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Twisted tensor products of field algebras

Ezio Vasselli

Abstract. Let $\mathcal{A}$ be a C*-algebra, $\boldsymbol{h}$ a Hilbert space, and $\mathcal{C}_{\boldsymbol{h}}$ the CAR algebra over $\boldsymbol{h}$. We construct a twisted tensor product of $\mathcal{A}$ by $\mathcal{C}_{\boldsymbol{h}}$ such that the two factors are not necessarily one in the relative commutant of the other. The resulting C*-algebra may be regarded as a generalized CAR algebra constructed over a suitable Hilbert $\mathcal{A}$-bimodule. As an application, we exhibit a class of fixed-time models where a free Dirac field (giving rise to the $\mathcal{C}_{\boldsymbol{h}}$ factor) in general is not relatively local to a free scalar field (which yields the $\mathcal{A}$ factor). In some of the models, gauge-invariant combinations of the two (not relatively local) fields form a local observable net.

1. Introduction

In quantum field theory, a common way to construct a field system is to perform the tensor product of Fock spaces and define for each factor the corresponding free field. This method is used in both exactly solvable models and perturbative theory, as a preliminary step to perform Wick or time-ordered products and get the corresponding Wightman or Green functions. At the algebraic level, the typical construction is given by the spatial tensor product $\mathcal{F} \doteq \mathcal{C}_{\boldsymbol{h}} \otimes \mathcal{A}$, where $\mathcal{C}_{\boldsymbol{h}}$ is the CAR algebra over a Hilbert space $\boldsymbol{h}$ (generalized free Fermi field) and $\mathcal{A}$ is a CCR algebra (generalized free bosonic field).

The starting point of the present paper is the remark that we may regard $\mathcal{F}$ as the C*-algebra generated by a Fermi field ψ intended in a broader sense, defined in terms of creation and annihilation operators living in a fermionic Fock bimodule $\mathfrak{F}_-(\mathfrak{h})$. Here, $\mathfrak{h} \doteq \boldsymbol{h} \otimes \mathcal{A}$ is the free Hilbert bimodule carrying the *trivial left action* uniquely determined by the relations $wA = Aw$, $w \in \boldsymbol{h}$, $A \in \mathcal{A}$. Adopting this point of view, $\mathcal{A}$ appears as part of the target space for the "non-commutative anticommutator function" of ψ.

This suggests a strategy to escape from tensor products and possibly produce interesting models. The idea is that of considering Hilbert bimodules carrying a non-trivial left action, with the aim of constructing field systems where the bosonic components do not necessarily commute with the fermionic ones. This is what happens, for example, in QED, where (in positive gauges) charged fields cannot be relatively local to the electromagnetic field [12].

The aim of the present paper is to provide an algebraic machinery to construct field C*-algebras having the above-mentioned property. We start by reviewing generic Hilbert

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bimodules, their Fock bimodules, and the corresponding GNS Hilbert spaces obtained by applying states of $\mathcal{A}$; see Section 2. On these spaces, there are well-defined annihilation and creation operators, and the "zero particle space" is the GNS space of $\mathcal{A}$ instead of $\mathbb{C}$ as in usual Fock space. Annihilation operators—while they perform the usual operation of annihilating states defined by $\mathfrak{h}$—modify (without annihilating) states in the GNS space of $\mathcal{A}$. Thus, in the present paper, the term *annihilation operator* should be intended in this broader sense.

To define fermionic spaces, one should introduce a permutation symmetry on Fock bimodules. Since in general this is an impossible task, we focus on free Hilbert bimodules for which it is possible to define a permutation symmetry in the obvious way; see Section 3. This does not necessarily lead to the trivial construction $\mathcal{F} = \mathcal{C}_{\boldsymbol{h}} \otimes \mathcal{A}$ because the crucial property to avoid it is non-triviality of the left $\mathcal{A}$-action. Having defined our fermionic Fock space, we face the fact that antisymmetry

$$f \otimes_{-} g = -g \otimes_{-} f, \quad f, g \in \mathfrak{h}, \tag{1.1}$$

does not hold in full generality. The reason relies on commutation properties of $\mathcal{A}$: namely, there are suitable support C*-subalgebras $\mathcal{A}(f)$, $\mathcal{A}(g)$ of $\mathcal{A}$, and a sufficient condition for (1.1) holding true is that

$$[\mathcal{A}(f), \mathcal{A}(g)] = 0$$

together with the fact that the left $\mathcal{A}(g)$-action on f is trivial and analogously for $\mathcal{A}(f)$ on g; see Remark 3.5. In this case, we say that f and g are *mutually free*.

Another consequence of the fact that we are dealing with Hilbert modules rather than Hilbert spaces is given by the non-stability of the fermionic Fock module both under the action of creation operators and the left $\mathcal{A}$-action. Both these problems are solved by considering a particular class of left $\mathcal{A}$-actions that we call *$\mathcal{G}$-twists*, where $\mathcal{G}$ is a group of unitary generators of $\mathcal{A}$ (Lemma 3.6). $\mathcal{G}$-twists are given by group morphisms $u : \mathcal{G} \to \mathcal{U}(\boldsymbol{h})$ that are used to perturb the trivial left $\mathcal{A}$-action; see Definition 3.2. We exhibit two classes of C*-algebras that naturally admit $\mathcal{G}$-twists: the first is given by Weyl algebras (giving rise to generalized free bosonic fields) (Example 3.3); the second, Example 3.4, is the universal C*-algebra of the electromagnetic field [8].

Finally, in Section 4, we construct our generalized CAR algebra having as input the free module $\mathfrak{h}$ and the twist u; see Theorem 4.3. Adding as a third ingredient a suitable conjugation κ acting on $\boldsymbol{h}$, we obtain what we call a *Dirac triple over* $\mathcal{A}$, from which we construct a Dirac field (4.11) and the corresponding field algebra (4.13). It is this field algebra that defines the desired twisted tensor product of $\mathcal{A}$ by $\mathcal{C}_{\boldsymbol{h}}$.

As an application, we exhibit a family of models in a fixed-time formulation, depending on a tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^3)$ that defines the twist. In our models, a free Dirac field (generating the C*-algebra $\mathcal{C}_{\boldsymbol{h}}$) and a free scalar field (generating a Weyl C*-algebra $\mathcal{W}$) give rise to the following situations: (1) $\mathcal{C}_{\boldsymbol{h}}$ and $\mathcal{W}$ do not commute but are relatively local (when σ is the Dirac delta); (2) $\mathcal{C}_{\boldsymbol{h}}$ and $\mathcal{W}$ are not relatively local (for σ having support with non-empty interior); (3) $\mathcal{W}$ is not relatively local to $\mathcal{C}_{\boldsymbol{h}}$ but is in the commutant of the fixed-point algebra of $\mathscr{C}_{\boldsymbol{h}}$ under the gauge action (when σ is the Lebesgue measure).

A discussion of our results, work in progress, and perspectives are given in the final section 5.

In the following points, we fix some conventions:

- Throughout the present paper, $\mathcal{A}$ will denote a C*-algebra with unit $\mathbf{1}$ and $\mathcal{S}(\mathcal{A})$ its state space. Unless otherwise stated, Hilbert space representations and *-morphisms are assumed to be non-degenerate (hence unital).
- We reserve Euler Fraktur fonts for Hilbert modules, related objects ($\mathfrak{h}$, $\mathfrak{B}(\mathfrak{h})$, $\mathfrak{F}(\mathfrak{h})$, and so on), and objects defined starting from Hilbert modules (for example, the Hilbert spaces $\mathfrak{h}^\omega$ for $\omega \in \mathcal{S}(\mathcal{A})$). Instead, unless otherwise stated, the calligraphic font is used for Hilbert spaces ($\mathcal{H}$) and C*-algebras ($\mathcal{A}$, $\mathcal{B}(\mathcal{H})$, …). As an exception to this rule, we adopt a lowercase bold font for "one-particle Hilbert spaces", which therefore are denoted by $\boldsymbol{h}$.
- Free Hilbert modules usually appear in literature with the notation $\mathfrak{h} = \boldsymbol{h} \otimes \mathcal{A}$, where $\boldsymbol{h}$ is a Hilbert space. To use the symbol $\otimes$ without ambiguities, in the sequel up to rare exceptions, we will write $\boldsymbol{h} \otimes \mathcal{A} \equiv \boldsymbol{h}\mathcal{A}$ and $v \otimes A \equiv vA$ for $v \in \boldsymbol{h}$ and $A \in \mathcal{A}$. The symbol $\otimes$ will be used for the internal tensor product of Hilbert bimodules or the Hilbert space tensor product.
- For generic elements of $\mathfrak{h} = \boldsymbol{h}\mathcal{A}$, we will often use the notation of implicit sum for repeated indices, $\sum_i v_i A_i \equiv v_i A_i$. The sum may be infinite, and in this case, our notation should be understood in terms of norm convergence in $\mathfrak{h}$.

2. Fock spaces over C*-algebras

In the present section, we collect some standard facts on Hilbert bimodules that we present in a form suitable for our purposes. References are Blackadar's book [5] and the papers [2, 15, 16, 18] for GNS-representations and Fock bimodules, respectively. For the use of Hilbert bimodules in mathematical physics, see [1, 3, 4, 10, 11, 17, 18] and the recent review [2].

Hilbert modules. A *(right Hilbert) $\mathcal{A}$-module* is a complex vector space $\mathfrak{h}$ carrying a right $\mathcal{A}$-module action and endowed with a $\mathcal{A}$-valued, strictly positive, and right $\mathcal{A}$-linear scalar product $\langle\cdot,\cdot\rangle$, which induces the norm $\|\langle\cdot,\cdot\rangle\|$ under which $\mathfrak{h}$ is a Banach space. It is worth noting that then

$$A\langle f, g\rangle = (\langle g, f\rangle A^*)^* = (\langle g, fA^*\rangle)^* = \langle fA^*, g\rangle$$

for all $f, g \in \mathfrak{h}$, $A \in \mathcal{A}$. An $\mathcal{A}$-module $\mathfrak{h}$ defines the C*-algebra $\mathfrak{B}(\mathfrak{h})$ of linear operators $T : \mathfrak{h} \to \mathfrak{h}$ such that there is $T^* \in \mathfrak{B}(\mathfrak{h})$ with $\langle T^* f, g\rangle = \langle f, Tg\rangle$, $f, g \in \mathfrak{h}$. This implies that $T \in \mathfrak{B}(\mathfrak{h})$ is right $\mathcal{A}$-linear, i.e., $T(fA) = (Tf)A$, and bounded. The C*-algebra of compact operators is given by the closed ideal $\mathfrak{K}(\mathfrak{h}) \subseteq \mathfrak{B}(\mathfrak{h})$ generated by the elementary operators $|f\rangle\langle g|h \doteq f\langle g,h\rangle$, $h \in \mathfrak{h}$. We say that $\mathfrak{h}$ is a *Hilbert $\mathcal{A}$-bimodule* whenever there is a *-morphism $\lambda : \mathcal{A} \to \mathfrak{B}(\mathfrak{h})$, called the left action. (Hilbert bimodules are also known as *C*-correspondences* [2].) In the present paper, we will assume that λ is faithful and non-degenerate, and following a standard notation, we will write

$$Af \equiv \lambda(A)f, \quad \forall f \in \mathfrak{h}, A \in \mathcal{A}.$$

Let $\omega \in \mathcal{S}(\mathcal{A})$ be a state. Then, $\langle f,g\rangle_\omega \doteq \omega(\langle f,g\rangle)$ is a scalar product and defines the Hilbert space $\mathfrak{h}^\omega$ whose elements $f^\omega \in \mathfrak{h}^\omega$ are defined in correspondence with $f \in \mathfrak{h}$. We have the representation

$$\pi^\omega : \mathfrak{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{h}^\omega), \quad T^\omega f^\omega \doteq (Tf)^\omega. \tag{2.1}$$

The argument to prove that (2.1) is well defined is standard, so we give just a sketch. Let $T \in \mathfrak{B}(\mathfrak{h})$ and $v \in \mathfrak{h}$ such that $v^\omega = 0$, i.e., $\langle v,v\rangle_\omega = 0$; then, $\|T\|^2 - T^*T$ is a positive element of $\mathfrak{B}(\mathfrak{h})$ and this implies that $\langle v, (\|T\|^2 - T^*T)v\rangle \in \mathcal{A}$ is positive for any $v \in \mathfrak{h}$. As a consequence,

$$0 = \|T\|^2\langle v,v\rangle_\omega \geq \langle v, T^*Tv\rangle_\omega = \langle Tv, Tv\rangle_\omega \geq 0,$$

implying $(Tv)^\omega = 0$, and T^ω is a well-defined linear operator on $\mathcal{B}(\mathfrak{h}^\omega)$ with norm $\leq \|T\|$. Note that (2.1) restricts to the representation of the left $\mathcal{A}$-action

$$\pi^\omega : \mathcal{A} \to \mathcal{B}(\mathfrak{h}^\omega), \quad A^\omega f^\omega \doteq (Af)^\omega. \tag{2.2}$$

We call (2.1) and (2.2) the GNS-representations induced by ω.

Example 2.1. We set $\mathfrak{h} \doteq \mathcal{A}$ with right (left) $\mathcal{A}$-module action given by right (left) multiplication and scalar product $\langle f,g\rangle \doteq f^*g$, $f,g \in \mathfrak{h}$. If $\omega \in \mathcal{S}(\mathcal{A})$, then it is readily seen that (2.2) is the GNS representation in the usual sense. For future use, we write v_A^ω to indicate the GNS vector corresponding to $A \in \mathcal{A}$ so that $\|v_A^\omega\|^2 = \omega(A^*A)$.

Example 2.2. Let $\mathfrak{h} \doteq \boldsymbol{h}\mathcal{A}$ denote the free right Hilbert $\mathcal{A}$-module, with right action $vB \doteq w \otimes AB$ and scalar product $\langle v,v'\rangle \doteq A^*A'\langle w,w'\rangle$, where $v \doteq w \otimes A$, $v' \doteq w' \otimes A'$, $w,w' \in \boldsymbol{h}$, $A, A', B \in \mathcal{A}$. If $\lambda : \mathcal{A} \to \mathfrak{B}(\mathfrak{h}) \simeq \mathcal{B}(\boldsymbol{h}) \otimes \mathcal{A}$ is a *-morphism, then we have the left $\mathcal{A}$-action $Af \doteq \lambda(A)f$. Given $\omega \in \mathcal{S}(\mathcal{A})$, the Hilbert space $\mathfrak{h}^\omega$ is isomorphic to $\boldsymbol{h} \otimes \boldsymbol{h}^\omega$, where $\boldsymbol{h}^\omega$ is the GNS space of ω.

The Fock bimodule. Let $n \in \mathbb{N}$ and $\mathfrak{h}$ denote a Hilbert bimodule. We consider the tensor product of complex vector spaces $\mathfrak{h}^{\odot n}$, which we endow with the scalar product

$$\langle v,w\rangle \doteq \langle v_n, \langle v_{n-1}, \langle \cdots \langle v_1, w_1\rangle \cdots\rangle w_{n-1}\rangle w_n\rangle, \tag{2.3}$$

where $v \doteq v_1 \otimes \cdots \otimes v_n$ and w are in $\mathfrak{h}^{\odot n}$. Note that the term $\langle v_{n-1}, \ldots, w_{n-1}\rangle$ appearing in the previous expression belongs to $\mathcal{A}$, so it makes sense to consider its left action on w_n, and analogously for the nested terms $\langle v_{n-k}, \dots, w_{n-k} \rangle$, $k = 2, \dots, n-1$. The completion obtained by (2.3) is denoted by $\mathfrak{h}^n$ and is called the *internal n-fold tensor product*: it is endowed with the $\mathcal{A}$-module actions

$$Av \doteq Av_1 \otimes \cdots \otimes v_n, \quad vA \doteq v_1 \otimes \cdots \otimes v_n A,$$

and therefore, it forms a Hilbert $\mathcal{A}$-bimodule. By construction, it turns out

$$\cdots \otimes v_k \otimes Av_{k+1} \otimes \cdots = \cdots \otimes v_k A \otimes v_{k+1} \otimes \cdots, \quad \forall A \in \mathcal{A}.$$

Remark 2.3. A delicate point of the tensor product of Hilbert bimodules is that an elementary tensor $v \in \mathfrak{h}^{\odot n}$ may have norm zero even when the all the factors $v_1, \dots, v_n$ do not. For example, if $\mathcal{A} = C(X)$ is commutative, $\mathfrak{h}$ is the module of sections of a vector bundle $E \to X$, and $Af = fA$ is defined by pointwise multiplication for all $A \in C(X)$ and $f \in \mathfrak{h}$, then $\langle v, v \rangle = \langle v_1, v_1 \rangle \langle v_2, v_2 \rangle = 0$ for $v \doteq v_1 \otimes v_2$ and $\mathrm{supp}(v_1) \cap \mathrm{supp}(v_2) = \emptyset$.

The direct sum of Hilbert $\mathcal{A}$-bimodules $\mathfrak{h}, \mathfrak{h}'$ is defined in the obvious way and yields the $\mathcal{A}$-bimodule $\mathfrak{h} \oplus \mathfrak{h}'$. Thus, we define the *Fock $\mathcal{A}$-bimodule*

$$\mathfrak{F}(\mathfrak{h}) \doteq \bigoplus_{n=0}^{\infty} \mathfrak{h}^n,$$

with $\mathfrak{h}^0 \doteq \mathcal{A}$ and $\mathfrak{h}^1 \doteq \mathfrak{h}$; any $v \in \mathfrak{F}(\mathfrak{h})$ is a sequence $v = \{v^n \in \mathfrak{h}^n\}$. Let now $f \in \mathfrak{h}$, $n \geq 1$, and $v \doteq v_1 \otimes \cdots \otimes v_n \in \mathfrak{h}^n$. We set

$$\langle f | v \doteq \langle f, v_1 \rangle v_2 \otimes \cdots \otimes v_n,$$

obtaining a right $\mathcal{A}$-linear operator $\langle f | : \mathfrak{h}^n \to \mathfrak{h}^{n-1}$. We define the annihilation operator

$$(a(f)v)^{n-1} \doteq \sqrt{n}\, \langle f | v^n, \quad n \geq 1,$$

with domain $\mathrm{Dom}(a(f)) \doteq \mathfrak{F}^{\#}(\mathfrak{h}) \doteq \{v \in \mathfrak{F}(\mathfrak{h}) : \exists \sum_n n \langle v^n, v^n \rangle \in \mathcal{A}\}$. Next, we define the creation operator

$$\begin{cases} (a^*(f)v)^{n+1} \doteq \sqrt{n+1}\, f \otimes v^n, & n \geq 1, \\ (a^*(f)v)^1 \doteq f v^0, & n = 0, \end{cases}$$

for $v \in \mathrm{Dom}(a^*(f)) \doteq \mathfrak{F}^{\#}(\mathfrak{h})$. Note that $v^0 \in \mathcal{A}$, so the expression $fv^0 \in \mathfrak{h}$ makes sense. Moreover, if $v^n = 0$ for all $n \geq 1$, then $a(f)v = 0$ so that $\mathcal{A} = \mathfrak{h}^0$ is in the kernel of $a(f)$. We have

$$a(f)a^*(g) = (n+1)\langle f, g \rangle, \quad a^*(g)a(f) = ng \otimes \langle f |, \quad f, g \in \mathfrak{h}.$$

The above notation suggests that $a^*(f)$ is the adjoint of $a(f)$, and, actually, for elementary tensors $v \in \mathfrak{h}^n$, $w \in \mathfrak{h}^{n-1}$, we have

$$\begin{aligned} \langle v, a^*(f)w \rangle &= \sqrt{n}\, \langle v, f \otimes w \rangle \\ &= \sqrt{n}\, \langle v_2 \otimes \cdots \otimes v_n, \langle v_1, f \rangle w \rangle \\ &= \sqrt{n}\, \langle \langle f, v_1 \rangle v_2 \otimes \cdots \otimes v_n, w \rangle \\ &= \langle a(f)v, w \rangle. \end{aligned}$$

Thus, $a^*(f)$ behaves as the adjoint of $a(f)$ on the common domain $\mathfrak{F}^{\#}(\mathfrak{h})$. Note that $a^*(f)$ and $a(f)$ are unbounded; thus, we should be careful when we use the term *adjoint*. Yet, since we are interested in the fermionic case where it will be shown that the creation and annihilation operators are bounded, we prefer to not discuss this point here.

Hilbert spaces in Hilbert modules. A *Hilbert space in* $\mathfrak{h}$ is given by a closed vector subspace $\boldsymbol{h} \subset \mathfrak{h}$ such that $\langle w, w' \rangle \in \mathbb{C}\mathbf{1}$, $\forall w, w' \in \boldsymbol{h}$. The proof of the following result is trivial; therefore, it is omitted.

Lemma 2.4. *Let* $\boldsymbol{h} \subset \mathfrak{h}$ *be a Hilbert space. Then, there is an injective linear mapping* $\mathcal{F}(\boldsymbol{h}) \to \mathfrak{F}(\mathfrak{h})$ *preserving the scalar product, where* $\mathcal{F}(\boldsymbol{h})$ *is the Fock space in the usual sense.*

GNS-Hilbert spaces of the Fock bimodule. Let now $\omega \in \mathcal{S}(\mathcal{A})$ and $\mathfrak{h}^{n,\omega}$ denote the Hilbert space obtained by completion of $\mathfrak{h}^n$ under the scalar product $\langle \cdot, \cdot \rangle_\omega$. Moreover, let $\mathfrak{F}^\omega(\mathfrak{h})$ denote the Hilbert space obtained by the analogous completion of $\mathfrak{F}(\mathfrak{h})$. Basic properties of this Hilbert space are resumed in the following result.

Proposition 2.5. *Let* $\mathfrak{h}$ *be a Hilbert* $\mathcal{A}$*-bimodule and* $\omega \in \mathcal{S}(\mathcal{A})$. *With the above notation, there is a decomposition*

$$\mathfrak{F}^\omega(\mathfrak{h}) \simeq \bigoplus_{n=0}^{\infty} \mathfrak{h}^{n,\omega}. \tag{2.4}$$

The component $\mathfrak{h}^{0,\omega}$ *is the usual GNS Hilbert space of* ω*, and there is a representation*

$$\widehat{\pi}^\omega : \mathcal{A} \to \mathcal{B}(\mathfrak{F}^\omega(\mathfrak{h})), \quad \widehat{\pi}^\omega = \bigoplus_n \pi^{n,\omega}, \tag{2.5}$$

where each $\pi^{n,\omega}$ *is defined as in* (2.2) *and*

$$\pi^{0,\omega} : \mathcal{A} \to \mathcal{B}(\mathfrak{h}^{0,\omega})$$

is the usual GNS representation. Given a Hilbert space $\boldsymbol{h} \subset \mathfrak{h}$*, the closed vector subspace of* $\mathfrak{F}^\omega(\mathfrak{h})$ *spanned by elements of the type* $(wA)^\omega$*,* $w \in \boldsymbol{h}^n$*,* $A \in \mathcal{A}$*,* $n \in \mathbb{N}$*, is a Hilbert space isomorphic to* $\mathcal{F}(\boldsymbol{h}) \otimes \mathfrak{h}^{0,\omega}$*, where* $\mathcal{F}(\boldsymbol{h})$ *is the Fock space in the usual sense so that there is an embedding*

$$\iota : \mathcal{F}(\boldsymbol{h}) \otimes \mathfrak{h}^{0,\omega} \to \mathfrak{F}^\omega(\mathfrak{h}), \quad \iota(w \otimes v_A^\omega) \doteq (wA)^\omega. \tag{2.6}$$

Proof. The direct sum decomposition (2.4) is obvious. About $\mathfrak{h}^{0,\omega}$, by definition, it is given by the completion of $\mathfrak{h}^0 = \mathcal{A}$ under the scalar product $\omega(A^*A')$, $A, A' \in \mathcal{A}$, so it is the GNS-space of ω. The decomposition (2.5) trivially follows by the previous points, as well as the fact that $\pi^{0,\omega}$ is the GNS representation of ω. Finally, the embedding (2.6) follows by the fact that, given elementary tensors $w, w' \in \boldsymbol{h}^n$, $A, A' \in \mathcal{A}$,

$$\langle wA, w'A' \rangle = A^* \prod_k \langle w_k, w'_k \rangle A',$$

having used the fact that

$$\langle w_n, \langle w_{n-1}, \langle \cdots \langle w_1, w_1' \rangle \cdots \rangle w_{n-1}' \rangle w_n' \rangle = \prod_k \langle w_k, w_k' \rangle.$$

(Recall that $\langle w_k, w_k' \rangle \in \mathbb{C}$ for all k). ∎

Remark 2.6 (Non-Fock nature of $\mathfrak{F}^\omega(\mathfrak{h})$). We emphasize that in general $\mathfrak{h}^{n,\omega}$ is not the n-fold tensor product $(\mathfrak{h}^\omega)^{\otimes n}$ in the sense of Hilbert spaces, and as a consequence, $\mathfrak{F}^\omega(\mathfrak{h})$ is not the Fock space of $\mathfrak{h}^\omega$. This is readily seen by comparing on elementary tensors $v, w \in \mathfrak{h}^n$ the scalar product obtained by (2.3),

$$\langle v^\omega, w^\omega \rangle \doteq \langle v_n, \langle v_{n-1}, \langle \cdots \langle v_1, w_1 \rangle \cdots \rangle w_{n-1} \rangle w_n \rangle_\omega$$

with the scalar product of $(\mathfrak{h}^\omega)^{\otimes n}$,

$$\langle v^\omega, w^\omega \rangle' \doteq \prod_k \langle v_k, w_k \rangle_\omega,$$

the latter corresponding to the Fock space of $\mathfrak{h}^\omega$.

Remark 2.7. Let $\boldsymbol{h} \subset \mathfrak{h}$ be a Hilbert space. Then, the embedding of $\mathscr{F}(\boldsymbol{h})$ into $\mathfrak{F}(\mathfrak{h})$ (Lemma 2.4) factorizes through the isometric mapping

$$\mathscr{F}(\boldsymbol{h}) \simeq \mathscr{F}(\boldsymbol{h}) \otimes \Omega \to \mathfrak{F}^\omega(\mathfrak{h}), \quad w \otimes \Omega \mapsto w^\omega,$$

where $\Omega \in \mathfrak{h}^{0,\omega}$ is the GNS-vector ("vacuum") obtained by (2.6) for $A \equiv \mathbf{1}$. Thus, $\mathscr{F}(\boldsymbol{h})$ embeds in any GNS space $\mathfrak{F}^\omega(\mathfrak{h})$.

Creation and annihilation operators in $\mathfrak{F}^\omega(\mathfrak{h})$. Let $f \in \mathfrak{h}$ and $n \in \mathbb{N}$. Then, the mapping $w \mapsto f \otimes w$ defines a right $\mathcal{A}$-linear operator $T(f) : \mathfrak{h}^n \to \mathfrak{h}^{n+1}$, and since

$$\langle T(f)w, v \rangle = \langle f \otimes w, v \rangle = \langle w, \langle f, v_1 \rangle (v_2 \otimes \cdots \otimes v_n) \rangle = \langle w, \langle f | v \rangle,$$

we find that $T(f)$ has adjoint $T(f)^* = \langle f |$ and, as a consequence, is bounded [5, Prop. 13.2.2]. (This also implies that $\langle f |$ is bounded and adjointable.) Therefore, the argument used to construct the representation (2.2) allows to define the linear operators

$$T(f)^\omega : \mathfrak{h}^{n,\omega} \to \mathfrak{h}^{n+1,\omega}, \quad \langle f |^\omega : \mathfrak{h}^{n+1,\omega} \to \mathfrak{h}^{n,\omega}.$$

Rescaling these operators by the $\sqrt{n}$ factors in dependence of the order of the tensor powers, we obtain the creation and annihilation operators

$$(a^\omega(f) v^\omega)^{n-1} \doteq \sqrt{n} \langle f |^\omega v^\omega = \sqrt{n} (\langle f | v)^\omega,$$

$$\begin{cases} (a^{*,\omega}(f) v^\omega)^{n+1} \doteq \sqrt{n+1} (f \otimes v^n)^\omega, & n \geq 1, \\ (a^{*,\omega}(f) v)^1 \doteq (f v^0)^\omega, & n = 0, \end{cases}$$

where $v^\omega = \{v^{n,\omega}\}$ is defined for $v \in \mathfrak{F}^\#(\mathfrak{h})$. Clearly, $a^{*,\omega}(f)$ is (contained in) the adjoint of $a^\omega(f)$ in the sense of Hilbert spaces:

$$\begin{aligned}\langle w^{\omega,n-1}, (a^\omega(f)v^\omega)^{n-1}\rangle_\omega &= \sqrt{n}\langle w^{n-1}, \langle f, v_1\rangle(v_2 \otimes \cdots \otimes v_n)\rangle_\omega \\ &= \sqrt{n}\langle f \otimes w^{n-1}, v_1 \otimes \cdots \otimes v_n\rangle_\omega \\ &= \langle (a^{*,\omega}(f)w^\omega)^n, v^\omega\rangle_\omega.\end{aligned}$$

Note that since $a^*(f)$ may have non-trivial kernel (Remark 2.3), the same is true for $a^{*,\omega}(f)$.

3. Twisted module actions and permutation symmetry

In the present section, we study the permutation symmetry on tensor powers of Hilbert bimodules or, to be honest, the factors that in general prevent the realization of this property. These factors are easily identified and are non-commutativity of $\mathcal{A}$ and the fact that in general $Af \neq fA$, $A \in \mathcal{A}$, $f \in \mathfrak{h}$. Anyway, even with the above limitations, we will give a version of the fermionic Fock space for free Hilbert bimodules and define the corresponding restrictions of the creation and annihilation operators. The tool used to perform these constructions is the one of *twist*, namely, a "perturbation" of the trivial left $\mathcal{A}$-module action defined using a group of unitary generators of $\mathcal{A}$. We remark that thanks to the Kasparov stabilization theorem [5] the choice of considering only free bimodules is not a severe requirement.

Let us now consider an elementary tensor $v = v_1 \otimes \cdots \otimes v_n \in \mathfrak{h}^n$. Given a permutation $\varrho \in \mathbb{P}(n)$, we would like to define the operator

$$U_\varrho v \doteq v_{\varrho(1)} \otimes \cdots \otimes v_{\varrho(n)}. \tag{3.1}$$

It is obvious that the above operator is well defined on the algebraic elementary tensors $v_1 \odot \cdots \odot v_n$; thus, the question is what happens when one applies the norm induced by $\mathcal{A}$. Here, we immediately encounter a problem, illustrated by the following example. Consider a Hilbert bimodule $\mathfrak{h}$ of the type Example 2.1 with $\mathfrak{h} = \mathcal{A} = \mathcal{O}_2$, the Cuntz algebra generated by mutually orthogonal isometries ψ_1, ψ_2; we take $v_1 \doteq \psi_1$, $v_2 \doteq \psi_2^*$ and evaluate the scalar products

$$\begin{aligned}\langle v_1 \otimes v_2, v_1 \otimes v_2\rangle &= \langle v_2, \langle v_1, v_1\rangle v_2\rangle = \psi_2\psi_1^*\psi_1\psi_2^* = \psi_2\psi_2^*, \\ \langle v_2 \otimes v_1, v_2 \otimes v_1\rangle &= \langle v_1, \langle v_2, v_2\rangle v_1\rangle = \psi_1^*\psi_2\psi_2^*\psi_1 = 0.\end{aligned}$$

This example shows that the above operators U_ϱ in general are ill-defined because they map zero norm algebraic elementary tensors into tensors with norm $\neq 0$. This also spoils the preservation of compositions of permutations in (3.1). For example, given the flip $\varrho(1) = 2$, $\varrho(2) = 1$, in the situation of the previous example, we find

$$\|U_\varrho(v_2 \otimes v_1)\| = \|v_1 \otimes v_2\| \neq 0, \quad U_\varrho U_\varrho(v_1 \otimes v_2) = 0, \quad U_{\varrho\varrho}(v_1 \otimes v_2) = v_1 \otimes v_2.$$

Free modules. In the language adopted in the present paper, a Hilbert bimodule $\mathfrak{h}$ is free if, and only if, there is a Hilbert space $\boldsymbol{h} \subset \mathfrak{h}$ such that

$$\mathfrak{h} = \boldsymbol{h}\mathcal{A} \doteq \text{closedspan}\{wA : w \in \boldsymbol{h},\ A \in \mathcal{A}\}. \tag{3.2}$$

Lemma 3.1. *Let $\mathfrak{h}$ be a free Hilbert bimodule. Then, there is an isomorphism of right Hilbert modules $\mathfrak{F}(\mathfrak{h}) \simeq \mathcal{F}(\boldsymbol{h})\mathcal{A}$ and, given a state $\omega \in \mathcal{S}(\mathcal{A})$, there is an isomorphism of Hilbert spaces $\mathfrak{F}^\omega(\mathfrak{h}) \simeq \mathcal{F}(\boldsymbol{h}) \otimes \mathfrak{h}^{0,\omega}$, where $\mathfrak{h}^{0,\omega}$ is the usual GNS Hilbert space. The left $\mathcal{A}$-module action λ defines by evaluation the representation*

$$\lambda^\omega : \mathcal{A} \to \mathcal{B}(\mathcal{F}(\boldsymbol{h}) \otimes \mathfrak{h}^{0,\omega}).$$

Proof. Let $n \in \mathbb{N}$ and $v = v_1 \otimes \cdots \otimes v_n \in \mathfrak{h}^n$; then, by (3.2), we may write $v_i = w_k A_{k;i}$, where $\{w_k\}$ is an orthonormal basis of $\boldsymbol{h}$, $A_{k;i} \doteq \langle w_k, v_i\rangle \in \mathcal{A}$, and the (possibly infinite) sum on the repeated index k is performed. Using the above decomposition, it is easily seen that $v \in \boldsymbol{h}^n\mathcal{A}$ and we conclude that $\mathfrak{h}^n \subseteq \boldsymbol{h}^n\mathcal{A}$. Since the opposite inclusion is obvious, we have $\mathfrak{h}^n = \boldsymbol{h}^n\mathcal{A}$, and this proves that $\mathfrak{h}^n \simeq \boldsymbol{h}^n\mathcal{A}$ implying $\mathfrak{F}(\mathfrak{h}) \simeq \mathcal{F}(\boldsymbol{h})\mathcal{A}$. Thus, $\mathfrak{F}^\omega(\mathfrak{h}) \simeq \mathcal{F}(\boldsymbol{h}) \otimes \mathfrak{h}^{0,\omega}$ by Proposition 2.5. ■

We will see in the following sections that in general $\lambda^\omega(\mathcal{A})$ is not in the commutant of $\mathcal{B}(\mathcal{F}(\boldsymbol{h}))$ in $\mathcal{B}(\mathcal{F}(\boldsymbol{h}) \otimes \mathfrak{h}^{0,\omega})$; moreover, operators of the type $\lambda^\omega(A)$, $A \in \mathcal{A}$, mix vectors of $\mathcal{F}(\boldsymbol{h})$ and $\mathfrak{h}^{0,\omega}$ in the sense that, given the GNS-vector $\Omega \doteq v_1^\omega \in \mathfrak{h}^{0,\omega}$, typically we will find $\lambda^\omega(A)(v \otimes \Omega) = \sum_k v_k \otimes v_k^0$ with $v_k \neq v$ and $v_k^0 \neq \Omega$. In contrast, let us define the left $\mathcal{A}$-module action:

$$A(wB) \doteq wAB, \quad w \in \boldsymbol{h},\ A, B \in \mathcal{A}. \tag{3.3}$$

(In standard notation, $A(w \otimes B) \doteq w \otimes AB$.) We call (3.3) the *trivial left action*. At the level of Fock space, it does not induce a mixing because $\lambda(A) = 1 \otimes A$, and clearly, $\lambda^\omega(\mathcal{A})$ and $\mathcal{B}(\mathcal{F}(\boldsymbol{h}))$ commute for any $\omega \in \mathcal{S}(\mathcal{A})$.

Twists and left actions. The following notion concerns a class of left actions well behaved with respect to the permutation symmetry, obtained by twisting the trivial one by means of a group action.

Definition 3.2. Let $\mathcal{G} \subseteq \mathcal{U}\mathcal{A}$ be a group generating $\mathcal{A}$ as C*-algebra and $\mathfrak{h} = \boldsymbol{h}\mathcal{A}$ a free bimodule. Then, the left $\mathcal{A}$-action λ is said to be $\mathcal{G}$*-twisted* whenever there is a group morphism $u : \mathcal{G} \to \mathcal{U}(\boldsymbol{h})$, which we call the $\mathcal{G}$*-twist*, such that

$$\lambda(\gamma)w \equiv \gamma w = (u_\gamma w)\gamma, \quad \forall \gamma \in \mathcal{G},\ w \in \boldsymbol{h}. \tag{3.4}$$

We say that the $\mathcal{G}$-twist is *trivial* whenever u is the trivial representation.

For reader's convenience, we check the consistence of the previous definition by verifying that the r.h.s. of (3.4) defines an adjointable (and as a consequence $\mathcal{A}$-linear and bounded) operator. We write $wA, w'A' \in \mathfrak{h}$ for $w, w' \in \boldsymbol{h}$, $A, A' \in \mathcal{A}$ and compute

$$\langle wA, \gamma w'A'\rangle = A^*\langle w, u_\gamma w'\rangle\gamma A' = (\gamma^*A)^*\langle u_\gamma^* w, w'\rangle A' = \langle \gamma^* wA, w'A'\rangle.$$

(Note that $\langle w, u_\gamma w' \rangle \in \mathbb{C}$.) Thus, (3.4) has adjoint γ^* as expected. We remark that since $\mathcal{G}$ generates $\mathcal{A}$ the left action λ is determined by u. Despite that, u may have a kernel even when λ does not (as will be evident in the following example, where $\mathcal{A}$ can be simple). Finally, we note that in the previous definition we did not assume strong continuity of u, so in general, it is not a unitary representation; therefore, we say that u is a *unitary morphism*.

Example 3.3 (Weyl algebras). Let $\mathcal{S}$ be a real vector space with a symplectic form η and $\mathcal{W}$ denote the associated Weyl C*-algebra generated by unitary symbols W_s, $s \in \mathcal{S}$. Let $\mathcal{G} \subseteq \mathcal{U}\mathcal{W}$ denote the group generated by $W_s e^{1/2i\theta}$ for $s \in \mathcal{S}$ and $\theta \in \mathbb{R}$. The Weyl relations imply that an Abelian quotient of $\mathcal{G}$ is given by $\mathcal{S}$ as an additive group. Thus, any unitary morphism $u_{ab} : \mathcal{S} \to \mathcal{U}(\boldsymbol{h})$ lifts to a morphism $u : \mathcal{G} \to \mathcal{U}(\boldsymbol{h})$ such that $u(W_s)u(W_{s'}) = u(W_{s+s'})$[1]. We then consider the free Hilbert module $\mathfrak{h} \doteq \boldsymbol{h}\mathcal{W}$ and set $W_s^\lambda(wB) \doteq (u(W_s)w)W_s B$, $s \in \mathcal{S}$, $w \in \boldsymbol{h}$, $B \in \mathcal{W}$, obtaining unitary operators $W_s^\lambda \in \mathfrak{B}(\mathfrak{h})$. Since

$$\begin{aligned} W_s^\lambda W_{s'}^\lambda wB &= (u(W_s)u(W_{s'})w)W_s W_{s'} B \\ &= e^{1/2i\eta(s,s')}(u(W_{s+s'})w)W_{s+s'} B \\ &= e^{1/2i\eta(s,s')} W_{s+s'}^\lambda wB, \end{aligned}$$

we have that W_s^λ fulfill the Weyl relations, so they define a *-morphism $\lambda : \mathcal{W} \to \mathfrak{B}(\mathfrak{h})$ that by construction is a left $\mathcal{W}$-action twisted by u.

Example 3.4 (The universal C*-algebra of the electromagnetic field). Let $k \in \mathbb{N}$ and $\mathcal{D}_k(\mathbb{R}^4)$ denote the vector space of smooth, compactly supported k-forms on $\mathbb{R}^4$ (with Minkowski metric). Let $\mathcal{C}_1(\mathbb{R}^4) \subset \mathcal{D}_1(\mathbb{R}^4)$ be the subspace of 1-forms $g = (g^\mu) \in \mathcal{D}_1(\mathbb{R}^4)$ such that $\delta g \doteq \partial_\mu g^\mu = 0$. Then, $\delta f \doteq -2\partial_\nu f^{\mu\nu}$ defines a 1-form $\delta f \in \mathcal{C}_1(\mathbb{R}^4)$ for any $f = (f^{\mu\nu}) \in \mathcal{D}_2(\mathbb{R}^4)$. With this notation, we define the C*-algebra $\mathcal{A}$ generated by the group $\mathcal{G}$ of unitary symbols $V(g)$, $g \in \mathcal{C}_1(\mathbb{R}^4)$, with relations

$$V(a_1 g)V(a_2 g) = V((a_1 + a_2)g), \qquad V(g)^* = V(-g), V(0) = 1, \tag{3.5}$$

$$V(\delta f_1)V(\delta f_2) = V(\delta f_1 + \delta f_2), \qquad \operatorname{supp} f_1 \perp \operatorname{supp} f_2, \tag{3.6}$$

$$[V(g_1), V(g_2)]_\bullet \in \mathcal{A} \cap \mathcal{A}', \qquad \operatorname{supp} g_1 \perp \operatorname{supp} g_2, \tag{3.7}$$

where $a_1, a_2 \in \mathbb{R}$. In the above expressions, we used the symbol $\perp$ to indicate spacelike separation, whilst $[U, V]_\bullet \doteq UVU^*V^*$ is the group commutator and $\mathcal{A} \cap \mathcal{A}'$ is the center of $\mathcal{A}$. Any Wightman field $F(f)$, $f \in \mathcal{D}_2(\mathbb{R}^4)$, fulfilling the Maxwell equations, defines a representation π of $\mathcal{A}$ such that $\pi(V(\delta f)) = e^{iF(f)}$ [8]. Note that since $\mathcal{C}_1(\mathbb{R}^4)$, as an additive group, is a quotient of $\mathcal{G}$ under the map $V(g) \mapsto g$, any unitary morphism of $\mathcal{C}_1(\mathbb{R}^4)$ induces a unitary morphism of $\mathcal{G}$. Let now $u : \mathcal{C}_1(\mathbb{R}^4) \to \mathcal{U}(\boldsymbol{h})$ be a unitary morphism; we set $\mathfrak{h} \doteq \boldsymbol{h}\mathcal{A}$ and define $V^\lambda(g)wA \doteq (u_g w)V(g)A$, $w \in \boldsymbol{h}$, $A \in \mathcal{A}$.

[1] As a matter of fact, such unitary morphisms can be easily obtained, for example, by exponentials of linear functionals $\rho : \mathcal{S} \to \mathbb{R}$.

A straightforward check shows that the operators $V^\lambda(g)$ are in $\mathfrak{B}(\mathfrak{h})$ and fulfil (3.5)–(3.7): for example, by (3.6), we have

$$\begin{aligned}V^\lambda(\delta f_1)V^\lambda(\delta f_2)wA &= (u_{\delta f_1}u_{\delta f_2}w)V(\delta f_1)V(\delta f_2)A\\ &= (u_{\delta f_1+\delta f_2}w)V(\delta f_1+\delta f_2)A\\ &= V^\lambda(\delta f_1+\delta f_2)wA.\end{aligned}$$

Thus, we have the left action $\lambda : \mathcal{A} \to \mathfrak{B}(\mathfrak{h})$, $\lambda(V(g)) \equiv V^\lambda(g)$, which by construction is $\mathcal{G}$-twisted.

Permutation symmetry. Lemma 3.1 allows to define a permutation symmetry in the obvious way by extending the one defined on $\mathcal{F}(\boldsymbol{h})$:

$$U_\varrho(wA) \doteq w_\varrho A, \varrho \in \mathbb{P}(n), \quad w \in \boldsymbol{h}^n,\ n \in \mathbb{N},$$

where $w_\varrho \in \boldsymbol{h}^n$ is the vector transformed under the usual permutation symmetry in Fock space. Of course, to apply the previous definition, we must express a tensor of the type $v_1 \otimes \cdots \otimes v_n$, $v_1, \ldots, v_n \in \mathfrak{h}$, in terms of tensors of the type $(w_1 \otimes \cdots \otimes w_n)A$, $w_1, \ldots,$ $w_n \in \boldsymbol{h}$, $A \in \mathcal{A}$. By definition, U_ϱ is unitary on $\mathfrak{F}(\mathfrak{h})$ in the sense of right Hilbert modules; thus, we get the spectral projection $P_- \doteq \bigoplus_n P_-^n \in \mathfrak{B}(\mathfrak{F}(\mathfrak{h}))$, $P_-^n \doteq (n!)^{-1}\sum_\varrho \varepsilon_\varrho U_\varrho$, where ε_ϱ is the sign of $\varrho \in \mathbb{P}(n)$. We write $\mathfrak{F}_-(\mathfrak{h}) \doteq P_-\mathfrak{F}(\mathfrak{h})$; it is obvious that $\mathfrak{F}_-(\mathfrak{h})$ is a right $\mathcal{A}$-module, and that it is free with $\mathfrak{F}_-(\mathfrak{h}) \simeq \mathcal{F}_-(\boldsymbol{h}) \otimes \mathcal{A}$, where $\mathcal{F}_-(\boldsymbol{h})$ is the ordinary fermionic Fock space. We have the decomposition

$$\mathfrak{F}_-(\mathfrak{h}) \doteq \bigoplus_{n\geq 0} \mathfrak{h}_-^n \simeq \bigoplus_{n\geq 0} (\boldsymbol{h}_-^n\mathcal{A}),$$

where $\mathfrak{h}_-^n \doteq P_-^n\mathfrak{h}^n$, $\boldsymbol{h}_-^n \doteq P_-^n\boldsymbol{h}^n$. Of course, $\mathfrak{h}_-^0 = \mathcal{A}$ and $\mathfrak{h}_-^1 = \mathfrak{h} \simeq \boldsymbol{h}\mathcal{A}$.

Remark 3.5 (The Pauli principle). In spite of the simplicity of our definition, some care is needed to handle anti-symmetric tensors. For example, if $f, g \in \mathfrak{h}$, then it is not ensured that

$$P_-(f \otimes g) = -P_-(g \otimes f),$$

unless f and g belong to $\boldsymbol{h}$. In particular, the Pauli principle

$$P_-(f \otimes f) = 0, \quad f \in \mathfrak{h},$$

does not hold in general, and a priori its validity is ensured only for $f \in \boldsymbol{h}$. A more general class of examples for which the Pauli principle holds is the following. Given $f \in \mathfrak{h}$, we define the *support*

$$\mathcal{A}(f) \doteq C^*\{\langle v, f\rangle \in \mathcal{A} : v \in \boldsymbol{h}\} \subseteq \mathcal{A}.$$

The support has the property that if $f = f_h A_h$, where $\{f_h\} \subset \boldsymbol{h}$ is an orthonormal set and $A_h \in \mathcal{A}$, then $A_h = \langle f_h, f\rangle \in \mathcal{A}(f)$ for all $h \in \mathbb{N}$. Given $f, g \in \mathfrak{h}$, we write

$$f \bowtie g \,\dot{\Leftrightarrow}\, fB' = B'f, \quad gB = Bg, \quad [B, B'] = 0$$

for all $B \in \mathcal{A}(f)$, $B' \in \mathcal{A}(g)$. In this case, we say that f and g are *mutually free*, and we find

$$\begin{aligned} P_-(f \otimes g) &= P_-(f_h \otimes A_h g) = P_-(f_h \otimes g) A_h \\ &= P_-(f_h \otimes g_k) A'_k A_h = -P_-(g_k \otimes f_h) A'_k A_h \\ &= -P_-(g_k \otimes f_h) A_h A'_k = -P_-(g_k \otimes f) A'_k \\ &= -P_-(g \otimes f) \end{aligned}$$

so that

$$P_-(f \otimes g) = -P_-(g \otimes f), \quad f \bowtie g.$$

We conclude that validity of the Pauli principle is related to commutation properties of $\mathcal{A}$ (*locality*, when $\mathcal{A}$ is the C*-algebra of a Haag–Kastler net) and properties of the left $\mathcal{A}$-action, that is, the eventuality that it is trivial on the involved vectors of $\mathfrak{h}$ and elements of $\mathcal{A}$. Note that for $f \in \boldsymbol{h}$ we have $\mathcal{A}(f) = \mathbb{C}$ so that $f, g \in \boldsymbol{h}$ implies $f \bowtie g$.

Fermionic creation and annihilation operators. To define and handle fermionic creation and annihilation operators, we make some remarks on elementary tensors, using the attention needed in the case of Hilbert bimodules.

We start by noting that we may arrange order n permutations by considering for any $k = 1, \ldots, n$ the set of those permutations that bring the k-th object at first position. Thus, for any elementary tensor $v = wA \in \mathfrak{h}^n$, $w \in \boldsymbol{h}^n$, $A \in \mathcal{A}$, we may write

$$v_- \doteq P_-^n v = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} w_k \otimes w_-^{(k)} A \in \mathfrak{h}_-^n, \tag{3.8}$$

$$w_-^{(k)} \doteq \cdots \otimes_- \hat{w}_k \otimes_- \cdots = \frac{1}{(n-1)!} \sum_{\varrho \in \mathbb{P}_{n-1,k}} \varepsilon_\varrho w_{\varrho(1)} \otimes \cdots \otimes w_{\varrho(n)} \in \boldsymbol{h}_-^{n-1},$$

where the notation $\hat{w}_k$ indicates that w_k does not appear in the tensor and $\mathbb{P}_{n-1,k}$ is understood as the permutation group of the set $\{1, \ldots, n\} \setminus \{k\}$; the term $(-1)^{k-1}$ in (3.8) is the sign of the transposition bringing w_k at first position. The above expression makes manifest that in general $\mathfrak{F}_-(\mathfrak{h})$ is not stable under the left $\mathcal{A}$-action because the terms $Bw_k = \sum_i w_i B_{ik}$, $B_{ik} \doteq \langle w_i, Bw_k \rangle \in \mathcal{A}$ typically belong to $\mathfrak{h}$ and induce a mixing in the tensor product. The point is that in general Bw_k does not belong to $\boldsymbol{h}$; thus, we must perform the operations of the proof of Lemma 3.1 to get a tensor of the form $Bv_- = \sum_i w'_i A'_i$ with $w'_i \in \boldsymbol{h}^n$. It is after this operation that we can apply the projection P_-^n and get $P_-^n Bv_- = \sum_i w'_{i,-} A_i$.

In the sequel, we will write $\mathfrak{h}_\circ \subseteq \mathfrak{h}$ for the vector space spanned by (finite linear combinations of) vectors of the type $w\gamma$, $w \in \boldsymbol{h}$, $\gamma \in \mathcal{G}$. By construction, $\mathfrak{h}_\circ$ is dense in $\mathfrak{h}$.

We now establish some properties of anti-symmetric tensors in case of a $\mathcal{G}$-twist. The most important is that, in spite of the previous remark, the fermionic space is stable both under the $\mathcal{G}$-action and anti-symmetric tensor products by vectors in $\mathfrak{h}_\circ$.

Lemma 3.6 (Twists and permutation symmetry). *Let $\mathfrak{h} = \boldsymbol{h}\mathcal{A}$ be a free Hilbert bimodule with $\mathcal{G}$-twist*

$$u : \mathcal{G} \to \mathcal{U}(\boldsymbol{h}).$$

Then, the following properties hold.

(1) *For any $\gamma \in \mathcal{G} \subseteq \mathcal{U}\mathcal{A}$, it turns out $\gamma\mathfrak{h}^n_- \in \mathfrak{h}^n_-$ so that $\mathfrak{F}_-(\mathfrak{h})$ is stable under the left module action by $\mathcal{G}$.*

(2) *Let $g = g_i\gamma_i \in \mathfrak{h}_\circ$ with $g_i \in \boldsymbol{h}$, $\gamma_i \in \mathcal{G}$. Then, for any tensor $w_-A \in \mathfrak{h}^n_-$ with*

$$w_- \doteq w_1 \otimes_- \cdots \otimes_- w_n \in \boldsymbol{h}^n_-, \quad A \in \mathcal{A},$$

we have

$$P^{n+1}_-(g \otimes w_-A) = g_i \otimes_- (u_{\gamma_i} w_1) \otimes_- \cdots \otimes_- (u_{\gamma_i} w_n) \cdot \gamma_i A \in \mathfrak{h}^n_-, \tag{3.9}$$

so $\mathfrak{F}_-(\mathfrak{h})$ is stable under anti-symmetric tensor product by elements of $\mathfrak{h}_\circ$.

(3) *For $g = g_i\gamma_i \in \mathfrak{h}_\circ$, it turns out*

$$P^{n+1}_-(g \otimes w_-A) = \frac{1}{n}\left(g \otimes w_-A - \sum_{k=1}^{n}(-1)^{k-1} w_k \otimes P^n_-(g \otimes w^{(k)}_- A)\right). \tag{3.10}$$

Proof. Point 1. Let $w_-A \in \mathfrak{h}^n_-$ with $w = w_1 \otimes_- \cdots \otimes_- w_n$ an elementary anti-symmetric tensor. Then, for any permutation ϱ, it turns out

$$\gamma w_{\varrho(1)} \otimes \cdots \otimes w_{\varrho(n)} = (u_\gamma w_{\varrho(1)}) \otimes \cdots \otimes (u_\gamma w_{\varrho(n)})\gamma.$$

Thus, defining $u^-_\gamma \doteq P^n_- \cdot \otimes^n u_\gamma \in \mathcal{U}(\boldsymbol{h}^n_-)$, we find that

$$\gamma w_- A = (u^-_\gamma w_-)\gamma A$$

belongs to $\mathfrak{h}^n_-$ as claimed.

Point 2. With the notation of the previous point, with $g = g_i\gamma_i$, we have

$$\begin{aligned} g \otimes w_-A &= \sum_\varrho \varepsilon_\varrho g_i \otimes \gamma_i w_{\varrho(1)} \otimes \cdots \otimes w_{\varrho(n)} A \\ &= \sum_\varrho \varepsilon_\varrho g_i \otimes (u_{\gamma_i} w_{\varrho(1)}) \otimes \cdots \otimes (u_{\gamma_i} w_{\varrho(n)})\gamma_i A \\ &= g_i \otimes (u_{\gamma_i} w_1) \otimes_- \cdots \otimes_- (u_{\gamma_i} w_n) \cdot \gamma_i A. \end{aligned}$$

Thus, using the fact that P^n_- is a right $\mathcal{A}$-linear operator, we conclude that

$$\begin{aligned} P^n_-(g \otimes w_-A) &= P^n_-(g_i \otimes ((u_{\gamma_i} w_1) \otimes_- \cdots \otimes_- (u_{\gamma_i} w_n)))\gamma_i A \\ &= g_i \otimes_- (u_{\gamma_i} w_1) \otimes_- \cdots \otimes_- (u_{\gamma_i} w_n)\gamma_i A. \end{aligned}$$

Point 3. We have, by applying (3.8) and (3.9),

$$\begin{aligned} P_-^{n+1}(g \otimes w_- A) &= (g_i \otimes_- u_{\gamma_i}^- w_-)\gamma_i A \\ &= 1/n \left(g_i \otimes u_{\gamma_i}^- w_- - \sum_{k=1}^n (-1)^{k-1} w_k \otimes (g_i \otimes_- u_{\gamma_i}^- w_-^{(k)}) \right) \gamma_i A \\ &= 1/n \left(g_i \gamma_i \otimes w_- - \sum_{k=1}^n (-1)^{k-1} w_k \otimes (g_i \gamma_i \otimes_- w_-^{(k)}) \right) A \\ &= 1/n \left(g \otimes w_- - \sum_{k=1}^n (-1)^{k-1} w_k \otimes (g \otimes_- w_-^{(k)}) \right) A. \end{aligned}$$

■

Now, by (3.8), given $f \in \mathfrak{h}$, we have

$$\langle f | v_- = \frac{1}{n} \sum_k (\pm 1)^{k-1} \langle f, w_k \rangle w_-^{(k)} A \in \mathfrak{h}^{n-1}; \tag{3.11}$$

thus, in general, we cannot say that $\langle f | v_- \in \mathfrak{h}_-^{n-1}$. Yet, we have the following property.

Lemma 3.7. *Assume that there is a $\mathcal{G}$-twist on $\mathfrak{h}$ and let $f \in \mathfrak{h}_\circ$. Then, $\langle f | v_- \in \mathfrak{h}_-^{n-1}$ for all $v_- = w_- A \in \mathfrak{h}_-^n$, and $\mathfrak{F}_-(\mathfrak{h})$ is stable under the action of the operator $\langle f |$.*

Proof. Starting from (3.11) and writing $f = f_i \gamma_i$, $f_i \in \boldsymbol{h}$, $\gamma_i \in \mathcal{G}$, we find

$$\begin{aligned} \langle f | v_- &= 1/n \sum_k (\pm 1)^{k-1} \langle f_i, w_k \rangle \gamma_i^* w_-^{(k)} A \\ &= 1/n \sum_k (\pm 1)^{k-1} \langle f_i, w_k \rangle (u_{\gamma_i}^* w_1) \otimes_- \cdots \otimes_- (u_{\gamma_i}^* w_n) \gamma_i^* A. \end{aligned}$$

Since $\langle f_i, w_k \rangle \in \mathbb{C}$, the last term belongs to $\mathfrak{h}_-^{n-1}$, as claimed. ■

The next computations will allow to evaluate the anti-commutation relations. By (3.10), for $f \in \mathfrak{h}$ and $g = g_i \gamma_i \in \mathfrak{h}_\circ$, we get

$$\langle f | P_-^{n+1}(g \otimes w_- A) = \frac{1}{n} \left(\langle f, g \rangle w_- A - \sum_{k=1}^n (-1)^{k-1} \langle f, w_k \rangle P_-^n (g \otimes w_-^{(k)}) A \right) \tag{3.12}$$

and

$$P_-^{n+1}(g \otimes \langle f | w_- A) = \frac{1}{n} \sum_k (-1)^{k-1} P_-^n (g \otimes \langle f, w_k \rangle w_-^{(k)}) A. \tag{3.13}$$

For the last equality, we note that due to the antisymmetrization operator P_-^n the factor $\langle f, w_k \rangle \in \mathcal{A}$ appears in i-th position of the involved elementary tensor for any $i = 1, \dots, n$. This implies that (3.13) may differ from the sum on the r.h.s. of (3.12) because in general $\langle f, w_k \rangle$ cannot freely shift on the left of the involved elementary tensor.

Let now $P_- \doteq \bigoplus_n P_-^n$, where P_-^0 and P_-^1 are the identity. We introduce the notation

$$\mathfrak{F}_-^\#(\mathfrak{h}) \doteq \mathfrak{F}_-(\mathfrak{h}) \cap \mathfrak{F}^\#(\mathfrak{h}),$$

and for any $f \in \mathfrak{h}_\circ$ define the fermionic annihilation and creation operators

$$\begin{cases} \boldsymbol{a}_-(f) : \mathfrak{F}_-^\#(\mathfrak{h}) \to \mathfrak{F}_-^\#(\mathfrak{h}), & \boldsymbol{a}_-(f) \doteq a(f) \restriction \mathfrak{F}_-^\#(\mathfrak{h}), \\ \boldsymbol{a}_-^*(f) : \mathfrak{F}_-^\#(\mathfrak{h}) \to \mathfrak{F}_-^\#(\mathfrak{h}), & \boldsymbol{a}_-^*(f) \doteq P_- a^*(f) \restriction \mathfrak{F}_-^\#(\mathfrak{h}). \end{cases} \tag{3.14}$$

Note that the property

$$\boldsymbol{a}_-(f)\mathfrak{F}_-^\#(\mathfrak{h}) \subseteq \mathfrak{F}_-^\#(\mathfrak{h}), \quad \boldsymbol{a}_-(f) = P_-\boldsymbol{a}_-(f), \quad \forall f \in \mathfrak{h}_\circ, \tag{3.15}$$

tacitly understood in (3.14) is a consequence of the hypothesis that there is a $\mathscr{G}$-twist and Lemma 3.7. Also, note that for the moment we do not know whether the fermionic creation and annihilation operators are bounded; thus, in (3.14), we make use of the domain $\mathfrak{F}_-^\#(\mathfrak{h})$. In the following result, we give an interpretation of the twist $u : \mathscr{G} \to \mathcal{U}(\boldsymbol{h})$ as an obstacle to make the creation and annihilation operators commute with elements of $\mathcal{A}$.

Lemma 3.8. *Let $v \in \boldsymbol{h}$ and $\gamma \in \mathscr{G}$. Then,*

$$\gamma \boldsymbol{a}_-^*(v) = \boldsymbol{a}_-^*(u_\gamma v)\gamma \quad \textit{and} \quad \gamma \boldsymbol{a}_-(u_\gamma^* v) = \boldsymbol{a}_-(v)\gamma.$$

Proof. Let $w_- A \in \mathfrak{h}_-^n$, $w_- \in \boldsymbol{h}_-^n$, $A \in \mathcal{A}$. By the argument of the proof of Lemma 3.6, Point 1, we find

$$\begin{aligned} \gamma \boldsymbol{a}_-^*(v) w_- A &= \gamma P_-(v \otimes w_-)A = (u_\gamma v \otimes_- u_\gamma^- w_-)\gamma A \\ &= \boldsymbol{a}_-^*(u_\gamma v)(u_\gamma^- w_-)\gamma A = \boldsymbol{a}_-^*(u_\gamma v)\gamma w_- A. \end{aligned}$$

With an analogous argument, the claim about $\boldsymbol{a}_-(v)$ is proved. ∎

Lemma 3.9. *Let $f \in \mathfrak{h}_\circ$. Then, $\boldsymbol{a}_-^*(f)$ is the adjoint of $\boldsymbol{a}_-(f)$ over the domain $\mathfrak{F}_-^\#(\mathfrak{h})$.*

Proof. By applying (3.14) and (3.15), we find

$$\langle v, \boldsymbol{a}_-(f)v' \rangle = \langle v, a(f)P_- v' \rangle = \langle P_- a^*(f)v, v' \rangle = \langle \boldsymbol{a}_-^*(f)v, v' \rangle$$

for all $v, v' \in \mathfrak{F}_-^\#(\mathfrak{h})$. This proves the lemma. ∎

Let $\omega \in \mathcal{S}(\mathcal{A})$. We define $\mathfrak{F}_-^\omega(\mathfrak{h})$ as the Hilbert space obtained by evaluation of $\mathfrak{F}_-(\mathfrak{h})$ over ω. In the previous lines, we proved that $\boldsymbol{a}_-(f)$ and $\boldsymbol{a}_-^*(f)$ are well defined, $\mathcal{A}$-linear and one the adjoint of the other in the sense of Hilbert modules (on $\mathfrak{F}_-^\#(\mathfrak{h})$); thus, by the argument used to construct $a^\omega(f)$ and $a^{*,\omega}(f)$, we define

$$\boldsymbol{a}_-^\omega(f), \quad \boldsymbol{a}_-^{*,\omega}(f), \quad f \in \mathfrak{h}_\circ,$$

as the evaluations of $\boldsymbol{a}_-(f)$ and $\boldsymbol{a}_-^*(f)$ over ω. The domain of these operators is clearly given by $\mathfrak{F}_-^{\#,\omega}(\mathfrak{h})$, defined by evaluation of vectors in $\mathfrak{F}_-^{\#}(\mathfrak{h})$.

4. CARs and fermionic fields

In the present section, we study the anti-commutation relations that our fermionic creation and annihilation operators fulfill and define the corresponding Dirac field assuming the presence of a suitable conjugation. We bring on the light some features that in general prevent anti-commutators to be expressed only in terms of the given $\mathcal{A}$-valued scalar product and, primarily, make anti-commutators non-local in the sense that they do not vanish even when the involved "spinors" are orthogonal. In fact, not surprisingly, to get anti-commutators of the usual form besides orthogonality, we must require mutual freeness Remark 3.5.

We proceed by maintaining the assumptions of the previous section so that $\mathfrak{h} = \boldsymbol{h}\mathcal{A}$ is free and there is a $\mathcal{G}$-twist $u : \mathcal{G} \to \mathcal{U}(\boldsymbol{h})$, $\mathcal{G} \subseteq \mathcal{U}\mathcal{A}$. Before computing our anti-commutation relations, for convenience, we give a notion of mutual freeness explicitly designed to handle vectors in $\mathfrak{h}_\circ$. Let $f = f_i\gamma_i$ and $g = g_h\gamma_h \in \mathfrak{h}_\circ$, with $f_i, g_h \in \boldsymbol{h}$, $\gamma_i, \gamma'_h \in \mathcal{G}$. We write $f \bowtie_\circ g$ whenever for all i, h it turns out

$$[\gamma_i, \gamma'_h] = 0, \quad u_{\gamma_i} g_h = g_h, \quad u_{\gamma'_h} f_i = f_i. \tag{4.1}$$

Lemma 4.1. *If $f, g \in \mathfrak{h}_\circ$ and $f \bowtie_\circ g$, then $f \bowtie g$.*

Proof. As a preliminary step, we note that any generator $\langle w, f\rangle$ of $\mathcal{A}(f)$, defined for $w \in \boldsymbol{h}$, is a linear combination in $\{\gamma_i\}$, so $\mathcal{A}(f)$ is contained in the C*-algebra generated by $\{\gamma_i\}$. Thus, the hypothesis $f \bowtie_\circ g$ and (4.1) imply $[A, B] = 0$ for all $A \in \mathcal{A}(f)$, $B \in \mathcal{A}(g)$; moreover, $\gamma_i g = (u_{\gamma_i} g_h)\gamma_i\gamma'_h = (u_{\gamma_i} g_h)\gamma'_h\gamma_i = g\gamma_i$, and analogously for f and γ'_h. This implies $f \bowtie g$, as claimed. ∎

Anticommutators of creation operators. Let $f \bowtie_\circ g \in \mathfrak{h}_\circ$ and $v_- = w_- A \in \mathfrak{h}_-^n$, $w_- \in \boldsymbol{h}_-^n$, $A \in \mathcal{A}$. Writing $f = f_i\gamma_i$, $g = g_h\gamma'_h$, we find

$$\begin{aligned}
\boldsymbol{a}_-^*(f)\boldsymbol{a}_-^*(g)w_-A &= P_-^{n+2}(f \otimes P_-^{n+1}(g \otimes w_-))A \\
&= P_-^{n+2}(f_i \otimes \gamma_i P_-^{n+1}(g_h \otimes u'_h w_-))\gamma'_h A \\
&= P_-^{n+2}(f_i \otimes P_-^{n+1}(g_h \otimes u_i u'_h w_-))\gamma_i\gamma'_h A \\
&= -P_-^{n+2}(g_h \otimes P_-^{n+1}(f_i \otimes u_i u'_h w_-))\gamma_i\gamma'_h A \\
&= -P_-^{n+2}(g_h \otimes P_-^{n+1}(f_i \otimes \gamma_i\gamma'_h w_-))A \\
&= -P_-^{n+2}(g_h \otimes P_-^{n+1}(f \otimes \gamma'_h w_-))A \\
&= -P_-^{n+2}(g \otimes P_-^{n+1}(f \otimes w_-))A \\
&= -\boldsymbol{a}_-^*(g)\boldsymbol{a}_-^*(f)w_-A.
\end{aligned}$$

In the previous computation, for any $n \in \mathbb{N}$, we defined the anti-symmetric tensor powers $u_i \doteq u(\gamma_i)^-$, $u'_h \doteq u(\gamma'_h)^- \in \mathcal{U}(\boldsymbol{h}^n_-)$; moreover, we used (3.9) and the fact that expressions of the type $P_-^{n+2}(f_i \otimes P_-^{n+1}(g_h \otimes u_i u'_h w_-))$ are ordinary anti-symmetric tensor powers on Hilbert space, so the Pauli principle applies and the minus sign appears when we exchange f_i and g_h. We conclude that

$$[\boldsymbol{a}^*_-(f), \boldsymbol{a}^*_-(g)]_+ = 0, \quad f \bowtie_\circ g \in \mathfrak{h}_\circ. \tag{4.2}$$

Note that the need to assume $f \bowtie_\circ g$ to make the above anti-commutators vanish is an aspect of non-validity of the Pauli principle in full generality (Remark 3.5).

Anticommutators of annihilation operators. It should not be a surprise that passing to the adjoint of (4.2) the anti-commutators of annihilation operators vanish. Anyway, it is instructive to perform the explicit computations, both to understand the interplay of elements of $\mathcal{G}$ with the relation $\bowtie_\circ$ and to keep in evidence details that may be useful to approach (unbounded) bosonic annihilation operators.

We maintain the hypothesis $f \bowtie_\circ g \in \mathfrak{h}_\circ$ and the notation for the orthonormal decompositions

$$f = f_i \gamma_i, \quad g = g_h \gamma'_h.$$

As a first step, we further apply (3.8) and get

$$w_- = \frac{1}{n(n-1)}\Big(\sum_{k<h}(-1)^{k+h-1} w_k \otimes w_h \otimes w^{(h,k)} + \sum_{k>h}(-1)^{k+h} w_h \otimes w_k \otimes w^{(h,k)}\Big) \tag{4.3}$$

$$= \frac{2}{n(n-1)} \sum_{k<h}(-1)^{k+h-1}(w_k \otimes_- w_h) \otimes w^{(h,k)}, \tag{4.4}$$

where

$$w^{(h,k)} \doteq \begin{cases} \cdots \hat{w}_k \otimes_- \cdots \otimes_- \hat{w}_h \cdots, & k < h, \\ \cdots \hat{w}_h \otimes_- \cdots \otimes_- \hat{w}_k \cdots, & k > h; \end{cases}$$

in (4.3), we used the fact that for $k < h$ the term w_h needs a cyclic permutation with order $h-2$ to shift on the left of the tensor $w_1 \otimes \cdots \hat{w}_k \cdots \otimes w_n$, whilst for $k > h$ the order is $h-1$. Using elementary properties of symmetric tensors, we find

$$w^{(h,k)} = -w^{(k,h)}, \quad h \neq k.$$

After these preparations, we get

$$\begin{aligned} \langle g|\langle f|v_- &= \frac{2}{n(n-1)} \sum_{k<h}(-1)^{k+h-1}\langle f \otimes g, w_k \otimes_- w_h\rangle w^{(h,k)} A \\ &= \frac{2}{n(n-1)} \sum_{k<h}(-1)^{k+h-1}\langle f \otimes g, w_k \otimes_- w_h\rangle w^{(h,k)} A. \end{aligned}$$

We analyze in details the terms

$$\begin{aligned}\langle f \otimes g, w_h \otimes_- w_k\rangle &= \langle g_h\gamma'_h, \langle f_i\gamma_i, w_h\rangle\, w_k\rangle - \langle g_h\gamma'_h, \langle f_i\gamma_i, w_k\rangle\, w_h\rangle \\ &= {\gamma'_h}^*\langle g_h\gamma_i, \langle f_i, w_h\rangle\, w_k\rangle - {\gamma'_h}^*\langle g_h\gamma_i, \langle f_i, w_k\rangle\, w_h\rangle \\ &= {\gamma'_h}^*\gamma_i^*(\langle f_i, w_h\rangle\langle g_h, w_k\rangle - \langle f_i, w_k\rangle\langle g_h, w_h\rangle) \\ &= {\gamma'_h}^*\gamma_i^*(\langle f_i, \langle g_h, w_k\rangle w_h\rangle - \langle f_i, \langle g_h, w_h\rangle w_k\rangle) \\ &= \langle f_i\gamma_i\gamma'_h, \langle g_h, w_k\rangle w_h\rangle - \langle f_i\gamma_i\gamma'_h, \langle g_h, w_h\rangle w_k\rangle \\ &= \langle \gamma'_h f, \langle g_h, w_k\rangle w_h\rangle - \langle \gamma'_h f, \langle g_h, w_h\rangle w_k\rangle \\ &= \langle f, \langle g, w_k\rangle w_h\rangle - \langle f, \langle g, w_h\rangle w_k\rangle \\ &= \langle g \otimes f, w_k \otimes w_h\rangle - \langle g \otimes f, w_h \otimes w_k\rangle \\ &= -\langle g \otimes f, w_h \otimes_- w_k\rangle\end{aligned}$$

and conclude that $(\langle g|\langle f| + \langle f|\langle g|)v_- = 0$, obtaining

$$[\boldsymbol{a}_-(f), \boldsymbol{a}_-(g)]_+ = 0, \quad f \bowtie_\circ g \in \mathfrak{h}_\circ. \tag{4.5}$$

Mixed Anticommutators. Finally, we analyze the anti-commutators $[\boldsymbol{a}_-(f), \boldsymbol{a}_-^*(g)]_+$. At a first stage, we consider $f, g \in \mathfrak{h}_\circ$ without further hypothesis, and vectors of the type $v_- = w_- A \in \mathfrak{h}_-^n$, with $w_- = w_1 \otimes_- \cdots \otimes_- w_n \in \boldsymbol{h}_-^n$, $A \in \mathcal{A}$; then, using (3.12) and (3.13), respectively, we compute

$$\boldsymbol{a}_-(f)\boldsymbol{a}_-^*(g)v_- = \langle f, g\rangle v_- - \sum_k (-1)^{k-1}\langle f, w_k\rangle P_-^n(g \otimes w_-^{(k)})A, \tag{4.6}$$

$$\boldsymbol{a}_-^*(g)\boldsymbol{a}_-(f)v_- = \sum_k (-1)^{k-1} P_-^n(g\langle f, w_k\rangle \otimes w_-^{(k)})A. \tag{4.7}$$

A quick look at the previous equalities is sufficient to realize that the terms

$$\langle f, w_k\rangle P_-^n(g \otimes w_-^{(k)})A, \quad P_-^n(g\langle f, w_k\rangle \otimes w_-^{(k)})A \tag{4.8}$$

may prevent the realization of the anti-commutation relations that one could expect. In fact, whilst in the case $\mathcal{A} = \mathbb{C}$, they eliminate each other leaving only the term $\langle f, g\rangle v_-$ present; in general, they could differ because the scalar products $\langle f, w_k\rangle$ are not free to shift through the elementary tensors. Thus, we adopt the usual hypothesis $f \bowtie_\circ g$ and analyze the terms in (4.8) in more detail.

As a first step, we write as usual $f = f_i\gamma_i$, $g = g_h\gamma'_h$ and, to be concise, we write $u_i \doteq u_{\gamma_i}$, $u'_h \doteq u_{\gamma'_h} \in \mathcal{U}(\boldsymbol{h})$, $\langle f, w_k\rangle = \gamma_i^* z_{ik} \in \mathcal{A}(f)$ with $z_{ik} \doteq \langle f_i, w_k\rangle \in \mathbb{C}$. With this notation, the hypothesis $f \bowtie_\circ g$ implies

$$[\gamma_i, \gamma'_h] = [\gamma_i^*, \gamma'_h] = 0$$

so that

$$[u_i, u'_h] = [u_i^*, u'_h] = 0.$$

For the first expression in (4.8), we compute

$$\begin{aligned}
&\langle f, w_k\rangle P_-^n(g \otimes w_-^{(k)})A \\
&\quad = (n-1)!^{-1} \sum_{\varrho \in \mathbb{P}_{n-1,k}} \varepsilon_\varrho \gamma_i^* z_{ik} P_-^n(g_h \gamma_h' \otimes w_{\varrho(1)} \otimes \cdots \otimes w_{\varrho(n)})A \\
&\quad = (n-1)!^{-1} \sum_{\varrho \in \mathbb{P}_{n-1,k}} \varepsilon_\varrho \gamma_i^* z_{ik} P_-^n(g_h \otimes u_h' w_{\varrho(1)} \otimes \cdots \otimes u_h' w_{\varrho(n)})\gamma_h' A.
\end{aligned}$$

The term $P_-^n(g_h \otimes u_h' w_{\varrho(1)} \otimes \cdots \otimes u_h' w_{\varrho(n)})$ is a linear combination of terms of the type

$$\varepsilon_\pi u_h' w_{\pi\rho(l)} \otimes \cdots g_h \cdots \otimes u_h' w_{\pi\rho(m)} \gamma_h' A,$$

where π is any permutation of the terms in the argument of P_-^n and g_h appears in position $\pi(1)$. Applying the operators $\langle f, w_k\rangle = \gamma_i^* z_{ik}$, we get terms of the type

$$\begin{aligned}
&\varepsilon_\pi \gamma_i^* z_{ik} u_h' w_{\pi\rho(l)} \otimes \cdots g_h \cdots \otimes u_h' w_{\pi\rho(m)} \gamma_h' A \\
&\quad = \varepsilon_\pi u_i^* u_h' w_{\pi\rho(l)} \otimes \cdots \gamma_i^* g_h \gamma_h' \cdots \otimes w_{\pi\rho(m)} z_{ik} A \\
&\quad = \varepsilon_\pi u_i^* u_h' w_{\pi\rho(l)} \otimes \cdots \gamma_i^* g \cdots \otimes w_{\pi\rho(m)} z_{ik} A \\
&\quad = \varepsilon_\pi u_i^* u_h' w_{\pi\rho(l)} \otimes \cdots g \gamma_i^* \cdots \otimes w_{\pi\rho(m)} z_{ik} A \\
&\quad = \varepsilon_\pi u_i^* u_h' w_{\pi\rho(l)} \otimes \cdots g_h \cdots \otimes u_h' u_i^* w_{\pi\rho(m)} z_{ik} \gamma_h \gamma_i^* A \\
&\quad = \varepsilon_\pi u_i^* u_h' w_{\pi\rho(l)} \otimes \cdots g_h \cdots \otimes u_i^* u_h' w_{\pi\rho(m)} z_{ik} \gamma_i^* \gamma_h A.
\end{aligned}$$

In conclusion,

$$\begin{aligned}
&\langle f, w_k\rangle P_-^n(g \otimes w_-^{(k)})A \\
&\quad = (n-1)!^{-1} \sum_{\varrho,\pi} \varepsilon_\varrho \varepsilon_\pi u_i^* u_h' w_{\pi\rho(l)} \otimes \cdots g_h \cdots \otimes u_i^* u_h' w_{\pi\rho(m)} z_{ik} \gamma_i^* \gamma_h A.
\end{aligned}$$

Finally, we evaluate the second term in (4.8),

$$\begin{aligned}
&P_-^n(g\langle f, w_k\rangle \otimes w_-^{(k)})A \\
&\quad = (n-1)!^{-1} \sum_{\varrho \in \mathbb{P}_{n-1,k}} \varepsilon_\varrho P_-^n(g_h \gamma_h' \gamma_i^* z_{ik} \otimes w_{\varrho(1)} \otimes \cdots \otimes w_{\varrho(n)})A \\
&\quad = (n-1)!^{-1} \sum_{\varrho \in \mathbb{P}_{n-1,k}} \varepsilon_\varrho P_-^n(g_h \gamma_i^* \gamma_h' z_{ik} \otimes w_{\varrho(1)} \otimes \cdots \otimes w_{\varrho(n)})A \\
&\quad = (n-1)!^{-1} \sum_{\varrho \in \mathbb{P}_{n-1,k}} \varepsilon_\varrho P_-^n(g_h \otimes u_i^* u_h' w_{\varrho(1)} \otimes \cdots \otimes u_i^* u_h' w_{\varrho(n)}) z_{ik} \gamma_i^* \gamma_h' A \\
&\quad = (n-1)!^{-1} \sum_{\varrho,\pi} \varepsilon_\varrho \varepsilon_\pi u_i^* u_h' w_{\varrho(1)} \otimes \cdots g_h \cdots \otimes u_i^* u_h' w_{\varrho(n)} z_{ik} \gamma_i^* \gamma_h' A,
\end{aligned}$$

concluding that

$$\langle f, w_k\rangle P_-^n(g \otimes w_-^{(k)})A = P_-^n(g\langle f, w_k\rangle \otimes w_-^{(k)})A$$

actually conspire to obtain, starting from (4.6) and (4.7), the anti-commutation relations

$$[\boldsymbol{a}_-(f), \boldsymbol{a}^*_-(g)]_+ = \langle f, g\rangle v_-, \quad f \bowtie_\circ g \in \mathfrak{h}_\circ. \tag{4.9}$$

Norm and generalized CARs. As a particular case, we now consider $f, g \in \boldsymbol{h}$ that clearly implies $f \bowtie_\circ g$; moreover, we have $\langle f, w\rangle \in \mathbb{C}$ for all $w \in \boldsymbol{h}$; thus, the undesirable terms in (4.9) vanish. As a consequence, we find

$$[\boldsymbol{a}_-(f), \boldsymbol{a}^*_-(g)]_+ = \langle f, g\rangle \in \mathbb{C}, \quad [\boldsymbol{a}^*_-(f), \boldsymbol{a}^*_-(g)]_+ = 0, \quad f, g \in \boldsymbol{h}.$$

Now, $\boldsymbol{a}_-(f)$ and $\boldsymbol{a}^*_-(g)$ act like the usual annihilation and creation operators when restricted to the Fock space $\mathcal{F}_-(\boldsymbol{h}) \subset \mathfrak{F}_-(\mathfrak{h})$: we denote the corresponding restrictions by $a_-(f)$, $a^*_-(g)$ (without bold font), and note that $\|a_-(f)\| = \|a^*_-(f)\| = \|f\|$ [7, Vol. 2, Prop. 5.2.2]. Given $w_- \in \mathcal{F}_-(\boldsymbol{h})$, $A \in \mathcal{A}$, by right $\mathcal{A}$-linearity, we have

$$\boldsymbol{a}_-(f)(w_-A) = (a_-(f)w_-)A, \boldsymbol{a}^*_-(f)(w_-A) = (a^*_-(f)w_-)A, \quad f \in \boldsymbol{h}$$

so that

$$\begin{aligned}\|\langle \boldsymbol{a}_-(f)(w_-A), \boldsymbol{a}_-(f)(w_-A)\rangle\| &= \|A^*\langle w_-, a^*_-(f)a_-(f)w_-\rangle A\| \\ &\le \|f\|^2\|A^*\langle w_-, w_-\rangle A\| \\ &= \|f\|^2\|w_-A\|^2,\end{aligned}$$

having used the fact that $\|f\|^2 - a^*_-(f)a_-(f)$ is a positive operator on $\mathcal{F}_-(\boldsymbol{h})$. Thus, we find

$$\|\boldsymbol{a}_-(f)\| = \|\boldsymbol{a}^*_-(f)\| = \|f\|$$

for all $f \in \boldsymbol{h}$. We use this property to prove the following, more general, result.

Lemma 4.2. *Let $f \in \mathfrak{h}_\circ$. Then, $\boldsymbol{a}_-(f)$ and $\boldsymbol{a}^*_-(f)$ are bounded.*

Proof. We start proving our assertion for the annihilation operator. Given $f = f_i\gamma_i$, $f_i \in \boldsymbol{h}$, $\gamma_i \in \mathcal{G}$ (finite sum), for the usual elementary tensors $w_-A \in \mathfrak{h}^n_-$, we compute

$$\begin{aligned}\boldsymbol{a}_-(f)w_-A &= \boldsymbol{a}_-(f_i\gamma_i)w_-A \\ &= \frac{1}{\sqrt{n}}\sum_k (-1)^{k-1}\gamma_i^*\langle f_i, w_k\rangle w_-^{(k)}A \\ &= \sum_i \gamma_i^*\boldsymbol{a}_-(f_i)w_-A.\end{aligned}$$

The previous relations say that $\boldsymbol{a}_-(f)$ is the sum of the operators $\gamma_i^*\boldsymbol{a}_-(f_i)$, where γ_i^* are regarded as unitary operators on $\mathfrak{h}$ and $\boldsymbol{a}_-(f_i)$ are, by the previous remarks, bounded. Thus, we conclude that $\boldsymbol{a}_-(f)$ is bounded for f (finite) linear combination in $\mathfrak{h}_\circ$. ∎

In the following result, we give a synthesis of (4.2), (4.5), (4.9), and Lemma 4.2.

Theorem 4.3. *Let $\mathfrak{h} = \boldsymbol{h}\mathcal{A}$ be a free Hilbert $\mathcal{A}$-bimodule with twist $u : \mathcal{G} \to \mathcal{U}(\boldsymbol{h})$, $\mathcal{G} \subseteq \mathcal{U}\mathcal{A}$. Then, for any $f, g \in \mathfrak{h}_\circ$, the creation and annihilation operators are bounded right $\mathcal{A}$-module operators on $\mathfrak{F}_-(\mathfrak{h})$, and the following properties hold.*

(1) *If $f \bowtie_\circ g$, then*

$$[\boldsymbol{a}_-(f), \boldsymbol{a}_-(g)]_+ = [\boldsymbol{a}_-^*(f), \boldsymbol{a}_-^*(g)]_+ = 0,$$

and

$$[\boldsymbol{a}_-(f), \boldsymbol{a}_-^*(g)]_+ = \langle f, g\rangle \in \mathcal{A}.$$

(2) *If $f, g \in \boldsymbol{h}$, then the previous anti-commutation relations hold with $\langle f, g\rangle \in \mathbb{C}$.*

Dirac fields. Let $\mathcal{U}_*(\boldsymbol{h})$ denote the set of anti-unitary operators on $\boldsymbol{h}$, and let $\kappa = \kappa^* \in \mathcal{U}_*(\boldsymbol{h})$ be a conjugation such that

$$[\kappa, u_\gamma] = 0, \quad \forall \gamma \in \mathcal{G} \subseteq \mathcal{U}\mathcal{A}. \tag{4.10}$$

Setting $\kappa(vA) \doteq (\kappa v)A^*$, $v \in \boldsymbol{h}$, $A \in \mathcal{A}$, we extend κ to the vector space spanned by elementary tensors in $\mathfrak{h}$, obtaining a densely defined anti-linear map. Note that in particular $\kappa(v\gamma) = (\kappa v)\gamma^*$, so κ is defined on $\mathfrak{h}_\circ$. In the following result, we check the compatibility of κ with the mutual freeness relation (4.1).

Lemma 4.4. *The following properties hold:*

(1) *$\mathcal{A}(f) = \mathcal{A}(\kappa f)$ for all $f \in \mathfrak{h}_\circ$;*

(2) *If $f, g \in \mathfrak{h}_\circ$ and $f \bowtie g$, then $\langle f, \kappa g\rangle = \langle g, \kappa f\rangle$;*

(3) *Let $f \bowtie_\circ g$; then, $f \bowtie_\circ \kappa g$, $\kappa g \bowtie_\circ f$, and $\kappa f \bowtie_\circ \kappa g$.*

Proof. (1) We can write $f = w_i A_i$ with $\{w_i\}$ an orthogonal base in $\boldsymbol{h}$ and $A_i \doteq \langle w_i, f\rangle \in \mathcal{A}$: this implies that $\mathcal{A}(f)$ is generated by the set $\{A_i\}$. On the other hand, $\kappa f = \kappa(w_i)A_i^*$, where also $\{\kappa w_i\}$ is a base of $\boldsymbol{h}$, implying that $\mathcal{A}(\kappa f)$ is generated by $\{A_i^*\}$. Thus, $\mathcal{A}(f) = \mathcal{A}(\kappa f)$, as claimed.

(2) Writing $g = v_h B_h$, $v_h \in \boldsymbol{h}$, $B_h \in \mathcal{A}(g)$, we get

$$\langle f, \kappa g\rangle = A_i^* B_h^* \langle w_i, \kappa v_h\rangle = B_h^* A_i^* \langle v_h, \kappa w_i\rangle = \langle v_h B_h, (\kappa w_i)A_i^*\rangle = \langle g, \kappa f\rangle,$$

having used the fact that $f \bowtie g$ implies $[A_i, B_h] = 0$.

(3) We write $f = f_i\gamma_i$, $g = g_h\gamma'_h$ and check that $f \bowtie_\circ \kappa g$:

$$\begin{aligned}
(\kappa g)\gamma_i &= \kappa(g_h\gamma'_h)\gamma_i \\
&= (\kappa g_h)\gamma_i {\gamma'_h}^* \\
&= \gamma_i(u_{\gamma_i}^* \kappa g_h){\gamma'_h}^* \\
&= \gamma_i(\kappa u_{\gamma_i}^* g_h){\gamma'_h}^* \\
&= \gamma_i(\kappa g_h){\gamma'_h}^* \\
&= \gamma_i(\kappa g).
\end{aligned}$$

The other cases are verified in an analogous way, so the lemma is proved. ∎

A *Dirac triple over* $\mathcal{A}$, written as $(\mathfrak{h}, u, \kappa)$, is given by a free Hilbert $\mathcal{A}$-bimodule $\mathfrak{h} = \boldsymbol{h}\mathcal{A}$ with $\mathcal{G}$-twist u and a conjugation $\kappa \in \mathcal{U}_*(\boldsymbol{h})$ fulfilling (4.10). The (self-dual) *Dirac field* associated with $(\mathfrak{h}, u, \kappa)$ is defined by

$$\widehat{\psi}(f) \doteq \frac{1}{\sqrt{2}}(\boldsymbol{a}_-^*(f) + \boldsymbol{a}_-(\kappa f)), \quad f \in \mathfrak{h}_\circ \subseteq \mathfrak{h}. \tag{4.11}$$

It yields operators $\widehat{\psi}(f) \in \mathfrak{B}(\mathfrak{F}_-(\mathfrak{h}))$, and by Lemma 3.9, we have

$$\widehat{\psi}^*(f) = \widehat{\psi}(\kappa f). \tag{4.12}$$

By applying Lemma 3.8, Theorem 4.3, and Lemma 4.4, we obtain

$$\begin{cases} [\widehat{\psi}(f), \widehat{\psi}(g)]_+ = \langle \kappa f, g\rangle \in \mathcal{A}, & f \bowtie_\circ g \in \mathfrak{h}_\circ, \\ \gamma\widehat{\psi}(w) = \widehat{\psi}(u_\gamma w)\gamma, & w \in \boldsymbol{h},\ \gamma \in \mathcal{G} \subseteq \mathcal{U}\mathcal{A}. \end{cases} \tag{4.13}$$

We denote the C*-algebra generated by the operators $\widehat{\psi}(f), \gamma \in \mathfrak{B}(\mathfrak{F}_-(\mathfrak{h}))$, $f \in \mathfrak{h}_\circ$, $\gamma \in \mathcal{G}$, by $\mathcal{F}_{\mathfrak{h},u,\kappa}$, and call it the *field C*-algebra* of $(\mathfrak{h}, u, \kappa)$. By construction, $\mathcal{F}_{\mathfrak{h},u,\kappa}$ contains $\lambda(\mathcal{A})$ and the CAR algebra $\mathcal{C}_{\boldsymbol{h}}$ and fulfills the relations (4.12) and (4.13)[2]. Any state $\omega \in \mathcal{S}(\mathcal{A})$ induces a Hilbert space representation of $\mathcal{F}_{\mathfrak{h},u,\kappa}$, defined as in (2.5). We remark that at the abstract level at which we worked no topology has been defined on $\mathfrak{h}_\circ$; thus, no continuity property is required for $\widehat{\psi}(f)$ at varying of $f \in \mathfrak{h}_\circ$.

A class of fixed-time models. We briefly present a family of models for the notion of Dirac triple and the associated Dirac field. A more detailed exposition of these and other models is postponed to a future publication.

We start by considering the symplectic space $\mathcal{S}$ given by pairs of compactly supported test functions $s = (s_0, s_1) \in \mathcal{S}(\mathbb{R}^3) \oplus \mathcal{S}(\mathbb{R}^3)$, with symplectic form

$$\eta(s, s') \doteq \int (s_1 s_0' - s_0 s_1')$$

(Lebesgue measure) and the associated Weyl C*-algebra $\mathcal{W}$. It is readily seen that $\mathcal{W}$ is the C*-algebra associated to the restriction at a fixed time of the free scalar field, with

$$W(s) = e^{i(\phi(s_0)+\dot{\phi}(s_1))}.$$

Here, the field $\phi(s_0)$ and its conjugate $\dot{\phi}(s_1)$, $s_0, s_1 \in \mathcal{S}(\mathbb{R}^3)$, are the initial conditions at time t_0 of the free scalar field [6, §8.4.A]. As explained in Example 3.3, any unitary morphism of $\mathcal{S}$ as an additive group yields a unitary morphism of the group $\mathcal{G}$ generated by Weyl unitaries and phases.

[2]Relations similar to (4.13) appeared in [13,14] in the special case where $\mathcal{A}$ is a Weyl algebra describing suitable asymptotic configurations of the electromagnetic field. Whilst we use the C*-norm induced by our Fock bimodule, the C*-algebra of the above reference is endowed with a maximal C*-norm, which is non-trivial because a (non-separable) representation is exhibited.

We then consider the Hilbert spaces $\boldsymbol{h}_+ \doteq L^2(\mathbb{R}^3, \mathbb{C}^4)$, $\boldsymbol{h}_- \doteq L^2(\mathbb{R}^3, \mathbb{C}^{4,*})$, where $\mathbb{C}^{4,*}$ is the conjugate space, and define $\boldsymbol{h} \doteq \boldsymbol{h}_+ \oplus \boldsymbol{h}_-$ with conjugation

$$\kappa \in \mathcal{U}_*(\boldsymbol{h}), \kappa(w_+ \oplus \bar{w}_-) \doteq w_- \oplus \bar{w}_+.$$

(Here, $\bar{w}$ is the conjugate map $\bar{w}(w') \doteq \langle w, w' \rangle$ defined by $w \in \boldsymbol{h}_+$.) Given a tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^3)$, we consider the unitary morphism

$$u_\sigma : \mathcal{S} \to \mathcal{U}(\boldsymbol{h}), u_{\sigma,s}(w_+ \oplus \bar{w}_-) \doteq e^{-i\sigma \star s_0} w_+ \oplus e^{i\sigma \star s_0} \bar{w}_-, \quad w \in \boldsymbol{h},$$

where $\sigma \star s_0 \in C^\infty(\mathbb{R}^3)$ is the convolution. (Note that s_1 does not come into play.) It is then clear that κ fulfills (4.10).

We are now in condition to form the free Hilbert bimodule $\mathfrak{h} = \boldsymbol{h}\mathcal{W}$ carrying the twisting defined by u_σ; we have $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ with obvious meaning of the symbols, and any $(\mathfrak{h}, u_\sigma, \kappa)$ is a Dirac triple over $\mathcal{W}$. With this input, we have the self-dual Dirac field $\widehat{\psi}(h)$, $h \in \mathfrak{h}_\circ$, from which for convenience we extract the electron field $\psi(f) \doteq \widehat{\psi}(f \oplus 0)$, $f \in \mathfrak{h}_{+,\circ}$, fulfilling the relations

$$\begin{cases} [\psi^*(f), \psi(g)]_+ = \langle f, g \rangle \in \mathcal{W}, & f \bowtie_\circ g \in \mathfrak{h}_{+,\circ}, \\ W(s)\psi(w) = \psi(e^{-i\sigma \star s_0} w) W(s), & w \in \boldsymbol{h}_+,\ s \in \mathcal{S}. \end{cases} \tag{4.14}$$

We denote the associated field C*-algebra by $\mathcal{F}_\sigma$. It is endowed with the gauge action

$$\beta : \mathbb{U}(1) \to \mathbf{aut}\, \mathcal{F}_\sigma, \quad \beta_z(\psi(f)) \doteq \bar{z}\psi(f).$$

Now, given $f = f_i W(s_i) \in \mathfrak{h}_{+,\circ}$, $f_i \in \boldsymbol{h}_+$, $s_i \in \mathcal{S}$, we define the support $\mathrm{supp}(f) \subset \mathbb{R}^3$ as the union of the "fermionic" and "bosonic" supports

$$\mathrm{supp}_\psi(f) \doteq \bigcup_i \mathrm{supp}(w_i), \quad \mathrm{supp}_{\mathcal{W}}(f) \doteq \bigcup_i \mathrm{supp}(s_i)$$

and introduce the C*-algebras $\mathcal{F}_\sigma(A)$ generated by those $\psi(f)$ having support in the open set $A \subset \mathbb{R}^3$. There are two subnets $\mathcal{C}_{\boldsymbol{h}}$ and $\mathcal{W}$ of $\mathcal{F}_\sigma$, the first defined by the operators $\psi(w)$, $w \in \boldsymbol{h}_+$, and the second given by the unitaries $W(s)$, $s \in \mathcal{S}$: the two subnets are defined by the free Dirac field and the free scalar field, respectively. In particular,

$$[\psi(w), \psi(w')]_+ = [\psi^*(w_1), \psi(w_2)]_+ = 0 \tag{4.15}$$

for all $w, w' \in \boldsymbol{h}_+$ and $w_1, w_2 \in \boldsymbol{h}_+$ such that $\mathrm{supp}(w_1) \cap \mathrm{supp}(w_2) = \emptyset$. We discuss the field net $\mathcal{F}_\sigma$ for several choices of $\sigma \in \mathcal{S}'(\mathbb{R}^3)$.

(1) σ *is the Dirac delta at the origin*. In this case, $\sigma \star s_0 = s_0$ and the unitaries $W(s)$ induce by adjoint action the local gauge transformations

$$\psi(w) \to \psi(e^{-is_0} w).$$

If $\mathrm{supp}(w) \cap \mathrm{supp}(s_0) = \emptyset$, then $u_{\sigma,s} w = w$ and $[\psi(w), W(s)] = 0$. Therefore, the subnets $\mathcal{C}_{\boldsymbol{h}}$ and $\mathcal{W}$ are relatively local, that is, $\mathcal{C}_{\boldsymbol{h}}(A) \subset \mathcal{W}(B)'$ for $A \cap B = \emptyset$, and $\mathcal{F}_\sigma$ is local in the sense that it fulfills normal commutation relations. For $A \cap B \neq \emptyset$, the second of (4.14) in general holds with $u_{\sigma,s} w \neq w$ and $[W(s), \psi(w)] \neq 0$. A sufficient condition to having $f \bowtie_\circ g$ is $\mathrm{supp}_a(f) \cap \mathrm{supp}_b(g) = \emptyset$ for all combinations in $a, b = \psi, \mathcal{W}$ different from $a = b = \psi$: in this case,

$$[\psi^*(f), \psi(g)]_+ = \sum_{ij} \langle f_i, g_j \rangle W(s'_j - s_i) \in \mathcal{W}, \tag{4.16}$$

having written $g = g_j W(s'_j)$. If f, g are not mutually free, then terms of the type (4.6)–(4.7), which are not in $\mathcal{W}$, appear in the corresponding anticommutator.

(2) *σ has support with a non-empty interior and contained in the 3-ball B_r, $r \in (0, \infty]$.* In this case, $\mathrm{supp}(\sigma \star s_0) \subset \mathrm{supp}(s_0) + B_r$. We may have $u_{\sigma,s} w = e^{-i\sigma \star s_0} w \neq w$ even for $\mathrm{supp}(s) \cap \mathrm{supp}(w) = \emptyset$, and

$$W(s)\psi(w) = \psi(e^{-i\sigma \star s_0} w) W(s)$$

implying that in general $\mathcal{C}_{\boldsymbol{h}}$ and $\mathcal{W}$ are not relatively local. If $f, g \in \mathfrak{h}_{+,\circ}$, then a sufficient condition to having $f \bowtie_\circ g$ is that $(\mathrm{supp}_a(f) + B_r) \cap (\mathrm{supp}_b(g) + B_r) = \emptyset$ for $(a, b) \neq (\psi, \psi)$; in that case, (4.16) holds. Again, terms of the type (4.6)–(4.7) appear for f, g not mutually free.

(2.1) *σ is the fundamental solution of the Poisson equation* [9, §9.4]. In this case,

$$(\sigma \star s_0)(\boldsymbol{x}) = \frac{1}{4\pi} \int \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} s_0(\boldsymbol{y}) d^3\boldsymbol{y}$$

in general has non-compact support and is non-constant (for example, take s_0 a non-negative bump function supported around the origin). $\mathcal{W}$ is not relatively local both to $\mathcal{C}_{\boldsymbol{h}}$ and the fixed-point subnet $\mathcal{C}_{\boldsymbol{h}}^\beta$; in particular,

$$W(s)\psi(w_1)\psi^*(w_2) = \psi(e^{-i\sigma \star s_0} w_1)\psi^*(e^{i\sigma \star s_0} w_2) W(s)$$

even when $\mathrm{supp}(s)$ is disjoint from the supports of w_1, w_2.

(2.2) *σ is the Lebesgue measure.* In this case,

$$(\sigma \star s_0)(\boldsymbol{x}) = \int s_0(\boldsymbol{x} - \boldsymbol{y}) d^3\boldsymbol{y} = -\langle s_0 \rangle \doteq - \int s_0$$

is constant and

$$W(s)\psi(w) = e^{i\langle s_0 \rangle} \psi(w) W(s) \tag{4.17}$$

for all $s \in \mathcal{S}$ and $w \in \boldsymbol{h}_+$. The above equality has a twofold interpretation. The first is that the Weyl unitaries $W(s)$ induce the global gauge transformations $e^{i\langle s_0 \rangle}$ on the charged fields $\psi(w)$. The second is that the operators $\psi(w)$ intertwine the identity and the automorphism $\alpha \in \mathbf{aut}\, \mathcal{W}$, $\alpha(W(s)) \doteq e^{-i\langle s_0 \rangle} W(s)$ so that (4.17) becomes

$$\alpha(W(s))\psi(w) = \psi(w) W(s). \tag{4.18}$$

The net $\mathscr{F}_\sigma$ is not local; in fact, $\mathscr{C}_{\boldsymbol{h}}(A)$ and $\mathscr{W}(B)$ are never one in the commutant of the other. Yet, by (4.17), we have that $\mathscr{W}$ is in the commutant of $\mathscr{C}_{\boldsymbol{h}}^\beta$ and $\mathscr{C}_{\boldsymbol{h}}$ is in the commutant of the subalgebra $\mathscr{W}^0$ generated by test functions s with $\langle s_0 \rangle = 0$. A local, gauge-invariant subnet $\mathscr{A}$ of $\mathscr{F}_\sigma$ is the one generated by operators of the type $A_{s,w_1,w_2} \doteq \psi(w_1)W(s)\psi^*(w_2)$, $w_1, w_2 \in \boldsymbol{h}_+$, $s \in \mathcal{S}$; in fact,

$$[A_{s,w_1,w_2}, A_{s',w_1',w_2'}] = 0$$

for $(\mathrm{supp}(s) \cup \mathrm{supp}(w_1) \cup \mathrm{supp}(w_2)) \cap (\mathrm{supp}(s') \cup \mathrm{supp}(w_1') \cup \mathrm{supp}(w_2')) = \emptyset$, having used (4.15), (4.17) and $[W(s), W(s')] = 0$. Note that $\mathscr{A}$ is local both to $\mathscr{C}_{\boldsymbol{h}}^\beta$ and $\mathscr{W}$.

5. Conclusions

In the present paper, we presented a construction based on Hilbert bimodules, in which the spatial tensor product of a CAR algebra by a C*-algebra is replaced with a twisted product. This allows to construct field systems with non-trivial commutation relations, as in (4.14) and (4.18). The technical obstacles concerning tensor products and (the absence of) permutation symmetry in Hilbert bimodules have been overcome by introducing the notion of twist, which yields a class of left actions for which these drawbacks are under control[3].

The models presented in the previous section are elementary, yet they pose questions that in our opinion deserve to be discussed. For example, the model (1) exhibits Weyl unitaries that induce local gauge transformations; thus, in regular representations of $\mathscr{W}$, we expect to find bosonic fields assuming the role usually played by the zero components of the Dirac current [19, §4.6.1] or the "longitudinal photon field" in Gupta–Bleuer gauge [19, §7.3.2]. In the model (2.1), σ is related to electrostatic potentials and not surprisingly it poses the problem of extracting a *local* observable subnet from which the initial (non-local) field net should be reconstructed. In this regard, the model (2.2) provides a simple illustration of the fact that this problem can be successfully solved in specific situations.

As a final remark, we point out that the physical understanding of the notion of twist is a topic that has not been discussed in the present paper, in which we used this object as a mathematical input. The correct interpretation should be obtained by a deeper discussion of our models, especially in regular representations of $\mathscr{W}$ [21]. In this regard, working in a fixed-time *régime* allows to avoid complications and easily produce examples, yet our aim is to construct and discuss models in Minkowski space, entering in this way in an explicitly relativistic scenario [20].

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[3] We remark that the same technique may be used to construct bosonic Fock bimodules and the corresponding fields: in such a scenario, the construction in [17] would be analogous to a bosonic Fock $\mathscr{A}$-bimodule, with $\mathscr{A}$ the finite d-dimensional Weyl algebra, $\boldsymbol{h} = L^2(\mathbb{R}^d)$, and $\mathfrak{h} = \boldsymbol{h}\mathscr{A}$ endowed with the trivial left action in the sense of the present paper.

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Ezio Vasselli
Dipartimento di Matematica, Universitá di Roma “Tor Vergata”, c/o Giuseppe Ruzzi,
Via della Ricerca Scientifica 1, 00133 Rome, Italy; ezio.vasselli@gmail.com