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A 2-complex with contracting non-positive immersions and positive maximal irreducible curvature

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Abstract. We prove that the 2-complex associated to the presentation $\langle a, b \mid b, bab^{-1}a^{-2}\rangle$ has contracting non-positive immersions and positive maximal irreducible curvature. This example shows that the contracting non-positive immersions property is not equivalent to the notion of non-positive irreducible curvature, answering a question raised by H. Wilton.

1. Introduction

In [11, 12], D. Wise introduced the notions of sectional curvature and non-positive immersions for 2-complexes in order to study properties of their fundamental groups such as coherence and local indicability. In [13], he investigated some variations of the non-positive immersions property. Here, we are concerned with the contracting variant. Recall that an immersion $Y \looparrowright X$ is a map between 2-complexes that is locally injective. A 2-complex X has *contracting non-positive immersions* if for any immersion $Y \looparrowright X$ with Y connected and compact, either $\chi(Y) \leq 0$ or Y is contractible. Here, $\chi(Y)$ denotes the Euler characteristic of Y. One of the main tools used in [11–13] is the theory of towers, due to J. Howie [4]. Wise remarked that the proofs of the non-positive immersions property for some 2-complexes are rather ad hoc and asked whether there exists an algorithm to recognize if a compact 2-complex has this property [13, Problem 1.9].

More recently H. Wilton [10] defined new curvature invariants of 2-complexes. These invariants are related to Wise's previous notions although Wilton's approach and techniques are different. The *average curvature* $\kappa(X)$ of a finite 2-complex X is defined as $\kappa(X) = \frac{\chi(X)}{\text{Area}(X)}$, where Area$(X)$ is the number of 2-cells, and the *maximal irreducible curvature* $\rho_+(X)$ is the supremum of the curvatures $\kappa(Y)$ among all finite irreducible branched 2-complexes Y admitting essential maps $Y \to X$ (see [10] for more details). In [9], it is proved that, when X is irreducible, this supremum is attained and is algorithmically computable. Wilton showed that if X has non-positive maximal irreducible curvature (i.e., $\rho_+(X) \leq 0$), then it has contracting non-positive immersions, and asked whether there is an example of a 2-complex with contracting non-positive immersions with $\rho_+(X) > 0$.

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We give here an example of such a 2-complex. The presentation $P = \langle a, b \mid b, bab^{-1}a^{-2}\rangle$ belongs to a family of balanced presentations of the trivial group introduced by Miller and Schupp [7]. Note that its associated 2-complex K_P is contractible by the Hurewicz and Whitehead theorems. This example appeared in Wilton's paper [10, Example 4.8] where it is proved that K_P is irreducible. Since, in this case, the identity map of K_P is an essential map from a finite irreducible branched 2-complex, then $\rho_+(K_P) \geq \kappa(K_P) = \frac{1}{2}$ (see [10, Section 8]). In [2], W. Fisher investigated Miller and Schupp's family of examples and showed that some of them fail to have the non-positive immersions property. However, his methods could not conclude for $P = \langle a, b \mid b, bab^{-1}a^{-2}\rangle$. We prove the following.

Theorem 1.1. *The 2-complex K_P associated to the presentation $\langle a, b \mid b, bab^{-1}a^{-2}\rangle$ has contracting non-positive immersions.*

Our approach uses techniques developed by Wilton in [10] and by Louder and Wilton in [6], where the immersions are built by performing foldings to the 2-complexes.

2. The proof

We work in the category of combinatorial 2-complexes and combinatorial maps (which send (open) n-cells homeomorphically to n-cells). We will use the notion of folding of 2-complexes introduced by Louder and Wilton in [6], which is a natural generalization of Stallings' foldings [8]. Given a combinatorial map between 2-complexes $X \to Y$, we first fold the 1-skeleton of X and then glue the 2-cells accordingly to obtain an immersion $Z \looparrowright Y$.

Given an immersion $X \looparrowright K_P$, the 2-cells of X which are mapped to the cell corresponding to the relator b are called of *type 1*, and the 2-cells mapped to the cell corresponding to $bab^{-1}a^{-2}$ are of *type 2*. By the work of Helfer and Wise [3] and, independently, by Louder and Wilton [5], it is known that torsion-free one-relator groups have contracting non-positive immersions. Thus, if an immersion $X \looparrowright K_P$ has only 2-cells of type 1 or of type 2, then $\chi(X) \leq 0$ or X is contractible. Therefore, we can reduce our study to immersions that use both types of 2-cells. We can also consider only immersions without free faces. Recall that a cell σ of a combinatorial 2-complex is a free face if it lies in the boundary of exactly one cell τ of higher dimension and τ hits σ only once. In that case, performing an elementary collapse (i.e., removing the interiors of σ and τ) does not change the homotopy type of the complex (see [1]). We will show that any immersion without free faces that uses both types of 2-cells is actually contractible.

In what follows, *coupling* a 2-cell to an immersion $X \looparrowright K_P$ will mean the immersion obtained by gluing a closed 2-cell along an edge to X and folding. From now on, the immersions $X \looparrowright K_P$ will be denoted with single letters (see the definitions of immersions D_i and C_i below). The immersion will be specified by giving orientations and labelings to the 1-skeleton of the domain.

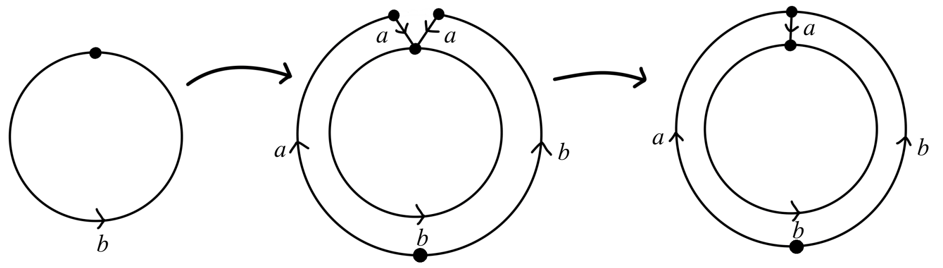

Figure 1. Coupling a 2-cell of type 2 to a 2-cell of type 1.

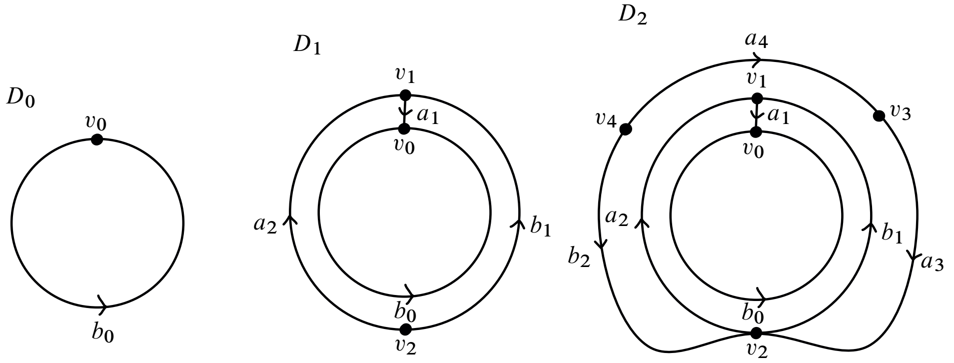

Figure 2. Discs D_0, D_1 and D_2 with their labelings.

We begin by defining two families of immersions D_i and C_i. Start with the immersion of a single 2-cell of type 1. We can couple a 2-cell of type 2 to this single 2-cell along the first b of $bab^{-1}a^{-2}$, or the second b (which appears as b^{-1}). We consider first the case when we couple it along the second appearance. In this case, we get a new disc that also has a free face labeled by b (see Figure 1).

We can once again couple a 2-cell of type 2 to this immersion along the second appearance of b. Again, we arrive to a disc with a free face labeled by b. We can continue with this process and obtain diagrams $D_0, D_1, D_2, \ldots$ as in Figure 2.

We number the labels of the edges of the D_i. The b in D_0 is labeled by b_0. At each step we are adding two a's and one b. The b added in step i is labeled by b_i, and the a's by a_{2i-1} and a_{2i} as shown in Figure 2. In addition, we label the vertices so that the edge labeled by a_i goes from v_i to v_{i-1}. With this numbering, we can give a more explicit description of the 1-skeleton of D_i.

Lemma 2.1. *The* 1*-skeleton of* D_i *has the following structure:*

- $2i + 1$ *vertices* $v_0, \ldots, v_{2i}$*;*
- $2i$ *oriented edges* $a_1, \ldots, a_{2i}$*, where* a_j *goes from* v_j *to* v_{j-1} $(1 \leq j \leq 2i)$*;*
- $i + 1$ *oriented edges* $b_0, \ldots, b_i$*, where* b_j *goes from* v_{2j} *to* v_j $(0 \leq j \leq i)$.

Proof. We prove this by induction on i. The cases $i = 0, 1$ are clear (see Figure 2). In order to obtain D_{i+1} from D_i, we couple a 2-cell of type 2 along the second appearance of b to b_i. After the folding is performed, this new 2-cell is glued along b_i and a_{i+1}. Then, in the 1-skeleton of D_{i+1} there are:

- two new vertices v_{2i+1} and v_{2i+2},
- two new edges labeled by a_{2i+1} and a_{2i+2} that go from v_{2i+1} to v_{2i} (which is the initial vertex of b_i) and from v_{2i+2} and v_{2i+1}, respectively, and
- a new edge labeled by b_{i+1} going from v_{2i+2} to v_{i+1} (which is the initial vertex of a_{i+1}). ■

Now we define the immersions C_i for $i \geq 1$. The 2-complex C_i is obtained from D_i by identifying b_i with b_0 and folding. Consider first the case $i = 1$. Note that, when we identify b_1 with b_0, the vertices v_1 and v_2 are automatically identified with v_0, and we must also identify a_1 with a_2 in order to obtain an immersion (see Figure 3). With all these identifications, we get $C_1 = K_P$, which is contractible. The next two lemmas describe C_i for every odd natural number i.

Lemma 2.2. *Let $i \geq 1$ be odd. Then the 1-skeleton of C_i has the following structure:*

- *i vertices $v_0, \ldots, v_{i-1}$;*
- *i oriented edges $a_1, \ldots, a_i$, where a_j goes from v_j to v_{j-1} if $1 \leq j \leq i-1$, and a_i goes from v_0 to v_{i-1};*
- *i oriented edges $b_0, \ldots, b_{i-1}$ where b_j goes from $v_{(2j \bmod i)}$ to v_j $(0 \leq j \leq i-1)$.*

As a topological space, C_i is homeomorphic to K_P and, in particular, is contractible.

Proof. After identifying b_i with b_0, v_i and v_{2i} become automatically identified with v_0. This implies that a_{2i} (which starts at the vertex v_{2i} in D_i) must be identified with a_i (whose starting vertex is v_i). Since the edges labeled by a form an oriented line, we must now identify a_{2i-k} with a_{i-k} for all $0 \leq k \leq i-1$. In particular, we also identify v_{2i-k} with v_{i-k} for all $1 \leq k \leq i-1$. We now have i oriented edges $a_1, \ldots, a_i$ and i vertices $v_0, \ldots, v_{i-1}$, where a_j goes from v_j to v_{j-1} if $1 \leq j \leq i-1$, and a_i goes from v_0 to v_{i-1}. Thus, no two edges labeled by a start at the same vertex, and no two edges labeled by a end at the same vertex. Similarly, after these identifications we have i oriented edges $b_0, \ldots, b_{i-1}$ where b_j goes from $v_{(2j \bmod i)}$ to v_j $(0 \leq j \leq i-1)$. Since, by hypothesis, i is odd, $v_{(2j \bmod i)} \neq v_{(2k \bmod i)}$ if $0 \leq j \neq k \leq i-1$, then no two edges labeled by b start at the same vertex, and no two edges labeled by b end at the same vertex. Hence, no more identifications are required.

Notice that the folding done to D_i corresponds to identifying the exterior stretch of the path of a's to the interior one. This procedure of identifications is analogous to the one for D_1, with the only difference being that instead of identifying a_2 with a_1 we are identifying whole stretches of a's (see Figure 3). Therefore, the 2-complex C_i is homeomorphic to $C_1 = K_P$ and, in particular, it is contractible. The homeomorphism is induced

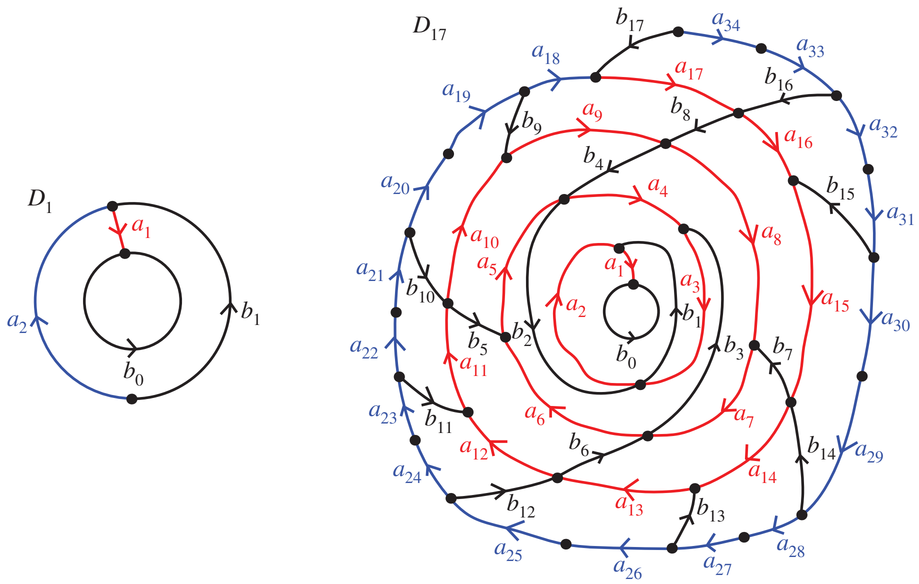

Figure 3. In D_{17} we identify b_{17} with b_0 and the blue path with the red one to get C_{17}. Similarly for D_1.

by the homeomorphism from D_1 to D_i which sends the edges b_0 and b_1 in D_1 to b_0 and b_i in D_i, and sends a_1 to the interior stretch of the path of a's (red path in Figure 3) and a_2 to the exterior one (blue path). ∎

Lemma 2.3. *Let $i \geq 1$ be odd. Then the following hold:*

(1) *Every edge labeled by a in C_i is a face of three 2-cells of type 2 (counted with multiplicity).*

(2) *Every edge labeled by b in C_i is adjacent to two distinct 2-cells of type 2.*

(3) *Every vertex of C_i has degree 4, and it is adjacent to two edges labeled with the letter a (with opposite orientations) and two edges (counted with multiplicity) labeled with the letter b (with opposite orientations).*

Proof. Item (3) of Lemma 2.3 follows directly from the description of the 1-skeleton of C_i of Lemma 2.2.

For items (1) and (2), we look at the structure of D_i and at the identifications made in Lemma 2.2. In D_i, the edges $a_1, \ldots, a_i$ are adjacent to two 2-cells of type 2 (counted with multiplicity), and the edges $a_{i+1}, \ldots, a_{2i}$ are adjacent to one 2-cell of type 2. After identifying these two stretches of a's, each edge $a_1, \ldots, a_i$ in C_i is adjacent to three 2-cells of type 2 (counted with multiplicity), and these 2-cells are not identified, since they correspond to the three distinct appearances of letter a in $bab^{-1}a^{-2}$. Similarly, in D_i the edges b_0 and b_i are adjacent to one 2-cell of type 2, so in C_i b_0 is adjacent to two distinct 2-cells of type 2. The edges $b_1, \dots, b_{i-1}$ in D_i are adjacent to two distinct 2-cells of type 2, and since they are not identified when folding, the edges $b_1, \dots, b_{i-1}$ in C_i are adjacent to two distinct 2-cells of type 2. ■

We now deal with the case where i is even. The following lemma implies that for every even natural number i, $C_i = C_{\text{odd}(i)}$, where $\text{odd}(i)$ is the largest odd divisor of i.

Lemma 2.4. *Let $i \geq 1$. Then $C_{2i} = C_i$.*

Proof. We start with D_{2i} and identify b_{2i} with b_0. After this identification, similarly as in Lemma 2.2, we must identify a_{4i-k} with a_{2i-k} for all $0 \leq k \leq 2i-1$, and v_{4i-k} with v_{2i-k} for all $0 \leq k \leq 2i$. However, now we must also identify b_i with b_0 since they both start at v_0. Similarly, we must also identify b_{i+k} with b_k for all $1 \leq k \leq i-1$. This is exactly the description of the 1-skeleton of the complex that we would obtain after starting with D_i and identifying b_i with b_0. Further identifications are needed to complete the folding, but at this point we have proved that $C_{2i} = C_i$. ■

If we start the process of constructing the immersions by coupling the first 2-cells of type 2 along the other b (the first appearance of b in $bab^{-1}a^{-2}$), we obtain immersions $\widetilde{D}_i$ and $\widetilde{C}_i$, where the only difference with the previous ones is that the orientations of all the a's are reversed (see Figure 4).

Lemma 2.5. *As combinatorial complexes, $\widetilde{D}_i = D_i$ and $\widetilde{C}_i = C_i$ for every $i \geq 1$. In particular, $\widetilde{C}_i$ is contractible ($i \geq 1$). As immersions, the only differences are that in $\widetilde{D}_i$, the edge a_j goes from v_{j-1} to v_j (for every $1 \leq j \leq 2i$), and for odd i, in $\widetilde{C}_i$, the edge a_j goes from v_{j-1} to v_j if $1 \leq j \leq i-1$, and a_i goes from v_{i-1} to v_0. Additionally, for every even natural number i, $\widetilde{C}_i = \widetilde{C}_{\text{odd}(i)}$.*

Proof. It is clear from construction that, as combinatorial complexes, $\widetilde{D}_i = D_i$ and $\widetilde{C}_i = C_i$. The only difference lies in the orientation of some of the edges. In $\widetilde{D}_i$, when coupling the jth 2-cell of type 2, the two new edges labeled by a_{2j-1} and a_{2j} go from v_{2j-2} to v_{2j-1} and from v_{2j-1} to v_{2j}, respectively. Therefore, the statement for $\widetilde{D}_i$ follows from Lemma 2.1. Similarly, for $i \geq 1$ odd, the only difference in the construction of the $\widetilde{C}_i$ is that, after identifying b_i with b_0 in $\widetilde{D}_i$, and the vertices v_i and v_{2i} with v_0, the edges a_{2i} and a_i must be identified because their final vertices (v_{2i} and v_i) are identified. In the same way, we must identify a_{2i-k} with a_{i-k} for all $0 \leq k \leq i-1$ because their final vertices are identified. Hence, the statement for $\widetilde{C}_i$ follows from Lemma 2.2.

Finally, given $i \geq 1$, by Lemma 2.4 and the relation between $\widetilde{C}_i$ and C_i (resp. $\widetilde{C}_{2i}$ and C_{2i}), we obtain that $\widetilde{C}_{2i} = \widetilde{C}_i$. ■

Note that Lemma 2.3 also holds for the $\widetilde{C}_i$ since, as complexes, they are equal to the C_i, and the orientation of the edges does not play any role in this result.

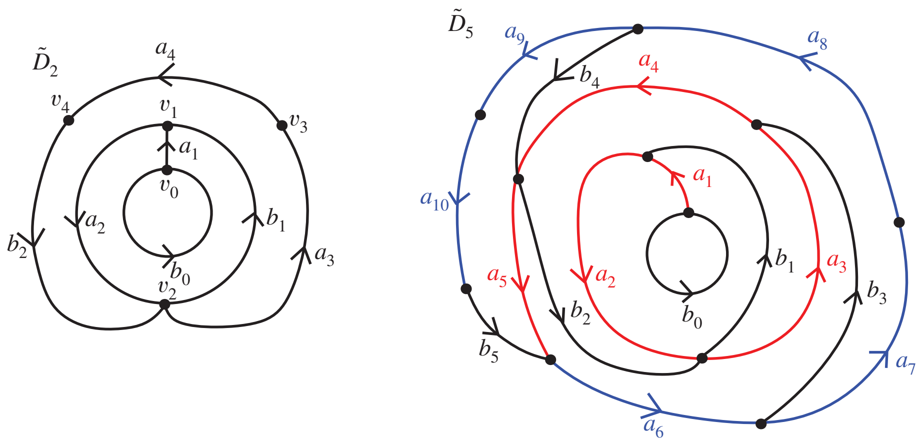

Figure 4. Discs $\widetilde{D}_2$ and $\widetilde{D}_5$. In $\widetilde{D}_5$ we identify b_5 with b_0 and the blue path with the red one to get $\widetilde{C}_5$.

Our goal now is to show that any immersion over K_P that uses 2-cells of both types and that does not have free faces must be a C_i or a $\widetilde{C}_i$ for some i, and therefore contractible. For that we need the following lemmas.

Lemma 2.6. *Let $i \geq 2$. If we identify two vertices of an immersion C_i (resp. $\widetilde{C}_i$) and fold, we obtain an immersion C_k (resp. $\widetilde{C}_k$) for some $k < i$.*

Proof. We prove the lemma for C_i since, by Lemma 2.5, C_i and $\widetilde{C}_i$ differ only in the orientations of the edges a_j, and the orientation does not play any role here. We proceed by induction on i. The base case is clear, since if we identify the two vertices of C_2 we obtain $K_P = C_1$. Let us assume that it holds for C_l with $l < i$. By Lemma 2.4, we can suppose that i is odd. Notice that the a's form an oriented cycle that traverses every vertex of C_i. Since we are folding afterwards, we can assume that we are identifying v_0 with some v_j with $0 < j < i$. Let $X \looparrowright K_P$ be the immersion obtained after identifying v_0 with v_j and folding. Then X admits an immersion from C_j, and therefore it has a quotient of C_j as a subcomplex, which by induction is some C_k. We know that X has a subcomplex isomorphic to C_k whose edges are labeled and oriented as in C_k, and that X is folded and connected. Therefore, by Lemma 2.3, if X had any edge or any 2-cell not belonging to said subcomplex, X would not be folded. So X is equal to C_k. ∎

Lemma 2.7. *Let $f : X \looparrowright K_P$ be an immersion. Suppose that there exists an immersion $g : C_i \looparrowright K_P$ (resp. $g : \widetilde{C}_i \looparrowright K_P$) that factors through f (i.e., there exists an immersion $h : C_i \looparrowright X$ (resp. $h : \widetilde{C}_i \looparrowright X$) such that $g = f \circ h$). Then $X = C_k$ (resp. $X = \widetilde{C}_k$) for some k.*

Proof. By Lemma 2.5, and since Lemma 2.3 also holds for the $\widetilde{C}_i$, it is enough to prove the result for the case where there exists an immersion $g : C_i \looparrowright K_P$. Since there exists an immersion $h : C_i \looparrowright X$, X has a quotient of C_i as a subcomplex. By Lemma 2.6, said subcomplex is some C_k for $k \leq i$. Now we apply the same argument as in the previous proof. Since X is folded and connected and has a subcomplex isomorphic to C_k whose edges are labeled and oriented as in C_k, by Lemma 2.3, X is equal to C_k. ■

Lemma 2.8. *If we identify b_i to another b_j in D_i (resp. $\widetilde{D}_i$) and fold, we obtain C_k (resp. $\widetilde{C}_k$) for some k.*

Proof. Again, since the only difference between D_i and $\widetilde{D}_i$ (and C_i and $\widetilde{C}_i$) is the orientation of the a's, it is enough to prove it for D_i. If we identify b_i with b_0, we obtain C_i by definition. Suppose we identify b_i with b_j with $j \neq 0$. Then we must identify v_m with v_n for $m \equiv n \bmod (i - j)$. As a consequence, we identify b_{i-j} with b_0. Let $X \looparrowright K_P$ be the immersion obtained after these identifications and foldings. Then there is an immersion from C_{i-j} to X, and by Lemma 2.7, $X = C_k$ for some k. ■

Lemma 2.9. *If we couple a 2-cell to b_i in D_i (resp. $\widetilde{D}_i$), we obtain D_i, D_{i+1} or C_i (resp. $\widetilde{D}_i$, $\widetilde{D}_{i+1}$ or $\widetilde{C}_i$).*

Proof. Once again, by Lemma 2.5, we only prove it for D_i. If the 2-cell is of type 1, then we must identify b_i with b_0 and we obtain C_i. If the 2-cell is of type 2 and we couple it to b_i, then depending on which appearance of b we use, we can get either D_i or D_{i+1}. ■

Now we can prove the main result of this note.

Proof of Theorem 1.1. Let $X \looparrowright K_P$ be an immersion of a finite, connected 2-complex without free faces. As we mentioned above, we can assume that it has 2-cells of both types. We will construct an immersion of some C_i (or $\widetilde{C}_i$) to K_P that factors through $X \looparrowright K_P$ as in Lemma 2.7. Since X has both types of 2-cells, it has at least one of type 1. We start to build our immersion from this 2-cell. Since X has no free faces, the boundary of this 2-cell must be in the boundary of another 2-cell of X and, since it is an immersion, the cell must be of type 2. We assume that it is attached along the second b (in the other case, we proceed similarly and end up with a $\widetilde{C}_i$). We couple it to the immersion to obtain D_1. Now D_1 has a free face labeled by b_1. Again, since X has no free faces, either this b_1 is identified with b_0 in X, or there is another 2-cell in X which has it as a face. If it is identified with b_0, we get C_1 and we are done. If it is not, then by Lemma 2.9, we get C_1 or D_2. If we get D_2, we continue with this process. Since X is finite and has no free faces, then by Lemmas 2.8 and 2.9, we must eventually arrive to some C_i. Therefore, by Lemma 2.7, X is a C_k, and hence, by Lemmas 2.2 and 2.4, it is contractible. ■

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