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Minimizing movements for forced anisotropic curvature flow of droplets

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Abstract. We study forced anisotropic curvature flow of droplets on an inhomogeneous horizontal hyperplane. As in Bellettini and Kholmatov [J. Math. Pures Appl. 117 (2018), 1–58], we establish the existence of smooth flow, starting from a regular droplet and satisfying the prescribed anisotropic Young's law, and also the existence of a $1/2$-Hölder continuous in time minimizing movement solution starting from a set of finite perimeter. Furthermore, we investigate various properties of minimizing movements, including comparison principles, uniform boundedness, and the consistency with the smooth flow.

Contents

1. Introduction

In this paper, we are interested in the forced anisotropic mean curvature flow of droplets on an inhomogeneous hyperplane. Representing the droplets by subsets of the half-plane

$$\Omega := \mathbb{R}^{n-1} \times (0, +\infty)$$

and the relative adhesion coefficient of the hyperplane

$$\partial\Omega := \mathbb{R}^{n-1} \times \{0\}$$

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by a function $\beta : \partial\Omega \to \mathbb{R}$, we write the corresponding evolution equation of droplets $\{E(t)\}_{t\in[0,T)}$ as

$$\begin{cases} v_{\Gamma(t)}(x) = -\kappa^{\Phi}_{\Gamma(t)}(x) - f(t,x) & \text{for } t \in (0,T) \text{ and } x \in \Gamma(t), \\ \partial\Gamma(t) \subset \partial\Omega & \text{for } t \in [0,T), \\ \nabla\Phi(\nu_{\Gamma(t)}(x)) \cdot \mathbf{e}_n = -\beta(x) & \text{for } t \in (0,T) \text{ and } x \in \partial\Gamma(t), \\ E(0) = E_0, \end{cases} \tag{1.1}$$

where $\Gamma(t) := \Omega \cap \partial E(t)$ is the free boundary of $E(t)$ in Ω, Φ is an *even anisotropy* in $\mathbb{R}^n$, i.e., a positively one-homogeneous even convex function in $\mathbb{R}^n$ satisfying

$$c_\Phi|x| \le \Phi(x) \le C_\Phi|x|, \quad x \in \mathbb{R}^n, \tag{1.2}$$

for some $0 < c_\Phi \le C_\Phi$, $v_{\Gamma(t)}$ and $\kappa^{\Phi}_{\Gamma(t)}$ are the normal velocity and the anisotropic mean curvature of $\Gamma(t)$, respectively, $\nu_{\Gamma(t)}$ is the unit normal, outer to $E(t)$, $f : \mathbb{R}^+_0 \times \Omega \to \mathbb{R}$ is a forcing term, $\mathbf{e}_n = (0,\dots,0,1)$, and E_0 is the initial droplet; here, $\mathbb{R}^+_0 := [0,+\infty)$. In the literature, the third equation in (1.1) is called the anisotropic Young's law or anisotropic contact-angle condition [16]. We refer to solutions of (1.1) as the Φ-curvature flow starting from E_0, with forcing f and anisotropic contact angle β.

The following result shows that equation (1.1) is well posed.

Theorem 1.1 (Existence of smooth flows). *Let Φ be an elliptic $C^{3+\alpha}$-anisotropy in $\mathbb{R}^n$, $f \in C^{\frac{\alpha}{2},\alpha}(\mathbb{R}^+_0 \times \overline{\Omega})$, $\beta \in C^{1+\alpha}(\partial\Omega)$ with $\|\beta\|_\infty < \Phi(\mathbf{e}_n)$, and $E_0 \subset \Omega$ a bounded set such that $\Gamma_0 := \Omega \cap \partial E_0$ is a $C^{2+\alpha}$-hypersurface with boundary satisfying*

$$\partial\Gamma_0 \subset \partial\Omega \quad \text{and} \quad \nabla\Phi(\nu_{\Gamma_0}) \cdot \mathbf{e}_n = \beta \quad \text{on } \partial\Gamma_0,$$

where $\alpha \in (0,1]$. Then, there exist a maximal time $T^\dagger > 0$ and a unique Φ-curvature flow $\{E(t)\}_{t\in[0,T^\dagger)}$ starting from E_0, with forcing f and anisotropic contact angle β.

In Section 4, we establish this theorem in a more general form (see Theorem 4.3) by following the arguments presented in [4, 27]. Specifically, we begin by introducing a convenient parametrization and linearize the problem near the initial and boundary conditions. Subsequently, we employ the Solonnikov method [37] to solve the linearized problem, and next, utilizing fixed-point arguments in the Hölder spaces, we solve the nonlinear problem for small time intervals. Finally, through iterative application of this short-time argument, we extend the solution till the maximal time. Since the fixed-point method is quite robust, the flow is stable with respect to small perturbations of initial condition, Φ, β, and f (Theorem 4.8). Moreover, as in the Euclidean case (see, e.g., [4, 36]), the smooth Φ-curvature flow satisfies a strong comparison principle (Theorem 4.9), which, in particular, shows the uniqueness of the flow. The stability of the smooth flow allows to flow tubular neighborhoods of initial sets (Theorem 4.12); we anticipate here that the evolution of tubular neighborhoods is an important ingredient in the proof of the consistency of GMM (Definition 1.2) with the smooth flow.

The evolution equation (1.1) can be seen as mean curvature flow of hypersurfaces with a prescribed Neumann-type boundary condition. There are quite a few results related to the well-posedness of the classical mean curvature flow with Neumann boundary condition; see, e.g., [2,4,8,22–24,27]. See also [21,25,34,38] for mean curvature flow with Dirichlet boundary conditions.

When $f \equiv 0$, the evolution equation (1.1) is a gradient flow for the functional

$$\mathcal{C}_\beta(E) := P_\Phi(E,\Omega) + \int_{\partial\Omega} \beta\chi_E \, d\mathcal{H}^{n-1}, \quad E \in BV(\Omega;\{0,1\}), \tag{1.3}$$

where for simplicity we drop the dependence of $\mathcal{C}_\beta$ on Φ,

$$P_\Phi(E,U) := \int_{U\cap\partial^* E} \Phi(\nu_E)\, d\mathcal{H}^{n-1}$$

is the Φ-perimeter of E in an open set U, and $\partial^* E$ and ν_E are the reduced boundary and the generalized outer unit normal of E. To maintain the $L^1(\Omega)$-lower semicontinuity and coercivity of the capillary functional, we always assume that

$$\exists \eta \in (0,1/2): \quad \|\beta\|_\infty \le (1-2\eta)\Phi(\mathbf{e}_n). \tag{1.4}$$

Under this assumption and a priori estimates (see (A.1) below),

$$\eta P_\Phi(E) \le \mathcal{C}_\beta(E) \le P_\Phi(E), \quad E \in \mathcal{S}. \tag{1.5}$$

In the literature, $\mathcal{C}_\beta$ is usually referred to as the anisotropic capillary functional. Originated in the work of Young, Laplace, Gauss, and others, this functional allows to consider more general classes of anisotropies Φ (such as crystalline) and relative adhesion coefficients β not necessarily constant (see, e.g., [16, 19, 30]). The global minimizers of this functional (usually under a volume constraint) are related to the equilibrium shapes of liquid or crystalline droplets in the container, which sometimes are called Winterbottom shapes [28, 30, 35]. Therefore, the problems, such as the existence of minimizers, the regularity of their free boundaries and contact sets, the validity of an anisotropic version of Young's contact-angle law, and the characterization of the shape of the minimizers, have been extensively investigated and addressed in numerous papers in the literature (see, e.g., [7, 9, 16, 19, 20, 26, 28, 30, 35] and the references therein).

To study a weak evolution of droplets, let

$$\mathcal{S} := \{E \in BV(\Omega;\{0,1\}) : E = E^{(1)}\}$$

be the metric space endowed with the $L^1(\Omega)$-distance $d(E,F) := |E\Delta F|$, where $E^{(1)}$ is the set of points of density 1 for E, i.e.,

$$E^{(1)} := \{x \in \mathbb{R}^n : \lim_{r\to 0^+} r^{-n}|B_r(x)\setminus E| = 0\},$$

and let

$$\mathcal{F}_{\beta,f}(E;E_0,\tau,k) := \begin{cases} |E\Delta E_0|, & k=0,\\ \mathcal{C}_\beta(E)+\frac{1}{\tau}\int_{E\Delta E_0} \mathrm{d}_{E_0}dx+\int_k^{k+1}ds\int_E f(\tau s,x)dx, & k\ge 1 \end{cases}$$

be the anisotropic capillary Almgren–Taylor–Wang functional with a nonautonomous (time-dependent) forcing, which generalizes the isotropic setting of [4], where $E, E_0 \in \mathcal{S}$, $\tau > 0$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathrm{d}_E(x) := \mathrm{dist}(x, \partial E)$, and f is a suitable forcing term. When $f \equiv 0$, we shortly write $\mathcal{F}_\beta$.

Definition 1.2 (Generalized minimizing movements [15]). (a) Given $\tau > 0$, a family $\{E(\tau,k)\}_{k\in\mathbb{N}_0} \subset \mathcal{S}$ is called a (discrete) *flat flow* starting from E_0 provided that $E(\tau,0) := E_0$,

$$E(\tau,k) \in \operatorname{argmin} \mathcal{F}_{\beta,f}(\cdot; E(\tau,k-1),\tau,k), \quad k \ge 1.$$

(b) A family $\{E(t)\}_{t\in\mathbb{R}_0^+}$ is called a *generalized minimizing movement* (GMM) starting from E_0 if there exist a sequence $\tau_i \to 0^+$ and flat flows $\{E(\tau_i,\cdot)\}$ such that

$$\lim_{i\to+\infty} |E(\tau_i, \lfloor t/\tau_i \rfloor)\Delta E(t)| = 0, \quad t \ge 0, \tag{1.6}$$

where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$.

The collection of all GMMs starting from E_0 and associated to $\mathcal{F}_{\beta,f}$ will be denoted by $\mathrm{GMM}(\mathcal{F}_{\beta,f}, E_0)$.

In applications, it is enough to establish (1.6) in any finite interval $[0,T)$. (Thus, different T may require different sequences $\tau_j \to 0^+$, but at the end, we can use a diagonal argument.)

Starting from the seminal papers [1, 15, 29], the minimizing movement approach has been employed in numerous papers, especially in proving the existence of weak (anisotropic) mean curvature flows (see, e.g., [11–13]). Moreover, the robustness of the method allows for applications in other settings, such as in (anisotropic) mean curvature evolution in Finsler geometry with forcing [10], a volume-preserving mean curvature flow [33], mean curvature flow with Dirichlet and Neumann-type boundary conditions [4, 32] (see also Theorem 1.3), and a mean curvature evolution of bounded Caccioppoli partitions, including anisotropies and forcing [3, 5].

The first main result of this paper is the following theorem.

Theorem 1.3 (Existence of generalized minimizing movements). *Assume that*

(a)

$$f \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+ \times \mathbb{R}^n) \quad \textit{and} \quad f^- \in L^1_{\mathrm{loc}}(\mathbb{R}_0^+; L^1(\mathbb{R}^n)), \tag{H1}$$

(b)

$$\forall T>0\, \exists \gamma_T > 0: \sup_{0<|A|<\omega_n\gamma_T^n, 0\le t\le T} \frac{1}{|A|^{\frac{n-1}{n}}}\int_A |f(t,x)|dx \le \frac{c_\Phi \eta n \omega_n^{1/n}}{4}, \tag{H2}$$

(c)

$$\limsup_{\tau\to0^+} \frac{1}{\tau} \int_0^\tau ds \int_{\mathbb{R}^n} |f(s,x)|dx \in [0,+\infty), \tag{H3}$$

(d) *for any* $T > 0$ *either*

$$c_T := \sup_{t\in[0,T]} \|f(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} < +\infty \tag{H4'}$$

or there exists $c_T > 0$ *such that*

$$\int_{\mathbb{R}^n} |f(s,x) - f(s+\tau,x)|dx \le c_T\tau, \quad s, s+\tau \in [0,T], \tau > 0. \tag{H4''}$$

Then, for any $E_0 \in \mathcal{S}$, $\mathrm{GMM}(\mathcal{F}_{\beta,f}, E_0)$ *is nonempty. Moreover, there exists* $C_0 > 0$ *depending only on* Φ, β, f, *and* E_0 *such that, for any* $E(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta,f}, E_0)$,

$$|E(t)\Delta E(s)| \le C_0|t-s|^{1/2}, \quad s,t > 0 \text{ with } |t-s| < 1. \tag{1.7}$$

If $|\partial E_0| = 0$, *then* (1.7) *holds for all* $s,t \ge 0$.

Furthermore, assume that E_0 *is bounded and for* $T > 0$,

$$\exists a_T, b_T > 0 : \quad f^-(t,x) \le a_T + b_T|x|, \quad t \in [0,T], |x| \in \mathbb{R}^n. \tag{H5}$$

Then, each $E(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta,f}, E_0)$ *is bounded in* $[0,T]$*; i.e., there exists* $\overline{R} > 0$ *such that* $E(t) \subset B_{\overline{R}}(0)$ *for any* $t \in [0,T]$.

Some comments on assumptions (a)–(d) are in order.

- The hypothesis (a) is necessary for the well definiteness of $\mathcal{F}_{\beta,f}$ and is related to the prescribed curvature functional.
- The condition (b) will be used in establishing uniform density estimates in Theorem 2.2 and also in proving the boundedness of minimizers.
- The hypothesis (c) is a technical assumption implying $\mathcal{F}_{\beta,f}(E_0; E_0, \tau, k) < +\infty$ for any $E_0 \in \mathcal{S}$ and will be necessary to estimate the forced capillary energies of flat flows $E(\tau,k)$ with that of $E(\tau,0)$ (see, e.g., (2.21), (2.25) and subsequent estimates).
- An example of forcing f satisfying (a)–(c) is

  $$f(t,x) = a(t)h(x)$$

  for some $a \in L^\infty(\mathbb{R}_0^+)$ and $h \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $p \ge n$.
- Assumptions (H4') and (H4") in (d) are two (in general different) sufficient conditions for the existence and local $1/2$-time Hölder continuity of GMMs.
- In [10], the authors established the local uniform boundedness of GMM for bounded forcing terms using comparison with balls. In this paper, we show that the same property holds also for forcing terms with at most linear growth, using comparison with Winterbottom shapes in place of balls (see Section 2.2).

To prove Theorem 1.3, we apply the already well-established machinery of Almgren–Taylor–Wang and Luckhaus–Sturzenhecker (see, e.g., [1, 4, 5, 10, 29, 33]). The main difficulty here is that as in [10] because of the time dependence of f, given a flat flow $\{E(\tau,k)\}$, the sequence

$$k \mapsto \mathcal{C}_\beta(E(\tau,k)) + \int_k^{k+1} ds \int_E f(\tau s, x)dx$$

is not necessarily nonincreasing. This creates numerous technical difficulties to bound the perimeter $P(E(\tau,k))$ uniformly in τ and k, which is important for the sequential compactness of $\{E(\tau,\lfloor t/\tau\rfloor)\}$ in τ. To overcome such an issue, we use assumption (d). It is worth to mention that under the assumption (H4") every GMM is globally $1/2$-Hölder continuous in time; i.e., in (1.7) the assumption $|t-s|<1$ is not necessary. This is true, for instance, in the case of an autonomous (time-independent) forcing f.

As in the Euclidean case without forcing [4, Section 6], minimizers of $\mathcal{F}_{\beta,f}$ satisfy various comparison principles (Theorem 3.1), which yield the following comparison principle for GMMs.

Theorem 1.4 (Comparison of GMMs). *Assume that $\beta_1 \ge \beta_2$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $E_0^1 \subset E_0^2$ and $f_1 \ge f_2$ a.e. in $\mathbb{R}_0^+ \times \Omega$. Then,*

(a) *for any $E^2(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta_2,f_2}, E_0^2)$ there exists $E^1(\cdot)_* \in \mathrm{GMM}(\mathcal{F}_{\beta_1,f_1}, E_0^1)$ such that*

$$E^1(t)_* \subset E^2(t) \quad \text{for all } t \ge 0; \tag{1.8}$$

(b) *for any $E^1(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta_1,f_1}, E_0^1)$ there exists $E^2(\cdot)^* \in \mathrm{GMM}(\mathcal{F}_{\beta_2,f_2}, E_0^2)$ such that*

$$E^1(t) \subset E^2(t)^* \quad \text{for all } t \ge 0.$$

Finally, we study the relation of GMM with the smooth flow solving (1.1).

Theorem 1.5 (Consistency). *Assume that $n \le 3$ and Φ is an elliptic $C^{3+\alpha}$-anisotropy in $\mathbb{R}^n$ for some $\alpha \in (0,1]$, or $n \le 4$ and Φ is Euclidean. Let $\beta \in C^{1+\alpha}(\partial\Omega)$, $f \in C^{\frac{\alpha}{2},\alpha}([0,+\infty)\times\overline{\Omega})$, and $\{E(t)\}_{t\in[0,T^\dagger)}$ be a smooth Φ-curvature flow starting from E_0, with forcing f and anisotropic contact angle β. Then, for any $F(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta,f}, E_0)$,*

$$E(t) = F(t), \quad t \in [0, T^\dagger).$$

Similar consistency result in the three-dimensional Euclidean case without forcing has been recently obtained in [36] using the techniques originated in [1]. To prove Theorem 1.5, we adapt those techniques adding anisotropy and also forcing. Note that the smallness of dimension n implies that the free boundary $\partial^\Omega E_\tau$ of minimizers E_τ of $\mathcal{F}_{\beta,f}(\cdot; E_0, \tau, k)$ is a C^2-hypersurface up to the boundary [16, 17], satisfying the anisotropic contact angle condition with β. This allows to establish smooth inner and outer barriers for minimizers of $\mathcal{F}_{\beta,f}$ in Proposition 3.3. To extend this proposition to higher dimensions, one need to show that a smooth hypersurface $\Gamma \subset \Omega$ with boundary in $\partial\Omega$ can be an outer or inner barrier for $\partial^\Omega E_\tau$ either only at points of the reduced boundary or only at regular points of $\partial^\Omega E_\tau$. Recall that the assertion for the reduced boundary is true since there are no singular minimizing cones for P_Φ containing a halfspace (see [1, Lemma 7.3]). However, currently not much seems known about such a behavior of singular minimizing cones for capillary functional $\mathcal{C}_\beta$.

The paper is organized as follows. In Section 2, we provide a full proof of Theorem 1.3. Various comparison results are established in Section 3. In Section 4, we establish the well-posedness of (1.1), proving Theorem 1.1 in more general form, and various properties of the smooth flows. Finally, we prove Theorem 1.5 in Section 5. We conclude the paper with an appendix, where we obtain some a priori estimates for capillary functional and a characterization of elliptic smooth anisotropies.

2. Existence of GMM

Notation

Given an anisotropy Φ in $\mathbb{R}^n$, the dual anisotropy is defined as

$$\Phi^o(x) := \max_{\Phi(y)=1} x \cdot y.$$

The following Young inequality holds:

$$x \cdot y \le \Phi^o(x)\Phi(y), \quad x, y \in \mathbb{R}^n.$$

The set $W^\Phi := \{\Phi^o(x) \le 1\}$ are called the *Wulff shape* for Φ. With a slight abuse of the definition, the translations and scalings $W_r^\Phi(x) = x + rW^\Phi$ of W^Φ are still called Wulff shapes.

We say an anisotropy Φ in $\mathbb{R}^n$ is $C^{k+\alpha}$ if $\Phi \in C^{k+\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. We denote by $\nabla\Phi$ and $\nabla^2\Phi$ the spatial gradient and Hessian of Φ. If there exists $\gamma > 0$ such that $x \mapsto \Phi(x) - \gamma|x|$ is also an anisotropy, Φ is called *elliptic*. By Proposition A.2, a $C^{k+\alpha}$-anisotropy with $k \ge 2$ is elliptic if and only if its dual is an elliptic $C^{k+\alpha}$-anisotropy.

Given an anisotropy Φ, we define

$$\mathrm{d}_E^\Phi(x) := \inf\{\Phi(x-y) : y \in \Omega \cap \partial^* E\}, \quad \mathrm{sd}_E^\Phi(x) := \begin{cases} \mathrm{d}_E^\Phi(x), & x \in \Omega \setminus E, \\ -\mathrm{d}_{E^c}^\Phi(x), & x \in E, \end{cases} \qquad x \in \Omega,$$

for $E \in \mathcal{S}$. When Φ is Euclidean, we write shortly d_E and sd_E.

To shorten the notation, we use

$$\partial^\Omega E := \Omega \cap \partial E$$

and

$$E \prec F \quad \Longleftrightarrow \quad E \subset F \quad \text{and} \quad \mathrm{dist}(\partial^\Omega E, \partial^\Omega F) > 0$$

for $E, F \in \mathcal{S}$. Note that

$$E \subset F \quad \Longleftrightarrow \quad \mathrm{sd}_E^\Phi \geq \mathrm{sd}_F^\Phi \quad \text{in } \Omega, \quad \textit{resp.}, \quad E \prec F \quad \Longleftrightarrow \quad \mathrm{sd}_E^\Phi > \mathrm{sd}_F^\Phi \quad \text{in } \Omega. \tag{2.1}$$

The following proposition shows the connection between the regular surfaces and distance functions (see also [36, Proposition 2.1]).

Proposition 2.1. *Let Γ be a $C^{2+\alpha}$-hypersurface (not necessarily connected and with or without boundary) in an open set $Q \subset \mathbb{R}^n$ for some $\alpha \in [0,1]$. Then,*

(a) *for any $x \in \Gamma$ there exists $r_x > 0$ such that*
- *Γ divides $B_{r_x}(x)$ into two connected components,*
- *$\mathrm{dist}(\cdot, \Gamma) \in C^{2+\alpha}(B_{r_x}(x) \setminus \Gamma)$;*

(b) *if Γ is compact and has no boundary, then $\inf_{x\in\Gamma} r_x > 0$, i.e., the radius r_x in (a) can be taken uniform in x;*

(c) *if $\Gamma = Q \cap \partial E$ for some $E \subset Q$ and Φ is an elliptic $C^{3+\alpha}$-anisotropy, then for any $x \in \Gamma$ there exists $r_x > 0$ such that $B_r(x) \subset Q$ and $\mathrm{sd}_E^\Phi \in C^{2+\alpha}(B_{r_x}(x))$. In this case,*

$$\nabla\, \mathrm{sd}_E^\Phi(x) = \nu_E(x)^{\Phi^o}$$

for any $x \in \Gamma$, where ν_E is the outer unit normal of E and

$$\eta^{\Phi^o} = \frac{\eta}{\Phi^o(\eta)}, \qquad 0 \neq \eta \in \mathbb{R}^n. \tag{2.2}$$

(d) *Assume that in (c), additionally, Q has a $C^{2+\alpha}$-boundary. Then, under assumptions of (b), for any $x_0 \in \partial Q \cap \partial\Gamma$ and for any $\eta \in \mathbb{S}^{n-1}$ with $\eta \cdot \nu_Q(x_0) < 0$, where $\nu_Q(x_0)$ is the outer unit normal of ∂Q at x_0, one has*

$$\mathrm{sd}_E^\Phi(x_0 + s\eta) - \mathrm{sd}_E^\Phi(x_0) = s\big(\nu_E(x_0)^{\Phi^o} \cdot \eta + o(1)\big) \quad \textit{as } s \to 0^+.$$

These assertions can be proven using the local geometry of Γ and Q, i.e., passing to the local coordinates (see also [18]).

Given an elliptic C^2-anisotropy Φ and a C^2-hypersurface $\Gamma \subset \mathbb{R}^n$ oriented by a unit normal ν_Γ, the Φ-*curvature* of Γ at $x \in \Gamma$ is defined as [1, Section 2.2]

$$\kappa_\Gamma^\Phi(x) := \mathrm{Tr}[\nabla^2\Phi^o(\nu_\Gamma(x))\nabla^2 R(x)],$$

where R is any C^2-function in a ball $B_r(x)$ with small radius $r > 0$ satisfying

$$B_r(x) \cap \Gamma = \{R = 0\} \quad \text{and} \quad \nabla R(x) = \nu_E(x).$$

Writing Γ as a graph near x, one can show that $\kappa_\Gamma^\Phi(x)$ is independent of the choice of R. When $\Gamma = \partial^\Omega E$ for some $E \subset \Omega$, we orient Γ along the outer unit normal of E and write

$$\kappa_E^\Phi := \kappa_\Gamma^\Phi.$$

By this convention, convex sets admit a nonnegative Φ-curvature. We also set

$$\|II_E\|_\infty := \sup_{x\in\Gamma} |II_\Gamma(x)|,$$

where II_Γ is the second fundamental form of Γ.

Recall that if

$$E_t = (\mathrm{I} + tX)(E)$$

is a C^1-perturbation of $E \subset \Omega$ for some $X \in C_c^1(U;\mathbb{R}^n)$ with $X\cdot \mathbf{e}_n = 0$ on $\partial\Omega$, then the first variation of the capillary functional $\mathcal{C}_\beta$ at E is computed as [26]

$$\begin{aligned}\frac{d}{dt}\mathcal{C}_\beta(E_t)\Big|_{t=0} = &\int_\Gamma \kappa_E^\Phi X\cdot \nu_E \, d\mathcal{H}^{n-1}\\ &+\int_{\partial\Omega\cap\partial\Gamma} X\cdot\big[\mathcal{R}_{\pi/2}(\pi(\nabla\Phi(\nu_\Gamma))) + \beta\mathbf{n}\big]d\mathcal{H}^{n-2},\end{aligned} \tag{2.3}$$

where $\Gamma := \partial^\Omega E$, ν_Γ is the outer unit normal to Γ, $\mathbf{n}$ is the conormal of Γ at its boundary points (i.e., tangent vector to Γ, but normal to $\partial\Gamma$), π is the orthogonal projection onto the hyperplane T spanned to $\{\nu_\Gamma, \mathbf{n}\}$ (both defined at $\partial\Gamma$), and $\mathcal{R}_\theta$ is the counterclockwise rotation in T by angle θ.

Recall that by [4, Lemma 2.1] for any $E \in \mathcal{S}$

$$\chi_E \in L^1(\partial\Omega) \quad \text{and} \quad E\in\mathcal{S}. \tag{2.4}$$

By (2.4) we can rewrite the anisotropic capillary functional (1.3) as

$$\mathcal{C}_\beta(F) = P_\Phi(F) + \int_{\partial\Omega} (\beta - \Phi^o(\mathbf{e}_n))\chi_F \, d\mathcal{H}^{n-1}.$$

Moreover, since $G\chi_G \in L^1(\Omega)$ for any $G\in\mathcal{S}$, up to an additive constant independent of E, we can write

$$\mathcal{F}_{\beta,f}(E;E_0,\tau,k) := \mathcal{C}_\beta(E) + \frac{1}{\tau}\int_E \mathrm{sd}_{E_0}\,dx + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}\int_E f(s,x)dxds, \quad k\ge 1.$$

2.1. Proof of Theorem 1.3

For the convenience of the reader, we divide the proof into smaller steps. In each step, we highlight which of the assumptions on f (mentioned in Theorem 1.3) will be used in that step.

2.1.1. Existence of minimizers. Given $E_0\in\mathcal{S}$, $\tau > 0$, and $k\in\mathbb{N}$, let $\{E_i\}$ be a minimizing sequence of $\mathcal{F}_{\beta,f}(\cdot;E_0,\tau,k)$. We may assume that

$$\mathcal{F}_{\beta,f}(E_i;E_0,\tau,k) \le \mathcal{F}_{\beta,f}(E_0\cap B_R;E_0,\tau,k)$$

for some $R > 0$ and for all $i \geq 1$. (We need such a truncation with $B_R(0)$ because a priori E_0 is not bounded, and thus, in general, the integral $\int_k^{k+1} ds \int_{E_0} f^+(\tau s, x)dx$ need not to be finite.) Then,

$$\begin{aligned}
&\mathcal{C}_\beta(E_i) + \frac{1}{\tau}\int_{E_i \setminus E_0} \mathrm{d}_{E_0} dx + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds \int_{E_i} f^+(\tau s, x)dx \\
&\leq \mathcal{F}_{\beta,f}(E_0 \cap B_R; E_0, \tau, k) + \frac{1}{\tau}\int_{E_0} \mathrm{d}_{E_0} dx \\
&\quad + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds \int_{\mathbb{R}^n} f^-(\tau s, x)dx := C_1.
\end{aligned} \tag{2.5}$$

In particular, by (1.5), $\{P_\Phi(E_i)\}$ is bounded, and hence, by $L^1_{\mathrm{loc}}(\Omega)$-compactness of $\mathcal{S}$, there exists $E_\infty \in BV_{\mathrm{loc}}(\Omega; \{0,1\})$ such that, up to a relabeled subsequence, $E_i \to E$ in $L^1_{\mathrm{loc}}(\Omega)$ as $i \to +\infty$. Moreover, for any bounded $U \subset \Omega$,

$$P_\Phi(E, U) \leq \liminf_{i\to+\infty} P_\Phi(E_i, U) \leq \liminf_{i\to+\infty} P_\Phi(E_i) \leq \frac{1}{\eta}\sup_i \mathcal{C}_\beta(E_i) \leq \frac{C_1}{\eta}.$$

Thus, letting $U \nearrow \mathbb{R}^n$, we get $P_\Phi(E) < +\infty$. Moreover, by the isoperimetric inequality

$$P_\Phi(E) \geq c_{\Phi,n}|E|^{\frac{n-1}{n}}, \quad c_{\Phi,n} = \frac{P_\Phi(W^\Phi)}{|W^\Phi|^{\frac{n-1}{n}}}, \tag{2.6}$$

for any bounded $U \subset \mathbb{R}^n$, we have

$$|U \cap E| = \lim_{i\to+\infty} |U \cap E_i| \leq \liminf_{i\to+\infty} |E_i| \leq c_{\Phi,n}^{-\frac{n}{n-1}} \liminf_{i\to+\infty} P_\Phi(E_i)^{\frac{n}{n-1}} \leq \left(\frac{C_1}{c_{\Phi,n}\eta}\right)^{\frac{n}{n-1}}.$$

Thus, $|E| < +\infty$, i.e., $E \in \mathcal{S}$. Then, the $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuity of $\mathcal{F}_{\beta,f}$ implies that E is a minimizer.

Notice that if E_τ is a minimizer, then as in (2.5)

$$\mathcal{C}_\beta(E_\tau) + \frac{1}{\tau}\int_{E_\tau \setminus E_0} \mathrm{d}_{E_0} dx + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds \int_{E_\tau} f^+(s, x)dx \leq C_1,$$

and hence, $f^+ \in L^1([k\tau, (k+1)\tau] \times E_\tau)$.

2.1.2. Density estimates for minimizers. In this section besides (H1), we assume (H2).

Theorem 2.2. *Let $E_0 \in \mathcal{S}$, $\tau > 0$, $k \in \mathbb{N}$, and E_τ be a minimizer of $\mathcal{F}_{\beta,f}(\cdot; E_0, \tau, k)$. Let $T > (k+1)\tau$. Then, there exists $\theta \in (0, 8^{-n})$ depending only on n, Φ, and η (see (1.5)) such that*

$$\sup_{x \in \overline{E_\tau \Delta E_0}} \mathrm{d}_{E_0}(x) \leq \frac{\sqrt{\tau}}{\theta} \tag{2.7}$$

provided $\theta\sqrt{\tau} \le \gamma_T$*, where* $\gamma_T > 0$ *is given by* (H2)*. Moreover, if* $x \in \partial E_\tau$ *and* $r \in (0, \theta\sqrt{\tau}]$*, then*

$$\theta \le \frac{|B_r(x) \cap E_\tau|}{|B_r(x)|} \le 1 - \theta, \tag{2.8}$$

$$\frac{P(E_\tau, B_r(x))}{r^{n-1}} \ge \theta. \tag{2.9}$$

In what follows, we refer to (2.8) and (2.9) as the uniform density estimates for E_τ.

Proof. For shortness, we write

$$h(\cdot) := \int_k^{k+1} f(\tau s, \cdot) ds.$$

Let us establish (2.7). For each $x \in E \Delta E_0$, let $p_x \in \overline{\partial^\Omega E_0}$ be such that

$$r_x := |p_x - x| = \mathrm{d}_{E_0}(x).$$

By the 1-lipschitzianity of d_{E_0},

$$|\mathrm{d}_{E_0}(x)| \le |\mathrm{d}_{E_0}(p_x)| + |x - p_x| = r_x.$$

Thus, we need to estimate r_x. Fix $r \in (0, r_x)$ and set $B_r := B_r(x)$.

Let $x \in E \setminus E_0$ so that $\mathrm{sd}_{E_0} \ge r_x - r$ in $B_r(x)$. Then, for a.e. $r \in (0, r_x)$ with $\mathcal{H}^{n-1}(\partial^* E_\tau \cap \partial B_r) = 0$, summing the equalities

$$\begin{aligned} P_\Phi(E_\tau \setminus B_r, \Omega) - P_\Phi(E_\tau, \Omega) &= \int_{E_\tau \cap \partial B_r} \Phi(\nu_{B_r}) d\mathcal{H}^{n-1} - P_\Phi(E_\tau, \Omega \cap B_r) \\ &= 2\int_{E_\tau \cap \partial B_r} \Phi(\nu_{B_r}) d\mathcal{H}^{n-1} - P_\Phi(E_\tau \cap B_r, \Omega), \end{aligned}$$

$$\int_{\partial\Omega} \beta\chi_{E_\tau \setminus B_r} d\mathcal{H}^{n-1} - \int_{\partial\Omega} \beta\chi_{E_\tau} d\mathcal{H}^{n-1} = -\int_{\partial\Omega} \beta\chi_{E_\tau \cap B_r} d\mathcal{H}^{n-1},$$

$$\int_{E_\tau \setminus B_r} \mathrm{sd}_{E_0}\, dx - \int_{E_\tau} \mathrm{sd}_{E_0}\, dx = -\int_{E_\tau \cap B_r} \mathrm{sd}_{E_0}\, dx,$$

$$\int_{E_\tau \setminus B_r} h dx - \int_{E_\tau} h dx = -\int_{E_\tau \cap B_r} h dx$$

and using the minimality of E, we find

$$\begin{aligned} 0 &\le \mathcal{F}_{\beta,f}(E_\tau \setminus B_r; E_0, \tau, k) - \mathcal{F}_{\beta,f}(E_\tau; E_0, \tau, k) \\ &= 2\int_{E_\tau \cap \partial B_r} \Phi(\nu_{B_r}) d\mathcal{H}^{n-1} - P_\Phi(E_\tau \cap B_r, \Omega) \\ &\quad - \int_{\partial\Omega} \beta\chi_{E_\tau \cap B_r} d\mathcal{H}^{n-1} - \frac{1}{\tau}\int_{E_\tau \cap B_r} \mathrm{sd}_{E_0}\, dx - \int_{E_\tau \cap B_r} h dx. \end{aligned}$$

Thus,

$$2\int_{E_\tau\cap\partial B_r}\Phi(\nu_{B_r})d\mathcal{H}^{n-1}\ge \mathcal{C}_\beta(E_\tau\cap B_r)+\frac{r_x-r}{\tau}|E_\tau\cap B_r|+\int_{E_\tau\cap B_r}h\,dx. \quad (2.10)$$

By (1.5), the definition of the Φ-perimeter, (1.2), and the Euclidean isoperimetric inequality

$$\mathcal{C}_\beta(E_\tau\cap B_r)\ge \eta P_\Phi(E_\tau\cap B_r)\ge \eta c_\Phi n\omega_n^{1/n}|E_\tau\cap B_r|^{\frac{n-1}{n}}.$$

On the other hand, if $r\le\gamma_T$, then by (H2)

$$\int_{E_\tau\cap B_r}|h|dx\le \frac{c_\Phi\eta n\omega_n^{1/n}}{4}|E_\tau\cap B_r|^{\frac{n-1}{n}},$$

and therefore, by (2.10) and (1.2),

$$\frac{3c_\Phi\eta n\omega_n^{1/n}}{4}|E_\tau\cap B_r|^{\frac{n-1}{n}}\le 2C_\Phi\mathcal{H}^{n-1}(E_\tau\cap\partial B_r).$$

Integrating this inequality, we get

$$|E_\tau\cap B_r|\ge\left(\frac{3c_\Phi\eta}{8C_\Phi}\right)^n\omega_n r^n,\qquad r\in[0,\gamma_T\wedge r_x]. \quad (2.11)$$

Inserting this in (2.10), we get

$$\frac{r_x-r}{\tau}\left(\frac{3c_\Phi\eta}{8C_\Phi}\right)^n\omega_n r^n\le 2C_\Phi n\omega_n r^{n-1},$$

and therefore,

$$r_x\le g(r):=r+\frac{C_2\tau}{r},\qquad r\in(0,\gamma_T\wedge r_x], \quad (2.12)$$

where

$$C_2:=2C_\Phi n\left(\frac{8C_\Phi}{3c_\Phi\eta}\right)^n.$$

On the other hand, if $x\in E_0\setminus E_\tau$, then using

$$\begin{aligned}
0&\le \mathcal{F}_{\beta,f}(E_\tau\cup B_r;E_0,\tau,k)-\mathcal{F}_{\beta,f}(E_\tau;E_0,\tau,k)\\
&=2\int_{E_\tau^c\cap\partial B_r}\Phi(\nu_{B_r})d\mathcal{H}^{n-1}-P_\Phi(E_\tau^c\cap B_r,\Omega)\\
&\quad+\int_{\partial\Omega}\beta\chi_{E_\tau^c\cap B_r}d\mathcal{H}^{n-1}-\frac{1}{\tau}\int_{E_\tau^c\cap B_r}\mathrm{sd}_{E_0}\,dx-\int_{E_\tau^c\cap B_r}h\,dx
\end{aligned}$$

for a.e. $r\in(0,r_x]$ with $r_x:=\mathrm{dist}(x,\partial E_0)$, we get

$$2\int_{E_\tau^c\cap\partial B_r}\Phi(\nu_{B_r})d\mathcal{H}^{n-1}\ge\mathcal{C}_{-\beta}(E_\tau^c\cap B_r)+\frac{r_x-r}{\tau}|E_\tau^c\cap B_r|+\int_{E_\tau^c\cap B_r}h\,dx.$$

Now, repeating the above arguments, we obtain

$$|E_\tau^c \cap B_r| \geq \left(\frac{3c_\Phi \eta}{8C_\Phi}\right)^n \omega_n r^n, \quad r \in [0, \gamma_T \wedge r_x], \tag{2.13}$$

and hence, again, (2.12) follows.

In the remaining part of the proof, we assume that $\sqrt{C_2\tau} \leq \gamma_T$. The function g in (2.12) admits its unique global minimum at $\sqrt{C_2\tau}$. Thus, if $r_x > \sqrt{C_2\tau}$, then $r_x \leq g(\sqrt{C_2\tau}) = 2\sqrt{C_2\tau}$. Therefore,

$$\sup_{x \in \overline{E_\tau \Delta E_0}} \mathrm{d}_{E_0}(x) = \sup_{x \in \overline{E_\tau \Delta E_0}} r_x \leq 2\sqrt{C_2\tau}.$$

Now, we prove density estimates. Fix any $x \in \partial E_\tau$, $r \in (0, \sqrt{C_2\tau}]$, and let $B_r := B_r(x)$. First, assume that $B_r \cap \overline{\partial^\Omega E_\tau} = \emptyset$. Then, B_r intersects only the flat part of ∂E_τ, and hence,

$$\mathcal{H}^{n-1}(B_r \cap \partial E_\tau) = \omega_{n-1} r^{n-1}, \quad |B_r \cap E_\tau| = \frac{\omega_n r^n}{2}.$$

Thus, consider the case $B_r \cap \overline{\partial^\Omega E_\tau} \neq \emptyset$ so that

$$\sup_{y \in B_r} \mathrm{d}_{E_0}(y) \leq r + \sup_{y \in B_r \cap [E \Delta E_0]} \mathrm{d}_{E_0}(y) \leq r + 2\sqrt{C_2\tau}. \tag{2.14}$$

By (H2),

$$\int_{E_\tau \cap B_r} |h| dx \leq \frac{c_\Phi \eta n \omega_n^{1/n}}{4} |E_\tau \cap B_r|^{\frac{n-1}{n}}, \quad \int_{E_\tau^c \cap B_r} |h| dx \leq \frac{c_\Phi \eta n \omega_n^{1/n}}{4} |E_\tau^c \cap B_r|^{\frac{n-1}{n}}.$$

Moreover, by (2.14),

$$\begin{aligned}\int_{E_\tau \cap B_r} \mathrm{d}_{E_0} dx &\leq (r + 2\sqrt{C_2\tau}) |E_\tau \cap B_r|^{\frac{n-1}{n}} |E_\tau \cap B_r|^{\frac{1}{n}} \\ &\leq \omega_n^{1/n} r (r + 2\sqrt{C_2\tau}) |E_\tau \cap B_r|^{\frac{n-1}{n}}\end{aligned}$$

and

$$\begin{aligned}\int_{E_\tau^c \cap B_r} \mathrm{d}_{E_0} dx &\leq (r + 2\sqrt{C_2\tau}) |E_\tau^c \cap B_r|^{\frac{n-1}{n}} |E_\tau^c \cap B_r|^{\frac{1}{n}} \\ &\leq \omega_n^{1/n} r (r + 2\sqrt{C_2\tau}) |E_\tau^c \cap B_r|^{\frac{n-1}{n}}.\end{aligned}$$

Thus, if we choose $r \leq C_3\sqrt{\tau}$, where C_3 satisfies

$$C_3(C_3 + 2\sqrt{C_2}) = \frac{c_\Phi \eta n}{4},$$

then

$$\int_{E_\tau \cap B_r} \left(\frac{\mathrm{d}_{E_0}}{\tau} + h\right) dx \leq \frac{c_\Phi \eta n \omega_n^{1/n}}{2} |E_\tau \cap B_r|^{\frac{n-1}{n}}$$

and

$$\int_{E_\tau^c \cap B_r} \left(\frac{\mathrm{d}_{E_0}}{\tau} + h\right) dx \le \frac{c_\Phi \eta n \omega_n^{1/n}}{2} |E_\tau^c \cap B_r|^{\frac{n-1}{n}}.$$

Thus, as in the proof of (2.11) and (2.13), we get

$$\left(\frac{c_\Phi \eta}{4C_\Phi}\right)^n \le \frac{|E_\tau \cap B_r|}{|B_r|} \le 1 - \left(\frac{c_\Phi \eta}{4C_\Phi}\right)^n, \quad r \in (0, C_3\sqrt{\tau}]. \tag{2.15}$$

Finally, (2.9) follows from (2.15) and the relative isoperimetric inequality for balls. ■

From the lower perimeter density estimate in Theorem 2.2 and a covering argument, we get the following corollary.

Corollary 2.3. *Under assumptions of Theorem 2.2, any minimizer E_τ of $\mathcal{F}_{\beta.f}$ satisfies*

$$\mathcal{H}^{n-1}(\partial E_\tau) = \mathcal{H}^{n-1}(\overline{E_\tau} \setminus \mathrm{Int}(E_\tau)) < +\infty \quad \text{and} \quad \mathcal{H}^{n-1}(\partial E_\tau \setminus \partial^* E_\tau) = 0.$$

In particular, E_τ may be assumed open.

Another corollary of density estimates is the following analog of the volume-distance inequality of [1].

Corollary 2.4. *Let $E_0 \in \mathcal{S}$, $\tau > 0$, and $k \in \mathbb{N}$ be such that*

$$P(E_0, B_r(x)) \ge \theta r^{n-1}, \quad r \in (0, \theta\sqrt{\tau}], \tag{2.16}$$

for some $\theta, \delta > 0$. Then, for any $p > 0$ and a minimizer E_τ of $\mathcal{F}_{\beta,f}(\cdot; E_0, \tau, k)$, we have

$$|E_\tau \Delta E_0| \le \frac{C_4}{p} \mathcal{C}_\beta(E_0)\tau + \frac{p}{\tau} \int_{E_\tau \Delta E_0} \mathrm{d}_{E_0} dx \tag{2.17}$$

provided $\tau < \theta^2 p^2$, where

$$C_4 := \frac{5^n \omega_n}{c_\Phi \theta \eta}$$

and $\eta > 0$ is given in (1.5).

Specific choices of p will be made in the proof of the almost-continuity of flat flows in the next section.

Proof. Let

$$A := \left\{x \in E_\tau \Delta E_0 : \mathrm{d}_{E_0}(x) < \frac{\tau}{p}\right\}, \quad B := \left\{x \in E_\tau \Delta E_0 : \mathrm{d}_{E_0}(x) \ge \frac{\tau}{p}\right\}.$$

By the Chebyshev inequality, we have

$$|B| \le \frac{p}{\tau} \int_{E_\tau \Delta E_0} \mathrm{d}_{E_0} dx.$$

The set A can be covered by balls $\{B_{\tau/p}(x)\}_{x\in\partial^\Omega E_0}$. By the Vitali covering lemma, there exists an at most countable disjoint family $\{B_{\tau/p}(x_i)\}_{i\ge 1}$ such that A is still covered by $\{B_{5\tau/p}(x_i)\}_{i\ge 1}$. Then, by (2.16) applied with $\tau/p\in(0,\theta\sqrt{\tau})$, we have

$$
\begin{aligned}
|A| &\le \sum_{i\ge 1}|B_{5\tau/p}(x_i)| \le \frac{5^n\omega_n\tau}{p}\sum_{i\ge 1}\Big(\frac{\tau}{p}\Big)^{n-1}\\
&\le \frac{5^n\omega_n\tau}{p\theta}\sum_{i\ge 1}P(E,B_{\tau/p}(x_i)) \le \frac{5^n\omega_n\tau}{c_\Phi p\theta}P_\Phi(E).
\end{aligned}
$$

Now, using (1.5) and the equality $|E_\tau\Delta E_0| = |A|+|B|$, we get (2.17). ∎

2.1.3. Flat flows. In this section besides (H1)–(H2), we assume (H3). Some further conditions on f will be assumed later.

Notice that under assumption (H1) for any $\tau>0$ and $E_0\in\mathcal{S}$, we can define a flat flow $\{E(\tau,k)\}$ starting from E_0. By Theorem 2.2, $E(\tau,k)$ for $k\ge 1$ satisfies the uniform lower perimeter estimates, and thus, by (2.17), for any $p>0$ and $1\le m_1<m_2$,

$$
\begin{aligned}
|E(\tau,m_1)\Delta E(\tau,m_2)| &\le \sum_{k=m_1+1}^{m_2}|E(\tau,k)\Delta E(\tau,k+1)|\\
&\le \frac{C_4}{p}\sum_{k=m_1+1}^{m_2}\mathcal{C}_\beta(E(\tau,k-1))\tau\\
&\quad+\frac{p}{\tau}\sum_{k=m_1+1}^{m_2}\int_{E(\tau,k-1)\Delta E(\tau,k)}\mathrm{d}_{E(\tau,k-1)}dx \qquad (2.18)
\end{aligned}
$$

whenever $\tau<\theta^2p^2$. Further, we will estimate both sums separately.

By the minimality of $E(\tau,k)$ and (H3) for $k\ge 1$,

$$
\begin{aligned}
&\mathcal{C}_\beta(E(\tau,k))+\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}ds\int_{E(\tau,k)}f(s,x)dx+\frac{1}{\tau}\int_{E(\tau,k-1)\Delta E(\tau,k)}\mathrm{d}_{E(\tau,k-1)}dx\\
&\quad\le \mathcal{C}_\beta(E(\tau,k-1))+\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}ds\int_{E(\tau,k-1)}f(s,x)dx. \qquad (2.19)
\end{aligned}
$$

To estimate the differences of forcing terms, we need some extra regularity conditions on f. Further, we fix $T>0$ and let τ be so small that

$$T>\max\{10\tau,(k+1)\tau\}$$

and $\frac{1}{\tau}\int_0^{2\tau}ds\int_{E_0}|f|dx$ is uniformly bounded (by assumption (H3)).

Condition 1: Forcing is bounded. Assume (H4'). Then, applying (2.17) with

$$p=\frac{1}{2(1+c_T)},$$

we get

$$
\begin{aligned}
&\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds\Big[\int_{E(\tau,k-1)} f(s,x)dx - \int_{E(\tau,k)} f(s,x)dx\Big]\\
&\quad\le c_T|E(\tau,k-1)\Delta E(\tau,k)|\\
&\quad\le 2C_4c_T(1+c_T)\mathcal{C}_\beta(E(\tau,k-1))\tau + \frac{1}{2\tau}\int_{E(\tau,k-1)\Delta E(\tau,k)} \mathrm{d}_{E(\tau,k-1)}dx
\end{aligned}
$$

provided $\tau<\theta^2/(2(1+c_T))^2$. Inserting this estimate in (2.19), we obtain

$$
\mathcal{C}_\beta(E(\tau,k)) + \frac{1}{2\tau}\int_{E(\tau,k-1)\Delta E(\tau,k)} \mathrm{d}_{E(\tau,k-1)}dx \le (1+a_T\tau)\mathcal{C}_\beta(E(\tau,k-1)), \tag{2.20}
$$

where $a_T:=2C_4c_T(1+c_T)$. By induction,

$$
\mathcal{C}_\beta(E(\tau,k)) \le (1+a_T\tau)^{k-1}\mathcal{C}_\beta(E(\tau,1)),\quad k\ge 1.
$$

As $k<\lfloor T/\tau\rfloor$, using the elementary inequality

$$
(1+a_T\tau)^{\lfloor T/\tau\rfloor-1} = \big((1+a_T\tau)^{\frac{1}{a_T\tau}}\big)^{a_T\tau(\lfloor T/\tau\rfloor-1)} \le e^{a_TT},
$$

we deduce

$$
\mathcal{C}_\beta(E(\tau,k)) \le e^{a_TT}\mathcal{C}_\beta(E(\tau,1)),\quad k=1,\dots,\lfloor T/\tau\rfloor-1. \tag{2.21}
$$

Moreover, given $1<m_1<m_2<\lfloor T/\tau\rfloor$, summing (2.20) in $k=m_1+1,\dots,m_2$, we get

$$
\begin{aligned}
&\frac{1}{2\tau}\sum_{k=m_1+1}^{m_2}\int_{E(\tau,k-1)\Delta E(\tau,k)} \mathrm{d}_{E(\tau,k-1)}dx\\
&\quad\le \mathcal{C}_\beta(E(\tau,m_1)) - \mathcal{C}_\beta(E(\tau,m_2)) + a_T\sum_{k=m_1+1}^{m_2}\mathcal{C}_\beta(E(\tau,k-1))\tau\\
&\quad\le e^{a_TT}\mathcal{C}_\beta(E(\tau,1)) + a_Te^{a_TT}\mathcal{C}_\beta(E(\tau,1))(m_2-m_1)\tau,
\end{aligned} \tag{2.22}
$$

where in the last inequality we used (2.21).

Next, fix $0<s<t<T$, and let $\tau>0$ be so small that $s>10\tau$ and $t-s>10\tau$. Applying (2.18) with $m_1=\lfloor s/\tau\rfloor$, $m_2=\lfloor t/\tau\rfloor$, and $p=|t-s|^{1/2}$, and using (2.21) and (2.22), we get

$$
\begin{aligned}
&|E(\tau,\lfloor s/\tau\rfloor)\Delta E(\tau,\lfloor t/\tau\rfloor)|\\
&\quad\le \frac{C_4e^{a_TT}\mathcal{C}_\beta(E(\tau,1))(t-s+\tau)}{|t-s|^{1/2}}\\
&\qquad + 2\big(e^{a_TT}\mathcal{C}_\beta(E(\tau,1)) + a_Te^{a_TT}\mathcal{C}_\beta(E(\tau,1))(t-s+\tau)\big)|t-s|^{1/2},
\end{aligned}
$$

and therefore,

$$|E(\tau,\lfloor s/\tau\rfloor)\Delta E(\tau,\lfloor t/\tau\rfloor)| \le C_5\mathcal{C}_\beta(E(\tau,1))\Big(|t-s|^{1/2}+|t-s|^{3/2}+\frac{\tau(|t-s|+1)}{|t-s|^{1/2}}\Big), \tag{2.23}$$

where

$$C_5 := (C_4+2+2a_T T)e^{a_T T}.$$

It remains to estimate $\mathcal{C}_\beta(E(\tau,1))$ uniform in τ. Applying (2.19) with $k=1$, we get

$$\begin{aligned}\mathcal{C}_\beta(E(\tau,1)) &\le \mathcal{C}_\beta(E_0)+\frac{1}{\tau}\int_\tau^{2\tau}ds\int_{E_0}f(s,x)dx-\frac{1}{\tau}\int_\tau^{2\tau}ds\int_{E_\tau}f(s,x)dx\\ &\le \mathcal{C}_\beta(E_0)+\frac{2}{\tau}\int_0^{2\tau}ds\int_{\mathbb{R}^n}f(s,x)dx := c_\tau,\end{aligned}$$

where by assumption (H4') c_τ is uniformly bounded as $\tau\to 0^+$. Owing to this and (2.23), and repeating the standard arguments in the existence of GMM (see, e.g., [4]), we conclude that $\mathrm{GMM}(\mathcal{F}_{\beta,f},E_0)\ne\emptyset$ and each GMM is locally $1/2$-Hölder continuous in time.

Condition 2: Forcing is locally time-Lipschitz. Assume (H4") and set

$$\sigma_k := \mathcal{C}_\beta(E(\tau,k))+\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}ds\int_{E(\tau,k)}f(s,x)dx,\quad k\ge 0.$$

By (H4") for all $1\le k\le\lfloor T/\tau\rfloor-1$, we have

$$\begin{aligned}&\int_{k\tau}^{(k+1)\tau}ds\int_{E(\tau,k)}f(s,x)dx-\int_{(k-1)\tau}^{k\tau}ds\int_{E(\tau,k-1)}f(s,x)dx\\ &\quad\le\int_{k\tau}^{(k+1)\tau}ds\int_{E(\tau,k-1)}|f(s,x)-f(s+\tau,x)|dx\le c_T\tau^2.\end{aligned}$$

Therefore, by (2.19),

$$\sigma_k+\frac{1}{\tau}\int_{E(\tau,k-1)\Delta E(\tau,k)}\mathrm{d}_{E(\tau,k-1)}dx\le\sigma_{k-1}+c_T\tau,$$

and summing these inequalities, we get

$$\sigma_k+\frac{1}{\tau}\sum_{i=1}^k\int_{E(\tau,i-1)\Delta E(\tau,i)}\mathrm{d}_{E(\tau,i-1)}dx\le\sigma_0+c_Tk\tau.$$

Let us rewrite this inequality as

$$\begin{aligned}&\mathcal{C}_\beta(E(\tau,k))+\frac{1}{\tau}\sum_{i=1}^k\int_{E(\tau,i-1)\Delta E(\tau,i)}\mathrm{d}_{E(\tau,i-1)}dx\\ &\qquad\le\sigma_0+c_Tk\tau-\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau}ds\int_{E(\tau,k)}f(s,x)dx.\end{aligned}\tag{2.24}$$

By (H4"),

$$\left|\frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds \int_{E(\tau,k)} f(s,x)dx - \frac{1}{\tau}\int_0^{\tau} ds \int_{E(\tau,k)} f(s,x)dx\right|$$
$$\leq \sum_{i=1}^{k}\int_i^{i+1} ds \int_{\mathbb{R}^n} |f(s\tau,x) - f(s\tau+\tau,x)| \leq c_T k\tau.$$

Inserting this in (2.24), for any $k < \lfloor T/\tau \rfloor$, we get

$$\mathcal{C}_\beta(E(\tau,k)) + \frac{1}{\tau}\sum_{i=1}^{k}\int_{E(\tau,i-1)\Delta E(\tau,i)} \mathrm{d}_{E(\tau,i-1)}dx$$
$$\leq \mathcal{C}_\beta(E_0) + 2c_T T + \frac{2}{\tau}\int_0^{2\tau} ds \int_{\mathbb{R}^n} |f(s,x)|dx =: c'_\tau, \tag{2.25}$$

where c'_τ is uniformly bounded for small τ. Now, take any $0 < s < t < T$, and let τ be so small that $t - s > 10\tau$ and $s > 10\tau$. Applying (2.18) with $m_1 := \lfloor s/\tau \rfloor$, $m_2 := \lfloor t/\tau \rfloor$ and $p = |t-s|^{1/2}$, and employing (2.25), we obtain

$$|E(\tau,\lfloor s/\tau\rfloor)\Delta E(\tau,\lfloor t/\tau\rfloor)| \leq (C_4+1)c'_\tau\Big(|t-s|^{1/2} + \frac{\tau}{|t-s|^{1/2}}\Big).$$

This implies that $\mathrm{GMM}(\mathcal{F}_{\beta,f},E_0) \neq \emptyset$ for any $E_0 \in \mathcal{S}$ and each GMM is $1/2$-Hölder continuous in time.

2.2. Uniform boundedness of GMM

In this section, we obtain L^∞-bounds for GMM starting from a bounded set E_0, assuming the growth condition (H5).

Recall that in the literature (see, e.g., [1, 10, 33]) without boundary conditions the following comparison can be established: *if $F_0 \subset W^\Phi_{r_0}$ and F_τ of the standard Almgren–Taylor–Wang functional with forcing f satisfies $F_\tau \subset W^\Phi_{r_\tau}$, then the following hold:*

- *if $f \equiv 0$, then (by truncation with convex sets* [1, 29]*) $r_\tau = r_0$,*
- *if $f \neq 0$, then (by truncation with balls* [10]*)*

$$\frac{r_\tau - r_0}{\tau} \leq c + \frac{c}{r_\tau}$$

for some constant $c > 0$.

Below we establish similar comparison principle, but due to the boundary term, we cannot apply Wulff shapes. Rather, we use Winterbottom shapes [28, 30, 35]: given a constant $\beta_0 \in (-\Phi(\mathbf{e}_n), \Phi(\mathbf{e}_n))$, the part of the Wulff shape $W_{\beta_0,R} := \Omega \cap W^\Phi_R(\beta_0 R\mathbf{e}_n)$ centered at $\beta_0 R\mathbf{e}_n$ of radius R, the so-called *Winterbottom shape*, satisfies

$$\mathcal{C}_{\beta_0}(E) \geq c_{\Phi,\beta_0,n}|E|^{\frac{n-1}{n}}, \quad c_{\Phi,\beta_0,n} := \frac{\mathcal{C}_{\beta_0}(W_{\beta_0,R})}{|W_{\beta_0,R}|^{\frac{n-1}{n}}}, \tag{2.26}$$

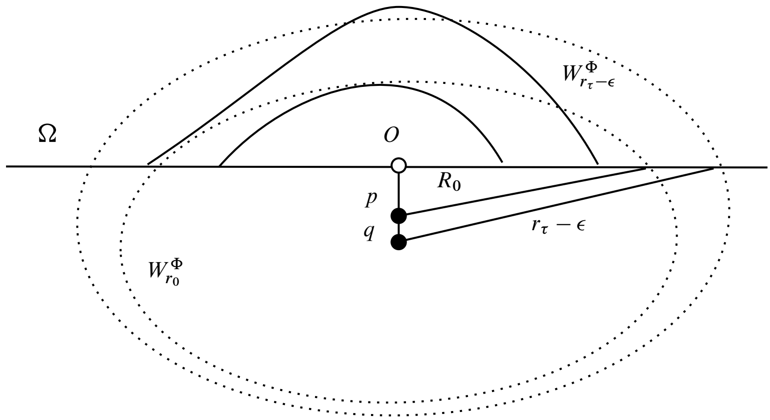

Figure 1. Comparison with Winterbottom shapes.

for all $E \in \mathcal{S}$. Note that the isoperimetric constant $c_{\Phi,\beta_0,n}$ is independent of R and horizontal translations of the Winterbottom shape. Recall that without forcing, in [4], we used a sort of "mean convex" sets (for capillary functional) to bound the minimizers of $\mathcal{F}_\beta$ uniformly.

Lemma 2.5. *Let $\tau > 0$, $k \in \mathbb{N}$, $F_0 \in \mathcal{S}$ be a bounded set, and F_τ be a minimizer of $\mathcal{F}_{\beta,f}(\cdot; F_0, \tau, k)$, which is bounded in view of (2.7). Let $-\Phi(\mathbf{e}_n) < \beta_0 < -(1-2\eta)\Phi(\mathbf{e}_n)$ be a constant, W_{β_0,r_0} contain F_0, and W_{β_0,r_τ} be the smallest Winterbottom shape containing F_τ. Then, either $r_\tau \le r_0$, or*

$$r_0 < r_\tau \le (1 + C_6\tau)r_0 + C_7\tau$$

for some constants C_6, C_7, depending only on Φ, β_0, and T, and for all $\tau < \frac{1}{4C_6}$.

Proof. Note that for each $r > 0$ there exists a unique Winterbottom shape $W_{\beta_0,r}$ whose center lies on the vertical line passing through the origin (see Figure 1), and $W_{\beta_0,r'} \subset W_{\beta_0,r''}$ whenever $r' < r''$. Therefore, if $r_\tau \le r_0$, we are done. Otherwise, fix $\varepsilon \in (0, r_\tau - r_0)$ and consider the Winterbottom shape $W_{\beta_0,r_\tau-\varepsilon}$. By the minimality of r_τ,

$$|F_\tau \setminus W_{\beta_0,r_\tau-\varepsilon}| \searrow 0 \quad \text{as } \varepsilon \to 0^+.$$

Let us estimate

$$\begin{aligned} 0 &\le \mathcal{F}_{\beta,f}(W_{\beta_0,r_\tau-\varepsilon} \cap F_\tau; F_0, \tau, k) - \mathcal{F}_{\beta,f}(F_\tau; F_0, \tau, k) \\ &= \mathcal{C}_\beta(W_{\beta_0,r_\tau-\varepsilon} \cap F_\tau) - \mathcal{C}_\beta(F_\tau) \\ &\quad - \frac{1}{\tau}\int_{F_\tau \setminus W_{\beta_0,r_\tau-\varepsilon}} \mathrm{sd}_{F_0}\, dx - \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds \int_{F_\tau \setminus W_{\beta_0,r_\tau-\varepsilon}} f dx \\ &=: I_1 - I_2 - I_3. \end{aligned}$$

Since

$$\mathrm{sd}_{E_0} = \mathrm{d}_{E_0} \geq c_\Phi \mathrm{d}_{E_0}^{\Phi^o} \geq c_\Phi \beta_0 (r_\tau - r_0 - \varepsilon) \quad \text{in } F_\tau \setminus W_{\beta_0, r_\tau - \varepsilon},$$

and, recalling $r_\tau > r_0$, by (H5) and (1.2),

$$\begin{aligned} |f(s,x)| &\leq a_T + b_T|x| \leq a_T + b_T(|x + \beta_0(r_\tau - \varepsilon)\mathbf{e}_n| + |\beta_0(r_\tau - \varepsilon)\mathbf{e}_n|) \\ &\leq a_T + \Big(b_T\beta_0 + \frac{b_T}{c_\Phi}\Big)(r_\tau - \varepsilon) \end{aligned}$$

for any $s \in [k\tau, (k+1)\tau]$ and $x \in F_\tau \setminus W_{\beta_0, r_\tau - \varepsilon}$. Therefore,

$$I_2 \geq \frac{c_\Phi \beta_0 (r_\tau - r_0 - \varepsilon)}{\tau} |F_\tau \setminus W_{\beta_0, r_\tau - \varepsilon}| \quad \text{and} \quad |I_3| \leq (a_T + c_T r_\tau)|F_\tau \setminus W_{\beta_0, r_\tau - \varepsilon}|.$$

Moreover, for a.e. ε using (2.26), we get

$$\begin{aligned} I_1 &= \mathcal{C}_{\beta_0}(W_{\beta_0, r_\tau - \varepsilon}) - \mathcal{C}_{\beta_0}(F_\tau \cup W_{\beta_0, r_\tau - \varepsilon}) + \int_{\partial\Omega} (\beta_0 - \beta)\chi_{F_\tau \cup W_{\beta_0, r_\tau - \varepsilon}} d\mathcal{H}^{n-1} \\ &\leq c_{\Phi, \beta_0, n}\Big(|W_{\beta_0, r_\tau - \varepsilon}|^{\frac{n-1}{n}} - |F_\tau \cup W_{\beta_0, r_\tau - \varepsilon}|^{\frac{n-1}{n}}\Big) \leq 0. \end{aligned}$$

Since $I_2 \leq I_1 + |I_3|$, from these estimates, we deduce

$$\frac{r_\tau - r_0 - \varepsilon}{\tau} \leq \frac{a_T + b_T r_\tau}{c_\Phi \beta_0}.$$

Thus, letting $\varepsilon \to 0^+$, we get

$$r_\tau \leq \frac{r_0}{1 - \frac{b_T}{c_\Phi \beta_0}\tau} + \frac{a_T \tau}{c_\Phi \beta_0 (1 - \frac{b_T}{c_\Phi \beta_0}\tau)}$$

provided $b_T \tau < c_\Phi \beta_0$. This implies the thesis with suitable C_6 and C_7 depending only on c_Φ, a_T, b_T, and β_0. ■

Now, consider any flat flow $\{E(\tau, k)\}$ starting from E_0, and let $W_{\beta_0, r(\tau,k)}$ be Winterbottom shapes, containing $E(\tau, k)$, such that $k \mapsto r(\tau, k)$ is nondecreasing and $r(\tau, 0) = r_0$. By Lemma 2.5 for each $k \geq 1$ we may assume either $r(\tau, k) = r(\tau, k-1)$ or

$$r(\tau, k-1) < r(\tau, k) < (1 + C_6\tau) r(\tau, k-1) + C_7\tau.$$

Then, applying an induction argument, we find

$$r(\tau, k) \leq (1 + C_6\tau)^k r_0 + C_7\tau \frac{(1 + C_6\tau)^k - 1}{(1 + C_6\tau) - 1} < \Big(r_0 + \frac{C_7}{C_6}\Big)(1 + C_6\tau)^k.$$

Thus, if $k \leq \lfloor T/\tau \rfloor$, then

$$(1 + C_6\tau)^k \leq \big((1 + C_6\tau)^{\frac{1}{C_6\tau}}\big)^{C_6\tau\lfloor T/\tau\rfloor} \leq e^{C_6 T}.$$

Therefore, for every $E(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta, f}, E_0)$ and $t \in [0, T)$, we get $E(t) \subset W_{\beta_0, e^{C_6 T}}$.

3. Some comparison principles

In this section, we establish some comparison principles as in [4, Section 6].

3.1. Discrete comparison and comparison of GMMs

We start this section with the following theorem.

Theorem 3.1 (Discrete comparison principle). *Let $\tau > 0$, $k \in \mathbb{N}$, β_i satisfy (1.4) and f_i satisfy (H1), and $E_0^i \in \mathcal{S}$, $i = 1, 2$.*

(a) *Assume that $\beta_1 \geq \beta_2$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $E_0^1 \subset E_0^2$ and $f_1 > f_2$ a.e. in $\mathbb{R}_0^+ \times \Omega$. Then, for any minimizer E_τ^i of $\mathcal{F}_{\beta_i, f_i}(\cdot; E_0^i, \tau, k)$,*

$$E_\tau^1 \subset E_\tau^2.$$

(b) *Assume that $\beta_1 \geq \beta_2$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $E_0^1 \prec E_0^2$ and $f_1 \geq f_2$ a.e. in $\mathbb{R}_0^+ \times \Omega$. Then, for any minimizer E_τ^i of $\mathcal{F}_{\beta_i, f_i}(\cdot; E_0^i, \tau, k)$,*

$$E_\tau^1 \subset E_\tau^2.$$

(c) *Assume that $\beta_1 \geq \beta_2$ $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$, $E_0^1 \subset E_0^2$ and $f_1 \geq f_2$ a.e. in $\mathbb{R}_0^+ \times \Omega$. Then, there exist minimizers $E_{\tau*}^1$ of $\mathcal{F}_{\beta_1, f_1}(\cdot; E_0^1, \tau, k)$ and $E_\tau^{(2)*}$ of $\mathcal{F}_{\beta_2, f_2}(\cdot; E_0^2, \tau, k)$ such that*

$$E_{\tau*}^1 \subset E_\tau^2 \quad \text{and} \quad E_\tau^1 \subset E_\tau^{(2)*}.$$

(d) *If $\beta_1 = \beta_2 =: \beta$, $f_1 = f_2 =: f$, and $E_0^1 = E_0^2 =: E_0$, then there exist minimizers $E_{\tau*}$ and E_τ^* of $\mathcal{F}_{\beta, f}(\cdot; E_0, \tau, k)$ such that every minimizer E_τ satisfies*

$$E_{\tau*} \subset E_\tau \subset E_\tau^*.$$

Setting

$$h_i := \frac{1}{\tau} \operatorname{sd}_{E_0^i} + \frac{1}{\tau} \int_{k\tau}^{(k+1)\tau} f_i(s, \cdot) ds, \quad i = 1, 2,$$

we observe that assumptions (a) and (b), resp., (c) imply $h_1 > h_2$, resp., $h_1 \geq h_2$. Since

$$\mathcal{F}_{\beta_i, f_i}(E; E_0^i, \tau, k) = \mathcal{C}_{\beta_i}(E) + \int_E h_i dx,$$

$\mathcal{F}_{\beta_i, f_i}$ is a sort of prescribed curvature functional, for which comparison principles are well established (see also [4, Section 6]). Therefore, we omit the proof.

We refer to $E_{\tau*}$ and E_τ^* as the minimal and maximal minimizers of $\mathcal{F}_{\beta, f}(\cdot; E_0, \tau, k)$.

Now, we are ready to establish the comparison principles between GMMs.

Proof of Theorem 1.4. (a) Take any $E^2(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta_2,f_2}, E_0^2)$ and let $\{E^2(\tau_i,k)\}$ be flat flows satisfying

$$\lim_{i\to+\infty} |E^2(\tau_i,\lfloor t/\tau_i\rfloor)\Delta E^2(t)| = 0 \quad \text{for any } t \ge 0; \tag{3.1}$$

here, $\tau_i \to 0^+$. For each i, let $\{E^1(\tau_i,k)_*\}$ be a flat flow starting from E_0^1, consisting of minimal minimizers. By the discrete comparison principle (Theorem 3.1 (c)),

$$E^1(\tau_i,k)_* \subset E^2(\tau_i,k) \quad \text{for any } k \ge 0. \tag{3.2}$$

Repeating the same arguments of the proof of Theorem 1.3, we can show that there exists a subsequence $\{\tau_{i_j}\}$ and a $E^1(\cdot)_* \in \mathrm{GMM}(\mathcal{F}_{\beta_1,f_1}, E_0^1)$ such that

$$\lim_{j\to+\infty} |E^1(\tau_{i_j},\lfloor t/\tau_{i_j}\rfloor)_*\Delta E^1(t)_*| = 0 \quad \text{for any } t \ge 0.$$

Then, (3.1) and (3.2) imply (1.8).

(b) is proven analogously using the maximal minimizers of $\mathcal{F}_{\beta_2,f_2}$. ■

3.2. Smooth inner and outer barriers

In this section, we assume that Φ is an elliptic C^3 anisotropy, $\beta \in C^1(\partial\Omega)$ satifies (1.4), and $f \in C^1(\mathbb{R}_0^+ \times \bar{\Omega})$.

Lemma 3.2. *Let $E_0 \in \mathcal{S}$ be a bounded set, for $\tau > 0$ and $k \in \mathbb{N}$ let E_τ be a (bounded) minimizer of $\mathcal{F}_{\beta,f}(\cdot; E_0,\tau,k)$, and let $\Gamma := \overline{\partial^\Omega E_\tau}$. Then,*

(a) *there exists a closed set $\Sigma \subset \Gamma$ with $\mathcal{H}^{n-3}(\Sigma) = 0$ such that $\Gamma \setminus \Sigma$ is a $C^{2+\alpha}$-hypersurface with boundary; if Φ is Euclidean, then $\mathcal{H}^{n-4}(\Sigma) = 0$;*

(b) *for every $x \in \Omega \cap (\Gamma \setminus \Sigma)$ from the first variation formula (2.3), it follows*

$$\frac{1}{\tau}\,\mathrm{sd}_{E_0}(x) + \kappa_{E_\tau}(x) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x)ds = 0;$$

(c) *at every $x \in \partial\Omega \cap (\Gamma \setminus \Sigma)$, the anisotropic contact angle condition holds:*

$$\nabla\Phi(\nu_{E_\tau}(x)) \cdot \mathbf{e}_n = -\beta.$$

Proof. The assertions (a) and (c) follow from [16, 17] while (b) follows from (a) and the regularity of f and the first variation formula (2.3). ■

The main result of this section is the following analog of [1, Lemma 7.3] (see also [36, Lemma 2.13]).

Proposition 3.3. *Assume either $n \le 3$ if Φ is any elliptic C^3-anisotropy or $n \le 4$ if Φ is Euclidean. Let $E_0 \in \mathcal{S}$ be a bounded set, and for $\tau > 0$, $k \in \mathbb{N}$, let E_τ be a minimizer of $\mathcal{F}_{\beta,f}(\cdot; E_0,\tau,k)$ and let G_0, G_τ be bounded sets with $C^{2+\alpha}$ free boundaries $\overline{\partial^\Omega G_0}$ and $\overline{\partial^\Omega G_\tau}$.*

(a) *Let* $E_0 \subset G_0$, $E_\tau \subset G_\tau$, G_τ *satisfy the anisotropic contact angle condition with* $\beta - s$ *for some* $s \in (0, \eta)$ *and*

$$\frac{\mathrm{sd}_{G_0}(x)}{\tau} + \kappa^{\Phi}_{G_\tau}(x) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x)ds > 0 \quad \text{on } \Omega \cap \partial G_\tau. \tag{3.3}$$

Then, $E_\tau \prec G_\tau$.

(b) *Let* $G_0 \subset E_0$, $G_\tau \subset E_\tau$, G_τ *satisfy the anisotropic contact angle condition with* $\beta + s$ *for some* $s \in (0, \eta)$ *and*

$$\frac{\mathrm{sd}_{G_0}(x)}{\tau} + \kappa^{\Phi}_{G_\tau}(x) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x)ds < 0 \quad \text{on } \Omega \cap \partial G_\tau.$$

Then, $G_\tau \prec E_\tau$.

Proof. (a) By the assumption on the dimension $\Omega \cap \partial^* E_\tau = \partial^\Omega E_\tau$. Thus, there exists $x_0 \in \Omega \cap \partial^\Omega E_\tau \cap \partial^\Omega G_\tau$; then, by assumption $E_0 \subset F_0$, we get $\mathrm{sd}_{E_0}(x_0) \geq \mathrm{sd}_{F_0}(x_0)$ and by assumption $E_\tau \subset F_\tau$, we get $\kappa^{\Phi}_{E_\tau}(x_0) \geq \kappa^{\Phi}_{F_\tau}(x_0)$. Therefore,

$$\begin{aligned} 0 &= \frac{\mathrm{sd}_{E_0}(x_0)}{\tau} + \kappa^{\Phi}_{E_\tau}(x_0) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x_0)ds \\ &\geq \frac{\mathrm{sd}_{F_0}(x_0)}{\tau} + \kappa^{\Phi}_{F_\tau}(x_0) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x_0)ds, \end{aligned}$$

which contradicts to (3.3). Hence, $\Omega \cap \partial E_\tau \cap \partial G_\tau = \emptyset$. Moreover, by Lemma 3.2 (a), $\partial^\Omega E_\tau$ satisfies the anisotropic contact angle condition with β at the boundary. Since $\partial^\Omega G_\tau$ satisfies this condition with $\beta - s$, we have also $\overline{\partial^\Omega E_\tau} \cap \overline{\partial^\Omega G_\tau} = \emptyset$. This implies $E_\tau \prec G_\tau$.

(b) is proven similarly. ∎

3.3. Comparison of flat flows with truncated Wulff shapes

In this section, we assume that f is bounded.

Theorem 3.4. *Let* $E_0 \in \mathcal{S}$, β *satisfy* (1.4) *and* $p \in \Omega$ *with* $R_0 := \mathrm{dist}(p, \partial^\Omega E_0) > 0$. *For* $\tau > 0$, *let* $\{E(\tau,k)\}$ *be flat flows starting from* E_0. *Then, for any* $\beta_0 \in (\|\beta\|_\infty, 1)$,

$$\Omega \cap W^{\Phi}_{R_0}(p) \subset E_0 \quad \Longrightarrow \quad \Omega \cap W^{\Phi}_{\frac{\beta_0 R_0}{16\Phi(\mathbf{e}_n)}}(p) \subset E(\tau,k), \tag{3.4}$$

$$W^{\Phi}_{R_0}(p) \cap E_0 = \emptyset \quad \Longrightarrow \quad W^{\Phi}_{\frac{\beta_0 R_0}{16\Phi(\mathbf{e}_n)}}(p) \cap E(\tau,k) = \emptyset \tag{3.5}$$

whenever $0 < \tau < \vartheta_0 R_0^2$ *and* $0 \leq k\tau \leq \frac{\vartheta_0 R_0^2}{R_0+1}$, *where* $\vartheta_0 \in (0,1)$ *is a constant depending only on* β_0 *and the constant* θ *of Theorem* 2.2.

Theorem 3.4 is a generalization of [1, Theorem 5.4] and [36, Theorem 2.11] in the anisotropic capillary setting. Notice that due to the presence of boundary terms in $\mathcal{F}_{\beta,f}$, we cannot argue as in the proof of [1, Theorem 5.4]. We postpone the proof after the following lemma.

Lemma 3.5. *Let E_0, F_0 be bounded sets of finite perimeter in Ω with $|\partial E_0| = |\partial F_0| = 0$, β_1, β_2 satisfy (1.4), f is bounded, and for $\tau > 0$ and $k \in \mathbb{N}$, let E_τ be a minimizer of $\mathcal{F}_{\beta_1,f}(\cdot; E_0, \tau, k)$.*

(a) *Let $F_0 \subseteq E_0$, $\beta_2 \geq \beta_1$, and $F_{\tau*}$ be the minimal minimizer of $\mathcal{F}_{\beta_2,f}(\cdot; F_0, \tau, k)$. Then, $F_{\tau*} \subseteq E_\tau$.*

(b) *Let $E_0 \cap F_0 = \emptyset$, $\beta_1 + \beta_2 \geq 0$, and $F_{\tau*}$ be the minimal minimizer of $\mathcal{F}_{\beta_2,-f}(\cdot; F_0, \tau, k)$. Then, $F_{\tau*} \cap E_\tau = \emptyset$.*

Proof. (a) follows from Theorem 3.1 (c). To prove (b), we sum the inequalities

$$\begin{aligned} \mathcal{F}_{\beta_1,f}(E_\tau; E_0, \tau, k) &\leq \mathcal{F}_{\beta_1,f}(E_\tau \setminus F_{\tau*}; E_0, \tau, k), \\ \mathcal{F}_{\beta_2,-f}(F_{\tau*}; F_0, \tau, k) &\leq \mathcal{F}_{\beta_2,-f}(F_{\tau*} \setminus E_\tau; F_0, \tau, k), \end{aligned}$$

and using

$$\begin{aligned} P(E_\tau, \Omega) + P(F_{\tau*}, \Omega) &\leq P(E_\tau \cap F_{\tau*}^c, \Omega) + P(F_{\tau*}^c \cup E_\tau, \Omega) \\ &= P(E_\tau \setminus F_{\tau*}, \Omega) + P(F_{\tau*} \setminus E_\tau, \Omega), \end{aligned}$$

we get

$$\frac{1}{\tau} \int_{F_{\tau*} \cap E_\tau} [\mathrm{sd}_{F_0} + \mathrm{sd}_{E_0}] dx + \int_{\partial\Omega} [\beta_1 + \beta_2] \chi_{F_{\tau*} \cap E_\tau} d\mathcal{H}^{n-1} \leq 0.$$

Since $E_0 \cap F_0 = \emptyset$ and $|\partial F_0| = |\partial E_0| = 0$, we have $\mathrm{sd}_{E_0} + \mathrm{sd}_{F_0} > 0$ a.e. in Ω. Therefore, recalling $\beta_1 + \beta_2 \geq 0$, we find that the last inequality holds if and only if

$$|E_\tau \cap F_{\tau*}| = 0.$$ ∎

Now, we are ready to prove the relations (3.4)–(3.5).

Proof of Theorem 3.4. We establish only (3.5), the proof of (3.4) being similar. We follow the arguments of [36, Theorem 2.11] and divide the proof into smaller steps. Fix $\beta_0 \in ((1 - 2\eta)\Phi(\mathbf{e}_n), \Phi(\mathbf{e}_n))$. Depending on the position of p, we distinguish three cases.

Case 1: $W_{R_0}^\Phi(p) \subset \Omega$. Before we proceed, we need some preliminaries. For shortness, set $W_r^\Phi := W_r^\Phi(p)$. Let $F_0 := \Omega \cap W_r^\Phi$ for some $r > 0$, and for $\tau > 0$ and $k \in \mathbb{N}$, let $F_{\tau*}$ be the minimal minimizer of $\mathcal{F}_{\beta_0,-f}(\cdot; F_0, \tau, k)$. Because of the forcing, in general, $F_{\tau*}$ is not necessarily a Wulff shape. By Theorem 2.2,

$$\sup_{F_{\tau*} \Delta F_0} \mathrm{d}_{F_0} \leq \frac{\sqrt{\tau}}{\theta},$$

thus, further assuming $0 < \tau < \frac{c_\Phi^2 \theta^2 r^2}{25}$ and using (1.2) and the definition of Φ^o, we get

$$\frac{1}{c_\Phi} \mathrm{dist}(\partial W_r^\Phi, \partial W_{4r/5}^\Phi) \geq \mathrm{dist}_{\Phi^o}(\partial W_r^\Phi, \partial W_{4r/5}^\Phi) = \frac{r}{5} > \frac{\sqrt{\tau}}{c_\Phi \theta},$$

and therefore, $W_{4r/5}^\Phi \subset F_{\tau*}$. Let W_ρ^Φ be the maximal Wulff shape such that $\Omega \cap W_\rho^\Phi \subset F_{\tau*}$. Clearly, $\rho \geq 4r/5$. We would like to estimate ρ from above.

Claim 1: either $\rho \geq r$ or

$$r > \rho \geq r - \left(C_8 + \frac{C_9}{r}\right)\tau \tag{3.6}$$

for some constants $C_8, C_9 > 0$ depending only on Φ, n, and $\|f\|_\infty$.

Indeed, assume that $\rho < r$ and fix any $\varepsilon \in (0, r-\rho)$. By the minimality of $F_{\tau*}$,

$$\begin{aligned}
0 &\leq \mathcal{F}_{\beta_0,-f}(F_{\tau*} \cup W^\Phi_{\rho+\varepsilon}; F_0, \tau, k) - \mathcal{F}_{\beta_0,-f}(F_{\tau*}; F_0, \tau, k) \\
&= P_\Phi(F_{\tau*} \cup W^\Phi_{\rho+\varepsilon}) - P_\Phi(F_{\tau*}) + \frac{1}{\tau}\int_{W^\Phi_{\rho+\varepsilon}\setminus F_{\tau*}} \mathrm{sd}_{F_0}\, dx \\
&\quad + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds \int_{W^\Phi_{\rho+\varepsilon}\setminus F_{\tau*}} f(s,x)dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Notice that by the maximality of ρ, $|W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}| \searrow 0$ as $\varepsilon \to 0^+$. Since f is bounded,

$$|I_3| \leq \|f\|_\infty |W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}|.$$

Moreover, by the assumption $\rho + \varepsilon < r$,

$$-\mathrm{sd}_{E_0} = \mathrm{d}_{E_0} \geq c_\Phi \mathrm{d}^{\Phi^o}_{E_0} \geq c_\Phi \operatorname{dist}_{\Phi^o}(\partial W^\Phi_r, \partial W^\Phi_{\rho+\varepsilon}) = c_\Phi(r-\rho-\varepsilon)$$

in $W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}$, and therefore,

$$-I_2 \geq \frac{c_\Phi(r-\rho-\varepsilon)}{\tau}|W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}|.$$

Finally, for a.e. $\varepsilon > 0$ with $\mathcal{H}^{n-1}(\partial W^\Phi_{\rho+\varepsilon} \cap \partial^* F_{\tau*}) = 0$ using the isoperimetric inequality (2.6), we get

$$\begin{aligned}
I_1 &= P_\Phi(W^\Phi_{\rho+\varepsilon}) - P_\Phi(W^\Phi_{\rho+\varepsilon} \cap F_{\tau*}) \\
&\leq c_{\Phi,n}\left(|W^\Phi_{\rho+\varepsilon}|^{\frac{n-1}{n}} - |W^\Phi_{\rho+\varepsilon} \cap F_{\tau*}|^{\frac{n-1}{n}}\right) \\
&= c_{\Phi,n}|W^\Phi_{\rho+\varepsilon}|^{\frac{n-1}{n}}\left(1 - \left|1 - \frac{|W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}|}{|W^\Phi_{\rho+\varepsilon}|}\right|^{\frac{n-1}{n}}\right) < \frac{c_{\Phi,n}|W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}|}{|W^\Phi_{\rho+\varepsilon}|^{1/n}} \\
&= \frac{c_{\Phi,n}}{|W^\Phi|^{1/n}(\rho+\varepsilon)}|W^\Phi_{\rho+\varepsilon} \setminus F_{\tau*}|,
\end{aligned}$$

where in the last inequality we used

$$(1-x)^\alpha > 1-x, \quad \alpha, x \in (0,1). \tag{3.7}$$

Now, using $-I_2 \leq I_1 + |I_3|$ and the above estimates for I_i, we get

$$\frac{c_\Phi(r-\rho-\varepsilon)}{\tau} \leq \frac{c_{\Phi,n}}{|W^\Phi|^{1/n}(\rho+\varepsilon)} + \|f\|_\infty.$$

Now, letting $\varepsilon \to 0^+$, and recalling $\rho \geq 4r/5$, we get (3.6).

Now, let $\{E(\tau,k)\}$ be any flat flow starting from E_0 and associated to $\mathcal{F}_{\beta,f}$, and let $\{F(\tau,k)_*\}$ be the flat flow starting from $F_0 := W^\Phi_{R_0}$ and associated to $\mathcal{F}_{\beta_0,-f}$, consisting of the minimal minimizers. By the choice of β_0, one has $\beta+\beta_0>0$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, and therefore, by Lemma 3.5 (b), $F(\tau,k)_*\cap E(\tau,k)=\emptyset$. Let

$$k\mapsto \rho(\tau,k)$$

be a nonincreasing sequence such that either $\rho(\tau,k)=\rho(\tau,k-1)$ or $W^\Phi_{\rho(\tau,k)}$ is the maximal Wulff shape contained in $F(\tau,k)_*$. By definition, $\rho(\tau,0)=R_0$. By Claim 1, for any $k\ge 1$, we may assume that

$$\rho(\tau,k-1)>\rho(\tau,k)\ge\rho(\tau,k-1)-\Big(C_8+\frac{C_9}{\rho(\tau,k-1)}\Big)\tau,\quad \tau\le\frac{c_\Phi^2\theta^2\rho(\tau,k-1)}{25}. \tag{3.8}$$

Let $k_0\ge 1$ be some element for which $\rho(\tau,k_0)\ge R_0/2$. By (3.8) for any $1\le k\le k_0$,

$$\rho(\tau,k)\ge\rho(\tau,0)-\sum_{i=0}^{k-1}\Big(C_8+\frac{C_9}{\rho(\tau,i)}\Big)\tau\ge R_0-\Big(C_8+\frac{2C_9}{R_0}\Big)k\tau.$$

Thus, if $\tau<\frac{c_\Phi^2\theta^2R_0^2}{100}$ and $k\tau\le\frac{R_0^2}{2C_8R_0+4C_9}$, then $\rho(\tau,k)\ge R_0/2$. In particular, by the definition of $F_{\tau*}$,

$$W^\Phi_{R_0/2}(p)\cap E(\tau,k)=\emptyset,\quad 0<\tau<\frac{c_\Phi^2\theta^2R_0^2}{100},\quad 0\le k\tau\le\frac{R_0^2}{2C_8R_0+4C_9}.$$

Step 2: $W^\Phi_{R_0}(p)\setminus\overline{\Omega}\ne\emptyset$, but $W^\Phi_{\lambda_0R_0}(p)\subset\Omega$, where $\lambda_0:=\frac{\beta_0}{8\Phi(\mathbf{e}_n)}$. By step 1 (applied with $R_0:=\lambda_0R_0$),

$$W^\Phi_{\lambda_0R_0/2}(p)\cap E(\tau,k)=\emptyset,\quad 0<\tau<\frac{c_\Phi^2\theta^2\lambda_0^2R_0^2}{100},\quad 0\le k\tau\le\frac{\lambda_0^2R_0^2}{2C_8\lambda_0R_0+4C_9}.$$

Step 3: $W^\Phi_{\lambda_0R_0}(p)\setminus\overline{\Omega}\ne\emptyset$, i.e., $p\cdot\mathbf{e}_3<\lambda_0R_0$. Fix any $\eta\in\partial\Phi(\mathbf{e}_n)$, i.e., any vector in $\mathbb{R}^n$ satisfying $\Phi^o(\eta)=1$ and $\eta\cdot\mathbf{e}_n=\Phi(\mathbf{e}_n)$. When Φ is smooth, $\eta=\nabla\Phi(\mathbf{e}_n)$ and is an outer normal to the Wulff shape W^Φ at $\mathbf{e}_n^{\Phi^o}$. For any $r>0$, let us define

$$W_r:=\Omega\cap W_r^\Phi\Big(\frac{\beta_0r\eta}{\Phi(\mathbf{e}_n)}\Big).$$

Since $\mathbf{e}_n\cdot\frac{\beta_0\eta}{\Phi(\mathbf{e}_n)}=\beta_0$, W_r is the horizontal translation of the Winterbottom shape $\Omega\cap W_r^\Phi(\beta_0r\mathbf{e}_n)$ with contact angle β_0, and thus, is itself a Winterbottom shape. For simplicity of the presentation, horizontally translating if necessary, we assume that $p=\lambda\eta$ for some $\lambda>0$. One can readily check that $W_r\subset W_{r'}$ if $r<r'$.

By the assumption of step 3, there exists a Winterbottom shape $W_r\subset W^\Phi_{R_0}(p)$. In the notation of Figure 2, let $W_{\rho_0}\subset W^\Phi_{R_0}(p)$ be the largest. Since W_{ρ_0} is the translation in the

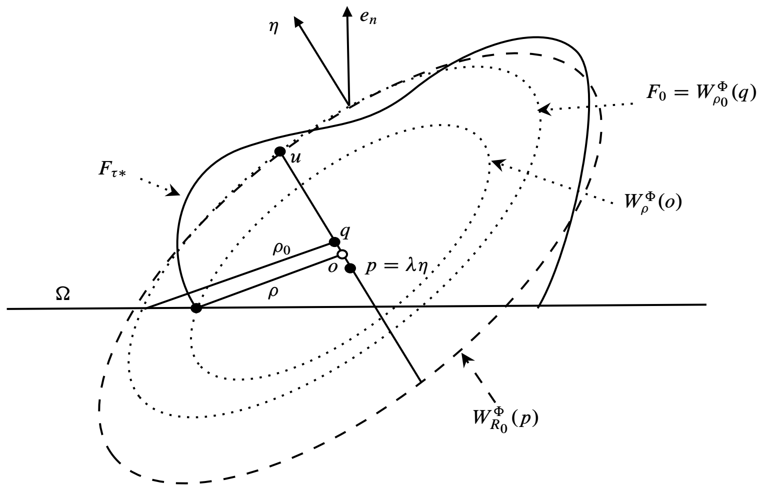

Figure 2. Winterbottom shapes contained in F_0 and $F_{\tau*}$.

η-direction of a Wulff shape $W^{\Phi}_{\rho_0}(p)$, we have

$$R_0 = \rho_0 + \Phi^o(q-p) \quad \Longrightarrow \quad \rho_0 = \frac{R_0 + \lambda}{1 + \frac{\beta_0}{\Phi(\mathbf{e}_n)}}. \tag{3.9}$$

Let $F_0 := W_{\rho_0}$ and $F_{\tau*}$ be the minimal minimizer of $\mathcal{F}_{\beta_0,-f}(\cdot; F_0, \tau, k)$. As in step 1 assuming $0 < \tau < \frac{\theta^2\rho_0^2}{25C_\Phi^2}$ and using (2.7), we can show that

$$\Omega \cap W^{\Phi}_{4\rho_0/5}(q) \subset F_{\tau*}. \tag{3.10}$$

Next, let W_ρ and $W_{\overline{\rho}}$ be the largest Winterbottom shapes contained in $F_{\tau*}$ and in $W^{\Phi}_{4\rho_0/5}(q)$, respectively. By (3.10) and (3.9) (applied with $R_0 := 4\rho_0/5$),

$$\rho \ge \overline{\rho} = \frac{4\rho_0/5 + \lambda}{1 + \frac{\beta_0}{\Phi(\mathbf{e}_n)}} \ge \frac{4\rho_0/5}{1 + \frac{\beta_0}{\Phi(\mathbf{e}_n)}}. \tag{3.11}$$

Claim 2: either $\rho \ge \rho_0$ or

$$\rho_0 > \rho \ge \rho_0 - \Big(C_{10} + \frac{C_{11}}{\rho_0}\Big)\tau, \tag{3.12}$$

where $C_{10}, C_{11} > 0$ are some constants depending only on Φ, β_0, n, and $\|f\|_\infty$.

Indeed, assume that $\rho < \rho_0$ and fix $\varepsilon \in (0, \rho_0 - \rho)$. Consider the Winterbottom shape $W_{\rho+\varepsilon}$. The maximality of ρ implies $|W_{\rho+\varepsilon} \setminus F_{\tau*}| \searrow 0$ as $\varepsilon \to 0^+$. As in step 1, by the minimality of $F_{\tau *}$,

$$
\begin{aligned}
0 &\le \mathcal{F}_{\beta_0,f}(F_{\tau *}\cup W_{\rho+\varepsilon}; F_0,\tau,k) - \mathcal{F}_{\beta_0,f}(F_{\tau *}; F_0,\tau,k)\\
&= \mathcal{C}_{\beta_0}(F_{\tau *}\cup W_{\rho+\varepsilon}) - \mathcal{C}_{\beta_0}(F_{\tau *}) - \frac{1}{\tau}\int_{W_{\rho+\varepsilon}\setminus F_{\tau *}} \mathrm{d}_{F_0}dx\\
&\quad + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} ds\int_{W_{\rho+\varepsilon}\setminus F_{\tau *}} f(s,x)dx\\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

By the boundedness of f,

$$
|I_3| \le \|f\|_\infty |W_{\rho+\varepsilon}\setminus F_{\tau *}|.
$$

Moreover, since the centers of the Winterbottom shapes W_{ρ_0} and W_ρ lie on the same line,

$$
-I_2 = \mathrm{d}_{F_0} \ge c_\Phi \mathrm{d}_{F_0}^{\Phi^o} \ge c_\Phi(\rho_0-\rho-\varepsilon-\Phi^o(q-q_\varepsilon)) \quad \text{in } W_{\rho+\varepsilon}\setminus F_{\tau *},
$$

where $q_\varepsilon\in\Omega$ and q are the centers of $W_{\rho+\varepsilon}$ and W_{ρ_0}. Since $q_\varepsilon = \frac{\beta_0(\rho+\varepsilon)\eta}{\Phi(\mathbf{e}_n)}$ and $q=\frac{\beta_0\rho_0\eta}{\Phi(\mathbf{e}_n)}$,

$$
\frac{1}{\tau}\int_{W_{\rho+\varepsilon}\setminus F_{\tau *}} \mathrm{d}_{F_0}dx \ge \frac{c_\Phi(\rho_0-\rho-\varepsilon)}{\tau}\Big(1-\frac{\beta_0}{\Phi(\mathbf{e}_n)}\Big)|W_{\rho+\varepsilon}\setminus F_{\tau *}|.
$$

Finally, for a.e. ε with $\mathcal{H}^{n-1}(\partial^\Omega W_{\rho+\varepsilon}\cap \partial^\Omega F_{\tau *})=0$ using (2.26) and (3.7), we get

$$
\begin{aligned}
I_1 &= \mathcal{C}_{\beta_0}(W_{\rho+\varepsilon}) - \mathcal{C}_{\beta_0}(W_{\rho+\varepsilon}\cap F_{\tau *}) \le c_{\Phi,\beta_0,n}\Big(|W_{\rho+\varepsilon}|^{\frac{n-1}{n}} - |W_{\rho+\varepsilon}\cap F_{\tau *}|^{\frac{n-1}{n}}\Big)\\
&\le c_{\Phi,\beta_0,n}|W_{\rho+\varepsilon}|^{\frac{n-1}{n}}\Bigg(1-\Bigg|1-\frac{|W_{\rho+\varepsilon}\setminus F_{\tau *}|}{|W_{\rho+\varepsilon}|}\Bigg|^{\frac{n-1}{n}}\Bigg) \le \frac{c_{\Phi,\beta_0,n}}{|W_1|^{1/n}(\rho+\varepsilon)}|W_{\rho+\varepsilon}\setminus F_{\tau *}|.
\end{aligned}
$$

Using $-I_2\le I_1+|I_3|$ and letting $\varepsilon\to 0$, we get

$$
\frac{c_\Phi(\rho_0-\rho)}{\tau}\Big(1-\frac{\beta_0}{\Phi(\mathbf{e}_n)}\Big) \le \frac{c_{\Phi,\beta_0,n}}{|W_1|^{1/n}\rho} + \|f\|_\infty.
$$

Now, in view of (3.11), we deduce (3.12) for suitable $C_{10}, C_{11}>0$ depending only on Φ, n, β_0, and $\|f\|_\infty$.

Now, take any flat flows $\{E(\tau,k)\}$ starting from E_0, and given ρ_0 in (3.9), let the numbers $\rho_0=\rho(\tau,0)\ge\rho(\tau,1)\ge\cdots$ be defined as follows. For each $k\ge 1$, if $\rho(\tau,k)<\rho(\tau,k-1)$, then $W_{\rho(\tau,k)}$ is the maximal Winterbottom shape staying inside the minimal minimizer of $\mathcal{F}_{\beta_0,-f}(\cdot; W_{\rho(\tau,k-1)},\tau,k)$. By the choice of R_0 and the definition of ρ_0,

$$
E(\tau,0)\cap W_{\rho(\tau,0)} = \emptyset.
$$

Therefore, applying Lemma 3.5 (b) inductively, we deduce

$$
E(\tau,k)\cap W_{\rho(\tau,k)} = \emptyset, \quad k=0,1,2,\dots. \tag{3.13}
$$

As in step 1, let $k_0 \geq 1$ be such that $\rho(\tau, k_0) \geq \rho_0/2$ and assume that $\tau < \frac{\theta^2 \rho_0^2}{100 C_\Phi^2}$. Then, $\tau < \frac{\theta^2 \rho(\tau,k-1)^2}{25 C_\Phi^2}$ for any $1 \leq k \leq k_0$ and hence, by Claim 2,

$$\rho(\tau, k) \geq \rho(\tau, k-1) - \left(C_{10} + \frac{C_{11}}{\rho(\tau, k-1)}\right)\tau, \quad k = 1, \dots, k_0.$$

From this inequality, we deduce

$$\rho(\tau, k) \geq \rho_0 - \left(C_{10} + \frac{2C_{11}}{\rho_0}\right)k\tau.$$

Thus, if we choose $0 \leq k\tau \leq \frac{\rho_0^2}{2C_{10}\rho_0 + 4C_{11}}$, then $\rho(\tau, k) \geq \rho_0/2$. Notice that by (3.13) for such τ and k, we have $E(\tau, k) \cap W_{\rho_0/2} = \emptyset$.

Let us show $W_{\lambda_0 R_0}(p) \subset W_{\rho_0/2}$. Since $p = \lambda\eta$ and the Wulff shape $W_{\rho_0/2}$ is centered at $\frac{\beta_0\rho_0}{2\Phi(\mathbf{e}_n)}\eta$, it suffices to show

$$\lambda + \lambda_0 R_0 \leq \frac{\beta_0 \rho_0}{2\Phi(\mathbf{e}_n)}.$$

By assumption of step 3 and the choice of p, the origin lies in $W^{\Phi}_{\lambda_0 R_0}(p)$, and therefore,

$$\lambda = \Phi^o(p - 0) \leq \lambda_0 R_0,$$

and hence, by the choice of ρ_0 and assumption $\beta_0 < \Phi(\mathbf{e}_n)$, we obtain

$$\frac{\beta_0 \rho_0}{2\Phi(\mathbf{e}_n)} > \frac{\beta_0 R_0}{4\Phi(\mathbf{e}_n)} = 2\lambda_0 R_0 \geq \lambda + \lambda_0 R_0.$$

Thus, $W^{\Phi}_{\lambda_0 R_0}(p) \subset W_{\rho_0/2}$.

Theorem is proved. ∎

Notice that when the forcing f is zero, then the coefficients C_9 and C_{11} in claim 1 and 2 can be taken 0.

4. Smooth anisotropic curvature flow of hypersurfaces with boundary

Throughout this section, $\alpha \in (0, 1]$ stands for a constant representing the Hölderinanity exponent, Φ is an elliptic at least $C^{3+\alpha}$-anisotropy in $\mathbb{R}^n$, $\beta \in C^{1+\alpha}(\partial\Omega)$ satisfying (1.4), and $f \in C^{1+\frac{\alpha}{2}, 1+\alpha}(\mathbb{R}_0^+ \times \overline{\Omega})$.

In this section, we prove that the evolution equation (1.1) is well posed and is solvable even in a more general setting.

Definition 4.1 (Φ-curvature flow of hypersurfaces). A family $\{\Gamma(t)\}_{t\in[0,T)}$ (for some $T > 0$) of smooth hypersurfaces in Ω with boundary is called a (smooth) Φ-*curvature flow,* *starting from a smooth hypersurface $\Gamma_0 \subset \Omega$, with forcing f and anisotropic contact angle β* provided that

$$\begin{cases} v_{\Gamma(t)} = -\kappa^{\Phi}_{\Gamma(t)} - f(t,\cdot) & \text{on } \Omega \cap \Gamma(t),\, t \in [0,T), \\ \partial\Gamma(t) \subset \partial\Omega & t \in [0,T), \\ \nabla\Phi(\nu_{\Gamma(t)}) \cdot \mathbf{e}_n = -\beta & \text{on } \partial\Gamma(t),\, t \in [0,T), \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{4.1}$$

4.1. Short-time existence

In this section, following the ideas of [4, 27], we prove the following short-time existence of the Φ-curvature flow.

Theorem 4.2. *Let $\Gamma_0 \subset \Omega$ be a bounded $C^{2+\alpha}$-hypersurface with boundary on $\partial\Omega$ oriented by a unit normal field ν_{Γ_0} and satisfying the anisotropic contact angle condition*

$$\nabla\Phi(\nu_{\Gamma_0}) \cdot \mathbf{e}_n = -\beta \quad \text{on } \partial\Gamma_0.$$

Let

$$H := \min_{x\in\Gamma_0,\, \nu_{\Gamma_0}(x)=\mathbf{e}_n} x \cdot \mathbf{e}_n.$$

Then, there exist $T > 0$, depending only on Φ, H, β, $\|II_{\Gamma_0}\|_\infty$, and a $C^{1+\frac{\alpha}{2}}$-in time family $\{\Gamma(t)\}_{t\in[0,T]}$ of $C^{2+\alpha}$-hypersurfaces in Ω which satisfies (4.1).

To prove this theorem, we first translate geometric PDE (4.1) into a nonlinear parabolic system using parametrizations. For the convenience of the reader, we divide the proof of the theorem into smaller steps.

4.1.1. Hölder spaces. For $T > 0$, an open set $\mathcal{U} \subset \mathbb{R}^{n-1}$, and a noninteger real number $\gamma > 0$, let $C_T^{\gamma/2,\gamma}$ be the Banach space $C^{\gamma/2,\gamma}([0,T] \times \overline{\mathcal{U}})$ of the Hölder continuous functions for which

$$\|w\|_{\gamma,T} := \sum_{0\le i\le[\gamma/2]} \left\| \frac{\partial^i w}{\partial t^i} \right\|_\infty + \sum_{0\le|\mu|\le[\gamma]} \left\| \frac{\partial^\mu w}{\partial x^\mu} \right\|_\infty + \left[\frac{\partial^{[\gamma/2]} w}{\partial t^{[\gamma/2]}} \right]_{t,\gamma/2-[\gamma/2]} + \sum_{|\mu|=[\gamma]} \left[\frac{\partial^{[\gamma]} w}{\partial x^{[\gamma]}} \right]_{x,\gamma-[\gamma]}$$

is finite. Here, $[x]$ is the integer part of $x \in \mathbb{R}$, $\mu = (\mu_1, \dots, \mu_{n-1}) \in \mathbb{N}_0^{n-1}$ is a multi-index and $|\mu| := \mu_1 + \cdots + \mu_{n-1}$,

$$\frac{\partial^\mu}{\partial x^\mu} = \frac{\partial^{|\mu|}}{\partial^{\mu_1} x_1 \cdots \partial^{\mu_{n-1}} x_{n-1}},$$

for a continuous function $f \in C^0([0,T] \times \overline{\mathcal{U}})$

$$\|f\|_\infty = \max_{[0,T]\times\overline{\mathcal{U}}} |f|,$$

and for $\theta \in (0,1]$ and $f \in C^0([0,T] \times \overline{\mathcal{U}})$,

$$[f]_{t,\theta} = \sup_{(t,x),(s,x)\in[0,T]\times\bar{U},s\neq t} \frac{|f(t,x) - f(s,x)|}{|t-s|^\theta},$$

and

$$[f]_{x,\theta} = \sup_{(t,x),(t,y)\in[0,T]\times\bar{U},x\neq y} \frac{|f(t,x) - f(t,y)|}{|x-y|^\theta}.$$

We consider Hölder spaces $C_T^{\gamma/2,\gamma}$ only for $\gamma = \alpha$ and $\gamma = 2+\alpha$ for some $\alpha \in (0,1]$. By $[C_T^{\gamma/2,\gamma}]^m$ we denote the Banach space of vectors $f = (f_1, \dots, f_m)$, where each $f_i \in C_T^{\gamma/2,\gamma}$, and the norm of f is given by

$$\|f\|_{\gamma,T} = \sum_{i=1}^m \|f_i\|_{\gamma,T}.$$

4.1.2. Introducing the parametrization. For simplicity, we assume that $\Gamma(t)$ are parametrized by a single chart $p : [0,T] \times \mathcal{U} \to \Omega$, where $\mathcal{U}$ is a bounded $C^{2+\alpha}$-open set in $\mathbb{R}^{n-1}$. In this case, we write

$$p = \begin{bmatrix} p^1 \\ \cdots \\ p^n \end{bmatrix} \quad \text{and} \quad p_x = \begin{bmatrix} p_x^1 \\ \cdots \\ p_x^n \end{bmatrix} = \begin{bmatrix} p_{x_1}^1 & \cdots & p_{x_{n-1}}^1 \\ & \cdots & \\ p_{x_1}^n & \cdots & p_{x_{n-1}}^n, \end{bmatrix}$$

and recall that $\{p_{x_i}\}_{i=1}^{n-1}$ is the set of basis vectors of the tangent hyperplane of $\Gamma(t)$,

$$\nu_\Gamma(p_x) := \frac{N(p_x)}{|N(p_x)|}, \quad N(p_x) = p_{x_1} \times \cdots \times p_{x_{n-1}} = \det \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ p_{x_1}^1 & \cdots & p_{x_1}^n \\ & \cdots & \\ p_{x_{n-1}}^1 & \cdots & p_{x_{n-1}}^n \end{bmatrix}, \tag{4.2}$$

is its "outward" unit normal field, where $\times$ is the vector product of two vectors,

$$g_{ij} = p_{x_i} \cdot p_{x_j}, \quad p_{x_i} = \frac{\partial p}{\partial x_i}, \quad i,j = 1,\dots,n-1,$$

are entries of the first fundamental form of $\Gamma(t)$, $\{g^{ij}\}$ is its inverse, and

$$h_{ij} = -\nu_{\Gamma(t)} \cdot p_{x_i x_j}, \quad p_{x_i x_j} = \frac{\partial^2 p}{\partial x_i \partial x_j}, \quad i,j = 1,\dots,n-1,$$

are the entries of the second fundamental form of $\Gamma(t)$. Under these notations, the Φ-curvature of $\Gamma(t) = p(\{t\} \times \mathcal{U})$ is represented as (see, e.g., [14])

$$\kappa_{\Gamma(t)}^\Phi = \sum_{i,j=1}^{n-1} g^{ij} [h_{\Gamma(t)}]_{ij},$$

where

$$[h_{\Gamma(t)}]_{ij} := \left(\nabla^2\Phi^o(\nu_{\Gamma(t)})\frac{\partial\nu_{\Gamma(t)}}{\partial x_i}\right)\cdot p_{x_j}, \quad i,j=1,\dots,n-1,$$

is the entries of the anisotropic version of the second fundamental form. Here, $\frac{\partial\nu_{\Gamma(t)}}{\partial x_i}$ is understood as a covariant derivative of $\nu_{\Gamma(t)}$ in $\mathbb{R}^n$ and can be defined as

$$\frac{\partial\nu_{\Gamma(t)}}{\partial x_i} = \sum_{k,l=1}^{n-1} h_{ik}g^{kl}p_{x_l} = -\sum_{k,l=1}^{n-1}\left(\nu_{\Gamma(t)}\cdot p_{x_ix_k}\right)g^{kl}p_{x_l}.$$

Then, the normal velocity of $\Gamma(t)$ is defined as

$$v_{\Gamma(t)} = -p_t\cdot\nu_{\Gamma(t)}$$

and the Φ-curvature is defined as

$$\kappa^\Phi_{\Gamma(t)} = \sum_{i,j,k,l=1}^{n-1} g^{ij}g^{kl}\left([\nabla^2\Phi(\nu_{\Gamma(t)})p_{x_l}]\cdot p_{x_j}\right)p_{x_ix_k}\cdot\nu_{\Gamma(t)}.$$

This suggests choosing the tangential velocity such that the equation $v_\Gamma = -\kappa^\Phi_\Gamma - f$ is represented by means of the parametrization p as

$$p_t = \sum_{i,j,k,l=1}^{n-1} g^{ij}g^{kl}\left([\nabla^2\Phi(\nu_{\Gamma(t)})p_{x_l}]\cdot p_{x_j}\right)p_{x_ix_k} + f\nu_{\Gamma(t)} \quad \text{on } \partial\Gamma(t).$$

The boundary condition $\partial\Gamma(t)\subset\partial\Omega$ is equivalent to $p\cdot\mathbf{e}_n=0$ on $\partial\mathfrak{U}$ and since $\nabla\Phi$ is positively 0-homogeneous, the anisotropic contact angle condition $\nabla\Phi(\nu_{\Gamma(t)})\cdot\mathbf{e}_n=-\beta$ on $\partial\Gamma(t)$ together with (4.2) becomes as $\nabla\Phi(N(p_x))\cdot\mathbf{e}_n=-\beta(p)$ on $\partial\mathfrak{U}$.

As in [4,27], to keep the presentation simpler, we assume that Γ_0 admits a parametrization $p^0:\mathfrak{U}\to\Omega$ with the property

$$\begin{cases} p^0(x)\cdot\mathbf{e}_n = 0,\\ \nabla\Phi(N(p^0_x(x)))\cdot\mathbf{e}_n = -\beta(p^0(x)),\\ \nabla[\nabla_n\Phi(N(p^0_x(x)))] = \mu_0(x)\sum_{i=1}^{n-1}\mathtt{n}^0_i(x)p^0_{x_i}(x)\end{cases} \tag{4.3}$$

for $x=(x_1,\dots,x_{n-1})\in\partial\mathfrak{U}$, where $\nabla_n\Phi=\nabla\Phi\cdot\mathbf{e}_n$, $\mathtt{n}^0:=(\mathtt{n}^0_1,\dots,\mathtt{n}^0_{n-1})$ is the outer unit normal to $\partial\mathfrak{U}$, and $\mu_0(x)$ is a scaling factor. The first condition in (4.3) maintains that $\partial\Gamma_0\subset\partial\Omega$, while the second one is the anisotropic contact angle condition. These two conditions are related solely to the geometry of Γ_0. The third condition in (4.3) is possible since $\nabla^2\Phi(N)N=\nabla[\nabla\Phi(N)]\cdot N=0$, which in particular implies

$$\nabla[\nabla_n\Phi(N(p^0_x(x)))]\cdot N = 0.$$

In view of the ellipticity, $\nabla^2_{nn}\Phi(N)>0$, and hence, $\mu_0>0$.

To make the problem well posed, we still need to impose $n-2$ conditions on the boundary which should determine the boundary tangential velocity of $\Gamma(t)$. Let $\tau_1^0, \dots, \tau_{n-2}^0$ be the basis of the tangent plane of $\partial\Omega \cap \partial\Gamma_0$. We assume that

$$\sum_{i=1}^{n-1} \mathrm{n}_i^0 p_{x_i} \cdot \tau_j^0(p^0) = \sum_{i=1}^{n-1} \mathrm{n}_i^0 p_{x_i}^0 \cdot \tau_j^0(p^0), \quad j = 1, \dots, n-2,$$

for $x = (x_1, \dots, x_{n-1}) \in \partial\mathcal{U}$.

Now, (4.1) is represented as

$$\begin{cases} p_t = \displaystyle\sum_{i,j,k,l=1}^{n-1} g^{ij}(p_x) g^{kl}(p_x) \big([\nabla^2\Phi(\nu(p_x)) p_{x_l}] \cdot p_{x_j}\big) p_{x_i x_k} & \\ \qquad + f(t,p)\nu(p_x) & \text{in } [0,T] \times \mathcal{U}, \\ p \cdot \mathbf{e}_n = 0 & \text{on } [0,T] \times \partial\mathcal{U}, \\ \nabla\Phi(N(p_x)) \cdot \mathbf{e}_n = -\beta(p) & \text{on } [0,T] \times \partial\mathcal{U}, \\ \displaystyle\sum_{j=1}^{n-1} \mathrm{n}_j^0 p_{x_j} \cdot \tau_i^0(p^0) = \sum_{j=1}^{n-1} \mathrm{n}_j^0 p_{x_j}^0 \cdot \tau_i^0(p^0) & \text{in } [0,T] \times \partial\mathcal{U},\ i = 1, \dots, n-2, \\ p(0,\cdot) = p^0, & \end{cases} \tag{4.4}$$

where $\nu := \nu_\Gamma$ is given as in (4.2).

Now, we linearize this system around the initial and boundary conditions, solve the linearized problem using Solonnikov theory [37], and then apply a fixed-point theory in Hölder spaces to show the solvability of (4.4).

4.1.3. Linearization. Using (4.3), we rewrite (4.4) as

$$[\mathcal{A}w, \mathcal{P}w, \mathcal{C}w, \mathcal{T}w, \mathcal{I}w] = [\bar{f}, 0, \bar{b}, 0, p^0] + \big[F(w,p^0), 0, B(w,p^0), 0, 0\big], \tag{4.5}$$

where $w \in [C_T^{1+\frac{\alpha}{2}, 2+\alpha}]^n$ for some $T > 0$, which will be chosen later,

$$\begin{aligned} \mathcal{A}w &:= w_t - \sum_{i,j,k,l=1}^{n-1} g^{ij}(p_x^0) g^{kl}(p_x^0) \big([\nabla^2\Phi(\nu(p_x^0)) p_{x_l}^0] \cdot p_{x_j}^0\big) w_{x_i x_k}, \\ \mathcal{P}w &= w \cdot \mathbf{e}_n, \\ \mathcal{C}w &= \nabla^2\Phi(N(p_x^0)) \nabla N(p_x^0)[w_x] \cdot \mathbf{e}_n + \nabla\beta(p^0) \cdot w, \\ \mathcal{T}w &= \sum_{j=1}^{n-1} \mathrm{n}_j^0 w_{x_j} \cdot \tau_i^0(p^0), \\ \mathcal{I}w &= w(0,\cdot) \end{aligned}$$

are homogeneous linear operators, a linearized part of the system (4.4), where

$$\nabla N(p_x^0)[w_x] = \left(\sum_{i=1}^{n}\sum_{j=1}^{n-1} \nabla_{p_{x_j}^i} N^1(p_x^0) w_{x_j}^i, \dots, \sum_{i=1}^{n}\sum_{j=1}^{n-1} \nabla_{p_{x_j}^i} N^n(p_x^0) w_{x_j}^i \right)^T,$$

the vector functions

$$\begin{aligned} \bar f &= f(\cdot, p^0)\nu(p_x^0), \\ \bar b &= \nabla^2\Phi(N(p_x^0))\nabla N(p_x^0)[p_x^0]\cdot \mathbf{e}_n + \nabla\beta(p^0)\cdot p^0 \end{aligned}$$

are the main parts of the right-hand side (independent of w) after linearization, and

$$\begin{aligned} F(w,p^0) = &\sum_{i,j,k,l=1}^{n-1} g^{ij}(w_x) g^{kl}(w_x)\big([\nabla^2\Phi(\nu(w_x))w_{x_l}]\cdot w_{x_j}\big) w_{x_i x_k} + f(t,w)\nu(w_x) \\ &- \sum_{i,j,k,l=1}^{n-1} g^{ij}(p_x^0) g^{kl}(p_x^0)\big([\nabla^2\Phi(\nu(p_x^0))p_{x_l}^0]\cdot p_{x_j}^0\big) w_{x_i x_k} - f(t,p^0)\nu(p_x^0), \\ B(w,p^0) = &-\big[\nabla\Phi(N(w_x)) - \nabla\Phi(N(p_x^0)) - \nabla^2\Phi(N(p_x^0))\nabla N(p_x^0)[w_x - p_x^0]\big]\cdot \mathbf{e}_n \\ &- \big[\beta(w) - \beta(p_x^0) - \nabla\beta(p_x^0)[w - p^0]\big]|N(p_x^0)| \end{aligned}$$

are nonlinear parts. Notice that $F(p^0,p^0) = 0$ and $B(p^0,p^0) = 0$.

4.1.4. Parabolicity of the linearized system. Let us show that the linear operator $\mathcal{A}$ in the system (4.5) is parabolic in the sense of Solonnikov [37, p. 9], the linear operators $[\mathcal{P},\mathcal{C},\mathcal{T}]$ satisfy the complementary conditions at the boundary $[0,T]\times\partial\mathcal{U}$ and at the initial time $t=0$ [37, pp. 11–12], and the boundary conditions and initial datum in the right-hand side of (4.5) are compatible of order 0 [37, p. 87].

Parabolicity of $\mathcal{A}$. For $(t,x)\in[0,T]\times\overline{\mathcal{U}}$, $z\in\mathbb{C}$, and $\xi\in\mathbb{C}^{n-1}$, let $\mathcal{A}(t,x,z,\xi)$ be the $n\times n$-diagonal matrix whose all diagonal entries are equal to

$$z - \sum_{i,j,k,l=1}^{n-1} g^{ij}(p_x^0) g^{kl}(p_x^0)\big([\nabla^2\Phi(\nu(p_x^0))p_{x_l}^0]\cdot p_{x_j}^0\big)\xi_i\xi_k,$$

and $L(t,x,z,\xi) := \det\mathcal{A}(t,x,z,\xi)$. Then, for any $\xi\in\mathbb{R}^{n-1}$, the equation $L(t,x,z,i\xi) = 0$ in $z\in\mathbb{C}$ has a unique solution (with multiplicity n)

$$z = -\sum_{i,j,k,l=1}^{n-1} g^{ij}(p_x^0) g^{kl}(p_x^0)\big([\nabla^2\Phi(\nu(p_x^0))p_{x_l}^0]\cdot p_{x_j}^0\big)\xi_i\xi_k.$$

Being a basis of the tangent hyperplane at $p^0(\cdot)$, $p_{x_i}^0(\cdot)$ are orthogonal to $\nu(p_x^0(\cdot))$, and hence, using the ellipticity of Φ and Proposition A.2 (b), we find

$$z = -\Big(\nabla^2\Phi(\nu(p_x^0))\Big[\sum_{k,l} g^{kl}\xi_k p_{x_l}^0\Big], \Big[\sum_{ij} g^{ij}\xi_i p_{x_j}^0\Big]\Big) \le -\gamma\Big|\sum_{ij} g^{ij}\xi_i p_{x_j}^0\Big|^2$$

for some $\gamma := \gamma(\Phi, n) > 0$. Since $p^0_{x_i} \cdot p^0_{x_j} = g_{ij}$, $\{g^{ij}\}$ is the inverse matrix to $\{g_{ij}\}$ and $\{g^{ij}\}$ is positive definite (by the linear independence of $\{p^0_{x_i}\}$),

$$\Big|\sum_{ij} g^{ij}\xi_i p^0_{x_j}\Big|^2 = \sum_{i,j,k,l} g^{ij}g^{kl}\xi_i\xi_k\big(p^0_{x_j}\cdot p^0_{x_l}\big) = \sum_{i,j,k,l} g^{ij}g^{kl}g_{jl}\xi_i\xi_k$$
$$= \sum_{k,l} g^{kl}\xi_k\xi_l \geq \overline{\gamma}|\xi|^2$$

for some $\overline{\gamma} > 0$ depending only on Γ_0. Thus, $z \leq -\gamma\overline{\gamma}|\xi|^2$ and $\mathcal{A}$ is (uniformly) parabolic.

Complementary condition for the boundary conditions. Let $\mathcal{B}_0(t, x, z, \xi)$ be the matrix, corresponding to the highest-order part of the boundary operator $[\mathcal{P}, \mathcal{B}, \mathcal{T}]$ whose entries are

$$B_{kl}(t, x, z, \xi) = \begin{cases} \delta_{ln}, & k = 1,\\ \sum_{i=1}^n \nabla^2_{ni}\Phi(N(p^0_x))\sum_{j=1}^{n-1}\nabla_{p^l_{x_j}} N^i(p^0_x)\xi_j, & k = 2,\\ \tau^{0,l}_{k-2}\mathrm{n}\cdot\xi, & i = 3, \dots, n, \end{cases}$$

where $\delta_{xy} = 1$ for $x = y$ and $= 0$ for $x \neq y$, and $l = 1, \dots, n$. By the definition (4.2) of N and the third relation in (4.3),

$$\sum_{i=1}^n \nabla^2_{ni}\Phi(N(p^0_x))\nabla_{p^l_{x_j}} N^i(p^0_x) = \det\begin{bmatrix} \nabla[\nabla_n\Phi(N(p^0_x))]\\ p^0_{x_1}\\ \dots\\ p_{x^0_{j-1}}\\ \mathbf{e}_l\\ p_{x^0_{j+1}}\\ \dots\\ p_{x^0_{n-1}} \end{bmatrix} = -\mu_0\mathrm{n}^0_j\det\begin{bmatrix} \mathbf{e}_l\\ p^0_{x_1}\\ \dots\\ p_{x^0_{n-1}} \end{bmatrix}$$
$$= -\mu_0\mathrm{n}^0_j N^l(p^0_x).$$

Therefore, we have also

$$\mathcal{B}_0(t, x, z, \xi) = \begin{bmatrix} \mathbf{e}_n\\ -\mu_0[\mathrm{n}^0\cdot\xi]N(p^0_x)\\ [\mathrm{n}^0\cdot\xi]\tau^0_1\\ \dots\\ [\mathrm{n}^0\cdot\xi]\tau^0_{n-2} \end{bmatrix}.$$

By [37, p. 11], the complementary conditions at the boundary holds iff at every $(t, x) \in [0, T] \times \partial\mathcal{U}$ and every tangent vector $\zeta(x) \in \mathbb{R}^{n-1}$ of $\partial\mathcal{U}$ at x, the rows of the matrix

$$\mathcal{D}(t, x, z, i(\zeta + \lambda\mathrm{n})) := \mathcal{B}_0(t, x, z, i(\zeta + \lambda\mathrm{n}))\widehat{\mathcal{A}}(t, x, z, i(\zeta + \lambda\mathrm{n}))$$

are linearly independent modulo the polynomial

$$M^+(t,x,z,\zeta;\lambda) := \big(\lambda - \lambda_s^+(t,x,z,\zeta)\big)^n,$$

where $\widehat{\mathcal{A}}(t,x,z,\xi) := L(t,x,z,\xi)\mathcal{A}(t,x,z,\xi)^{-1}$, $\lambda_s^\pm(t,x,z,\zeta)$ are the zeros (of multiplicity n) of $L(t,x,z,i(\zeta+\lambda\mathrm{n})) = 0$ in λ for $\Re(z) \ge -\delta_1|\zeta|^2$ and $|z|^2+|\zeta|^2>0$ with $\delta_1>0$. In our case, $\widehat{\mathcal{A}}$ is the identity matrix multiplied by $(\lambda-\lambda_s^+)^{n-1}(\lambda-\lambda_s^-)^{n-1}$, and hence, in view of the explicit expression of $\mathcal{B}_0$, the compatibility condition is equivalent to the linear independence of the vectors

$$\mathbf{e}_n, \quad N(p_x^0), \quad \tau_1^0, \quad \ldots, \quad \tau_{n-2}^0. \tag{4.6}$$

Take $c_1,\ldots,c_n \in \mathbb{R}$ such that

$$c_1\mathbf{e}_n + c_2 N(p_x^0) + \sum_{i=3}^n c_i\tau_{i-2}^0 = 0.$$

By definition, $\tau_j^0\cdot\mathbf{e}_n = 0$ and $\tau_j^0\cdot N(p_x^0) = 0$, and hence, from the linear independence of τ_j^0 (being a basis), $c_i = 0$ for $i\ge 3$. Moreover, if $c_1\ne 0$ (hence, $c_2\ne 0$), then

$$\mathbf{e}_n = -\frac{c_2}{c_1}N(p_x^0),$$

and therefore, by the angle-condition (the second equality in (4.3)) and the evenness of Φ,

$$-\beta = \nabla\Phi(N(p_x^0))\cdot\mathbf{e}_n = \frac{-c_2}{c_1}\Phi(N(p_x^0)) = \frac{\operatorname{sign} c_2}{\operatorname{sign} c_1}\Phi\Big(\frac{-c_2}{c_1}N(p_x^0)\Big) = \frac{\operatorname{sign} c_2}{\operatorname{sign} c_1}\Phi(\mathbf{e}_n).$$

However, in view of (1.4), this equality cannot happen, and therefore, $c_1 = c_2 = 0$; i.e., the vectors in (4.6) are linearly independent.

Complementary conditions for the initial datum. Let $\mathcal{C}$ be the identity matrix, which corresponds to the operator $\mathscr{C}$. By [37, p. 12], the complementary condition for the initial datum is read as follows: for each $x\in\mathcal{U}$, the rows of the matrix

$$\widetilde{\mathcal{D}}(x,z) := \mathcal{C}(x,0,z)\widehat{\mathcal{A}}(0,x,z,0)$$

are linearly independent modulo polynomial z^n. As we have seen above, $\widehat{\mathcal{A}}(0,x,z,0)$ is identity matrix multiplied by z^{n-1}, and hence, by the definition of $\mathcal{C}$, so is $\widetilde{\mathcal{D}}(x,z)$. Then, clearly, the rows of $\widetilde{\mathcal{D}}(x,z)$ are linearly independent modulo z^n.

Compatibility conditions. Notice that while linearizing we obtained the identity

$$[\mathscr{P}p^0, \mathscr{C}p^0, \mathscr{T}p^0] = [0, b + B(p^0,p^0), 0],$$

which reads as the 0-order compatibility of the boundary datum in the right-hand side of (4.5) with the initial datum in the sense of [37, p. 87].

4.1.5. Solvability of the linearized system. For $L > 1$ and $T > 0$, let $X_{L,T}$ be the collection of all $w \in C_T^{1+\frac{\alpha}{2},2+\alpha}$ such that

(1) $w(0,\cdot) = p^0(\cdot)$ in $\overline{\mathcal{U}}$,

(2) $w \cdot \mathbf{e}_n = 0$ in $[0,T] \times \partial\mathcal{U}$,

(3) $\sum_{i=1}^{n-1} n_i^0 w_{x_i} \cdot \tau_j^0(p^0) = 0$ in $[0,T] \times \partial\mathcal{U}$,

(4) the vectors $\{w_{x_i}\}_{i=1}^{n-1}$ are linearly independent,

(5) $\|w\|_{C^{1+\frac{\alpha}{2},1+\alpha}} \le L$.

Clearly, $X_{L,T} \neq \emptyset$, since conditions (1), (2), (3), and (5) allow to construct w first in the neighborhood of $\partial\mathcal{U}$, and then to extend to interior of $\mathcal{U}$. The condition (4) holds at least for small T; in fact, since

$$w_x(t,x) = p_x^0(x) + \int_0^t w_{tx}(s,x)ds, \quad w \in X_L(T),$$

$\|p_x^0 - w_x(t,\cdot)\|_\infty \le LT$ whenever $t \le T$. Thus,

$$\det(\{w_{x_i} \cdot w_{x_j}\}) = \det(\{p_{x_i}^0 \cdot p_{x_j}^0\}) - C_1 T,$$

where $C_1 > 0$ depends only on n, $\|p_x^0\|_\infty$, and L. Thus, if we choose

$$T < T_1 := \frac{\det(\{p_{x_i}^0 \cdot p_{x_j}^0\})}{C_1},$$

then w_{x_i} are linearly independent. We can also show that $X_{L,T}$ is a closed convex subspace of $C_T^{1+\frac{\alpha}{2},2+\alpha}$.

Notice that for any $w \in X_{L,T}$ the vectors $[\bar{f} + F(w,p^0), 0, \bar{b} + B(w,p^0), 0, p^0]$ satisfy the 0-order compatibility condition, and therefore, there exists a unique $\mathcal{S}_w \in C_T^{1+\frac{\alpha}{2},2+\alpha}$ such that

$$[\mathcal{A}[\mathcal{S}_w], \mathcal{P}[\mathcal{S}_w], \mathcal{C}[\mathcal{S}_w], \mathcal{T}[\mathcal{S}_w], \mathcal{I}[\mathcal{S}_w]] = [\bar{f} + F(w,p^0), 0, \bar{b} + B(w,p^0), 0, p^0] \quad (4.7)$$

and

$$\|\mathcal{S}_w\|_{2+\alpha,T} \le C_0\big(\|\bar{f} + F(w,p^0)\|_{\alpha,T} + \|\bar{b} + B(w,p^0)\|_{\alpha,T} + \|p^0\|_{\alpha,T}\big) \quad (4.8)$$

for some $C_0 > 0$ (continuously) depending only on β, Φ, p^0, and also on $\mathcal{U}$ and n. By uniqueness and linearity, from (4.8) for any $w_1, w_2 \in X_{L,T}$, we have

$$\begin{aligned}\|\mathcal{S}_{w_1} - \mathcal{S}_{w_2}\|_{2+\alpha,T} &= \|\mathcal{S}_{w_1-w_2}\|_{2+\alpha,T} \\ &\le C_0\big(\|F(w_1,p^0) - F(w_2,p^0)\|_{\alpha,T} + \|B(w_1,p^0) - B(w_2,p^0)\|_{\alpha,T}\big).\end{aligned}$$

Using the explicit expressions of F and B, the definition of $X_{L,T}$, and the equality

$$u(t,x) = p^0(x) + \int_0^t u_t(s,x)ds, \quad w \in C^{1,0}([0,T] \times \bar{U}), \quad (4.9)$$

we can compute

$$\|F(w_1, p^0) - F(w_2, p^0)\|_{\alpha,T} \le C_1 T \|w_1 - w_2\|_{2+\alpha,T}$$

and

$$\|B(w_1, p^0) - B(w_2, p^0)\|_{\alpha,T} \le C_1 T \|w_1 - w_2\|_{2+\alpha,T}$$

for some C_1 depending on L but not on T, w_1 and w_2. Thus, if we choose

$$T < T_2 := \frac{1}{C_0 C_1},$$

then $w \mapsto \mathcal{S}_w$ is a contraction. To apply a fixed-point theorem, it remains to show that $\mathcal{S}_w \in X_{L,T}$ whenever $w \in X_{L,T}$. The equalities (1)–(3) for $\mathcal{S}_w$ follow from the system (4.7). Moreover, since $T < T_1$, the vectors $\{(\mathcal{S}_w)_{x_i}\}$ are also linearly independent. It remains to check condition (5). Consider the estimate (4.8). By definition of F and B (they are somehow estimated by a power of L times the norm of $w - p^0$),

$$\|F(w, p^0)\|_{\alpha,T} \le C_2 L^{10} T, \quad \|B(w, p^0)\|_{\alpha,T} \le C_2 L^{10} T,$$

where C_2 does not depend on $T > 0$ and $L > 1$, and hence, by (4.9),

$$\begin{aligned} &\|\bar{f} + F(w, p^0)\|_{\alpha,T} + \|\bar{b} + B(w, p^0)\|_{\alpha,T} + \|p^0\|_{\alpha,T} \\ &\le \|\bar{f}\|_{\alpha,T} + \|\bar{b}\|_{\alpha,T} + \|p^0\|_{\alpha,T} + 2C_2 L^{10} T. \end{aligned}$$

Now, if we choose

$$L := 1 + 2C_0[\|\bar{f}\|_{\alpha,T} + \|\bar{b}\|_{\alpha,T} + \|p^0\|_{\alpha,T}],$$

then $\mathcal{S}_w \in X_{L,T}$ provided

$$T \le T_3 := \frac{1 + C_0[\|\bar{f}\|_{\alpha,T} + \|\bar{b}\|_{\alpha,T} + \|p^0\|_{\alpha,T}]}{2C_0 C_2 L^{10}}.$$

Now, the Banach fixed-point theorem implies that there exists a unique $w \in X_{L,T}$ which satisfies $\mathcal{S}_w = w$. Then, (4.7) implies that w is a solution of (4.4) for small $T > 0$.

4.2. Long-time evolution

Applying Theorem 4.2 inductively, we obtain the following generalization of Theorem 1.1.

Theorem 4.3. *Let $\Gamma_0 \subset \Omega$ be a bounded $C^{2+\alpha}$-hypersurface with boundary satisfying*

$$\partial\Gamma_0 \subset \partial\Omega \quad \text{and} \quad \nabla\Phi(\nu_{\Gamma_0}) \cdot \mathbf{e}_n = -\beta \text{ on } \partial\Omega.$$

Then, there exist a maximal time $T^\dagger > 0$ and a smooth Φ-curvature flow $\{\Gamma(t)\}_{t\in[0,T^\dagger)}$ starting from Γ_0, with the forcing f and anisotropic contact angle β.

The term "maximal" refers to the fact that there is no smooth Φ-curvature flow $\{\Gamma(t)\}_{t\in[0,T')}$ for any $T' > T^\dagger$. Notice that at the maximal time $T^\dagger$ for the set $\Gamma(T^\dagger)$ (defined, for instance, as a Kuratowski limit of $\Gamma(t)$ as $t \nearrow T^\dagger$) at least one of the following holds (otherwise applying Theorem 4.2 we would extend the flow slightly after $T^\dagger$):

- $\Gamma(T^\dagger)$ is not C^2-anymore (the curvature blows up),
- $\overline{\Gamma(T^\dagger)}$ is not injective,
- some interior point of $\Gamma(T^\dagger)$ touches to $\partial\Omega$ (because of the forcing).

In this paper, we do not deal with the singularity analysis.

Remark 4.4. The Φ-curvature flow equation is represented by means of the signed distances as

$$\frac{\partial}{\partial t}\,\mathrm{sd}_{E(t)}(x) = \kappa^{\Phi}_{E(t)}(x) + f(t,x), \quad t \in [0,T^\dagger), x \in \partial^\Omega E(t). \tag{4.10}$$

4.3. Stability of the anisotropic curvature flow

The classical mean curvature flow of boundaries has the following remarkable stability property: *if $\{E(t)\}_{t\in[0,T^\dagger)}$ is the smooth mean curvature flow of $C^{2+\alpha}$-sets, then for every $0 < T < T^\dagger$ there exists $\varepsilon > 0$ such that if $F(0)$ is such that $\partial F(0)$ belongs to the $C^{2+\alpha}$-neighborhood of $\partial E(0)$, then there exists a unique mean curvature flow $\{F(t)\}_{t\in[0,T')}$ starting from $F(0)$ and $T' > T$* (see, e.g., [1, Theorem 7.1]).

In this section, we prove that the flow solving (1.1) admits such a stability property. As in [36], we are mainly interested in droplets with nonempty contact on $\partial\Omega$, and therefore, it is natural to restrict ourselves to the regular droplets without connected components "hanging" in Ω.

Definition 4.5 (Admissibility). (a) We say a bounded set $E \subset \Omega$ is *admissible* provided there exist a bounded $C^{2+\alpha}$-open set $\mathfrak{U} \subset \mathbb{R}^{n-1}$ and a $C^{2+\alpha}$-diffeomorphism $p \in C^{2+\alpha}(\overline{\mathfrak{U}};\mathbb{R}^n)$ satisfying

$$p[\mathfrak{U}] = \Gamma, \quad p[\partial\mathfrak{U}] = \partial\Gamma, \quad p\cdot\mathbf{e}_n > 0 \text{ in } \mathfrak{U}, \quad \text{and} \quad p\cdot\mathbf{e}_n = 0 \text{ on } \partial\mathfrak{U},$$

where $\Gamma := \partial^\Omega E$. Any such map p is called a *parametrization* of Γ.

(b) We say E is *admissible with anisotropic contact angle β* if E is admissible and

$$\nabla\Phi(\nu_E)\cdot\mathbf{e}_n = -\beta \quad \text{on } \partial\Omega\cap\overline{\Gamma}. \tag{4.11}$$

We call the number

$$h_E := \min_{x\in\overline{\Gamma},\nu_E(x)=\mathbf{e}_n} x\cdot\mathbf{e}_n \tag{4.12}$$

the *minimal height* of E. Since E satisfies (4.11) and β satisfies (1.4), $h_E > 0$.

(c) Let Q be a compact set in $\mathbb{R}^m$ for some $m \ge 1$. We say a family $\{E[q]\}_{q\in Q}$ of bounded subsets of Ω is *admissible* if there exist $\alpha \in (0,1]$, a bounded $C^{2+\alpha}$-open set

$\mathcal{U} \subset \mathbb{R}^{n-1}$, and a map $p \in C^{2+\alpha,2+\alpha}(Q \times \overline{\mathcal{U}}; \mathbb{R}^n)$ such that $p[q,\cdot]$ is a parametrization of $\partial^\Omega E[q]$.

(d) We say a family $\{E[q,t]\}_{q\in Q, t\in[0,T)}$ of bounded subsets of Ω *admissible* if for any $T' \in (0,T)$ there exist $\alpha \in (0,1]$, a bounded $C^{2+\alpha}$-open set $\mathcal{U} \subset \mathbb{R}^{n-1}$, and a map $p \in C^{2+\alpha,1+\frac{\alpha}{2},2+\alpha}(Q \times [0,T'] \times \overline{\mathcal{U}}; \mathbb{R}^n)$ such that $p[q,t,\cdot]$ is a parametrization of $\partial^\Omega E[q,t]$.

Remark 4.6. (a) By definition, if E is an admissible set, then the $C^{2+\alpha}$-surface $\Gamma := \partial^\Omega E$ is diffeomorphic to a bounded smooth open set in $\mathbb{R}^{n-1}$ and not necessarily connected. (Clearly, boundaries of two connected components do not touch.) In particular, Γ cannot not have "hanging" components compactly contained in Ω. Moreover, its boundary $\partial\Gamma$ lies on $\partial\Omega$ and the relative interior of Γ does not touch to $\partial\Omega$.

(b) When Q is empty in Definition 4.5 (d), then we simply write $\{E[t]\}_{t\in[0,T)}$ to denote the corresponding admissible family.

(c) If

$$p \in C^{2+\alpha,1+\frac{\alpha}{2},2+\alpha}(Q \times [0,T'] \times \overline{\mathcal{U}}; \mathbb{R}^n)$$

is a parametrization of $\{E[q,t]\}_{q\in Q,t\in[0,T)}$ and $\mathcal{U}' \subset \mathbb{R}^{n-1}$ is a bounded $C^{2+\alpha}$ open set diffeomorphic to $\mathcal{U}$ via a map $\psi : \mathcal{U}' \to \mathcal{U}$, then $p[q,t,\psi(\cdot)]$ is also a parametrization of $E[q,t]$.

Remark 4.6 (c) allows to introduce the closeness of the free boundaries of two droplets.

Definition 4.7. For any two admissible sets E_1 and E_2, we write

$$\bar{d}(E_1, E_2) = \inf_{p_1,p_2} \|p_1 - p_2\|_{C^{2+\alpha}(\overline{\mathcal{U}})},$$

where $p_i \in C^{2+\alpha}(\mathcal{U}; \mathbb{R}^n)$ is a parametrization of $\partial^\Omega E_i$. Similarly, if $\{E_1[q,t]\}_{q\in Q,t\in[0,T)}$ and $\{E_2[q,t]\}_{q\in Q,t\in[0,T)}$ are two admissible families, we write

$$\bar{d}(E_1(t), E_2(t)) = \inf_{p_1,p_2} \|p_1 - p_2\|_{C^{2+\alpha,1+\frac{\alpha}{2},2+\alpha}(Q\times[0,T']\times\overline{\mathcal{U}})},$$

where $p_i \in C^{2+\alpha,1+\frac{\alpha}{2},2+\alpha}(Q \times [0,T'] \times \overline{\mathcal{U}}; \mathbb{R}^n)$ is a parametrization of $E_i[\cdot,\cdot]$.

One can readily check that the infimum in the definition of $\bar{d}$ is in fact a minimum.

As we have observed in the proof of Theorem 4.2, the constants C_0, C_1, C_2 and bounds T_1, T_2, T_3 for local time T continuously depend on $\|f\|_{C^{\alpha/2,\alpha}(\mathbb{R}_0^+\times\overline{\Omega})}$, $\|\beta\|_{C^{1+\alpha}(\partial\Omega)}$, $\|\Phi\|_{C^{3+\alpha}(\mathbb{S}^{n-1})}$, and $\|p^0\|_{C^{2+\alpha}(\overline{\mathcal{U}})}$. This implies the following stability of the flow which generalizes [1, Theorem 7.1].

Theorem 4.8 (Stability of Φ-curvature flow). *Let Φ_0 be an elliptic $C^{3+\alpha}$-anisotropy, $\beta_0 \in C^{1+\alpha}(\partial\Omega)$ satisfy (1.4), $f_0 \in C^{\frac{\alpha}{2},\alpha}(\mathbb{R}_0^+ \times \overline{\Omega})$, and $\{E_0(t)\}_{t\in[0,T_0)}$ be a bounded smooth Φ_i-curvature flow with forcing f_i and anisotropic contact angle β_i for some $T_0 > 0$. Then, for any $T \in (0,T_0)$, there exist $\varepsilon_0 > 0$ and a nondecreasing function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\psi(0) = 0$ with the following property. For $i = 1,2$, let Φ_i be an elliptic $C^{3+\alpha}$-anisotropy, $\beta_i \in C^{1+\alpha}(\partial\Omega)$ satisfying (1.4), and $f_i \in C^{\frac{\alpha}{2},\alpha}(\mathbb{R}_0^+ \times \overline{\Omega})$ and Φ_i-curvature* *flow* $\{E_i(t)\}_{t\in t\in[0,T_i]}$ *with forcing* f_i *and anisotropic contact angle* β_i *for some* $T_i > 0$ *be such that*

$$\|\Phi_i - \Phi_0\|_{C^{3+\alpha}(\overline{B_2(0)}\setminus B_{1/2}(0))} + \|\beta_i - \beta_0\|_{C^{1+\alpha}(\partial\Omega)} \\ + \|f_i - f_0\|_{C^{\alpha/2,\alpha}([0,T_0]\times\overline{\Omega})} + \bar{d}(E_i(0, E_0(0))) \le \varepsilon_0.$$

Then, $T_i > T$ *and*

$$\bar{d}(E_1(t), E_2(t)) \le \psi(\bar{d}(E_1(0), E_2(0))), \quad t \in [0, T]. \tag{4.13}$$

In what follows, we refer to (4.13) as *smooth dependence on the initial condition*.

Let us consider some applications of the stability.

4.3.1. Comparison principle for anisotropic curvature flows. The main result of this section is the following.

Theorem 4.9 (Strong comparison). *Let* Φ *be an elliptic* $C^{3+\alpha}$*-anisotropy,* $\beta_i \in C^{1+\alpha}(\partial\Omega)$ *satisfy* (1.4) *and* $f_i \in C^{\frac{\alpha}{2},\alpha}(\mathbb{R}_0^+ \times \overline{\Omega})$*,* $\{E_i(t)\}_{t\in[0,T)}$ *be a bounded smooth* Φ*-curvature flow with forcing* f_i *and anisotropic contact angle* β_i*,* $i = 1, 2$*. Then,*

$$\beta_1 \ge \beta_1, \quad f_1 \ge f_2, \quad E_1(0) \prec E_2(0) \quad \Longrightarrow \quad E_1(t) \prec E_2(t), \quad t \in [0, T). \tag{4.14}$$

In other words, $\overline{\partial^\Omega E_1(t)} \cap \overline{\partial^\Omega E_2(t)} = \emptyset$ *for all* $t \in [0, T)$ *if so at* $t = 0$*.*

Further, we refer to assertion (4.14) as the *strong comparison principle*.

Proof. In view of Theorem 4.8, decreasing f_i and β_i a bit, it is enough to prove (4.14) when the inequalities between β_i and f_i are strict. For $t \in [0, T]$, let

$$a_\Phi^i(t, x) := \mathrm{sd}_{E_i(t)}^\Phi(x), \quad a^i(t, x) := \mathrm{sd}_{E_i(t)}(x),$$

and

$$d_\Phi(t) := \min\{x \in \overline{\Omega} : a_\Phi^1(t, x) - a_\Phi^2(t, x)\}, \quad d(t) := \min\{x \in \overline{\Omega} : a^1(t, x) - a^2(t, x)\}.$$

Since $E(0) \prec F(0)$, by (2.1) $d(0), d_\Phi(0) > 0$. By contradiction, assume that there exists $t_0 \in (0, T)$ such that $d(t), d_\Phi(t) > 0$ in $(0, t_0)$ and $d(t_0) = d_\Phi(t_0) = 0$. Thus, for some $x_0 \in \overline{\partial^\Omega E_1(t_0)} \cap \overline{\partial^\Omega E_2(t_0)}$, $d_\Phi(t_0) = a_\Phi^1(t_0, x_0) - a_\Phi^2(t_0, x_0) = 0$.

First, assume that $x_0 \in \partial\Omega$ and let $\eta := \nabla\Phi(\nu_{E_1(t_0)}(x_0)) - \nabla\Phi(\nu_{E_2(t_0)}(x_0))$. Since $-\mathbf{e}_n$ is the outer unit normal to Ω, by the anisotropic contact angle condition, $\eta \cdot (-\mathbf{e}_n) = \beta_1 - \beta_2 > 0$. Thus, applying Proposition 2.1 (d) with $-\eta$ and recalling the definitions of x_0 and $d_\Phi(t_0)$, we find

$$\begin{aligned} 0 &\le a_\Phi^1(t_0, x_0 - s\eta) - a_\Phi^1(t_0, x_0) - a_\Phi^2(t_0, x_0 - s\eta) + a_\Phi^2(t_0, x_0) \\ &= -s\big(\big[\nu_{E_1(t_0)}^{\Phi^o} - \nu_{E_2(t_0)}^{\Phi^o}\big] \cdot \eta + o(1)\big) \end{aligned}$$

as $s \to 0^+$, where η^{Φ^o} is defined in (2.2). Since $\nabla\Phi(\cdot)$ is strictly maximal monotone (as a subdifferential of convex functions) and positively 0-homogeneous,

$$0 \le -\big[\nu^{\Phi^o}_{E_1(t_0)} - \nu^{\Phi^o}_{E_2(t_0)}\big] \cdot \eta = -\big[\nu^{\Phi^o}_{E_1(t_0)} - \nu^{\Phi^o}_{E_2(t_0)}\big] \cdot \big[\nabla\Phi(\nu^{\Phi^o}_{E_1(t_0)}) - \nabla\Phi(\nu^{\Phi^o}_{E_2(t_0)})\big] < 0,$$

a contradiction. Thus, $x_0 \in \Omega$. By the time-smoothness of the flows $E_i(\cdot)$, there exists $\delta > 0$ such that for any $t \in [t_0 - \delta, t_0]$ minimum points of $f(t,\cdot) - g(t,\cdot)$ lie in Ω (basically the minimizers belong to a union of half-lines in Ω starting from $\partial\Omega$ and crossing both $\partial^\Omega E_1(t_0)$ and $\partial^\Omega E_2(t_0)$ orthogonally). Therefore, using a Hamilton-type trick (see, e.g., [31, Chapter 2]), we can show that

$$d'(t) = \frac{\partial}{\partial t}\,\mathrm{sd}_{E_1(t)}(y_t) - \frac{\partial}{\partial t}\,\mathrm{sd}_{E_2(t)}(y_t), \quad t \in [t_0 - \delta, t_0],$$

where $y_t \in \partial^\Omega E_2(t)$ is any point satisfying $d(t) = \mathrm{sd}_{E_1(t)}(y_t) - \mathrm{sd}_{E_2(t)}(y_t)$. Let $z_t \in \partial^\Omega E_1(t)$ and $u_t \in \partial^\Omega E_2(t)$ be such that $d(t) = \mathrm{d}_{E_1(t)}(y_t) = |y_t - z_t|$ and $\mathrm{d}_{E_2(t)}(y_t) = |y_t - u_t|$. By the minimality of y_t, $\nu_{E_1(t)}(z_t) = \nu_{E_2(t)}(u_t) =: \nu_0$ and y_t, z_t, u_t lie on the same straight line parallel to $\nu_{E_2(t)}(u_t)$. Now, applying (4.10), we find

$$d'(t) = \kappa^\Phi_{E_1(t)}(z_t) - \kappa^\Phi_{E_2(t)}(u_t) + f_1(t, z_t) - f_2(t, u_t).$$

By the minimality of $y_t \in \partial^\Omega E_2(t)$ and smoothness and the ellipticity of Φ, translating $E_1(t)$ along $\nu_{E_2(t)}(u_t)$ until we reach to $u_t \in \partial E_2(t)$ we deduce that $\widetilde{E}_1(t) \subset E_2(t)$ and $\partial\widetilde{E}_1(t)$ is tangent to $\partial E_2(t)$ at u_t, where $\widetilde{E}_1(t)$ is the translated $E_1(t)$. Then, $\kappa^\Phi_{E_1(t)}(z_t) = \kappa^\Phi_{\widetilde{E}_1(t)}(u_t) \ge \kappa^\Phi_{E_2(t)}(u_t)$, and therefore, by the $C^{\alpha/2,\alpha}$-regularity of f_2,

$$\begin{aligned} d'(t) &\ge f_1(t, z_t) - f_2(t, u_t) = f_1(t, z_t) - f_2(t, z_t + d(t)\nu_0) \\ &\ge f_1(t, z_t) - f_2(t, z_t) - C_{f_2} d(t)^\alpha, \end{aligned}$$

where C_{f_2} is the Hölder constant of f_2. Since $\{E_i(t)\}$ is bounded uniformly in $t \in [t_0 - \delta, t_0]$ and by assumption $f_1 > f_2$, there exists $\gamma_0 > 0$ independent of t such that $f_1(t, z_t) - f_2(t, z_t) \ge \gamma_0$. Thus, recalling the continuity of $d(\cdot)$ and assumption $d(t_0) = 0$ possibly decreasing δ a bit, we get $d'(t) > \gamma_0/2$ for any $t \in [t_0 - \delta, t_0]$. Therefore, d is strictly increasing in this interval so that $0 = d(t_0) > d(t_0 - \delta) > 0$, a contradiction.

These contradictions show that

$$\overline{\partial^\Omega E(t)} \cap \overline{\partial^\Omega F(t)} = \emptyset$$

for any $t \in [0, T)$. Hence, $E(t) \prec F(t)$. ∎

4.3.2. Evolution of tubular neighborhoods. Recall that a crucial part in the proof of the consistency in [1, Theorem 7.4] is the evolution of tubular neighborhoods [1, Corollary 7.2] which is given by the level sets of signed distance functions. Unfortunately, in our setting, due to the contact angle condition we cannot use signed distances. Therefore, as in [36], we construct a sort of tubular neighborhoods, which possess similar properties of the true tubular neighborhoods in case of without boundary, important in the proof of the consistency.

To this aim, in the following lemma, we define a “foliation” of a tubular neighborhood of the boundary of an admissible set, consisting of boundaries of admissible families with a prescribed anisotropic contact angle.

Lemma 4.10 (Foliations). *Let E_0 be an admissible set with anisotropic contact angle β. Then, there exist positive numbers $\rho \in (0,1)$ and $\sigma \in (0,\eta)$, depending only on $\|II_{E_0}\|_\infty$ and h_{E_0} (see (4.12)), and admissible families $\{G_0^\pm[r,s]\}_{(r,s)\in[0,\rho]\times[0,\sigma]}$ such that*

$$G_0^\pm[0,0] = E_0$$

and for all $(r,s) \in [0,\rho]\times[0,\sigma]$:

(a) $\operatorname{dist}(\partial^\Omega G_0^\pm[r,s], \partial^\Omega E_0) \ge r+s$ *and*

$$\begin{aligned} &G_0^-[r,s] \subset E_0 \subset G_0^+[r,s],\\ &\operatorname{dist}(\partial^\Omega G_0^\pm[r,s], \partial^\Omega G_0^\pm[0,s]) = r,\\ &\operatorname{dist}(\partial^\Omega G_0^\pm[0,s], \partial^\Omega E_0) = s; \end{aligned}$$

(b) $G_0^\pm[r,s]$ *is admissible with anisotropic contact angle $\beta \mp s$.*

Note that we ignore the dependence on α and η. Since this lemma is a generalization of [35, Lemma 2.4] to the anisotropic setting which can be done along the same lines, we omit the proof.

Corollary 4.11. *Let $\{E[t]\}_{t\in[0,T)}$ be an admissible family contact angle β. Then, for any $T' \in (0,T)$, there exist $\rho \in (0,1)$ and $\sigma \in (0,\eta)$ depending only $\sup_{t\in[0,T']}\|II_{E[t]}\|_\infty$ and $\inf_{t\in[0,T']} h_{E[t]}$, and admissible families $\{G_0^\pm[r,s,a]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T']}$ such that $G_0^\pm[0,0,a] = E[a]$ and for all $(r,s,a) \in [0,\rho]\times[0,\sigma]\times[0,T']$:*

(a) $\operatorname{dist}(\partial^\Omega G_0^\pm[r,s,a], \partial^\Omega E[a]) \ge r+s$ *and*

$$\begin{aligned} &G_0^-[r,s,a] \subset E[a] \subset G_0^+[r,s,a],\\ &\operatorname{dist}(\partial^\Omega G_0^\pm[r,s,a], \partial^\Omega G_0^\pm[0,s,a]) = r,\\ &\operatorname{dist}(\partial^\Omega G_0^\pm[0,s,a], \partial^\Omega E[a]) = s; \end{aligned}$$

(b) $G_0^\pm[r,s,a]$ *is admissible with anisotropic contact angle $\beta \mp s$.*

By the definition of the admissibility, $G_0^\pm[r,s,a]$ is close to $E[a]$ in the sense of Definition 4.7. Therefore, applying Theorem 4.8, we deduce the following.

Theorem 4.12. *Let a family $\{E[t]\}_{t\in[0,T^\dagger)}$ of admissible sets be a Φ-curvature flow with forcing f and anisotropic contact angle β, and let $T \in (0,T^\dagger)$. Let $\rho \in (0,1)$ and $\sigma \in (0,\eta)$, and for $a \in [0,T)$, the families $\{G_0^\pm[r,s,a]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T']}$ be given by Corollary 4.11. Then (possibly decreasing ρ and σ slightly, depending only on $\{E(t)\}$),* *there exist unique admissible families* $\{G^{\pm}[r,s,a,t]\}_{(r,s,a)\in[0,\rho]\times[0,\sigma]\times[0,T'],t\in[a,T]}$ *such that*

- $G^{\pm}[r,s,a,a] = G_0^{\pm}[r,s,a]$,
- $G^{\pm}[r,s,a,t]$ *is admissible with anisotropic contact angle* $\beta \mp s$,
- "$G^{\pm}[r,s,a,t]$ *solves the equation*

$$v_{G^{\pm}[r,s,a,t]}(x) = -\kappa_{G^{\pm}[r,s,a,t]}(x) - f(t,x) \pm s \tag{4.15}$$

for $t \in (a,T)$ *and* $x \in \partial^{\Omega} G^{\pm}[r,s,a,t]$.

Furthermore,

(a) $G^{\pm}[0,0,a,t] = E[t]$ *for all* $t \in [a,T]$*;*

(b) *there exists an increasing continuous function*

$$g : [0,+\infty) \to [0,+\infty)$$

with $g(0) = 0$ *such that*

$$\max_{x\in\partial^{\Omega} G^{\pm}[0,s,a,t]} \operatorname{dist}(x, \partial^{\Omega} G^{\pm}[0,0,a,t]) \le g(s)$$

for all $s \in [0,\sigma]$, $a \in [0,T]$ *and* $t \in [0,T]$*;*

(c) *there exists* $t^* \in (0,\rho/64)$ *(independent of r, s and a) such that*

$$G_0^{+}[\rho/2,s,a] \subset G^{+}[\rho,s,a,a+t'] \quad \text{and} \quad G_0^{-}[\rho/2,s,a] \supset G^{-}[\rho,s,a,a+t'] \tag{4.16}$$

for all $t' \in [0,t^*]$ *with* $a + t' \le T$.

Notice that the assertions (a)–(c) follow from the continuous dependence of $G^{\pm}$ on $[r,s,a,t]$. In view of (4.10), we can represent (4.15) as

$$\frac{\partial}{\partial t}\operatorname{sd}_{G^{\pm}[r,s,a,t]}(x) = \kappa_{G^{\pm}[r,s,a,t]}(x) + f(t,x) \mp s$$

for $t \in (a,T)$ and $x \in \partial^{\Omega} G^{\pm}[r,s,a,t]$.

Proposition 4.13. *For any* $s \in (0,\sigma]$*, there exists* $\tau_0(s) > 0$ *such that, for any* $r \in [0,\rho]$*,* $a \in [0,T)$*,* $\tau \in (0,\tau_0)$*, and* $t \in [a+\tau,T]$*,*

$$\frac{\operatorname{sd}_{G^{+}[r,s,a,t-\tau]}(x)}{\tau} + \kappa_{G^{+}[r,s,a,t]}(x) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x)ds > \frac{s}{2}, \quad x \in \partial^{\Omega} G^{+}[r,s,a,t],$$

and

$$\frac{\operatorname{sd}_{G^{-}[r,s,a,t-\tau]}(x)}{\tau} + \kappa_{G^{-}[r,s,a,t]}(x) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s,x)ds < -\frac{s}{2}, \quad x \in \partial^{\Omega} G^{+}[r,s,a,t],$$

where $k := \lfloor t/\tau \rfloor$.

This result is proven along the same lines of [36, Proposition 2.7]. Therefore, we omit it.

5. Consistency of GMM with smooth anisotropic curvature flow

In this section, we prove Theorem 1.5.

Let $\{E(t)\}_{t\in[0,T^\dagger)}$ be a smooth Φ-curvature flow starting from E_0, with a bounded forcing f and anisotropic contact angle β. Given $T \in (0, T^\dagger)$, let ρ, σ, $\{G_0^\pm[r,s,a]\}$, $\{G^\pm[r,s,a,t]\}$, and $t^* > 0$ be as in Theorem 4.12. Let also $F(\cdot) \in \mathrm{GMM}(\mathcal{F}_{\beta,f}, E_0)$, $\tau_j \searrow 0$, and $\{F(\tau_j, k)\}$ be such that

$$\lim_{j\to+\infty} |F(\tau_j, \lfloor t/\tau_j\rfloor)\Delta F(t)| = 0 \quad \text{for all } t \ge 0. \tag{5.1}$$

We show that

$$E(t) = F(t) \quad \text{for any } 0 < t < T. \tag{5.2}$$

We start with an ancillary technical lemma. For $s \in (0,\sigma]$, let $\tau_0(s) > 0$ be given by Proposition 4.13, and for $\beta_0 := \frac{\|\beta\|_\infty + \Phi(\mathbf{e}_n)}{2} \in (\|\beta\|_\infty, \Phi(\mathbf{e}_n))$, let ϑ_0 be given by Theorem 3.4. We may assume that $\tau_j < \vartheta_0\rho^2/64^2$ for all j.

Lemma 5.1. *Assume that $t_0 \in [0,T)$ and $k_0 \in \mathbb{N}_0$ are such that*

$$G_0^-[0,s,t_0] \subset F(\tau_j, k_0) \subset G_0^+[0,s,t_0]. \tag{5.3}$$

Then, there exists $\bar t \in (0, t^*]$ depending only on t^* and ρ such that

$$G^-[0,s,t_0,t_0+k\tau_j] \subset F(\tau_j, k_0+k) \subset G^+[0,s,t_0,t_0+k\tau_j]$$

for all $s \in (0,\sigma]$, $j \ge 1$ with $\tau_j \in (0,\tau_0(s))$ and $k = 0,1,\dots,\lfloor \bar t/\tau_j\rfloor$ with $t_0 + k\tau_j < T$. Moreover, let $t_0 + \bar t < T$, the increasing continuous function g be given by Theorem 4.12 (b), and $\bar\sigma \in (0,\sigma/2)$ be such that $4g(2\bar\sigma) < \sigma$. Then, for any $s \in (0,\bar\sigma)$, there exists $\bar j(s) > 1$ such that

$$G_0^-[0,4g(2s),t_0+\bar t] \subset F(\tau_j, k_0+\bar k_j) \subset G_0^+[0,4g(2s),t_0+\bar t] \tag{5.4}$$

whenever $j > \bar j(s)$, where $\bar k_j := \lfloor \bar t/\tau_j\rfloor$.

Proof. The proof runs along the similar lines of [36, Lemma 3.1]. By Corollary 4.11 (a),

$$\mathrm{dist}(\partial^\Omega G_0^\pm[\rho/4,s,t_0], \partial^\Omega G_0^\pm[0,s,t_0]) = \rho/4,$$

and therefore, by (5.3),

$$G_0^-[\rho/4,s,t_0] \prec G_0^-[0,s,t_0] \subset F(\tau_j,k_0) \subset G_0^+[0,s,t_0] \prec G_0^+[\rho/4,s,t_0] \tag{5.5}$$

and by (5.5) $B_{\rho/4}(x) \subset F(\tau_j,k_0)$ if $x \in G_0^-[\rho/4,s,t_0]$ and $B_{\rho/4}(x) \cap F(\tau_j,k_0) = \emptyset$ if $x \in \Omega \setminus G_0^+[\rho/4,s,t_0]$. Therefore, using Theorem 3.4 (with $R_0 = \rho/4$ and $\beta_0 := \frac{\Phi(\mathbf{e}_n)+\|\beta\|_\infty}{2}$) and again (5.3), we obtain

$$\begin{cases} B_{\frac{\beta_0\rho}{64\Phi(\mathbf{e}_n)}}(x) \subset F(\tau_j,k_0+k), & x \in G_0^-[\rho/4,s,t_0], \\ B_{\frac{\beta_0\rho}{64\Phi(\mathbf{e}_n)}}(x) \cap F(\tau_j,k_0+k) = \emptyset, & x \in \Omega\setminus G_0^+[\rho/4,s,t_0], \end{cases} \quad 0 \le k \le \lfloor t^{**}/\tau_j\rfloor, \tag{5.6}$$

where

$$t^{**} := \frac{\vartheta_0 \rho^2}{16}.$$

By (5.6) and Corollary 4.11 (b),

$$\begin{aligned} G_0^-\Big[\frac{\rho}{2}, s, t_0\Big] \subset G_0^-\Big[\frac{\rho}{4} - \frac{\beta_0\rho}{64}, s, t_0\Big] \subset F(\tau_j, k_0 + k) \\ \subset G_0^+\Big[\frac{\rho}{4} - \frac{\beta_0\rho}{64}, s, t_0\Big] \subset G_0^+\Big[\frac{\rho}{4}, s, t_0\Big] \end{aligned} \tag{5.7}$$

for all $0 \le k \le \lfloor t^{**}/\tau_j \rfloor$. Set

$$\bar t := \min\{t^*, t^{**}\},$$

where t^* is given by Theorem 4.12 (c). Then, by (4.16) and (5.7),

$$G_0^-[\rho, s, t_0 + k\tau_j] \subset F(\tau_j, k_0 + k) \subset G_0^+[\rho, s, t_0 + k\tau_j], \quad k = 0, 1, \dots, \lfloor \bar t/\tau_j \rfloor, \tag{5.8}$$

with $t_0 + k\tau_j < T$. We claim for such k and $j \ge 1$ with $\tau_j \in (0, \tau_0(s))$

$$G^-[0, s, t_0, t_0 + k\tau_j] \subset F(\tau_j, k_0 + k) \subset G^+[0, s, t_0, t_0 + k\tau_j].$$

Indeed, let

$$\begin{aligned} \bar r := \inf\big\{r \in [0, \rho] : F(\tau_j, k_0 + k) \subset G^+[r, s, t_0, t_0 + k\tau_j], \\ k = 0, 1, \dots, \lfloor \bar t/\tau_j \rfloor,\ t_0 + k\tau_j < T\big\}. \end{aligned}$$

By (5.8), the infimum is taken over a nonempty set. By contradiction, assume that $\bar r > 0$. In view of the continuity of $G^+[r, s, t_0, t_0 + k\tau_j]$ at $r = \bar r$, there exists the smallest integer $k \le \lfloor \bar t/\tau_j \rfloor$ (clearly, $k > 0$ by (5.5)) such that

$$\overline{\partial^\Omega F(\tau_j, k_0 + k)} \cap \overline{\partial^\Omega G^+[\bar r, s, t_0, t_0 + k\tau_j]} \ne \emptyset. \tag{5.9}$$

By the minimality of $k \ge 1$,

$$F(\tau_j, k_0 + k - 1) \subset G^+[\bar r, s, t_0, t_0 + (k-1)\tau_j], \quad F(\tau_j, k_0 + k) \subset G^+[\bar r, s, t_0, t_0 + k\tau_j].$$

Moreover, by construction, $G^+[\bar r, s, t_0, t_0 + k\tau_j]$ satisfies the contact angle condition with $\beta - s$ and by Proposition 4.13 applied with $\tau = \tau_j \in (0, \tau_0(s))$:

$$\frac{\mathrm{sd}_{G^+[\bar r, s, t_0, t_0 + (k-1)\tau_j]}(x)}{\tau_j} + \kappa_{G^+[\bar r, s, t_0, t_0 + k\tau_j]}(x) + \frac{1}{\tau}\int_{k\tau}^{(k+1)\tau} f(s, x)ds > \frac{s}{2}$$

for all $x \in \partial^\Omega G^+[\bar r, s, t_0, t_0 + k\tau_j]$. However, in view of Proposition 3.3 (a), these properties imply $F(\tau_j, k_0 + k) \prec G^+[\bar r, s, t_0, t_0 + k\tau_j]$, which contradicts (5.9). Thus, $\bar r = 0$. Analogous contradiction argument based on Proposition 3.3 (b) shows $G^-[0, s, t_0, t_0 + k\tau_j] \subset F(\tau_j, k_0 + k)$ for all $0 \le k \le \lfloor \bar t/\tau_j \rfloor$.

Finally, let us prove (5.4). Recall that by construction $G_0^-[0,2s,t_0] \prec G_0^-[0,s,t_0]$ and $G_0^+[0,s,t_0] \prec G_0^+[0,2s,t_0]$; therefore, by the strong comparison principle (Theorem 4.9), $G^-[0,2s,t_0,t] \prec G^-[0,s,t_0,t]$ and $G^+[0,s,t_0,t] \prec G^+[0,2s,t_0,t]$ for all $t \in [t_0,T]$. Now, by the continuity of $G^\pm[0,s,t_0,t]$ on its parameters, we could find $\bar{j} = \bar{j}(s) > 1$ such that, for all $j > \bar{j}$,

$$\begin{aligned} G^-[0,2s,t_0,t_0+\bar{t}\,] &\prec G^-[0,s,t_0,t_0+\bar{k}_j\tau_j] \\ &\subset F(\tau_j,\bar{k}_j) \subset G^+[0,s,t_0,t_0+\bar{k}_j\tau_j] \\ &\prec G^+[0,2s,t_0,t_0+\bar{t}\,]. \end{aligned} \tag{5.10}$$

By the definition of g,

$$\max_{x\in\partial^\Omega G^\pm[0,2s,t_0,t_0+\bar{t}\,]} \operatorname{dist}(x,\partial^\Omega E(t_0+\bar{t}\,)) \le g(2s), \tag{5.11}$$

and therefore, by construction in Corollary 4.11 (a),

$$\operatorname{dist}(\partial^\Omega G_0^\pm[0,4g(2s),t_0+\bar{t}\,],\partial^\Omega E(t_0+\bar{t}\,)) = 4g(2s) > 0.$$

Combining this with (5.11) and the construction of $G_0^\pm$, we deduce

$$G_0^-[0,4g(2s),t_0+\bar{t}\,] \prec G^-[0,2s,t_0,t_0+\bar{t}\,]$$

and

$$G^+[0,2s,t_0,t_0+\bar{t}\,] \prec G_0^+[0,4g(2s),t_0+\bar{t}\,].$$

These inclusions together with (5.10) imply (5.4). ∎

Now, we are ready to prove the equality (5.2). Let $\bar{t}$ be given by Lemma 5.1 as follows:

$$N := \lfloor T/\bar{t}\rfloor + 1,$$

and let $\sigma_0 \in (0,\sigma/16)$ be such that the numbers

$$\sigma_l = 4g(2\sigma_{l-1}), \quad l = 1,\dots,N,$$

satisfy $\sigma_l \in (0,\sigma/16)$. By the monotonicity and continuity of g together with $g(0)=0$, such choice of σ_0 is possible.

Fix any $s \in (0,\sigma_0)$ and let

$$a_0(s) := s, \quad a_l(s) := 4g(2a_{l-1}(s)), \quad l = 1,\dots,N.$$

Note that $a_l(s) \in (0,\sigma_l)$. In particular, the numbers $\bar{j}_l^s := \bar{j}(a_l(s))$, given by the last assertion of Lemma 5.1, are well defined. Let also

$$\tilde{j}_l^s := \max\{j \ge 1 : \tau_j \notin (0,\tau_0(a_l(s)))\}$$

and

$$\bar{j}_s := 1 + \max_{l=0,\dots,N} \max\{\bar{j}^s_l, \tilde{j}^s_l\}.$$

By Corollary 4.11 (a),

$$G^-_0[0,s,0] \subset E(0) = E_0 = F(\tau_j,0) \subset G^+_0[0,s,0]$$

for all $j > \bar{j}_s$. Therefore, by Lemma 5.1 applied with $k_0 = 0$ and $t_0 = 0$, we find

$$G^-[0,s,0,k\tau_j] \subset F(\tau_j,k) \subset G^+[0,s,0,k\tau_j], \quad k = 0,1,\dots,\bar{k}_j,$$

where $\bar{k}_j := \lfloor \bar{t}/\tau_j \rfloor$. Moreover, since $s \in (0,\sigma_0,)$ by the last assertion of Lemma 5.1,

$$G^-_0[0,a_1(s),\bar{t}\,] \subset F(\tau_j,\bar{k}_j) \subset G^+_0[0,a_1(s),\bar{t}\,]$$

for all $j \ge \bar{j}_s$. Hence, we can reapply Lemma 5.1 with $s := a_1(s)$, $t_0 = \bar{t}$, and $k_0 = \bar{k}_j$ to find

$$G^-[0,a_1(s),0,\bar{t}+k\tau_j] \subset F(\tau_j,\bar{k}_j+k) \subset G^+[0,a_1(s),0,\bar{t}+k\tau_j], \quad k = 0,1,\dots,\bar{k}_j.$$

In particular, since $j > \bar{j}_s > \bar{j}(a_1(s))$, again by the last assertion of Lemma 5.1, we deduce

$$G^-_0[0,a_2(s),2\bar{t}\,] \subset F(\tau_j,2\bar{k}_j) \subset G^+_0[0,a_2(s),2\bar{t}\,].$$

Repeating this argument at most N times, for all $j \ge \bar{j}_s$, we find

$$\begin{aligned} G^-[0,a_l(s),0,l\bar{t}+k\tau_j] &\subset F(\tau_j,l\bar{k}_j+k) \\ &\subset G^+[0,a_l(s),0,l\bar{t}+k\tau_j], \quad k = 0,1,\dots,\bar{k}_j \end{aligned} \tag{5.12}$$

whenever $l = 0,\dots,N$ and $l\bar{t}+k\tau_j \le T$.

Now, take any $t \in (0,T)$, and let $l := \lfloor t/\bar{t} \rfloor$ and $k = \lfloor t/\tau_j \rfloor - l\bar{k}_j$ so that $l\bar{k}_j + k = \lfloor t/\tau_j \rfloor$. By means of l and k, as well as the definition of $\bar{k}_j$, we represent (5.12) as

$$\begin{aligned} &G^-\Big[0,a_l(s),0,l\bar{t}+\tau_j\Big\lfloor \frac{t}{\tau_j}\Big\rfloor - l\tau_j\Big\lfloor \frac{\bar{t}}{\tau_j}\Big\rfloor\Big] \\ &\quad \subset F\Big(\tau_j,\Big\lfloor \frac{t}{\tau_j}\Big\rfloor\Big) \subset G^+\Big[0,a_l(s),0,l\bar{t}+\tau_j\Big\lfloor \frac{t}{\tau_j}\Big\rfloor - l\tau_j\Big\lfloor \frac{\bar{t}}{\tau_j}\Big\rfloor\Big] \end{aligned} \tag{5.13}$$

for all $j > \bar{j}_s$. Since

$$\lim_{j\to+\infty} \Big(l\bar{t}+\tau_j\Big\lfloor \frac{t}{\tau_j}\Big\rfloor - l\tau_j\Big\lfloor \frac{\bar{t}}{\tau_j}\Big\rfloor\Big) = t,$$

by the continuous dependence of $G^\pm$ on its parameters, as well as the convergence (5.1) of the flat flows, letting $j \to +\infty$ in (5.13), we obtain

$$G^-[0,a_l(s),0,t] \subset F(t) \subset G^+[0,a_l(s),0,t], \tag{5.14}$$

where due to the L^1-convergence the inclusions in (5.1) are up to some negligible sets. Now, we let $s \to 0^+$ and recalling that $a_l(s) \to 0$ (by the continuity of g and assumption $g(0) = 0$), from (5.14), we deduce

$$G^-[0,0,0,t] \subset F(t) \subset G^+[0,0,0,t].$$

Then, by Theorem 4.12 (a),

$$F(t) = G^\pm[0,0,0,t] = E(t).$$

A. Some useful results

The following lemma extends analogous results in the Euclidean case [4, Sections 2 and 3].

Lemma A.1 (A priori estimates for capillary functional). *Let $\beta \in L^\infty(\partial\Omega)$. Then,*

(a) *for any $E \in BV(\Omega;\{0,1\})$,*

$$\frac{\Phi(\mathbf{e}_n) + \inf\beta}{2\Phi(\mathbf{e}_n)} P_\Phi(E) \le \mathcal{C}_\beta(E) \le \max\Big\{\frac{\sup\beta}{\Phi(\mathbf{e}_n)}, 1\Big\} P_\Phi(E); \tag{A.1}$$

(b) *$\mathcal{C}_\beta$ is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous if and only if $\|\beta\|_\infty \le \Phi(\mathbf{e}_n)$.*

Proof. (a) The upper bound is clear. To prove the lower bound, let

$$\beta_o := \inf\beta.$$

By the anisotropic minimality of the half-spaces (see, e.g., [6, Example 2.4]),

$$P_\Psi(E) \ge \Psi(\mathbf{e}_n) \int_{\partial\Omega} \chi_E \, d\mathcal{H}^{n-1} \tag{A.2}$$

for any anisotropy Ψ in $\mathbb{R}^n$. Thus, if $\beta_o \ge 0$, then by (A.2)

$$\begin{aligned} \mathcal{C}_\beta(E) &\ge \gamma P_\Phi(E,\Omega) + (1-\gamma) P_\Phi(E,\Omega) + \beta_o \int_{\partial\Omega} \chi_E \, d\mathcal{H}^{n-1} \\ &\ge \gamma P_\Phi(E,\Omega) + \Big(1 - \gamma + \frac{\beta_o}{\Phi(\mathbf{e}_n)}\Big)\Phi(\mathbf{e}_n) \int_{\partial\Omega} \chi_E \, d\mathcal{H}^{n-1}, \end{aligned}$$

and hence, choosing $\gamma = \frac{\Phi(\mathbf{e}_n)+\beta_o}{2\Phi(\mathbf{e}_n)}$, we deduce (A.1).

Assume that $\beta_o < 0$. Then, $\Psi := \frac{\Phi(\mathbf{e}_n)-\beta_o}{2\Phi(\mathbf{e}_n)}\Phi$ is an anisotropy. By (A.2),

$$\int_{\Omega\cap\partial^* E} \Psi(\nu_E) d\mathcal{H}^{n-1} \ge \Psi(\mathbf{e}_n) \int_{\partial\Omega} \chi_E \, d\mathcal{H}^{n-1} = \frac{\Phi(\mathbf{e}_n) - \beta_o}{2} \int_{\partial\Omega} \chi_E \, d\mathcal{H}^{n-1}.$$

Thus,

$$\begin{aligned}
\mathcal{C}_\beta(E) &= \int_{\Omega\cap\partial^*E} \frac{\Phi(\mathbf{e}_n)+\beta_o}{2\Phi(\mathbf{e}_n)}\Phi(\nu_E)d\mathcal{H}^{n-1} + \int_{\Omega\cap\partial^*E} \Psi(\nu_E)d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial\Omega} \beta\chi_E\, d\mathcal{H}^{n-1} \\
&\geq \frac{\Phi(\mathbf{e}_n)+\beta_o}{2\Phi(\mathbf{e}_n)} P_\Phi(E,\Omega) + \int_{\partial\Omega} \frac{\Phi(\mathbf{e}_n)-\beta_o+2\beta}{2}\chi_E\, d\mathcal{H}^{n-1} \\
&\geq \frac{\Phi(\mathbf{e}_n)+\beta_o}{2\Phi(\mathbf{e}_n)} P_\Phi(E).
\end{aligned}$$

(b) Repeat arguments of [4, Lemma 3.5]. ■

The following proposition provides a characterization of elliptic C^2-norms.

Proposition A.2. *For any $C^{k+\alpha}$-anisotropy Φ with $k \geq 2$ and $\alpha \in [0,1]$, the following assertions are equivalent:*

(a) *Φ is elliptic;*

(b) *there exists $\gamma > 0$ such that*

$$\nabla^2\Phi(x)y^T \cdot y^T \geq \frac{\gamma}{|x|} \quad \textit{for any } x \in \mathbb{R}^n \setminus \{0\},\ y \in \mathbb{S}^{n-1} \textit{ with } x\cdot y = 0;$$

(c) *there exists $\gamma > 0$ such that*

$$\nabla^2\Phi(x)y^T \cdot y^T \geq \frac{\gamma}{|x|} \quad \textit{for any } x \in \mathbb{R}^n \setminus \{0\},\ y \in \mathbb{S}^{n-1} \textit{ with } \nabla\Phi(x)\cdot y = 0;$$

(d) *there exists $\gamma > 0$ such that*

$$\nabla^2\Phi(x)y^T \cdot y^T \geq \frac{\gamma}{|x|}\Big|y - \Big(y\cdot\frac{x}{|x|}\Big)\frac{x}{|x|}\Big|^2 \quad \textit{for any } x \in \mathbb{R}^n \setminus \{0\},\ y \in \mathbb{R}^n;$$

(e) *for any segment $[x,y]$, lying on the line not passing through the origin, the second derivative of the function $t \mapsto h(t) := \Phi(x+t(y-x))$ is strictly positive in $[0,1]$;*

(f) *Φ^o is $C^{k+\alpha}$ and elliptic;*

(g) *the principal curvature of the boundary of W^Φ is strictly positive;*

(h) *there exists $r \in (0,1)$ such that for any $z \in \partial W^\Phi$ there exist $x_z, y_z \in \mathbb{R}^n$ such that*

$$B_r(x_z) \subset W^\Phi \subset \overline{B_{1/r}(y_z)} \quad \textit{and} \quad \partial B_r(x_z)\cap\partial W^\Phi = \partial B_{1/r}(y_z)\cap\partial W^\Phi = \{z\}.$$

Proof. Since Φ is C^2 and

$$\nabla^2\Phi(x)x^T = 0, \quad x \in \mathbb{R}^n \setminus \{0\},$$

the ellipticity of Φ is equivalent to the strict positivity of its Hessian $\nabla^2\Phi(x)$ on $T_x := \{y : x \cdot y = 0\}$. Thus, passing to local coordinates, one can show (a)$\Rightarrow$(g)$\Rightarrow$(h)$\Rightarrow$(g)$\Rightarrow$(a). Moreover, the assertions

$$\text{(a)}\Rightarrow\text{(b)}\Rightarrow\text{(c)}\Rightarrow\text{(b)}\Rightarrow\text{(d)}\Rightarrow\text{(e)}\Rightarrow\text{(b)}\Rightarrow\text{(a)}$$

follow directly from the definition of ellipticity.

Finally, let us show (a)$\Rightarrow$(f). Since ∂W^Φ does not contain segments and

$$\nabla\Phi(x)\cdot x = \Phi(x) \quad \text{and} \quad \Phi^o(\nabla\Phi(x)) = 1, \quad x \in \mathbb{R}^n,$$

Φ^o is differentiable on $\mathbb{R}^n \setminus \{0\}$. Hence, by convexity, Φ^o is C^1. The implication (g)$\Leftrightarrow$(h) follows from the fact that the second fundamental form of ∂W^Φ is bounded from below and from above by that of ball. Similar arguments can be used in the proof of the implication (a)$\Rightarrow$(h) using the strict positive definiteness of $\nabla^2\Phi(x)$ on T_x (in view of the convexity of $x \mapsto \Phi(x) - \gamma|x|$).

Finally, we prove (b)$\Rightarrow$(f). Since ∂W^Φ has no segments, Φ^o is differentiable in $\mathbb{R}^n \setminus \{0\}$, and by convexity, $\nabla\Phi^o$ is continuous. Since the map $x \in \partial W^\Phi \mapsto \nabla\Phi(x) \in \partial W^{\Phi^o}$ is a homeomorphism. By (b) and (c),

$$\nabla^2\Phi(x)y^T \cdot y^T \geq \frac{\gamma}{|x|}$$

for any $x \in \partial W^\Phi$ and $y \in \mathbb{S}^{n-1}$ with $x \cdot y = 0$. This implies that $\nabla^2\Phi$ maps the tangent plane of ∂W^Φ at x to the tangent plane of ∂W^{Φ^o} at $\nabla\Phi(x)$. Thus, by the inverse mapping theorem, the $\nabla\Phi$ is a $C^{k-1+\alpha}$-homeomorphism. In particular, ∂W^{Φ^o} is locally a $C^{k+\alpha}$-manifold, and hence, Φ^o is $C^{k+\alpha}$. Finally, to prove the ellipticity, it is enough to observe

$$\nabla^2\Phi^o(x)y^T \cdot y^T > 0$$

for any $x \in \partial W^{\Phi^o}$ and $y \in \mathbb{S}^{n-1}$ with $y \cdot \nabla\Phi^o(x) = 0$; thus, assertion (c) holds, and hence, Φ^o is also elliptic. ∎

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References

[1] F. Almgren, J. E. Taylor, and L. Wang, Curvature-driven flows: A variational approach. *SIAM J. Control Optim.* **31** (1993), no. 2, 387–438 Zbl 0783.35002 MR 1205983

[2] S. J. Altschuler and L. F. Wu, Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. *Calc. Var. Partial Differential Equations* **2** (1994), no. 1, 101–111 Zbl 0812.35063 MR 1384396

[3] G. Bellettini, A. Chambolle, and S. Kholmatov, Minimizing movements for forced anisotropic mean curvature flow of partitions with mobilities. *Proc. Roy. Soc. Edinburgh Sect. A* **151** (2021), no. 4, 1135–1170 Zbl 1471.53071 MR 4284515

[4] G. Bellettini and S. Y. Kholmatov, Minimizing movements for mean curvature flow of droplets with prescribed contact angle. *J. Math. Pures Appl. (9)* **117** (2018), 1–58 Zbl 1398.49044 MR 3841505

[5] G. Bellettini and S. Y. Kholmatov, Minimizing movements for mean curvature flow of partitions. *SIAM J. Math. Anal.* **50** (2018), no. 4, 4117–4148 Zbl 1396.53089 MR 3831241

[6] G. Bellettini, M. Novaga, and S. Y. Kholmatov, Minimizers of anisotropic perimeters with cylindrical norms. *Commun. Pure Appl. Anal.* **16** (2017), no. 4, 1427–1454 Zbl 1366.49045 MR 3637919

[7] J. Berthier and K. Brakke, *The physics of microdroplets*. John Wiley & Sons, New Jersey, 2012 Zbl 1347.76002

[8] X. Bian, Y. Giga, and H. Mitake, A level-set method for a mean curvature flow with a prescribed boundary. 2023, arXiv:2306.14218

[9] L. A. Caffarelli and A. Mellet, Capillary drops on an inhomogeneous surface. In *Perspectives in nonlinear partial differential equations*, pp. 175–201, Contemp. Math. 446, American Mathematical Society, Providence, RI, 2007 Zbl 1200.76053 MR 2373730

[10] A. Chambolle, D. D. Gennaro, and M. Morini, Minimizing movements for anisotropic and inhomogeneous mean curvature flows. [v1] 2022, [v2] 2023, arXiv:2212.05027v2, to appear in *Adv. Calc. Var.*

[11] A. Chambolle, M. Morini, M. Novaga, and M. Ponsiglione, Existence and uniqueness for anisotropic and crystalline mean curvature flows. *J. Amer. Math. Soc.* **32** (2019), no. 3, 779–824 Zbl 1420.53073 MR 3981988

[12] A. Chambolle, M. Morini, and M. Ponsiglione, Nonlocal curvature flows. *Arch. Ration. Mech. Anal.* **218** (2015), no. 3, 1263–1329 Zbl 1328.35166 MR 3401008

[13] A. Chambolle, M. Morini, and M. Ponsiglione, Existence and uniqueness for a crystalline mean curvature flow. *Comm. Pure Appl. Math.* **70** (2017), no. 6, 1084–1114 Zbl 1366.53047 MR 3639320

[14] U. Clarenz and H. von der Mosel, On surfaces of prescribed F-mean curvature. *Pacific J. Math.* **213** (2004), no. 1, 15–36 Zbl 1058.53003 MR 2040248

[15] E. De Giorgi, New problems on minimizing movements. In *Boundary value problems for partial differential equations and applications*, pp. 81–98, RMA Res. Notes Appl. Math. 29, Masson, Paris, 1993 Zbl 0851.35052 MR 1260440

[16] G. De Philippis and F. Maggi, Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.* **216** (2015), no. 2, 473–568 Zbl 1321.35158 MR 3317808

[17] G. De Philippis and F. Maggi, Dimensional estimates for singular sets in geometric variational problems with free boundaries. *J. Reine Angew. Math.* **725** (2017), 217–234 Zbl 1366.49057 MR 3630122

[18] M. C. Delfour and J.-P. Zolésio, *Shapes and geometries. Metrics, analysis, differential calculus, and optimization*. Second edn., Adv. Des. Control 22, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011 Zbl 1251.49001 MR 2731611

[19] R. Finn, *Equilibrium capillary surfaces*. Grundlehren Math. Wiss. 284, Springer, New York, 1986 Zbl 0583.35002 MR 0816345

[20] P.-G. D. Gennes, F. Brochard-Wyart, and D. Quéré, *Capillarity and wetting phenomena: Drops, bubbles, pearls, waves*. Springer, New York, 2004 Zbl 1139.76004

[21] Y. Giga, F. Onoue, and K. Takasao, A varifold formulation of mean curvature flow with Dirichlet or dynamic boundary conditions. *Differential Integral Equations* **34** (2021), no. 1-2, 21–126 Zbl 1474.35020 MR 4198540

[22] Y. Giga and M.-H. Sato, Generalized interface evolution with the Neumann boundary condition. *Proc. Japan Acad. Ser. A Math. Sci.* **67** (1991), no. 8, 263–266 Zbl 0786.35076 MR 1137924

[23] B. Guan, Mean curvature motion of nonparametric hypersurfaces with contact angle condition. In *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, pp. 47–56, A K Peters, Wellesley, MA, 1996 Zbl 0870.35048 MR 1417947

[24] S. Hensel and T. Laux, BV solutions for mean curvature flow with constant contact angle: Allen–Cahn approximation and weak-strong uniqueness. *Indiana Univ. Math. J.* **73** (2024), no. 1, 111–148 Zbl 1539.53108 MR 4721314

[25] G. Huisken, Nonparametric mean curvature evolution with boundary conditions. *J. Differential Equations* **77** (1989), no. 2, 369–378 Zbl 0686.34013 MR 0983300

[26] X. Jia, G. Wang, C. Xia, and X. Zhang, Alexandrov's theorem for anisotropic capillary hypersurfaces in the half-space. *Arch. Ration. Mech. Anal.* **247** (2023), no. 2, article no. 25 Zbl 1514.53104 MR 4562813

[27] M. Katsoulakis, G. T. Kossioris, and F. Reitich, Generalized motion by mean curvature with Neumann conditions and the Allen–Cahn model for phase transitions. *J. Geom. Anal.* **5** (1995), no. 2, 255–279 Zbl 0827.35003 MR 1341031

[28] L. Kreutz and B. Schmidt, A note on the Winterbottom shape. 2024, arXiv:2402.03281

[29] S. Luckhaus and T. Sturzenhecker, Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations* **3** (1995), no. 2, 253–271 Zbl 0821.35003 MR 1386964

[30] F. Maggi, *Sets of finite perimeter and geometric variational problems. An introduction to geometric measure theory*. Cambridge Stud. Adv. Math. 135, Cambridge University Press, Cambridge, 2012 Zbl 1255.49074 MR 2976521

[31] C. Mantegazza, *Lecture notes on mean curvature flow*. Progr. Math. 290, Birkhäuser/Springer Basel AG, Basel, 2011 Zbl 1230.53002 MR 2815949

[32] U. Massari and N. Taddia, Generalized minimizing movements for the mean curvature flow with Dirichlet boundary condition. *Ann. Univ. Ferrara Sez. VII (N.S.)* **45** (1999), 25–55 (2000) Zbl 1011.49030 MR 1802474

[33] L. Mugnai, C. Seis, and E. Spadaro, Global solutions to the volume-preserving mean-curvature flow. *Calc. Var. Partial Differential Equations* **55** (2016), no. 1, article no. 18 Zbl 1336.53082 MR 3455792

[34] V. I. Oliker and N. N. Uraltseva, Evolution of nonparametric surfaces with speed depending on curvature. III. Some remarks on mean curvature and anisotropic flows. In *Degenerate diffusions (Minneapolis, MN, 1991)*, pp. 141–156, IMA Vol. Math. Appl. 47, Springer, New York, 1993 Zbl 0794.35087 MR 1246345

[35] Sh. Kholmatov, On the minimality of the Winterbottom shape. 2024, arXiv:2402.02179

[36] Sh. Kholmatov, Consistency of minimizing movements with smooth mean curvature flow of droplets with prescribed contact-angle in $\mathbb{R}^3$. [v1] 2024, [v2] 2024, arXiv:2401.06307v2

[37] V. A. Solonnikov, On boundary value problems for linear parabolic systems of differential equations of general form. *Trudy Mat. Inst. Steklov.* **83** (1965), 3–163 Zbl 0164.12502 MR 0211083

[38] B. White, Mean curvature flow with boundary. *Ars Inven. Anal.* (2021), article no. 4 Zbl 1485.53109 MR 4462472

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