Ann. Inst. H. Poincaré C
Anal. Non Linéaire (Online first)
DOI 10.4171/AIHPC/140

Published by EMS Press

Mountain pass frozen planet orbits in the helium atom model

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Abstract. We seek frozen planet orbits for the helium atom through an application of the mountain pass lemma to the Lagrangian action functional. Our method applies to a wide class of gravitational-like interaction potentials, thus generalising the results in Cieliebak, Frauenfelder, and Volkov [Ann. Inst. H. Poincaré C Anal. Non Linéaire 40 (2023), 379–455]. We also let the charges of the two electrons tend to zero and perform an asymptotic analysis to prove convergence to a limit trajectory having a collision-reflection singularity between the electrons.

1. Introduction

Frozen planet orbits are motions of a one-dimensional helium atom model, where the nucleus is fixed and both electrons move on the same side of a line. These motions are periodic, in the sense that one electron keeps bouncing against the nucleus, whereas the second one slowly oscillates far from it (the "frozen planet"), running along a brake trajectory. The existence of these orbits has been the focus of recent mathematical literature (see [5–7,10]), also for its relevance in the semiclassical analysis of the helium atom model (cf, [9]). The quoted papers deal with the existence of periodic solutions of the following equations:

$$\begin{cases} \ddot{q}_1 = -\dfrac{2}{q_1^2} - \dfrac{1}{(q_2-q_1)^2}, \\ \ddot{q}_2 = -\dfrac{2}{q_2^2} + \dfrac{1}{(q_2-q_1)^2}. \end{cases} \tag{1.1}$$

This system models the behaviour of two collinear electrons (q_1, q_2) in the helium atom, assuming that the nucleus is fixed at the origin. Each of the two particles is subject to an attractive force $-\frac{2}{q_i^2}$ towards the nucleus, and to a repulsive force $\pm\frac{1}{(q_2-q_1)^2}$, which pushes the electrons apart. While [10] adopts a shooting method, [5–7] propose a variational approach to search for such orbits. Intrigued by the indirect and somewhat cumbersome approach pursued there, we wondered about the possibility of directly applying some critical point theorem, paving the way for the treatment of a broader class of models.

Mathematics Subject Classification 2020: 47J30 (primary); 34C25, 70F10, 34D15, 81V45 (secondary).
Keywords: mountain pass lemma, helium atom, frozen planet orbits.

Indeed, the aim of this paper is to study periodic solutions of the following system of one-dimensional second-order non-linear equations:

$$\begin{cases} \ddot{q}_1 = f'(|q_1|) + g'(|q_2 - q_1|), \\ \ddot{q}_2 = f'(|q_2|) - g'(|q_2 - q_1|), \end{cases} \tag{1.2}$$

where f and g are real functions, both having a singularity at the origin. Here, f represents the attraction force to the nucleus and g the repulsive interaction between the particles.

In our paper we will deal with a much larger class of potentials than the one in [5–7,10] and we will make the following natural assumptions on the functions $f, g\colon \mathbb{R}^+ \to \mathbb{R}^+$ involved in (1.2). We assume that $f, g \in C^2(\mathbb{R}^+)$ and

$$f(s), f'(s), g(s), g'(s) \to 0, \quad \text{as } s \to +\infty, \tag{1.3}$$

$$f'(s), g'(s) \leq 0 \text{ and } g''(s), f''(s) \geq 0 \ \forall s > 0, \tag{1.4}$$

$$\exists \alpha \in (0,2) \text{ s.t. } sf'(s) + \alpha f(s) \geq 0, sg'(s) + \alpha g(s) \leq 0 \ \forall s > 0, \tag{1.5}$$

$$\exists \bar{s} > 0 \text{ such that } 0 < g(\bar{s}) < f(\bar{s}). \tag{1.6}$$

Note that potentials $f(s) = a/s^\alpha$, $g(s) = b/s^\beta$ always fulfil these assumptions provided $\alpha \in (0,2)$, $\beta \geq \alpha$ and, whenever $\alpha = \beta$, $b < a$. In particular, system (1.1) also does. The class of admissible potentials, however, is clearly much broader.

Assumption (1.5) is a classical homogeneity condition already present in the literature (see e.g. [1,2]). Condition (1.6) ensures that, independently of the initial position of the electrons, the attractive singularity prevails over the repulsive one, when the outer electron q_2 is far enough.

Notice that the first equation of (1.2) may be singular at the origin, even if with a weak force, since $\alpha \in (0,2)$. Therefore, as q_1 has one degree of freedom, *any bounded solution* has a collision with the origin. Thus, we will construct *generalised* periodic solutions, for which we allow collisions between the first electron and the nucleus. For a given $T > 0$, we seek solutions (q_1, q_2) of (1.2) satisfying $q_1(t) < q_2(t)$ for all $t \in [0, T]$ and such that

$$\begin{cases} q_1(0) = 0 = \dot{q}_2(0), \\ \dot{q}_1(T) = 0 = \dot{q}_2(T). \end{cases} \tag{1.7}$$

Thus, extending the trajectory in an even way as $(x_1, x_2)\colon [-T, T] \to \mathbb{R}^2$,

$$(x_1(t), x_2(t)) = \begin{cases} (q_1(-t), q_2(-t)) & \text{if } t \in [-T, 0], \\ (q_1(t), q_2(t)) & \text{if } t \in (0, T], \end{cases}$$

we obtain a brake generalised periodic solution of period $2T$, with a unique collision at $t = 0$. Such a solution will be called a *frozen planet orbit of period* $2T$. Our main results are the following:

Theorem 1.1. *For any functions f, g satisfying assumptions* (1.3)–(1.6) *and for any $T > 0$, there exists a* frozen planet orbit, *i.e. a generalised solution of* (1.2) *defined in $[-T, T]$ and periodic of period $2T$, which satisfies the conditions given by* (1.7) *and has no other singularities than the one at $t = 0$.*

Remark 1.2. Of course, this theorem applies also to (1.1). See also Corollary 3.5 further ahead for the fixed energy problem.

In the last section of the paper we investigate the behaviour of (1.2) in the case in which the repulsive singularity is damped. To model this situation, we introduce a parameter $\mu \in [0, 1]$ and consider the following singularly perturbed system:

$$\begin{cases} \ddot{q}_1 = f'(|q_1|) + \mu g'(|q_2 - q_1|), \\ \ddot{q}_2 = f'(|q_2|) - \mu g'(|q_2 - q_1|). \end{cases} \tag{1.8}$$

When $\mu = 0$ we see that (1.8) decouples into two Kepler-like equations. We shall prove the following result:

Theorem 1.3. *As $\mu \to 0$ theμ-frozen planet orbits constructed in Theorem* 1.1 *converge uniformly on $[-T, T]$ and $\mathcal{C}^2_{\mathrm{loc}}([-T, 0) \cup (0, T])$ to a function $q = (q_1, q_2)$. Each of the components of q_i is a segment of the brake orbit of period $2T$ of the f-Kepler problem:*

$$\ddot{x} = f'(x).$$

Moreover, $q_1(0) = 0$, $\dot{q}_2(0) = 0$, $\dot{q}_1(\pm T) = -\dot{q}_2(\pm T) > 0$ and $q_1(T) = q_2(T)$.

Remark 1.4. Theorem 1.3 can be framed in the context of *degenerate billiards* introduced in [4] (see also [8] and references therein). In our case, the singularity set is the positive diagonal $\{(q_1, q_2) \in \mathbb{R}^2 : q_1, q_2 \geq 0,\ q_1 = q_2\}$ and the billiard trajectory is the periodic trajectory obtained by *regularising* the collision at $q_1 = 0$ of the limit curve q obtained in Theorem 1.3. A possible application to (1.8) of the results in [4], in order to prove existence of periodic solutions for small values of μ, would involve regularising the singularity of f and verifying an appropriate non-degeneracy condition on the limiting profile. Then trajectories of (1.8) could be obtained as perturbations of trajectories of two decoupled Kepler problems. Consequently, according with the technique proposed there, it could be possible to construct branches of solutions of (1.8) emanating from billiard trajectories reflecting on the singularity set. However, although we believe it possible, in this paper we will not use the perturbative approach to prove the existence of solutions for μ small, since we do develop a global variational approach, valid for all values of μ compatible with our assumptions. Indeed, thanks to our global approach, we show this one-parameter family of solutions can be continued for values of μ belonging to the whole interval $(0, 1]$.

As stated, the approach we follow in this paper is variational. We wish to characterise solutions of (1.2) as critical points of the Lagrangian action functional $\mathcal{A}$ defined on the open set $\mathcal{U} = \{(q_1, q_2) \in H^1([0, T], \mathbb{R}^2) : q_1 \neq q_2\}$ of the Hilbert space $H^1([0, T], \mathbb{R}^2)$.

Indeed, non-collision solutions of (1.2) correspond to critical points of

$$\mathcal{A}(q_1, q_2) = \int_0^T \frac{1}{2}(|\dot{q}_1|^2 + |\dot{q}_2|^2) + f(|q_1|) + f(|q_2|) - g(|q_2 - q_1|).$$

Unfortunately, we must cope with the presence of singularities on both f and g, so the notion of critical point must be suitably generalised. Collisions of the first electron with the nucleus, in particular, cannot be avoided since, as already mentioned, solutions of (1.2) with initial conditions (1.7) are expected to have collisions. In order to deal with this problem, we introduce a family of modified functionals $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ depending on two small parameters $\varepsilon_1, \varepsilon_2 > 0$. They are defined as

$$\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q_1, q_2) = \int_0^T \frac{1}{2}(|\dot{q}_1|^2 + |\dot{q}_2|^2) + f_{\varepsilon_1}(q_1) + f_{\varepsilon_1}(q_2) - g_{\varepsilon_2}(|q_1 - q_2|),$$

where f_{ε_1} is a smooth function which approximates the singularity at the origin, while g_{ε_2} is a penalisation of g with a *strong force* term concentrated in an ε_2-neighbourhood of collisions (precise definitions are given in Section 2.1, and their main properties are listed in Lemma 2.1).

For any $\varepsilon_1, \varepsilon_2$ small enough, we will seek critical points of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ in

$$\mathcal{D} := \{(q_1, q_2) \in \mathcal{U} : q_1(0) = 0,\ q_1 < q_2\}.$$

From a variational perspective, such critical points are always saddles, and existence is proved via a slight variant of the *mountain pass lemma* (see Lemma B.1). Thanks to some suitable a priori estimates on the energy and on the H^1 norms of solutions, we prove that up to subsequence they converge uniformly to a solution of (1.2) satisfying (1.7). Compared with the variational approach developed in [6,7], ours has the advantage of avoiding the regularisation argument and, being more direct, of giving more information on the action level of the solutions, which will come in handy in the analysis of the asymptotics as $\mu \to 0$. Finally, we believe it is amenable to an extension to the multi-electron case, which we will deal with in a subsequent study.

The structure of the paper is the following. We first introduce the smoothings f_{ε_1} and g_{ε_2} and then verify the hypotheses of Lemma B.1. In Section 2 we prove the existence of frozen planet orbits as critical points for $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$. In Section 3 we prove some a priori bounds on solutions and some of their qualitative properties. Finally, Section 4 is devoted to the proof of Theorem 1.3.

2. Existence of critical points for $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$

In this section we introduce, for $\varepsilon_1, \varepsilon_2 > 0$ small enough, the smoothed functional $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$. After that, we establish the existence of *frozen planet orbits* for each $\varepsilon_1, \varepsilon_2$ via a mountain pass lemma.

2.1. Construction and properties of f_{ε_1} and g_{ε_2}

We start by introducing the functions f_{ε_1}. They approximate the function f, smoothing the Keplerian singularity at 0. Let $p_{\varepsilon_1,f}$ be the line tangent to f at $s = \varepsilon_1$, namely,

$$p_{\varepsilon_1,f}(s) = f(\varepsilon_1) + f'(\varepsilon_1)(s - \varepsilon_1).$$

Let $\tilde{p}_{\varepsilon_1,f}$ be the degree-two polynomial having the same value and derivative at 0 as $p_{\varepsilon_1,f}$ and a maximum in $-\varepsilon_1$, which reads

$$\tilde{p}_{\varepsilon_1,f}(s) = f(\varepsilon_1) - f'(\varepsilon_1)\varepsilon_1 + f'(\varepsilon_1)s + \frac{f'(\varepsilon_1)}{2\varepsilon_1}s^2.$$

Also define f_{ε_1} as follows:

$$f_{\varepsilon_1}(s) = \begin{cases} f(s) & \text{if } s \geq \varepsilon_1, \\ p_{\varepsilon_1,f}(s) & \text{if } s \in [0, \varepsilon_1], \\ \tilde{p}_{\varepsilon_1,f}(s) & \text{if } s \in [-\varepsilon_1, 0], \\ \tilde{p}_{\varepsilon_1,f}(-\varepsilon_1) & \text{if } s \leq -\varepsilon_1. \end{cases} \tag{2.1}$$

Moreover, consider a smooth monotone decreasing function $\psi_{\varepsilon_2}\colon \mathbb{R}_0^+ \to \mathbb{R}$ such that

(1) $\psi_{\varepsilon_2}|_{[\varepsilon_2,+\infty)} \equiv 0$;
(2) $\psi_{\varepsilon_2}(0) = \frac{1}{2}$;
(3) ψ_{ε_2} is convex on $[0, \varepsilon_2]$.

Then we define g_{ε_2} as

$$g_{\varepsilon_2}(s) = g(s) + \frac{\psi_{\varepsilon_2}(s)}{s^2},$$

where the second addendum is a *strong force* perturbation of the original repulsive singularity. In the following result we collect the main properties of the functions f_{ε_1} and g_{ε_2}.

Lemma 2.1. *For all $\varepsilon_1, \varepsilon_2 > 0$, the following properties hold:*

(1) $f_{\varepsilon_1} \in C^{1,1}(\mathbb{R})$*;*
(2) f_{ε_1} *is monotone decreasing on* $\mathbb{R}$ *and convex on* $[0, +\infty)$*;*
(3) $sf'_{\varepsilon_1}(s) + \alpha f_{\varepsilon_1}(s) \geq 0$ *for all s;*
(4) g_{ε_2} *is monotone decreasing on* $(0, +\infty)$ *and convex;*
(5) $sg'_{\varepsilon_2}(s) + \alpha g_{\varepsilon_2}(s) \leq 0$ *for all* $s > 0$.

Proof. Properties (1) and (2) follow by construction. Let us prove point (3). The inequality holds on $[\varepsilon_1, \infty)$ since f_{ε_1} coincides with f and on $(-\infty, -\varepsilon_1]$ since f_{ε_1} is constant. When $s \in [0, \varepsilon_1]$, f_{ε_1} coincides with p_{ε_1}, so it is a consequence of assumptions (1.4)–(1.5) on f.

On the other hand, when $s \in [-\varepsilon_1, 0]$, a direct computation shows that

$$(\alpha \tilde{p}_{\varepsilon_1,f}(s) + s\tilde{p}'_{\varepsilon_1,f}(s))' = (\alpha+1)\tilde{p}'_{\varepsilon_1,f}(s) + s\tilde{p}''_{\varepsilon_1,f}(s)$$
$$= (\alpha+1)f'(\varepsilon_1) + (\alpha+2)\frac{f'(\varepsilon_1)}{\varepsilon_1}s,$$

and so the maximum of $\alpha \tilde{p}_{\varepsilon_1,f}(s) + s\tilde{p}'_{\varepsilon_1,f}(s)$ is achieved at $\bar{s} = -\frac{\alpha+1}{\alpha+2}\varepsilon_1$, which is an internal point. Since $\alpha \tilde{p}_{\varepsilon_1,f}(s) + s\tilde{p}'_{\varepsilon_1,f}(s)$ is greater than the minimum of its values at 0 and $-\varepsilon_1$, which are both positive, we are done.

Points (4) and (5) hold for g by assumption so we have to check them for $\frac{\psi_{\varepsilon_2}(s)}{s^2}$. A straightforward computation shows that for $s > 0$,

$$\left(\frac{\psi_{\varepsilon_2}}{s^2}\right)' = \frac{s\psi'_{\varepsilon_2} - 2\psi_{\varepsilon_2}}{s^3} \leq 0,$$
$$\left(\frac{\psi_{\varepsilon_2}}{s^2}\right)'' = \frac{s^2\psi''_{\varepsilon_2} - 4s\psi'_{\varepsilon_2} + 6\psi_{\varepsilon_2}}{s^4} \geq 0,$$

proving (4). Finally, note that proving (5) for $\frac{\psi_{\varepsilon_2}(s)}{s^2}$ is equivalent to showing that $s^\alpha \frac{\psi_{\varepsilon_2}(s)}{s^2} = s^{\alpha-2}\psi_{\varepsilon_2}(s)$ is non-increasing. This property is obvious, since both $s^{\alpha-2}$ and $\psi_{\varepsilon_2}(s)$ are non-negative and non-increasing for s positive. ∎

2.2. Mountain pass geometry

We will now consider the problem of finding critical points of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q_1,q_2)$ on the set $\mathcal{D} = \{(q_1,q_2) : q_1(0) = 0,\ q_1 < q_2\}$. The basic tool we will use is a mountain-pass-type theorem. In this section we show how to set up a mountain-pass-type geometry.

For every $\varepsilon_1 > 0$, denote by q_{ε_1} the brake orbit of the smoothed Kepler problem having period T. It coincides with the minimiser of:

$$\mathcal{F}_{\varepsilon_1}(q) = \int_0^T \frac{1}{2}|\dot{q}|^2 + f_{\varepsilon_1}(q) \tag{2.2}$$

on $\{q \in H^1([0,T]) : q(0) = 0\}$. Let a_{ε_1} be the value of $\mathcal{F}_{\varepsilon_1}(q_{\varepsilon_1})$ (see Lemma A.1 for details). Fix some constants $c, C \in \mathbb{R}$ such that $C > c > q_{\varepsilon_1}(T)$ and let us define the following family of continuous paths $\gamma\colon [0,1] \to \mathcal{D}$:

$$\Gamma_c^C(\varepsilon_1) = \left\{\gamma\colon [0,1] \to \mathcal{D} \text{ such that } \gamma(0) = (q_{\varepsilon_1}, c) \text{ and } \gamma(1) = (q_{\varepsilon_1}, C)\right\}.$$

To ease the notation, we will write γ_s in place of $\gamma(s)$. Let us prove the following:

Lemma 2.2. *For ε_1, ε_2 small enough, there exist $\delta_1 > \delta_2 > 0$ and $\bar{c}, \bar{C} \in (q_{\varepsilon_1}(T), +\infty)$ such that, for every $c \in (q_{\varepsilon_1}(T), \bar{c})$, $C \geq \bar{C}$ and for any path $\gamma \in \Gamma_c^C(\varepsilon_1)$, we have*

$$\max\{\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_0), \mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_1)\} \leq a_{\varepsilon_1} + \delta_2, \qquad \max_{s\in[0,1]} \mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_s) > a_{\varepsilon_1} + \delta_1.$$

Proof. The first step is to identify the values of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_0)$ and $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_1)$. Recall that $f_{\varepsilon_1}(s) = f(s)$ whenever $s > \varepsilon_1$. For $c > \varepsilon_1$, we have that

$$\begin{aligned}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q_{\varepsilon_1}, c) &= \int_0^T \frac{1}{2}\dot{q}_{\varepsilon_1}^2 + f_{\varepsilon_1}(q_{\varepsilon_1}) + f_{\varepsilon_1}(c) - g_{\varepsilon_2}(c - q_{\varepsilon_1}) \\ &= \mathcal{F}_{\varepsilon_1}(q_{\varepsilon_1}) + \int_0^T f(c) - g_{\varepsilon_2}(c - q_{\varepsilon_1}).\end{aligned}$$

Since q_{ε_1} solves the equation $\ddot{q}_{\varepsilon_1} = f'_{\varepsilon_1}(q_{\varepsilon_1})$ on $[0, T]$, we can compute its Taylor expansion at $t = T$. Recalling that $\dot{q}_{\varepsilon_1}(T) = 0$, for ε_1 sufficiently small this yields

$$q_{\varepsilon_1}(t) = q_{\varepsilon_1}(T) + f'(q_{\varepsilon_1}(T))(T - t)^2 + O((T - t)^3).$$

Thus, $c - q_{\varepsilon_1}(t) = c - q_{\varepsilon_1}(T) + f'(q_{\varepsilon_1})(T - t)^2$. By definition of g_{ε_2}, the function $g_{\varepsilon_2}((T - t)^2)$ is not in L^1. It follows that

$$\liminf_{c\to q_{\varepsilon_1}(T)} \int_0^T g_{\varepsilon_2}(c - q_{\varepsilon_1}(t))\,dt \geq \int_0^T g_{\varepsilon_2}(q_{\varepsilon_1}(T) - q_{\varepsilon_1}(t))\,dt = +\infty,$$

and so

$$\lim_{c\to q_{\varepsilon_1}(T)} \mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_0) = -\infty.$$

Concerning the other end point, for C large enough we have

$$\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_1) = \mathcal{A}_{\varepsilon_1,\varepsilon_2}(q_{\varepsilon_1}, C) = a_{\varepsilon_1} + \int_0^T f(C) - g(C - q_{\varepsilon_1}).$$

Since both f and g go to zero at infinity thanks to (1.3), $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_1)$ is arbitrarily close to a_{ε_1}. This concludes the proof of the first claim of our statement.

Now we have to show that the maximum of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ along any path is greater than $a_{\varepsilon_1} + \delta_1$ for some δ_1. For $d > 0$, let us consider the set of paths $(q_1, q_2) \in \mathcal{D}$ at distance at least d:

$$\mathcal{V}_d = \{(q_1, q_2) \in \mathcal{D} : \min_{t\in[0,T]}(q_2(t) - q_1(t)) = d\}.$$

Since any path $\gamma \in \Gamma_c^C(\varepsilon_1)$ joins (q_{ε_1}, c) to (q_{ε_1}, C), by continuity, it has to cross $\mathcal{V}_d$ as soon as $d < C - q_{\varepsilon_1}(T)$. Note that, since C can be taken arbitrarily large, d can also be. The next step is to estimate from below, for d sufficiently large, the quantity

$$m_{\varepsilon_1} \doteq \min_{q\in\mathcal{V}_d} \mathcal{A}_{\varepsilon_1,\varepsilon_2}(q).$$

Without loss of generality, let us assume that $d > \max\{\varepsilon_1, \varepsilon_2, \bar{s}\}$, where $\bar{s}$ was introduced in (1.6). Since $\min_{t\in[0,T]}(q_2(t) - q_1(t)) = d$, we have $g_{\varepsilon_2}(q_2 - q_1) = g(q_2 - q_1)$ and hence m_{ε_1} does not actually depend on ε_2.

First of all notice that, as d grows, the first component q_1 of any minimiser must remain bounded. Indeed, setting $D = d + q_{\varepsilon_1}(T)$, testing $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ on (q_{ε_1}, D) and recalling that $a_{\varepsilon_1} \leq a_0$ and $g \geq 0$, we have

$$\begin{aligned} m_{\varepsilon_1} \leq \mathcal{A}_{\varepsilon_1,\varepsilon_2}(q_{\varepsilon_1}, D) &= a_{\varepsilon_1} + \int_0^T f(D) - g(D - q_{\varepsilon_1}(t)) \\ &\leq a_0 + Tf(D) \leq a_0 + Tf(\bar{s}). \end{aligned}$$

Moreover, for any $q = (q_1, q_2) \in \mathcal{V}_d$ we have $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q) \geq \frac{1}{2}\|\dot{q}_1\|_2^2 - T\, g(d)$ and so, for minimisers in $\mathcal{V}_d$, $\|\dot{q}_1\|_2$ and $|q_1|$ are bounded independently of d, thanks to our choice $d > \bar{s}$. In particular, for any minimiser q, note that

$$\|q_1\|_\infty \leq \sqrt{2T}\sqrt{a_0 + T(f(\bar{s}) + g(\bar{s}))} \doteq \gamma.$$

Let us assume that the quantity $f(q_2) - g(q_2 - q_1) \leq 0$ for some $q_1 \leq \gamma$, $q_2 \geq d$ and $q_2 - q_1 \geq d$. Thanks to assumption (1.5)–(1.6), we have the following two inequalities on f and g for $s \geq \bar{s}$:

$$f(s) \geq f(\bar{s})\Big(\frac{\bar{s}}{s}\Big)^\alpha, \quad g(s) \leq g(\bar{s})\Big(\frac{\bar{s}}{s}\Big)^\alpha.$$

As soon as $q_2 - q_1$ and q_2 are greater than $\bar{s}$, this implies that

$$f(q_2) - g(q_2 - q_1) \geq f(\bar{s})\Big(\frac{\bar{s}}{q_2}\Big)^\alpha - g(\bar{s})\Big(\frac{\bar{s}}{q_2 - q_1}\Big)^\alpha.$$

The left-hand side is negative, so setting $\nu_\alpha = (\frac{f(\bar{s})}{g(\bar{s})})^{\frac{1}{\alpha}} > 1$ we obtain the following inequality:

$$q_1 \geq \frac{\nu_\alpha - 1}{\nu_\alpha} q_2.$$

Therefore, further assuming $d > \frac{\gamma(2\nu_\alpha - 1)}{\nu_\alpha - 1}$, we find $q_1 > \gamma$, which is a contradiction.

At this point, it is enough to choose $d > \max\{\bar{s} + \gamma, \frac{\gamma(2\nu_\alpha - 1)}{\nu_\alpha - 1}\}$ so that, on minimisers, $f(q_2) > g(q_2 - q_1)$ and so $m_{\varepsilon_1} > a_{\varepsilon_1}$. Indeed, let $q = (q_1, q_2) \in \mathcal{V}_d$ be a minimiser; we have

$$\begin{aligned} \mathcal{A}_{\varepsilon_1,\varepsilon_2}(q) &\geq \min_{q_1 : q_1(0) = 0} \mathcal{F}_{\varepsilon_1}(q_1) + \int_0^T \frac{1}{2}|\dot{q}_2| + f(q_2) - g(q_2 - q_1) \\ &\geq a_{\varepsilon_1} + \int_0^T f(q_2) - g(q_2 - q_1). \end{aligned}$$

On the other hand,

$$\int_0^T f(q_2) - g(q_2 - q_1) \geq T \min_{t \in [0,T]} (f(q_2) - g(q_2 - q_1)) > 0,$$

proving the lemma. ∎

The mountain pass lemma also requires that the functional $\mathcal{A}$ is unbounded on the boundary $\partial\mathcal{D}$ (see Lemma B.1). Notice that

$$\partial\mathcal{D} = \big\{(q_1, q_2) \in H^1([0,T], \mathbb{R}^2) : q_1(0) = 0,\ q_1(t) = q_2(t),\ \text{for } t \in [0,T]\big\},$$

and, by construction, g_{ε_2} behaves like a strong force close to ∂D. The following lemma is a straightforward modification of a classical argument needed to show that, in the strong force case, the action blows up at collisions (see e.g. [3, Lemma 5.3]).

Lemma 2.3. *If $q \in \partial\mathcal{D}$ then $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q) = -\infty$.*

Proof. By construction, $g_{\varepsilon_2}(s) \geq \psi_{\varepsilon_2}(s)/s^2$. It follows that, for any $[t,s] \subseteq [0,T]$,

$$\int_0^T g_{\varepsilon_2}(q_2 - q_1) \geq \int_t^s \frac{\psi_{\varepsilon_2}(q_2 - q_1)}{(q_2 - q_1)^2}.$$

Let $q = (q_1, q_2) \in \partial\mathcal{D}$. Assume without loss of generality that $q_1(s) = q_2(s)$ and $q_1(w) < q_2(w)$ for all $w \in [t,s)$. Up to choosing a bigger t, we can assume that $\psi_{\varepsilon_2}(q_2 - q_1) \geq 1/4$. We have

$$\begin{aligned} |\log((q_2 - q_1)(w)) - \log((q_2 - q_1)(t))| &\leq \int_t^w \frac{|\dot{q}_2 - \dot{q}_1|}{q_2 - q_1} \\ &\leq \left(\int_t^w \frac{1}{(q_2 - q_1)^2}\right)^2 \|\dot{q}\|_2^2 \\ &\leq 16\left(\int_0^T g_{\varepsilon_2}(q_2 - q_1)\right)^2 \|\dot{q}\|_2^2. \end{aligned}$$

Taking the limit as $w \to s$ we obtain that $g_{\varepsilon_2} \notin L^1[0,T]$. Since f_{ε_1} is bounded we conclude that $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\partial\mathcal{D}) = -\infty$. ∎

2.3. Palais–Smale condition

It remains to show that the action functional $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ satisfies the Palais–Smale condition. Let us define the candidate critical value of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ as

$$c_{\varepsilon_1,\varepsilon_2} = \inf_{\gamma \in \Gamma_c^C(\varepsilon_1)} \max_{s \in [0,1]} \mathcal{A}_{\varepsilon_1,\varepsilon_2}(\gamma_s) > a_{\varepsilon_1}.$$

We recall that $(u_n) \subseteq \mathcal{D}$ is a Palais–Smale sequence, (PS) for short, at level $c_{\varepsilon_1,\varepsilon_2}$ if $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(u_n) \to c_{\varepsilon_1,\varepsilon_2}$ and $d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2} \to 0$ in the H^{-1} norm.

Proposition 2.4. *Any Palais–Smale sequence at level $c_{\varepsilon_1,\varepsilon_2}$ in $\mathcal{D}$ admits a strongly convergent subsequence.*

Proof. First of all, let us show that any Palais–Smale sequence is bounded in H^1. To ease the notation, for $q = (q_1, q_2) \in \mathcal{D}$, define the total potential

$$U(q) = f_{\varepsilon_1}(q_1) + f_{\varepsilon_1}(q_2) - g_{\varepsilon_2}(q_2 - q_1).$$

For a (PS) sequence $(u_n) = (q_1^n, q_2^n) \subseteq \mathcal{D}$ and a test function v, we can compute

$$d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(v) = \int_0^T \langle \dot{u}_n, \dot{v}\rangle + \langle \nabla U(u_n), v\rangle,$$
$$\nabla U(u_n) = \begin{pmatrix} f'_{\varepsilon_1}(q_1^n) \\ f'_{\varepsilon_2}(q_2^n) \end{pmatrix} + \begin{pmatrix} g'_{\varepsilon_2}(q_2^n - q_1^n) \\ -g'_{\varepsilon_2}(q_2^n - q_1^n) \end{pmatrix}.$$

In particular, choosing $v = u_n$ and using points (3) and (5) of Lemma 2.1, we have that

$$\int_0^T \left\langle \begin{pmatrix} f'_{\varepsilon_1}(q_1^n) \\ f'_{\varepsilon_2}(q_2^n) \end{pmatrix}, \begin{pmatrix} q_1^n \\ q_2^n \end{pmatrix} \right\rangle = \int_0^T f'_{\varepsilon_1}(q_1^n)q_1^n + f'_{\varepsilon_2}(q_2^n)q_2^n$$
$$\geq -\alpha \int_0^T f_{\varepsilon_1}(q_1^n) + f_{\varepsilon_1}(q_2^n)$$

and

$$\int_0^T \left\langle \begin{pmatrix} g'_{\varepsilon_2}(q_2^n - q_1^n) \\ g'_{\varepsilon_2}(q_2^n - q_1^n) \end{pmatrix}, \begin{pmatrix} q_1^n \\ q_2^n \end{pmatrix} \right\rangle = -\int_0^T g'_{\varepsilon_2}(q_2^n - q_1^n)(q_2^n - q_1^n)$$
$$\geq \alpha \int_0^T g_{\varepsilon_2}(q_2^n - q_1^n).$$

It follows that

$$d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(u_n) \geq \|\dot{u}_n\|_2^2 - \alpha \int_0^T U(u_n) = \frac{2+\alpha}{2}\|\dot{u}_n\|_2^2 - \alpha \mathcal{A}_{\varepsilon_1,\varepsilon_2}(u_n). \tag{2.3}$$

Since (u_n) is a (PS) sequence, we know that

$$\mathcal{A}_{\varepsilon_1,\varepsilon_2}(u_n) = \frac{1}{2}\|\dot{u}_n\|_2^2 + \int_0^T U(u_n) \to c_{\varepsilon_1,\varepsilon_2} > 0.$$

Assume by contradiction that $(\dot{u}_n)$ is unbounded. Combining with (2.3), for n large enough we obtain

$$d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(u_n) > 0.$$

Being a (PS) sequence also implies that $d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(u_n)/\|u_n\|_{H^1} \to 0$ and so we deduce that

$$\frac{\|\dot{u}_n\|_2^2}{\|u_n\|_{H^1}} \to 0. \tag{2.4}$$

Up to a subsequence, assume that $\|\dot{u}_n\|_2 \to +\infty$. In order for (2.4) to hold, we must have that

$$\frac{\|u_n\|_2}{\|\dot{u}_n\|_2^2} \to +\infty. \tag{2.5}$$

Since $q_1^n(0) = 0$, a Poincaré inequality holds for q_1^n and so $\|q_1^n\|_2 \leq \sqrt{T}\|\dot{q}_1^n\|_2$.

Note that a *modified* Poincaré inequality holds for q_2^n as well. Let us define the function $x_n(t) = q_2^n(T-t)$ which, for any $s, w \in [0,T]$, using the Jensen inequality, satisfies

$$(x_n(s) - x_n(w))^2 = \left(-\int_w^s \dot{q}_2^n(T-t)\right)^2 \leq \int_0^T (\dot{q}_2^n(T-t))^2.$$

Integrating with respect to s, this implies that

$$\|q_2^n - q_2^n(w)\|_2^2 = \int_0^T (q_2^n(s) - q_2^n(w))^2\, ds \leq T\|\dot{q}_2^n\|_2^2.$$

Thus, for any $w \in [0,T]$ and for a positive constant C depending only on T, we have obtained

$$\|q_2^n\|_2 \leq C(\|\dot{q}_2^n\|_2 + |q_2^n(w)|).$$

On the other hand, the unboundedness of $\|\dot{u}_n\|$ implies that there exists some $(w_n) \subseteq [0,T]$ such that $q_2^n(w_n) - q_1^n(w_n) \to 0$. Indeed, since both the value of the action and f_{ε_1} are bounded over the (PS) sequence, there exists a constant $C > 0$ such that

$$\left|\frac{1}{2}\|\dot{u}_n\|_2^2 - \int_0^T g_{\varepsilon_2}(q_2^n - q_1^n)\right| \leq C.$$

Thus $\|\dot{u}_n\|_2^2$ explodes if and only if q_1^n and q_2^n get closer and closer. Thus, for some constant $C_1 > 0$ we have

$$\|q_1^n\|_2 + \|q_2^n\|_2 \leq \sqrt{T}\|\dot{q}_1^n\|_2 + C(\|\dot{q}_2^n\|_2 + |q_2^n(w)|) \leq C_1(\|\dot{q}_1^n\|_2 + \|\dot{q}_2^n\|_2)$$

and so $\|u_n\|_2$ is controlled by $\|\dot{u}_n\|_2$, a contradiction with (2.5).

So far, we have shown that $\|\dot{u}_n\|_2$ and $\|q_1^n\|_2$ are bounded. To prove the H^1 boundedness of (PS) sequences we have to show that q_2^n is bounded in L^2 too. Let us assume by contradiction that this is not the case. There exists a sequence $c_n \to +\infty$ such that $q_2^n \geq c_n$ and this implies that

$$\nabla U(u_n) - \begin{pmatrix} f_{\varepsilon_1}'(q_1^n) \\ 0 \end{pmatrix} \to 0 \quad \text{uniformly as } n \to +\infty.$$

Testing $d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ on $(0, q_2^n - q_2^n(0))$, one finds that $\|\dot{q}_2^n\|_2 \to 0$ and so $q_2^n - q_2^n(0)$ converges strongly to 0.

Moreover, let $\mathcal{F}_{\varepsilon_1}$ be the functional defined in (2.2). Then, for any $v \in H^1$, $v(0) = 0$ we have

$$d_{q_1^n}\mathcal{F}_{\varepsilon_1}(v) = \langle \dot{q}_1^n, \dot{v}_1\rangle + \int_0^T f_{\varepsilon_1}'(q_1^n)v_1 \quad \text{and} \quad \|d_{q_1^n}\mathcal{F}_{\varepsilon_1}\| \to 0.$$

Thus, we have obtained that (q_1^n) is a (PS) sequence for $\mathcal{F}_{\varepsilon_1}$. Thanks to Lemma A.1, it converges to a brake orbit q_{ε_1} for a smoothed Kepler problem and so we conclude that

$$\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q_1^n, q_2^n) \to a_{\varepsilon_1} < c_{\varepsilon_1,\varepsilon_2},$$

a contradiction.

So far, we have proved that u_n is bounded in H^1 and that

$$\min_{t\in[0,T]} |q_1^n(t) - q_2^n(t)| > d_\varepsilon.$$

Up to a subsequence, u_n admits a weak limit u. In particular, $u_n \to u$ uniformly and in L^2. Thanks to uniform convergence and the bound on the distance between (q_1^n, q_2^n), dominated convergence implies that

$$\int_0^T \langle \nabla U(u_n), (u - u_n)\rangle \to 0.$$

By hypothesis, $d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2} \to 0$, and so we obtain strong convergence since

$$d_{u_n}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(u - u_n) = \langle \dot{u}_n, \dot{u} - \dot{u}_n\rangle + o(1) \to 0.$$ ■

2.4. Existence of critical points

We are now in a position to prove this result:

Proposition 2.5. *For any $\varepsilon_1, \varepsilon_2 > 0$ small enough and any $T > 0$, the functional $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ has a critical point at level $c_{\varepsilon_1,\varepsilon_2}$ in the set*

$$\mathcal{D} = \left\{(q_1, q_2) \in H^1([0,T];\mathbb{R}^2) : q_1 < q_2,\ q_1(0) = 0\right\}.$$

In particular, there exists a collisionless solution of

$$\begin{cases} \ddot{q}_1 = f'_{\varepsilon_1}(q_1) + g'_{\varepsilon_2}(q_2 - q_1), \\ \ddot{q}_2 = f'_{\varepsilon_1}(q_2) - g'_{\varepsilon_2}(q_2 - q_1), \end{cases} \tag{2.6}$$

satisfying $q_1(0) = \dot{q}_1(T) = 0$ and $\dot{q}_2(0) = \dot{q}_2(T) = 0$.

Proof. This is an application of Lemma B.1. In Lemma 2.2 we proved that $c_{\varepsilon_1,\varepsilon_2} > a_{\varepsilon_1}$, for suitable choices of C and c in the definition of $\Gamma_c^C(\varepsilon_1)$. Moreover, in Proposition 2.4 we showed that the (PS) condition holds at level $c_{\varepsilon_1,\varepsilon_2}$. Finally, Lemma 2.3 verifies that $\mathcal{A}_{\varepsilon_1,\varepsilon_2}(\partial\mathcal{D}) = -\infty$. ■

3. Existence of critical points for $\mathcal{A}$

In this section we exploit a limit argument in order to prove that our main problem (1.2) actually admits solutions. To this extent, the first step is to provide suitable a priori estimates on the H^1 norm and on the energy of each solution of (2.6).

3.1. A priori estimates and finer properties of solutions

To ease notation we will denote solutions of (2.6) by $q^\varepsilon = (q_1^\varepsilon, q_2^\varepsilon)$. In this first result, we detect some useful properties on the monotonicity of such solutions and their derivatives.

Lemma 3.1. *For every $\varepsilon_1, \varepsilon_2 > 0$, the following hold:*

(1) *q_1^ε and $q_2^\varepsilon + q_1^\varepsilon$ are concave. In particular, they are positive for all $t \le T$ and admit a maximum at T;*

(2) *$q_2^\varepsilon - q_1^\varepsilon$ is convex with a minimum at $t = T$;*

(3) *$|\dot q_2^\varepsilon| \le |\dot q_1^\varepsilon|$;*

(4) *q_1^ε is monotone increasing and q_2^ε is monotone decreasing.*

Proof. By the properties listed in Lemma 2.1 and (2.6), q_1^ε and $q_2^\varepsilon + q_1^\varepsilon$ are concave. Thus they are monotone with a maximum point at $t = T$ since $\dot q_1^\varepsilon(T) = 0 = \dot q_2^\varepsilon(T)$. Since f_{ε_1} is convex on $[0, \infty)$, f'_{ε_1} is increasing and $f'_{\varepsilon_1}(q_2) - f'_{\varepsilon_2}(q_1) \ge 0$ provided $q_2 \ge q_1$. It follows that $\ddot q_2^\varepsilon - \ddot q_1^\varepsilon \ge 0$ and $q_2^\varepsilon - q_1^\varepsilon$ is convex. By the boundary conditions, $t = T$ is a critical point and thus a minimum.

Assertion (3) follows from the fundamental theorem of calculus. Indeed,

$$\dot q_i^\varepsilon(t) = -\int_t^T \ddot q_i^\varepsilon(s)\,ds, \quad i = 1, 2,$$

and, observing that $\ddot q_1^\varepsilon + \ddot q_2^\varepsilon \le 0$ and $\ddot q_2^\varepsilon - \ddot q_1^\varepsilon \ge 0$, we conclude that $-\dot q_1^\varepsilon \le \dot q_2^\varepsilon \le \dot q_1^\varepsilon$.

The last assertion is proved as follows. We already know that q_1^ε is strictly increasing since it is strictly concave. Let us assume that there is a point $t^* < T$ which is a strict minimum for q_2^ε. In particular, $q_2^\varepsilon(t) > q_2^\varepsilon(t^*)$ for all $t > t^*$ small enough and $\ddot q_2^\varepsilon(t^*) \ge 0$. By Lemma 2.1, f'_{ε_1} is monotone increasing, and so

$$f'_{\varepsilon_1}(q_2^\varepsilon(t)) - f'_{\varepsilon_1}(q_2^\varepsilon(t^*)) \ge 0, \quad \text{if } q_2^\varepsilon(t) > q_2^\varepsilon(t^*).$$

Similarly, since $q_2^\varepsilon - q_1^\varepsilon$ is decreasing (and this follows from (2)), we have that $-g'_{\varepsilon_2}((q_2^\varepsilon - q_1^\varepsilon)(t)) \ge -g'_{\varepsilon_2}((q_2^\varepsilon - q_1^\varepsilon)(t^*))$ and so, plugging in (2.6), we find that $\ddot q_2^\varepsilon(t) \ge 0$ as long as $q_2^\varepsilon(t) \ge q_2^\varepsilon(t^*)$. Thus q_2 is convex on $[t^*, T]$ and there can be no stationary point at $t = T$. A contradiction. ∎

The following proposition deals with the boundedness of solutions of (2.6) and their energies.

Proposition 3.2. *Let $q^\varepsilon = (q_1^\varepsilon, q_2^\varepsilon)$ be a critical point of $\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ at level $c_{\varepsilon_1,\varepsilon_2}$ and let h_ε be the corresponding total energy value. Then q^ε is uniformly bounded in H^1 and h_ε is uniformly bounded in $\mathbb{R}$, for $\varepsilon_1, \varepsilon_2$ sufficiently small. Moreover, we have*

$$-\frac{c_{\varepsilon_1,\varepsilon_2}}{T} \le h_\varepsilon \le \frac{(\alpha - 2)\,c_{\varepsilon_1,\varepsilon_2}}{(2+\alpha)T} < 0.$$

Proof. Since q^ε is a critical point, $d_{q^\varepsilon}\mathcal{A}_{\varepsilon_1,\varepsilon_2} = 0$. In particular, we have that

$$d_{q_\varepsilon}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q^\varepsilon) = \|\dot q^\varepsilon\|_2^2 + \int_0^T \big(q_1^\varepsilon f'_{\varepsilon_1}(q_1^\varepsilon) + q_2^\varepsilon f'_{\varepsilon_1}(q_2^\varepsilon) - (q_2^\varepsilon - q_1^\varepsilon) g'_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon)\big)\, dt.$$

By points (3) and (5) of Lemma 2.1 we have that

$$0 = d_{q_\varepsilon}\mathcal{A}_{\varepsilon_1,\varepsilon_2}(q^\varepsilon) \geq \|\dot q^\varepsilon\|_2^2 - \alpha \int_0^T \big(f_{\varepsilon_1}(q_1^\varepsilon) + f_{\varepsilon_1}(q_2^\varepsilon) - g_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon)\big)\, dt.$$

Rewriting the inequality as in the proof of Proposition 2.4 (see in particular (2.3)), we obtain that

$$\|\dot q^\varepsilon\|_2^2 \leq \frac{2\alpha\, c_{\varepsilon_1,\varepsilon_2}}{2+\alpha}. \tag{3.1}$$

Clearly, $c_{\varepsilon_1,\varepsilon_2}$ is uniformly bounded in ε_1, ε_2 and so is $\|\dot q^\varepsilon\|_2^2$.

Reasoning as in Proposition 2.4, q_1^ε is bounded in H^1 if and only if $\dot q_1^\varepsilon$ is bounded in L^2. It remains to show that q_2^ε is bounded in L^2. Indeed, assume by contradiction that $q_2^\varepsilon(T) \to \infty$. Testing $d_{q^\varepsilon}\mathcal{A}_{\varepsilon_1,\varepsilon_2}$ on the variation $(0, q_2^\varepsilon)$, we find that $\dot q_2^\varepsilon$ goes to zero in L^2 and thus, as in Proposition 2.4, $c_{\varepsilon_1,\varepsilon_2}$ approaches the level of a brake orbit. A contradiction.

Let us prove the bound on the energy. Integrating over $[0, T]$ we have that

$$\begin{aligned} T h_\varepsilon &= \frac{1}{2}\|\dot q^\varepsilon\|_2^2 - \int_0^T U(q^\varepsilon)\, dt \\ &= \|\dot q^\varepsilon\|_2^2 - \mathcal{A}_{\varepsilon_1,\varepsilon_2}(q^\varepsilon) = \|\dot q^\varepsilon\|_2^2 - c_{\varepsilon_1,\varepsilon_2}. \end{aligned}$$

On the other hand, by equation (3.1), we have that

$$\frac{(\alpha-2)\, c_{\varepsilon_1,\varepsilon_2}}{2+\alpha} \geq \|\dot q^\varepsilon\|_2^2 - c_{\varepsilon_1,\varepsilon_2} = T h_\varepsilon.$$

Finally, we have that $h_\varepsilon \geq -\frac{1}{T} c_{\varepsilon_1,\varepsilon_2}$. ∎

We need to guarantee that we are not approaching a total collision, where both the electrons collapse into the nucleus. This is proved in the following.

Proposition 3.3 (No total collision). *For any $\varepsilon_1, \varepsilon_2 > 0$ there exist constants $C_1, C_2 > 0$ not depending on ε_1, ε_2 such that*

$$\|\dot q^\varepsilon\|_2 \geq C_1, \quad q_1^\varepsilon(T) \geq C_2.$$

Proof. Let us show that, if $\dot q_1^\varepsilon$ goes to zero in L^2, solutions q^ε of (2.6) converge uniformly to zero. Since $q_1^\varepsilon(0) = 0$, the Poincaré inequality shows that if $\dot q_1^\varepsilon$ converges to zero in L^2, then $q_1^\varepsilon(T)$ goes to zero as well. Moreover, from point (3) of Lemma 3.1, we easily see that $\dot q_2^\varepsilon$ converges to 0 in L^2 as well. Thus q_2^ε converges uniformly to a constant curve.

The energies of solutions are uniformly bounded by Proposition 3.2, and at T read

$$h_\varepsilon = -f_{\varepsilon_1}(q_1^\varepsilon(T)) - f_{\varepsilon_1}(q_2^\varepsilon(T)) + g_{\varepsilon_2}(q_2^\varepsilon(T) - q_1^\varepsilon(T));$$

therefore, $q_2^\varepsilon(T)$ must converge to zero, otherwise $h_\varepsilon \to -\infty$. So q_1^ε and q_2^ε converge uniformly to 0.

Having proved this claim, let us assume by contradiction that q_1^ε converges uniformly to 0. We have that

$$\dot{q}_1^\varepsilon(t) = \int_t^T -f'(q_1^\varepsilon) - g'_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon),$$

$$q_1^\varepsilon(T) = -\int_0^T \int_0^T \chi_{[t,T]}(s)(f'(q_1^\varepsilon)(s) + g'_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon)(s))\, ds\, dt,$$

where $\chi_{[t,T]}$ denotes the characteristic function of $[t, T]$. Applying the Fatou lemma we obtain that

$$0 \geq \int_0^T \int_0^T \liminf_{\varepsilon\to 0}\big(-\chi_{[t,T]}(s)(f'(q_1^\varepsilon)(s) + g'_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon)(s))\big)\, ds\, dt = +\infty,$$

which is clearly not possible. ∎

3.2. Existence of a solution of (1.2)

We are now equipped with all the tools and properties needed to show the main result of this section.

Theorem 3.4 (Convergence of q^ε). *For any ε_1, ε_2 small enough, there exists a subsequence of q^ε which converges in the C^2 topology on any compact subset $[\delta, T]$, with $\delta > 0$, to a solution q of*

$$\begin{cases} \ddot{q}_1 = f'(q_1) + g'(q_2 - q_1), \\ \ddot{q}_2 = f'(q_2) - g'(q_2 - q_1), \end{cases}$$

with energy h given by

$$h = -f(q_1) - f(q_2) + g(q_2 - q_1)|_{t=T} < 0,$$

satisfying $\dot{q}_2(0) = \dot{q}_1(T) = \dot{q}_2(T) = 0$ and $q_1(0) = 0$.

As a consequence, the function $(x_1, x_2)\colon [-T, T] \to \mathbb{R}^2$ defined as

$$(x_1(t), x_2(t)) = \begin{cases} (q_1(-t), q_2(-t)) & \text{if } t \in [-T, 0], \\ (q_1(t), q_2(t)) & \text{if } t \in (0, T], \end{cases}$$

is a frozen planet orbit *of period $2T$.*

Proof. Recall that the total energy of a solution q^ε is given by $h_\varepsilon = -U(q^\varepsilon(T)) < 0$ (see Proposition 3.2) where $U(q)$ stands for the total potential energy:

$$U(q) = f_{\varepsilon_1}(q_1) + f_{\varepsilon_1}(q_2) - g_{\varepsilon_2}(q_2 - q_1).$$

Since $q_1^\varepsilon(T)$ and $q_2^\varepsilon(T)$ are uniformly bounded away from 0, we can assume that ε_1 is so small that $f_{\varepsilon_1}(q_i^\varepsilon(T)) = f(q_i^\varepsilon(T))$. Since the total energy and the contribution of $f(q_1^\varepsilon(T)) + f(q_2^\varepsilon(T))$ are bounded, so is the contribution of $g_{\varepsilon_2}(q_2^\varepsilon(T) - q_1^\varepsilon(T))$. This implies that there exists a constant d, independent of $\varepsilon_1, \varepsilon_2$ (provided that they are small enough), such that $q_2^\varepsilon(T) - q_1^\varepsilon(T) \geq d$. Notice that, thanks to point (2) of Lemma 3.1, this implies that $q_2^\varepsilon - q_1^\varepsilon \geq d$. In particular, we have

$$h_\varepsilon = g(q_2^\varepsilon(T) - q_1^\varepsilon(T)) - f(q_1^\varepsilon(T)) - f(q_2^\varepsilon(T)) < 0.$$

By Proposition 3.2, q^ε admits a weakly convergent subsequence and thus converges uniformly and in L^2 to some limit function $q = (q_1, q_2)$. Since we have established that $q_2^\varepsilon - q_1^\varepsilon$ and q_2^ε are bounded from below by some positive constants, the right-hand side of (2.6) converges uniformly to the bounded function $f'(q_2) - g'(q_2 - q_1)$. This fact implies uniform convergence of $\dot{q}_2^\varepsilon$ too (by the Ascoli–Arzelà theorem).

Differentiating (2.6) we obtain

$$\dddot{q}_2^\varepsilon = f''_{\varepsilon_1}(q_2^\varepsilon)\dot{q}_2^\varepsilon - g''_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon)(\dot{q}_2^\varepsilon - \dot{q}_1^\varepsilon).$$

Again, $f''_{\varepsilon_1}(q_2^\varepsilon)$ and $g''_{\varepsilon_2}(q_2^\varepsilon - q_1^\varepsilon)$ converge uniformly. By Proposition 3.2, $\dot{q}_1^\varepsilon$, $\dot{q}_2^\varepsilon$ are bounded in L^2 and so we obtain that the $\ddot{q}_2^\varepsilon$ are equi-continuous and so they converge uniformly. Thus the limit q_2 is C^2 and a classical solution of (2.6).

Let us now consider the convergence of q_1^ε. As already mentioned, we have a uniform limit q_1. We have to show that $\dot{q}_1^\varepsilon$ converges uniformly on compact sets of the form $[\delta, T]$, for $\delta > 0$ small. To this aim, let us show that, for any $\delta > 0$ there exists $C > 0$ such that, for any $\varepsilon_1, \varepsilon_2$ sufficiently small, $q_1^\varepsilon|_{[\delta,T]} \geq C$. If that were not the case, there would be a subsequence $q_1^{\varepsilon_n}$ such that $q_1^{\varepsilon_n}(\delta) \to 0$. In particular, there would exist $(\omega_n) \subseteq [0, \delta]$ such that

$$\frac{q_1^{\varepsilon_n}(\delta) - q_1^{\varepsilon_n}(0)}{\delta} = \dot{q}_1^{\varepsilon_n}(\omega_n) > \dot{q}_1^{\varepsilon_n}(s) \geq 0,$$

for any $s > \delta$. Therefore, $\dot{q}_1^{\varepsilon_n}$ would converge uniformly to 0 on $[\delta, T]$ and, by the Poincaré inequality, $q_1^{\varepsilon_n}(T) \to 0$ as well, contradicting Proposition 3.3.

The claim we just proved implies that the right-hand side of (2.6) converges uniformly to the bounded function $f'(q_1) + g'(q_2 - q_1)$. Applying the Ascoli–Arzelà theorem again, we see that q_1 is C^2 on any $[\delta, T]$ and q_1^ε converges to q_1 in the C^2 topology. Thus q_1 is a classical solution of (2.6) on $[\delta, T]$, for any $\delta > 0$. ∎

We can make the result above specific to the case of homogeneous potentials. Let us take $f(s) = 1/s$ and $g(s) = \mu/s$ with $\mu \in (0, 1)$. We obtain the following result.

Corollary 3.5. *For any negative value of the energy h there exists a solution $q = (q_1, q_2)\colon [0,T] \to \mathbb{R}^2$ of*

$$\begin{cases} \ddot{q}_1 = -\dfrac{1}{q_1^2} - \dfrac{\mu}{(q_2-q_1)^2}, \\ \ddot{q}_2 = -\dfrac{1}{q_2^2} + \dfrac{\mu}{(q_2-q_1)^2}, \end{cases}$$

satisfying $\dot{q}_1(T) = q_1(0) = 0$ and $\dot{q}_2(T) = \dot{q}_2(0) = 0$.

As a consequence, the function

$$(x_1(t), x_2(t)) = \begin{cases} (q_1(-t), q_2(-t)) & \text{if } t \in [-T, 0], \\ (q_1(t), q_2(t)) & \text{if } t \in (0, T], \end{cases}$$

*is a μ-*frozen planet orbit *of period $2T$.*

Proof. We only need to show that this result holds for any negative value of the energy. Looking at the statement of Proposition 3.2 and observing that for these choices of f and g assumption (1.5) is satisfied with $\alpha = 1$, we get $h_\varepsilon < -c_{\varepsilon_1,\varepsilon_2}/T$. Thus also the energy of the limit obtained in Theorem 3.4 is negative.

Since now the potentials are homogeneous, scaling a solution as $\lambda^{-1}q(\lambda^{3/2}t)$ rescales the energy by a factor λ, yielding solutions for any negative value of the energy. ∎

4. Zero charge case

Take $\mu \in [0,1]$ and consider the ODE system

$$\begin{cases} \ddot{q}_1 = f'(q_1) + \mu g'(q_2 - q_1), \\ \ddot{q}_2 = f'(q_2) - \mu g'(q_2 - q_1). \end{cases}$$

Theorem 3.4 implies that, for any $\mu > 0$ there exists a solution $q^\mu = (q_1^\mu, q_2^\mu)$ having $q_1^\mu(0) = \dot{q}_1^\mu(T) = 0$ and $\dot{q}_2^\mu(0) = \dot{q}_2^\mu(T) = 0$. For $\mu = 0$, the system decouples and reduces to two independent f-Kepler problems:

$$\begin{cases} \ddot{q}_1 = f'(q_1), \\ \ddot{q}_2 = f'(q_2). \end{cases}$$

Let $\hat{q}$ be a brake orbit for the f-Kepler problem on $[0, 2T]$ having $\hat{q}(0) = 0$. Define a curve $q = (q_1, q_2)$ in $\mathbb{R}^2$ as

$$q(t) = (q_1(t), q_2(t)) = (\hat{q}(t), \hat{q}(2T - t)), \quad t \in [0, T]. \tag{4.1}$$

By construction, we have $\dot{q}_2(0) = q_1(0) = 0$ and $q_1(T) = q_2(T)$. The next proposition shows that, as the charge parameter μ tends to 0, there exists a sequence of solutions q^μ converging strongly to $\hat{q}$. Note that, applying this result to $q^\mu(t)$ and $q^\mu(-t)$, one finally obtains Theorem 1.3.

Theorem 4.1 (Convergence to segments of brake). *There exists a subsequence of q^μ which converges uniformly in the C^2 topology to the curve q given in (4.1) on any interval $[\delta, T-\delta]$, $\delta > 0$. Moreover, there exists a constant $C > 0$ such that*

$$q_2^\mu(T) - q_1^\mu(T) \geq C\sqrt[\alpha]{\mu}.$$

Proof. First let us observe that $q_2^\mu(T) - q_1^\mu(T) \to 0$ as $\mu \to 0$. Indeed, if that were not the case, we could find a subsequence of (q_2^μ) which would converge in the C^2 topology to a solution of $\ddot{x} = f'(x)$, having $\dot{x}(0) = \dot{x}(T) = 0$, which does not exist since f is strictly monotone decreasing. Indeed, if $f'(x_0) = 0$ for some x_0, then f is constant on the whole half-line and thus is zero, thanks to (1.3)–(1.4). However, as we have already observed in the proof of Lemma 2.2, $f(s) \geq f(\bar{s})(\frac{\bar{s}}{s})^\alpha > 0$, where $\bar{s}$ is introduced in (1.6).

Next, let us show that $\|\dot{q}^\mu\|_2^2$ is bounded in L^2 and consequently q^μ in H^1, since $q_1^\mu(0) = 0$ and $q_2^\mu(T) - q_1^\mu(T) \to 0$. Indeed, the value of the action on each q^μ is uniformly bounded in μ by some constant c due to the mountain pass structure (see the proofs of Propositions 3.2 and 2.4). Therefore,

$$0 = d_{q^\mu}\mathcal{A}_\mu(q^\mu) \geq \|\dot{q^\mu}\|_2^2 - \alpha\int_0^T U_\mu(q^\mu) \geq \left(1 + \frac{\alpha}{2}\right)\|\dot{q}^\mu\|_2^2 - \alpha c,$$

and so $(\dot{q}^\mu)$ is uniformly bounded in L^2. Up to a subsequence, we can assume that the sequence of solutions (q^μ) converges uniformly and in L^2 on $[0, T]$ to a function q.

Now we prove the C^2 convergence on intervals of the form $[\delta, T-\delta]$. Let us observe that q_1^μ are strictly concave and there exists a constant $C > 0$, independent on μ, such that $\ddot{q}_1^\mu \leq -C$. Moreover, thanks to point (4) of Lemma 3.1, q_2^μ is monotone decreasing and so

$$q_1^\mu(t) < q_1^\mu(T) \leq q_2^\mu(T) \leq q_2^\mu(t).$$

Thus, the sequence of functions $(q_2^\mu - q_1^\mu)$ cannot converge uniformly to zero on any subinterval of $[0, T]$. A slight modification of the argument given in Theorem 3.4 or in Proposition 3.3 shows that q_1^μ does not converge uniformly to 0 on $[0, \delta]$ either. Applying the Ascoli–Arzelà theorem on the intervals $[\delta, T-\delta]$, we obtain convergence in the C^2 topology to two solutions $q = (q_1, q_2)$ of $\ddot{x} = f'(x)$.

We have yet to show that q_i are brake orbits. Let us observe that $q_1^\mu + q_2^\mu$ solves the equation

$$\ddot{q}_1^\mu + \ddot{q}_2^\mu = f'(q_1^\mu) + f'(q_2^\mu).$$

The right-hand side is bounded on intervals $[\delta, T]$. This means that we can assume that $q_1^\mu + q_2^\mu$ converges to $q_1 + q_2$ in the C^2 topology on $[\delta, T]$. Thus $\dot{q}_1(T) = -\dot{q}_2(T)$. Finally, the same argument implies that q_2^μ converges uniformly in the C^2 topology on $[0, T-\delta]$ and so $\dot{q}_2(0) = 0$.

Thus, the curve $\hat{q}$ defined as

$$\hat{q}(t) = \begin{cases} q_1(t) & \text{if } t \leq T, \\ q_2(2T - t) & \text{if } t \geq T, \end{cases}$$

determines a C^1 trajectory on $[0, 2T]$ having $\dot{\hat{q}}(2T) = 0$. It follows that $\hat{q}$ is the brake orbit of period $2T$.

It remains to show that there exists $C > 0$ such that $q_2^\mu(T) - q_1^\mu(T) \geq C\mu^{1/\alpha}$. Since we have established convergence of the energies (h^μ) of the sequence (q^μ) to the energy h of q, we have

$$\mu g(q_2^\mu(T) - q_1^\mu(T)) \to \dot{q}_1^2(T) > 0.$$

Moreover, thanks to (1.5), the function g satisfies the inequality $g(s)s^\alpha \geq g(1)$ for $s \leq 1$. Thus, there exists a constant $C > 0$ such that

$$q_2^\mu(T) - q_1^\mu(T) \geq C \sqrt[\alpha]{\mu}. \qquad \blacksquare$$

A. Brake orbits for the f_{ε_1}-Kepler problem

In this section we briefly discuss the properties of the solution of

$$\begin{cases} \ddot{q} = f'_{\varepsilon_1}(q), \\ q(0) = 0,\ \dot{q}(T) = 0, \end{cases} \tag{A.1}$$

where f_{ε_1} was defined in (2.1) and approximates the attractive potential f. Let us consider the space $\mathcal{V} = \{q \in H^1([0,T];\mathbb{R}) : q(0) = 0\}$, the family of functionals $\mathcal{F}_{\varepsilon_1}$ and $\mathcal{F}$ defined as

$$\mathcal{F}_{\varepsilon_1}(q) = \int_0^T \frac{1}{2}\dot{q}^2 + f_{\varepsilon_1}(q), \quad \mathcal{F}(q) = \int_0^T \frac{1}{2}\dot{q}^2 + f(|q|).$$

Minimisers of $\mathcal{F}$ and $\mathcal{F}_{\varepsilon_1}$ on $\mathcal{V}$ are called *brake orbits* and satisfy (A.1).

Lemma A.1. *The following assertions hold true:*

(1) *$\mathcal{F}$ admits a unique minimiser $\bar{q}$ in $\mathcal{V}$ which is of class $C^2((0,T])$. This is the unique half-brake orbit with minimal period $2T$.*

(2) *For any $\varepsilon_1 > 0$, $\mathcal{F}_{\varepsilon_1}$ has a unique minimiser q_{ε_1} in $\mathcal{V}$ which is of class $C^2([0,T])$. This is the unique half-brake orbit with minimal period $2T$.*

(3) *The family $\{q_{\varepsilon_1}\}$ is bounded in H^1 and converges strongly to $\bar{q}$.*

(4) *Denote $a_0 = \mathcal{F}(\bar{q})$ and $a_{\varepsilon_1} = \mathcal{F}_{\varepsilon_1}(q_{\varepsilon_1})$. Then, $a_{\varepsilon_1} \to a_0$.*

(5) *Each $\mathcal{F}_{\varepsilon_1}$ satisfies the Palais–Smale condition at any level $c > 0$.*

Proof. We first prove points (1) and (2). From Lemma 2.1, the functions f_{ε_1} are $C^{1,1}$ on $\mathbb{R}$, the functionals $\mathcal{F}_{\varepsilon_1}$ are C^1 and so their critical points are C^2. Moreover, the functionals are coercive. Indeed, since f_{ε_1} is positive, we have

$$\mathcal{F}_{\varepsilon_1}(q) \geq \frac{1}{2}\int_0^1 |\dot{q}|^2\, dt,$$

and $\|q\|_{H^1} \to +\infty$ if and only if $\|\dot{q}\|_2 \to +\infty$ by the Poincaré inequality (recall that $q(0) = 0$ in $\mathcal{V}$). The functional $\mathcal{F}_{\varepsilon_1}$ is also weakly lower semicontinuous and so minimisers exist for any $\varepsilon_1 > 0$ by direct methods. Notice that, since we have fixed the starting point and the end point is free, any critical point q_{ε_1} must satisfy $\dot{q}_{\varepsilon_1}(T) = 0$. The same argument shows that there exists at least a minimiser for $\mathcal{F}$.

For the uniqueness part, let us observe that T can be written in terms of the final position $w = q_{\varepsilon_1}(T)$. Indeed, integrating the energy equation we get

$$\begin{aligned} T(w) &= \frac{1}{\sqrt{2}} \int_0^{T(w)} \frac{\dot{q}_{\varepsilon_1}\, dt}{\sqrt{f_{\varepsilon_1}(q_{\varepsilon_1}) - f_{\varepsilon_1}(w)}} = \frac{1}{\sqrt{2}} \int_0^{w} \frac{dq}{\sqrt{f_{\varepsilon_1}(q) - f_{\varepsilon_1}(w)}} \\ &= \frac{1}{\sqrt{2}} \int_0^{1} \frac{w\, dq}{\sqrt{f_{\varepsilon_1}(wq) - f_{\varepsilon_1}(w)}}. \end{aligned}$$

Thus, computing the derivative with respect to the final position w, we obtain

$$\partial_w T = \frac{1}{2\sqrt{2}} \int_0^1 \frac{2 f_{\varepsilon_1}(wq) - 2 f_{\varepsilon_1}(w) - q w f'_{\varepsilon_1}(wq) + w f'_{\varepsilon_1}(w)}{(f_{\varepsilon_1}(wq) - f_{\varepsilon_1}(w))^{3/2}}.$$

So T is strictly monotone in w if the function $\psi(s) = 2 f_{\varepsilon_1}(s) - s f'_{\varepsilon_1}(s)$ has the same property. Computing its derivative we obtain $\psi'(s) = f'_{\varepsilon_1}(s) - s f''_{\varepsilon_1}(s)$, which is always negative in our case.

Let us prove (3). By construction, for any curve $q \in \mathcal{V}$, $\mathcal{F}_{\varepsilon_1}(q) \leq \mathcal{F}(q)$. It follows that

$$a_{\varepsilon_1} = \min_{q \in \mathcal{V}} \mathcal{F}_{\varepsilon_1}(q) \leq \min_{q \in \mathcal{V}} \mathcal{F}(q) = a_0.$$

Moreover, since f_{ε_1} is positive, we have

$$\frac{1}{2} \|\dot{q}_{\varepsilon_1}\|_2^2 \leq \frac{1}{2} \|\dot{q}_{\varepsilon_1}\|_2^2 + \int_0^T f_{\varepsilon_1}(q_{\varepsilon_1}) \leq a_0.$$

Thus, minimisers of $\mathcal{F}_\varepsilon$ form a bounded subset of H^1 and so are weakly pre-compact. It remains to prove that (q_{ε_1}) converges strongly in H^1 to $\bar{q}$, which is the unique minimiser of $\mathcal{F}$. This is a consequence of the Fatou lemma and uniform convergence:

$$\begin{aligned} \frac{1}{2} \|\dot{\bar{q}}\|_2^2 + \int_0^T f(\bar{q}) = a_0 &\geq \liminf_{\varepsilon_1} \left(\frac{1}{2} \|\dot{q}_{\varepsilon_1}\|_2^2 + \int_0^T f_{\varepsilon_1}(q_{\varepsilon_1}) \right) \\ &\geq \liminf_{\varepsilon_1} \frac{1}{2} \|\dot{q}_{\varepsilon_1}\|_2^2 + \int_0^T f(\bar{q}). \end{aligned}$$

This implies that $\|\dot{\bar{q}}\|_2^2 = \liminf_{\varepsilon_1} \|\dot{q}_{\varepsilon_1}\|_2^2$, proving strong convergence in H^1.

Let us show (4). Denoting by h_{ε_1} the energy of q_{ε_1}, it is not difficult to see that $a_\varepsilon = \|\dot{q}_{\varepsilon_1}\|_2^2 - h_{\varepsilon_1} T$. By the argument given in Proposition 3.3 and uniform convergence, we have that $h_{\varepsilon_1} \to \bar{h}$ and so $a_{\varepsilon_1} \to a_0$.

It remains to show (5). Let u_n be a (PS) sequence for $\mathcal{F}_{\varepsilon_1}$ at a positive level $c > 0$. The critical point equation implies

$$d_{u_n}\mathcal{F}_{\varepsilon_1}(u_n) = \|\dot{u}_n\|_2^2 + \int_0^T u_n f_\varepsilon'(u_n) \geq \|\dot{u}_n\|_2^2 - \alpha \int_0^T f_\varepsilon(u_n)$$
$$\geq \Big(1 + \frac{\alpha}{2}\Big)\|\dot{u}_n\|^2 - \alpha \mathcal{F}_{\varepsilon_1}(u_n),$$

where the first inequality follows from point (3) of Lemma 2.1. Since $d_{u_n}\mathcal{F}_\varepsilon(u_n)/\|u_n\|_2 \to 0$ and we have a Poincaré inequality, (u_n) is a bounded sequence in H^1. Thus, up to subsequence, there exists a weak, uniform and L^2 limit u. To show that it is actually a strong limit, it is enough to notice that

$$d_{u_n}\mathcal{F}_\varepsilon(u - u_n) = \langle \dot{u}_n, \dot{u} - \dot{u}_n\rangle + \int_0^T u_n f_{\varepsilon_1}'(u_n)(u - u_n).$$

By the uniform convergence of u_n and the continuity of f_{ε_1}', the integral goes to zero and so $\|\dot{u}_n\|_2 \to \|\dot{u}\|_2$, yielding strong convergence. ■

B. A mountain pass lemma

In this appendix we state and prove an ad hoc version of the mountain pass lemma.

Lemma B.1. *Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{D} \subset \mathcal{H}$ an open set having smooth boundary and consider a C^1 functional $\mathcal{A}\colon \mathcal{D} \to \mathbb{R}$ having Lipschitz gradient. Assume that $\mathcal{A}$ can be extended to $\overline{\mathcal{D}}$ and $\mathcal{A}(\partial\mathcal{D}) = -\infty$. Let $p, q \in \mathcal{D}$, consider the class of paths*

$$\Gamma = \big\{\gamma\colon [0,1] \to \mathcal{D} : \gamma(0) = p,\ \gamma(1) = q\big\}$$

and define the value

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{A}(\gamma(t)).$$

Moreover, let us assume that

(1) $c > \max\{\mathcal{A}(p), \mathcal{A}(q)\}$,

(2) *$\mathcal{A}$ satisfies the (PS)-condition at level c.*

Then c is a critical value of $\mathcal{A}$.

Proof. By contradiction, let us assume that c is not a critical value. Since $\mathcal{A}$ satisfies the (PS) condition, this implies that there is no (PS) sequence at level c. This means that there exists $\delta > 0$ such that, for any u_n with $\mathcal{A}(u_n) \to c$, we have $\|d_{u_n}\mathcal{A}\| > \delta$. As a consequence, there exists some $\varepsilon > 0$ such that $\|d_u\mathcal{A}\| > 2\varepsilon$ for any $u \in \mathcal{A}^{-1}([c-\varepsilon, c+\varepsilon])$. Let us consider the sets

$$X = \mathcal{A}^{-1}([c - 2\varepsilon, c + 2\varepsilon]), \quad Y = \mathcal{A}^{-1}([c - \varepsilon, c + \varepsilon]).$$

They are both closed and $\partial\mathcal{D} \cap X = \emptyset$ since $\mathcal{A}(\partial\mathcal{D}) = -\infty$ by assumption. This implies that the function

$$\psi(u) = \frac{\operatorname{dist}(u, X^c)}{\operatorname{dist}(u, X^c) + \operatorname{dist}(u, Y)}$$

is Lipschitz, vanishes on X^c and is equal to 1 on Y. Define the vector field and the associated ODE:

$$V = -\frac{\psi \nabla \mathcal{A}}{\|\nabla \mathcal{A}\|}, \quad \dot{\eta} = V(\eta).$$

By construction, V is bounded and locally Lipschitz. Thus there exists a well-defined continuous flow Φ. It leaves $\mathcal{D}$ invariant since $V \equiv 0$ outside X and $\mathcal{A}$ is decreasing on solutions since

$$\frac{d}{dt}\mathcal{A}(\eta(t)) = -\langle \nabla \mathcal{A}, \dot{\eta}\rangle = -\psi \|\nabla \mathcal{A}\| \leq 0.$$

Take γ_n a minimising sequence approaching c. For ε sufficiently small, $p, q \in X^c$ and $\Phi_t(\gamma_n)$ still belongs to Γ. Let u_n be a point in which the maximum is realised. For n sufficiently large, $u_n \in Y$. By definition of c and for n sufficiently large, $c + \varepsilon > \mathcal{A}(\Phi(u_n)) \geq c$; however in this case,

$$\mathcal{A}(\Phi(u_n)) - \mathcal{A}(u_n) = -\int_0^1 \|\nabla \mathcal{A}(\Phi_s(u_n))\| \leq -2\varepsilon.$$

This implies that $\mathcal{A}(\Phi(u_n)) < c - \varepsilon$, a contradiction. ∎

Funding. The authors are partially supported by INDAM-GNAMPA research group. The first author is partially supported by the PRIN 2022 project 2022FPZEES – *Stability in Hamiltonian dynamics and beyond*. The third author is partially supported by the PRIN 2022 project 20227HX33Z – *Pattern formation in nonlinear phenomena*.

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Received 2 April 2024; revised 28 July 2024; accepted 30 July 2024.

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