us study the volume form on $M \times N$: this is given by

$$\mathrm{Vol}_{M\times N}(x, t, y) = \det(g_{ij}(x,t)) \cdot \det(h_{rs}(y))\, dx^I \wedge dt \wedge dy^J.$$

Observe that $\tilde{t}_f$ is a diffeomorphism with its image. This means that its fiber volume is null outside the image of $\tilde{t}_f$. Moreover, on $\mathrm{im}(\tilde{t}_f)$, the fiber volume is given by

$$[\tilde{t}_f^{-1}]^*\Big(\frac{\mathrm{Vol}_{f^*T^\delta Y}}{\tilde{t}_f^*(\mathrm{Vol}_{M\times N})}\Big).$$

Notice that, in (x_0, t_0, q), we have

$$\frac{\mathrm{Vol}_{f^*T^\delta Y}}{\tilde{t}_f^*(\mathrm{Vol}_{M\times N})}(x_0, t_0, \mu^j) = \big[\det\big(h_{rs}(\psi_\epsilon(t_0)\cdot y)\big)\big]^{-1} \frac{1}{\psi_\epsilon(t_0)^n}.$$

This means that, for each (x, t, q) in $\mathrm{im}(\tilde{t}_f)$,

$$\mathrm{Vol}_{\tilde{t}_f}(x, t, q) \leq C \cdot \frac{1}{\psi_\epsilon(t)^n}. \tag{4.2}$$

Let us decompose M as the union of $\mathcal{N}_X(\frac{3}{10}\delta)$ and $M_0 := M \setminus \mathcal{N}_X(\frac{3}{10}\delta)$. We are choosing $\frac{3}{10}\delta$ because on $\mathcal{N}_X(\frac{3}{10}\delta)$ the inequality $\phi(x,t) = \psi_\epsilon(t) \leq \frac{1}{2}$ holds. This inequality will be useful later.

However, we can see the fiber volume of $\tilde{p}_f$ as the integral

$$\mathrm{Vol}_{\tilde{p}_f}(q) = \int_{\mathcal{N}_X(\frac{3}{10}\delta)} \mathrm{Vol}_{\tilde{t}_f}(p,q)\, d\mu_M + \int_{M_0} \mathrm{Vol}_{\tilde{t}_f}(p,q)\, d\mu_M.$$

Notice that, thanks to inequality (4.2), on M_0 the fiber volume of $\tilde{t}_f$ is uniformly bounded by a constant C_ψ since $\psi_\epsilon(t)$ is bounded from below by $\phi(x, \frac{3}{10}\delta) = \psi_\epsilon(\frac{3}{10}\delta)$. Moreover, since f is uniformly proper, then also $\pi \circ f$ and $\tilde{p}_f$ are uniformly proper and so this means that there is a number C (which does not depend on q) such that

$$\operatorname{diam}\big(\pi(\tilde{p}_f^{-1}(q))\big) \leq C.$$

Since M is an open subset of bounded geometry, we obtain a constant K such that

$$\mu_M\big(\pi(\tilde{p}_f^{-1}(q)) \cap M_0\big) \leq \mu_M\big(\pi(\tilde{p}_f^{-1}(q))\big) \leq K.$$

Finally, observe that $\operatorname{im}(\tilde{t}_f) \cap [M \times \{q\}] = \pi(\tilde{p}_f^{-1}(q)) \times \{q\}$ and we obtain

$$\int_{M_0} \operatorname{Vol}_{\tilde{t}_f}(p,q)\, d\mu_M \leq \int_{\pi(\tilde{p}_f^{-1}(q))} \operatorname{Vol}_{\tilde{t}_f}(p,q)\, d\mu_M \leq K \cdot C_\psi.$$

We only have to prove the boundedness of the integral over $\mathcal{N}_X(\frac{3}{10}\delta)$.

Recall that if (x,t,q) is not in $\operatorname{im}(\tilde{t}_f)$, then the fiber volume of $\operatorname{Vol}_{\tilde{t}_f}(x,t,q) = 0$. Observe that $\operatorname{im}(\tilde{t}_f)$ is given by the elements (x,t,q) of $M \times N$ such that there is a tangent vector $w_{f(x,t)}$ on $f(x,t)$ whose norm is less than or equal to $\delta \cdot \phi(t)$ and which satisfies

$$\tilde{p}_f(w_{f(x,t)}) = \exp_{f(x,t)}(w_{f(x,t)}) = q = f(x_0,t_0).$$

This means that, if (x,t,q) is in $\operatorname{im}(\tilde{t}_f)$, then

$$\begin{aligned} d_M((x_0,t_0),(x,t)) = d_N(f(x_0,t_0), f(x,t)) &= d_N(q, f(x,t)) \\ &\leq \|w_{f(x,t)}\| \leq \delta \cdot \psi_\epsilon(t). \end{aligned} \tag{4.3}$$

Observe that for each (x,t) in $\mathcal{N}_X(r_0)$

$$|t - t_0| = d_M((x_0,t_0),(x_0,t)) \leq d_M((x_0,t_0),(x,t));$$

indeed, if (x_1,t) is a point such that $d_M((x_0,t_0),(x_1,t)) < |t-t_0|$ then $d((x_1,t),\partial M) < t$ which is a contradiction with Lemma 1.2.

Moreover, on $\mathcal{N}_X(\frac{3}{10}\delta)$, we also have that $\psi_\epsilon(t) \leq \frac{1}{2}t$ for each (x,t). Then, because of the inequality (4.3), if (x,t,q) is in $\operatorname{im}(\tilde{t}_f) \cap \mathcal{N}_X(\frac{3}{10}\delta) \times N$, then

$$|t - t_0| \leq \delta \cdot \psi_\epsilon(t) \leq \frac{\delta}{2}t \Longrightarrow \frac{2}{2+\delta}t_0 \leq t \leq \frac{2}{2-\delta}t_0, \tag{4.4}$$

and so, since ψ_ϵ is a monotone polynomial of degree 7, we obtain

$$d_M((x_0,t_0),(x,t)) \leq \delta \cdot \psi_\epsilon(t) \leq \delta \cdot \psi_\epsilon\Big(\frac{2}{2-\delta}t_0\Big) \leq \delta \cdot \frac{2^7}{(2-\delta)^7} \cdot \psi_\epsilon(t_0).$$

This means that $\pi(\tilde{p}_f^{-1}(q))$ is contained in a ball on M of radius $\delta \cdot \frac{2^7}{(2-\delta)^7} \cdot \psi_\epsilon(t_0)$, where $t_0 = d(q, \partial N)$, and so

$$\mu_M\big(\pi(\tilde{p}_f^{-1}(q))\big) \leq J_1 \cdot \psi_\epsilon(t_0)^n.$$

Moreover, if (x, t, q) is an element of $\mathrm{im}(\tilde{t}_f) \cap \mathcal{N}_X(\frac{3}{10}\delta) \times N$, thanks to formula (4.4) we obtain

$$\frac{1}{\psi_\epsilon(t)^n} \leq \frac{1}{\psi_\epsilon(\frac{2}{2+\delta} t_0)^n} = \frac{(2+\delta)^{7n}}{2^{7n}} \cdot \frac{1}{\psi_\epsilon(t_0)^n} \leq J_2 \cdot \frac{1}{\psi_\epsilon(t_0)^n}.$$

So we conclude by observing that

$$\int_{\mathcal{N}_X(\frac{3}{10}\delta)} \mathrm{Vol}_{\tilde{t}_f}(x_0, t_0, q)\, d\mu_M \leq J_1 \cdot \phi(t_0)^n \cdot J_2 \cdot \frac{1}{\phi(t_0)^n} \leq B. \qquad \blacksquare$$

4.2 – *The pullback operator*

Fix a uniform map $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ which is isometric on the unbounded ends. Let ω be the Mathai–Quillen–Thom form of f^*TY with support contained in $f^*T^\delta Y$ defined in Section 3.3. Then we define the pullback operator as

$$T_f(\alpha) := \int_{B^\delta} \tilde{p}_f^* \alpha \wedge \omega.$$

Proposition 4.2. *Denote by $d_{\min,Y}$ and $d_{\max,Y}$ (resp. $d_{\min,X}$ and $d_{\max,X}$) be the minimal and maximal extension of the exterior derivative operator d on Y (resp. on X). The operator T_f satisfies the following properties:*

(1) *T_f is $\mathcal{L}^2$-bounded,*

(2) *$T_f(\mathrm{dom}(d_{\min,Y})) \subset \mathrm{dom}(d_{\min,X})$ and $T_f(\mathrm{dom}(d_{\max,Y})) \subset \mathrm{dom}(d_{\max,X})$,*

(3) *$T_f d_{\max,Y} = d_{\max,X} T_f$.*

Proof. Let χ_N and $\chi_{Y\setminus N}$ be the characteristic functions relative to M and to $X \setminus M$. If α is a smooth $\mathcal{L}^2$-form, then $\chi_N\alpha$ and $\chi_{Y\setminus N}\alpha$ are $\mathcal{L}^2$-forms and

$$|\alpha|^2_{\mathcal{L}^2(Y,h)} = |\chi_N\alpha|^2_{\mathcal{L}^2(Y,h)} + |\chi_{Y\setminus N}\alpha|^2_{\mathcal{L}^2(Y,h)}.$$

Observe that

$$T_f(\chi_{Y\setminus N}\alpha) = \chi_{X\setminus M} f^*\alpha$$

and so, since f on $X \setminus M$ is an isometry, we obtain

$$|T_f(\chi_{Y\setminus N}\alpha)|^2_{\mathcal{L}^2(X,g)} = |\chi_{Y\setminus N}\alpha|^2_{\mathcal{L}^2(Y,h)}.$$

Let us focus on $\chi_N\alpha$. Notice that

$$T_f(\chi_N\alpha) = \mathrm{pr}_{X\star} \circ e_\omega \circ \tilde{p}_f^*(\chi_N\alpha),$$

where $e_\omega(\beta) = \beta \wedge \omega$ and $\mathrm{pr}_{X\star}$ is the integration along the fiber of $f^*(T^\delta Y)$.

We already know that $\tilde{p}_f^*$ is an R.–N.–Lipschitz map and so this means that $\tilde{p}_f^*$ is an $\mathcal{L}^2$-bounded operator. In particular, $\tilde{p}_f^*\alpha$ is a smooth form on $f^*T^\delta Y_{|M}$ and it is null on $f^*T^\delta Y_{|X\setminus M}$. Moreover, thanks to Proposition 3.5, we also know that $\mathrm{pr}_{X\star}$ is $\mathcal{L}^2$-bounded. On the other hand, since the norm $|\omega|_{(p,w_{f(p)})}$ could be not uniformly bounded, the operator e_ω can be not $\mathcal{L}^2$-bounded. However, observe that on $\pi^{-1}(M)$ the norm $|\omega|_{(p,w_{f(p)})}$ actually is uniformly bounded since M and N are open subsets of bounded geometry. So, if we restrict $e_\omega\colon \mathcal{L}^2(\bar{M}) \to \mathcal{L}^2(\bar{M})$, we obtain an $\mathcal{L}^2$-bounded operator. Moreover, we also have that $\tilde{p}_f^*(\chi_N\alpha)$ has support contained in $\pi^{-1}(\bar{M})$ and so this means that

$$|T_f(\chi_N\alpha)|^2_{\mathcal{L}^2(X,g)} \leq C \cdot |\chi_N\alpha|^2_{\mathcal{L}^2(Y,h)}.$$

Observe that $T_f(\chi_N\alpha) = 0$ on $X \setminus M$ and $T_f(\chi_{Y\setminus N}\alpha) = 0$ on M. Then we obtain

$$\begin{aligned}
|T_f\alpha|^2_{\mathcal{L}^2(X,g)} &= |T_f\chi_N\alpha|^2_{\mathcal{L}^2(X,g)} + |T_f\chi_{Y\setminus N}\alpha|^2_{\mathcal{L}^2(X,g)} \\
&\leq K \cdot |\chi_N\alpha|^2_{\mathcal{L}^2(Y,h)} + |\chi_{Y\setminus N}\alpha|^2_{\mathcal{L}^2(Y,h)} \\
&\leq \max(K,1)(|\chi_N\alpha|^2_{\mathcal{L}^2(Y,h)} + |\chi_{Y\setminus N}\alpha|^2_{\mathcal{L}^2(Y,h)}) \\
&= \max(K,1)|\alpha|^2_{\mathcal{L}^2(Y,h)}.
\end{aligned}$$

So we have proved (1).

In order to prove points (2) and (3) we apply Lemma 2.2.

Notice that, since $\tilde{p}_f$ is a map of class C^3, the pullback along $\tilde{p}_f\colon f^*T^\delta N \to N$ of a smooth form lies in $C^2(\Lambda^*(f^*T^\delta N))$, which is the space of differential forms of class C^2 over $f^*T^\delta N$. Observe that on this space the exterior derivative of a form α is defined exactly as it is defined for smooth forms (see [8, p. 549]) and all the properties of d are the same. In particular, $\tilde{p}_f^*$ and the exterior derivative operator commute.

The operator e_ω also commutes with the exterior differential and it preserves $C^2(\Lambda^*(f^*T^\delta N))$. Finally, if the support of β is vertically compact, the integration along the fibers $\pi_\star$ of $f^*T^\delta N$ also satisfies $\pi_\star d\beta = d\pi_\star\beta$ (the proof is the same as the classical one: see for example [2, Proposition 6.14.1]) and, moreover, if β is a C^k-form, the same holds for $\pi_\star\beta$.

So we obtain that if α is a smooth form on Y, then $T_f\alpha$ is a C^2-form on X and $T_f d\alpha = dT_f\alpha$. Moreover, since $\tilde{p}_f$ is uniformly proper, if α is an element of $\Omega_c^*(N)$, then $T_f\alpha$ is compactly supported. Finally, observe that $T_f\alpha$ is smooth on $X \setminus \partial M$.

Observe that $T_f^\dagger = \tilde{p}_{f\star} \circ r_\omega \circ \pi^*$, where $\tilde{p}_{f\star}$ is the integration along the fibers of $\tilde{p}_f$ and $r_\omega(\beta) = \omega \wedge \beta$. Since $\tilde{p}_f$ is a C^3-map, then for each q in N and for each p in $f^*T^\delta N$ there are a couple of C^3-charts $(U, x^1, \ldots., x^n, y^1, \ldots, y^n)$ and $(V, y^1, \ldots, y^n)$ such that $\tilde{p}_f(x^1, \ldots., x^n, y^1, \ldots, y^n) = (y^1, \ldots, y^n)$. This implies, by a partition of unity argument, that the integral along the fibers of $\tilde{p}_f$ of a smooth form is a C^3-form. So $dT_f^\dagger \gamma$ is a compactly supported C^3-form. Then we can conclude by applying Lemma 2.2. ∎

4.3 – *The isomorphisms induced by the pullback*

Let $(X = M \cup E_X, g)$ and $(Y = N \cup E_Y, h)$ be two manifolds of bounded geometry with unbounded ends and let $f\colon (X, g) \to (Y, h)$ be a uniform homotopy equivalence isometric on the ends. In this section we introduce a couple of operators $y\colon \mathcal{L}^2(Y, h) \to \mathcal{L}^2(Y, h)$ and $z\colon \mathcal{L}^2(X, g) \to \mathcal{L}^2(X, g)$ such that $y(\mathrm{dom}(d_{\max,Y})) \subseteq \mathrm{dom}(d_{\max,Y})$, $y(\mathrm{dom}(d_{\min,Y})) \subseteq \mathrm{dom}(d_{\min,Y})$ and $z(\mathrm{dom}(d_{\max,X})) \subseteq \mathrm{dom}(d_{\max,X})$, $z(\mathrm{dom}(d_{\min,X})) \subseteq \mathrm{dom}(d_{\min,X})$ and

$$1 \pm T_f^\dagger T_f = dy + yd \quad \text{and} \quad 1 \pm T_f T_f^\dagger = dz + zd \tag{4.5}$$

on the maximal domain of the exterior derivative operator. In this formula (4.5), there is a $+$ if f reverses the orientations and a $-$ otherwise.

In order to define these operators, we need a metric on the bundle $f^*(TY) \oplus f^*(TY)$ over X. We fix the generalized Sasaki metric g_S induced by g, $h \oplus h$ and $\nabla_h^{\mathrm{LC}} \oplus \nabla_h^{\mathrm{LC}}$. In general, even if $f_1 = f_2 = f$, we will denote the bundle $f^*(TY) \oplus f^*(TY)$ by $f_1^*(TY) \oplus f_2^*(TY)$. Moreover, given $i = 1, 2$, we denote by $\mathrm{pr}_i\colon f_1^*(TY) \oplus f_2^*(TY) \to f^*(TY)$ the projection on the i-th component, i.e. $\mathrm{pr}_i(w_{f_1(p)} \oplus w_{f_2(p)}) := w_{f_i(p)}$ and so we obtain the maps $\tilde{p}_{f,i} := \tilde{p}_f \circ \mathrm{pr}_i$ and $\pi_i := \pi \circ \mathrm{pr}_i$.

Finally, we denote by

$$\mathcal{B} := \big\{ w_{f_1(p)} \oplus w_{f_2(p)} \in f_1^*(TY) \oplus f_2^*(TY) \text{ s.t. } |w_{f_1(p)}|_h \leq \delta,\ |w_{f_2(p)}|_h \leq \delta \big\}.$$

LEMMA 4.3. *Assume that $(X = M \cup E_X, g)$ and $(Y = N \cup E_Y, h)$ are two Riemannian manifolds and let $f\colon (X, g) \to (Y, h)$ be a smooth uniform map isometric on the unbounded ends of X. Then there are two $\mathcal{L}^2$-bounded operators $y_0\colon \mathcal{L}^2(Y, h) \to \mathcal{L}^2(Y, h)$ and $z_0\colon \mathcal{L}^2(X, g) \to \mathcal{L}^2(X, g)$ such that they preserve the minimal and the maximal domains of the exterior derivative operators*

$$\tilde{p}_{f,2}^* - \tilde{p}_{f,1}^* = dy_0 + y_0 d \quad \textit{and} \quad \pi_2^* - \pi_1^* = dz_0 + z_0 d.$$

PROOF. Let $\int_{0,\mathcal{L}}^1 \colon \Omega^*(\mathcal{B} \times [0,1]) \to \Omega^*(\mathcal{B})$ be the operator defined in [14, Lemma 4.13]. It is defined as follows: if α is a 0-form with respect to [0, 1], then

$$\int_{0,\mathcal{L}}^1 \alpha := \int_{0,\mathcal{L}}^1 g(x,t) p^*\omega = 0,$$

and, if α is a 1-form with respect to [0, 1],

$$\int_{0,\mathcal{L}}^1 \alpha := \left(\int_0^1 f(x,t)\, dt \right) \omega.$$

This is an $\mathcal{L}^2$-bounded operator and it sends compactly supported C^1-forms to $C_c^1(\Lambda^*(T^*\mathcal{B}))$. Moreover, if α is a C^1-form on $\mathcal{B} \times [0,1]$, then

$$j_1^*\alpha - j_0^*\alpha = d \int_{0,\mathcal{L}}^1 \alpha + \int_{0,\mathcal{L}}^1 d\alpha,$$

where $j_i \colon \mathcal{B} \to \mathcal{B} \times [0,1]$ is defined as $j_i(x) := (x,i)$.

Let ϕ_1 and ϕ_0 be two R.–N.–Lipschitz maps which are uniformly homotopic with a uniformly proper, R.–N.–Lipschitz homotopy $H \colon (\mathcal{B} \times [0,1], g_S + dt) \to (Y,h)$. Then, as a consequence of Lemma 2.2, the operator $\int_{0,\mathcal{L}}^1 \circ H$ is an $\mathcal{L}^2$-bounded operator which preserves the minimal and the maximal domains of the exterior derivative operator and, on the maximal domain,

$$\phi_1^* - \phi_0^* = d\left(\int_{0,\mathcal{L}}^1 \circ H \right) + \left(\int_{0,\mathcal{L}}^1 \circ H \right) d.$$

So, in order to conclude the proof, it is sufficient to find a couple of uniformly proper, R.–N.–Lipschitz homotopies.

Let $A \colon (\mathcal{B} \times [0,1], g_S + dt) \to (f^*T^\delta Y, g_f)$ be defined as

$$A(w_{f(p)} \oplus v_{f(p)}, s) := s \cdot w_{f(p)} + (1-s) v_{f(p)}.$$

It is an easy exercise to prove that A is a uniformly proper R.–N.–Lipschitz map: a proof of this fact can also be found in the work of the author [15]. Then we can conclude by observing that $\tilde{p}_f \circ A$ is a homotopy between $\tilde{p}_{f_1}$ and $\tilde{p}_{f_2}$ and $\pi \circ A$ is a homotopy between π_1 and π_2. ∎

The proof of the following lemma is very similar to that given in [15]; however, for the sake of completeness, it is proved here.

LEMMA 4.4. *Let $(Y = N \cup E_Y, g)$ be a manifold of bounded geometry with some possibly unbounded ends. Fix $\tilde{p}_{\mathrm{id}} \colon T^\delta Y \to Y$ and let ω be a Thom form of the bundle*

$\pi\colon TY \to Y$, where $\pi(v_p) = p$, such that $\mathrm{supp}(\omega) \subset T^\delta Y$*. Then for all q in N,*

$$\int_{F_q} \omega = 1,$$

where F_q is the fiber of $\tilde{p}_{\mathrm{id}}$.

Proof. For all q in N the fiber F_q is an oriented compact submanifold with boundary. The same also holds for B^δ_q, which is the fiber of the projection $\pi\colon T^\delta N \to N$ defined as $\pi(v_q) := q$.

Define $H\colon T^\delta N \times [0,1] \to N$ as

$$H(v_p, s) = \tilde{p}_{\mathrm{id}}(s \cdot v_p).$$

Since H is a proper submersion, the fiber along H given by $F_{H,q}$ is a submanifold of $T^\delta N \times [0,1]$. Its boundary, in particular, is

$$\partial F_{H,q} = B^\delta_q \times \{0\} \sqcup F_q \times \{1\} \cup A,$$

where A is contained in

$$S^\delta N := \{v_p \in TN \mid |v_p| = \delta\}.$$

If ω is a Thom form of TN whose support is contained in $T^\delta N$, then

$$0 = \int_{F_{H_q}} d\omega = d\int_{F_{H_q}} \omega + \int_{\partial F_{H_q}} \omega. \tag{4.6}$$

Observe that ω is a k-form and $\dim(F_{H_q}) = k+1$. Then the first integral on the right-hand side of equality (4.6) is 0. Moreover, we obtain that ω is null on A, and so

$$\int_A \omega = 0.$$

Then, by equality (4.6),

$$0 = \mp\int_{B^\delta} \omega \pm \int_{F_q} \omega,$$

and we conclude. ∎

Lemma 4.5. *Let $(X = M \cup E_X, g)$ and $(Y = N \cup E_Y, h)$ be two manifolds of bounded geometry with unbounded ends and let $f\colon (X,g) \to (Y,h)$ be a uniform homotopy equivalence isometric on the unbounded ends. Then there are a couple* *of operators* $y\colon \mathcal{L}^2(Y,h) \to \mathcal{L}^2(Y,h)$ *and* $z\colon \mathcal{L}^2(X,g) \to \mathcal{L}^2(X,g)$ *such that they preserve the minimal and maximal domains of the exterior derivative operators and*

$$1 \pm T_f^\dagger T_f = dy + yd \quad \textit{and} \quad 1 \pm T_f T_f^\dagger = dz + zd$$

on the maximal domain of the exterior derivative operator. In this formula, there is a $+$ *if* f *reverses the orientations and* $-$ *otherwise.*

Proof. First we focus on z. We have the diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\widetilde{P}_{f,1}} & f_2^*(T^\delta Y) \\ \big\downarrow \widetilde{P}_{f,2} & & \big\downarrow \tilde{p}_f \\ f_1^*(T^\delta Y) & \xrightarrow{\tilde{p}_f} & Y, \end{array}$$

where $\widetilde{P}_{f,1}$ and $\widetilde{P}_{f,2}$ are the bundle maps induced by $\tilde{p}_f$. Moreover, we also have

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\Pi_1} & f_2^*(T^\delta Y) \\ \big\downarrow \Pi_2 & & \big\downarrow \pi \\ f_1^*(T^\delta Y) & \xrightarrow{\pi} & X, \end{array}$$

where Π_1 and Π_2 are the bundle maps induced by the projection of the bundle $\pi\colon f^*T^\delta Y \to X$.

Denote $\omega_i := \Pi_i^*\omega$, where ω is the Mathai–Quillen–Thom form of $f^*T^\delta Y$.

Fix α and β in $\Omega_c^k(X)$ for some k in $\mathbb{N}$. Then

$$\begin{aligned}
\langle T_f T_f^\dagger \alpha; \beta\rangle_{\mathcal{L}^2(X,g)} &= \langle T_f^\dagger \alpha; \tau T_f^\dagger \tau\beta\rangle_{\mathcal{L}^2(Y,h)} \\
&= \int_N \left(\int_{F_2} \omega_2 \wedge \pi_2^*\alpha\right) \wedge \left(\int_{F_1} \omega_1 \wedge \pi_1^* \star \beta\right) \\
&= \deg(f) \int_{f_1^*TN} \left(\int_{F_2} \omega_2 \wedge \pi_2^*\alpha\right) \wedge \omega_1 \wedge \pi_1^* \star \beta \\
&= \deg(f)(-1)^{j(n+j)} \int_{f_1^*TN} \omega_1 \wedge \pi_1^* \star \beta \wedge \left(\int_{F_2} \omega_2 \wedge \pi_2^*\alpha\right) \\
&= \deg(f)(-1)^{j(n+j)} \int_{\mathcal{B}} \omega_1 \wedge \pi_1^* \star \beta \wedge \omega_2 \wedge \pi_2^*\alpha \\
&= \deg(f) \int_{\mathcal{B}} \pi_2^*\alpha \wedge \pi_1^* \star \beta \wedge \omega_1 \wedge \omega_2.
\end{aligned}$$

Then, as a consequence of $\pi_2^* = \pi_1^* + dz_0 + z_0 d$, we obtain

$$\begin{aligned}\langle T_f T_f^\dagger \alpha; \beta\rangle_{\mathcal{L}^2(X,g)} &= \deg(f)\int_{\mathcal{B}} \pi_1^*\alpha \wedge \pi_1^* \star \beta \wedge \omega_1 \wedge \omega_2 \\ &\quad + \deg(f)\int_{\mathcal{B}} (dz_0 + z_0 d)\alpha \wedge \pi_1^* \star \beta \wedge \omega_1 \wedge \omega_2 \\ &= \langle \deg(f)\cdot 1\alpha; \beta\rangle_{\mathcal{L}^2(X,g)} + \langle [d(-z) + (-z)d]\alpha; \beta\rangle_{\mathcal{L}^2(X,g)},\end{aligned}$$

where $z := -\deg(f)\cdot \pi_\star \circ e_\omega \circ \Pi_{2\star} \circ e_{\Pi_2^*\omega} \circ z_0$.

Observe that z is a composition of $\mathcal{L}^2$-bounded operator and $z(\Omega_c^*(X)) \subseteq C_c^2(\Lambda^*(T^*X))$. Moreover,

$$z^\dagger = -\deg(f) z_0^\dagger \circ r_{\Pi_2^*\omega} \circ \Pi_2^* \circ r_\omega \circ \pi^*$$

is an $\mathcal{L}^2$-bounded operators and $-z^\dagger(\Omega_c^*(Y)) \subseteq C_c^2(\Lambda^*(T^*Y))$. So by Lemma 2.2, we know that z preserves the minimal and the maximal domains of d and

$$1 \pm T_f T_f^\dagger = dz + zd,$$

where we have a $-$ if f preserves the orientations and a $+$ otherwise.

Let us focus on the operator y. Given α and β in $\Omega_c^*(Y)$, we obtain

$$\begin{aligned}\langle T_f^\dagger T_f \alpha, \beta\rangle_{\mathcal{L}^2(Y,h)} &= \langle T_f\alpha, \tau T_f \tau\beta\rangle_{\mathcal{L}^2(X,g)} \\ &= \int_M \left(\int_{B_2^\delta} \tilde{p}_{f,2}^*\alpha \wedge \omega_2\right) \wedge \left(\int_{B_1^\delta} \tilde{p}_{f,1}^* \star \beta \wedge \omega_1\right) \\ &= \int_{f^*(T^\delta Y)_1} (\int_{B_2^\delta} \tilde{p}_{f,2}^*\alpha \wedge \omega_2) \wedge \tilde{p}_{f,1}^* \star \beta \wedge \omega_1 \\ &= (-1)^{(n+j)j} \int_{f^*(T^\delta Y)_1} \tilde{p}_{f,1}^* \star \beta \wedge \omega_1 \wedge \left(\int_{B_2^\delta} \tilde{p}_{f,2}^*\alpha \wedge \omega_2\right) \\ &= (-1)^{(n+j)j} \int_{\mathcal{B}} \tilde{p}_{f,1}^* \star \beta \wedge \omega_1 \wedge \tilde{p}_{f,2}^*\alpha \wedge \omega_2 \\ &= (-1)^{(n+j)j}(-1)^{(n+j)j} \int_{\mathcal{B}} \tilde{p}_{f,2}^*\alpha \wedge \tilde{p}_{f,1}^* \star \beta \wedge \omega_1 \wedge \omega_2 \\ &= \int_{\mathcal{B}} \tilde{p}_{f,2}^*\alpha \wedge \tilde{p}_{f,1}^* \star \beta \wedge \omega_1 \wedge \omega_2.\end{aligned}$$

Then, thanks to Lemma 4.3, we obtain

$$\begin{aligned}\langle T_f^\dagger T_f \alpha, \beta\rangle_{\mathcal{L}^2(Y,h)} &= \int_{\mathcal{B}} \tilde{p}_{f,1}^*(\alpha \wedge \star\beta) \wedge \omega_1 \wedge \omega_2 \\ &\quad + \int_{\mathcal{B}} (dy_0 + y_0 d)\alpha \wedge \tilde{p}_{f,1}^* \star \beta \wedge \omega_1 \wedge \omega_2\end{aligned}$$

$$
\begin{aligned}
&= \deg(f) \int_{TN} \tilde{p_{\mathrm{id}}}^*(\alpha \wedge \star\beta) \wedge \omega \\
&\quad + \langle (dy + yd)\alpha, \beta\rangle_{\mathcal{L}^2(Y,h)} \\
&= \langle \deg(f) \cdot 1(\alpha); \beta\rangle_{\mathcal{L}^2(Y,h)} + \langle (dy + yd)\alpha, \beta\rangle_{\mathcal{L}^2(Y,h)},
\end{aligned}
$$

where $y := p_{f,\star} \circ e_\omega \circ \Pi_{1,\star} \circ e_{\Pi_1^*\omega} \circ y_0$. Notice that y is a composition of $\mathcal{L}^2$-bounded operators and $y(\Omega_c^*(Y)) \subseteq C_c^2(\Lambda^*(T^*Y))$. Moreover,

$$y^\dagger = y_0^\dagger \circ r_{\Pi_1^*\omega} \circ \Pi_1^* \circ r_\omega \circ p_f^*,$$

and so $y^\dagger(\Omega_c^*(X)) \subseteq C_c^2(\Lambda^*(T^*X))$. So, as a consequence of Lemma 2.2, we obtain that y preserves the maximal and the minimal domains of d and

$$1 \pm T_f^\dagger T_f = dy + yd. \qquad \blacksquare$$

THEOREM 4.6. *Let $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a uniform homotopy equivalence quasi-isometric on the unbounded ends. Then, for each k in $\mathbb{N}$, the operators T_f and $T_f^\dagger$ induce the following isomorphisms:*

$$
\begin{aligned}
H^k_{2,\max}(X, g) &\cong H^k_{2,\max}(Y, h), \\
\bar{H}^k_{2,\max}(X, g) &\cong \bar{H}^k_{2,\max}(Y, h), \\
H^k_{2,\min}(X, g) &\cong H^k_{2,\min}(Y, h), \\
\bar{H}^k_{2,\min}(X, g) &\cong \bar{H}^k_{2,\min}(Y, h).
\end{aligned}
$$

PROOF. The proof immediately follows by Lemma 4.5. ■

5. Consequences

5.1 – *A mapping cone for the L^2-cohomology*

In this subsection, given a Riemannian manifold (X, g), we will denote $\Omega^*_{m\setminus\mathcal{M}}(X, g) := \mathrm{dom}(d_{\min\setminus\max,X})$ and $d_{X,m\setminus\mathcal{M}} := d_{\min\setminus\max,X}$.

Let $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a uniform map quasi-isometric on the unbounded ends. It is not required that f is a uniform homotopy equivalence. Thanks to Proposition 4.2, we know that there is an $\mathcal{L}^2$-bounded operator $T_f\colon \mathcal{L}^2(N, h) \to \mathcal{L}^2(M, g)$ such that $T_f(\Omega^*_{m\setminus\mathcal{M}}(M, g)) \subseteq \Omega^*_{m\setminus\mathcal{M}}(N, h)$. In this subsection we will define the L^2-mapping cone of a map f and we will see some properties of this cone.

Definition 5.1. Let $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a uniform map quasi-isometric on the unbounded ends between two Riemannian manifold and let us denote by $(\Omega^*_{2,\min\backslash\max}(f), d_{f,\min\backslash\max})$ the cochain complexes

$$0 \longrightarrow \Omega^0_{2,m\backslash\mathcal{M}}(f) \xrightarrow{d_{f,m\backslash\mathcal{M},0}} \Omega^1_{2,m\backslash\mathcal{M}}(f) \xrightarrow{d_{f,m\backslash\mathcal{M},1}} \Omega^3_{2,m\backslash\mathcal{M}}(f) \xrightarrow{d_{f,m\backslash\mathcal{M},2}} \cdots,$$

where $\Omega^*_{2,m\backslash\mathcal{M}}(f) := \Omega^*_{m\backslash\mathcal{M}}(Y,h) \oplus \Omega^{*-1}_{m\backslash\mathcal{M}}(X,g)$ and $d_{f,m\backslash\mathcal{M}}(\alpha,\beta) := (-d_{m\backslash\mathcal{M},Y}\alpha; T_f\alpha - d_{m\backslash\mathcal{M},X}\beta)$.

Remark 15. Notice that, since T_f and the exterior derivative operator commute, then $d^2_{f,\min\backslash\max} = 0$.

Definition 5.2. The *k-th group of the L^2-mapping cone of f* is the cohomology group of the L^2-mapping cone, i.e.

$$H^k_{2,m\backslash\mathcal{M}}(f) := \frac{\ker(d_{f,m\backslash\mathcal{M},k})}{\operatorname{im}(d_{f,m\backslash\mathcal{M},k})}.$$

The *reduced k-th group of the L^2-mapping cone of f* is the group defined as

$$\overline{H}^k_{2,m\backslash\mathcal{M}}(f) := \frac{\ker(d_{f,m\backslash\mathcal{M},k})}{\overline{\operatorname{im}(d_{f,m\backslash\mathcal{M},k})}}.$$

Exactly as the mapping cone in the de Rham case, we have a short exact sequence

$$0 \to \Omega^{*-1}_{2,m\backslash\mathcal{M}}(X,g) \xrightarrow{A} \Omega^*_{2,m\backslash\mathcal{M}}(f) \xrightarrow{B} \Omega^*_{2,m\backslash\mathcal{M}}(Y,h) \to 0,$$

where $A(\omega) := (0,\omega)$ and $B(\alpha,\omega) := \alpha$. This sequence induces a long exact sequence on cohomology, and so

$$0 \to H^0_{2,m\backslash\mathcal{M}}(f) \to H^0_{2,m\backslash\mathcal{M}}(X,g) \xrightarrow{\delta} H^0_{2,m\backslash\mathcal{M}}(Y,h) \to H^1_{2,m\backslash\mathcal{M}}(f) \to \cdots,$$

where δ is the connecting homomorphism.

Following the proof given in Bott and Tu [2, p. 78], we obtain that $\delta[\omega] := [T_f\omega]$. So we obtain the following proposition.

Proposition 5.1. *Let $f\colon (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a uniform map quasi-isometric on the unbounded ends of M and N. Then the following two statements are equivalent:*

(1) *the morphism induced by T_f on L^2-cohomology is an isomorphism,*

(2) *all the cohomology groups of $\Omega^*_{2,m\backslash\mathcal{M}}(f)$ are null.*

Proof. This is a classical proof which holds for each cochain morphism $T_f: A \to B$ between cochain complexes on an additive category. ■

Remark 16. The author used a *bounded geometry version* of Proposition 5.1 in [15] in order to prove the invariance of the Roe index of the signature operator of a manifold of bounded geometry under uniform homotopy equivalences which preserve the orientations. As a consequence of Proposition 5.1, it could be compelling to generalize this result to a broader context, for example in the case of complete Riemannian manifolds with uniform homotopy equivalence which are quasi-isometric on the unbounded ends.

In the reduced case we also obtain a long sequence

$$0 \to \bar{H}^0_{2,m\setminus\mathcal{M}}(f) \xrightarrow{B^*} \bar{H}^0_{2,m\setminus\mathcal{M}}(M,g) \xrightarrow{T_f} \bar{H}^0_{2,m\setminus\mathcal{M}}(N,h) \xrightarrow{A^*} \bar{H}^1_{2,m\setminus\mathcal{M}}(f) \to \cdots,$$

but it is not exact this time. Indeed it is exact only on $\bar{H}^k_{2,m\setminus\mathcal{M}}(f)$ and on $\bar{H}^k_{2,m\setminus\mathcal{M}}(M,g)$, while on $\bar{H}^k_{2,m\setminus\mathcal{M}}(N,h)$ it is just *weakly exact*, which means that $\ker(T_f) = \overline{\operatorname{im}(B^*)}$.

5.2 – *Uniform homotopy invariance of signature*

Let (X,g) be a complete Riemannian manifold. Recall that in this case there is exactly one closure for the operator d. This means that the maximal and minimal L^2-cohomology groups coincide, both the reduced and the unreduced ones. In the next pages we will denote the unique closed extension of the exterior derivative operator by d.

In this subsection we introduce the L^2-signature σ_M of a manifold (X,g) with $\dim(X) = 4k$ such that $\bar{H}^{2k}_2(X,g)$ is finite-dimensional. This L^2-signature is the signature of a pairing defined on $\bar{H}^{2k}_2(X,g)$. Consider the operator $d + d^*$. This operator switches the eigenspaces of the chiral operator τ, so it is possible to define the L^2-signature operator $(d + d^*)^+$ as the restriction of $d + d^*$ to the $+1$-eigenspace of τ.

We obtain that $(d + d^*)^+$ is a Fredholm operator and its index equals the L^2-signature. The definitions and the proofs of all these facts can be found in the work of Bei [1].

Proposition 5.2. *Let $f: (X = M \cup E_X, g) \to (Y = N \cup E_Y, h)$ be a uniform homotopy equivalence quasi-isometric on the unbounded ends. Assume that (X,g) and (Y,h) are complete Riemannian manifolds:*

(1) *The operator T_f is well behaved with respect to the pairings on M and N, i.e. $\langle T_f[\alpha], T_f[\beta]\rangle_{\mathcal{L}^2(X,g)} = \deg(f) \cdot \langle[\alpha],[\beta]\rangle_{\mathcal{L}^2(Y,h)}$.*

(2) $\sigma_M = \deg(f)\cdot\sigma_N$. *This implies that the index of the L^2-signature operator is invariant under uniform homotopy equivalences quasi-isometric on the unbounded ends which preserve the orientations.*

Proof. In order to prove the first point, we need to introduce the pairing $\langle\cdot;\cdot\rangle_{\mathcal{L}^2(X,g)}$. This is defined in [1] as

$$\langle\cdot,\cdot\rangle_{\mathcal{L}^2(X,g)}\colon \overline{H}^i_2(X,g)\times\overline{H}^i_2(X,g)\longrightarrow\mathbb{R},$$
$$([\eta],[\omega])\longrightarrow\int_X\eta\wedge\omega.$$

Let id be the identity map on Y and let $\tilde{p}_{\mathrm{id}}\colon T^\delta Y\to Y$ be the submersion related to id. Because of Lemma 4.5, we know that this is an R.–N.–Lipschitz map and so T_{id} is an L^2-bounded operator. Let us denote $K := \int_{0\mathcal{L}}^1\circ\tilde{p}_h^*$, where $\tilde{p}_h\colon T^\delta Y\times[0,1]\to Y$ is defined as $\tilde{p}_h(w_p,s) := \tilde{p}_{\mathrm{id}}(s\cdot w_p)$. By applying the same proof as [14, Lemma 4.11], we obtain that $\tilde{p}_h$ is an R.–N.–Lipschitz map and so K is an L^2-bounded operator such that

$$T_{\mathrm{id}}-\mathrm{Id} = dK + Kd$$

for each smooth form α. Then, as a consequence of Lemma 4.5, we obtain that T_{id} and K both preserve the domain of d.

Observe that the operator $T_f = f^*\circ T_{\mathrm{id}}$: this directly follows by the definition of T_f and by $\int_{F_0'}F^*\alpha = f^*\int_{F_0}\alpha$. Here, F^* is the bundle map induced by f, and F_0 and F_0' are the fibers of TY and of f^*TY. Observe that $T_{\mathrm{id}}\alpha$ is a smooth form on $Y\setminus\partial N$ and so $f^*T_{\mathrm{id}}\alpha$ is smooth on $X\setminus\partial M$.

Then we can easily conclude as follows:

$$\begin{aligned}
\langle[T_f\alpha],[T_f\beta]\rangle_{\mathcal{L}^2(X,g)} &= \int_X T_f\alpha\wedge T_f\beta\\
&= \int_X f^*(T_{\mathrm{id}}\alpha)\wedge f^*(T_{\mathrm{id}}\beta) = \int_X f^*(T_{\mathrm{id}}\alpha\wedge T_{\mathrm{id}}\beta)\\
&= \deg(f)\cdot\int_Y T_{\mathrm{id}}\alpha\wedge T_{\mathrm{id}}\beta\\
&= \deg(f)\int_Y(\alpha+d\eta)\wedge(\beta+d\nu)\\
&= \deg(f)\int_Y\alpha\wedge\beta+\deg(f)\int_N d(\alpha\wedge\nu)\\
&\quad+\deg(f)\int_Y d(\eta\wedge\beta)+\deg(f)\int_Y d(\eta\wedge d\nu)\\
&= \deg(f)\int_Y\alpha\wedge\beta+0 = \deg(f)\langle[\alpha],[\beta]\rangle_{\mathcal{L}^2(Y,h)}.
\end{aligned}$$

We have proved point (1). The proof of point (2) is a direct consequence of point (1) and [1, Theorem 4.2]. ■

5.3 – *Compact manifolds with unbounded ends*

Notice that Theorem 4.6 is coherent with Lott in [9, Proposition 5]. In Lott's work, it is proved that if two complete Riemannian manifolds (X, g) and (Y, h) are isometric outside a compact set, then, for each k in $\mathbb{N}$,

$$\dim(\bar{H}_2^k(X, g)) = +\infty \iff \dim(\bar{H}_2^k(Y, h)) = +\infty.$$

Then, as a consequence of Theorem 4.6, we can say something more if we add some assumptions on f on the compact subsets. On the other hand, we are not assuming the completeness of (X, g) and (Y, h) and we also relax the assumptions on f on the ends.

Corollary 5.3. *Let (X, g) and (Y, h) be two (possibly not complete) oriented Riemannian manifolds. Let K be a compact subset of M. We denote $K' := f(K)$. Assume the existence of a homotopy equivalence $f: (X, g) \to (Y, h)$ such that $f_{|X \setminus K}: (X \setminus K, g) \to (Y \setminus K', h)$ is a quasi-isometry. Then both the minimal and maximal L^2-cohomology groups are isomorphic. The same also happens for the reduced L^2-cohomology groups.*

Proof. Thanks to Proposition 1.5 we can assume that $f_{|K}$ is an isometry. Fix a number $r > 0$ and denote by $B_r(K)$ (resp. $B_r(K')$) the subset of the points whose distance from K (resp. K') is less than r. Let δ be a number small enough such that $B_\delta(K)$ and $B_\delta(K')$ are two tubular neighborhoods of K and K'. Then $B_\delta(K)$ and $B_\delta(K')$ are two open subsets of bounded geometry and f is a uniform homotopy equivalence isometric on the unbounded ends. Then we conclude by applying Theorem 4.6. ■

Example 5.1. Let X be the three punctured sphere $S^2 \setminus \{p_1, p_2, p_3\}$ and let Y be a punctured torus $S^1 \times S^1 \setminus \{q\}$. It is a well-known fact X and Y are homotopy equivalent even if they are not homeomorphic.

In particular, there exists a homotopy equivalence $f: X \to Y$ such that, given some neighborhoods U_{p_i} and $U(q)$ of p_i and of q respectively and given θ in $(0, 2\pi)$,

$$\begin{cases} f(U_{p_1}) = U(q), \\ f(U_{p_2}) = (0, \theta) \times S^1, \\ f(U_{p_3}) = S^1 \times (0, \theta). \end{cases}$$

Denote $K = X \setminus [\bigsqcup_{i=1,2,3} U_{p_i}]$ and $K' = Y \setminus [U(q) \cup (0,\theta) \times S^1 \cup S^1 \times (0,\theta)]$. We can also assume that $f_{|U_{p_i}}$ is a local diffeomorphism and that $f(K) \subseteq K'$.

Let g and h be two Riemannian metrics on X and Y such that, for each $i = 1,2,3$, we have that f^*h and g are quasi-isometric on U_{p_i}. For example, we can fix $f^*h = g$ since f is a local diffeomorphism on U_{p_i}.

Observe that, since Y is not compact, the L^2-cohomology groups do depend on the choice of the metric h around q.

Thanks to Corollary 5.3, we know that the maximal and minimal cohomology groups of (X, g) and (Y, h) are isomorphic, in both their reduced and unreduced versions.

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