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Global solutions for time-space fractional fully parabolic Keller–Segel system

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Abstract. We show the existence of a global solution to time-space fractional fully parabolic Keller–Segel system:

$$\begin{cases} {}_0^cD_t^\beta u + (-\Delta)^{\alpha/2}u + \nabla\cdot(u\nabla v) = 0, & x\in\mathbb{R}^n,\ t>0,\\ {}_0^cD_t^\beta v + (-\Delta)^{\alpha/2}v - u = 0, & x\in\mathbb{R}^n,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\mathbb{R}^n, \end{cases}$$

under the smallness condition on the initial data, where $0<\beta<1$, $1<\alpha\le 2$ and $n\ge 2$, u and v denote the cell density and the concentration of the chemoattractant, respectively, and ${}_0^cD_t^\beta$ denotes the Caputo fractional derivative of order β with respect to time t. The nonlocal operator $(-\Delta)^{\alpha/2}$, defined with respect to the space variable x, is known as the Laplacian of order $\frac{\alpha}{2}$. We establish the existence of weak solution to the above system by fixed-point arguments under suitable conditions on u_0 and v_0.

1. Introduction

In recent years, chemotaxis has gained significant interest due to its important role in various biological phenomena; see, for instance, [1,6,14,32,33,36,37]. The mathematical analysis on chemotaxis models has provided a foundation for much of this work. Because of its natural simplicity, analytical tractability, and extent to replicate key behavior of chemotactic populations, the applications of this model have produced a huge literature on fascinating problems on the global existence of solutions, blow-up, and asymptotic behavior of solutions.

The theoretical and mathematical modeling of chemotaxis originates from the pioneering works of Patlak in 1950s [45] and Keller and Segel in 1970s [32]. Let us recall the earlier works on the Keller–Segel systems to motivate and demonstrate our results in the right perspective. The very first mathematical model of chemotaxis given by

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v),\\ v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

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is well known as Keller–Segel (K-S) system; see [32]. This system describes the chemotactic interaction between amoebae as considered in [38], where u is the unknown density and v is the signal concentration. Model (1.1) is well studied for the global boundedness and blow-up criterion of positive solutions in radial domains. For a quick review of the developments on Keller–Segel systems, we refer to [3, 25, 26]. There are numerous chemotaxis models, which can be described in a more general form as follows:

$$\begin{cases} u_t = \nabla\cdot(S(u,v,|\nabla u|)\nabla u) - \chi\nabla\cdot(uD(u,v,|\nabla v|)\nabla v), \\ \tau v_t = a\Delta v + k(u,v) - h(u,v)v, \end{cases} \tag{1.2}$$

where u denotes the density of cells in a domain and v represents the concentration of chemical signal, $S(u,v,|\nabla u|)$ is the mobility function describing the diffusivity of cells, and $D(u,v,|\nabla v|)$ is called the chemotactic sensitivity. The kinetic functions k and h act for the generation and degradation of a chemical signal, respectively, and $\tau\in\{0,1\}$.

We refer to [19, 20, 57] for the Keller–Segel models with cross-diffusion term depending on a function of u, $-\nabla\cdot(u\phi(u)\nabla v)$ and the chemotaxis systems with corresponding parabolic equation given by

$$u_t = \nabla\cdot(S(u)\nabla u) - \chi\nabla\cdot(uD(u)\nabla v).$$

We refer to [23, 56], where, for certain choices of $S(u)$ and $D(u)$, the existence of global and bounded solution has been established. There is substantial research on chemotaxis models with nonlinear signal production; see [22, 51, 61, 62], where the second equation of (1.2) is considered as

$$\tau v_t = \Delta v - v + k(u), \quad \tau\in\{0,1\}.$$

The Keller–Segel systems with gradient dependent chemotactic coefficients are investigated in many papers; see, for instance, [5, 7, 8, 30, 44, 53, 58].

There also have been developments on the existence and qualitative behavior of solutions to the following systems in $\mathbb{R}^n$:

$$\begin{cases} u_t = \Delta u - \nabla\cdot\big(\chi(u,v)\nabla v\big), & x\in\mathbb{R}^n,\ t>0, \\ \tau v_t = \Delta v - \gamma v + \beta u, & x\in\mathbb{R}^n,\ t>0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{1.3}$$

Let $\gamma=\beta=1$ and $\chi(u,v)=u$ in (1.3); then, the global strong solution to the corresponding system in $\mathbb{R}^n$, $n\ge 3$, has been proved in [38] with suitable smallness conditions on the initial data $u_0\in L_w^{n/2}(\mathbb{R}^n)$ and $v_0\in BMO$. Authors have employed the method built on the perturbation of linearization along with the L^p-L^q estimates of the heat semigroup. They have also discussed the stability of solution of (1.3). Kozono and Sugiyama [39] considered the Keller–Segel system (1.3) with $\tau=0$, $\beta=1$, and $\chi(u,v)=u$ in dimension 2. They proved the existence of a mild solution to the system for every $u_0\in L^1(\mathbb{R}^2)$. They also established the finite time blow-up of strong solutions under the assumption

$\|u_0\|_{L^1} > 8\pi$ and $\|x^2u_0\|_{L^1} < \frac{1}{\gamma} \cdot g(\frac{\|u_0\|_{L^1}}{8\pi})$, where $g(s)$ is an increasing function of $s > 1$.

Next, we also mention a few recent works which deal with the existence of solutions to the following more general class of Keller–Segel system:

$$\begin{cases} u_t = \nabla \cdot \big(\nabla u^m - \chi(u,v)\nabla v\big), & x \in \mathbb{R}^n,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

The above Keller–Segel system with power-type nonlinearity has been studied by many authors. For instance, Sugiyama and Kunii [50] established the L^r-decay property, $1 \le r < \infty$, of solutions to (1.4) with $\tau = 0$ and $\chi(u,v) = u^{q-1}$, when $q \ge m + \frac{2}{n}$. The L^∞-decay property of the same system has been obtained in [27]. Ishida and Yokota [28] studied the global existence of weak solutions to (1.4) with $\chi(u,v) = u^{q-1}$ under the condition $q < m + \frac{2}{n}$. They have also showed the global existence of weak solution to the same system with small initial data in [29]. Sugiyama [49] proved the global existence of a weak solution to (1.4) with $\chi(u,v) = u$, if either $m \ge 2$ for large initial data or $1 < m \le 2 - \frac{2}{n}$ for small initial data. She also discussed the decay properties of the solution when the initial data is small. Antonio Carrillo and Lin [18] studied the global existence and blow-up of weak solutions to the following degenerate chemotaxis model with two species and two stimuli in dimension $n \ge 3$:

$$\begin{cases} u_t = \nabla u^{m_1} - \nabla \cdot (u\nabla v) & x \in \mathbb{R}^n,\ t > 0, \\ -\nabla v = w, & x \in \mathbb{R}^n,\ t > 0, \\ w_t = \nabla w^{m_2} - \nabla \cdot (w\nabla z), & x \in \mathbb{R}^n,\ t > 0, \\ -\nabla z = u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $m_1, m_2 > 1$ denote the constants. They demonstrated that the qualitative behavior of the solutions is determined by critical curves. More precisely, they have obtained two critical curves intersecting at one point which separate the global existence and blow-up of weak solutions to the problem. Ulusoy [54] studied the existence and blow-up of solutions to the gradient flow problems in higher dimensions. He established the existence of a critical value of a parameter in the equation below which there is a global-in-time energy solution and above which there exist blowing-up energy solutions. We also refer to [17] for further properties of gradient flow problems.

There have been considerable efforts to study fractional Keller–Segel systems. Fractional derivative in time is used to model complex behaviors, like particle sticking and trapping phenomena. These phenomena involve intricate interactions over time, where traditional integer-order derivatives may not capture all the nuances. Fractional derivative in space is used to model situations where particles undergo long jumps or exhibit anomalous diffusion. Such behavior is often observed in systems with underlying complexity, such as disordered media or environments with obstacles. As it is well known that the behavior of most biological systems has memory and aftereffects; therefore understanding the behavior of these systems with memory effects is crucial for improving the accuracy of mathematical models in describing real-world phenomena. Because of these novel characteristics, the biological systems with fractional derivatives have become more captivating in recent years. We point out that there are very few works dealing with time-fractional Keller–Segel systems. See, for instance, [4], where Azvedo et al. investigated the global existence of solutions to fractional Keller–Segel system in $\mathbb{R}^n$, $n \geq 2$. They assumed that the initial data is small enough and belongs to a class of Besov–Morrey spaces, that is, $u_0 \in \mathcal{N}^{-b}_{r,\lambda,\infty}$, $v_0 \in \dot{B}^0_{\infty,\infty}$. They used the iteration method to obtain the self-similar solutions of the system. Cuevas et al. [21] focused on the well-posedness of solutions to the same system considering specific initial conditions $u_0 \in L^N \cap L^{\frac{N}{2}} \cap L^\infty$, $N \geq 2$, and $v_0 \in \dot{B}^0_{\infty,\infty}$. They also explored the asymptotic behavior and regularity properties of the solutions in suitable Lebesgue spaces.

In the context of time-fractional partial differential equations (PDEs), to demonstrate and apply the compactness theorem, Li and Liu [42] examined the following system:

$$\begin{cases} {}^c_t D^\alpha u + \nabla \cdot (u \nabla v) = \Delta u, & x \in \mathbb{R}^2,\ t > 0, \\ -\Delta v = u, & x \in \mathbb{R}^2,\ t > 0, \end{cases}$$

and proved the existence and uniqueness of weak solutions using mollifiers and iteration method. In this case, the authors also assumed that $u_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and nonnegative. The authors in [2] proved the existence of nonnegative solution to time-fractional Keller–Segel system

$$\begin{cases} {}^c_t D^\alpha u = d_1 \Delta u - \nabla \cdot \big(\chi(u,v) \nabla v\big), & x \in \Omega,\ t > 0, \\ {}^c_t D^\alpha v = d_2 \Delta v - \gamma v + \beta u, & x \in \Omega,\ t > 0, \end{cases}$$

with Dirichlet boundary conditions by using Galerkin's approximation technique. They also discussed the existence of solutions to the system with Neumann boundary conditions. Zhang et al. [60] considered a fractional parabolic-elliptic Keller–Segel system

$$\begin{cases} u_t + (-\Delta)^s u + \chi \nabla \cdot (u \nabla v) = u(a - bu), & t > 0,\ x \in \mathbb{R}^n, \\ 0 = (\Delta - I) v + u, & t > 0,\ x \in \mathbb{R}^n, \end{cases}$$

with a logistic source on $\mathbb{R}^n$ and $s \in (0,1)$. They established the regularity of weak solutions to the system. They also proved the existence and uniqueness of classical solutions by semigroup method. With different choices of $B(u)$, many authors discussed the existence and related qualitative questions to the following system:

$$\begin{cases} u_t = -(-\Delta)^{s/2} u - \chi \nabla \cdot (u B(u)) + f(u), \\ \tau v_t = \Delta v + g(u,v). \end{cases} \tag{1.5}$$

For instance, the local existence and uniqueness of solutions to (1.5) with $\tau = 0$, $B(u) = \nabla \cdot (\Delta^{-1} u)$, and $f(u) = 0$ have been studied in [40] under $u_0 \in L^p(\mathbb{R}^2) \cap H^m(\mathbb{R}^2)$ for some p with $1 < p < 2$ and $m > 3$. In [11], the conditions for the local and global existence of positive weak solutions to this system in dimensions 2 and 3 have been obtained. Also, the local existence of solution to the same system with $u_0 \in B^{1-s}_{2,r}(\mathbb{R}^2)$, $r \in [1, \infty)$, and $1 < s < 2$ has been established by Biler and Wu [13]. For $n \geq 2$ and $u_0 \in L^p(\mathbb{R}^n)$, the existence of unique mild solution to (1.5) with $B(u) = \nabla \cdot ((-\Delta)^{-\theta/2} u)$ is proved in [12] under the conditions that $1 < s \leq 2$, $1 < \theta \leq n$, and $\max\{\frac{n}{s+\theta-2}, \frac{2n}{n-\theta-1}\} < p \leq s$. Biler et al. [10] derived the blow-up criteria for the solutions of (1.5) with $B(u) = \nabla \cdot ((\gamma I - \Delta)^{-1} u)$, $\gamma \geq 0$, in terms of Morrey spaces when $n = 2$. Wu and Zheng [59] established the well-posedness of solutions to the following space fractional parabolic-parabolic Keller–Segel system:

$$\begin{cases} u_t + (-\Delta)^{s_1/2} u + \nabla \cdot (\chi u \nabla v) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ v_t + (-\Delta)^{s_2/2} v = u, & t > 0,\ x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

with small initial data in the Fourier–Herz spaces. Wang et al. [55] showed the well-posedness and decay of global solutions of (1.6) in dimension 3, where $s_1/2 = s_2/2 \in (2/3, 1)$. Burczak and Belinchon [15] considered the fractional Keller–Segel system with logistic term

$$\begin{cases} u_t + \mu(-\Delta)^{s_1/2} u = \nabla \cdot (u(-\Delta)^{\frac{s_2-1}{2}} v) + ru(1-u), & t > 0,\ x \in \mathbb{R}^n, \\ \tau v_t + \nu(-\Delta)^{s_2/2} v = u - \lambda v, & t > 0,\ x \in \mathbb{R}^n \end{cases}$$

in dimension one and proved the well-posedness in subcritical and critical cases. Moreover, they discussed the dynamics properties of the system. When $n \leq 3$ and $0 < s_1, s_2 < 2$, the decay estimates for the following Poisson–Nernst–Planck system:

$$\begin{cases} u_t + (-\Delta)^{s_1/2} u + \nabla \cdot (\chi u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ v_t + (-\Delta)^{s_2/2} v + \nabla \cdot (\chi v \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \Delta \psi = u - v, & t > 0,\ x \in \mathbb{R}^n, \end{cases}$$

have been obtained in [24] in suitable spaces. We refer to [31] for a hyperbolic Keller–Segel system with degenerate nonlinear fractional diffusion. We remark that it will be of interest to see the existence and blow-up results to the above Keller–Segel systems, where $(-\Delta)^{\alpha/2} u$ and $(-\Delta)^{\alpha/2} v$ are replaced by $(-\Delta)^{\alpha/2} f(u)$ and $(-\Delta)^{\alpha/2} g(v)$, respectively, under suitable conditions on f and g in the spirit of [31].

Recently, using the fixed-point arguments, Li et al. [42] proved the existence and uniqueness of solutions to the following time-space fractional Keller–Segel system:

$$\begin{cases} {}^c_0 D^\beta_t u + (-\Delta)^{\alpha/2} u + \nabla \cdot (u B(u)) = 0, & (x, t) \in \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $0<\beta<1$, $1<\alpha\le 2$ and $B(u)=\nabla((-\Delta)^{-\gamma/2}u)$, $1<\gamma\le n$. They also established the non-negativity of the solution and blow-up behaviors.

Motivated by [42] and the above works on the Keller–Segel systems and importance of these problems in biology, we are interested in discussing the existence of solutions to the following time-space fractional parabolic-parabolic Keller–Segel system:

$$\begin{cases} {}_0^cD_t^\beta u+(-\Delta)^{\alpha/2}u+\nabla\cdot(u\nabla v)=0, & x\in\mathbb{R}^n,\ t>0,\\ {}_0^cD_t^\beta v+(-\Delta)^{\alpha/2}v-u=0, & x\in\mathbb{R}^n,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.7}$$

where $0<\beta<1$, $1<\alpha\le 2$, and $n\ge 2$, u and v denote the cell density and the concentration of the chemoattractant, respectively, ${}_0^cD_t^\beta$ denotes the Caputo fractional derivative of order β with respect to t, and $(-\Delta)^{\alpha/2}$ is defined as follows:

$$(-\Delta)^{\alpha/2}u(x)=\mathcal{F}^{-1}\big(|\xi|^\alpha\hat{u}(\xi)\big)(x),$$

where

$$\hat{u}(\xi)=\mathcal{F}u(x)=\int_{\mathbb{R}^n}u(x)e^{-ix\xi}\,\mathrm{d}x \tag{1.8}$$

is the Fourier transform of $u(x)$. To the best of our knowledge, we are not aware of the existence results for (1.7). System (1.7) describes the biological phenomenon chemotaxis with both anomalous diffusion and memory effects.

We mention that (1.7) generalizes the following classical Keller–Segel system (with $\tau=1$):

$$\begin{cases} u_t=\Delta u-\nabla\cdot\big(u\nabla v\big), & x\in\mathbb{R}^n,\ t>0,\\ \tau v_t=\Delta v+u, & x\in\mathbb{R}^n,\ t>0,\\ u(x,0)=u_0(x),\ v(x,0)=v_0(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.9}$$

which was one of the main motivations to consider (1.7). System (1.9) is called the simplified Keller–Segel system. This describes the evolution of cell density governed by the diffusion and the impact of a chemoattractant. Such systems appear not only in biological sciences but are also used, for example, in astrophysics to depict the evolution of clouds of self-gravitating particles; see [9, 47, 48]. From the mild formulation of (1.7), we have

$$\begin{aligned} u(t)&=S_\alpha^\beta(t)u_0-\int_0^t\nabla\cdot(T_\alpha^\beta(t-s)(u(s)Lu(s)))\,\mathrm{d}s\\ &\quad-\int_0^t\nabla\cdot(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0))\,\mathrm{d}s\\ &=S_\alpha^\beta(t)u_0+B(u,u)+H(u), \end{aligned}$$

where B is the bilinear form and H is the linear operator on suitable Banach spaces, which are explicitly mentioned later. The operators S_α^β and T_α^β are defined next. We employ the fixed-point arguments to find the solution u of

$$u(t) = S_\alpha^\beta(t)u_0 + B(u,u) + H(u).$$

We would like to point out that, because of the parabolic-parabolic fractional Keller–Segel system, the second equation of (1.7) leads to the additional term $H(u)$ in the above equation. The following facts make our problem quite challenging:

(i) System (1.7) is parabolic-parabolic fractional Keller–Segel system.

(ii) The corresponding solution of the second equation has a representation in the integral form, where the boundedness of v is not handy.

(iii) To prove the boundedness of the bilinear form B, there is a challenge as Hardy–Littlewood–Sobolev inequality is not applicable.

(iv) The presence of the initial data v_0 poses an additional difficulty in proving the boundedness of v in L^p space.

We overcome these challenges by the following ideas:

(i) In order to find the L^q estimates of ∇v, we use Hölder's inequality and the properties of the fundamental solutions are utilized, efficiently.

(ii) The boundedness of bilinear form is proved with the help of the L^q estimates of ∇v as we do in (i).

1.1. Results

The purpose of this section is to present our results on the local existence, uniqueness, and global existence of solutions to the Cauchy problem (1.7). First, we perform the scaling analysis to find suitable L^p spaces to study (1.7). Let (u,v) be the solution of (1.7). Let us consider the mass preserving scaling as follows:

$$u_\rho(x,t) := \rho^n u(\rho x, \rho^b t) \quad \text{and} \quad v_\rho(x,t) := \rho^{n-\alpha} v(\rho x, \rho^b t).$$

Then, $(-\Delta)^{\frac{\alpha}{2}} u_\rho = \rho^{n+\alpha}(-\Delta)^{\frac{\alpha}{2}} u$ and $\nabla\cdot(u_\rho \nabla v_\rho) = \rho^{2n+2-\alpha}\nabla\cdot(u\nabla v)$. Now, if $n+\alpha > 2n+2-\alpha$, i.e., $n < 2\alpha - 2$, then the diffusion is stronger. Since we have assumed $\alpha \in (1,2)$ and $n \geq 2$, this will not be possible. Now, if $n > 2\alpha - 2$, the aggregation term can be strong and this case is referred to as the super-critical case (in terms of mass concentration or diffusion). Now, it is easy to see that the scaling

$$u^\rho(x,t) := \rho^{2\alpha-2} u(\rho x, \rho^{\frac{\alpha}{\beta}} t) \quad \text{and} \quad v^\rho(x,t) := \rho^{\alpha-2} v(\rho x, \rho^{\frac{\alpha}{\beta}} t)$$

also satisfies the system (1.7) with the initial data

$$u_0^\rho = \rho^{2\alpha-2} u_0(\rho x) \quad \text{and} \quad v_0^\rho = \rho^{\alpha-2} v_0(\rho x).$$

Under the transformation $u \to u^\rho$, the $L^{\frac{n}{2\alpha-2}}$-norm is invariant. Thus, the critical index should be $p_c := \frac{n}{2\alpha-2}$ and L^{p_c} is the critical space.

The main results of this paper are the following theorems, which we will prove in ensuing sections following the arguments in [42].

Next, Theorem 1.1 states that system (1.7) has a unique mild solution for small time.

Theorem 1.1. *Let* $n \geq 2$, $0 < \beta < 1$, *and* $1 < \alpha \leq 2$. *Let* $p \in (p_c, \infty)$, $q \in (2p_c, \infty)$, *and* $0 < \frac{1}{p} - \frac{1}{q} < \frac{\alpha-1}{n}$. *Then, for any* $u_0 \in L^p(\mathbb{R}^n)$ *and* $\nabla v_0 \in L^q(\mathbb{R}^n)$, *there exists* $T > 0$ *such that* (1.7) *admits a unique mild solution* (u, v) *satisfying* $u \in C([0,T]; L^p(\mathbb{R}^n))$ *and* $\nabla v \in C([0,T]; L^q(\mathbb{R}^n))$ *with initial value* u_0 *and* v_0, *respectively, in the sense of Definition* 3.1. *Let*

$$T_m := \sup\big\{T > 0 : (1.7) \text{ has a unique solution } (u,v) \text{ with } u \in C([0,T]; L^p(\mathbb{R}^n)) \\ \text{and } \nabla v \in C([0,T]; L^q(\mathbb{R}^n))\big\}.$$

Then, if $T_m < \infty$, *we have*

$$\limsup_{t \to T_m^-} \|u(\cdot,t)\|_p = +\infty$$

and

$$\limsup_{t \to T_m^-} \|\nabla v(\cdot,t)\|_q = +\infty.$$

In the next theorem, we have the existence of global-in-time solution of (1.7).

Theorem 1.2. *Let* $n \geq 2$, $0 < \beta < 1$, *and* $1 < \alpha \leq 2$. *Let* p, q, *and* ℓ *satisfy* $1 \leq \frac{p_c}{2} < \ell < p_c < \frac{n\ell}{n+\ell(1-2\alpha)}$, $p = \frac{p_c \ell}{p_c - \ell}$, *and* $0 < \frac{1}{p} - \frac{1}{q} < \frac{\alpha-1}{n}$. *Then, there exists* $\delta > 0$ *such that, for* $u_0 \in L^{p_c}(\mathbb{R}^n)$ *with* $\|u_0\|_{p_c} \leq \delta$ *and* $v_0 = 0$, *system* (1.7) *admits a mild solution* (u, v) *with* $u \in C([0,\infty); L^{p_c}(\mathbb{R}^n))$ *and* $\nabla v \in C([0,\infty); L^p(\mathbb{R}^n)) \cap C([0,\infty); L^q(\mathbb{R}^n))$ *with initial value* $(u_0, 0)$, *satisfying*

$$\|u(t)\|_{p_c} \leq 2\delta, \quad \forall t > 0,$$

and $u \in C((0,\infty); L^p(\mathbb{R}^n))$. *Further,* u *is unique in*

$$X_T := \big\{u \in C([0,T]; L^{p_c}(\mathbb{R}^n)) \cap C([0,T]; L^p(\mathbb{R}^n)) \mid \|u\|_{p_c,p,T} < \infty\big\}, \quad T \in (0,\infty),$$

and hence, v *is also unique.*

In the next theorem, we establish the integrability of solution to (1.7), when $u_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and $\nabla v_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$.

Theorem 1.3. *Let* $n \geq 2$, $0 < \beta < 1$, *and* $1 < \alpha \leq 2$. *Let* $p \in (p_c, \infty)$, $q \in (2p_c, \infty)$, *and* $0 < \frac{1}{p} - \frac{1}{q} < \frac{\alpha-1}{n}$. *Suppose* $u_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, *and* $\nabla v_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. *Then, we have the following.*

(i) *There exists* $T > 0$ *such that* (1.7) *admits a unique mild solution* (u, v) *with* $u \in X := C([0,T]; L^1(\mathbb{R}^n)) \cap C([0,T]; L^p(\mathbb{R}^n))$ *and*

$$\nabla v \in Y := C([0,T]; L^1(\mathbb{R}^n)) \cap C([0,T]; L^q(\mathbb{R}^n))$$

with initial values (u_0, v_0). *Further,*

$$\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^n} v(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} v_0(x)\,\mathrm{d}x + \frac{t^\beta}{\beta\Gamma(\beta)} \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x.$$

(ii) *Let*

$$T_m = \sup\{T > 0 \mid (1.7) \text{ has a unique solution } (u,v) \text{ with } u \in X \text{ and } \nabla v \in Y\}.$$

If $T_m < \infty$, *then we have*

$$\lim_{t\to T_m^-} \sup(\|u(\cdot,t)\|_1 + \|u(\cdot,t)\|_p) = +\infty$$

and

$$\lim_{t\to T_m^-} \sup(\|\nabla v(\cdot,t)\|_1 + \|\nabla v(\cdot,t)\|_p) = +\infty.$$

Next result gives the solution in the weighted spaces for small time.

Theorem 1.4. *For* $n \geq 2$, $0 < \beta < 1$, $1 < \alpha \leq 2$, *let* $u_0 \in L^\infty_{n+\alpha}(\mathbb{R}^n)$ *and* $\nabla v_0 \in L^\infty_{n+\alpha}(\mathbb{R}^n)$. *Then, there exists* $T > 0$ *such that* (1.7) *has a unique mild solution* (u, v) *with* $u \in L^\infty([0,T]; L^\infty_{n+\alpha}(\mathbb{R}^n))$ *and* $\nabla v \in L^\infty([0,T]; L^\infty_{n+\alpha}(\mathbb{R}^n))$ *satisfying*

$$\int_{\mathbb{R}^n} u\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0\,\mathrm{d}x$$

and

$$\int_{\mathbb{R}^n} v(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} v_0(x)\,\mathrm{d}x + \frac{t^\beta}{\beta\Gamma(\beta)} \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x.$$

Let $T_m^\alpha = \sup\{T > 0 \mid (1.7)$ *has a unique mild solution*$\}$. *If* $T_m^\alpha < \infty$, *then we have*

$$\limsup_{t\to T_m^{\alpha-}} \|u(\cdot,t)\|_{L^\infty_{n+\alpha}} = +\infty$$

and

$$\limsup_{t\to T_m^{\alpha-}} \|\nabla v(\cdot,t)\|_{L^\infty_{n+\alpha}} = +\infty.$$

Further, this solution is the same as in Theorem 1.3 *on* $[0, T_m^\alpha)$, *and* $u \in C([0,T_m^\alpha), L^p(\mathbb{R}^n))$, $\forall p \in [1,\infty)$.

1.2. Organization of the article

The paper is organized as follows. In Section 2, we recall useful preliminaries and list the important results which are used in ensuing sections. Section 3 deals with the proofs of main results. Finally, in appendix, we provide a proof of integral representation of the solution to the given system.

2. Preliminaries

Let us recall the important definitions and auxiliary results. For any Banach space X, $L^p(0,T;X)$ consists of all strongly measurable functions $u : [0,T] \to X$ with

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_X^p \, dt \right)^{1/p} < \infty$$

for $1 \le p < \infty$ and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{0 \le t \le T} \|u\|_X < \infty.$$

Definition 2.1 (The Gamma function [46]). Let us recall the Gamma function $\Gamma(z)$, which is defined as follows:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt.$$

The above integral converges in the right half of the complex plane $\mathrm{Re}(z) > 0$, $z \in \mathbb{C}$.

Definition 2.2 (The Mittag–Leffler function [35]). The one-parameter Mittag–Leffler function $E_\alpha(z)$ is defined as follows:

$$E_\alpha(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \ \mathrm{Re}(\alpha) > 0.$$

The two-parameter Mittag–Leffler function is described by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \beta \in \mathbb{C}, \ \mathrm{Re}(\alpha) > 0.$$

In particular, it is easy to see that, for $\beta = 1$, we have $E_{\alpha,1}(z) = E_\alpha(z)$.

The following definitions and auxiliary results are borrowed from [42]. For the convenience of the reader and better exposition, we re-write the same here. For the details on these, we refer to [42].

Definition 2.3 ([42]). Suppose that X is a Banach space and $u \in L^1_{loc}((0,T);X)$ is a locally integrable function. If there exists $u_0 \in X$ such that

$$\lim_{t \to 0+} \frac{1}{t} \int_0^t \|u(s) - u(0)\|_X \, ds = 0,$$

we say u_0 is the right limit of u at $t = 0$, denoted as $u(0+) = u_0$. Similarly, $u(T-)$ is the constant $u_T \in X$ such that

$$\lim_{t \to T-} \frac{1}{T-t} \int_t^T \|u(s) - u_T\|_X \, ds = 0.$$

As in [41], we also use the following distributions $\{g_\alpha\}$ as the convolution kernels for $\alpha > 1$:

$$g_\alpha = \begin{cases} \frac{\theta(t)}{\Gamma(\alpha)} t^{\alpha-1}, & \alpha > 0, \\ \delta(t), & \alpha = 0, \\ \frac{1}{\Gamma(1+\alpha)} D(\theta(t)t^\alpha), & \alpha \in (-1,0). \end{cases}$$

Here, $\theta(t)$ is the standard Heaviside step function and D stands for the distributional derivative. For a locally integrable function, the weak Caputo derivative is defined as follows.

Definition 2.4 ([42]). *Suppose that X is a Banach space and $u \in L^1_{loc}([0,T);X)$, $u_0 \in X$. We define the weak Caputo derivative of u associated with initial data u_0 to be ${}^c_0D^\alpha_t u \in \mathcal{D}'$ (space of all distributions) such that, for any test function $\varphi \in C^\infty_c((-\infty,T);\mathbb{R})$,*

$$\langle {}^c_0D^\alpha_t u, \varphi \rangle = \int_0^T (u - u_0) {}^c_0\widetilde{D}^\alpha_t \varphi \, \mathrm{d}t,$$

where ${}^c_0\widetilde{D}^\alpha_t u$ denotes the right Caputo derivative of u associated with u_T. If $u(0+) = u_0$ in the sense of Definition 2.3, we say that ${}^c_0D^\alpha_t u$ is the Caputo derivative.

Let $X = \mathbb{R}^n$. Then, the Caputo derivative is given by

$${}^c_0D^\alpha_t u = g_{-\alpha} * ((u - u_0)\theta(t)).$$

We define the functions $P_1(x,t)$ and $P_2(x,t)$ for $0 < \alpha \le 2$ and $0 < \beta < 1$ as follows:

$$\mathcal{F} P_1(\cdot,t) = E_\beta(-|\xi|^\alpha t^\beta) \quad \text{and} \quad \mathcal{F} P_2(\cdot,t) = E_{\beta,\beta}(-|\xi|^\alpha t^\beta),$$

where $\mathcal{F}$ denotes the Fourier transform defined in (1.8). Also, define

$$Z(x,t) := t^{\beta-1} P_2(x,t).$$

Let $A = (-\Delta)^{\alpha/2}$ and consider the operators $S^\beta_\alpha(t)$, $T^\beta_\alpha(t)$ as follows:

$$f(x) \to S^\beta_\alpha(t) f(x) = E_\beta(-t^\beta A) f(x) = P_1(\cdot,t) * f(x), \tag{2.1}$$

$$f(x) \to T^\beta_\alpha(t) f(x) = t^{\beta-1} E_{\beta,\beta}(-t^\beta A) f(x) = Z(\cdot,t) * f(x). \tag{2.2}$$

The pair (P_1, Z) is the fundamental solution of (1.7); see [34] for the details on it. As a next result, we recall the L^r-L^q estimates of the fundamental solutions or, in other words, the solution operators. These estimates are very crucial in our analysis. We refer to [42, Proposition 3.3] for the details.

Proposition 2.5 ([42, Proposition 3.3]). *Let $0 < \beta < 1$ and $1 < \alpha \le 2$. Then, the following estimates hold.*

(i) *We have*

$$\left\|S_\alpha^\beta(t)u\right\|_\infty \leq \|u\|_\infty, \quad \left\|T_\alpha^\beta(t)u\right\|_\infty \leq \frac{1}{\Gamma(\beta)}t^{\beta-1}\|u\|_\infty, \quad \left\|\nabla S_\alpha^\beta(t)u\right\|_p \leq C\|\nabla u\|_p$$

$$\left\|\nabla S_\alpha^\beta(t)u\right\|_\infty \leq Ct^{-\frac{\beta}{\alpha}}\|u\|_\infty, \quad \left\|\nabla T_\alpha^\beta(t)u\right\|_\infty \leq Ct^{-\frac{\beta}{\alpha}+\beta-1}\|u\|_\infty.$$

(ii) *Let $q \in [1,\infty)$. We define $\theta_1 = \frac{qn}{n-q\alpha}$ if $n > q\alpha$ and $\theta_1 = \infty$, otherwise. Then, for any $r \in [1,\theta_1)$, we have*

$$\|S_\alpha^\beta(t)u\|_r \leq Ct^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{r})}\|u\|_q.$$

If $r = q$, the constant can be chosen to be 1. If $n < q\alpha$, then the above also holds for $r = \theta_1 = \infty$.

(iii) *Let $q \in [1,\infty)$. We define $\theta_2 = \frac{qn}{n-2q\alpha}$ if $n > 2q\alpha$ and $\theta_2 = \infty$, otherwise. Then, for any $r \in [1,\theta_2)$, we have*

$$\|T_\alpha^\beta(t)u\|_r \leq Ct^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{r})+\beta-1}\|u\|_q.$$

If $r = q$, the constant can be chosen as $\frac{1}{\Gamma(\beta)}$. If $n < 2q\alpha$, then the above also holds for $r = \theta_2 = \infty$.

(iv) *Let $q \in [1,\infty)$. We define $\theta_3 = \frac{qn}{n+q(1-\alpha)}$ if $n > q(\alpha-1)$ and $\theta_3 = \infty$, otherwise. Then, for any $r \in [q,\theta_3)$, there is $C > 0$ satisfying*

$$\|\nabla S_\alpha^\beta(t)u\|_r \leq Ct^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{r})-\frac{\beta}{\alpha}}\|u\|_q.$$

If $n < q(\alpha-1)$, then the estimate also holds for $r = \theta_3 = \infty$.

(v) *Let $q \in [1,\infty)$. Let $\theta_4 = \frac{qn}{n+q(1-2\alpha)}$ if $n > q(2\alpha-1)$ and $\theta_4 = \infty$, otherwise. Then, for $r \in [q,\theta_4)$, there is $C > 0$ satisfying*

$$\|\nabla T_\alpha^\beta(t)u\|_r \leq Ct^{-\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{r})-\frac{\beta}{\alpha}+\beta-1}\|u\|_q.$$

If $n < q(2\alpha-1)$, the estimate also holds for $r = \theta_4 = \infty$.

Next, we recall the weighted estimates of the fundamental solutions.

Proposition 2.6 ([42, Proposition 3.5]). *Assume $0 < \beta < 1$, $1 < \alpha \leq 2$, and*

$$u_0 \in L^\infty_{n+\alpha}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

Then, there is $C > 0$ such that

$$\left\|S_\alpha^\beta(t)u_0\right\|_{L^\infty_{n+\alpha}} \leq C\left\|u_0\right\|_{L^\infty_{n+\alpha}} + Ct^\beta\left\|u_0\right\|_1,$$
$$\left\|\nabla T_\alpha^\beta(t)u_0\right\|^\infty_{L^\infty_{n+\alpha}} \leq Ct^{-\frac{\beta}{\alpha}+\beta-1}\left\|u_0\right\|_{L^\infty_{n+\alpha}} + Ct^{2\beta-\frac{\beta}{\alpha}-1}\left\|u_0\right\|_1.$$

Next, we mention a fixed-point theorem on the existence of solutions of equations with continuous bilinear mappings. This theorem is crucial to obtain existence and uniqueness of solutions.

Lemma 2.7 ([16, Lemma 5], [59, Lemma 3.2]). *Let X be the Banach space, $H : X \to X$ a linear operator such that, for any $x \in X$,*

$$\|H(x)\|_X \leq \tau \|x\|_X,$$

and $B : X \times X \to X$ a bilinear mapping such that, for any $x_1, x_2 \in X$,

$$\|B(x_1, x_2)\|_X \leq \eta \|x_1\|_X \|x_2\|_X$$

for some constant η; then, for any τ with $0 \leq \tau < 1$ and for any $y \in X$ such that

$$4\eta\|y\| < (1-\tau)^2,$$

the equation

$$x = y + B(x, x) + H(x)$$

has a solution x in X. In particular, the solution is such that

$$\|x\|_X \leq \frac{2\|y\|_X}{1-\tau},$$

and it is unique in $B(0, \frac{1-\tau}{2\eta})$.

3. Existence of mild solution

The results in this section are proved using fixed-point arguments for a bilinear operator in Banach space. Denote $A = (-\Delta)^{\alpha/2}$. Then, following [52] (see Appendix), taking the Laplace transform of (1.7), solution of (1.7) is given by the following Duhamel's-type integral equation:

$$\begin{cases} u(t) = E_\beta(-t^\beta A)u_0 - \beta \displaystyle\int_0^t (t-\tau)^{\beta-1} E'_\beta(-(t-s)^\beta A)(\nabla \cdot (u\nabla v))(s)\,\mathrm{d}s \\ \qquad = E_\beta(-t^\beta A)u_0 - \displaystyle\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)(\nabla \cdot (u\nabla v))(s)\,\mathrm{d}s, \\ v(t) = E_\beta(-t^\beta A)v_0 + \displaystyle\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)u(s)\,\mathrm{d}s, \end{cases} \tag{3.1}$$

where $\beta E'_\beta(t) = E_{\beta,\beta}(t)$. See [34, 42] for the properties of fundamental solutions of the Cauchy problem

$$D_t^\beta u + (-\Delta)^{\alpha/2} u = f(x,t)$$

with $0 < \beta < 1$ and $1 < \alpha \leq 2$. From (2.1), (2.2), and (3.1), the mild formulation of (1.7) can be written as follows:

$$\begin{cases} u(t) = S_\alpha^\beta(t)\, u_0 - \displaystyle\int_0^t \nabla \cdot (T_\alpha^\beta(t-s)(u(s)\nabla v(s)))\,\mathrm{d}s, \\ v(t) = S_\alpha^\beta(t)\, v_0 + \displaystyle\int_0^t T_\alpha^\beta(t-s)u(s)\,\mathrm{d}s. \end{cases} \tag{3.2}$$

Therefore, we can write

$$\begin{aligned} u(t) &= S_\alpha^\beta(t)u_0 - \int_0^t \nabla\cdot(T_\alpha^\beta(t-s)(u(s)Lu(s)))\,\mathrm{d}s \\ &\quad - \int_0^t \nabla\cdot(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0))\,\mathrm{d}s \\ &= S_\alpha^\beta(t)u_0 + B(u,u) + H(u), \end{aligned}$$

where the bilinear form B and the linear operators L and H are defined as

$$\begin{aligned} B(u,z) &:= -\int_0^t \nabla\cdot(T_\alpha^\beta(t-s)(u(s)Lz(s)))\,\mathrm{d}s, \\ Lz(t) &:= -\int_0^t \nabla T_\alpha^\beta(t-s)z(s)\,\mathrm{d}s, \\ H(u) &:= -\int_0^t \nabla\cdot(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0))\,\mathrm{d}s. \end{aligned}$$

This linear operator L gives the information about the solution of the second equation in (1.7). Next, following [42], we recall the definition of the mild solution of (1.7).

Definition 3.1. Let X and Y be the Banach spaces over space and time. Then, $u \in X$ and $v \in Y$ is called a mild solution of (1.7) if u and v satisfy the integral equation (3.1).

Once we have u satisfying the first equation in (3.2), we get v from the second equation of (3.2). Therefore, to show that (u,v) satisfies (3.1), it is enough to prove that u satisfies first equation in (3.2). Subsequently, using properties of the operators S_α^β and T_α^β, first, we establish L^q and L^p, estimates for operators L, B, and H, respectively.

3.1. Proof of Theorem 1.1

The proof consists in constructing solutions to (3.1) by Lemma 2.7 with $y = S_\alpha^\beta(t)u_0$ and with the associated bilinear form B and linear operator H. By (i) of Proposition 2.5, $S_\alpha^\beta u_0 \in C([0,T];L^p(\mathbb{R}^n))$ and

$$\|S_\alpha^\beta u_0\|_{C([0,T];L^p(\mathbb{R}^n))} \le \|u_0\|_p.$$

Assume $1 \le p \le q < \theta_4$ and $0 < \frac{1}{p} - \frac{1}{q} < \frac{\alpha-1}{n}$. Then, for $p \in [1,\infty)$ and $q \in [p,\theta_4)$, by (v) of Proposition 2.5, we have

$$\begin{aligned} \|Lz(t)\|_q &\le \int_0^t \left\|\nabla T_\alpha^\beta(t-s)z(s)\right\|_q \mathrm{d}s \\ &\le C\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta-1}\|z(s)\|_p\,\mathrm{d}s \\ &\le C\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta-1}\sup_{0<s\le t}\|z(s)\|_p\,\mathrm{d}s \\ &\le Ct^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta}\sup_{0<s\le t}\|z(s)\|_p, \end{aligned} \tag{3.3}$$

since $-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta>0$. The relation $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}<1$ defines the exponent $r\in(1,p)$. Let us fix the exponents p and q such that $p>p_c$ and $q>\frac{n}{\alpha-1}$. This will imply that $\theta_4=\infty$ and

$$-\frac{n\beta}{\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{\beta}{\alpha}+\beta=-\frac{n\beta}{\alpha}\left(\frac{1}{q}\right)-\frac{\beta}{\alpha}+\beta>0. \tag{3.4}$$

Next, we obtain the estimate for the bilinear form B. Again, we use (v) of Proposition 2.5 with $r\in(1,p)$ and $p\in[r,\theta_4')$, where

$$\theta_4'=\frac{rn}{n+r(1-2\alpha)}$$

if $n>r(2\alpha-1)$ or $\theta_4'=\infty$, otherwise. In either case, $\theta_4'\le\theta_4$:

$$\begin{aligned}\|B(u,z)(t)\|_p&\le\int_0^t\left\|\nabla\cdot(T_\alpha^\beta(t-s)(u(s)Lz(s)))\right\|_p\,\mathrm{d}s\\&\le C\int_0^t(t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}\|u(s)Lz(s)\|_r\,\mathrm{d}s.\end{aligned}$$

Then, applying Hölder's inequality and using (3.3), we get

$$\begin{aligned}&\|B(u,z)(t)\|_p\\&\quad\le C\int_0^t(t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}\|u(s)\|_p\|Lz(s)\|_q\,\mathrm{d}s\\&\quad\le C\int_0^t(t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}s^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta}\|u(s)\|_p\sup_{0<s\le t}\|z(s)\|_p\,\mathrm{d}s.\end{aligned}\tag{3.5}$$

From (3.4) and $\frac{1}{p}-\frac{1}{q}<\frac{\alpha-1}{n}$, we obtain the following estimate:

$$\begin{aligned}&\int_0^t(t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}s^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta}\,\mathrm{d}s\\&\quad=\mathscr{B}\left(-\frac{n\beta}{\alpha}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\beta}{\alpha}+\beta+1,-\frac{n\beta}{\alpha}\left(\frac{1}{q}\right)-\frac{\beta}{\alpha}+\beta\right)\\&\qquad\times t^{-\frac{n\beta}{\alpha}\left(\frac{1}{p}\right)-\frac{2\beta}{\alpha}+2\beta},\end{aligned}\tag{3.6}$$

where $\mathscr{B}$ denotes the beta function and is defined by

$$\int_a^b(t-a)^{x-1}(b-t)^{y-1}\,\mathrm{d}t=(b-a)^{x+y-1}\mathscr{B}(x,y),\quad x>0\text{ and }y>0.$$

Since $p>\frac{n}{2\alpha-2}$ implies that $-\frac{n\beta}{\alpha}(\frac{1}{p})-\frac{2\beta}{\alpha}+2\beta>0$, taking supremum on both sides of (3.5) and then using (3.6) yields

$$\sup_{0<t\le T}\|B(u,z)\|_p\le C\|u\|_{C([0,T];L^p(\mathbb{R}^n))}\|z\|_{C([0,T];L^p(\mathbb{R}^n))}T^{-\frac{n\beta}{\alpha}\left(\frac{1}{p}\right)-\frac{2\beta}{\alpha}+2\beta}.$$

Similarly, we have

$$\begin{aligned}\|H(u)\|_p &\leq \int_0^t \|\nabla\cdot(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0))\|_p\,\mathrm{d}s\\ &\leq C\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}\|u(s)\nabla S_\alpha^\beta(s)v_0\|_r\,\mathrm{d}s.\end{aligned}$$

Then, applying Hölder's inequality and Proposition 2.5, we get

$$\begin{aligned}\|H(u)\|_p &\leq \int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}\|u(s)\|_p\|\nabla_\alpha^\beta v_0\|_q\,\mathrm{d}s\\ &\leq C\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1}\|u(s)\|_p\|\nabla v_0\|_q\,\mathrm{d}s\\ &\leq C\|\nabla v_0\|_q\|u(s)\|_p t^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta}.\end{aligned}\tag{3.7}$$

Hence,

$$\sup_{0<t\leq T}\|H(u)\|_p \leq C\|\nabla v_0\|_q\|u\|_{C([0,T];L^p(\mathbb{R}^n))}T^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta}.$$

Next, we claim that $B(u,z)\in C([0,T],L^p(\mathbb{R}^n))$. Let r be as above and using Hölder's inequality, we get

$$w(s) = u(s)Lz(s)\in C([0,T];L^r(\mathbb{R}^n)).$$

Now, for some $t>0$ and $\delta>0$, select $0\leq t<t+\delta\leq T$ and $\delta_1>0$, and then, we have

$$\begin{aligned}&\|B(u,z)(t+\delta)-B(u,z)(t)\|_p\\ &\quad=\left\|\int_0^{t+\delta}\nabla T_\alpha^\beta(t+\delta-s)w(s)\,\mathrm{d}s-\int_0^t\nabla T_\alpha^\beta(t-s)w(s)\,\mathrm{d}s\right\|_p\\ &\quad\leq\left\|\int_{\max\{0,t-\delta_1\}}^{t+\delta}\nabla T_\alpha^\beta(t+\delta-s)w(s)\,\mathrm{d}s\right\|_p\\ &\qquad+\left\|\int_{\max\{0,t-\delta_1\}}^{t}\nabla T_\alpha^\beta(t-s)w(s)\,\mathrm{d}s\right\|_p\\ &\qquad+\left\|\int_0^{\max\{0,t-\delta_1\}}\big(\nabla T_\alpha^\beta(t+\delta-s)-\nabla T_\alpha^\beta(t-s)\big)w(s)\,\mathrm{d}s\right\|_p.\end{aligned}\tag{3.8}$$

The first two terms in the right-hand side of (3.8) can be estimated as in (3.5) and controlled by $C\|w\|_{C([0,T];L^r(\mathbb{R}^n))}(\delta+\delta_1)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta}$. The third term $\to 0$ as $\delta\to 0$. Hence, $B(u,z)\in C([0,T],L^p(\mathbb{R}^n))$. Similarly, we can show that

$$H(u)\in C([0,T],L^p(\mathbb{R}^n)).$$

Now, we by an application of Lemma 2.7, we get the existence of a mild for small T. Note that, from (3.3), we get $v\in C([0,T];L^q(\mathbb{R}^n))$.

Now, suppose that T_m is the maximum existence time and $T_m < \infty$; then, the contradiction argument as in [42] yields

$$\limsup_{t\to T_m^-} \|u(\cdot,t)\|_p = +\infty$$

and

$$\limsup_{t\to T_m^-} \|\nabla v(\cdot,t)\|_q = +\infty.$$

This completes the proof. ∎

3.2. Global existence

Note that the L^r-L^q estimate of S_β^α in Proposition 2.5 implies that

$$\sup_{0\le t\le T} \left(\|S_\beta^\alpha(t)u\|_p + t^{\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{r})}\|S_\beta^\alpha(t)u\|_r\right) \le C\|u\|_p$$

for $r \in [p,\theta_1)$. Using this, we define the modified norm for $u \in C([0,T];L^p(\mathbb{R}^n))$ as follows:

$$\|u\|_{p,r;T} := \sup_{0\le t\le T} \left(\|S_\beta^\alpha(t)u\|_p + t^{\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{r})}\|S_\beta^\alpha(t)u\|_r\right) \le C\|u\|_p. \tag{3.9}$$

Proof of Theorem 1.2. Let us fix $T \in (0,\infty)$ and consider the space $X := X_T$ with the norm

$$\|\cdot\|_X = \|\cdot\|_{p_c,p;T}.$$

One can easily check that this is a Banach space.

By (i) of Proposition 2.5, we have $S_\alpha^\beta u_0 \in C([0,T];L^{p_c}(\mathbb{R}^n))$ for any $T > 0$, and by (3.9), we find that

$$\|S_\alpha^\beta u_0\|_X \le C\|u_0\|_{p_c} \le C\delta.$$

Hence, $S_\alpha^\beta u_0 \in X$. Now, we show that the bilinear form B is continuous. For that, we use (v) of Proposition 2.5:

$$\begin{aligned}\|B(u,z)(t)\|_{p_c} &\le \int_0^t \left\|\nabla\cdot(T_\alpha^\beta(t-s)(u(s)Lz(s)))\right\|_{p_c} \mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{\ell}-\frac{1}{p_c})-\frac{\beta}{\alpha}+\beta-1}\|u(s)Lz(s)\|_\ell \,\mathrm{d}s\\ &\le C\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{l}-\frac{1}{p_c})-\frac{\beta}{\alpha}+\beta-1}\|u(s)\|_{p_c}\|Lz(s)\|_p \,\mathrm{d}s.\end{aligned}$$

Here, we need $\frac{n\ell}{n+\ell(1-2\alpha)} = \theta_4 > p_c > \ell \ge 1$. Note that $p_c > \ell \ge 1$ ensures Hölder's inequality, where

$$p = \frac{p_c l}{p_c - l}.$$

Assume $p > p_c$, i.e., $\ell > \frac{p_c}{2}$. Again, using (v) of Proposition 2.5, we have

$$\begin{aligned}
\|L(z(t))\|_p &\leq \int_0^t \|\nabla T_\alpha^\beta(t-s)z(s)\|_p \,\mathrm{d}s \\
&\leq C \int_0^t (t-s)^{\frac{-\beta}{\alpha}+\beta-1} \|z(s)\|_p \,\mathrm{d}s \\
&\leq C \int_0^t (t-s)^{\frac{-\beta}{\alpha}+\beta-1} \sup_{0\leq s\leq t} \|z(s)\|_p \,\mathrm{d}s \\
&\leq C t^{\beta-\frac{\beta}{\alpha}} \sup_{0\leq s\leq t} \|z(s)\|_p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|B(u,z)\|_{p_c} &\leq C \int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{l}-\frac{1}{p_c})-\frac{\beta}{\alpha}+\beta-1} s^{\beta-\frac{\beta}{\alpha}} \|u(s)\|_{p_c} \sup_{0\leq s\leq t} \|z(s)\|_p \,\mathrm{d}s \\
&\leq C\|z(s)\|_X \|u(s)\|_X \int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{l}-\frac{1}{p_c})-\frac{\beta}{\alpha}+\beta-1} s^{\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{p_c})+\beta-\frac{\beta}{\alpha}} \,\mathrm{d}s \\
&\leq C\|z(s)\|_X \|u(s)\|_X.
\end{aligned}$$

Note that $\frac{p_c}{2} < \ell < p_c$ implies that $\frac{1}{p} - \frac{1}{p_c} < \frac{1}{p_c}$, and hence, $-\frac{n\beta}{\alpha}(\frac{1}{\ell} - \frac{1}{p_c}) - \frac{\beta}{\alpha} + \beta > 0$. Also, since $p > p_c$, it is true that $\frac{n\beta}{\alpha}(\frac{1}{p} - \frac{1}{p_c}) + \beta - \frac{\beta}{\alpha} > -1$. This ensures that the integrals with respect to s converge.

Now, multiplying both sides of inequality (3.5) by $t^{\frac{n\beta}{\alpha}(\frac{1}{p_c}-\frac{1}{p})}$, we get

$$\begin{aligned}
&t^{\frac{n\beta}{\alpha}(\frac{1}{p_c}-\frac{1}{p})}\|B(u,z)(t)\|_p \\
&\quad\leq C t^{\frac{n\beta}{\alpha}(\frac{1}{p_c}-\frac{1}{p})} \int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1} s^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta} \|u(s)\|_p \sup_{0<s\leq t} \|z(s)\|_p \,\mathrm{d}s \\
&\quad\leq C\left(t^{\frac{n\beta}{\alpha}(\frac{1}{p_c}-\frac{1}{p})} \sup_{0\leq s\leq t} \|z(s)\|_p\right) \\
&\qquad\times\left(\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1} s^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta} \|u(s)\|_p \,\mathrm{d}s\right).
\end{aligned}$$

In order to make sure the above inequalities hold, we need p and q to satisfy the conditions of Theorem 1.1. However, as we know $p > p_c$, these conditions are satisfied automatically. We then find that

$$\begin{aligned}
&t^{\frac{n\beta}{\alpha}(\frac{1}{p_c}-\frac{1}{p})}\|B(u,z)(t)\|_p \\
&\quad\leq C\|u(s)\|_X\|z(s)\|_X\left(\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1} s^{-\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{q})-\frac{\beta}{\alpha}+\beta} s^{\frac{n\beta}{\alpha}(\frac{1}{p}-\frac{1}{p_c})} \,\mathrm{d}s\right) \\
&\quad\leq C\|u(s)\|_X\|z(s)\|_X\left(\int_0^t (t-s)^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta-1} s^{\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p_c})-\frac{\beta}{\alpha}+\beta} \,\mathrm{d}s\right).
\end{aligned}$$

For this integral to converge, we need $-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta>0$, and $\frac{n\beta}{\alpha}(\frac{1}{q}-\frac{1}{p_c})-\frac{\beta}{\alpha}+\beta>-1$. With our assumptions on p, q, and r, both inequalities hold.

The proof of $B(u,z)\in C([0,T],L^{p_c}(\mathbb{R}^n))\cap C([0,T],L^p(\mathbb{R}^n))$ is similar to the one in Theorem 1.1. Therefore, we omit the details here. Since we are assuming $v_0=0$, therefore $H=0$. Now, using Lemma 2.7, we get the required existence and uniqueness of the mild solution. ∎

Proof of Theorem 1.3. Consider the space

$$X=C([0,T];L^1(\mathbb{R}^n))\cap C([0,T];L^p(\mathbb{R}^n))$$

with the norm

$$\|u\|_X=\sup_{0\le t\le T}(\|u\|_1+\|u\|_p).$$

Then, X is a Banach space. It is easy to see that, for any $r\in[1,p]$, $\|u\|_{C([0,T];L^r(\mathbb{R}^n))}\le\|u\|_X$. By Proposition 2.5, we get

$$\|S_\alpha^\beta(t)u_0\|_X=\sup_{0\le t\le T}\left(\|S_\alpha^\beta(t)u_0\|_1+\|S_\alpha^\beta(t)u_0\|_p\right)\le\|u_0\|_1+\|u_0\|_p.$$

Let $p_1\in[\frac{p}{p-1},\ p]$, which implies that $\frac{p_1}{p_1-1}\in[\frac{p}{p-1},\ p]$. Then, by Proposition 2.5, we have

$$\begin{aligned}\|Lz(t)\|_{\frac{p_1}{p_1-1}}&\le\int_0^t\left\|\nabla T_\alpha^\beta(t-s)z(s)\right\|_{\frac{p_1}{p_1-1}}\mathrm{d}s\\&\le C\int_0^t(t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|z(s)\|_{\frac{p_1}{p_1-1}}\mathrm{d}s\\&\le C\int_0^t(t-s)^{-\frac{\beta}{\alpha}+\beta-1}\sup_{0<s\le t}\|z(s)\|_{\frac{p_1}{p_1-1}}\mathrm{d}s\\&\le Ct^{-\frac{\beta}{\alpha}+\beta}\sup_{0<s\le t}\|z(s)\|_{\frac{p_1}{p_1-1}}.\end{aligned}\tag{3.10}$$

Then, for any $0\le t\le T$, using Proposition 2.5 with $q=r=1$, we obtain

$$\begin{aligned}\|B(u,z)\|_1&\le\int_0^t\left\|\nabla\cdot T_\alpha^\beta(t-s)u(s)Lz(s)\right\|_1\mathrm{d}s\\&\le C\int_0^t(t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|u(s)Lz(s)\|_1\,\mathrm{d}s.\end{aligned}$$

Then, applying Hölder's inequality and using (3.10), we get

$$\begin{aligned}\|B(u,z)\|_1&\le C\int_0^t(t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|u(s)\|_{p_1}\|Lz(s)\|_{\frac{p_1}{p_1-1}}\mathrm{d}s\\&\le C\int_0^t(t-s)^{-\frac{\beta}{\alpha}+\beta-1}s^{-\frac{\beta}{\alpha}+\beta}\|u(s)\|_{p_1}\sup_{0<s\le t}\|z(s)\|_{\frac{p_1}{p_1-1}}\mathrm{d}s\\&\le C\|u\|_X\|z\|_XT^{-\frac{2\beta}{\alpha}+2\beta}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|H(u)\|_1 &\le \int_0^t \|\nabla\cdot(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0))\|_1\,\mathrm{d}s\\ &\le \int_0^t (t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|u(s)\nabla S_\alpha^\beta(s)v_0\|_1\,\mathrm{d}s\\ &\le C\|\nabla v_0\|\int_0^t (t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|u(s)\|_1\,\mathrm{d}s\\ &\le C\|\nabla v_0\|\|u\|_X T^{\beta-\frac{\beta}{\alpha}}.\end{aligned}$$

Note that the constraint $\theta_4 > 1$ is automatically satisfied here.

Now, assume $1 \le p \le q < \theta_4$, $0 < \frac{1}{p} - \frac{1}{q} < \frac{\alpha-1}{n}$, and $p \in (p_c, \infty)$, $q \in (2p_c, \infty)$. Then, using (3.3) through (3.6), we get

$$\|B(u,z)\|_p \le C\|u\|_X\|z\|_X T^{-\frac{n\beta}{\alpha}(\frac{1}{p})-\frac{2\beta}{\alpha}+2\beta}.$$

Similarly, using (3.7), we have

$$\|H(u)\|_p \le C\|\nabla v_0\|_q\|u(s)\|_X T^{-\frac{n\beta}{\alpha}(\frac{1}{r}-\frac{1}{p})-\frac{\beta}{\alpha}+\beta}.$$

The claim that $H(u)$ and $B(u,z) \in C([0,T], L^1(\mathbb{R}^n)) \cap C([0,T], L^p(\mathbb{R}^n))$ can be proved as in the proof of Theorem 1.1. For the sake of brevity, we omit the details here. Then, $H(u) \in X$ and $B(u,z) \in X$, and we get

$$\|B(u,z)\|_X \le CT^\delta\|u\|_X\|v\|_X$$

and

$$\|H(u)\|_X \le CT^\sigma\|u\|_X$$

for some positive numbers δ and σ. Now, we can use Lemma 2.7 to get the existence and uniqueness for small time. Now, integrating the first equation of (3.2) yields

$$\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} S_\alpha^\beta u_0\,\mathrm{d}x - \int_{\mathbb{R}^n}\int_0^t \nabla\cdot(T_\alpha^\beta(t-s)(u\nabla v))\,\mathrm{d}s\,\mathrm{d}x.$$

But by [42, Lemma 3.1], we have

$$\int_{\mathbb{R}^n} S_\alpha^\beta u_0\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0\,\mathrm{d}x.$$

Using the density of C_c^∞-space in the L^1-space, one can approximate $u\nabla v$ with some sequence $\{\zeta_k\} \subset C_c^\infty((0,t]\times\mathbb{R}^n)$. Now, Green's identity implies that

$$\int_{\mathbb{R}^n}\int_0^t \nabla\cdot(T_\alpha^\beta(t-s)(\zeta_k))\,\mathrm{d}s\,\mathrm{d}x = 0$$

for each $k \geq 1$. Hence,

$$\int_{\mathbb{R}^n} \int_0^t \nabla \cdot (T_\alpha^\beta(t-s)(u\nabla v))\, ds\, dx = 0.$$

Thus,

$$\int_{\mathbb{R}^n} u(x,t)\, dx = \int_{\mathbb{R}^n} u_0(x)\, dx.$$

Again, using [42, Lemma 3.1], we infer

$$\begin{aligned}
\int_{\mathbb{R}^n} v(x,t)\, dx &= \int_{\mathbb{R}^n} E_\beta(-t^\beta A)v_0\, dx + \int_{\mathbb{R}^n}\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)u(s)\, ds\, dx \\
&= \int_{\mathbb{R}^n} v_0\, dx + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\left(\int_{\mathbb{R}^n} u(x,t)\, dx\right) ds \\
&= \int_{\mathbb{R}^n} v_0\, dx + \frac{t^\beta}{\beta\Gamma(\beta)} \int_{\mathbb{R}^n} u_0\, dx.
\end{aligned}$$

This proves that the integrals are preserved. The proof of statement (ii) follows the similar lines of proof as Theorem 1.1. Here, we skip the details of proofs of both statements. ■

3.3. Existence in the weighted space

Here, we study the existence of mild solutions to (1.7) in the weighted spaces. Define

$$\begin{aligned}
L_\nu^\infty(\mathbb{R}^n) &:= \{v \in L^\infty(\mathbb{R}^n) \mid \|v\|_{L_\nu^\infty} := \|(1+|x|)^\nu v(x)\|_\infty < \infty\}, \\
X_T &:= L^\infty([0,T], L_{n+\alpha}^\infty(\mathbb{R}^n)).
\end{aligned}$$

Proof of Theorem 1.4. The proof of this theorem follows exactly on the similar lines of proof of [42, Theorem 4.5]. The only difference is the L^∞-estimate for $L(z)(t)$. For the sake of completeness, we write the necessary details. Note that

$$\|u\|_{L_{n+\alpha}^\infty} < \infty \text{ implies that } u \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

This inequality together with Proposition 2.6 implies that

$$\|S_\alpha^\beta(\cdot)u_0\| \leq C(1+T^\beta)\|u_0\|_{L_{n+\alpha}^\infty}.$$

Using (i) of Proposition 2.5, we have

$$\begin{aligned}
|L(z)(t)| &\leq \left| \int_0^t \nabla T_\alpha^\beta(t-s)z(s)\, ds \right| \\
&\leq \int_0^t |\nabla T_\alpha^\beta(t-s)z(s)|\, ds \\
&\leq C\int_0^t (t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|z\|_\infty\, ds \\
&\leq Ct^{-\frac{\beta}{\alpha}+\beta}\|z\|_{X_T}.
\end{aligned}$$

This inequality leads us to the following estimate for B:

$$\begin{aligned}\|B(u,z)\|_{X_T} &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} (1+|x|)^{n+\alpha}\left|\int_0^t \nabla\cdot\left(T_\alpha^\beta(t-s)(u(s)Lz(s))\right)\mathrm{d}s\right| \\ &\le \int_0^t \left\|\nabla\cdot\left(T_\alpha^\beta(t-s)(u(s)Lz(s))\right)\right\|_{L^\infty_{n+\alpha}}\mathrm{d}s \\ &\le C\int_0^t \left((t-s)^{-\frac{\beta}{\alpha}+\beta-1}\|uLz(s)\|_{L^\infty_{n+\alpha}} + (t-s)^{-\frac{\beta}{\alpha}+2\beta-1}\|uLz(s)\|_1\right)\mathrm{d}s \\ &\le Ct^{2\beta-\frac{2\beta}{\alpha}}\|u\|_{X_T}\|z\|_{X_T} + Ct^{3\beta-\frac{2\beta}{\alpha}}\|u\|_{X_T}\|z\|_{X_T}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\|H(u)\|_{X_T} \\ &= \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} (1+|x|)^{n+\alpha}\left|\int_0^t \nabla\cdot\left(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0)\right)\mathrm{d}s\right| \\ &\le \int_0^t \left\|\nabla\cdot\left(T_\alpha^\beta(t-s)(u(s)\nabla S_\alpha^\beta(s)v_0)\right)\right\|_{L^\infty_{n+\alpha}}\mathrm{d}s \\ &\le C\int_0^t \left((t-s)^{-\frac{\beta}{\alpha}+\beta-1}\left\|u\nabla S_\alpha^\beta(s)v_0\right\|_{L^\infty_{n+\alpha}} + (t-s)^{-\frac{\beta}{\alpha}+2\beta-1}\left\|u\nabla S_\alpha^\beta(s)v_0\right\|_1\right)\mathrm{d}s.\end{aligned}$$

Now, we have

$$\begin{aligned}\left\|u\nabla S_\alpha^\beta(s)v_0\right\|_{L^\infty_{n+\alpha}} &= \left\|(1+|x|)^{n+\alpha}u(s)\nabla S_\alpha^\beta(s)v_0\right\|_\infty \\ &\le C\|u\|_{X_T}\|\nabla v_0\|_\infty\end{aligned}$$

and

$$\begin{aligned}\left\|u\nabla S_\alpha^\beta(s)v_0\right\|_{L^\infty_{n+\alpha}} &\le C\|u\|_1\left\|\nabla S_\alpha^\beta(s)v_0\right\|_\infty \\ &\le C\|u\|_{X_T}\|\nabla v_0\|.\end{aligned}$$

Thus,

$$\|H(u)\|_{X_T} \le Ct^{\beta-\frac{\beta}{\alpha}}\|u\|_{X_T} + Ct^{2\beta-\frac{\beta}{\alpha}}\|u\|_{X_T}.$$

Now, we use Lemma 2.7 to get the existence of solution. We refer to the proof of [42, Theorem 4.5] for the remaining details of this theorem. ■

Remark 3.2. Note that if the initial values u_0 and v_0 of (1.7) are nonnegative, then it implies that the solution (u, v) obtained in Theorem 1.1 (or Theorem 1.2, Theorem 1.3, Theorem 1.4) remains nonnegative. The proof of this statement follows the similar lines of proof of [42, Theorem 5.1]. The only change we need to make is to replace $B(u)$ in [42, Theorem 5.1] with $L(u) - \nabla S_\alpha^\beta(\cdot)v_0$. Due to this change, we cannot use the Hardy–Littlewood–Sobolev inequality to control $L(u) - \nabla S_\alpha^\beta(\cdot)v_0$. Instead, one can use the Hölder's inequality and the L^q estimates of ∇v to control $L(u) - \nabla S_\alpha^\beta(\cdot)v_0$.

Next, in the appendix, we re-collect the integral representation of solutions to fractional parabolic-parabolic Keller–Segel system, which is easy to see. Some special cases of it are used by several researchers in different contexts and can be found out in the literature. For the sake of completeness, we write the details.

A. Appendix

Following [52], using Laplace transform, we show the integral representation of solution to the system. For this, let us consider the first equation of (1.7) with $A = (-\Delta)^{\alpha/2}$ and $f = \nabla \cdot (u\nabla v)$. Taking Laplace transform on both sides and invoking

$$\mathcal{L}({}_t^c D_t^\beta u)(s) = s^\beta \mathcal{L}u(s) - s^{\beta-1}u(0),$$

we get

$$s^\beta \mathcal{L}u(s) + A\mathcal{L}u(s) = s^{\beta-1}u_0 + \mathcal{L}f(s).$$

This implies that

$$\mathcal{L}u(s) = s^{\beta-1}(s^\beta + A)^{-1}u_0 + (s^\beta + A)^{-1}\mathcal{L}f(s). \tag{A.1}$$

Now, applying the inverse Laplace transform to (A.1), we get

$$u(t) = \mathcal{L}^{-1}\left[\frac{s^{\beta-1}}{s^\beta + A}\right]u_0 + \mathcal{L}^{-1}\left[\frac{1}{s^\beta + A}\mathcal{L}f(s)\right]. \tag{A.2}$$

It is well known that (see [43])

$$\begin{aligned} \int_0^\infty e^{-st}t^{m\alpha+\beta-1}E_{\alpha,\beta}^{(m)}(\pm at^\alpha)\,\mathrm{d}t &= \frac{m!s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}, \\ \mathcal{L}^{-1}\left[\frac{s^{\alpha\gamma-\beta}}{(s^\alpha + a)^\gamma}\right] &= t^{\beta-1}E_{\alpha,\beta}^\gamma(-at^\alpha), \end{aligned} \tag{A.3}$$

where $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$. Using these equalities and the convolution theorem, (A.2) becomes

$$u(t) = E_\beta(-t^\beta A)u_0 + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-(t-s)^\beta A)f(s)\,\mathrm{d}s.$$

Similarly, taking Laplace transform of the second equation of (1.7), we get

$$s^\beta \mathcal{L}v(s) + A\mathcal{L}v(s) = s^{\beta-1}v(0) + \mathcal{L}u(s).$$

After simplification, it follows that

$$\mathcal{L}v(s) = \frac{s^{\beta-1}}{s^\beta + A}v_0 + \frac{1}{s^\beta + A}\mathcal{L}u(s). \tag{A.4}$$

Then, using (A.3), the integral representation of the solution to the second equation of (1.7) can be rewritten as

$$v(t) = E_\beta(-t^\beta A)v_0 + \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-(t-s)^\beta A)u(s)\,\mathrm{d}s$$

by applying the inverse Laplace transform to (A.4).

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References

[1] J. Adler, Chemotaxis in bacteria. *Annu. Rev. Biochem.* **44** (1975), no. 15, 341–356

[2] A. Akilandeeswari and J. Tyagi, Nonnegative solutions to time fractional Keller–Segel system. *Math. Methods Appl. Sci.* **44** (2021), no. 2, 1812–1830 Zbl 1470.35385 MR 4185347

[3] G. Arumugam and J. Tyagi, Keller–Segel chemotaxis models: A review. *Acta Appl. Math.* **171** (2021), article no. 6 Zbl 1464.35001 MR 4188348

[4] J. Azevedo, C. Cuevas, and E. Henriquez, Existence and asymptotic behaviour for the time-fractional Keller–Segel model for chemotaxis. *Math. Nachr.* **292** (2019), no. 3, 462–480 Zbl 1419.35200 MR 3932757

[5] N. Bellomo, A. Bellouquid, J. Nieto, and J. Soler, Multiscale biological tissue models and flux-limited chemotaxis for multicellular growing systems. *Math. Models Methods Appl. Sci.* **20** (2010), no. 7, 1179–1207 Zbl 1402.92065 MR 2673415

[6] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler, Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues. *Math. Models Methods Appl. Sci.* **25** (2015), no. 9, 1663–1763 Zbl 1326.35397 MR 3351175

[7] N. Bellomo and M. Winkler, A degenerate chemotaxis system with flux limitation: maximally extended solutions and absence of gradient blow-up. *Comm. Partial Differential Equations* **42** (2017), no. 3, 436–473 Zbl 1430.35166 MR 3620894

[8] N. Bellomo and M. Winkler, Finite-time blow-up in a degenerate chemotaxis system with flux limitation. *Trans. Amer. Math. Soc. Ser. B* **4** (2017), 31–67 Zbl 1367.35044 MR 3664719

[9] P. Biler, M. Cannone, I. A. Guerra, and G. Karch, Global regular and singular solutions for a model of gravitating particles. *Math. Ann.* **330** (2004), no. 4, 693–708 Zbl 1078.35097 MR 2102308

[10] P. Biler, T. Cieślak, G. Karch, and J. Zienkiewicz, Local criteria for blowup in two-dimensional chemotaxis models. *Discrete Contin. Dyn. Syst.* **37** (2017), no. 4, 1841–1856 Zbl 1360.35290 MR 3640577

[11] P. Biler, T. Funaki, and W. A. Woyczynski, Interacting particle approximation for nonlocal quadratic evolution problems. *Probab. Math. Statist.* **19** (1999), no. 2, 267–286 Zbl 0985.60091 MR 1750904

[12] P. Biler and G. Karch, Blowup of solutions to generalized Keller–Segel model. *J. Evol. Equ.* **10** (2010), no. 2, 247–262 Zbl 1239.35177 MR 2643796

[13] P. Biler and G. Wu, Two-dimensional chemotaxis models with fractional diffusion. *Math. Methods Appl. Sci.* **32** (2009), no. 1, 112–126 Zbl 1163.35490 MR 2477103

[14] J. T. Bonner, *The cellular slime molds*. Princeton University Press, Princeton, 1967

[15] J. Burczak and R. Granero-Belinchón, On a generalized doubly parabolic Keller–Segel system in one spatial dimension. *Math. Models Methods Appl. Sci.* **26** (2016), no. 1, 111–160 Zbl 1337.35153 MR 3417726

[16] M. Cannone, Harmonic analysis tools for solving the incompressible Navier–Stokes equations. In *Handbook of mathematical fluid dynamics. Vol. III*, pp. 161–244, North-Holland, Amsterdam, 2004 Zbl 1222.35139 MR 2099035

[17] E. A. Carlen and S. Ulusoy, Dissipation for a non-convex gradient flow problem of a Patlack–Keller–Segel type for densities on $\mathbb{R}^d$, $d \geq 3$. *Nonlinear Anal.* **208** (2021), article no. 112314 Zbl 1462.35072 MR 4222396

[18] J. A. Carrillo and K. Lin, Sharp conditions on global existence and blow-up in a degenerate two-species and cross-attraction system. *Adv. Nonlinear Anal.* **11** (2022), no. 1, 1–39 Zbl 1473.35064 MR 4280573

[19] T. Cieślak and C. Morales-Rodrigo, Quasilinear non-uniformly parabolic-elliptic system modelling chemotaxis with volume filling effect. Existence and uniqueness of global-in-time solutions. *Topol. Methods Nonlinear Anal.* **29** (2007), no. 2, 361–381 Zbl 1128.92005 MR 2345067

[20] T. Cieślak and M. Winkler, Finite-time blow-up in a quasilinear system of chemotaxis. *Nonlinearity* **21** (2008), no. 5, 1057–1076 Zbl 1136.92006 MR 2412327

[21] C. Cuevas, C. Silva, and H. Soto, On the time-fractional Keller–Segel model for chemotaxis. *Math. Methods Appl. Sci.* **43** (2020), no. 2, 769–798 Zbl 1445.35301 MR 4056463

[22] F. Dai and B. Liu, Asymptotic stability in a quasilinear chemotaxis-haptotaxis model with general logistic source and nonlinear signal production. *J. Differential Equations* **269** (2020), no. 12, 10839–10918 Zbl 1458.35052 MR 4150361

[23] M. Freitag, Blow-up profiles and refined extensibility criteria in quasilinear Keller–Segel systems. *J. Math. Anal. Appl.* **463** (2018), no. 2, 964–988 Zbl 1516.35115 MR 3785493

[24] R. Granero-Belinchón, On a drift-diffusion system for semiconductor devices. *Ann. Henri Poincaré* **17** (2016), no. 12, 3473–3498 Zbl 1361.82038 MR 3568023

[25] D. Horstmann, From 1970 until present: The Keller–Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein.* **105** (2003), no. 3, 103–165 Zbl 1071.35001 MR 2013508

[26] D. Horstmann, From 1970 until present: the Keller–Segel model in chemotaxis and its consequences. II. *Jahresber. Deutsch. Math.-Verein.* **106** (2004), no. 2, 51–69 Zbl 1072.35007 MR 2073515

[27] S. Ishida, L^∞-decay property for quasilinear degenerate parabolic-elliptic Keller–Segel systems. *Discrete Contin. Dyn. Syst.* (2013), no. Dynamical systems, differential equations and applications. 9th AIMS Conference. Suppl., 335–344 Zbl 1439.35268 MR 3462380

[28] S. Ishida and T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic-parabolic type. *J. Differential Equations* **252** (2012), no. 2, 1421–1440 Zbl 1237.35093 MR 2853545

[29] S. Ishida and T. Yokota, Global existence of weak solutions to quasilinear degenerate Keller–Segel systems of parabolic-parabolic type with small data. *J. Differential Equations* **252** (2012), no. 3, 2469–2491 Zbl 1241.35118 MR 2860626

[30] A. Jaiswal, P. Rani, and J. Tyagi, Global weak solutions of a parabolic-elliptic Keller–Segel system with gradient dependent chemotactic coefficients. *Discrete Contin. Dyn. Syst. Ser. B* **28** (2023), no. 7, 4144–4166 Zbl 1512.92005 MR 4553731

[31] K. H. Karlsen and S. Ulusoy, On a hyperbolic Keller–Segel system with degenerate nonlinear fractional diffusion. *Netw. Heterog. Media* **11** (2016), no. 1, 181–201 Zbl 1348.35289 MR 3461741

[32] E. F. Keller and L. A. Segel, Model for chemotaxis. *J. Theor. Biol.* **30** (1971), no. 2, 225–234 Zbl 1170.92307

[33] E. F. Keller and L. A. Segel, Traveling bands of chemotactic bacteria: A theoretical analysis. *J. Theor. Biol.* **30** (1971), no. 2, 235–248 Zbl 1170.92308

[34] J. Kemppainen, J. Siljander, and R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations. *J. Differential Equations* **263** (2017), no. 1, 149–201 Zbl 1366.35218 MR 3631303

[35] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*. North-Holland Math. Stud. 204, Elsevier Science B.V., Amsterdam, 2006 Zbl 1092.45003 MR 2218073

[36] A. Kiselev and L. Ryzhik, Biomixing by chemotaxis and enhancement of biological reactions. *Comm. Partial Differential Equations* **37** (2012), no. 2, 298–318 Zbl 1236.35190 MR 2876833

[37] A. Kiselev and X. Xu, Suppression of chemotactic explosion by mixing. *Arch. Ration. Mech. Anal.* **222** (2016), no. 2, 1077–1112 Zbl 1351.35233 MR 3544323

[38] H. Kozono and Y. Sugiyama, The Keller–Segel system of parabolic-parabolic type with initial data in weak $L^{n/2}(\mathbb{R}^n)$ and its application to self-similar solutions. *Indiana Univ. Math. J.* **57** (2008), no. 4, 1467–1500 Zbl 1158.35101 MR 2440871

[39] H. Kozono and Y. Sugiyama, Local existence and finite time blow-up of solutions in the 2-D Keller–Segel system. *J. Evol. Equ.* **8** (2008), no. 2, 353–378 Zbl 1162.35040 MR 2407206

[40] D. Li, J. L. Rodrigo, and X. Zhang, Exploding solutions for a nonlocal quadratic evolution problem. *Rev. Mat. Iberoam.* **26** (2010), no. 1, 295–332 Zbl 1195.35182 MR 2666316

[41] L. Li and J.-G. Liu, A generalized definition of Caputo derivatives and its application to fractional ODEs. *SIAM J. Math. Anal.* **50** (2018), no. 3, 2867–2900 Zbl 1401.26013 MR 3809535

[42] L. Li, J.-G. Liu, and L. Wang, Cauchy problems for Keller–Segel type time-space fractional diffusion equation. *J. Differential Equations* **265** (2018), no. 3, 1044–1096 Zbl 1427.35329 MR 3788635

[43] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-order systems and controls: Fundamentals and applications*. Adv. Ind. Control, Springer, London, 2010 Zbl 1211.93002 MR 3012798

[44] M. Negreanu and J. I. Tello, On a parabolic-elliptic system with gradient dependent chemotactic coefficient. *J. Differential Equations* **265** (2018), no. 3, 733–751 Zbl 1391.35186 MR 3788622

[45] C. S. Patlak, Random walk with persistence and external bias. *Bull. Math. Biophys.* **15** (1953), 311–338 Zbl 1296.82044 MR 0081586

[46] I. Podlubny, *Fractional differential equations*. Math. Sci. Eng. 198, Academic Press, San Diego, CA, 1999 Zbl 0924.34008 MR 1658022

[47] A. Raczyński, Existence of solutions for a model of self-gravitating particles with external potential. In *Nonlocal elliptic and parabolic problems*, pp. 263–275, Banach Center Publ. 66, Institute of Mathematics of the Polish Academy of Sciences, Warsaw, 2004 Zbl 1235.35150 MR 2143370

[48] A. Raczyński, Weak-L^p solutions for a model of self-gravitating particles with an external potential. *Studia Math.* **179** (2007), no. 3, 199–216 Zbl 1113.35030 MR 2291731

[49] Y. Sugiyama, Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis. *Differential Integral Equations* **20** (2007), no. 2, 133–180 Zbl 1212.35241 MR 2294463

[50] Y. Sugiyama and H. Kunii, Global existence and decay properties for a degenerate Keller–Segel model with a power factor in drift term. *J. Differential Equations* **227** (2006), no. 1, 333–364 Zbl 1102.35046 MR 2235324

[51] X. Tao, S. Zhou, and M. Ding, Boundedness of solutions to a quasilinear parabolic-parabolic chemotaxis model with nonlinear signal production. *J. Math. Anal. Appl.* **474** (2019), no. 1, 733–747 Zbl 1483.35111 MR 3912925

[52] M. E. Taylor, Remarks on fractional diffusion equations. In *Diffusion processes and other random processes*, pp. 1–44, 2018

[53] J. I. Tello, Blow up of solutions for a Parabolic–Elliptic chemotaxis system with gradient dependent chemotactic coefficient. *Comm. Partial Differential Equations* **47** (2022), no. 2, 307–345 Zbl 1491.35071 MR 4378609

[54] S. Ulusoy, A Keller–Segel type system in higher dimensions. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **34** (2017), no. 4, 961–971 Zbl 1435.35205 MR 3661866

[55] X. Wang, Z. Liu, and L. Zhou, Asymptotic decay for the classical solution of the chemotaxis system with fractional Laplacian in high dimensions. *Discrete Contin. Dyn. Syst. Ser. B* **23** (2018), no. 9, 4003–4020 Zbl 1410.35255 MR 3927586

[56] M. Winkler, Does a 'volume-filling effect' always prevent chemotactic collapse? *Math. Methods Appl. Sci.* **33** (2010), no. 1, 12–24 Zbl 1182.35220 MR 2591220

[57] M. Winkler, Global existence and slow grow-up in a quasilinear Keller–Segel system with exponentially decaying diffusivity. *Nonlinearity* **30** (2017), no. 2, 735–764 Zbl 1382.35048 MR 3604361

[58] M. Winkler, A critical blow-up exponent for flux limitation in a Keller–Segel system. *Indiana Univ. Math. J.* **71** (2022), no. 4, 1437–1465 Zbl 1501.35094 MR 4481090

[59] G. Wu and X. Zheng, On the well-posedness for Keller–Segel system with fractional diffusion. *Math. Methods Appl. Sci.* **34** (2011), no. 14, 1739–1750 Zbl 1229.35314 MR 2833827

[60] W. Zhang, Z. Liu, and L. Zhou, Global existence and asymptotic behavior of classical solutions to a fractional logistic Keller–Segel system. *Nonlinear Anal.* **189** (2019), article no. 111624 Zbl 1425.35080 MR 4007559

[61] P. Zheng, C. Mu, X. Hu, and Y. Tian, Boundedness of solutions in a chemotaxis system with nonlinear sensitivity and logistic source. *J. Math. Anal. Appl.* **424** (2015), no. 1, 509–522 Zbl 1307.35069 MR 3286576

[62] M. Zhuang, W. Wang, and S. Zheng, Boundedness in a fully parabolic chemotaxis system with logistic-type source and nonlinear production. *Nonlinear Anal. Real World Appl.* **47** (2019), 473–483 Zbl 1415.35067 MR 3894330

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