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Bifurcation for a sharp interface generation problem

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Abstract. As opposed to the widely studied bifurcation phenomena for maps or PDE problems, we are concerned with bifurcation for stationary points of a nonlocal variational functional defined not on functions but on sets of finite perimeter, and involving a nonlocal term. This sharp interface model (1.2), arised as the Γ-limit of the FitzHugh–Nagumo energy functional in a (flat) square torus in $\mathbb{R}^2$ of size T, possesses lamellar stationary points of various widths with well-understood stability ranges and exhibits many interesting phenomena of pattern formation as well as wave propagation. We prove that when the lamella loses its stability, bifurcation occurs, leading to a two-dimensional branch of nonplanar stationary points. Thinner nonplanar structures, achieved through a smaller T, or multiple layered lamellae in the same-sized torus, are more stable. To the best of our knowledge, bifurcation for nonlocal problems in a geometric measure theoretic setting is an entirely new result.

1. Introduction

Self-organization mechanisms are of great importance in pattern generation. As been observed in many fields of science, self-generated patterns [2, 12, 13, 16, 17, 43, 46] may exist for a wide range of parameters and are robust under certain conditions [3,4,14,15,50], for instance, in investigating nerve pulses in biological systems, concentration drops in chemical systems, filament current in physical systems, and copolymers in material science [4, 18, 35, 37–39, 41, 44].

Equilibrium models [2, 5, 17, 19, 35, 50] have recently garnered attention as a result of their self-generated patterns. Such systems are characterized by the presence of coexisting phases induced by the two potential wells, and the resulting structure of sharp transition interfaces defines the pattern. These patterns are metastable in certain ranges of the parameters and can undergo morphological instabilities [47, 49], leading to the formation of more complex patterns.

One of the well-known equilibrium models is governed by the variational functional

$$\int_{\mathbb{T}} \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \varepsilon^{-1} F(u)\Big)\, d\mathbf{x} + \frac{\sigma}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} u(\mathbf{x})(-\Delta + \kappa^2)^{-1} u(\mathbf{z})\, d\mathbf{z}\, d\mathbf{x} \qquad (1.1)$$

in a periodic torus $\mathbb{T}$. Here, $u \in W^{1,2}_{\text{per}}$ is a scalar $\mathbb{T}$-periodic function typically denoting

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density or chemical concentration. The function F represents either a standard double-well potential (but then α does not show up) or a slightly unbalanced version as [3, equation (1.6)]

$$F(u) = \frac{u^2(u-1)^2}{4} + \frac{\alpha\varepsilon}{\sqrt{2}}\left(\frac{u^3}{3} - \frac{u^2}{2}\right)$$

with $\varepsilon > 0$ measuring the deviation from balanced potential wells. With a small ε in (1.1), there exist spatial patterns resulting from the competition between rival forces operating at different length scales. In this energy-driven model, the Green's function G associated with $-\Delta + \kappa^2$ represents a screened Coulomb kernel, so called because the constant κ has the physical meaning of the inverse of the Debye screening length [38, 39]. The critical points of (1.1) also represent steady-state solutions of FitzHugh–Nagumo equations. Following a fascinating idea of Turing [49], reaction-diffusion systems have been employed as models for studying pattern formation and wave propagation [9–11, 14, 16, 43, 46, 51]. Historically, the original model [26, 40] was derived as a simplification of the Hodgkin–Huxley equations [30] for nerve impulse propagation. With many new phenomena being discovered in recent years, the FitzHugh–Nagumo model has been extensively studied as a paradigmatic activator-inhibitor system [10, 21, 25]. Such systems are of great interest to the scientific community as breeding grounds to study the generation of localized structures; these lead to a deeper understanding of the complicated dynamics of reaction-diffusion systems of activator-inhibitor type. For instance, even when fixing all physical parameters, reaction-diffusion waves exhibit front propagation in both directions between two distinct equilibrium states so that both the high and low energy states can be invaded by one another; at the same time, traveling fronts and pulses co-exist with different speed. Such localized waves are referred to as dissipative solitons [30, 35, 40], which are characterized by sharp interfaces.

Γ-convergence is a useful tool in studying pattern formation, in particular, to detect the location and shape of an interface or free boundary in congregates of distinct phases. Let $\mathbb{T} := [0, T]^N$ be a (flat) torus in $\mathbb{R}^N$. By setting $\kappa = 1$ and passing $\varepsilon \to 0$ in (1.1), the Γ-limit is a sharp interface model with a free energy functional $J : \mathcal{A} \to (-\infty, \infty]$ defined by

$$J(E) = \mathcal{P}_{\mathbb{T}}(E) - \alpha|E| + \frac{\sigma}{2}\int_E \mathcal{N}_E \, d\mathbf{x}. \tag{1.2}$$

Here, α, σ are given positive parameters, $\mathcal{A} := \{E \subset \mathbb{T} : E \text{ is Lebesgue measurable}\}$, the set E has a Lebesgue measure $|E|$, and its (possibly infinite) perimeter in $\mathbb{T}$ is denoted by $\mathcal{P}_{\mathbb{T}}(E)$. If E is of class $\mathcal{C}^1$, then $\mathcal{P}_{\mathbb{T}}(E)$ is the surface measure of its (periodic) boundary ∂E. The last term on the right of (1.2) represents a long range nonlocal effect; $\mathcal{N}$ is an operator that assigns for each set E the solution of the modified Helmholtz equation

$$-\Delta\mathcal{N}_E + \mathcal{N}_E = \chi_E \text{ in } \mathbb{T}, \quad \mathcal{N}_E \text{ is periodic in } \mathbb{T}.$$

The operator $\mathcal{N}$ is well defined, since $\mathcal{N}_E$ is the unique $\mathbb{T}$-periodic minimizer of

$$v \mapsto \int_{\mathbb{T}} \left(\frac{|Dv|^2}{2} + \frac{v^2}{2} - v\chi_E\right) d\mathbf{x}.$$

The Γ-limit version for a similar problem, namely, the Ohta–Kawasaki model for treating diblock polymers, has been studied in [4]; this sharp interface model shares some similarity with (1.2) as well as reflects certain significant difference (see the survey article [1]). In particular, the imposed volumetric constraint requires a determination of the Lagrange multiplier in a highly nonlinear fashion; its resulting Euler–Lagrange equation bears similarity with ours.

Both the empty set and the full torus are critical points of J. The nonlocal interaction term of J contains a positive parameter σ, and its effect favors an identically zero solution as a minimizer; on the other hand, the positive parameter α measures the driving force towards a non-zero state. When α, σ are of comparable magnitudes, the competing mechanisms can result in a simple periodic configuration known as lamella. With suitable parameters α, σ in a large torus, it has been shown [2] that a lamella has a lower energy than both the empty set and the full torus. On the other hand, according to [2, Proposition 1.5], let $\sigma > 0$ and c_N be the isoperimetric constant in the N-dimensional torus. If $\alpha < \sigma/2$ satisfies $\alpha \leq c_N \sqrt[N]{2}/T$, the unique global minimizer of J is the empty set, and reciprocally if $\alpha > \sigma/2$ satisfies $\alpha \geq \sigma - c_N \sqrt[N]{2}/T$, the unique global minimizer is the full torus. A local minimizer or a critical point of J is called a stationary, or critical, set.

To understand stability of critical sets, we recall the definition of the variations of our functional J at a smooth set $E \subset \mathbb{T}$. Let $X : \mathbb{T} \to \mathbb{T}$ be a $\mathcal{C}^m$ vector field and consider the associated flow $\Phi : \mathbb{T} \times (-\infty, \infty) \to \mathbb{T}$ defined by

$$\begin{cases} \frac{\partial \Phi}{\partial t}(\mathbf{z}, t) = X(\Phi(\mathbf{z}, t)), \\ \Phi(\mathbf{z}, 0) = \mathbf{z}. \end{cases} \tag{1.3}$$

The global existence and uniqueness of Φ follow from X being smooth and bounded; see Lemma 2.1. Suppose $\partial E \in \mathcal{C}^m$ (which, by definition, is the same as saying $E \in \mathcal{C}^m$) and define

$$E_t := \Phi(E, t).$$

The kth variation of J at E with respect to the flow induced by X, denoted by $J^{(k)}(E)[X]$, is defined as $\frac{d^k}{dt^k} J(E_t)\big|_{t=0}$. A critical set E of J satisfies $J'(E)[X] = 0$ for any X; a direct computation [3, Proposition 3.1] of its first variation yields, in a weak sense, the associated Euler–Lagrange equation

$$\mathcal{K}\big|_{\partial E} - \alpha + \sigma \mathcal{N}_E = 0 \quad \text{on } \partial E, \tag{1.4}$$

where $\mathcal{K}$ denotes the mean curvature (defined as the sum of all principal curvatures, and is taken to be positive for a sphere) at $\partial E \cap \mathbb{T}$; see, for example, [12, 13]. By elliptic regularity, see Section 2, sufficiently regular solutions of (1.4) enjoy higher regularity properties.

Let ν be the unit outward normal at ∂E. For a regular critical E, it has been shown in [3, Proposition 3.1] that $J''(E)[X]$ depends only on the normal component $\eta = X \cdot \nu$ on ∂E. For any $\eta \in W^{1,2}_{\text{per}}(\partial E) := \{u \in W^{1,2}(\partial E) : u \text{ is } \mathbb{T}\text{-periodic}\}$, it is therefore natural to write

$$J''(E)[\eta] = \int_{\partial E} \left(|\nabla_\tau \eta|^2 - \|B_{\partial E}\|^2 \eta^2\right) d\mathcal{H}^{N-1} + \sigma \int_{\partial E}\int_{\partial E} \mathbf{G}(\mathbf{x},\mathbf{y})\eta(\mathbf{x})\eta(\mathbf{y})\, d\mathcal{H}^{N-1}_{\mathbf{x}}\, d\mathcal{H}^{N-1}_{\mathbf{y}} + \sigma \int_{\partial E} (\nabla \mathcal{N}_E \cdot \nu)\eta^2\, d\mathcal{H}^{N-1}. \tag{1.5}$$

Here, $\|B_{\partial E}\|^2$ is the sum of the squares of the principal curvatures at ∂E; $\mathbf{G}$ is the Green's function for the Helmholtz operator in $\mathbb{T}$ with periodic boundary conditions, and ∇_τ is the tangential gradient on ∂E. The local stability of E can be inferred from $J''(E)[\eta]$. As a reminder, there are additional terms in $J''(E)[X]$ that depend on the tangential component of X if E is not stationary; see [3, Proposition 3.1].

Let $\mathbf{b}_i$ be the unit vector along the ith-coordinate axis. As $J(E) = J(E + t\mathbf{b})$ for any constant $X = \mathbf{b} \in \mathbb{R}^N$ and $t \in \mathbb{R}$, it is immediate that $J''(E)[\eta] = 0$ for $\eta = \nu_E^{(i)} := \mathbf{b}_i \cdot \nu$, and $J''(E)[\eta] = 0$ for $\eta \in \mathcal{M}(\partial E) := \operatorname{span}\{\nu_E^{(1)}, \ldots, \nu_E^{(N)}\} \subset W^{1,2}_{\text{per}}(\partial E)$. The subspace $\mathcal{M}(\partial E)$ can in some circumstances have a dimension less than N as in the case of a lamella. Now, let

$$\mathcal{M}^\perp(\partial E) := \left\{\eta \in W^{1,2}_{\text{per}}(\partial E) : \int_{\partial E} \eta \nu_E^{(i)}\, d\mathcal{H}^{N-1} = 0,\ i = 1, \ldots, N\right\},$$

and we have $W^{1,2}_{\text{per}}(\partial E) = \mathcal{M}(\partial E) \oplus \mathcal{M}^\perp(\partial E)$.

Definition 1.1. A regular critical point E of J is stable if

$$J''(E)[\eta] > 0 \quad \text{for all } \eta \in \mathcal{M}^\perp(\partial E) \setminus \{0\}.$$

Define the L^1 distance between sets modulo translations by

$$\delta(E, F) := \min_\tau |E \triangle (F + \tau)|.$$

When E is a stable critical set in the above sense, it follows from [4, Theorem 1.1], [3, Theorem 3.5] that E is a strict local minimizer of J, isolated in the δ distance sense.

Besides lamellar critical sets, one natural question is how to find periodic configurations with different shapes of interfaces; that is, to look for the existence of configurations with higher-dimensional patterns rather than one dimensional. This seems challenging; nevertheless, the bifurcation method provides a way to reach the goal and this approach is usually accompanied by stability analysis. In many discrete and continuous models, there exist multiple solutions with distinct qualitative features; transition of one kind into another, or merging of two kinds, usually takes place when the stability of such a state changes. These phenomena are known as (local) bifurcations, and they have been extensively studied within the framework of models described by maps and differential equations: the corresponding bifurcations have been investigated for decades [6, 22–24].

There are far fewer references on bifurcation of geometric problems involving volumes and perimeters of sets; most of them concentrate on the subject of minimal surfaces and constant mean curvature surfaces [27, 32, 46, 52]. As been shown (see, e.g., [7, 31, 37]), without nonlocal interaction in such kind of systems, some interface dynamics are governed by mean curvature flow, and thus, on convex domains, evolution does not generate stable steady-state patterns [8, 36]. On the other hand, we show that this nonlocal term plays an important role in inducing nontrivial periodic structures. In this paper, we consider the bifurcation problem for (1.4), a geometric energy functional involving a nonlocal term; this seems to be a new venture in treating sharp interfaces. Though we develop formulas to check possible bifurcation in general circumstances, we do focus on a 1-lamella that allows explicit verification of bifurcation criteria. Recall that a 1-lamella consists of exactly 1 vertical lamella and 1 empty wedge (space) in the whole torus $\mathbb{T}$. The stabilities of both the lamella and the non-planar configurations resulting from this bifurcation will be investigated.

First, we recall facts about this 1-lamella by setting $k = 1$ in [3, Proposition 2.2] (which summarizes results from [2]). Let

$$c := 1 - 2\alpha/\sigma. \tag{1.6}$$

For a lamella $\mathbb{L}$ to be stationary, it has to satisfy (1.4); its zero curvature compels $\mathcal{N}_{\mathbb{L}}\big|_{\partial\mathbb{L}} = \alpha/\sigma = (1-c)/2$. Since $0 < \mathcal{N}_{\mathbb{L}} < 1$ for any lamella which is not an empty set or a full torus, see [2, Remark 1.1, Lemma 2.5], it is necessary that $-1 < c < 1$. We therefore assume $c \in (-1, 1)$. For such c, there is a stationary 1-lamella $\mathbb{L} := [0, x_0] \times [0, T]^{N-1} \subset \mathbb{T} \subset \mathbb{R}^N$ for all $\sigma > 0$; its thickness is given by (see [3, (2.3)])

$$x_0 = \frac{T}{2} - \sinh^{-1}\left(c \sinh \frac{T}{2}\right); \tag{1.7}$$

clearly, x_0 is a strictly decreasing function of c and $x_0(-c) = T - x_0(c)$. To prove certain assertions later, it will sometimes be advantageous to replace the dependence on x_0 with the dependence on other constants, all equivalent. In particular, for every fixed T, using c is equivalent to using x_0, and we can also use a, λ, where

$$\mathcal{T} := \frac{T}{2}, \quad a := \frac{x_0}{T}, \quad \lambda := 1 - 2a = \frac{T - 2x_0}{T} = \frac{\mathcal{T} - x_0}{\mathcal{T}}. \tag{1.8}$$

In addition, we have (see [3, equation (2.4)])

$$\mathcal{N}_{\mathbb{L}}\big|_{\partial\mathbb{L}} = \frac{1}{2\sinh\frac{T}{2}}\left(\sinh\frac{T}{2} - \sinh\frac{T-2x_0}{2}\right) = \frac{\sinh\mathcal{T} - \sinh(\lambda\mathcal{T})}{2\sinh\mathcal{T}} = \frac{1-c}{2} \tag{1.9}$$

and [3, equation (2.5)]

$$d_0 := -\nabla\mathcal{N}_{\mathbb{L}} \cdot \nu\big|_{\partial\mathbb{L}} = \frac{1}{\sinh\frac{T}{2}} \sinh\frac{T-x_0}{2} \sinh\frac{x_0}{2}. \tag{1.10}$$

Note that these quantities identifying stationary 1-lamellae depend only on c (or any of a, x_0, λ), but not on σ; however, stability of stationary lamellae depends on both parameters. For small σ, this 1-lamella $\mathbb{L}$ is always stable [3, Theorem 5.1]. There is then a critical σ_{crit} such that $\mathbb{L}$ is stable for $\sigma < \sigma_{\text{crit}}$ and unstable for $\sigma > \sigma_{\text{crit}}$. This lamella loses its stability at $\sigma = \sigma_{\text{crit}}$ when $J''(\mathbb{L})[\eta] = 0$ for some appropriate choice of ($\mathbb{T}$-periodic) $\eta = X \cdot \nu$. This can also be interpreted as follows: a certain linearized operator, named $D_t\mathcal{F}(\mathbb{L}, \sigma_{\text{crit}})$ in the future, has a (double) zero eigenvalue with corresponding eigenfunctions η on $\partial\mathbb{L}$, i.e., $D_t\mathcal{F}(\mathbb{L}, \sigma_{\text{crit}})[\eta] = 0$. In the future, depending on which of the quantities in (1.8) we are using, we will denote this critical σ_{crit} as σ_c or σ_λ or other unmistakable notation. The most delicate case for stability is $c = 0$: indeed, by [3, Theorem 5.13], whenever the critical 1-lamella corresponding to a certain value of c is stable, then the same is true also for $|c'| > |c|$.

In Sections 2 and 3, we study the derivatives of the nonlocal term and curvature terms for general dimension N, general sets E, and all values of c; beginning with Section 4, we specialize to lamellae in the case $N = 2$ and flow fields X which are independent of the first coordinate, but still for any value of c; the bifurcation analysis is carried out in Sections 5 and 6 for the delicate case $c = 0$, and Section 7 will deal with the case $c \neq 0$.

We generally write $\mathbf{x} = (x, y)$ to designate a point on the 2-dimensional torus, though in later sections when there is a need to designate two distinct points $\mathbf{x}, \mathbf{y}$ on the 2-dimensional torus, we will employ the notation $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Also, to save vertical space, we write all vectors and vector fields as rows, even when they should be written as columns. If $\mathbb{L} = [0, x_0] \times [0, T]$ is the critical lamella, we denote its sides as

$$\partial\mathbb{L} = L_1 \cup L_2 \quad \text{with } L_1 \text{ along } x = 0 \text{ and } L_2 \text{ along } x = x_0.$$

For any function f defined on L_i, we will simply write $f \llcorner L_i$ instead of $f\mathcal{H}^1 \llcorner L_i$.

It is crucial to understand the structure of the destabilizing eigenfunctions on $\partial\mathbb{L}$: the lamella $\mathbb{L}$ becomes unstable due to the zero-eigenvalue mode

$$\eta := X \cdot \nu = -\psi(y)\llcorner L_1 + \psi(y)\llcorner L_2, \quad \psi \in \operatorname{span}\left\{\sin\frac{2\pi y}{T}, \cos\frac{2\pi y}{T}\right\};$$

see [3, pp. 580–582] where our η was denoted by ζ; the fact that η at L_1 is exactly the opposite as η at L_2 comes from the fact that $(-1, 1)$ is the sole eigenvector of the 2×2 matrix $\mathcal{A}^{(1)}$ on [3, p. 581] corresponding to its smallest eigenvalue $d_0^{(1)}$ we will document in (4.5).

Suppose $\partial\mathbb{L}_t = (L_1)_t \cup (L_2)_t$. Since $\eta_i = X \cdot \nu$ on the interface L_i, $i = 1, 2$, to leading order in time evolutions of $(L_1)_t$ and $(L_2)_t$ are governed by such eigenfunctions so that they are 'anti-symmetric' with respect to their outward normal directions; i.e., they form 'parallel' curves with a translational shift of $x_0 + o(1)$ in the x-direction; up to vertical translations, they are also symmetric with respect to the center of $\mathbb{L}$. For $c = 0$, we will demonstrate that there is exact parallelism for the non-planar solutions resulting from bifurcation; on the other hand, parallelism between $(L_1)_t$ and $(L_2)_t$ is lost when $c \neq 0$, but symmetry about the center of $\mathbb{L}$ is kept. Since lamella stability changes at $\sigma = \sigma_{\rm crit}$, it is natural to seek these non-planar solutions of (1.4) in the vicinity of $(\mathbb{L}, \sigma_{\rm crit})$ as a result of a local bifurcation.

Functions in $\operatorname{span}\{\sin \frac{2\pi y}{T}, \cos \frac{2\pi y}{T}\}$ may be thought of as vertical (i.e., in the y-direction) translations of multiples of $\sin \frac{2\pi y}{T}$. The latter, besides being periodic, are odd with respect to y in the sense that

$$f(y) = -f(T - y).$$

If f is odd, the flow (1.3) associated with $X(x, y) = (f(y), 0)$ transforms $\mathbb{L}$ into configurations with particular symmetry properties, which we will call "y-odd".

In Section 5, we prove the following results.

Theorem 1.2. *Let $c = 0$ and $N = 2$ so that $\mathbb{L} = [0, T/2] \times [0, T]$ is a critical 1-lamella. Suppose that σ_0 is the critical value in (4.3) at which $\mathbb{L}$ loses its stability for $\sigma > \sigma_0$. Then, a 2-dimensional bifurcation branch arises at $(\mathbb{L}, \sigma_0)$ of non-planar critical points $(\mathbb{L}_\sigma, \sigma)$ of J, which is obtained by vertical translations from a 1-dimensional branch of volume-preserving y-odd configurations.*

Precisely, for small $t \in \mathbb{R}$, there exist two real functions $\beta(t)$ and $w(y, t)$ of class $\mathscr{C}^2$ with $\beta(0) = \beta'(0) = 0$ and with $w(\cdot, t)$ orthogonal to $\sin \frac{2\pi y}{T}$ in $L^2[0, T]$ and y-odd such that for every vertical shift s if

$$\begin{aligned} \sigma(t) &= \sigma_0 + \beta(t), \\ \mathbb{L}(t) &= \left\{\left(x + t \sin \frac{2\pi y}{T} + t w(y, t), y\right) : (x, y) \in \mathbb{L}\right\}, \\ \mathbb{L}(t, s) &= \mathbb{L}(t) + (0, s), \end{aligned}$$

then for all (t, s) the non-planar configuration $\big(\mathbb{L}(t, s), \sigma(t)\big)$ is a critical point of J.

Theorem 1.3. *Under the assumptions of Theorem 1.2, there exists a function $\mathcal{S}_0(T)$ such that if $\mathcal{S}_0(T) > 0$, bifurcation is supercritical and the branch $(\mathbb{L}_\sigma, \sigma)$ is stable; if $\mathcal{S}_0(T) < 0$, bifurcation is subcritical and the branch is unstable.*

Section 6 is devoted to the study of the sign of $\mathcal{S}_0(T)$: we prove that it is negative both for T small and large, and numerics indicate that $\mathcal{S}_0(T) < 0$ for all T, so bifurcation is always subcritical. In Section 7, we prove a bifurcation theorem in the general case $c \neq 0$; however, the type of bifurcation phenomenon would require a lengthy further study. Bifurcation for $c \neq 0$ is not attained by y-odd configurations; we remark that y-odd configurations are also (a special case of) symmetric configurations with respect to the center of $\mathbb{L}$: if we call such configurations "center symmetric" (see Section 7 for a precise definition), we have the following theorem.

Theorem 1.4. *Let $N = 2$, $|c| < 1$ and x_0 given by (1.7) so that $\mathbb{L} = [0, x_0] \times [0, T]$ is a critical 1-lamella. Suppose that σ_c is the critical value in (4.2) at which $\mathbb{L}$ loses its stability for $\sigma > \sigma_c$. Then, a 2-dimensional bifurcation branch arises at $(\mathbb{L}, \sigma_c)$ of* *non-planar critical points* $(\mathbb{L}_\sigma, \sigma)$ *of* J*, which is obtained by vertical translations from a* 1*-dimensional branch of (not necessarily volume preserving) center symmetric configurations.*

Finally, in Section 8, we extend all results to the case of a multiple lamella (a "k-lamella"), with surprising asymptotic results.

Theorem 1.5. *Let* $k \geq 2$*; Theorems* 1.2 *and* 1.4 *remain valid also for a* k*-lamella. In addition, in the case* $c = 0$*, the* k*-lamella undergoes a supercritical bifurcation for small values of* T *and a subcritical bifurcation for large values of* T*.*

Remark 1.6. In a periodic torus, there is no radially symmetric solution. In the whole space $\mathbb{R}^N$ [12, 13], only saddle-node-type bifurcations take place among all radially symmetric solution; however, bifurcation into non-radially symmetric solutions has not been performed. On the other hand, with an imposed volumetric constraint, bifurcation from radially symmetric into non-spherical shape has been shown in [28] with nuclear fission being the motivation behind this study. Similar bifurcation into non-radially symmetric solution for (1.1)—which is not a sharp interface model—has been treated in [45].

2. Preliminary material and derivatives of the nonlocal term

We begin by making sure smooth domains remain smooth under smooth flows.

Lemma 2.1. *Let* $m \in \mathbb{N}$ *and* $0 \leq \omega < 1$*. Under the flow* (1.3) *with an autonomous vector field* $X \in \mathcal{C}^{m,\omega}(\mathbb{T})$*, we have* $\Phi \in \mathcal{C}^{m,\omega}(\mathbb{T} \times \mathbb{R})$ *with respect to* $(\mathbf{z}, t)$*. In fact,* $\Phi \in \mathcal{C}^{m+1,\omega}(\mathbb{T} \times \mathbb{R})$ *except for* $(m+1)$*th partial derivatives entirely computed with respect to the components of* $\mathbf{z}$*. In particular,* $\Phi(\cdot, t)$ *is a* $\mathcal{C}^{m,\omega}$*-diffeomorphism from* $\mathbb{T}$ *to* $\mathbb{T}$ *for all* t*, and* $\partial E_t = (\partial E)_t \in \mathcal{C}^{m,\omega}$ *for each small* t*.*

Proof. We focus only on $m = 1$ and $\omega = 0$. That Φ is $\mathcal{C}^1(\mathbb{T} \times \mathbb{R})$ follows from [20, Theorem 7.5] or [29, p. 95, Theorem 3.1]. The right side of (1.3) is $\mathcal{C}^1$ with respect to $(\mathbf{z}, t)$ so that $\frac{\partial^2 \Phi}{\partial t^2}$ is $\mathcal{C}^2$. This is also true for the mixed second derivatives $\frac{\partial^2 \Phi}{\partial t \partial z_i}$, $i = 1, 2, \ldots, N$; see [29, p. 97, Corollary 3.2]. However, $\frac{\partial^2 \Phi}{\partial z_i \partial z_j}$ may not exist.

To prove the last assertion, without loss of generality, let $\mathbf{z} = 0 \in \partial E$ with ∂E being represented by $z_N = f(z_1, \ldots, z_{N-1})$, $f \in \mathcal{C}^1$, $f(0) = 0$, $\nabla f(0) = 0$, in a small neighborhood around $0 \in \mathbb{R}^{N-1}$. Define $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ such that

$$h(\mathbf{x}, t) := \Phi_N(\mathbf{x}, -t) - f(\Phi_1(\mathbf{x}, -t), \ldots, \Phi_{N-1}(\mathbf{x}, -t));$$

as a result, $h \in \mathcal{C}^1$ and $h(0, 0) = 0$. Suppose $\mathbf{x} \in \partial E_t$. Since

$$\frac{\partial h}{\partial x_N}(0, 0) = \frac{\partial \Phi_N}{\partial x_N}(0, 0) = 1,$$

the implicit function theorem tells us that x_N is a $\mathcal{C}^1$ function of $(x_1, \ldots, x_{N-1}, t)$ in a neighborhood of $(0, 0) \in \mathbb{R}^{N-1} \times \mathbb{R}$; in particular, $\partial E_t = (\partial E)_t \in \mathcal{C}^1$ for each small t. ∎

For any $0 < \omega < 1$, choose a large enough $p > 1$ so that $\mathcal{N}_E \in W^{2,p}(\mathbb{T}) \subset \mathcal{C}^{1,\omega}(\mathbb{T})$. If a critical set E is a local minimizer of J, elliptic regularity for (1.4) ensures that the interface $\partial E \in \mathcal{C}^{3,\omega}$ when $N \leq 7$; see, for example, [4, Theorem 2.8]. We call such a critical E regular, and it always satisfies (1.4) in the classical sense. In this paper, the non-planar critical sets $\mathbb{L}_t$ created through local bifurcation may not be local minimizers. However, they are obtained as a result of suitable smooth flows from a lamella so that they are regular critical sets; in fact, they can be smoother than $\mathcal{C}^{3,\omega}$ when the velocity fields are very smooth.

To analyze the bifurcation of (1.4), define $\mathcal{U} := \{E \subset \mathbb{T} : E \text{ is a } \mathcal{C}^5 \text{ domain}\}$, and $\mathcal{F} : \mathcal{U} \times \mathbb{R} \times (-1, 1) \to \mathcal{C}^3(\partial E)$ such that for each $E \in \mathcal{U}$

$$\mathcal{F}(E, \sigma, c) := \mathcal{K}\big|_{\partial E} - \frac{(1-c)\sigma}{2} + \sigma \mathcal{N}_E\big|_{\partial E}. \tag{2.1}$$

Let $E_t = \Phi(E, t)$ due to a prescribed $\mathcal{C}^5$ field X. By Lemma 2.1, it is clear that Φ is $\mathcal{C}^5$ in $(\mathbf{x}, t)$ and E_t is a $\mathcal{C}^5$ domain. We need to compute up to the third derivatives of $\mathcal{F}(E_t, \sigma)$ with respect to (t, σ) at $t = 0$. With $\partial E_t \in \mathcal{C}^5$ in $(\mathbf{z}, t) \in \partial E \times (-\varepsilon, \varepsilon)$, we have $\mathcal{K}\big|_{\partial E_t} \in \mathcal{C}^3(\partial E \times (-\varepsilon, \varepsilon))$ and $\mathcal{N}_{E_t}\big|_{\partial E_t} \in \mathcal{C}^5$ in t for fixed $z \in \partial E$; the latter smoothness is due to the differentiability requirement (2.2) and (2.6) when expanding Φ and $J\Phi(\mathbf{z}, t)$ in power series of t; see the proof of Lemma 2.2 in which $X \in \mathcal{C}^3$ gives a third-order derivative of the nonlocal term.

For the rest of this paper, observe that the Green's function $\mathbf{G}(\mathbf{x}, \mathbf{y})$ for the modified Helmholtz operator in a square torus depends only on $\mathbf{x} - \mathbf{y}$; therefore, we write

$$\mathbf{G}(\mathbf{x}, \mathbf{y}) =: \widehat{G}(\mathbf{x} - \mathbf{y})$$

for some $\widehat{G} : \mathbb{T} \to \mathbb{R}$. At the same time, D_V denotes the usual directional derivative along $V \in \mathbb{R}^2$. For example, if $Z := D_X X = DX[X]$, we mean $Z_i = X_j D_j X_i$ with summing over repeated index; similarly,

$$Y := D_X Z = D_X D_X X.$$

It is clear that

$$\Phi(\mathbf{z}, t) = \mathbf{z} + tX(\mathbf{z}) + \frac{t^2}{2} Z(\mathbf{z}) + \frac{t^3}{6} Y(\mathbf{z}) + O(t^4). \tag{2.2}$$

Having finished with preliminaries, we compute up to the third derivatives of the nonlocal term $\mathcal{N}_E$ at any regular set E with respect to any prescribed smooth flow field X and express the first 2 derivatives as integrals on the boundary of E. For simplicity in representation, we sum over repeated indices in the following.

Lemma 2.2 (Derivatives of nonlocal term). *Suppose $E, X \in \mathcal{C}^3(\mathbb{T})$, and $E_t = \Phi(E, t)$ due to the field X. Let $\mathbf{x} \in \partial E$. Then,*

$$\frac{\partial}{\partial t}\left(\mathcal{N}_{E_t}\big|_{\partial E_t}\right)\Big|_{t=0} = (X \cdot \nabla \mathcal{N}_E)\big|_{\partial E} + \int_{\partial E} \widehat{G}(\mathbf{x} - \mathbf{y})\ (X \cdot \nu)(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}}, \tag{2.3}$$

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\left(\mathcal{N}_{E_t}|_{\partial E_t}\right)\Big|_{t=0} = \frac{1}{2}\int_{\partial E} D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))X_j(\mathbf{y})\,\nu_j(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}}$$
$$-\frac{1}{2}\int_{\partial E} D_i\widehat{G}(\mathbf{x}-\mathbf{y})X_i(\mathbf{x})(X_j(\mathbf{x})-X_j(\mathbf{y}))\nu_j(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}}$$
$$+\left(\frac{1}{2}Z\cdot\nabla\mathcal{N}_E\right)\Big|_{\partial E}$$
$$+\frac{1}{2}\int_{\partial E}\widehat{G}(\mathbf{x}-\mathbf{y})X\cdot\nu(\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}}, \tag{2.4}$$

$$\frac{1}{6}\frac{\partial^3}{\partial t^3}\left(\mathcal{N}_{E_t}|_{\partial E_t}\right)\Big|_{t=0} = \textit{sum of RHS of } (2.9),\ (2.12), (2.15),\ \textit{and } (2.16). \tag{2.5}$$

Proof. To reduce integrals on ∂E_t to integrals on ∂E for comparison, we will use the Jacobian determinant $J\Phi$ being

$$J\Phi(\mathbf{y},t) = 1 + t\,\mathrm{div}\,X(\mathbf{y}) + \frac{t^2}{2}\,\mathrm{div}((\mathrm{div}\,X)X)(\mathbf{y})$$
$$+\frac{t^3}{6}\,\mathrm{div}(\mathrm{div}((\mathrm{div}\,X)X)X)(\mathbf{y}) + O(t^4). \tag{2.6}$$

Note that $J\Phi \in \mathcal{C}(\mathbb{T}\times(-\varepsilon,\varepsilon))$ since $X\in\mathcal{C}^3(\mathbb{T})$. We fix a point $\mathbf{x}\in\partial E$; since formulas will generally be very long, in the proof of (2.3), we will employ the following shorthand, where $\mathbf{y}$ will denote another generic point in ∂E:

$$\delta\mathbf{x} := \mathbf{x}-\mathbf{y}, \quad \delta\Phi := \Phi(\mathbf{x},t)-\Phi(\mathbf{y},t), \quad \delta X := X(\mathbf{x})-X(\mathbf{y})$$

and the same for δY and δZ. We have, for fixed $\mathbf{x}\in\partial E$,

$$\mathcal{D} := \mathcal{N}_{E_t}(\Phi(\mathbf{x},t)) - \mathcal{N}_E(\mathbf{x}) = \int_{E_t} G(\Phi(\mathbf{x},t),\mathbf{y})\,d\mathbf{y} - \int_E G(\mathbf{x},\mathbf{y})\,d\mathbf{y}$$
$$= \int_E G(\Phi(\mathbf{x},t),\Phi(\mathbf{y},t))\,J\Phi(\mathbf{y},t)\,d\mathbf{y} - \int_E G(\mathbf{x},\mathbf{y})\,d\mathbf{y}$$
$$= \int_E \left(\widehat{G}(\delta\Phi)-\widehat{G}(\delta\mathbf{x})\right)d\mathbf{y} + t\int_E \widehat{G}(\delta\Phi)\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y}$$
$$+\frac{t^2}{2}\int_E \widehat{G}(\delta\Phi)\,\mathrm{div}((\mathrm{div}\,X)X)(\mathbf{y})\,d\mathbf{y}$$
$$+\frac{t^3}{6}\int_E \widehat{G}(\delta\Phi)\,\mathrm{div}(\mathrm{div}((\mathrm{div}\,X)X)X)(\mathbf{y})\,d\mathbf{y} + O(t^4)$$
$$:= I + t\,II + \frac{t^2}{2}\,III + \frac{t^3}{6}IV + O(t^4).$$

We now treat the different terms.

Step 1. The first derivative of the first term is straightforward:

$$I = \int_E \left(\widehat{G}(\delta\Phi)-\widehat{G}(\delta\mathbf{x})\right)d\mathbf{y} = \int_E d\mathbf{y}\int_0^1 \frac{d}{d\tau}\widehat{G}(\delta\mathbf{x}+\tau(\delta\Phi-\delta\mathbf{x}))\,d\tau$$

$$= \int_E d\mathbf{y} \int_0^1 D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x}))(\delta\Phi - \delta\mathbf{x})\, d\tau$$
$$= \int_E d\mathbf{y} \int_0^1 D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x}))\Big(t\delta X + \frac{t^2}{2}\delta Z + \frac{t^3}{6}\delta Y + O(t^4)\Big)\, d\tau.$$

Hence,

$$\frac{\partial I}{\partial t}\bigg|_{t=0} = \lim_{t\to 0}\frac{I}{t} = \int_E D\widehat{G}(\delta\mathbf{x})\,\delta X\, d\mathbf{y}. \tag{2.7}$$

Step 2. From (2.7), we have

$$I - t\frac{\partial I}{\partial t}\bigg|_{t=0} = t\int_E d\mathbf{y}\int_0^1 \big\{D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x})) - D\widehat{G}(\delta\mathbf{x})\big\}\,\delta X\, d\mathbf{y}$$
$$+ \frac{t^2}{2}\int_E d\mathbf{y}\int_0^1 D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x}))\,\delta Z\, d\mathbf{y} + O(t^3).$$

With $\widehat{G}(\mathbf{z}) = O(\log|\mathbf{z}|)$, $D\widehat{G}(\mathbf{z}) = O(1/|\mathbf{z}|)$, $D^2\widehat{G}(\mathbf{z}) = O(1/|\mathbf{z}|^2)$, we have

$$\frac{1}{2}\frac{\partial^2 I}{\partial t^2}\bigg|_{t=0} = \lim_{t\to 0}\frac{I - t\frac{\partial I}{\partial t}\big|_{t=0}}{t^2}$$
$$= \lim_{t\to 0}\int_E d\mathbf{y}\int_0^1 \frac{D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x})) - D\widehat{G}(\delta\mathbf{x})}{t}\,\delta X\, d\tau$$
$$+ \frac{1}{2}\int_E D\widehat{G}(\delta\mathbf{x})\,\delta Z\, dy$$
$$= \int_E d\mathbf{y}\int_0^1 D^2\widehat{G}(\delta\mathbf{x})[\tau\delta X, \delta X]\, d\tau + \frac{1}{2}\int_E D\widehat{G}(\delta\mathbf{x})\,\delta Z\, d\mathbf{y}$$
$$= \frac{1}{2}\int_E D^2\widehat{G}(\delta\mathbf{x})[\delta X, \delta X]\, d\mathbf{y} + \frac{1}{2}\int_E D\widehat{G}(\delta\mathbf{x})\,\delta Z\, d\mathbf{y}. \tag{2.8}$$

Note that the two integrands on the right are bounded functions due to the smoothness of X and Z; both factors $\delta X = X(\mathbf{x}) - X(\mathbf{y})$ and $\delta Z = Z(\mathbf{x}) - Z(\mathbf{y})$ are of $O(|\mathbf{x} - \mathbf{y}|)$.

Step 3. The next item requires a more involved calculation:

$$I - t\frac{\partial I}{\partial t}\bigg|_{t=0} - \frac{t^2}{2}\frac{\partial^2 I}{\partial t^2}\bigg|_{t=0} = t\int_E d\mathbf{y}\int_0^1 \big(D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x})) - D\widehat{G}(\delta\mathbf{x})\big)\,\delta X\, d\tau$$
$$+ \frac{t^2}{2}\int_E d\mathbf{y}\int_0^1 \big(D\widehat{G}(\delta\mathbf{x}+\tau(\delta\Phi-\delta\mathbf{x})) - D\widehat{G}(\delta\mathbf{x})\big)\,\delta Z\, d\tau$$
$$+ \frac{t^3}{6}\int_E d\mathbf{y}\int_0^1 D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x}))\,\delta Y\, d\tau$$
$$- \frac{t^2}{2}\int_E D^2\widehat{G}(\delta\mathbf{x})[\delta X, \delta X]\, d\mathbf{y} + O(t^4)$$
$$:= tA_1 + \frac{t^2}{2}A_2 + \frac{t^3}{6}A_3 - \frac{t^2}{2}A_4 + O(t^4).$$

Note that

$$\begin{aligned} tA_1 - \frac{t^2}{2}A_4 \\ &= t^2 \int_E d\mathbf{y} \int_0^1 \tau \, d\tau \int_0^1 D^2\widehat{G}(\delta\mathbf{x} + \alpha\tau(\delta\Phi - \delta\mathbf{x}))\Big[\delta X + \frac{t}{2}\delta Z + O(t^2), \delta X\Big] \, d\alpha \\ &\quad - t^2 \int_E d\mathbf{y} \int_0^1 \tau \, d\tau \int_0^1 D^2\widehat{G}(\delta\mathbf{x})[\delta X, \delta X] \, d\alpha, \end{aligned}$$

which leads to

$$\begin{aligned} \lim_{t\to 0} \frac{tA_1 - \frac{t^2}{2}A_4}{t^3} &= \int_E d\mathbf{y} \int_0^1 \tau \, d\tau \int_0^1 \alpha\tau D^3\widehat{G}(\delta\mathbf{x})[\delta X, \delta X, \delta X] \, d\alpha \\ &\quad + \frac{1}{4}\int_E D^2\widehat{G}(\delta\mathbf{x})[\delta Z, \delta X] \, d\mathbf{y} \\ &= \frac{1}{6}\int_E D^3\widehat{G}(\delta\mathbf{x})[\delta X, \delta X, \delta X] \, d\mathbf{y} + \frac{1}{4}\int_E D^2\widehat{G}(\delta\mathbf{x})[\delta Z, \delta X] \, d\mathbf{y}. \end{aligned}$$

Since

$$\frac{t^2}{2}A_2 = \frac{t^2}{2}\int_E d\mathbf{y} \int_0^1 \tau \, d\tau \int_0^1 D^2\widehat{G}(\delta\mathbf{x} + \alpha\tau(\delta\Phi - \delta\mathbf{x}))[\delta\Phi - \delta\mathbf{x}, \delta Z] \, d\alpha$$

and $\delta\Phi - \delta\mathbf{x} = t\,\delta X + O(t^2)$, it can now be verified that

$$\begin{aligned} \frac{1}{6}\frac{\partial^3 I}{\partial t^3}\Big|_{t=0} &= \lim_{t\to 0}\frac{1}{t^3}\Big(tA_1 + \frac{t^2}{2}A_2 + \frac{t^3}{6}A_3 - \frac{t^2}{2}A_4 + O(t^4)\Big) \\ &= \frac{1}{6}\int_E D^3\widehat{G}(\delta\mathbf{x})[\delta X, \delta X, \delta X] \, d\mathbf{y} \\ &\quad + \frac{1}{2}\int_E D^2\widehat{G}(\delta\mathbf{x})[\delta X, \delta Z] \, d\mathbf{y} + \frac{1}{6}\int_E D\widehat{G}(\delta\mathbf{x})\,\delta Y \, d\mathbf{y}. \end{aligned} \tag{2.9}$$

Step 4. Observe $(t\,II)\big|_{t=0} = 0$ so that

$$\frac{\partial(t\,II)}{\partial t}\Big|_{t=0} = \lim_{t\to 0}\frac{t\,II}{t} = \int_E \widehat{G}(\delta\mathbf{x}) \operatorname{div} X(\mathbf{y}) \, d\mathbf{y}. \tag{2.10}$$

Step 5. It is immediate that

$$\begin{aligned} t\,II - t\frac{\partial(t\,II)}{\partial t}\Big|_{t=0} &= t\int_E \big(\widehat{G}(\delta\Phi) - \widehat{G}(\delta\mathbf{x})\big) \operatorname{div} X(\mathbf{y}) \, d\mathbf{y} \\ &= t^2 \int_E d\mathbf{y} \int_0^1 D\widehat{G}(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x}))(\delta X + O(t)) \operatorname{div} X(\mathbf{y}) \, d\tau, \end{aligned}$$

which yields

$$\frac{1}{2}\frac{\partial^2(t\,II)}{\partial t^2}\Big|_{t=0} = \lim_{t\to 0}\frac{t\,II - t\frac{\partial(t\,II)}{\partial t}\big|_{t=0}}{t^2} = \int_E D\widehat{G}(\delta\mathbf{x})\,\delta X \operatorname{div} X(\mathbf{y}) \, d\mathbf{y}. \tag{2.11}$$

Step 6. Pushing further,

$$
\begin{aligned}
t\,II - t\,\frac{\partial(t\,II)}{\partial t}\bigg|_{t=0} - \frac{t^2}{2}\frac{\partial^2(t\,II)}{\partial t^2}\bigg|_{t=0}
&= t\int_E \big(\widehat{G}(\delta\Phi) - \widehat{G}(\delta\mathbf{x})\big)\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y} - t^2\int_E D\widehat{G}(\delta\mathbf{x})\,\delta X\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y} \\
&= t^2\int_E d\mathbf{y}\int_0^1 \mathrm{div}\,X(\mathbf{y})D\widehat{G}\big(\delta\mathbf{x} + \tau(\delta\Phi - \delta\mathbf{x})\big)\Big(\delta X + \frac{t}{2}\delta Z + O(t^2)\Big)\,d\tau \\
&\quad - t^2\int_E \mathrm{div}\,X(\mathbf{y})D\widehat{G}(\delta\mathbf{x})\,\delta X\,d\mathbf{y} \\
&= t^3\int_E d\mathbf{y}\int_0^1 \tau\,d\tau\int_0^1 \mathrm{div}\,X(\mathbf{y})D^2\widehat{G}(\delta\mathbf{x})[\delta X, \delta X]\,d\alpha \\
&\quad + \frac{t^3}{2}\int_E \mathrm{div}\,X(\mathbf{y})D\widehat{G}(\delta\mathbf{x})\,\delta Z\,d\mathbf{y} + O(t^4)
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{1}{6}\frac{\partial^3(t\,II)}{\partial t^3}\bigg|_{t=0} &= \lim_{t\to 0}\frac{1}{t^3}\bigg(t\,II - t\,\frac{\partial(t\,II)}{\partial t}\bigg|_{t=0} - \frac{t^2}{2}\frac{\partial^2(t\,II)}{\partial t^2}\bigg|_{t=0}\bigg) \\
&= \frac{1}{2}\int_E \mathrm{div}\,X(\mathbf{y})\big(D^2\widehat{G}(\delta\mathbf{x})[\delta X, \delta X] + D\widehat{G}(\delta\mathbf{x})\,\delta Z\big)\,d\mathbf{y}. \qquad (2.12)
\end{aligned}
$$

Step 7. Observe that $(\frac{t^2}{2}\,III)\big|_{t=0} = 0$; thus, it follows by similar calculations that

$$
\frac{\partial(\frac{t^2}{2}\,III)}{\partial t}\bigg|_{t=0} = \lim_{t\to 0}(\frac{t}{2}III) = 0, \qquad (2.13)
$$

$$
\frac{1}{2}\frac{\partial^2(\frac{t^2}{2}III)}{\partial t^2}\bigg|_{t=0} = \lim_{t\to 0}\frac{\frac{t^2}{2}III}{t^2} = \frac{1}{2}\int_E \widehat{G}(\delta\mathbf{x})\,\mathrm{div}((\mathrm{div}\,X)X)(\mathbf{y})\,d\mathbf{y}, \qquad (2.14)
$$

$$
\begin{aligned}
\frac{1}{6}\frac{\partial^3(\frac{t^2}{2}\,III)}{\partial t^3}\bigg|_{t=0} &= \lim_{t\to 0}\frac{1}{t^3}\bigg(\frac{t^2}{2}\,III - t\,\frac{\partial(\frac{t^2}{2}\,III)}{\partial t}\bigg|_{t=0} - \frac{t^2}{2}\frac{\partial^2(\frac{t^2}{2}\,III)}{\partial t^2}\bigg|_{t=0}\bigg) \\
&= \frac{1}{2}\int_E D\widehat{G}(\delta\mathbf{x})\,\delta X\,\mathrm{div}((\mathrm{div}\,X)X)(\mathbf{y})\,d\mathbf{y}. \qquad (2.15)
\end{aligned}
$$

Step 8. We have

$$
\frac{t^3}{6}IV\bigg|_{t=0} = \frac{\partial(\frac{t^3}{6}IV)}{\partial t}\bigg|_{t=0} = \frac{\partial^2(\frac{t^3}{6}IV)}{\partial t^2}\bigg|_{t=0} = 0,
$$

leading to

$$
\frac{1}{6}\frac{\partial^3(\frac{t^2}{2}\,IV)}{\partial t^3}\bigg|_{t=0} = \lim_{t\to 0}\frac{\frac{t^3}{6}IV}{t^3} = \frac{1}{6}\int_E \widehat{G}(\delta\mathbf{x})\,\mathrm{div}(\mathrm{div}((\mathrm{div}\,X)X)X)(\mathbf{y})\,d\mathbf{y}. \qquad (2.16)
$$

We now abandon the shorthand with δ. By employing (2.7), (2.10), and (2.13), we now derive

$$\begin{aligned}
\frac{\partial}{\partial t}\big(\mathcal{N}_{E_t}|_{\partial E_t}\big)\big|_{t=0} &= \frac{\partial \mathcal{D}}{\partial t}\bigg|_{t=0} \\
&= \int_E D\widehat{G}(\mathbf{x}-\mathbf{y})(X(\mathbf{x})-X(\mathbf{y}))\,d\mathbf{y} + \int_E \widehat{G}(\mathbf{x}-\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y} \\
&= X\cdot\nabla_{\mathbf{x}}\Big(\int_E \widehat{G}(\mathbf{x}-\mathbf{y})\,d\mathbf{y}\Big) \\
&\quad + \int_E \widehat{G}(\mathbf{x}-\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y} + \int_E \nabla_{\mathbf{y}}\big(\widehat{G}(\mathbf{x}-\mathbf{y})\big)\cdot X(\mathbf{y})\,d\mathbf{y}.
\end{aligned}$$

There is a sign change in the last term above since $\nabla_{\mathbf{x}}\widehat{G}(\mathbf{x}-\mathbf{y}) = -\nabla_{\mathbf{y}}(\widehat{G}(\mathbf{x}-\mathbf{y}))$. An application of the divergence theorem to the last 2 terms yields (2.3).

For the second derivative, we obtain

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\big(\mathcal{N}_{E_t}|_{\partial E_t}\big)\big|_{t=0} = \text{sum of RHS of (2.8), (2.11), and (2.14).} \tag{2.17}$$

To cast these terms in an alternate form, observe that

$$\begin{aligned}
&-\frac{1}{2}\int_{\partial E}\big[D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))\big](X_j(\mathbf{x})-X_j(\mathbf{y}))\nu_j(\mathbf{y})\,d\mathcal{H}^1 \\
&\quad= \frac{1}{2}\int_E \frac{\partial}{\partial \mathbf{y}_j}\big\{\big[-D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))\big](X_j(\mathbf{x})-X_j(\mathbf{y}))\big\}\,d\mathbf{y} \\
&\quad= \frac{1}{2}\int_E \big\{D_{ij}\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))(X_j(\mathbf{x})-X_j(\mathbf{y})) \\
&\qquad\quad + D_i\widehat{G}(\mathbf{x}-\mathbf{y})D_jX_i(\mathbf{y})(X_j(\mathbf{x})-X_j(\mathbf{y})) \\
&\qquad\quad + D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))\,\mathrm{div}\,X(\mathbf{y})\big\}\,d\mathbf{y}.
\end{aligned}$$

Combining with (2.17), we end up with

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial t^2}\big(\mathcal{N}_{E_t}|_{\partial E_t}\big)\big|_{t=0} \\
&\quad= -\frac{1}{2}\int_{\partial E}\big[D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))\big](X_j(\mathbf{x})-X_j(\mathbf{y}))\nu_j(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}} \\
&\qquad - \frac{1}{2}\int_E D_i\widehat{G}(\mathbf{x}-\mathbf{y})D_jX_i(\mathbf{y})(X_j(\mathbf{x})-X_j(\mathbf{y}))\,d\mathbf{y} \\
&\qquad + \frac{1}{2}\int_E D\widehat{G}(\mathbf{x}-\mathbf{y})[(Z(\mathbf{x})-Z(\mathbf{y})) + (X(\mathbf{x})-X(\mathbf{y}))\,\mathrm{div}\,X(\mathbf{y})]\,d\mathbf{y} \\
&\qquad + \frac{1}{2}\int_E \widehat{G}(\mathbf{x}-\mathbf{y})\,\mathrm{div}((\mathrm{div}\,X)X)(\mathbf{y})\,d\mathbf{y},
\end{aligned}$$

which further simplifies after we observe that

$$-\int_E D\widehat{G}(\mathbf{x}-\mathbf{y})X(\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y} = \int_E \frac{\partial}{\partial \mathbf{y}_j}(\widehat{G}(\mathbf{x}-\mathbf{y}))X_j(\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y}$$

combines with the last term to form a boundary integral and we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial t^2}\big(\mathcal{N}_{E_t}\big|_{\partial E_t}\big)\big|_{t=0}\\
&\quad = -\frac{1}{2}\int_{\partial E}\big[D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))\big](X_j(\mathbf{x})-X_j(\mathbf{y}))\nu_j(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}}\\
&\qquad -\frac{1}{2}X_jD_i\int_E\widehat{G}(\mathbf{x}-\mathbf{y})D_jX_i(\mathbf{y})\,d\mathbf{y} + \frac{1}{2}X_iD_i\int_E\widehat{G}(\mathbf{x}-\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y}\\
&\qquad + \left(\frac{1}{2}Z\cdot\nabla\mathcal{N}_E\right)\bigg|_{\partial E} + \frac{1}{2}\int_{\partial E}\widehat{G}(\mathbf{x}-\mathbf{y})X\cdot\nu(\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}}.
\end{aligned} \tag{2.18}$$

Now, we see that the sum of the integrals in the first two lines may be written as a boundary integral by employing the following.

Lemma 2.3. *Let E and X be regular, $\mathbf{x}\in\partial E$, and $f:\mathbb{R}^2\to\mathbb{R}$ be a function of class $\mathcal{C}^2$ symmetric with respect the origin. Then,*

$$\begin{aligned}
&-\frac{1}{2}\int_{\partial E}[D_if(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x})-X_i(\mathbf{y}))](X_j(\mathbf{x})-X_j(\mathbf{y}))\nu_j(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}}\\
&-\frac{1}{2}X_jD_i\int_E f(\mathbf{x}-\mathbf{y})D_jX_i(\mathbf{y})\,d\mathbf{y} + \frac{1}{2}X_iD_i\int_E f(\mathbf{x}-\mathbf{y})\,\mathrm{div}\,X(\mathbf{y})\,d\mathbf{y}\\
&= \frac{1}{2}\int_{\partial E}D_if(\mathbf{x}-\mathbf{y})\big(X_j(\mathbf{y})[X_i(\mathbf{x})-X_i(\mathbf{y})]-X_i(\mathbf{x})[X_j(\mathbf{x})-X_j(\mathbf{y})]\big)\nu_j(\mathbf{y})d\mathcal{H}^1_{\mathbf{y}}.
\end{aligned}$$

Proof. Explicitly writing $D_{\mathbf{x}_i}$ or $D_{\mathbf{y}_i}$ when differentiation is taken with respect to the ith direction of the $\mathbf{x}$ or $\mathbf{y}$ variables, we have

$$\begin{aligned}
D_{\mathbf{x}_i}\int_E f(\mathbf{x}-\mathbf{y})D_{\mathbf{y}_j}X_j(\mathbf{y})\,d\mathbf{y} &= D_{\mathbf{x}_i}\int_E D_{\mathbf{y}_j}[f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y})]\,d\mathbf{y}\\
&\quad - D_{\mathbf{x}_i}\int_E D_{\mathbf{y}_j}(f(\mathbf{x}-\mathbf{y}))X_j(\mathbf{y})\,d\mathbf{y}.
\end{aligned} \tag{2.19}$$

We work on the last integral: by the symmetry of f,

$$\begin{aligned}
&-D_{\mathbf{x}_i}\int_E D_{\mathbf{y}_j}(f(\mathbf{x}-\mathbf{y}))X_j(\mathbf{y})\,d\mathbf{y}\\
&\quad = D_{\mathbf{x}_i}\int_E D_{\mathbf{x}_j}(f(\mathbf{x}-\mathbf{y}))X_j(\mathbf{y})\,d\mathbf{y} = D_{\mathbf{x}_i}\int_E D_{\mathbf{x}_j}(f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y}))\,d\mathbf{y}\\
&\quad = D_{\mathbf{x}_j}\int_E D_{\mathbf{x}_i}(f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y}))\,d\mathbf{y} = D_{\mathbf{x}_j}\int_E D_{\mathbf{x}_i}(f(\mathbf{x}-\mathbf{y}))X_j(\mathbf{y})\,d\mathbf{y}\\
&\quad = -D_{\mathbf{x}_j}\int_E D_{\mathbf{y}_i}(f(\mathbf{x}-\mathbf{y}))X_j(\mathbf{y})\,d\mathbf{y}\\
&\quad = -D_{\mathbf{x}_j}\int_E D_{\mathbf{y}_i}(f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y}))\,d\mathbf{y} + D_{\mathbf{x}_j}\int_E f(\mathbf{x}-\mathbf{y})D_{\mathbf{y}_i}X_j(\mathbf{y})\,d\mathbf{y}
\end{aligned}$$

so that (2.19) gives

$$
\begin{aligned}
D_{\mathbf{x}_i}\int_E f(\mathbf{x}-\mathbf{y})D_{\mathbf{y}_j}X_j(\mathbf{y})\,d\mathbf{y} = D_{\mathbf{x}_i}\int_E D_{\mathbf{y}_j}[f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y})]\,d\mathbf{y}\\
- D_{\mathbf{x}_j}\int_E D_{\mathbf{y}_i}(f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y}))\,d\mathbf{y}\\
+ D_{\mathbf{x}_j}\int_E f(\mathbf{x}-\mathbf{y})D_{\mathbf{y}_i}X_j(\mathbf{y})\,d\mathbf{y},
\end{aligned}
$$

and therefore (we switch indexes in the last two terms below),

$$
\begin{aligned}
X_i(\mathbf{x})D_{\mathbf{x}_i}\int_E f(\mathbf{x}-\mathbf{y})D_{\mathbf{y}_j}X_j(\mathbf{y})\,d\mathbf{y} = X_i(\mathbf{x})\int_E D_{\mathbf{y}_j}[D_{\mathbf{x}_i}f(\mathbf{x}-\mathbf{y})X_j(\mathbf{y})]\,d\mathbf{y}\\
- X_j(\mathbf{x})\int_E D_{\mathbf{y}_j}(D_{\mathbf{x}_i}f(\mathbf{x}-\mathbf{y})X_i(\mathbf{y}))\,d\mathbf{y}\\
+ X_j(\mathbf{x})D_{\mathbf{x}_i}\int_E f(\mathbf{x}-\mathbf{y})D_{\mathbf{y}_j}X_i(\mathbf{y})\,d\mathbf{y}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
&-\frac{1}{2}X_jD_i\int_E f(\mathbf{x}-\mathbf{y})D_jX_i(\mathbf{y})\,d\mathbf{y}+\frac{1}{2}X_iD_i\int_E f(\mathbf{x}-\mathbf{y})\operatorname{div}X(\mathbf{y})\,d\mathbf{y}\\
&=\frac{1}{2}\int_{\partial E}D_{\mathbf{x}_i}f(\mathbf{x}-\mathbf{y})[X_j(\mathbf{y})X_i(\mathbf{x})-X_i(\mathbf{y})X_j(\mathbf{x})]\nu_j(\mathbf{y})d\mathcal{H}^1_{\mathbf{y}}.
\end{aligned}
$$

The end of the proof is algebraic, since writing $a_h = X_h(\mathbf{x})$ and $b_h = X_h(\mathbf{y})$ gives

$$
-(a_i-b_i)(a_j-b_j)+a_ib_j-a_jb_i = a_i(b_j-a_j)+b_j(a_i-b_i). \qquad \blacksquare
$$

Then, (2.4) follows from (2.18) by approximating $\widehat{G}$ with regular functions and applying Lemma 2.3. Finally, the third derivative is given by (2.5), thus completing the proof of Lemma 2.2. ∎

3. Derivatives of mean curvature

The first and second derivatives of the area functional $\mathcal{P}_{\mathbb{T}}(E)$ at a critical set E are well known [34, Chapter 1]; they are represented by $\int_{\partial E}\mathcal{K}\eta\,d\mathcal{H}^{N-1}$ and the first line on RHS of (1.5), respectively, where $\eta = X\cdot\nu$. As the curvature $\mathcal{K}$ appears in the first derivative of the area functional, it is not a surprise that one can derive the first derivative of $\mathcal{K}$ at E from the second derivative of the area functional. However, we need up to the third derivative of curvature for a bifurcation analysis. Since we cannot find a ready reference, we compute these results in this section. Some concepts and tools in geometry will be needed; for the convenience of non-geometers, they are documented at the beginning of this section, and the new parts begin after Lemma 3.2. Summing over repeated indices is assumed throughout this section.

Recall that $E \subset \mathbb{T}$ evolves into E_t due to the field X. Let $\mathbf{r} : U \to \mathbb{R}^N$ be a parametrization of a neighborhood of $p \in \partial E$ with $U \subset \mathbb{R}^{N-1}$, $\partial_i := \frac{\partial \mathbf{r}}{\partial u^i}$ for $i = 1, 2, \ldots, N-1$ so that $\{\partial_i\}_{i=1}^{N-1}$ is a basis of the tangent space. Without loss of generality, $\{\partial_i\big|_p\}_{i=1}^{N-1}$ is orthonormal (but not at other points in ∂E). Let $p_t := \Phi(p,t) \in \partial E_t$ and $U_t := \Phi(U,t) \subset \partial E_t$. It is clear that $\Phi(\mathbf{r}(\cdot), t)$ is a parametrization of U_t; in particular,

$$e_i\big|_{p_t} := \frac{\partial \Phi(\mathbf{r}(\cdot), t)}{\partial u^i} = D_z \Phi(p,t)\partial_i,$$

and $\{e_i|_{p_t}\}_{i=1}^{N-1}$ is a basis of the tangent space $T_{p_t}\partial E_t$.

Suppose that $f : \mathbb{R}^N \to \mathbb{R}$ is a scalar function; then,

$$\frac{\partial}{\partial u^i} f(\Phi(\mathbf{r}(\cdot), t)) = (Df)e_i\big|_{p_t} = D_{e_i} f\big|_{p_t},$$

which depends only on the values of f on ∂E_t; similarly, $\frac{\partial}{\partial t} f(\Phi(\mathbf{r}(\cdot), t)) = (Df)X\big|_{p_t} = D_X f\big|_{p_t}$. Thus,

$$D_X D_{e_i} f\big|_{p_t} = \frac{\partial}{\partial t}\frac{\partial}{\partial u^i} f(\Phi(\mathbf{r}(\cdot), t)) = \frac{\partial}{\partial u^i}\frac{\partial}{\partial t} f(\Phi(\mathbf{r}(\cdot), t)) = D_{e_i} D_X f\big|_{p_t}.$$

Similarly, $\frac{\partial}{\partial t}\frac{\partial}{\partial u^i}\Phi(\mathbf{r}(\cdot), t) = \frac{\partial}{\partial u^i}\frac{\partial}{\partial t}\Phi(\mathbf{r}(\cdot), t)$, which leads to

$$D_X e_i = D_{e_i} X. \tag{3.1}$$

Evaluating at $t = 0$ and $p \in \partial E$, this last relation says that how $\partial_i\big|_p$ evolves in t depends only on X in a neighborhood of p on ∂E. Next, recall that we have employed $\{\mathbf{b}_i\}_{i=1}^N$ in representing the standard basis in $\mathbb{R}^N$. Write $Q = Q^i \mathbf{b}_i$ for any vector field Q on the torus; it follows that

$$D_X D_{e_i} Q = (D_X D_{e_i} Q^j)e_j = (D_{e_i} D_X Q^j)e_j = D_{e_j} D_X Q.$$

The first fundamental form is given by $g_{ij} := \langle e_i, e_j\rangle$ on ∂E_t. Let (g^{ij}) be the inverse of the matrix (g_{ij}) and $e^i := g^{ij}e_j$. It is readily verified that $\langle e^i, e_j\rangle = \delta_{ij}$ and $g^{ij} = \langle e^i, e^j\rangle$. With our chosen coordinate system, $g_{ij}\big|_p = \delta_{ij}$; this relation is false at $p_t \in \partial E_t$ when $t > 0$. Let $\nu\big|_{p_t}$ be the unit outward normal vector at $p_t \in \partial E_t$. For every t and every $p_t \in \partial E_t$, define the shape operator (Weingarten map) $S : T_{p_t}\partial E_t \to T_{p_t}\partial E_t$ by $SV := -D_V \nu$ for any tangent vector $V \in T_{p_t}\partial E_t$ [42, p. 406]. Note that we should have written S_{t,p_t}, but not to overload the notation we omitted the subscript—anyway the time and point will always be clear from the context, and no time derivative of S will be taken. Clearly, SV is a tangent vector as $\langle SV, \nu\rangle = -\langle D_V \nu, \nu\rangle = -\frac{1}{2} D_V\langle \nu, \nu\rangle = 0$; at the same time, $S(V^i e_i) = V^i S e_i$ even when V^i depends on $p_t \in \partial E_t$. In addition,

$$\langle Se_i, e_j\rangle = -\langle D_{e_i}\nu, e_j\rangle = \langle \nu, D_{e_i} e_j\rangle = \left\langle \nu, \frac{\partial^2 \Phi(\mathbf{r}(\cdot), t)}{\partial u_i \partial u_j}\right\rangle = \langle Se_j, e_i\rangle;$$

as a result, $\langle SV, W\rangle = \langle V, SW\rangle$ for any tangent vectors $V, W \in T_{p_t}\partial E_t$. In other words, the operator S is self-adjoint. Consequently, there are eigenvalues $\{-\rho_i\}_{i=1}^{N-1}$ and corresponding orthonormal eigenvectors $\{V_i\}_{i=1}^{N-1}$ such that $SV_i = -\rho_i V_i$; $\{\rho_i\}_{i=1}^{N-1}$ and $\{V_i\}_{i=1}^{N-1}$ are known as the principal curvatures and the principal directions, respectively. Without loss of generality, we can assume that $\{\partial_i\}_{i=1}^{N-1}$ are the principal directions at $p \in \partial E$ when $t = 0$. Note that this is only true at a point, rather than for a whole neighborhood of p.

The mean curvature is given by

$$\mathcal{K} := \sum_{i=1}^{N-1} \rho_i = -\langle Se_i, e^i\rangle := -\operatorname{trace} S \tag{3.2}$$

if $\{e_i\}_{i=1}^{N-1}$ are the orthonormal principal directions of the tangent space. But $\langle Se_i, e^i\rangle$ is independent of coordinate changes; hence, (3.2) is valid for general $\{e_i\}_{i=1}^{N-1}$. To justify this last claim, suppose that the surface ∂E_t is described by 2 parametrizations $\{u_i\}_{i=1}^{N-1}$ and $\{\tilde{u}_i\}_{i=1}^{N-1}$ with $e_i = \frac{\partial \mathbf{r}}{\partial u^i}$, $\tilde{e}_i = \frac{\partial \mathbf{r}}{\partial \tilde{u}^i}$, $H_{\alpha i} = \frac{\partial u^\alpha}{\partial \tilde{u}^i}$, $g_{ij} = \langle e_i, e_j\rangle$, $\tilde{g}_{ij} = \langle \tilde{e}_i, \tilde{e}_j\rangle$, $e^i = g^{ij}e_j$, $\tilde{e}^i = \tilde{g}^{ij}\tilde{e}_j$. Then, $\tilde{e}_i = H_{\alpha i}e_\alpha$, and we obtain $\tilde{g}_{ij} = g_{\alpha\beta}H_{\alpha i}H_{\beta j}$, or in term of matrices $\tilde{g} = H^{\mathrm{T}} g H$. This gives $g^{-1} = H\tilde{g}^{-1}H^{\mathrm{T}}$. On recalling $(g^{ij}) = g^{-1}$ and $(\tilde{g}^{ij}) = \tilde{g}^{-1}$, the claim follows from

$$\langle S\tilde{e}_i, \tilde{e}^i\rangle = \tilde{g}^{ij}\langle S\tilde{e}_i, \tilde{e}_j\rangle = H_{\alpha i}H_{\beta j}\tilde{g}^{ij}\langle Se_\alpha, e_\beta\rangle = g^{\alpha\beta}\langle Se_\alpha, e_\beta\rangle = \langle Se_\alpha, e^\alpha\rangle.$$

From (3.2) and the definition of S, we derive

$$\mathcal{K} = \operatorname{div}_\tau \nu = \langle D_{e_i}\nu, e^i\rangle \tag{3.3}$$

once we recall $\operatorname{div}_\tau W := \langle D_{e_i}W, e^i\rangle$ for any, including non-tangential, vector field W on the torus; see [48, Chapter 2, subsection 4.20].

Lemma 3.1. *Let (g^{ij}) be the inverse matrix of (g_{ij}) and $e^i = g^{ij}e_j$; we have*

$$\frac{\partial e_i}{\partial t} = D_X e_i = D_{e_i}X, \tag{3.4}$$

$$\frac{\partial \nu}{\partial t} = D_X\nu = -\langle \nu, D_{e_j}X\rangle e^j, \tag{3.5}$$

$$\frac{\partial e^i}{\partial t} = D_X e^i = g^{ij}\langle \nu, D_{e_j}X\rangle\nu - \langle e^i, D_{e_j}X\rangle e^j,$$

$$\frac{\partial g^{ij}}{\partial t} = D_X g^{ij} = -g^{kj}\langle D_{e_k}X, e^i\rangle - g^{ki}\langle D_{e_k}X, e^j\rangle. \tag{3.6}$$

Proof. It is immediate that (3.4) is the same as (3.1). Next, observe that $\frac{\partial \nu}{\partial t} = D_X\nu$ is a tangent vector. As $V = \langle V, e_j\rangle e^j$ for any tangent vector V, it follows that

$$D_X\nu = \langle D_X\nu, e_j\rangle e^j = -\langle \nu, D_X e_j\rangle e^j = -\langle \nu, D_{e_j}X\rangle e^j,$$

which is (3.5). Since $\langle e^i, e_j\rangle = \delta_{ij}$, we have

$$0 = D_X\langle e^i, e_j\rangle = \langle D_X e^i, e_j\rangle + \langle e^i, D_X e_j\rangle = \langle D_X e^i, e_j\rangle + \langle e^i, D_{e_j}X\rangle.$$

This leads to

$$\frac{\partial e^i}{\partial t} = D_X e^i = \langle D_X e^i, \nu\rangle \nu + \langle D_X e^i, e_j\rangle e^j = -\langle e^i, D_X \nu\rangle \nu - \langle e^i, D_X e_j\rangle e^j$$
$$= g^{ij}\langle \nu, D_{e_j} X\rangle \nu - \langle e^i, D_{e_j} X\rangle e^j .$$

Moreover,

$$\frac{\partial g^{ij}}{\partial t} = D_X\langle e^i, e^j\rangle = \langle D_X e^i, e^j\rangle + \langle e^i, D_X e^j\rangle$$

which leads to (3.6). ∎

We need further concepts from geometry to represent results of higher curvature derivatives. Let $\mathfrak{X}(\partial E)$ represent the collection of all (smooth enough) vector fields on the manifold ∂E. The second fundamental form $\overrightarrow{II} : \mathfrak{X}(\partial E) \times \mathfrak{X}(\partial E) \to \mathrm{span}\{\nu\}$ is a symmetric bilinear form defined by

$$\overrightarrow{II}(W, V) := \langle D_W V, \nu\rangle \nu$$

for any tangent vectors $W, V \in T_{p_t}\partial E_t$; its symmetry follows from

$$\overrightarrow{II}(W, V) = \langle D_W V, \nu\rangle \nu = -\langle V, D_W \nu\rangle \nu = \langle V, SW\rangle \nu = \langle W, SV\rangle \nu = \overrightarrow{II}(V, W).$$

Note that $\overrightarrow{II}(W, V)$ is just the normal component of $D_W V$. The tangential component is called the covariant derivative $\nabla_W V$ given by

$$\nabla_W V := D_W V - \overrightarrow{II}(W, V).$$

Immediately, we see that $D_W V - D_V W = \nabla_W V - \nabla_V W$ is a tangent vector and

$$D_W V = \nabla_W V + \langle SW, V\rangle \nu.$$

Since $D_{e_i} e_j = D_{e_j} e_i = \frac{\partial^2 \Phi(\mathbf{r}(\cdot), t)}{\partial u_i \partial u_j}$, consequently, $\nabla_{e_i} e_j = \nabla_{e_j} e_i$. Since

$$D_{e_i} D_{e_j} f = D_{e_j} D_{e_i} f$$

for any smooth scalar function f, we obtain $\nabla_{e_i}\nabla_{e_j} f = \nabla_{e_j}\nabla_{e_j} f$ if one defines $\nabla_{e_i} f := D_{e_i} f$. Moreover, the product rule for covariant derivative holds:

$$\nabla_W\langle V, Q\rangle = \langle D_W V, Q\rangle + \langle V, D_W Q\rangle = \langle \nabla_W V, Q\rangle + \langle V, \nabla_W Q\rangle$$

for any tangent vectors W, V, Q.

For any $\mathcal{C}^2$ scalar function ϕ on ∂E_t, its tangential gradient is $\nabla_\tau \phi := (\nabla_{e_i}\phi)e^i$. Its (tangential) Hessian at $p_t \in \partial E_t$ is given by

$$\mathrm{Hess}_\tau\, \phi : \mathfrak{X}(\partial E_t) \to \mathfrak{X}(\partial E_t) \quad \text{with } (\mathrm{Hess}_\tau\, \phi)\big|_{p_t} W := \nabla_W \nabla_\tau \phi$$

for any tangent field W; note that it depends only on information of ϕ on the manifold ∂E_t. Thus,

$$\begin{aligned}\langle(\mathrm{Hess}_\tau\,\phi)e_i, e_j\rangle &= \langle\nabla_{e_i}\nabla_\tau\phi, e_j\rangle = \nabla_{e_i}\langle\nabla_\tau\phi, e_j\rangle - \langle\nabla_\tau\phi, \nabla_{e_i}e_j\rangle \\ &= \nabla_{e_i}\nabla_{e_j}\phi - \langle\nabla_\tau\phi, \nabla_{e_j}e_i\rangle = \langle e_i, (\mathrm{Hess}_\tau\,\phi)e_j\rangle\end{aligned}$$

so that $\mathrm{Hess}_\tau\,\phi$ is self-adjoint. In addition,

$$\mathrm{trace}(\mathrm{Hess}_\tau\,\phi) = \langle(\mathrm{Hess}_\tau\,\phi)e_i, e^i\rangle = \langle\nabla_{e_i}\nabla_\tau\phi, e^i\rangle = \mathrm{div}_\tau\,\nabla_\tau\phi =: \Delta_\tau\phi.$$

We note that Δ_τ is known as the tangential Laplacian.

By a $\mathbb{R}$-linear operator $B : \mathfrak{X}(\partial E) \to \mathfrak{X}(\partial E)$ we mean $B(c_1V + c_2W) = c_1BV + c_2BW$ for any tangent vectors $V, W \in \mathfrak{X}(\partial E)$ and constants c_1, c_2. If $B(f_1V + f_2W) = f_1BV + f_2BW$ for any scalar functions $f_1, f_2 \in C^1(\partial E)$ as well; then, we say B is $\mathcal{C}^1(\partial E)$-linear if B involves only first derivatives (see the analogous definition $\mathcal{C}^\infty(\partial E)$-linear in [33, p. 262]).

Example. Let B be the covariant derivative in the e_1 direction, i.e., $BV := \nabla_{e_1}V$ for $V \in \mathfrak{X}(\partial E)$; then, B is $\mathbb{R}$-linear but not $C^1(\partial E)$-linear, because $B(fV) = f\nabla_{e_i}V + (\nabla_{e_i}f)V$ when f is a scalar field on ∂E. On the other hand, the shape operator S is $\mathcal{C}^1(\partial E)$-linear; this is what we will need later on. Another example of $\mathcal{C}^1(\partial E)$-linear operator is $\mathrm{Hess}_\tau\,\phi$. ∎

Lemma 3.2. *Let $A : \mathfrak{X}(\partial E) \to \mathfrak{X}(\partial E)$ be $\mathbb{R}$-linear, and let $B : \mathfrak{X}(\partial E) \to \mathfrak{X}(\partial E)$ be $\mathcal{C}^1(\partial E)$-linear, both not necessarily self-adjoint. Then,*

$$\langle Ae_i, e^k\rangle\langle Be_k, e^j\rangle = \langle BAe_i, e^j\rangle, \tag{3.7}$$

and $\langle V, e^k\rangle\langle Be_k, W\rangle = \langle BV, W\rangle$ for any tangent vectors V, W.

Proof. Write $Ae_i = A_i^\alpha e_\alpha$ and $Be_i = B_i^\beta e_\beta$. One readily checks that both the LHS and RHS of (3.7) yield $A_i^k B_k^j$. This is the first statement. Now, take A to be the identity operator, $V = V^i e_i$ and $W = W_j e^j$; the second statement follows. ∎

With the above geometry preliminaries, we are ready to compute derivatives of the curvature $\mathcal{K}$. Suppose $\Phi \in \mathcal{C}^5(\mathbb{T} \times \mathbb{R})$ and, as before, $\mathbf{z} = \mathbf{r}(\mathbf{u})$ with $\mathbf{u} \in U \subset \mathbb{R}^{N-1}$ represents a parametrization of the manifold ∂E. From (2.2), it is immediate that

$$e_i = \frac{\partial}{\partial u^i}\Phi(\mathbf{r}(\mathbf{u}), t) = \partial_i + tD_{\partial_i}X + \frac{t^2}{2}D_{\partial_i}Z + O(t^3), \quad i = 1, 2, \ldots, N-1,$$

with X and Z in the above RHS evaluated at ∂E. Take any $p \in \partial E$, and let $\{\partial_i|_p\}_{i=1}^{N-1}$ be orthonormal (and in the principal directions). Assuming $X = \eta\nu_0$ on ∂E for simplicity, we have $D_{\partial i}X = (D_{\partial_i}\eta)\nu_0 - \eta S\partial_i$ so that

$$\langle\partial_j, D_{\partial_i}X\rangle = -\eta\langle\partial_j, S\partial_i\rangle = \langle\partial_i, D_{\partial_j}X\rangle.$$

Since Z can have a tangential component Z^τ even when X does not, it is necessary to write $Z = \xi\nu_0 + Z^\tau$ on ∂E. Hence, at $p_t \in \partial E_t$,

$$\begin{aligned}
g_{ij}\big|_{p_t} &= \langle e_i, e_j\rangle = \delta_{ij} + t(\langle \partial_i, D_{\partial_j} X\rangle + \langle \partial_j, D_{\partial_i} X\rangle)\\
&\quad + \frac{t^2}{2}(\langle \partial_i, D_{\partial_j} Z\rangle + \langle \partial_j, D_{\partial_i} Z\rangle + 2\langle D_{\partial_i} X, D_{\partial_j} X\rangle) + O(t^3)\\
&= \delta_{ij} - 2t\eta\langle \partial_j, S\partial_i\rangle + \frac{t^2}{2}\big\{-2\xi\langle \partial_j, S\partial_i\rangle + \langle \partial_i, \nabla_{\partial_j} Z^\tau\rangle + \langle \partial_j, \nabla_{\partial_i} Z^\tau\rangle\big\}\\
&\quad + t^2\big\{D_{\partial_i}\eta D_{\partial_j}\eta + \eta^2\langle S\partial_i, S\partial_j\rangle\big\} + O(t^3),
\end{aligned}$$

where the above RHS is evaluated at p. If we now define $(g^{ij}) := (g_{ij})^{-1}$, recalling that $(I + A)^{-1} = I - A + A^2 + \cdots$ for any (small) matrix A, on using Lemma 3.2 we have

$$\begin{aligned}
g^{ij}\big|_{p_t} &= \delta_{ij} + 2t\eta\langle \partial_j, S\partial_i\rangle + t^2\big\{\xi\langle \partial_j, S\partial_i\rangle - D_{\partial_i}\eta D_{\partial_j}\eta + 3\eta^2\langle S^2\partial_i, \partial_j\rangle\big\}\\
&\quad - \frac{t^2}{2}\big\{\langle \partial_i, \nabla_{\partial_j} Z^\tau\rangle + \langle \partial_j, \nabla_{\partial_i} Z^\tau\rangle\big\} + O(t^3).
\end{aligned}$$

Using these Taylor's expansions for g^{ij} and e_j, one calculates

$$e^i\big|_{p_t} = g^{ij} e_j\big|_{p_t} = \partial_i + tA_i^{(1)} + \frac{t^2}{2}A_i^{(2)} + O(t^3),$$

where

$$\begin{aligned}
A_i^{(1)} &= (D_{\partial_i}\eta)\nu_0 + \eta S\partial_i,\\
A_i^{(2)} &= D_{\partial_i} Z + 4\eta\langle \partial_j, S\partial_i\rangle D_{\partial_j} X + 2\xi S\partial_i - 2(D_{\partial_i}\eta)\nabla_\tau\eta + 6\eta^2 S^2\partial_i\\
&\quad - \big\{\langle \partial_i, \nabla_{\partial_j} Z^\tau\rangle + \langle \partial_j, \nabla_{\partial_i} Z^\tau\rangle\big\}\partial_j;
\end{aligned}$$

the above RHS is again evaluated at p.

Turning our attention to the study of the unit normal vector,

$$\nu = \nu_0 + t\frac{\partial\nu}{\partial t}\bigg|_{t=0} + \frac{t^2}{2}\frac{\partial^2\nu}{\partial t^2}\bigg|_{t=0} + O(t^3).$$

From Lemma 3.1, we see that $\frac{\partial\nu}{\partial t} = -\langle \nu, D_{e_j} X\rangle e^j$; this leads to

$$\frac{\partial^2\nu}{\partial t^2} = -\left\langle \frac{\partial\nu}{\partial t}, D_{e_j} X\right\rangle e^j - \langle \nu, D_{e_j} Z\rangle e^j - \langle \nu, D_{e_j} X\rangle\frac{\partial e^j}{\partial t}.$$

When $X = \eta\nu_0$ on ∂E, a simple computation yields

$$B_1 := \frac{\partial\nu}{\partial t}\bigg|_{t=0} = -\nabla_\tau\eta;$$

at the same time,

$$B_2 := \frac{\partial^2\nu}{\partial t^2}\bigg|_{t=0} = -2\eta S\nabla_\tau\eta - \langle \nu_0, D_{\partial_j} Z\rangle\,\partial_j - |\nabla_\tau\eta|^2\nu_0.$$

In other words,

$$\nu = \nu_0 + tB_1 + \frac{t^2}{2}B_2 + O(t^3).$$

As a result,

$$\begin{aligned} D_{e_i}\nu = \frac{\partial \nu}{\partial u^i} &= \frac{\partial \nu_0}{\partial u^i} + t\frac{\partial B_1}{\partial u^i} + \frac{t^2}{2}\frac{\partial B_2}{\partial u^i} + O(t^3) \\ &= D_{\partial_i}\nu_0 + tD_{\partial_i}B_1 + \frac{t^2}{2}D_{\partial_i}B_2 + O(t^3) \end{aligned}$$

with RHS evaluated at p.

Theorem 3.3. *Suppose $E \in \mathcal{C}^5$ (not necessarily a critical set), $X \in \mathcal{C}^5(\mathbb{T})$, $X = \eta\nu_0$ on ∂E, $Z = Z^\tau + \xi\nu_0$ on ∂E, where Z^τ and $\xi\nu_0$ are its tangential and normal components, respectively. Then, on ∂E at $t = 0$,*

$$\left.\frac{\partial \mathcal{K}}{\partial t}\right|_{t=0} = -\eta\,\mathrm{trace}\,S^2 - \Delta_\tau\eta, \tag{3.8}$$

$$\begin{aligned} \left.\frac{\partial^2 \mathcal{K}}{\partial t^2}\right|_{t=0} &= -\,\mathrm{div}_\tau(S\nabla_\tau\eta^2) - \mathrm{div}_\tau(SZ^\tau) - \Delta_\tau\xi - |\nabla_\tau\eta|^2\,\mathcal{K} \\ &\quad - 2\eta^2\,\mathrm{trace}\,S^3 + g^{ij}\langle S\partial_i, \nabla_{\partial_j}Z^\tau\rangle \\ &\quad - 2\eta g^{ij}\langle(\mathrm{Hess}_\tau\,\eta)\partial_i, S\partial_j\rangle - \xi\,\mathrm{trace}\,S^2. \end{aligned} \tag{3.9}$$

Proof. From (3.3), we have

$$\begin{aligned} \mathcal{K} &= \langle D_{e_i}\nu, e^i\rangle \\ &= \left\langle D_{\partial_i}\nu_0 + tD_{\partial_i}B_1 + \frac{t^2}{2}D_{\partial_i}B_2,\ \partial_i + t\,A_i^{(1)} + \frac{t^2}{2}A_i^{(2)}\right\rangle + O(t^3) \\ &= \mathcal{K}_0 + t\big\{\langle D_{\partial_i}\nu_0, A_i^{(1)}\rangle + \langle D_{\partial_i}B_1, \partial_i\rangle\big\} \\ &\quad + \frac{t^2}{2}\big\{\langle D_{\partial_i}\nu_0, A_i^{(2)}\rangle + \langle D_{\partial_i}B_2, \partial_i\rangle + 2\langle D_{\partial_i}B_1, A_i^{(1)}\rangle\big\} + O(t^3) \\ &:= \mathcal{K}_0 + t\big\{\langle D_{\partial_i}\nu_0, A_i^{(1)}\rangle + \langle D_{\partial_i}B_1, \partial_i\rangle\big\} + \frac{t^2}{2}\{I + II + III\} + O(t^3). \end{aligned}$$

Hence,

$$\begin{aligned} \left.\frac{\partial \mathcal{K}}{\partial t}\right|_{t=0} &= \langle D_{\partial_i}\nu_0, A_i^{(1)}\rangle + \langle D_{\partial_i}B_1, \partial_i\rangle = -\langle S\partial_i, \eta S\partial_i\rangle - \mathrm{div}_\tau\,\nabla_\tau\eta \\ &= -\eta\,\mathrm{trace}\,S^2 - \Delta_\tau\eta, \\ \left.\frac{\partial^2 \mathcal{K}}{\partial t^2}\right|_{t=0} &= I + II + III. \end{aligned}$$

Employing Lemma 3.2, we have

$$
\begin{aligned}
I &:= \langle D_{\partial_i}\nu_0, A_i^{(2)}\rangle \\
&= -\langle S\partial_i,\ D_{\partial_i}Z\rangle + 4\eta^2\langle\partial_j, S\partial_i\rangle\langle S\partial_i, S\partial_j\rangle - 2\xi\langle S\partial_i, S\partial_i\rangle + 2\langle S\nabla_\tau\eta, \nabla_\tau\eta\rangle \\
&\quad - 6\eta^2\langle S\partial_i, S^2\partial_i\rangle + \big(\langle\partial_j, \nabla_{\partial_i}Z^\tau\rangle + \langle\partial_i, \nabla_{\partial_j}Z^\tau\rangle\big)\langle S\partial_i, \partial_j\rangle \\
&= \langle S\partial_i,\ D_{\partial_i}Z^\tau\rangle - \xi \operatorname{trace} S^2 - 2\eta^2 \operatorname{trace} S^3 + 2\langle S\nabla_\tau\eta, \nabla_\tau\eta\rangle; \\
II &:= \langle D_{\partial_i}B_2, \partial_i\rangle = \operatorname{div}_\tau B_2 = \operatorname{div}_\tau\big(-2\eta\, S\nabla_\tau\eta - \langle\nu_0, D_{\partial_j}Z\rangle\,\partial_j - |\nabla_\tau\eta|^2\nu_0\big) \\
&= -\operatorname{div}_\tau\big(S\nabla_\tau(\eta^2)\big) + \operatorname{div}_\tau\big(\langle D_{\partial_j}\nu_0, Z^\tau\rangle\,\partial_j\big) - \Delta_\tau\xi - |\nabla_\tau\eta|^2\,\mathcal{K} \\
&= -\operatorname{div}_\tau\big(S\nabla_\tau(\eta^2)\big) - \operatorname{div}_\tau(SZ^\tau) - \Delta_\tau\xi - |\nabla_\tau\eta|^2\,\mathcal{K}; \\
III &:= 2\langle D_{\partial_i}B_1, A_i^{(1)}\rangle = -2\langle D_{\partial_i}\nabla_\tau\eta, (D_{\partial_i}\eta)\nu_0 + \eta\, S\partial_i\rangle \\
&= 2\langle\nabla_\tau\eta, D_{\partial_i}((D_{\partial_i}\eta)\nu_0)\rangle - 2\eta\langle(\operatorname{Hess}_\tau\eta)\partial_i, S\partial_i\rangle \\
&= -2\langle\nabla_\tau\eta, (D_{\partial_i}\eta)\, S\partial_i\rangle - 2\eta\langle(\operatorname{Hess}_\tau\eta)\partial_i, S\partial_i\rangle \\
&= -2\langle\nabla_\tau\eta, S\nabla_\tau\eta\rangle - 2\eta\langle(\operatorname{Hess}_\tau\eta)\partial_i, S\partial_i\rangle.
\end{aligned}
$$

On adding up I, II, III, we recover Theorem 3.3 provided $\{\partial_i\}_{i=1}^{N-1}$ is orthonormal. However, both (3.8) and (3.9) remain valid in general coordinates as they are coordinate-invariant. ■

We now combine results for derivatives of the nonlocal and curvature terms in the following.

Corollary 3.4. *Suppose that* $E \in \mathcal{C}^5$ *is a critical set; i.e., it satisfies* $\mathcal{F}(\cdot,\sigma,c) = 0$ *on* ∂E*;* $X = \eta\nu_0 + X^\tau \in \mathcal{C}^5(\mathbb{T})$ *with* $\eta = X\cdot\nu_0$ *and a tangential field* X^τ*;* $Z = \xi\nu_0 + Z^\tau$ *on* ∂E *with* $\xi = Z\cdot\nu_0$ *and a tangential field* Z^τ*. Then,*

$$
\begin{aligned}
D_t\mathcal{F}(E,\sigma,c)[X] &:= \frac{d}{dt}\mathcal{F}(E_t,\sigma,c)\Big|_{t=0} \\
&= -\Delta_\tau\eta - \|B_{\partial E}\|^2\eta \\
&\quad + \sigma\int_{\partial E}\widehat{G}(\mathbf{x}-\mathbf{y})\eta(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}} + \sigma(\nabla\mathcal{N}_E\cdot\nu_0)\eta\Big|_{\partial E}.
\end{aligned} \tag{3.10}
$$

Proof. If E is a critical set, then $D_{X^\tau}\mathcal{F}(E,\sigma,c) = 0$ on ∂E. Hence, we have

$$
D_t\mathcal{F}(E,\sigma,c)[X] = \frac{d}{dt}\mathcal{F}(E_t,\sigma,c)\Big|_{t=0} = D_X\mathcal{F}(E,\sigma,c) = D_{\eta\nu}\mathcal{F}(E,\sigma,c),
$$

which leads to (3.10) once we employ (2.3), (3.8), and $\|B_{\partial E}\|^2 = \operatorname{trace} S^2$. ■

Remark 3.5. Suppose E is a critical set; from Corollary 3.4, we see that $D_t\mathcal{F}(E,\sigma,c)[X]$ is a linear operator acting on only the normal component $\eta = X\cdot\nu_0$. We therefore write $D_t\mathcal{F}(E,\sigma,c)[\eta] := D_t\mathcal{F}(E,\sigma,c)[X]$. The same clearly applies to $D_{\sigma t}\mathcal{F}(E,\sigma,c)[X]$. This notation will be handy to investigate eigenvalues and eigenfunctions of the operator $D_t\mathcal{F}(E,\sigma,c)$ at a critical set later on. Similarly, one can write $D_{t\sigma}\mathcal{F}(E,\sigma)[\eta]$ for $D_{t\sigma}\mathcal{F}(E,\sigma)[X]$.

On the other hand, $D_{tt}\mathcal{F}(E,\sigma,c)[X,X]$ depends not only on η but the tangential component of X as well. Thus, when we take another t-derivative of $D_t\mathcal{F}(E,\sigma,c)[\eta]$ in Section 5, we record that as $D_{tt}\mathcal{F}(E,\sigma,c)[X,X]$.

4. Analyzing nonlocal and curvature terms at critical lamellae

Before we start analyzing the contributions of the nonlocal and curvature terms to the bifurcation phenomena, it is clear that checking explicitly any bifurcation criteria requires a knowledge of σ_{crit} (which we write as σ_λ or σ_c below to emphasize its dependence on these parameters); the definitions of the quantities c, x_0, $\mathcal{T}$, λ, and d_0 were given in (1.6)–(1.10). This critical value of σ at which the stationary 1-lamella with thickness $0 < x_0 < T$ switches stabilities can be deduced by setting $k = 1$ and $h(\theta_1) - h(\theta_0) = 0$ on [3, p. 587 in the proof of Theorem 5.13], from which, recalling the notation introduced in (1.8), the condition to be satisfied is

$$\begin{aligned}\frac{4\pi^2}{T^2} &= \frac{4}{T^2}(\theta_1 - \theta_0)\\ &= \frac{\sigma_\lambda T}{4}\left(\frac{\cosh\big(\lambda\sqrt{\pi^2+T^2/4}\big)}{\sqrt{\pi^2+T^2/4}\sinh\big(\sqrt{\pi^2+T^2/4}\big)} - \frac{\cosh\big(\sqrt{\pi^2+T^2/4}\big)}{\sqrt{\pi^2+T^2/4}\sinh\big(\sqrt{\pi^2+T^2/4}\big)}\right.\\ &\qquad\left. - \frac{\cosh(\lambda T/2)}{(T/2)\sinh(T/2)} + \frac{\cosh(T/2)}{(T/2)\sinh(T/2)}\right).\end{aligned}$$

It is useful to introduce the functions

$$\begin{aligned}\Omega(s,m,\mathcal{T}) &:= \frac{\cosh\big(s\sqrt{m^2+\mathcal{T}^2}\big)}{\sqrt{m^2+\mathcal{T}^2}\sinh\big(\sqrt{m^2+\mathcal{T}^2}\big)},\\ \delta(s,m,\mathcal{T}) &:= \Omega(s,m,\mathcal{T}) - \Omega(1,m,\mathcal{T}),\end{aligned} \tag{4.1}$$

so we may write

$$\sigma_\lambda = \frac{2\pi^2}{\mathcal{T}^3\big(\delta(\lambda,\pi,\mathcal{T}) - \delta(\lambda,0,\mathcal{T})\big)}; \tag{4.2}$$

in particular, when $c = 0$,

$$\begin{aligned}\sigma_{c=\lambda=0} &:= 8\pi^2\cdot\left[T^3\left(\frac{\tanh(T/4)}{T} - \frac{\tanh(\sqrt{T^2+4\pi^2}/4)}{\sqrt{T^2+4\pi^2}}\right)\right]^{-1}\\ &= 2\pi^2\cdot\left[\mathcal{T}^3\left(\frac{\tanh(\mathcal{T}/2)}{\mathcal{T}} - \frac{\tanh(\sqrt{\mathcal{T}^2+\pi^2}/2)}{\sqrt{\mathcal{T}^2+\pi^2}}\right)\right]^{-1};\end{aligned} \tag{4.3}$$

see [3, Theorem 1.2]. This is one of the ingredients that govern the shape of the bifurcation curves.

That the 1-lamella loses stability above this value $\sigma_{\rm crit}$ is due to the fact that a certain matrix depending on σ gets a zero eigenvalue. Precisely, (we define here ρ_2, C_1, C_2 that we use only at a later stage) let

$$\rho_1 = \frac{4\pi^2}{T^2}, \quad \rho_2 = \frac{16\pi^2}{T^2}, \quad C_i = \frac{1}{2\sqrt{1+\rho_i}\,\sinh(\frac{T}{2}\sqrt{1+\rho_i})}, \tag{4.4}$$

$$d_0^{(1)} = \frac{1}{\sinh(\frac{T\sqrt{1+\rho_1}}{2})} \sinh \frac{(T-x_0)\sqrt{1+\rho_1}}{2} \sinh \frac{x_0\sqrt{1+\rho_1}}{2}; \tag{4.5}$$

the matrix [3, p. 581]

$$\mathcal{A}^{(1)} = \frac{1}{2\sinh\big(\frac{T}{2}\sqrt{1+\rho_1}\big)} \begin{pmatrix} \cosh\big(\frac{T}{2}\sqrt{1+\rho_1}\big) & \cosh\big((T/2-x_0)\sqrt{1+\rho_1}\big) \\ \cosh\big((T/2-x_0)\sqrt{1+\rho_1}\big) & \cosh\big(\frac{T}{2}\sqrt{1+\rho_1}\big) \end{pmatrix}$$

has $d_0^{(1)}$ as its least eigenvalue [3, p. 582], with corresponding eigenvector $(-1, 1)$; the vector components represent displacement magnitude in the outward normal directions on L_1 and L_2, respectively. Since $\nu\big|_{L_1} = -\nu\big|_{L_2}$, to leading order this destabilizing mode leads to ‘parallel’ configuration between the two deformed boundaries.

Let $-1 < c < 1$ and $\mathbb{L} = [0, x_0] \times [0, T]$ be the critical 1-lamella described in (1.6), (1.7). Recall that $\mathbf{x} = (x, y)$ designates a point on the 2D torus $\mathbb{T}$. Fix the physical parameter c. Suppose that a local bifurcation occurs at a certain σ_c, resulting in a non-planar lamellar structure $\mathbb{L}_t$ for some $\sigma \approx \sigma_c$; it represents a small perturbation of the original $\mathbb{L}$ with $\partial\mathbb{L}_t = (L_1)_t \cup (L_2)_t$. Clearly, the new shape $(L_1)_t$ can be attained by a suitable flow field $X_t(x, y) = (g_1(y), 0)$ defined in a neighborhood of L_1 for a small time t; similarly, a different flow field $X_t(x, y) = (g_2(y), 0)$ governs the new shape $(L_2)_t$. When $c = 0$, it turns out that the super-symmetry between space and wedge will allow us to seek a non-planar solution created through bifurcation with $g_1 = g_2$ in the next section. Thus, one can just let $X_t = (X_1(y), 0)$ in the whole torus, resulting in $(L_2)_t = (L_1)_t + (T/2, 0)$, see Lemma 5.1 below; i.e., $(L_2)_t$ is a translation of $(L_1)_t$ to the right by an amount $T/2$ in the x-direction; moreover, whenever (1.4) is satisfied on $(L_1)_t$, the same will automatically be true at $(L_2)_t$; see Section 5. This resulting ‘anti-symmetric’ configuration (with respect to their outward normal directions on the boundaries) simplifies the analysis, as it leads to a bifurcation due to a simple eigenvalue [22] when $c = 0$. In the following theorem, we therefore study a flow field $X_t = (X_1(y), 0)$ in the whole torus to look for anti-symmetric non-planar solutions with the additional constraint that X_1 is y-odd. It is noted that the flow fields away from neighborhoods of $L_1 \cup L_2$ have really no impact on the shape of $\mathbb{L}_t$. Define

$$\varphi_1(y) := \sin\frac{2\pi y}{T}, \quad \phi_1 := \varphi_1 \llcorner L_2 - \varphi_1 \llcorner L_1. \tag{4.6}$$

Let $X(x, y) = (\varphi_1(y), 0)$. Since $\nu_0 = (1, 0)$ at L_2 and $(-1, 0)$ at L_1, we have $X = \eta\nu_0$ at $\partial\mathbb{L}$ when $t = 0$ with $\eta = X \cdot \nu_0 = \phi_1$.

Lemma 4.1. *Suppose that the flow field is* $X(x,y) = (\varphi_1(y), 0)$. *Then,*

$$\frac{\partial}{\partial t}(\mathcal{N}_{\mathbb{L}_t}|_{\partial \mathbb{L}_t})|_{t=0} = \left(\frac{d_0^{(1)}}{\sqrt{1+\rho_1}} - d_0\right)\phi_1(y). \tag{4.7}$$

Proof. On setting $E = \mathbb{L}$ and $X(x,y) = (\varphi_1(y), 0)$ in (2.3), we have $\eta = X \cdot \nu_0 = \phi_1$ on $\partial\mathbb{L}$ and

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{N}_{\mathbb{L}_t}|_{\partial \mathbb{L}_t})|_{t=0} &= \left(\eta \frac{\partial \mathcal{N}_{\mathbb{L}}}{\partial \nu}\right)\Big|_{\partial\mathbb{L}} + \int_{\partial\mathbb{L}} \widehat{G}(\mathbf{x}-\mathbf{y})\eta(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}} \\ &= -d_0\phi_1|_{\partial\mathbb{L}} + \int_{\partial\mathbb{L}} \widehat{G}(\mathbf{x}-\mathbf{y})\phi_1(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}}.\end{aligned}$$

In the notation of [3], the above integral term equals $V_Z(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x})$, where V_i is given in [3, equation (5.7)] with $m = 1, \alpha_1^1 = -1, \alpha_1^2 = 1, \ell_1 = 0, \ell_2 = x_0$, resulting in

$$\begin{aligned}V_Z|_{\partial\mathbb{L}} &= C_1\left(\cosh\left(\frac{T}{2}\sqrt{1+\rho_1}\right) - \cosh\left(\left(\frac{T}{2} - x_0\right)\sqrt{1+\rho_1}\right)\right)\phi_1(y) \\ &= \frac{d_0^{(1)}}{\sqrt{1+\rho_1}}\phi_1(y),\end{aligned}$$

which yields (4.7). ∎

Lemma 4.2. *Suppose that the flow field is* $X(x,y) = (X_1(y), 0)$ *with* X_1 *being* T*-periodic (note that* X_1 *needs not be* y*-odd). Then,*

$$\int_{\partial\mathbb{L}} \phi_1 \frac{\partial^2}{\partial t^2}(\mathcal{N}_{\mathbb{L}_t}|_{\partial\mathbb{L}})|_{t=0}\, d\mathcal{H}^1 = 0.$$

For the case $c = 0$*, we have a stronger statement:*

$$\frac{\partial^2}{\partial t^2}(\mathcal{N}_{\mathbb{L}_t}|_{\partial\mathbb{L}})|_{t=0} = 0. \tag{4.8}$$

Proof. With $Z = \operatorname{div} X = 0$ and our chosen $X = (X_1(y), 0)$, it follows from (2.4) that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(\mathcal{N}_{E_t}|_{\partial E_t})|_{t=0} &= \int_{\partial E} D_i\widehat{G}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x}) - X_i(\mathbf{y}))X_j(\mathbf{y})\,\nu_j(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}} \\ &\quad - \int_{\partial E} D_i\widehat{G}(\mathbf{x}-\mathbf{y})X_i(\mathbf{x})(X_j(\mathbf{x}) - X_j(\mathbf{y}))\nu_j(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}} \\ &= \int_{\partial E} D_1\widehat{G}(\mathbf{x}-\mathbf{y})(X_1(\mathbf{x}) - X_1(\mathbf{y}))X_1(\mathbf{y})\,\nu_1(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}} \\ &\quad - \int_{\partial E} D_1\widehat{G}(\mathbf{x}-\mathbf{y})X_1(\mathbf{x})(X_1(\mathbf{x}) - X_1(\mathbf{y}))\nu_1(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}} \\ &= -\int_{\partial E} D_1\widehat{G}(\mathbf{x}-\mathbf{y})(X_1(\mathbf{x}) - X_1(\mathbf{y}))^2\nu_1(\mathbf{y})\, d\mathcal{H}^1_{\mathbf{y}} =: I(\mathbf{x}).\end{aligned} \tag{4.9}$$

Now, we specialize to $E = \mathbb{L}$ and we split I from (4.9) at any point $\mathbf{x} \in \partial\mathbb{L}$ as

$$I(\mathbf{x}) = I_S(\mathbf{x}) + I_O(\mathbf{x}),$$

where I_S is the integral with $\mathbf{y}$ belonging to the same side of $\mathbb{L}$ as $\mathbf{x}$ and $I_O(\mathbf{x})$ is the integral with $\mathbf{y}$ belonging to the different side on $\mathbb{L}$ as $\mathbf{x}$.

Before we proceed, we introduce the following notation which will be useful in this proof: we denote by $\mathbf{z}$ points in L_1 and by $\hat{\mathbf{z}}$ points in L_2; moreover, if $\mathbf{z} \in L_1$, we denote by $\widehat{\tau}(\mathbf{z})$ the point in L_2 with the same y-coordinate, namely, $\widehat{\tau}(\mathbf{z}) = \mathbf{z} + (x_0, 0)$, and conversely, $\tau(\hat{\mathbf{z}}) \in L_1$ has the same y-coordinate as $\hat{\mathbf{z}} \in L_2$.

The Green function $G(\mathbf{x},\mathbf{y}) = \widehat{G}(\mathbf{x}-\mathbf{y})$ associated with the modified Helmholtz operator in $\mathbb{T}$ has symmetry properties due to the periodicity: namely,

$$\begin{aligned}
&\text{the values } \widehat{G}(\pm x, \pm y) \text{ are the same},\\
&\widehat{G}(x,y) = \widehat{G}(T-x,y) = \widehat{G}(x,T-y),\\
&D_1\widehat{G}(x,y) = -D_1\widehat{G}(-x,y), && (4.10)\\
&D_{11}\widehat{G}(x,y) = D_{11}\widehat{G}(-x,y), && (4.11)\\
&D_1\widehat{G}(0,y) = D_1\widehat{G}(T/2,y) = 0. && (4.12)
\end{aligned}$$

Now, we rewrite for $\mathbf{x} \in L_1$

$$\begin{aligned}
I(\mathbf{x}) &= I_S(\mathbf{x}) + I_O(\mathbf{x})\\
&= \int_{L_1} D_1\widehat{G}(\mathbf{x}-\mathbf{y})(X_1(\mathbf{x}) - X_1(\mathbf{y}))^2\, d\mathcal{H}^1_{\mathbf{y}}\\
&\quad - \int_{L_2} D_1\widehat{G}(\mathbf{x}-\hat{\mathbf{y}})(X_1(\mathbf{x}) - X_1(\hat{\mathbf{y}}))^2\, d\mathcal{H}^1_{\hat{\mathbf{y}}}\\
\text{[by (4.12)]} \quad &= -\int_{L_2} D_1\widehat{G}(\mathbf{x}-\hat{\mathbf{y}})(X_1(\mathbf{x}) - X_1(\hat{\mathbf{y}}))^2\, d\mathcal{H}^1_{\hat{\mathbf{y}}} && (4.13)\\
\text{[by (4.10) and } X_1 = X_1(x_2)] \quad &= \int_{L_2} D_1\widehat{G}(\widehat{\tau}(\mathbf{x}) - \tau(\hat{\mathbf{y}}))\big[X_1(\widehat{\tau}(\mathbf{x})) - X_1(\tau(\hat{\mathbf{y}}))\big]^2 d\mathcal{H}^1_{\hat{\mathbf{y}}}\\
&= \int_{L_1} D_1\widehat{G}(\widehat{\tau}(\mathbf{x}) - \mathbf{y})\big[X_1(\widehat{\tau}(\mathbf{x})) - X_1(\mathbf{y})\big]^2\, d\mathcal{H}^1_{\mathbf{y}}\\
&= I_O(\widehat{\tau}(\mathbf{x})) = I(\widehat{\tau}(\mathbf{x})); && (4.14)
\end{aligned}$$

this also proves that $I(\hat{\mathbf{x}}) = I\big(\tau(\hat{\mathbf{x}})\big)$. In the case $c = 0$, the first coordinate of $\mathbf{x} - \hat{\mathbf{y}}$ is $T/2$ so $I(\mathbf{x}) = 0$ by (4.12) and (4.13), which proves (4.8) by (4.9). In the general case,

$$\int_{\partial\mathbb{L}} \phi_1 \frac{\partial^2}{\partial t^2}\big(\mathcal{N}_{\mathbb{L}_t}\big|_{\partial\mathbb{L}}\big)\big|_{t=0}\, d\mathcal{H}^1 = \int_{L_2} \varphi_1(\hat{x}_2)\, I_O(\hat{\mathbf{x}})\, d\mathcal{H}^1_{\hat{\mathbf{x}}} - \int_{L_1} \varphi_1(x_2)\, I_O(\mathbf{x})\, d\mathcal{H}^1_{\mathbf{x}} = 0$$

by (4.14). ∎

Remark 4.3. Taking $X = (X_1(y), 0)$ on $\mathbb{L}$, we see from (4.9) that $\frac{\partial^2}{\partial t^2}(\mathcal{N}_{E_t}|_{\partial E_t})|_{t=0}$ depends only on X_1, which is $\eta = X \cdot \nu_0$ up to a sign. General flow fields can lead to dependency on input other than η. For example, Z can be non-zero and it makes a non-zero contribution to the second derivative by (2.4).

To investigate if the bifurcation is supercritical or subcritical, we need to calculate

$$\mathcal{A}_\lambda := \int_{\partial\mathbb{L}} \phi_1(\mathbf{x}) \frac{\partial^3}{\partial t^3}(\mathcal{N}_{E_t}|_{\mathbf{x}\in\partial\mathbb{L}})|_{t=0}\, d\mathcal{H}^1_{\mathbf{x}}. \tag{4.15}$$

Here, the subscript λ is a value related to the lamella thickness, as indicated in (1.8).

Lemma 4.4. *Let $X(x, y) = (X_1(y), 0) = (\varphi_1(y), 0)$. Then,*

$$\mathcal{A}_\lambda = 2\Big(\int_{\mathbf{z}\in L_2} - \int_{\mathbf{z}\in L_1}\Big)\frac{\partial^2 \widehat{G}}{\partial x_1^2}(\mathbf{z})\, d\mathcal{H}^1_{\mathbf{z}} \int_{\mathbf{x}\in L_2} (\varphi_1(x_2) - \varphi_1(x_2 - z_2))^3\, \varphi_1(x_2)\, d\mathcal{H}^1_{\mathbf{x}}.$$

Proof. Since $\operatorname{div} X = Z = Y = 0$, the RHS of (2.12), (2.15), and (2.16) are all zero; the same is true for the 2nd and 3rd terms on RHS of (2.9). From (2.5),

$$\begin{aligned}
&\frac{\partial^3}{\partial t^3}(\mathcal{N}_{\mathbb{L}_t}|_{\partial\mathbb{L}_t})|_{t=0}\\
&\quad= \int_{\mathbb{L}} D^3\widehat{G}(\delta\mathbf{x})[\delta X, \delta X, \delta X]\, d\mathbf{y}\\
&\quad= \int_{\mathbb{L}} \frac{\partial^3\widehat{G}}{\partial x_i \partial x_j \partial x_k}(\mathbf{x}-\mathbf{y})(X_i(\mathbf{x}) - X_i(\mathbf{y}))(X_j(\mathbf{x}) - X_j(\mathbf{y}))(X_k(\mathbf{x}) - X_k(\mathbf{y}))\, d\mathbf{y}\\
&\quad= \int_{\mathbb{L}} \frac{\partial^3\widehat{G}}{\partial x_1^3}(\mathbf{x}-\mathbf{y})(X_1(\mathbf{x}) - X_1(\mathbf{y}))^3\, d\mathbf{y}\\
&\quad= -\int_{\mathbb{L}} \frac{\partial}{\partial y_1}\left\{\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{x}-\mathbf{y})\right\}(X_1(\mathbf{x}) - X_1(\mathbf{y}))^3\, d\mathbf{y}\\
&\quad= -\int_{\mathbb{L}} \frac{\partial}{\partial y_1}\left\{\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{x}-\mathbf{y})(X_1(\mathbf{x}) - X_1(\mathbf{y}))^3\right\} d\mathbf{y}\\
&\quad= \Big(\int_{L_1} - \int_{L_2}\Big)\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{x}-\mathbf{y})(X_1(\mathbf{x}) - X_1(\mathbf{y}))^3\, d\mathcal{H}^1_{\mathbf{y}}\\
&\quad= \Big(\int_{\mathbf{x}-L_1} - \int_{\mathbf{x}-L_2}\Big)\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{z})(X_1(\mathbf{x}) - X_1(\mathbf{x}-\mathbf{z}))^3\, d\mathcal{H}^1_{\mathbf{z}}\\
&\quad= \begin{cases} \big(\int_{L_2} - \int_{L_1}\big)\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{z})(\varphi_1(x_2) - \varphi_1(x_2 - z_2))^3\, d\mathcal{H}^1_{\mathbf{z}} & \text{if } \mathbf{x} \in L_2,\\ \big(\int_{L_1} - \int_{L_2}\big)\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{z})(\varphi_1(x_2) - \varphi_1(x_2 - z_2))^3\, d\mathcal{H}^1_{\mathbf{z}} & \text{if } \mathbf{x} \in L_1 \end{cases}
\end{aligned}$$

because due to periodicity of the integrands and (4.11), upon change of variables from $\mathbf{y}$ to $\mathbf{z}$, we have $\int_{\mathbf{x}-L_1} = \int_{L_2}$ and $\int_{\mathbf{x}-L_2} = \int_{L_1}$ when $\mathbf{x} \in L_2$; and $\int_{\mathbf{x}-L_1} = \int_{L_1}$ and $\int_{\mathbf{x}-L_2} = \int_{L_2}$

when $\mathbf{x} \in L_1$. In particular,

$$\frac{\partial^3}{\partial t^3}\big(\mathcal{N}_{\mathbb{L}_t}\big|_{\mathbf{x}\in L_2}\big)\big|_{t=0} = -\frac{\partial^3}{\partial t^3}\big(\mathcal{N}_{\mathbb{L}_t}\big|_{\hat{\mathbf{x}}\in L_1}\big)\big|_{t=0} \quad \text{with } \mathbf{x} = (x_0, s) \text{ and } \hat{\mathbf{x}} = (0, s)$$

for some $s \in [0, T)$. Hence,

$$\mathcal{A}_\lambda := \int_{\partial\mathbb{L}} \phi_1(\mathbf{x})\frac{\partial^3}{\partial t^3}\big(\mathcal{N}_{E_t}\big|_{\mathbf{x}\in\partial\mathbb{L}}\big)\big|_{t=0}\, d\mathcal{H}^1_{\mathbf{x}} = 2\int_{L_2} \varphi_1(\mathbf{x})\frac{\partial^3}{\partial t^3}\big(\mathcal{N}_{E_t}\big|_{\mathbf{x}\in L_2}\big)\big|_{t=0}\, d\mathcal{H}^1_{\mathbf{x}},$$

which is the same as (4.15). ∎

The value of $\mathcal{A}_\lambda$ will play an important role in the shape of the bifurcation curve and the stability of the resulting non-planar solutions later on; we therefore simplify (4.15) as much as possible, starting with the following.

Lemma 4.5. *Let $s \in \mathbb{R}$. Then,*

$$\int_0^T \varphi_1(y)(\varphi_1(y) - \varphi_1(y-s))^3\, dy = 3T \sin^4 \frac{\pi s}{T}.$$

Proof. With $\varphi_1(y) = \sin(2\pi y/T)$, we carry out a direct computation as follows:

$$\begin{aligned}
\text{LHS} &= 8\sin^3\frac{\pi s}{T}\int_0^T \sin\frac{2\pi y}{T}\cos^3\frac{2\pi(y-s/2)}{T}\, dy \\
&= 8\sin^3\frac{\pi s}{T}\int_{-s/2}^{T-s/2} \sin\frac{2\pi(\xi+s/2)}{T}\cos^3\frac{2\pi\xi}{T}\, d\xi \\
&= 8\sin^3\frac{\pi s}{T}\int_0^T \sin\frac{2\pi(\xi+s/2)}{T}\cos^3\frac{2\pi\xi}{T}\, d\xi \\
&= 8\sin^4\frac{\pi s}{T}\int_0^T \cos^4\frac{2\pi\xi}{T}\, d\xi = 3T\sin^4\frac{\pi s}{T}.
\end{aligned}$$ ∎

Using the above lemma, we obtain

$$\begin{aligned}
\mathcal{A}_\lambda &= 6T\Big(\int_{L_2} - \int_{L_1}\Big)\frac{\partial^2\widehat{G}}{\partial x_1^2}(\mathbf{z})\sin^4(\frac{\pi z_2}{T})\, d\mathcal{H}^1_{\mathbf{z}} \\
&= 6T\int_0^T\Big\{\frac{\partial^2\widehat{G}}{\partial z_1^2}(x_0, z_2) - \frac{\partial^2\widehat{G}}{\partial z_1^2}(0^+, z_2)\Big\}\Big(\frac{3}{8} - \frac{1}{2}\cos\Big(\frac{2\pi z_2}{T}\Big) + \frac{1}{8}\cos\Big(\frac{4\pi z_2}{T}\Big)\Big)\, dz_2 \\
&= 6T\Big\{\frac{\partial^2}{\partial z_1^2}\int_0^T \widehat{G}(z_1, z_2)\Big(\frac{3}{8} - \frac{1}{2}\cos\Big(\frac{2\pi z_2}{T}\Big) + \frac{1}{8}\cos\Big(\frac{4\pi z_2}{T}\Big)\Big)\, dz_2\Big\}\Big|_{z_1=0^+}^{z_1=x_0}.
\end{aligned}$$

From (4.11), [3, equation (3.9)] and the notation in [3], we see that

$$\begin{aligned}
\int_0^T \widehat{G}(\mathbf{z})\, dz_2 &= \int_0^T \mathbf{G}(\mathbf{z}, \mathbf{0})\, dz_2 = G_{1D}(z_1, 0) = \mathcal{G}(|z_1|_T) = \mathcal{G}(z_1) \\
&= \frac{1}{2\sinh(T/2)}\cosh\Big(z_1 - \frac{T}{2}\Big).
\end{aligned}$$

Let u_1, u_2 be 1D T-periodic functions satisfying

$$-u_i'' + (1+\rho_i)u_i = \delta_{z_1}, \quad i = 1, 2$$

so that (see [3, p. 580])

$$u_i(y_1) = C_i \cosh(\sqrt{1+\rho_i}(|y_1 - z_1|_T - T/2)).$$

Then,

$$\int_0^T \widehat{G}(\mathbf{z}) \cos \frac{2\pi z_2}{T}\, dz_2 = \int_0^T \mathbf{G}(\mathbf{z}, \mathbf{0}) \cos \frac{2\pi z_2}{T}\, dz_2 = \int_0^T \mathbf{G}(\mathbf{0}, \mathbf{z}) \cos \frac{2\pi z_2}{T}\, dz_2$$
$$= u_1(0)\cos 0 = C_1 \cosh(\sqrt{1+\rho_1}(z_1 - T/2))$$

because the last integral above equals to V_1, where

$$-\Delta V_1(\mathbf{x}) + V_1(\mathbf{x}) = \cos(2\pi x_2/T) \llcorner \{x_1 = z_1\},$$

resulting in $V_1(\mathbf{x}) = u_1(x_1)\cos(2\pi x_2/T)$. Similarly,

$$\int_0^T \widehat{G}(\mathbf{z}) \cos \frac{4\pi z_2}{T}\, dz_2 = u_2(0)\cos 0 = C_2 \cosh(\sqrt{1+\rho_2}(z_1 - T/2)).$$

After substituting and carrying the differentiation with respect to z_1, recalling (4.1), we obtain

$$\begin{aligned}
\mathcal{A}_\lambda &= 6T\left\{\frac{3}{16\sinh(T/2)}\cosh(z_1 - \frac{T}{2}) - \frac{C_1(1+\rho_1)}{2}\cosh(\sqrt{1+\rho_1}(z_1 - T/2))\right. \\
&\quad \left. + \frac{C_2(1+\rho_2)}{8}\cosh(\sqrt{1+\rho_2}(z_1 - T/2))\right\}\Bigg|_{z_1=0^+}^{z_1=x_0} \\
&= 12\mathcal{T}\left[\frac{3\mathcal{T}}{16}\delta(\lambda,0,\mathcal{T}) - \frac{\mathcal{T}}{4}\left(1+\frac{\pi^2}{\mathcal{T}^2}\right)\delta(\lambda,\pi,\mathcal{T}) + \frac{\mathcal{T}}{16}\left(1+\frac{4\pi^2}{\mathcal{T}^2}\right)\delta(\lambda,2\pi,\mathcal{T})\right] \\
&= 3\mathcal{T}^2\left[\frac{3}{4}\delta(\lambda,0,\mathcal{T}) - \left(1+\frac{\pi^2}{\mathcal{T}^2}\right)\delta(\lambda,\pi,\mathcal{T}) + \frac{1}{4}\left(1+\frac{4\pi^2}{\mathcal{T}^2}\right)\delta(\lambda,2\pi,\mathcal{T})\right].
\end{aligned} \tag{4.16}$$

Introducing the functions

$$\begin{aligned}
\Delta_{1,0}(\lambda,\mathcal{T}) &:= \delta(\lambda,\pi,\mathcal{T}) - \delta(\lambda,0,\mathcal{T}), \\
\Delta_{2,0}(\lambda,\mathcal{T}) &:= \delta(\lambda,2\pi,\mathcal{T}) - \delta(\lambda,0,\mathcal{T}), \\
\Delta_{2,1}(\lambda,\mathcal{T}) &:= \delta(\lambda,2\pi,\mathcal{T}) - \delta(\lambda,\pi,\mathcal{T}),
\end{aligned} \tag{4.17}$$

we rewrite (4.16) as (we drop $\lambda, \mathcal{T}$ in the arguments in the first line for clarity)

$$\begin{aligned}
\mathcal{A}_\lambda &= 3\mathcal{T}^2\left(\frac{3}{4}\delta(0) - \delta(\pi) + \frac{1}{4}\delta(2\pi) - \frac{\pi^2}{\mathcal{T}^2}\delta(\pi) + \frac{\pi^2}{\mathcal{T}^2}\delta(2\pi)\right) \\
&= 3\mathcal{T}^2\left(\frac{1}{4}\Delta_{2,0}(\lambda,\mathcal{T}) - \Delta_{1,0}(\lambda,\mathcal{T}) + \frac{\pi^2}{\mathcal{T}^2}\Delta_{2,1}(\lambda,\mathcal{T})\right).
\end{aligned} \tag{4.18}$$

Note that we have not emphasized the dependence of $\mathcal{A}_\lambda$ on λ and $\mathcal{T}$ for a simpler representation. This new notation allows us to rewrite (4.2) as

$$\sigma_\lambda = \frac{2\pi^2}{\mathcal{T}^3 \Delta_{1,0}(\lambda, \mathcal{T})}. \tag{4.19}$$

For any lamella, the curvature $\mathcal{K} = 0$ and the shape operator $S = 0$ on $\partial\mathbb{L}$. From Theorem 3.3, we obtain $\frac{\partial \mathcal{K}}{\partial t}\big|_{t=0} = -\Delta_\tau \eta$ and $\frac{\partial^2 \mathcal{K}}{\partial t^2}\big|_{t=0} = -\Delta_\tau \xi$ for any general flow field X. In the bifurcation analysis for the critical lamellae below, we need to focus only when $X(x, y) = (g(y), 0)$ on the whole torus for some T-periodic and smooth function g; this leads to additional simplification. In particular, $\eta = (X \cdot \nu_0)\big|_{t=0} = g(y)\llcorner L_2 - g(y)\llcorner L_1$, $Z = D_X X = 0$, and $\xi = (Z \cdot \nu_0)\big|_{t=0} = 0$; thus, $\frac{\partial \mathcal{K}}{\partial t}\big|_{t=0} = -g''(y)\llcorner L_2 + g''(y)\llcorner L_1$ and $\frac{\partial^2 \mathcal{K}}{\partial t^2}\big|_{t=0} = 0$ at $L_1 \cup L_2$. It turns out that we will need the third derivative of the curvature with respect to t in the bifurcation analysis; for simplicity, we compute this only for the case of a lamella with the special flow field in the following lemma.

Lemma 4.6. *Suppose that $\mathbb{L}$ is a lamella (not necessarily critical) and $X(x, y) = (g(y), 0)$. Then,*

$$\frac{\partial \mathcal{K}}{\partial t}\bigg|_{t=0} = -g''(y)\llcorner L_2 + g''(y)\llcorner L_1, \quad \frac{\partial^2 \mathcal{K}}{\partial t^2}\bigg|_{t=0} = 0 \quad \text{at } L_1 \cup L_2, \tag{4.20}$$

$$\frac{1}{6}\frac{\partial^3 \mathcal{K}}{\partial t^3}\bigg|_{t=0} = \frac{3}{2} g''(y)(g'(y))^2 \llcorner L_2 - \frac{3}{2} g''(y)(g'(y))^2 \llcorner L_1. \tag{4.21}$$

Proof. If $\mathbf{x} \in \mathbb{T}$, then under the flow (1.3) one has $\Phi(\mathbf{x}, t) = \mathbf{x} + t(g(y), 0)$; therefore, the line L_2 evolves into $\{(x, y) : x = H(y) := x_0 + t\, g(y)\}$. Treating it as a graph over the y-axis, its curvature is

$$\mathcal{K} = -\frac{H''(y)}{(1 + H'(y)^2)^{3/2}} = -t\, g''(y)\left\{1 - \frac{3}{2} t^2 (g'(y))^2 + O(t^4)\right\}; \tag{4.22}$$

the sign in the above curvature formula has been chosen so that it is consistent with our choice that a circle has positive curvature. In other words, the initial point (x_0, y) with zero curvature moves to the new location $(H(y), y)$ with curvature given by the above RHS at time t. This Taylor expansion immediately gives (4.20) and (4.21) at L_2. For the corresponding results at L_1, the additional negative sign arises since the outward normal at L_1 points in the negative x-direction. ■

Lemma 4.7. *Let $X(x, y) = (\varphi_1(y), 0)$ with φ_1 given by (4.6). Then,*

$$\mathcal{B} := \int_{\partial\mathbb{L}} \phi_1(\mathbf{x}) \frac{\partial^3 \mathcal{K}}{\partial t^3}\bigg|_{t=0} d\mathcal{H}^1_{\mathbf{x}} = -\frac{9T}{4}\left(\frac{2\pi}{T}\right)^4 = -\frac{9\pi^4}{2\mathcal{T}^3}. \tag{4.23}$$

Proof. Substituting $g = \varphi_1 = \sin \frac{2\pi y}{T}$ in (4.21), we obtain

$$\frac{\partial^3 \mathcal{K}}{\partial t^3}\bigg|_{t=0} = -9\left(\frac{2\pi}{T}\right)^4 \varphi_1(y) \cos^2 \frac{2\pi y}{T} \llcorner L_2 + 9\left(\frac{2\pi}{T}\right)^4 \varphi_1(y) \cos^2 \frac{2\pi y}{T} \llcorner L_1.$$

A direct computation then yields

$$\begin{aligned}\mathcal{B} &= 2\int_{L_2} \varphi_1(\mathbf{x}_2)\frac{\partial^3 \mathcal{K}}{\partial t^3}\bigg|_{t=0} d\mathcal{H}^1_{\mathbf{x}} \\ &= -18\left(\frac{2\pi}{T}\right)^4 \int_0^T \sin^2\frac{2\pi y}{T}\cos^2\frac{2\pi y}{T}\,dy = -\frac{9T}{4}\left(\frac{2\pi}{T}\right)^4. \qquad \blacksquare\end{aligned}$$

When we study bifurcation for $c = \lambda = 0$ in the coming Section 5, the sign of the scalar $\mathcal{B} + \sigma_0 \mathcal{A}_0$ will be important.

5. Bifurcation analysis when $c = 0$

So far, there is no difference in calculating derivatives of curvature and nonlocal term at critical lamellae when subject to a prescribed flow corresponding to any $c \in (-1, 1)$. In this section, we fix $c = 0$ and adopt the simpler notation $\mathcal{F}(\mathbb{L}_t, \sigma)$ instead of $\mathcal{F}(\mathbb{L}_t, \sigma, c)$ to study the bifurcation of the Euler–Lagrange equation $\mathcal{F}(\mathbb{L}_t, \sigma) = 0$, where $\mathcal{F}$ is given in (2.1), in the vicinity of the configuration $(\mathbb{L}, \sigma_0)$, where the critical lamella $\mathbb{L}$ loses its stability due to the first eigenmode ϕ_1 on $\partial\mathbb{L}$. The resulting non-planar solutions coming from a perturbation of critical laminae can be attained by specifying a suitable flow field around the neighborhood of $\partial\mathbb{L}$. Recall that the thickness of the lamella is exactly $T/2$; thus, volume (which is preserved through the flow) is equally divided among the (evolved) lamella and the empty space. One may expect that there should be equal importance of either set even when the configuration breaks into a non-lamellar mode. This prompts us to look for solutions when $(L_2)_t$ is a translation of $(L_1)_t$ by a distance of $T/2$ in the x-direction for all (small) t. In other words, the imposed flow field $X(x, y) = (X_1(y), 0)$ around L_1 is the same as that around L_2, and we can employ $X(x, y) = (X_1(y), 0)$ in the whole torus; the resulting configuration is antisymmetric with respect to the outward normal direction.

Lemma 5.1. *Suppose $c = 0$ and $X(x, y) = (g(y), 0)$ in the whole torus with $g \in \mathcal{C}^2$ being T-periodic. Then,*

(a) $(L_2)_t = (L_1)_t + (T/2, 0)$;

(b) *if the Euler–Lagrange equation (2.1) is satisfied on one side of $\mathbb{L}_t$, it will be satisfied on the other;*

(c) *for points with the same y-coordinate at the left and the right boundaries, the three quantities $\mathcal{K}$, $\mathcal{N}_{\mathbb{L}_t} - 1/2$, and $\mathcal{F}(\mathbb{L}_t, \sigma)$ are opposite in sign;*

(d) *if, moreover, g is y-odd, then*

$$\mathcal{K}, \quad \mathcal{N}_{\mathbb{L}_t} - 1/2, \quad \mathcal{F}(\mathbb{L}_t, \sigma)$$

are y-odd at each boundary.

Proof. If $\mathbf{x} \in \mathbb{T}$, then under the flow (1.3)

$$\mathbf{x}_t := \Phi(\mathbf{x}, t) = \mathbf{x} + t(g(y), 0).$$

We adopt a notation similar to what was used in Lemma 4.2: define

$$\mathbf{t} := (T/2, 0), \quad \mathbf{c} := (T/4, T/2),$$

and for every point $\mathbf{x} \in L_1$, let

$$\hat{\mathbf{x}} := \mathbf{x} + \mathbf{t}, \quad \tilde{\mathbf{x}} := (0, T) - \mathbf{x},$$

so $\hat{\mathbf{x}} \in L_2$ has the same vertical coordinate as $\mathbf{x}$, while $\tilde{\mathbf{x}} \in L_1$ is the symmetric of $\mathbf{x}$ with respect to the horizontal mid-line of $\mathbb{T}$. Both $\tilde{\mathbf{x}}$ and $\mathbf{c}$, the center of $\mathbb{L}$, will be used only to prove (d). Also, set

$$\widehat{\mathbb{L}} := \mathbb{L} + \mathbf{t}.$$

Clearly,

$$\hat{\mathbf{x}}_t = \mathbf{x}_t + \mathbf{t}, \quad \widehat{\mathbb{L}}_t = \mathbb{L}_t + \mathbf{t};$$

therefore, the function $\mathcal{N}_{\widehat{\mathbb{L}}_t}$ is a translate by $\pm\mathbf{t}$ of $\mathcal{N}_{\mathbb{L}_t}$ (the sign does not matter due to periodicity):

$$\mathcal{N}_{\widehat{\mathbb{L}}_t}(\mathbf{z}) = \mathcal{N}_{\mathbb{L}_t}(\mathbf{z} \pm \mathbf{t}).$$

But $\mathcal{N}_{\widehat{\mathbb{L}}_t} + \mathcal{N}_{\mathbb{L}_t} = \mathcal{N}_{\mathbb{T}} \equiv 1$, so for every point $\mathbf{x} \in L_1$,

$$\mathcal{N}_{\mathbb{L}_t}(\mathbf{x}_t) + \mathcal{N}_{\mathbb{L}_t}(\hat{\mathbf{x}}_t) = \mathcal{N}_{\mathbb{L}_t}(\mathbf{x}_t) + \mathcal{N}_{\widehat{\mathbb{L}}_t}(\mathbf{x}_t) = 1$$

or equivalently

$$[\mathcal{N}_{\mathbb{L}_t} - 1/2](\mathbf{x}_t) = -[\mathcal{N}_{\mathbb{L}_t} - 1/2](\hat{\mathbf{x}}_t). \tag{5.1}$$

We already know from (4.20) that

$$\mathcal{K}(\mathbf{x}_t) = -\mathcal{K}(\hat{\mathbf{x}}_t), \tag{5.2}$$

and the first three assertions are proved since in the case $c = 0$

$$\mathcal{F}(\mathbb{L}_t, \sigma) = \mathcal{K} + \sigma[\mathcal{N}_{\mathbb{L}_t} - 1/2].$$

To prove the last, remark that if g is odd and periodic, then $g(0) = g(T/2) = g(T) = 0$; thus, $\mathbf{c}_t \equiv \mathbf{c}$ and $\mathbb{L}_t$ is symmetric with respect to $\mathbf{c}$, so the same applies to $\mathcal{K}$ and $\mathcal{N}_{\mathbb{L}_t}$. In particular, since $\tilde{\mathbf{x}}_t$ is the symmetric of $\hat{\mathbf{x}}_t$,

$$\mathcal{K}(\tilde{\mathbf{x}}_t) = \mathcal{K}(\hat{\mathbf{x}}_t), \quad \mathcal{N}_{\mathbb{L}_t}(\tilde{\mathbf{x}}_t) = \mathcal{N}_{\mathbb{L}_t}(\hat{\mathbf{x}}_t),$$

which proves that $\mathcal{K}$ and $\mathcal{N}_{\mathbb{L}_t}$ are y-odd at each boundary due to (5.1), (5.2). ∎

It is not a surprise that one needs information about the Taylor expansion of $\mathcal{F}$ evaluated at the (possible) bifurcation point $(\mathbb{L}, \sigma_0, 0)$. With $\mathcal{F}$ being composed of both the nonlocal and the curvature terms, we can extract the necessary derivatives of $\mathcal{F}$ from the previous sections. As $c = 0$, we adopt the simpler notation $\mathcal{F}(\mathbb{L}_t, \sigma)$ from now on; in addition, with $\mathbb{L}$ being critical, $D_t\mathcal{F}[\eta] = D_t\mathcal{F}(\mathbb{L}, \sigma)[X] := (\frac{\partial}{\partial t}\mathcal{F}(\mathbb{L}_t, \sigma))\big|_{t=0}$, where $\eta = X \cdot \nu_0$, and similar notations for higher derivatives as well.

Lemma 5.2. *Let $c = 0$ and suppose $X(x, y) = (\varphi_1(y), 0)$; we have*

$$D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] = 0, \tag{5.3}$$

$$D_\sigma\mathcal{F}(\mathbb{L}, \sigma) = 0 \qquad \textit{for all } \sigma, \tag{5.4}$$

$$D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma)[\phi_1] = \left\{\frac{d_0^{(1)}}{\sqrt{1 + \rho_1}} - d_0\right\}\phi_1(y) \llcorner \partial\mathbb{L} \qquad \textit{for all } \sigma, \tag{5.5}$$

$$D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] = -\frac{\rho_1}{\sigma_0}\phi_1(y) \llcorner \partial\mathbb{L}. \tag{5.6}$$

Proof. Using (2.1), (4.20), and (4.7), we have

$$D_t\mathcal{F}(\mathbb{L}, \sigma)[\phi_1] = \left\{\rho_1 + \frac{\sigma d_0^{(1)}}{\sqrt{1 + \rho_1}} - \sigma d_0\right\}\phi_1(y) \llcorner \partial\mathbb{L}. \tag{5.7}$$

When $\sigma = \sigma_0$, we have $\{\cdots\} = 0$; see [3, equation (5.12)]; this leads to (5.3). At the same time in view of (1.9), for all $\sigma > 0$,

$$D_\sigma\mathcal{F}(\mathbb{L}, \sigma) = -\frac{1}{2} + \mathcal{N}_{\mathbb{L}}\big|_{\partial\mathbb{L}} = 0,$$

which is (5.4). We now take a derivative of (5.7) with respect to σ to obtain (5.5). In turn, (5.5) gives (5.6). ∎

We summarize what we know so far for the case $c = 0$. For all $\sigma > 0$, the 1-lamella $\mathbb{L} = [0, T/2] \times [0, T]$ is a critical point of the geometric variational functional J, leading to $\mathcal{F}(\mathbb{L}, \sigma) = 0$. By Remark 3.5, we write $D_t\mathcal{F}(\mathbb{L}, \sigma)[\eta]$ with $\eta = X \cdot \nu_0$ instead of $D_t\mathcal{F}(\mathbb{L}, \sigma)[X]$ at the critical set $\mathbb{L}$. This lamella loses its stability at $\sigma = \sigma_0$ through a double eigenvalue $\Lambda = 0$, i.e., $D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\eta] = 0$ with, recalling (4.6),

$$\begin{aligned}\eta \in \operatorname{span}&\left\{-\sin\frac{2\pi y}{T} \llcorner L_1 + \sin\frac{2\pi y}{T} \llcorner L_2, -\cos\frac{2\pi y}{T} \llcorner L_1 + \cos\frac{2\pi y}{T} \llcorner L_2\right\} \\ &= \operatorname{span}\{\phi_1(\cdot),\ \phi_1(\cdot + T/4)\};\end{aligned}$$

this 2-dimensional null space arises from the fact that a translation in the y-direction of the eigenfunction ϕ_1 remains an eigenfunction due to periodicity.

The fact that $\dim(\ker(D_t\mathcal{F}(\mathbb{L}, \sigma_0))) = 2$ induces technical difficulty in local bifurcation analysis. To restore bifurcation due to a simple eigenvalue, we focus only on the bifurcation for an antisymmetric perturbed lamella consisting of y-odd periodic functions (this closed subspace is itself a Banach space); this eliminates translation in the y-direction and results in $\ker(D_t\mathcal{F}(\mathbb{L},\sigma_0)) = \mathrm{span}\{\phi_1\}$ in the closed subspace so that

$$\dim(\ker(D_t\mathcal{F}(\mathbb{L},\sigma_0))) = 1.$$

Fix $\omega \in (0,1)$ from now on (say $\omega = 1/2$), and let the flow field be $X = (X_1(x,y),0)$ with $X_1(x,y) = g(y) := \varphi_1(y) + w(y)$ for all $(x,y) \in \mathbb{T}$ and

$$w \in W := \left\{u \in \mathcal{C}^{2+\omega}_{\mathrm{per}}[0,T] : u(y) = -u(T-y);\ \text{for } 0 \le y \le T, \int_0^T u(y)\varphi_1(y)\,dy = 0\right\}$$

is a y-odd function with zero average $\int_0^T w(y)\,dy = 0$. This ensures that, under the flow X, the profile $(L_2)_t$ is a translate of $(L_1)_t$ by a distance of x_0 in the x-direction for all (small) $t \ge 0$; moreover, both $(L_1)_t$ and $(L_2)_t$ are y-odd. This flow field can generate any perturbed antisymmetric profile at any t starting from the lamella. Antisymmetry allows us to focus only on satisfying the Euler–Lagrange equation at $(x,y) \in (L_2)_t$.

We now closely follow the idea in Crandall–Rabinowitz simple eigenvalue bifurcation theorem [6,22–24]. With our choice of X_1 so that $\eta = X\cdot\nu_0 = \phi_1 + \mathfrak{w} := \phi_1 + w\llcorner L_2 - w\llcorner L_1$, let $G : W\times\mathbb{R}\times\mathbb{R} \to Z_1 \oplus \mathrm{span}\{\phi_1\}$ such that

$$G(w,\beta,t) := \begin{cases} \frac{\mathcal{F}(\mathbb{L}_t,\sigma_0+\beta)}{t} & \text{for } t \neq 0,\\ D_t\mathcal{F}(\mathbb{L},\sigma_0+\beta)[\phi_1+\mathfrak{w}] & \text{for } t = 0.\end{cases}$$

Here,

$$W_1 := \left\{u \in \mathcal{C}^{\omega}_{per}[0,T] : u(y) = -u(T-y) \text{ for } 0 \le y \le T, \int_0^T u(y)\,\varphi_1(y)\,dy = 0\right\},$$
$$Z_1 := \{-u\llcorner L_1 + u\llcorner L_2 : u \in W_1\}.$$

That the range of G lies in $Z_1 \oplus \mathrm{span}\{\phi_1\}$ is a direct consequence of assertion (d) in Lemma 5.1. Now, $G \in \mathcal{C}^2$ since $\mathcal{F}(\mathbb{L}_t,\sigma)$ is in $\mathcal{C}^3$. Note that $G = 0$ for $t \neq 0$ corresponds to non-lamellar solution of the Euler–Lagrange equation $\mathcal{F} = 0$. In fact, for $t \neq 0$,

$$\begin{aligned} &G(w,\beta,t)\\ &= \frac{1}{t}\left\{\mathcal{F}(\mathbb{L},\sigma_0+\beta) + tD_t\mathcal{F}(\mathbb{L},\sigma_0+\beta)[X] + \frac{t^2}{2}D_{tt}\mathcal{F}(\mathbb{L},\sigma_0+\beta)[X,X] + \cdots\right\}\\ &= D_t\mathcal{F}(\mathbb{L},\sigma_0+\beta)[X] + \frac{t}{2}D_{tt}\mathcal{F}(\mathbb{L},\sigma_0+\beta)[X,X] + \cdots. \end{aligned}$$

When $w = 0$, we have $X(x,y) = (\varphi_1(y),0)$ so that $G(0,0,0) = 0$ by (5.3); we next take a derivative of G with respect to w by using (2.3) and (4.20) so that at $\partial\mathbb{L}$

$$\begin{aligned} D_wG(0,0,0)\hat{w} &= D_t\mathcal{F}(\mathbb{L},\sigma_0)[\hat{\mathfrak{w}}]\\ &= \sigma_0\begin{pmatrix}\hat{\mathfrak{w}}\\0\end{pmatrix}\cdot\nabla\mathcal{N}_{\mathbb{L}} + \sigma_0\int_{\partial\mathbb{L}}\widehat{G}(\mathbf{x}-\mathbf{y})\begin{pmatrix}\hat{\mathfrak{w}}\\0\end{pmatrix}\cdot\nu_0(\mathbf{y})\,d\mathcal{H}^1_y - \hat{\mathfrak{w}}'' \end{aligned} \quad (5.8)$$

for all $\hat{w} \in W$ and $\hat{\mathfrak{w}} := \hat{w}\llcorner L_2 - \hat{w}\llcorner L_1$. At the same time, from (5.6),

$$D_\beta G(0,0,0)\widehat{\beta} = \widehat{\beta} D_{t\sigma}\mathcal{F}(\mathbb{L},\sigma_0)[\phi_1] = -\frac{\widehat{\beta}\rho_1}{\sigma_0}\phi_1$$

for any $\widehat{\beta} \in \mathbb{R}$. Since

$$\begin{aligned} D_{(w,\beta)}G(0,0,0)[\hat{w},\widehat{\beta}] &= D_w G(0,0,0)\hat{w} + D_\beta G(0,0,0)\widehat{\beta} \\ &= D_w G(0,0,0)\hat{w} - \frac{\widehat{\beta}\rho_1}{\sigma_0}\phi_1, \end{aligned} \tag{5.9}$$

we demonstrate that $D_{(w,\beta)}G(0,0,0)$ is invertible.

First, we check that it is injective. Set $D_{(w,\beta)}G(0,0,0)[\hat{w},\widehat{\beta}] = 0$. Using a separation of variable argument, we know that $D_w G(0,0,0)\hat{w}$ is orthogonal to ϕ_1 with $L^2(\partial\mathbb{L})$ inner product when $\hat{w} \in W_1$. Hence, by taking the inner product of $D_{(w,\beta)}G(0,0,0)[\hat{w},\widehat{\beta}]$ with ϕ_1, one immediately sees that $\widehat{\beta} = 0$. This leaves behind $D_w G(0,0,0)\hat{w} = 0$. We claim that this implies $\hat{w} = 0$. Indeed, multiplying by $\eta = \hat{\mathfrak{w}}$ and integrating over $\partial\mathbb{L}$, we obtain [3, RHS of (3.2)] for $\sigma = \sigma_0$. We can split $\eta = \mu + \zeta$, the mean part and the zero-average part, respectively. The mean part $\mu = 0$ because $\int_0^T w(y)\,dy = 0$, leaving behind $\zeta = \hat{\mathfrak{w}}$. Now, consider [3, RHS of (3.2)] when η is replaced by ζ. As ζ represents 2nd or higher modes, this inner product $\int_{\partial\mathbb{L}} \eta(D_w G(0,0,0)\hat{w})\,d\mathcal{H}^1 > 0$ unless $\hat{w} = 0$; see calculation in proving [3, equation (5.16)].

Next, we check surjectivity. For any $(u,\lambda) \in W_1 \times \mathbb{R}$, we have to find $(\hat{w},\widehat{\beta}) \in W \times \mathbb{R}$ such that $D_{(w,\beta)}G(0,0,0)[\hat{w},\widehat{\beta}] = -u\llcorner L_1 + u\llcorner L_2 + \lambda\phi_1 \in Z_1 \oplus \operatorname{span}\{\phi_1\}$, which reduces to

$$\widehat{\beta} = -\frac{\sigma_0}{\rho_1}\lambda$$

and

$$\sigma_0\begin{pmatrix}\hat{\mathfrak{w}}\\0\end{pmatrix}\cdot\nabla\mathcal{N}_{\mathbb{L}} + \sigma_0\int_{\partial\mathbb{L}}\widehat{G}(\mathbf{x}-\mathbf{y})\begin{pmatrix}\hat{\mathfrak{w}}\\0\end{pmatrix}\cdot\nu_0(\mathbf{y})\,d\mathcal{H}^1_y - \hat{\mathfrak{w}}'' = -u\llcorner L_1 + u\llcorner L_2.$$

Both sides just change sign on switching from L_1 to L_2; it suffices to solve the equation on L_2 alone. This is the same as solving on $[0,T]$

$$-d_0\sigma_0\hat{w} + \sigma_0\int_{\partial\mathbb{L}}\widehat{G}(\mathbf{x}-\mathbf{y})\hat{w}(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}} - \hat{w}'' = u \tag{5.10}$$

or

$$(1-D^2)\hat{w} - (1+d_0\sigma_0)\hat{w} + \sigma_0\int_{\partial\mathbb{L}}\widehat{G}(\mathbf{x}-\mathbf{y})\hat{w}(\mathbf{y})\,d\mathcal{H}^1_{\mathbf{y}} = u;$$

employing the operator $(1-D^2)^{-1}$ on both sides, one sees that the Fredholm alternative applies, and the already-proven injectivity ensures surjectivity; thus, the operator $D_{(w,\beta)}G(0,0,0)$ is invertible and therefore continuous.

Now, the implicit function theorem gives unique $w_t \in W$ and $\beta_t \in \mathbb{R}$, which are $\mathcal{C}^2$ in t, such that $G(w_t,\beta_t,t) = 0$ for small $|t|$ with $w_t\big|_{t=0} = 0$ and $\beta_t\big|_{t=0} = 0$. This represents a bifurcation at a simple eigenvalue, resulting in a unique non-lamellar branch which is y-odd. Any translation in the y-direction of the non-lamella solution gives a distinct non-odd solution; hence, they form a 2-dimensional space of bifurcated solutions in a neighborhood of the lamella $\mathbb{L}$ at $\sigma = \sigma_0$, and the proof of Theorem 1.2 is completed, as all configurations obtained by horizontal flows depending only on the vertical variable are obviously volume-preserving. Note that we have not excluded the possibility of non-odd non-planar solutions of (2.1) that cannot be obtained by such vertical translation.

Remark 5.3. Let E_σ be a critical (non-lamella) set which depends on σ, and (E_σ, σ) be described by a 1-dimensional smooth curve. At some $\sigma = \sigma_0$ and $E = E_0$, let $D_t\mathcal{F}(E_0, \sigma_0)$ have a simple eigenvalue $\Lambda = 0$ with an eigenfunction ϕ_1, i.e., $D_t\mathcal{F}(E_0, \sigma_0)[\phi_1] = 0$. At the same time, we assume $D_\sigma\mathcal{F}(E_0, \sigma_0) = 0$. Since $D_{\sigma\sigma}\mathcal{F}(E, \sigma) = 0$ is always valid for $\mathcal{F}$ defined in (2.1), a Taylor expansion of $\mathcal{F}(E_t, \sigma_0 + \beta_t)$ as in (5.12) with

$$\beta_t = t\beta_1 + O(t^2)$$

due to a flow field $X = \phi_1\nu + t\mathfrak{w}_1\nu + O(t^2)$ leads to

$$D_t\mathcal{F}(E_0, \sigma_0)[\mathfrak{w}_1] + \beta_1 D_{\sigma t}\mathcal{F}(E_0, \sigma_0)[\phi_1] = -\frac{1}{2}D_{tt}\mathcal{F}(E_0, \sigma_0)[X_0, X_0] + O(t),$$

where $X_0 = \phi_1\nu$. If one can verifies that the above linear equation is uniquely solvable for any RHS in the domain and range that we impose, then a bifurcation due to a simple eigenvalue takes place. One can make this argument rigorous by using implicit function theorem as in the proof of Theorem 1.2.

One can regard $\mathcal{F}(E_t, \sigma)$ as a $\mathcal{C}^3$ function of $(\partial E, t, \sigma)$ when $X \in \mathcal{C}^5$ is prescribed; this claim comes from the discussion just below (2.1). Let E be a critical set; by Remark 3.5, both $D_t\mathcal{F}(E, \sigma)$ and $D_{t\sigma}\mathcal{F}(E, \sigma)$ act only on the normal component $\eta = X \cdot \nu\big|_{\partial E}$. Another important observation can be drawn from (2.3) and (3.8):

$$\int_{\partial E} \eta_1 D_t\mathcal{F}(E, \sigma)[\eta_2]\, d\mathcal{H}^1 = \int_{\partial E} \eta_2 D_t\mathcal{F}(E, \sigma)[\eta_1]\, d\mathcal{H}^1 \tag{5.11}$$

for any smooth (normal components) η_1, η_2. In other words, $D_t\mathcal{F}(E, \sigma)$ at a critical E is a symmetric operator with respect to the inner product over ∂E.

On the other hand, $D_{tt}\mathcal{F}(E, \sigma)$ may depend on X and not just on its normal component; see Remark 3.5. Though in the special case when $X = (X_1, 0)$ and $E = \mathbb{L}$, $D_{tt}\mathcal{F}(E, \sigma)$ acts on η only; we still prefer to write that as $D_{tt}\mathcal{F}(\mathbb{L}, \sigma)[X, X]$ below.

Upon bifurcation, we have from the above proof of Theorem 1.2 that $w_t = tw_1 + \frac{t^2}{2}w_2 + o(t^2)$ with $w_1, w_2 \in W$ and $\beta_t = t\beta_1 + \frac{t^2}{2}\beta_2 + o(t^2)$. For later use, we define $\mathfrak{w}_i := w_i \llcorner L_2 - w_i \llcorner L_1$, $i = 1, 2$, on $\partial\mathbb{L}$. We restrict our attention to the flow field $X_t(x, y, t) = (\varphi_1(y) + w_t(y), 0)$. Putting $g = \varphi_1 + tw_1 + \frac{t^2}{2}w_2 + o(t^2)$ in (4.22), we obtain

$$\mathcal{K} = t\rho_1\varphi_1 - t^2 w_1'' - \frac{t^3}{2}w_2'' + \frac{3}{2}t^3(\varphi_1')^2\varphi_1'' + o(t^3)$$

if we view it as a graph over the y-axis. After taking into account the outward normal direction in deciding the sign of curvature, we have

$$\mathcal{K} = t\rho_1\phi_1 - t^2\mathfrak{w}_1'' - \frac{t^3}{2}\mathfrak{w}_2'' - \rho_1\frac{3}{2}t^3(\phi_1')^2\phi_1 + o(t^3) \quad \text{on } \partial\mathbb{L}.$$

Remark 5.4. Let $s \in \mathbb{R}$ and think of $X(x, y) = (\varphi_1(y) + sw_1(y) + \frac{s^2}{2}w_2(y) + o(s^2), 0)$ as an autonomous flow field. Putting this in $\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)$ results in a Taylor polynomial in t and s. Now, set $s = t$ to get a polynomial in t only. This polynomial is the same as that obtained by expanding $\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)$ with a flow field $(\varphi_1 + tw_1 + \frac{t^2}{2}w_2 + o(t^2), 0)$ below. However, this latter procedure is easier to carry out.

Since $w_t = tw_1 + \frac{t^2}{2}w_2 + o(t^2)$ with $w_1, w_2 \in W$, and $\beta_t = t\beta_1 + \frac{t^2}{2}\beta_2 + o(t^2)$, Lemma 5.2 and (4.8) give

$$\begin{aligned} 0 &= \mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t) \\ &= \mathcal{F}(\mathbb{L}, \sigma_0) + tD_t\mathcal{F}(\mathbb{L}, \sigma_0)[X] + \beta_t D_\sigma\mathcal{F}(\mathbb{L}, \sigma_0) + \frac{t^2}{2}D_{tt}\mathcal{F}(\mathbb{L}, \sigma_0)[X, X] \\ &\quad + t\beta_t D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma_0)[X] + \frac{\beta_t^2}{2}D_{\sigma\sigma}\mathcal{F}(\mathbb{L}, \sigma_0) + \frac{t^3}{6}D_{ttt}\mathcal{F}(\mathbb{L}, \sigma_0)[X, X, X] \\ &\quad + \frac{t^2\beta_t}{2}D_{\sigma tt}\mathcal{F}(\mathbb{L}, \sigma_0)[X, X] + \frac{t\beta_t^2}{2}D_{\sigma\sigma t}\mathcal{F}(\mathbb{L}, \sigma_0)[X] + \frac{\beta_t^3}{6}D_{\sigma\sigma\sigma}\mathcal{F}(\mathbb{L}, \sigma_0) + \cdots \\ &= t^2 D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\mathfrak{w}_1 + \frac{t}{2}\mathfrak{w}_2] + t^2(\beta_1 + \frac{t}{2}\beta_2)D_{\sigma t}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1 + t\mathfrak{w}_1] \\ &\quad + \frac{t^3}{6}D_{ttt}\mathcal{F}(\mathbb{L}, \sigma_0)[X, X, X] + \frac{t^3\beta_1}{2}D_{\sigma tt}\mathcal{F}(\mathbb{L}, \sigma_0)[X, X] + o(t^3). \end{aligned} \tag{5.12}$$

Considering only second-order terms gives

$$D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\mathfrak{w}_1] + \beta_1 D_{\sigma t}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] = 0; \tag{5.13}$$

multiplying the above equation by ϕ_1 and integrating over $\partial\mathbb{L}$, we get

$$\int_{\partial\mathbb{L}} \phi_1\{D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\mathfrak{w}_1] + \beta_1 D_{\sigma t}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1]\}\, d\mathcal{H}^1 = 0.$$

The first term on LHS is zero because

$$\int_{\partial\mathbb{L}} \phi_1 D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\mathfrak{w}_1]\, d\mathcal{H}^1 = \int_{\partial\mathbb{L}} \mathfrak{w}_1 D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1]\, d\mathcal{H}^1 = 0.$$

Since

$$\hat{a} := \int_{\partial\mathbb{L}} \phi_1 D_{\sigma t}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1]\, d\mathcal{H}^1 = -\frac{\rho_1}{\sigma_0}\int_{\partial\mathbb{L}} \phi_1^2\, d\mathcal{H}^1 = -\frac{\rho_1 T}{\sigma_0} \neq 0, \tag{5.14}$$

we have $\beta_1 = 0$, so (5.13) gives $w_1 = 0$ by the same argument in showing $D_{(w,\beta)}G(0,0,0)$ is injective.

Setting $O(t^3)$ terms to zero in (5.12) with $\beta_1 = w_1 = 0$, we obtain for $t = 0$

$$0 = \frac{1}{2} D_t \mathcal{F}(\mathbb{L}, \sigma_0)[w_2] + \frac{1}{2}\beta_2 D_{\sigma t} \mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] + \frac{1}{6} D_{ttt} \mathcal{F}(\mathbb{L}, \sigma_0)[\varphi_1, \varphi_1, \varphi_1].$$

(In the last term, we wrote φ_1 instead of $(\varphi_1, 0)$ to shorten notation.) We prove that β_2 and w_2 are solvable from this linear equation: the source term $D_{ttt}\mathcal{F}(\mathbb{L}, \sigma_0)[\varphi_1, \varphi_1, \varphi_1]$ is known; now, multiply by ϕ_1 and integrate over $\partial\mathbb{L}$, so the first term drops out and it follows that

$$\beta_2 = -\frac{1}{3\hat{a}} \int_{\partial\mathbb{L}} \phi_1 D_{ttt} \mathcal{F}(\mathbb{L}, \sigma_0)[\varphi_1, \varphi_1, \varphi_1]\, d\mathcal{H}^1, \tag{5.15}$$

where $\hat{a}$ is defined in (5.14) above. This agrees with a corresponding equation on [6, p. 22]. (Note that $\beta_2 = \frac{d^2\sigma}{dt^2}\big|_{t=0}$, and the quantity called "c" in [6] is our β_2 here.) Suppose $\beta_2 \neq 0$; since $\beta_1 = 0$, we have $\beta = \frac{t^2}{2}\beta_2 + O(t^3)$ so that

$$t = \pm\sqrt{\frac{2\beta}{\beta_2}} + O(\beta^{3/2}).$$

We know non-lamella solution exists. If $\beta_2 > 0$, this can only be true when $\beta > 0$. We call this phenomenon supercritical bifurcation. On the other hand, when $\beta_2 < 0$, we need $\beta < 0$, corresponding to a subcritical bifurcation. In both cases, the perturbed lamella $(L_1)_t$ is represented by $x_2 = g_t(x_1) := t\varphi(x_2) + \frac{t^3}{6} w_2(x_2) + O(t^4)$, while $(L_2)_t$ is just $T/2$ translate of $(L_1)_t$ in the x_1-direction.

Next, we examine the stability of solutions arising from bifurcation. With

$$J'(\mathbb{L}_t; \sigma_0 + \beta_t)[X] = \int_{\mathbb{L}_t} \mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)(X \cdot \nu)\, d\mathbf{x},$$

a critical point $\mathbb{L}_t$ of J is given by the Euler–Lagrange equation $\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t) = 0$. Using this information, another derivative leads to

$$J''(\mathbb{L}_t; \sigma_0 + \beta_t)[X] = \int_{\partial\mathbb{L}_t} D_t \mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[X]\, \eta\, d\mathcal{H}^1,$$

where $\eta = X \cdot \nu\big|_{\partial\mathbb{L}}$. From (3.10), $D_t\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[X]$ is a scalar function depending only on η (including its tangential derivatives) on $\partial\mathbb{L}_t$, but not the tangential component of X. By an eigenvalue μ_t and a corresponding (non-zero) eigenfunction η_t of $D_t\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)$ we mean $D_t\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[\eta_t] = \mu_t \eta_t$.

One can decompose η into the mean-value part and the zero-average part [3, Section 4]. The mean-value part is always stable. (In fact, from anti-symmetry in our case, the mean-value part is just a pure x-direction translation.) For zero-average part, eigenmodes are untangled from one another [3, Section 5]; thus, as long as it is stable for each eigenmode, we have overall stability. We therefore focus only on the cases when η is a zero-average eigenfunction.

Let us first look at how the lamella $\mathbb{L}$ changes its stability. When $\beta < 0$, all eigenvalues of $D_t\mathcal{F}(\mathbb{L}, \sigma_0 + \beta)$ are positive so that $J''(\mathbb{L}, \sigma_0 + \beta) > 0$; in such a case, the lamella $\mathbb{L}$ is stable as $\mathbb{L}_t$ represents a local energy minimizer. For $\beta = 0$, there is a simple zero eigenvalue with an odd eigenfunction $\eta = \phi_1 \in W$. When $\beta > 0$, this eigenvalue becomes negative, which leads to an unstable $\mathbb{L}$. Denote this eigenvalue by Λ_β. Thus, $\Lambda_\beta > 0$ when $\beta < 0$; $\Lambda_0 = 0$; and $\Lambda_\beta < 0$ when $\beta > 0$. Let the corresponding eigenfunction be ξ_β, which is the normal component on $\partial\mathbb{L}$ of a vector field $(\xi_\beta, 0)$, which is a T-periodic function on both L_1 and L_2. In addition, ξ_β is normalized so that $\xi_0 = \phi_1$ and $\int_{\partial\mathbb{L}} \xi_\beta \phi_1 \, d\mathcal{H}^1 = T$. Using the implicit function theorem, the simple eigenvalue Λ_β and the normalized eigenfunction $\xi_\beta\big|_{\partial\mathbb{L}}$ are C^2 functions of β. Since

$$D_t\mathcal{F}(\mathbb{L}, \sigma_0 + \beta)[\xi_\beta] = \Lambda_\beta \xi_\beta$$

taking a derivative with respect to β and then evaluating at $\beta = 0$, we obtain

$$D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] + D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\xi'_\beta(0)] = \Lambda'_\beta(0)\phi_1.$$

Next, multiply by ϕ_1 and integrate over $\partial\mathbb{L}$. The second term on the LHS drops out by (5.11), and from (5.14), we obtain $T\Lambda'_\beta(0) = -\rho_1 T/\sigma_0$ so that $\Lambda'_\beta(0) = -\rho_1/\sigma_0 < 0$.

Along the non-lamellar branch with critical points represented by $\mathbb{L}_t$, continuous dependence of the positive eigenvalues for $\mathbb{L}$ at $\beta_t = 0$ ensures that they remain stable modes of $\mathbb{L}_t$ for small enough $|t|$. How the zero-eigenvalue mode evolves along the non-lamellar branch determines its stability. Denote this β_t-dependent eigenvalue by μ_t along the non-lamellar branch so that $\mu_0 = 0$; it is a real and simple eigenvalue of $D_t\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)$ with a corresponding eigenfunction ψ_t on $\partial\mathbb{L}_t$, which is the normal component of a vector field Y_t with $Y_0 = (\varphi_1, 0)$ at $\partial\mathbb{L}$. Here, ψ_t is a T-periodic function of x_2 and $\psi_t\big|_{(L_2)_t} = -\psi_t\big|_{(L_1)_t}$. We normalize this eigenfunction so that $\int_{\partial\mathbb{L}_t} \psi_t\phi_1 \, dx_2 = T$ with $\psi_0 = \phi_1$. This normalization uniquely determines the eigenfunction ψ_t for small $|t|$; moreover, the simple eigenvalue μ_t and the normalized eigenfunction $\psi_t\big|_{\partial\mathbb{L}_t}$ are C^2 functions of t. Since

$$D_t\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[\psi_t] = \mu_t\psi_t,$$

taking a derivative with respect to t

$$\begin{aligned} D_{tt}\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[Y_t, Y_t] + \beta'_t D_{t\sigma}\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[\psi_t] + D_t\mathcal{F}(\mathbb{L}_t, \sigma_0 + \beta_t)[\psi'_t] \\ = \mu'_t\psi_t + \mu_t\mu'_t. \end{aligned} \tag{5.16}$$

Recall Remark 3.5 on the notation of $\mathcal{F}_{tt}$; evaluating at $t = 0$, we obtain

$$D_{tt}\mathcal{F}(\mathbb{L}, \sigma_0)[Y_0, Y_0] + \beta'_t(0) D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] + D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\psi'_t(0)] = \mu'_t(0)\phi_1.$$

The 1st and 2nd terms on LHS are zero due to (4.8) and $\beta'_t(0) = \beta_1 = 0$. Multiplying by ϕ_1 and integrating over $\partial\mathbb{L}$, we see that $\mu'_t(0) = 0$. Putting these piece of information back into the above equation, we see that $D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\psi'_t(0)] = 0$. In other words, $\psi'_t(0) \in \operatorname{span}\{\phi_1\}$.

We now compute another derivative of (5.16) and set $t = 0$:

$$D_{ttt}\mathcal{F}(\mathbb{L}, \sigma_0)[Y_0, Y_0, Y_0] + 3D_{tt}\mathcal{F}(\mathbb{L}, \sigma_0)[(D_t Y_t)\big|_{t=0}, Y_0] \\ + \beta_t''(0) D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1] + D_t\mathcal{F}(\mathbb{L}, \sigma_0)[\psi_t''(0)] = \mu_t''(0)\phi_1.$$

The 2nd term on the LHS is zero because $\psi_t'(0)$ is parallel to ϕ_1 and (4.8). Multiplying by ϕ_1 and integrating over $\partial\mathbb{L}$, we get

$$\int_{\partial\mathbb{L}} \phi_1 D_{ttt}\mathcal{F}(\mathbb{L}, \sigma_0)[Y_0, Y_0, Y_0]\, d\mathcal{H}^1 + \beta_t''(0)\int_{\partial\mathbb{L}} \phi_1 D_{t\sigma}\mathcal{F}(\mathbb{L}, \sigma_0)[\phi_1]\, d\mathcal{H}^1 \\ = \mu_t''(0)\int_{\partial\mathbb{L}} \phi_1^2\, d\mathcal{H}^1$$

which, by (5.15) and (5.14), simplifies to

$$-3\hat{a}\beta_2 + \beta_2\hat{a} = \mu_t''(0)T$$

leading to

$$\mu_t''(0) = -\frac{2\hat{a}\beta_2}{T} = \frac{2\rho_1}{\sigma_0}\beta_2.$$

Thus, when $\beta_2 > 0$ (i.e., on supercritical bifurcation branch), we have $\mu_t = \frac{\mu_t''(0)}{2}t^2 + o(t^2) > 0$. Therefore, $\mathbb{L}_t$ is stable (for small non-zero t, irrespective of its sign). Similarly, a subcritical bifurcation is unstable.

Let $\mathcal{S}_0 := \sigma_0\mathcal{A}_0 + \mathcal{B}$. From (5.15) and $\hat{a} < 0$, we see that

$$\text{sign of } \beta_2 = \text{sign of } \int_{\partial\mathbb{L}} \phi_1 D_{ttt}\mathcal{F}(\mathbb{L}, \sigma_0)[Y_0, Y_0, Y_0]\, d\mathcal{H}^1 = \text{sign of } \mathcal{S}_0.$$

This sign determines the type of bifurcation and the stability of the non-planar branch, and also the proof of Theorem 1.3 is completed. One can build an intuitive feel on the sign of $\mathcal{S}_0$ by comparing energy levels of $\mathbb{L}$ and $\mathbb{L}_t$; details can be found in the appendix since it involves a somewhat lengthy calculation.

6. Behavior of $\mathcal{S}_0$ as the torus size is small or large

The bifurcation analysis in the previous section pinpoints the importance of the sign for $\mathcal{S}_0 = \sigma_0\mathcal{A}_0 + \mathcal{B}$. From (4.18) and (4.19), we see the complicated dependence of both $\mathcal{A}_\lambda$ and σ_λ on the parameters λ and $\mathcal{T}$, which is to some extent simplified in the case $\lambda = c = 0$. A complete understanding of such dependence turns out to be difficult, and we therefore estimate the terms $\mathcal{A}_0$ and σ_0 as $T \to +\infty$ and as $T \to 0$.

Case 1: $T \to \infty$. We have the trigonometric identity

$$\frac{1 - \cosh x}{\sinh x} = -\tanh(x/2).$$

When x is large, on denoting by $\mathcal{E}(x)$ any sum of terms decreasing exponentially,

$$\tanh(x/2) = \frac{1-e^{-x}}{1+e^{-x}} = 1 + \mathcal{E}(x)$$

so that

$$\frac{1-\cosh x}{x \sinh x} = -\frac{1}{x} + \mathcal{E}(x).$$

Recall definition (4.1); these allow us to write

$$\begin{aligned} \delta(0,0,\mathcal{T}) &= -\frac{1}{\mathcal{T}} + \mathcal{E}(\mathcal{T}), \\ \delta(0,\pi,\mathcal{T}) &= -\frac{1}{\sqrt{\pi^2+\mathcal{T}^2}} + \mathcal{E}(\mathcal{T}), \\ \delta(0,2\pi,\mathcal{T}) &= -\frac{1}{\sqrt{4\pi^2+\mathcal{T}^2}} + \mathcal{E}(\mathcal{T}). \end{aligned}$$

But for small y,

$$\frac{1}{\sqrt{1+y^2}} = 1 - \frac{y^2}{2} + \frac{3y^4}{8} - \frac{5y^6}{16} + o(y^7);$$

thus,

$$\begin{aligned} \frac{1}{\sqrt{\pi^2+\mathcal{T}^2}} &= \frac{1}{\mathcal{T}} - \frac{\pi^2}{2\mathcal{T}^3} + \frac{3\pi^4}{8\mathcal{T}^5} - \frac{5\pi^6}{16\mathcal{T}^7} + o(1/\mathcal{T}^8), \\ \frac{1}{\sqrt{4\pi^2+\mathcal{T}^2}} &= \frac{1}{\mathcal{T}} - \frac{2\pi^2}{\mathcal{T}^3} + \frac{6\pi^4}{\mathcal{T}^5} - \frac{20\pi^6}{\mathcal{T}^7} + o(1/\mathcal{T}^8). \end{aligned}$$

Upon recalling (4.17), we have

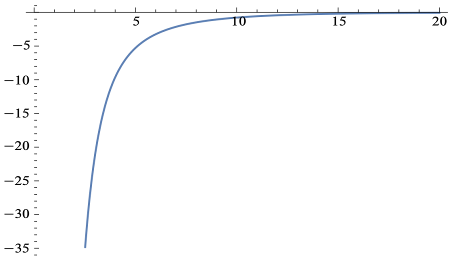
$$\begin{aligned} \Delta_{1,0}(0,\mathcal{T}) &= \frac{\pi^2}{2\mathcal{T}^3} - \frac{3\pi^4}{8\mathcal{T}^5} + \frac{5\pi^6}{16\mathcal{T}^7} + o(1/\mathcal{T}^8), \\ \Delta_{2,0}(0,\mathcal{T}) &= \frac{2\pi^2}{\mathcal{T}^3} - \frac{6\pi^4}{\mathcal{T}^5} + \frac{20\pi^6}{\mathcal{T}^7} + o(1/\mathcal{T}^8), \\ \Delta_{2,1}(0,\mathcal{T}) &= \frac{3\pi^2}{2\mathcal{T}^3} - \frac{45\pi^4}{8\mathcal{T}^5} + \frac{315\pi^6}{16\mathcal{T}^7} + o(1/\mathcal{T}^8). \end{aligned}$$

From (4.19), we may write

$$\sigma_0 = \frac{4}{1-(3\pi^2/4\mathcal{T}^2) + o(1/\mathcal{T}^3)} = 4 + \frac{3\pi^2}{\mathcal{T}^2} + o(1/\mathcal{T}^3) \tag{6.1}$$

and $\mathcal{A}_0$ from (4.18) as

$$\mathcal{A}_0 = 3\mathcal{T}^2\left(\frac{1}{4}\Delta_{2,0} - \Delta_{1,0} + \frac{\pi^2}{\mathcal{T}^2}\Delta_{2,1}\right) = \frac{9\pi^4}{8\mathcal{T}^3} - \frac{45\pi^6}{16\mathcal{T}^5} + o(1/\mathcal{T}^6).$$



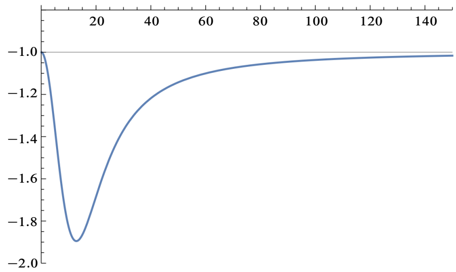


Figure 1. On the left, plot of $\mathcal{S}_0$ versus T. On the right, plot of $\mathcal{S}_0 \cdot (T^3/S_0 + T^5/S_\infty)$.

Joining (6.1), we obtain

$$\sigma_0 \cdot \mathcal{A}_0 = \frac{9\pi^4}{2\mathcal{T}^3} - \frac{63\pi^6}{8\mathcal{T}^5} + o(1/\mathcal{T}^6),$$

and finally, by (4.23),

$$\mathcal{S}_0 = \mathcal{B} + \sigma_0 \cdot \mathcal{A}_0 = -\frac{63\pi^6}{8\mathcal{T}^5} + o(1/\mathcal{T}^6) = -\frac{252\pi^6}{T^5} + o(1/T^6) \approx -242000\, T^{-5}.$$

Thus, the sign of $\mathcal{S}_0$ as the size T of the torus goes to infinity is negative. We define $S_\infty := 252\pi^6$.

Case 2: $T \to 0$. Behavior near $T = 0$ is far more complicated: while $\mathcal{T}^3\mathcal{B} \equiv -9\pi^4/2$, one may tediously compute the Taylor expansions near zero and check that

$$T^3\mathcal{S}_0 = T^3(\mathcal{B} + \sigma_0\mathcal{A}_0) \to 12\pi^4\left(-7 + 4\frac{\pi - \frac{\cosh(2\pi)-1}{\sinh(2\pi)}}{\pi - 2\frac{\cosh\pi - 1}{\sinh\pi}}\right) =: -S_0 \approx -509.$$

The first graph in Figure 1 depicts $\mathcal{S}_0$ as a function of T; the second depicts $\mathcal{S}_0$ multiplied by $(T^3/S_0) + (T^5/S_\infty)$, a positive quantity.

7. Bifurcation analysis when $c \neq 0$

Due to the balance between space and wedge (empty space) when $c = 0$, the two curve boundaries of a deformed lamella are ‘parallel’ to one another with a distance of $T/2$ apart in the x-direction; this also implies that the volume of the lamella (which is equal to the volume of its complementary set) is preserved. One does not expect the same phenomenon to hold for general c, since either lamella or empty space might expand at the expense of the other: thus, one may not restrict only to the y-odd component and need to consider also a y-even component. However, there is another symmetry that we can exploit; namely, (to eliminate pure translations in the x-direction) we may fix the center of the lamella. To simplify notation, we change coordinate system for this section so that

$$\mathbb{T} = [-T/2, T/2] \times [-T/2, T/2], \quad \mathbb{L} = [-x_0/2, x_0/2] \times [-T/2, T/2],$$
$$L_1 = \{x = -x_0/2\}, \;\; L_2 = \{x = x_0/2\}.$$

In view of the lemma below, which adapts Lemma 4.2, we impose that the deformed profile is symmetric with respect to the (new) origin. If the right profile $(L_2)_t$ is the graph over L_2 of a (periodic and smooth) function $t \cdot g(y)$, so its equation is $x = tg(y) + x_0/2$; we want $(L_1)_t$ to be the graph with equation $x = -tg(-y) - x_0/2$: such a choice eliminates translations. This is equivalent to saying that if we split g into its odd and even parts g_o, g_e, then $(L_2)_t$ is the image of L_2 through the flow associated with $X^{(2)}(x, y) = (g_e(y) + g_o(y), 0)$ while $(L_1)_t$ is the image of L_1 through the flow associated with $X^{(1)}(x, y) = (-g_e(y) + g_o(y), 0)$.

Whenever we are in such a situation, we may choose any smooth periodic field X_g such that

$$X_g \equiv X^{(i)} \text{ in a neighborhood of } L_i; \tag{7.1}$$

the bifurcation result will not depend on such extension, as it only considers the behavior of X_g near L_i. We also define, for all periodic functions g,

$$\tilde{g}(y) = -g(-y) = -g_e(y) + g_o(y). \tag{7.2}$$

Lemma 7.1. *Let g be smooth and periodic and X_g be as in (7.1). Then,*

(a) *$(L_2)_t = -(L_1)_t$; in particular, if $\mathbf{z} \in \partial\mathbb{L}_t$, then $-\mathbf{z} \in \partial\mathbb{L}_t$ and there is no translation in the x-direction;*

(b) *for every $\mathbf{z} \in \partial\mathbb{L}_t$,*

$$\mathcal{K}(\mathbf{z}) = \mathcal{K}(-\mathbf{z}), \quad \mathcal{N}_{\mathbb{L}_t}(\mathbf{z}) = \mathcal{N}_{\mathbb{L}_t}(-\mathbf{z}),$$

so if the Euler–Lagrange equation is satisfied on one side of $\partial\mathbb{L}_t$, it is satisfied on the other.

The statements are apparent since $\mathbb{L}_t$ is invariant by rotation about the origin by 180°. Due to the imposed symmetry about the origin,

$$\int_{L_1} X \cdot \nu \, d\mathcal{H}^1 = \int_{L_2} X \cdot \nu \, d\mathcal{H}^1$$

is not necessarily zero, which leads to a perturbed lamella with volumetric change.

As we have seen in the previous section, $D_t\mathcal{F}(\mathbb{L}, \sigma_c, c)[\eta] = 0$ through a double eigenvalue $\Lambda = 0$ with eigenfunctions $\eta \in \operatorname{span}\{\phi_1(\cdot), \phi_1(\cdot + T/4)\}$. If we restrict our attention to the subspace symmetric about the origin, then $\eta \in \operatorname{span}\{\phi_1\}$. This is because all functions $\eta \in \operatorname{span}\{\phi_1(\cdot), \phi_1(\cdot + T/4)\}$ satisfy $\eta\big|_{L_2}(y) = -\eta\big|_{L_1}(y)$, but symmetry about the origin entails $\eta\big|_{L_1}(y) = \eta_{L_2}(-y)$; thus, $\eta\big|_{L_2}(y) = -\eta\big|_{L_2}(-y)$. The last statement will be the same at L_1. This 1-dimensional eigenspace allows us to employ the Crandall–Rabinowitz simple eigenvalue bifurcation theorem again.

To prepare us for a bifurcation analysis, analogously to what we did in Section 5, for every (periodic and smooth) function w let X_g be the field defined in (7.1) with $g(y) = \varphi_1(y) + w(y)$, let Φ be the associated flow, and $\mathbb{L}_t = \Phi(\mathbb{L}, t)$ so that $\mathcal{F}(\mathbb{L}_t, \sigma_c, c)$ is well defined. Recalling (7.2), define

$$\mathcal{W} := \left\{ u \in \mathcal{C}^{2+\omega}_{per}[-T/2, T/2] : \int_{-T/2}^{T/2} u(y)\varphi_1(y)\,dy = 0 \right\},$$

$$\mathcal{W}_1 := \left\{ u \in \mathcal{C}^{\omega}_{per}[-T/2, T/2] : \int_{-T/2}^{T/2} u(y)\varphi_1(y)\,dy = 0 \right\},$$

$$\mathcal{Z}_1 := \left\{ -\tilde{u} \llcorner L_1 + u \llcorner L_2 : u \in \mathcal{W}_1 \right\}$$

(remark that $u \in \mathcal{W}_1 \iff \tilde{u} \in \mathcal{W}_1$) and for every $w \in \mathcal{W}$ define

$$\mathfrak{w} := -\tilde{w} \llcorner L_1 + w \llcorner L_2.$$

We may finally define $G : \mathcal{W} \times \mathbb{R} \times \mathbb{R} \to \mathcal{Z}_1 \oplus \mathrm{span}\{\phi_1\}$ as

$$G(w, \beta, t) := \begin{cases} \frac{\mathcal{F}(\mathbb{L}_t, \sigma_c + \beta, c)}{t} & \text{for } t \neq 0, \\ D_t \mathcal{F}(\mathbb{L}, \sigma_c + \beta, c)[\phi_1 + \mathfrak{w}] & \text{for } t = 0. \end{cases}$$

The fact that the range of G lies in $\mathcal{Z}_1 \oplus \mathrm{span}\{\phi_1\}$ is an immediate consequence of Lemma 7.1. Now, we may repeat verbatim the argument in Section 5 to obtain as in (5.9)

$$D_{(w,\beta)} G(0,0,0)[\hat{w}, \widehat{\beta}] = D_w G(0,0,0)\hat{w} - \frac{\widehat{\beta}\rho_1}{\sigma_c}\phi_1$$

with $D_w G(0,0,0)\hat{w}$ still given by (5.8), clearly with σ_c instead of σ_0. To check injectivity, we prove that $D_{(w,\beta)} G(0,0,0)[\hat{w}, \widehat{\beta}] = 0$ implies $\hat{w} = 0$ and $\widehat{\beta} = 0$, but the latter is immediate by multiplying by ϕ_1, and therefore, $D_w G(0,0,0)\hat{w} = 0$. Again, following Section 5, we multiply by $\hat{\mathfrak{w}}$ and the RHS of [3, equation (3.2)] decouples into the contributions from the mean part μ and zero-average part ζ of $\hat{\mathfrak{w}}$: they are given by [3, equations (3.12) and (3.13)], but in [3, Section 4], we proved that the contribution of μ is positive for all non-zero translation-free perturbations, and in [3, Section 5], we proved that the contribution of ζ is positive for all non-zero ζ orthogonal to ϕ_1; thus, $\mu = \zeta = \hat{w} = 0$.

For surjectivity we consider

$$D_{(w,\beta)} G(0,0,0)[\hat{w}, \widehat{\beta}] = z_1 + \lambda\phi_1$$

for any $\lambda \in \mathbb{R}$ and $z_1 \in \mathcal{Z}_1$. As the RHS is symmetric about the origin, there is no translation mode in the x-direction. The same argument employed in Section 5 leads to (5.10) with σ_0 replaced by σ_c to be solved on L_2. This leads to solvability of $(\hat{w}, \widehat{\beta})$ using Fredholm alternative. Thus, the implicit function theorem gives rise to Theorem 1.4.

8. Bifurcation analysis for the k-lamella

Up to now, we considered the case of one lamella inside the torus; the bifurcation study can be carried out also for multi-lamellar sets: a critical k-lamella in our square torus $\mathbb{T}$ is composed [2, Proposition 2.6] of k equal width vertical lamellae separated by equal wedges. The arguments employed in the preceding sections may be adapted with little effort (but a heavier notation) to the k-lamella case, due to a simple observation. Let $\mathbb{L}_k$ be a k-lamella, composed of the single lamellae L_k^i, for $i = 0, \dots, k-1$, and denote by L_1^i and L_2^i the left and right sides of $\mathbb{L}_k^i$; also, denote by L_1 and L_2 the unions of left and right sides. The total thickness x_0 for a stationary k-lamella is given through

$$\frac{x_0}{k} = \frac{T}{2k} - \operatorname{arc\,sinh}\left(c \sinh \frac{T}{2k}\right)$$

(note that this new value of x_0 and those of d_0, $d_0^{(1)}$, and C_i below depend on k) and the outward normal derivative of $\mathcal{N}$ at $\partial\mathbb{L}_k$ is $-d_0$ with

$$d_0 = \frac{1}{\sinh \frac{T}{2k}} \sinh \frac{T - x_0}{2k} \sinh \frac{x_0}{2k},$$

and we replace (4.4) and (4.5) by

$$d_0^{(1)} = \frac{1}{\sinh(\frac{T\sqrt{1+\rho_1}}{2k})} \sinh \frac{(T - x_0)\sqrt{1+\rho_1}}{2k} \sinh \frac{x_0\sqrt{1+\rho_1}}{2k},$$

$$C_i = \frac{1}{2\sqrt{1+\rho_i}\,\sinh(\frac{T}{2k}\sqrt{1+\rho_i})}, \qquad i = 1, 2.$$

Call $\mathbb{T}_k$ the rectangular torus $[0, T/k] \times [0, T]$ and $\mathbf{G}_k(\mathbf{x}, \mathbf{y}) = \widehat{G}_k(\mathbf{x} - \mathbf{y})$ the Green function for the Helmholtz operator in $\mathbb{T}_k$. Then, recalling that $\mathbf{b}_1 = (1, 0)$, we have

$$\mathbf{G}_k(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{k-1} \mathbf{G}(\mathbf{x}, \mathbf{y} + (iT/k)\mathbf{b}_1).$$

Indeed, if f is any $\mathbb{T}_k$-periodic function, for any $\mathbf{x} \in \mathbb{T}_k$, then

$$\begin{aligned}
\int_{\mathbb{T}_k} \mathbf{G}_k(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} &= \int_{\mathbb{T}} \mathbf{G}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \\
&= \sum_{i=0}^{k-1} \int_{\mathbb{T}_k + (iT/k)\mathbf{b}_1} \mathbf{G}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} \\
&= \int_{\mathbb{T}_k} \sum_{i=0}^{k-1} \mathbf{G}(\mathbf{x}, \mathbf{y} + (iT/k)\mathbf{b}_1) f(\mathbf{y})\, d\mathbf{y}.
\end{aligned}$$

We prove the generalization of Lemma 4.1—remark that due to our notation we may still write $\phi_1 = -\varphi_1 \llcorner L_1 + \varphi_1 \llcorner L_2$.

Lemma 8.1. *Suppose that the flow field is $X(x, y) = (\varphi_1(y), 0)$. Then,*

$$\frac{\partial}{\partial t}\left(\mathcal{N}_{(\mathbb{L}_k)_t}\big|_{\partial(\mathbb{L}_k)_t}\right)\big|_{t=0} = \left(\frac{d_0^{(1)}}{\sqrt{1+\rho_1}} - d_0\right)\phi_1.$$

In particular, the LHS takes the same value at all points $\mathbf{x} \in L_1^k$ with the same y-coordinate (and the opposite value at points in L_2^k).

Proof. On setting $E = \mathbb{L}_k$ and $X(x, y) = (\varphi_1(y), 0)$ in (2.3), we have $\eta = X \cdot \nu_0 = \phi_1$ on $\partial\mathbb{L}_k$ and for every $\mathbf{x} \in \partial\mathbb{L}_k^0$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\mathcal{N}_{(\mathbb{L}_k)_t}\big|_{\partial(\mathbb{L}_k)_t}\right)\big|_{t=0}(\mathbf{x}) &= \left(\eta\frac{\partial\mathcal{N}_{\mathbb{L}_k}}{\partial\nu}\right)\bigg|_{\partial\mathbb{L}_k} + \int_{\partial\mathbb{L}_k} \widehat{G}(\mathbf{x}-\mathbf{y})\eta(\mathbf{y})\, d\mathcal{H}_{\mathbf{y}}^1 \\ &= -d_0\phi_1\big|_{\partial\mathbb{L}_k^0} + \int_{\partial\mathbb{L}_k^0} \widehat{G}_k(\mathbf{x}-\mathbf{y})\phi_1(\mathbf{y})\, d\mathcal{H}_{\mathbf{y}}^1;\end{aligned}$$

the value of the RHS will be the same at $\mathbf{x}' = \mathbf{x} + (iT/k)\mathbf{b}_1$. The rest of the proof follows exactly Lemma 4.1. ∎

All other proofs follow similar lines; as an example, Lemma 4.2 holds; in the proof, one simply replaces the integral on the "same" side with the sum of integrals on the k "same" sides, and analogously for the "other" side; the fact that $I_S = 0$ if $\mathbf{x} \in \partial\mathbb{L}_k^0$, or any $\partial\mathbb{L}_k^i$, depends on the fact that the sum of integrals of $D_1\widehat{G}$ on the "same" sides is the integral of $D_1\widehat{G}_k$ on the first of such sides, which is zero by (4.12), replaced in our case by

$$D_1\widehat{G}_k(0, x_2) = D_1\widehat{G}_k(T/2k, x_2) = 0;$$

this last equation will be used again to obtain the finer result in the case $c = 0$.

Bifurcation occurs exactly as in the case of a 1-lamella, namely, by parallel (and T/k periodic in x) y-odd configurations when $c = 0$, and by (suitably defined) center symmetric x-periodic configurations in the case $c \neq 0$.

The difference comes when analyzing the shape of the bifurcation curve. We only examine the case $c = 0$; the validity of Lemma 4.2 again forces us to compute the third derivatives to obtain the quantities

$$\mathcal{A}_0^k, \quad \sigma_0^k, \quad \mathcal{B}^k,$$

analogous to those already studied with $k = 1$ (the superscript k does not represent a power). Clearly,

$$\mathcal{B}^k = k\mathcal{B} = -\frac{36k\pi^4}{T^3}$$

as there are $2k$ surfaces in $\partial\mathbb{L}_k$ instead of just two. The number σ_0^k may be found by setting to zero for RHS of [3, equation (5.20)], giving

$$\sigma_0^k = \frac{32\pi^2}{T^3}\left[\frac{\tanh(T/4k)}{T/4} - \frac{\tanh(\sqrt{T^2+4\pi^2}/4k)}{\sqrt{T^2+4\pi^2}/4}\right]^{-1}. \tag{8.1}$$

The nonlocal term requires retracing the proof of (4.16) with extensive use of grouping left and right sides, and passing to $\widehat{G}_k$, to obtain

$$\mathcal{A}_0^k = -6k\left(\frac{3T}{16}\tanh\frac{T}{4k} - \frac{\sqrt{T^2+4\pi^2}}{4}\tanh\frac{\sqrt{T^2+4\pi^2}}{4k} + \frac{\sqrt{T^2+16\pi^2}}{16}\tanh\frac{\sqrt{T^2+16\pi^2}}{4k}\right). \tag{8.2}$$

It is of interest to study the sign of $\mathcal{S}_0^k = \mathcal{B}^k + \sigma_0^k \mathcal{A}_0^k$; again, a positive $\mathcal{S}_0^k$ designates a supercritical bifurcation with stable non-planar solutions while a negative value represents an unstable subcritical bifurcation.

We first examine this sign for fixed T as k increases. After developing in powers of k for fixed T, one has as $k \to \infty$

$$\mathcal{A}_0^k \sim \frac{3\pi^4}{8k^2}, \quad \sigma_0^k \sim \frac{32\cdot 12k^3}{T^3} \quad \Rightarrow \quad \mathcal{S}_0^k \sim \frac{(12^2-36)\pi^4}{T^3}k;$$

thus, for any given torus size, the k-lamella undergoes a supercritical bifurcation for large k. Other interesting studies are the asymptotic behaviors for fixed k as $T \to \infty$ and as $T \to 0$, as we did in Section 6: in the former case when we develop in powers of T for fixed k, we get

$$\mathcal{A}_0^k \sim \frac{9\pi^4}{T^3}k - \frac{90\pi^6}{T^5}k, \quad \sigma_0^k \sim 4 + \frac{12\pi^2}{T^2} \quad \Rightarrow \quad \mathcal{S}_0^k \sim \frac{-252\pi^6}{T^5}k; \tag{8.3}$$

thus, for any given k, the k-lamella with $c = 0$ undergoes a subcritical bifurcation for large torus size T. Instead as $T \to 0$ for any given k, we have very simply from (8.2)

$$\mathcal{A}_0^k \to -6k\left(\frac{\pi}{4}\tanh\frac{\pi}{k} - \frac{\pi}{2}\tanh\frac{\pi}{2k}\right) = 3k\pi\left(\tanh\frac{\pi}{2k} - \frac{1}{2}\tanh\frac{\pi}{k}\right).$$

Equally easily since

$$\frac{\tanh(T/4k)}{T/4} - \frac{\tanh\left(\sqrt{T^2+4\pi^2}/4k\right)}{\sqrt{T^2+4\pi^2}/4} \to \frac{1}{k} - \frac{2}{\pi}\tanh\frac{\pi}{2k},$$

we deduce from (8.1) that as $T \to 0$

$$\sigma_0^k \sim \frac{16\pi^3/T^3}{\frac{\pi}{2k} - \tanh\frac{\pi}{2k}};$$

thus,

$$\begin{aligned}\mathcal{S}_0^k &\sim -\frac{36k\pi^4}{T^3} + \frac{48k\pi^4}{T^3}\frac{\tanh\frac{\pi}{2k} - \frac{1}{2}\tanh\frac{\pi}{k}}{\frac{\pi}{2k} - \tanh\frac{\pi}{2k}} \\ &= \frac{12k\pi^4}{T^3}\left(-3 + 4\frac{\tanh\frac{\pi}{2k} - \frac{1}{2}\tanh\frac{\pi}{k}}{\frac{\pi}{2k} - \tanh\frac{\pi}{2k}}\right) =: \frac{1}{T^3}s_k.\end{aligned}$$

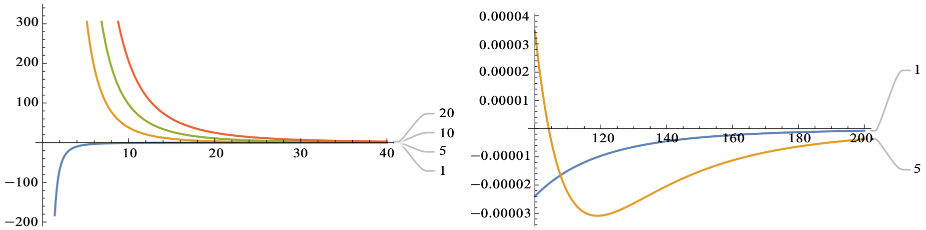


Figure 2. Plot of $\mathcal{S}_0^k$ versus T for $k = 1, 5, 10, 20$. On the right, same with $k = 1$ and 5 which shows the intersection of the two curves at large enough T.

We prove that the quantity inside brackets is positive for $k \geq 2$. Indeed, calling $t = \pi/2k$, the assertion is the same as

$$\frac{\tanh t - (1/2)\tanh(2t)}{t - \tanh t} > \frac{3}{4} \quad \text{for } 0 < t \leq \frac{\pi}{4},$$

but as both numerator and denominator vanish at $t = 0$, by using the Cauchy's mean value theorem, there exists $0 < \tau < \pi/4$ such that

$$\frac{\tanh t - (1/2)\tanh(2t)}{t - \tanh t} = \left.\frac{[\tanh t - (1/2)\tanh(2t)]'}{[t - \tanh t]'}\right|_{t=\tau} = \frac{\tanh^2(2\tau) - \tanh^2 \tau}{\tanh^2 \tau}$$

and

$$\frac{\tanh^2(2\tau)}{\tanh^2 \tau} - 1 > \frac{3}{4} \iff \frac{\tanh(2\tau)}{\tanh \tau} = \frac{2}{1 + \tanh^2 \tau} > \frac{\sqrt{7}}{2} \iff \tanh^2 \tau < \frac{4}{\sqrt{7}} - 1 \sim 0.5.$$

But $\tanh(\pi/4) \sim 0.6$ and the assertion is proved. Note that for $k = 1$ we have $s_1 < 0$, as we already know from Section 6, so it is clear that (apart for the 1-lamella) bifurcation is supercritical for small T and subcritical for large T when $c = 0$; see Figures 2 and 3, thus proving Theorem 1.5.

We can make a finer comparison between k'-lamella and k''-lamella as follows.

Proposition 8.2. *The sequence s_k is strictly increasing, so if $k' < k''$ [and $k' \geq 2$ for the first inequality], then*

$$[0 <]\, \mathcal{S}_0^{k'} < \mathcal{S}_0^{k''} \text{ for } T \leq T_1(k', k''), \qquad 0 > \mathcal{S}_0^{k'} > \mathcal{S}_0^{k''} \text{ for } T \geq T_2(k', k''). \tag{8.4}$$

Proof. For small T, the assertion is equivalent to proving that

$$t \mapsto \frac{\tanh t - (1/2)\tanh(2t)}{t - \tanh t} = \frac{\frac{\tanh^3 t}{1+\tanh^2 t}}{t - \tanh t} = \frac{\frac{1}{1+\tanh^2 t}}{\frac{t - \tanh t}{\tanh^3 t}}$$

is decreasing. Since numerator and denominator are positive and the former is decreasing, it is enough to prove that the denominator is increasing. Taking the inverse of the

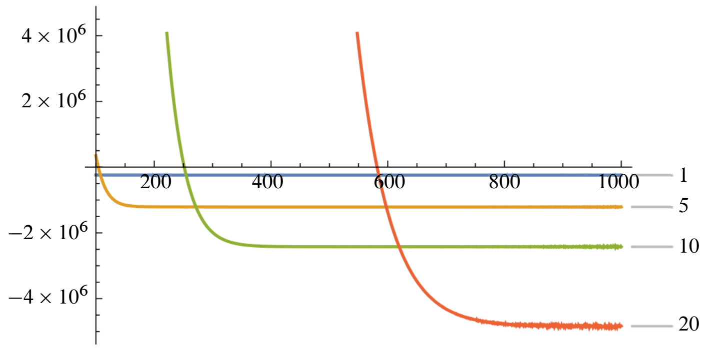


Figure 3. Plot of $\mathcal{S}_0^k \cdot T^5$ versus T for large T and $k = 1, 5, 10, 20$.

hyperbolic tangent, which is increasing, we set for $t \geq 0$

$$\tanh t = x \quad \text{so that } t = f(x) := \tanh^{-1} x \text{ with } f'(x) = \frac{1}{1-x^2}$$

and we prove that $x \mapsto x^{-3} f(x) - x^{-2}$ is increasing: indeed its derivative is

$$-3x^{-4} f(x) + x^{-3} \frac{1}{1-x^2} + 2x^{-3} = x^{-4} \left(\frac{3x - 2x^3}{1-x^2} - 3f(x) \right) =: x^{-4} \tilde{f}(x);$$

the function $\tilde{f}$ vanishes for $x = 0$ and its derivative is

$$\frac{(3-6x^2)(1-x^2) + 2x(3x-2x^3)}{(1-x^2)^2} - \frac{3}{1-x^2} = \frac{2x^4}{(1-x^2)^2},$$

so $\tilde{f}(x) > 0$ for $x > 0$ and the proof of the monotonicity of s_k is completed, thus entailing the first part of (8.4); the second part for large T was included in (8.3). ■

A. Appendix

Let $c = 0$. A positive sign of $\mathcal{S}_0 = \sigma_0 \mathcal{A}_0 + \mathcal{B}$, which is related to $J''(\mathbb{L}_t)$, implies that $\mathbb{L}_t$ is stable; while a negative sign implies instability. The terms $\sigma_0 \mathcal{A}_0$ and $\mathcal{B}$ come from the nonlocal and the perimeter (or curvature) terms, respectively. It is known that $\mathcal{B} < 0$, see (4.23); one can also prove $\mathcal{A}_0 > 0$. Therefore, the nonlocal term stabilizes while the perimeter term destabilizes $\mathbb{L}_t$. Their competition results in stability being controlled by the sign of $\mathcal{S}_0$.

It is difficult to build intuition on the sign of the second derivative. However, it can be shown that a bifurcated $\mathbb{L}_t$ is stable iff $J(\mathbb{L}_t, \sigma_0 + \beta_t) < J(\mathbb{L}, \sigma_0 + \beta_t)$; thus,

$$\text{supercritical bifurcation} \iff \text{stable } \mathbb{L}_t \iff J(\mathbb{L}_t, \sigma_0 + \beta_t) < J(\mathbb{L}, \sigma_0 + \beta_t).$$

It is far easier to build intuition on stability by comparing energies J of $\mathbb{L}$ versus $\mathbb{L}_t$. The perimeter of $\mathbb{L}_t$ is larger than that of $\mathbb{L}$; this term makes $\mathbb{L}_t$ more likely to be unstable. This reconciles with the fact $\mathcal{B} < 0$.

We now claim that the nonlocal term for $\mathbb{L}_t$ has a lower energy than that of $\mathbb{L}$, which validates $\mathcal{A}_0 > 0$. Indeed, recall that a flow field $X_t(x,y) = (\varphi_1(y) + w_t(y), 0)$ for some suitable $w_t = tw_1 + \frac{t^2}{2}w_2 + o(t^2)$ generates $\mathbb{L}_t$ when $\sigma = \sigma_0 + \beta_t$. Since we need to compute energy of $\mathbb{L}_t$ to leading order of accuracy for small t, it suffices to set $X_t(x,y) = (\varphi_1(y), 0) = (\sin\frac{2\pi y}{T}, 0)$ on the torus $\mathbb{T} = (0,T) \times (-T/2, T/2)$ with the lamella $\mathbb{L}$ being located at $(0, T/2) \times (-T/2, T/2)$.

For small $t > 0$, let region

$$E_1 := \Big\{(x,y) : 0 < x < t\sin\frac{2\pi y}{T},\ 0 < y < T/2\Big\}, \quad E_2 := E_1 + (T/2, 0)$$

be a shift in the x-direction, $E_3 := \{(x,y) : t\sin\frac{2\pi y}{T} < x < 0,\ -T/2 < y < 0\}$, and $E_4 := E_3 + (T/2, 0)$. Hence, $\mathbb{L}_t = \mathbb{L} \cup E_2 \cup E_3 \setminus (E_1 \cup E_4)$ to leading order. To compare the nonlocal energies, it suffices to compute

$$\begin{aligned}
&\int_{\mathbb{T}} (\chi_{\mathbb{L}} - \chi_{E_1} + \chi_{E_2} + \chi_{E_3} - \chi_{E_4})(\mathcal{N}_{\mathbb{L}} - \mathcal{N}_{E_1} + \mathcal{N}_{E_2} + \mathcal{N}_{E_3} - \mathcal{N}_{E_4})\, d\mathbf{x} - \int_{\mathbb{T}} \chi_{\mathbb{L}} \mathcal{N}_{\mathbb{L}}\, d\mathbf{x} \\
&= \int_{\mathbb{T}} \big\{\chi_{\mathbb{L}}(-\mathcal{N}_{E_1} + \mathcal{N}_{E_2} + \mathcal{N}_{E_3} - \mathcal{N}_{E_4}) + \mathcal{N}_{\mathbb{L}}(-\chi_{E_1} + \chi_{E_2} + \chi_{E_3} - \chi_{E_4})\big\}(1 + O(t))\, d\mathbf{x} \\
&= 2\int_{\mathbb{T}} \mathcal{N}_{\mathbb{L}}(-\chi_{E_1} + \chi_{E_2} + \chi_{E_3} - \chi_{E_4})(1 + O(t))\, d\mathbf{x}.
\end{aligned}$$

Note that E_1 and E_4 are inside $\mathbb{L}$, while E_2 and E_3 are outside $\mathbb{L}$. As (the 1D) $\mathcal{N}_{\mathbb{L}}$ attains its maximum at $x = T/4$ and strictly decreases on both sides with $|x - T/4|$, it is clear that $\int_{\mathbb{T}} \mathcal{N}_{\mathbb{L}}(\chi_{E_1} + \chi_{E_4})\, d\mathbf{x} > \int_{\mathbb{T}} \mathcal{N}_{\mathbb{L}}(\chi_{E_2} + \chi_{E_3})\, d\mathbf{x}$. Such observation leads us to conclude that

$$\int_{\mathbb{T}} (\chi_{\mathbb{L}} - \chi_{E_1} + \chi_{E_2} + \chi_{E_3} - \chi_{E_4})(\mathcal{N}_{\mathbb{L}} - \mathcal{N}_{E_1} + \mathcal{N}_{E_2} + \mathcal{N}_{E_3} - \mathcal{N}_{E_4})\, d\mathbf{x} < \int_{\mathbb{T}} \chi_{\mathbb{L}} \mathcal{N}_{\mathbb{L}}\, d\mathbf{x},$$

which proves our above claim.

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