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Von Neumann entropy of the angle operator between a pair of intermediate subalgebras

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Abstract. Given a pair of intermediate C^*-subalgebras of a unital inclusion of simple C^*-algebras equipped with a conditional expectation of finite Watatani index, we discuss the corresponding 'angle operator' and its Fourier transform. We provide a calculable formula for the von Neumann entropy of the (Fourier) dual angle operator for a large class of quadruples of simple C^*-algebras.

1. Introduction

The study of the lattice structure of von Neumann subalgebras was initiated by Murray and von Neumann. The lattice of intermediate von Neumann subalgebras of a subfactor with finite Jones index was studied by many researchers. In recent years, the study of the symmetries of an inclusion of C^*-algebras has become an active area of research (for instance, [4, 5, 10, 15] and references therein). In this short article, we focus on unital inclusion of simple C^*-algebras $B \subset A$ with a conditional expectation of finite *Watatani index* (a generalization of the Jones index) and a pair of intermediate C^*-subalgebras $B \subset C, D \subset A$. Ino and Watatani (in [9, Corollary 3.9]) have shown that every irreducible pair of simple unital C^*-algebras with a conditional expectation of finite index has only finitely many intermediate C^*-subalgebras. In [2,5], we have introduced a notion of angle between intermediate subalgebras C and D to obtain an upper bound on the cardinality of this set (see also [3] for improvement).

Motivated by [16], we may also consider the corresponding *angle operator* Θ between the intermediate subalgebras C and D. Moreover, using the *Fourier transform* on the *relative commutants* of $B \subset A$, as developed in [5], we may consider the Fourier dual $\mathcal{F}(\Theta)$ of the angle operator. It seems to be an apt place to mention that the Fourier transform on the relative commutants has been used in [4, 11] to obtain a noncommutative uncertainty principle. Our goal in this article is to compute the *von Neumann entropy* of the angle operator and its Fourier dual. We observe that the subalgebras C and D are 'orthogonal'; in other words, $B \subset C, D \subset A$ form a 'commuting square', if and only if the von Neumann entropy of the angle operator vanishes. The main result of this article is the following theorem (see Theorem 3.12).

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Theorem A. *Let $B \subset A$ be an irreducible inclusion of simple unital C^*-algebras with a conditional expectation of index-finite type, and suppose that C, D are two intermediate unital C^*-subalgebras. Then, we have the following:*

$$H\big(\big|\mathcal{F}(\Theta)\big|^2\big) = \frac{2}{\sqrt{[A : B]_0}} \eta\big(\delta \,\mathrm{tr}(e_C e_D)\big),$$

where $\delta = \sqrt{[A : B]_0}$ and $\eta : [0, \infty) \to \mathbb{R}$ is the continuous function defined by

$$\eta(t) = \begin{cases} -t \log t & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

We remark that the generalization of the above formula beyond the irreducible situation seems difficult at the moment. However, in the case of 'co-commuting squares' (a dual notion of commuting square), we have an explicit formula (see Theorem 3.15).

Theorem B. *Let $(B \subset C,\ D \subset A)$ be a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type. If the quadruple is a co-commuting square, then we have the following:*

$$H\big(\big|\mathcal{F}(\Theta)\big|^2\big) = \frac{2}{\sqrt{[A : B]_0}} \eta\bigg(\frac{\sqrt{[A : B]_0}}{[A : C]_0 [A : D]_0}\bigg).$$

2. Preliminaries

We first briefly recall Watatani's C^*-index theory for a unital inclusion of simple C^*-algebras following [17] and then touch upon the Fourier theory developed in [5] for the relative commutants of that inclusion. The Fourier transformation on the higher relative commutants is a crucial tool in the theory of subfactors. It was introduced by Ocneanu (for more see [13]). Consider a unital inclusion of C^*-algebras $B \subset A$ and a conditional expectation $E : A \to B$. E is said to be of index-finite type if there exists a finite set $\{\lambda_1, \dots, \lambda_n\} \subset A$ such that every $x \in A$ can be written as

$$x = \sum_{i=1}^{n} E(x\lambda_i)\lambda_i^* = \sum_{i=1}^{n} \lambda_i E(\lambda_i^* x).$$

The finite set $\{\lambda_1, \dots, \lambda_n\} \subset A$ is called quasi-basis for E (see [17]). In this case, the Watatani index of E is given by $\mathrm{Ind}_w(E) = \sum_{i=1}^{n} \lambda_i \lambda_i^*$. For an inclusion $B \subset A$ of simple C^*-algebras with a conditional expectation of index-finite type, there always exists a minimal conditional expectation E_B^A. The minimal index of $B \subset A$, denoted by $[A : B]_0$, is defined as $\mathrm{Ind}_w(E_B^A)$. In the sequel, we always deal with an inclusion of simple unital C^*-algebras. We denote the C^*-basic construction of the inclusion $B \subset A$ by A_1. Iterating the C^*-basic construction, we get a tower of simple unital C^*-algebras $B \subset A \subset A_1 \subset A_2 \subset$

$\cdots \subset A_k \subset \cdots$ and obtain the unique minimal condition expectation $E^{A_k}_{A_{k-1}} : A_k \to A_{k-1}$ for each $k \geq 0$. Let e_{k+1} denote the Jones projection that implements the basic construction of the inclusion $A_{k-1} \subset A_k$ with respect to $E^{A_k}_{A_{k-1}}$. The relative commutants $B' \cap A_k$ and $A' \cap A_k$ are finite-dimensional. For each $k \geq 0$, $E^A_B \circ E^{A_1}_A \circ \cdots \circ E^{A_k}_{A_{k-1}}|_{B'\cap A_k}$ is a faithful tracial state on $B' \cap A_k$, to be denoted by tr_k. We often drop 'k' and denote tr_k simply by tr. The unique trace preserving conditional expectation $E^{\mathrm{tr}}_k : B' \cap A_k \to A' \cap A_k$ is given by

$$E^{\mathrm{tr}}_k(x) = \frac{1}{[A:B]_0} \sum_i \lambda_i x \lambda_i^*.$$

On the higher relative commutants of any extremal subfactor, a formula for the Fourier transform was given by Bisch in [6]. Following [5] (see also [4]), we quickly recall the notion of 'Fourier transform' and 'convolution' product on the relative commutants for C^*-inclusion. For each $k \geq 0$, the *Fourier transform* $\mathcal{F}_k : B' \cap A_k \to A' \cap A_{k+1}$ is defined as

$$\mathcal{F}_k(x) = \delta^{k+2} E^{\mathrm{tr}}_k(x e_{k+1} e_k \cdots e_2 e_1) \quad \text{for all } x \in B' \cap A_k;$$

and the *inverse Fourier transform* $\mathcal{F}_k^{-1} : A' \cap A_{k+1} \to B' \cap A_k$ is defined as

$$\mathcal{F}_k^{-1}(x) = \delta^{k+2} E^{A_{k+1}}_{A_k}(x e_1 e_2 \cdots e_k e_{k+1}) \quad \text{for all } x \in A' \cap A_{k+1}.$$

Here, we use the term 'inverse' in the sense that $\mathcal{F}_k \circ \mathcal{F}_k^{-1} = \mathrm{id}_{A'\cap A_{k+1}}$ and $\mathcal{F}_k^{-1} \circ \mathcal{F}_k = \mathrm{id}_{B'\cap A_k}$. For simplicity, we shall use $\mathcal{F}$ for $\mathcal{F}_1$.

Theorem 2.1 ([5, Theorem 3.5]). *$\mathcal{F}$ and $\mathcal{F}^{-1}$ are isometries with respect to the norm defined by $||x||_2 = \mathrm{tr}(x^*x)$, where the* tr *on $A' \cap A_2$ is the restriction of* tr *on $B' \cap A_2$.*

For $x, y \in B' \cap A_1$, the *convolution* product of x and y, denoted by $x \star y$, is defined as follows:

$$x \star y := \mathcal{F}^{-1}\big(\mathcal{F}(y)\mathcal{F}(x)\big).$$

Similarly, for $w, z \in A' \cap A_2$, their *convolution* product is defined as

$$w \star z := \mathcal{F}\big(\mathcal{F}^{-1}(z)\mathcal{F}^{-1}(w)\big).$$

The convolution product is associative [5, Lemma 3.20]. For $x, y \in B' \cap A_1$, by [4, Proposition 3.8], we have

$$(x \star y)^* = x^* \star y^*;$$

and similarly for $w, z \in A' \cap A_2$, we have

$$(w \star z)^* = w^* \star z^*. \tag{1}$$

Suppose that C is an intermediate simple unital C^*-subalgebra of $B \subset A$. By [10], there exist minimal conditional expectations E^C_B from C onto B and E^A_C from A onto C such that $E^A_B = E^C_B \circ E^A_C$. Indeed, E^C_B is given by restricting E^A_B onto C. Let C_1 denote the

C^*-basic construction of the inclusion $C \subset A$ with Jones projection e_C corresponding to the minimal conditional expectation E^A_C. Below we list some useful results, the proofs of which follow along the same line of argument as in [5, Section 4].

Lemma 2.2 ([5, Lemma 4.2]). *Let $B \subset C \subset A$ be as discussed above. Then, we have the following:*

(i) *$C_1 \subset A_1$ is an inclusion of simple unital C^*-algebras with common identity. The dual conditional expectations $E^{A_1}_A$ and $E^{C_1}_A$ are minimal. $E^{A_1}_A$ and $E^{C_1}_A$ must satisfy $E^{A_1}_A = E^{C_1}_A E^{A_1}_{C_1}$, and hence $E^{A_1}_A|_{C_1} = E^{C_1}_A$.*

(ii) *The unique* tr*-preserving conditional expectation from $B' \cap A_1$ onto $B' \cap C_1$ is given by $E^{A_1}_{C_1}|_{B'\cap A_1}$.*

(iii) *The tracial state on the relative commutant $C' \cap C_1$ induced by the inclusion $C \subset A \subset C_1$ is the restriction of the tracial state on $B' \cap A_1$ induced by the inclusion $B \subset A \subset A_1$.*

Proposition 2.3 ([5, Lemma 4.4 (2) and Proposition 4.6]). *Let $B \subset C \subset A$ be as in Lemma 2.2. Then, we have the following:*

(i) $[A_1 : C_1]_0 = [C : B]_0$;

(ii) $E^{A_1}_{C_1}(e_1) = \frac{1}{[C:B]_0} e_C$;

(iii) $\mathcal{F}(e_C) = \alpha_c e_{C_1}$, *where* $\alpha_c = \frac{\sqrt{[A:B]_0}}{[A:C]_0}$. *In particular,* $\mathcal{F}(e_1) = \frac{1}{\sqrt{[A:B]_0}}$.

Notation 2.4. *Now we will fix the notations, which will be used frequently in the sequel.*

(i) *The minimal index $[A : B]_0$ will be denoted by δ^2. Also, $r = \frac{[C:B]_0}{[A:D]_0} = \frac{[D:B]_0}{[A:C]_0}$, $\tau = \frac{1}{[A:B]_0}$, and $\tau_C = \frac{1}{[A:C]_0}$.*

(ii) $\kappa_0^+ = \min\{\mathrm{tr}(p) : p \in \mathcal{P}(B' \cap A)\}$, $\kappa_0^- = \min\{\mathrm{tr}(q) : q \in \mathcal{P}(A' \cap A_1)\}$, *and* $\kappa_0 = \sqrt{\kappa_0^+ \kappa_0^-}$.

(iii) *A quadruple of C^*-algebras is the following diagram:*

$$\begin{array}{ccc} D & \subset & A \\ \cup & & \cup \\ B & \subset & C. \end{array}$$

For notational convenience, we denote this by $(B \subset C,\ D \subset A)$.

3. Fourier transform of the angle operator and its von Neumann entropy

Throughout this section, $(B \subset C,\ D \subset A)$ denotes a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type. Given the quadruple $(B \subset C,\ D \subset A)$, there exist unique mininimal conditional expectations E^A_C

and E^A_D. Let e_C and e_D be the Jones projection corresponding to E^A_C and E^A_D, respectively. By Lemma 2.2, the new quadruple $(A \subset C_1, D_1 \subset A_1)$ again forms a quadruple of simple unital C^*-algebras, where B_1, C_1, and D_1 are C^*-basic constructions corresponding to E^A_B, E^A_C, and E^A_D, respectively. The quadruple $(B \subset C, D \subset A)$ will be called *commuting square* if $E^A_C E^A_D = E^A_D E^A_C = E^A_B$. The same quadruple will be called *co-commuting* if the dual quadruple $(A \subset C_1, D_1 \subset A_1)$ is a commuting square.

Motivated by [12, 16], we define the angle operator between a pair of intermediate simple C^*-subalgebras as follows.

Definition 3.1. Let $B \subset A$ be an inclusion of simple unital C^*-algebras, and let C, D be two intermediate (unital) simple C^*-algebras. The angle operator, denoted by Θ, is defined as follows:

$$\Theta = e_C e_D.$$

Definition 3.2 ([11]). Let $B \subset A$ be an inclusion of simple unital C^*-algebras and B_1 is the C^*-basic construction. For $x \in B' \cap A_1$, the von Neumann entropy of $|x|^2$ is defined as the following quantity:

$$H\big(|x|^2\big) = \operatorname{tr}\big(\eta(|x|^2)\big),$$

where η is the continuous function $\eta(t) = -t \log t$.

Proposition 3.3. *Suppose that $(B \subset C,\ D \subset A)$ is a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type. Then,*

$$H\big(|\Theta|^2\big) + H\big(\big|\mathcal{F}(\Theta)\big|^2\big) \geq \frac{2\kappa_0}{\delta}\eta\left(\frac{\delta}{\kappa_o}\operatorname{tr}(e_C e_D)\right),$$

where $\kappa_0^+ = \min\{\operatorname{tr}(p) : p \in \mathcal{P}(B' \cap A)\}$, $\kappa_0^- = \min\{\operatorname{tr}(q) : q \in \mathcal{P}(A' \cap A_1)\}$, and $\kappa_0 = \sqrt{\kappa_0^+ \kappa_0^-}$.

Proof. Use [4, Theorem 4.10]. ∎

Proposition 3.4. *If $(B \subset C,\ D \subset A)$ is a commuting square of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type, then $H(|\Theta|^2) = 0$. Furthermore, in this case, $H(|\mathcal{F}(\Theta)|^2) = \eta([A : B]_0^{-1})$.*

Proof. Since $e_C e_D = e_D e_C = e_1$, we get that $|\Theta|^2 = e_1$. Therefore, $\eta(|\Theta|^2) = 0$ and so, $H(|\Theta|^2) = 0$. Also, as $\mathcal{F}(e_1) = \frac{1}{\sqrt{[A:B]_0}}$, we have

$$\big|\mathcal{F}(\Theta)\big|^2 = \frac{1}{[A : B]_0}.$$ ∎

The converse of the above theorem is also true.

Proposition 3.5. *Let $(B \subset C, D \subset A)$ be an irreducible inclusion of simple unital C^*-algebras with a conditional expectation E from A onto B of index-finite type. If $H(|\Theta|^2) = 0$, then*

$$e_C e_D = e_D e_C = e_{C \cap D}.$$

Proof. Observe that $0 \le e_D e_C e_D \le 1$ and η is positive in $[0,1]$. Now, $H(|\Theta|^2) = 0$ implies that $\eta(e_D e_C e_D) = 0$, and by [14, p. 74], we conclude that $e_D e_C e_D$ is a projection in $B' \cap A_1$. Since $B' \cap A_1$ is finite-dimensional, $(e_D e_C e_D)^n$ converges to $e_C \wedge e_D$ in the norm topology. Also since $e_C \wedge e_D = e_{C\cap D}$ [3, Proposition 4.1], we immediately get that $e_D e_C e_D = e_{C\cap D}$, and hence $e_D e_C = e_C e_D = e_{C\cap D}$. ∎

To provide a formula for the von Neumann entropy of the dual angle operator beyond the commuting square situation, let us recall the auxiliary operators associated with a quadruple of simple C^*-algebras, along with various properties of them, as developed in [5].

Suppose that $\{\gamma_i : 1 \le i \le m\}$ and $\{\delta_j : 1 \le j \le n\}$ are quasi-bases of E^C_B and E^D_B, respectively. We can define two auxiliary operators $p(C,D) \in C' \cap D_1$ and $q(C,D) \in D' \cap C_1$ by

$$p(C,D) := \sum_{i,j} \gamma_i \delta_j e_1 \delta_j^* \gamma_i^* \quad \text{and} \quad q(C,D) := \sum_{i,j} \delta_j \gamma_i e_1 \gamma_i^* \delta_j^*. \tag{2}$$

Lemma 3.6 ([5, Proposition 5.15]). *Let $B \subset A$ be an irreducible inclusion of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type. If C and D are two intermediate unital C^*-subalgebras of $B \subset A$, then $\frac{1}{t} p(C,D)$ and $\frac{1}{t} q(C,D)$ both are projections, where $t = [A:B]_0 \operatorname{tr}(e_C e_D)$.*

Lemma 3.7. *If $(B \subset C, D \subset A)$ is a commuting square of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type, then $p(C,D)$ and $q(C,D)$ both are projections.*

Proof. The proof follows along the same line of argument as in [2, Proposition 2.20] and hence we omit it. ∎

The following result was proved in [5] in the irreducible case only.

Lemma 3.8. *If $(B \subset C, \ D \subset A)$ is a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type, then*

$$p(C,D) = [D:B]_0 E^{A_1}_{D_1}(e_C) \quad \text{and} \quad q(C,D) = [C:B]_0 E^{A_1}_{C_1}(e_D).$$

Proof. Recall that the trace on $B' \cap A_1$ is $E^A_B \circ E^{A_1}_A|_{B'\cap A_1}$ and e_1 is the Jones projection that implements the basic construction of the inclusion $B \subset A$ with respect to the minimal conditional expectation E^A_B. By [5, Proposition 4.2 (4)], we know that $E^{A_1}_{C_1}|_{B'\cap A_1}$ is the unique trace preserving conditional expectation from $B' \cap A_1$ onto $B' \cap C_1$. For any $w = \sum_i x_i e_C y_i \in B' \cap C_1$, where $x_i, y_i \in A$ for all i, we have

$$\begin{aligned}
\operatorname{tr}(e_D w) &= \operatorname{tr}\Big(e_D\Big(\sum_i x_i e_C y_i\Big)\Big) \\
&= \operatorname{tr}\Big(\Big(\sum_k \delta_k e_1 \delta_k^*\Big)\Big(\sum_i x_i e_C y_i\Big)\Big) && \text{(by [5, Lemma 5.11])} \\
&= \operatorname{tr}\Big(\sum_{k,i} \delta_k e_1 e_C \delta_k^* x_i e_C y_i\Big)
\end{aligned}$$

$$
\begin{aligned}
&= E^A_B E^{C_1}_A E^{A_1}_{C_1}\Big(\sum_{k,i} \delta_k e_1 E^A_C(\delta_k^* x_i) y_i\Big)\\
&= E^A_B E^{C_1}_A \Big(\sum_{k,i} \delta_k E^{A_1}_{C_1}(e_1) E^A_C(\delta_k^* x_i) y_i\Big)\\
&= \frac{1}{[C:B]_0} E^A_B E^{C_1}_A \Big(\sum_{k,i} \delta_k e_C\, E^A_C(\delta_k^* x_i) y_i\Big) \qquad \big(\text{by [5, Lemma 4.4 (2)]}\big)\\
&= \frac{1}{[C:B]_0} E^A_B \Big(\sum_{k,i} \delta_k E^{C_1}_A(e_C) E^A_C(\delta_k^* x_i) y_i\Big)\\
&= \frac{1}{[C:B]_0[A:C]_0} E^A_B \Big(\sum_{k,i} \delta_k E^A_C(\delta_k^* x_i) y_i\Big) \quad \big(\text{since } E^{C_1}_A(e_C) = [A:C]_0^{-1}\big).
\end{aligned}
$$

On the other hand, by [5, Lemma 5.11], we have the following:

$$
\begin{aligned}
\operatorname{tr}\big(q(C,D)w\big) &= \operatorname{tr}\Big(\Big(\sum_k \delta_k e_C \delta_k^*\Big)\Big(\sum_i x_i e_C y_i\Big)\Big)\\
&= \operatorname{tr}\Big(\sum_{k,i} \delta_k e_C \delta_k^* x_i e_C y_i\Big)\\
&= E^A_B E^{A_1}_A \Big(\sum_{k,i} \delta_k E^A_C(\delta_k^* x_i) e_C y_i\Big)\\
&= E^A_B \Big(\sum_{k,i} \delta_k E^A_C(\delta_k^* x_i) E^{A_1}_A(e_C) y_i\Big)\\
&= \frac{1}{[A:C]_0} E^A_B \Big(\sum_{k,i} \delta_k E^A_C(\delta_k^* x_i) y_i\Big).
\end{aligned}
$$

Therefore, we get the following:

$$
\operatorname{tr}\big(q(C,D)w\big) = [C:B]_0 \operatorname{tr}(e_D w).
$$

By Lemma 2.2, the tr on the left-hand side which is the trace on $B' \cap C_1$ is equal to the restriction of the tr on $B' \cap A_1$. Thus, we have

$$
q(C,D) = [C:B]_0\, E^{A_1}_{C_1}(e_D).
$$

The proof for $p(C,D)$ is similar. ∎

Corollary 3.9. *If $(B \subset C,\ D \subset A)$ is a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type, then*

$$
\operatorname{tr}\big(p(C,D)\big) = \operatorname{tr}\big(q(C,D)\big) = r,
$$

where $r = \frac{[C:B]_0}{[A:D]_0} = \frac{[D:B]_0}{[A:C]_0}$.

Proof. By Lemma 3.8, we have $q(C,D) = [C:B]_0\, E^{A_1}_{C_1}(e_D)$. Thus,

$$\operatorname{tr}\big(q(C,D)\big) = [C:B]_0 \operatorname{tr}\big(E^{A_1}_{C_1}(e_D)\big) = [C:B]_0 \operatorname{tr}(e_D) = r.$$

The proof for $p(C,D)$ is similar. ∎

Proposition 3.10. *If $(B \subset C,\ D \subset A)$ is a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type, then*

$$p(C,D) = \delta e_C \star e_D \quad \text{and} \quad q(C,D) = \delta e_D \star e_C.$$

Proof. By [4, Lemma 4.15], we know that for any $x, y \in B' \cap A_1$, we have

$$E^{A_2}_{A_1}\big(\mathcal{F}(x) y \mathcal{F}(x)^*\big) = \frac{1}{\delta} y \star x x^*.$$

Putting $x = e_D$ and $y = e_C$, we immediately obtain

$$\delta e_C \star e_D = [D:B]_0^2 E^{A_2}_{A_1}(e_{D_1} e_C e_{D_1}),$$

thanks to Proposition 2.3. Since $e_{D_1} e_C e_{D_1} = E^{A_1}_{D_1}(e_C) e_{D_1}$ and $E^{A_2}_{A_1}(e_{D_1}) = [A_1 : D_1]_0^{-1}$ $= [D:B]_0^{-1}$ (see Proposition 2.3), we observe that

$$\delta e_C \star e_D = [D:B]_0 E^{A_1}_{D_1}(e_C).$$

The proof is complete once we apply Lemma 3.8. The proof for $q(C,D)$ is obtained by interchanging C and D. ∎

Lemma 3.11. *If $(B \subset C,\ D \subset A)$ is a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type, then*

$$\operatorname{tr}(e_{C_1} e_{D_1}) = \frac{[A:C]_0}{[D:B]_0} \operatorname{tr}(e_C e_D).$$

Proof. Using Proposition 2.3 and by the multiplicativity of the minimal index, we observe that the following equalities hold true:

$$\begin{aligned}
\operatorname{tr}(e_{C_1} e_{D_1}) &= \langle e_{C_1}, e_{D_1} \rangle \\
&= \frac{[A:C]_0 [A:D]_0}{[A:B]_0} \big\langle \mathcal{F}(e_C), \mathcal{F}(e_D) \big\rangle \\
&= \frac{[A:C]_0}{[D:B]_0} \langle e_C, e_D \rangle \quad \big(\text{by [5, Theorem 3.5]}\big) \\
&= \frac{[A:C]_0}{[D:B]_0} \operatorname{tr}(e_C e_D),
\end{aligned}$$

which completes the proof. ∎

Theorem 3.12. *Let $B \subset A$ be an irreducible inclusion of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type and suppose that C, D are two intermediate (unital) C^*-subalgebras. Then,*

$$H\big(|\mathcal{F}(\Theta)|^2\big) = \frac{2}{\delta}\eta\big(\delta\,\mathrm{tr}(e_C e_D)\big),$$

where $\delta = \sqrt{[A:B]_0}$, $r = \frac{[C:B]_0}{[A:D]_0} = \frac{[D:B]_0}{[A:C]_0}$, and η is the continuous function $\eta(t) = -t\log t$.

Proof. First note that $\mathcal{F}(\Theta) = re_{D_1} \star e_{C_1}$, and therefore

$$\begin{aligned} H\big(|\mathcal{F}(\Theta)|^2\big) &= H\big(\mathcal{F}(\Theta)^*\mathcal{F}(\Theta)\big) \\ &= H\big((re_{D_1} \star e_{C_1})^*(re_{D_1} \star e_{C_1})\big) \\ &= H\big(r^2(e_{D_1}^* \star e_{C_1}^*)(e_{D_1} \star e_{C_1})\big) \quad \big(\text{by equation (1)}\big) \\ &= H\big(r^2(e_{D_1} \star e_{C_1})^2\big). \end{aligned}$$

By Proposition 3.10, we have $q_1 := q(C_1, D_1) = \delta\, e_{D_1} \star e_{C_1}$. Since $B \subset A$ is an irreducible inclusion, it follows that the inclusion $A \subset A_1$ is also irreducible (see [5, Proposition 3.2], for instance). Now, using [5, Proposition 5.15], we know that

$$q_1^2 = [A_1 : A]_0\,\mathrm{tr}(e_{C_1}e_{D_1})q_1.$$

Thus, using Lemma 3.11, we have the following:

$$\begin{aligned} (\delta e_{D_1} \star e_{C_1})^2 &= [A:B]_0\delta\,\mathrm{tr}(e_{C_1}e_{D_1})e_{D_1} \star e_{C_1} \\ &= \frac{[A:C]_0[A:B]_0^{\frac{3}{2}}}{[D:B]_0}\,\mathrm{tr}(e_C e_D)e_{D_1} \star e_{C_1}. \end{aligned}$$

In other words, we see that $(e_{D_1} \star e_{C_1})^2 = \frac{[A:C]_0\sqrt{[A:B]_0}}{[D:B]_0}\,\mathrm{tr}(e_C e_D)e_{D_1} \star e_{C_1}$. If we put $f = \frac{\delta e_{D_1} \star e_{C_1}}{[A:C]_0[A:D]_0\,\mathrm{tr}(e_C e_D)}$, then a straightforward calculation yields

$$r^2(e_{D_1} \star e_{C_1})^2 = \big(\delta\,\mathrm{tr}(e_C e_D)\big)^2 f. \tag{3}$$

Using Lemma 3.11, we can easily see that

$$f = \frac{\delta e_{D_1} \star e_{C_1}}{[A_1:A]_0\,\mathrm{tr}(e_{C_1}e_{D_1})}.$$

Thus, by [5, Proposition 5.15] and Proposition 3.10, we conclude that f is a projection. Furthermore,

$$\begin{aligned} \mathrm{tr}(f) &= \frac{[C_1:A]_0}{[A_1:D_1]_0[A:C]_0[A:D]_0\,\mathrm{tr}(e_C e_D)} && (\text{by Corollary 3.9}) \\ &= \frac{1}{[A:B]_0\,\mathrm{tr}(e_C e_D)} && (\text{by Proposition 2.3}). \end{aligned}$$

Therefore, using equation (3), we easily obtain the following equalities:

$$\begin{aligned} H\big(\left|\mathcal{F}(\Theta)\right|^2\big) &= H\big(r^2(e_{D_1} \star e_{C_1})^2\big) \\ &= H\big(\big(\delta\, \mathrm{tr}(e_C e_D)\big)^2 f\big) \\ &= \mathrm{tr}\, \eta\big(\big(\delta\, \mathrm{tr}(e_C e_D)\big)^2 f\big). \end{aligned}$$

As $\eta(\alpha f) = \eta(\alpha) f$ for f a projection, and $\eta(\alpha^2) = 2\alpha\eta(\alpha)$ for any scalar α, we get the following:

$$\begin{aligned} H\big(\left|\mathcal{F}(\Theta)\right|^2\big) &= \mathrm{tr}\left(\eta\big(\big(\delta\, \mathrm{tr}(e_C e_D)\big)^2\big) f\right) \\ &= \eta\big(\big(\delta\, \mathrm{tr}(e_C e_D)\big)^2\big)\, \mathrm{tr}(f) \\ &= 2\delta\, \mathrm{tr}(e_C e_D)\eta\big(\delta\, \mathrm{tr}(e_C e_D)\big)\, \mathrm{tr}(f) \\ &= 2\delta\, \mathrm{tr}(e_C e_D)\eta\big(\delta\, \mathrm{tr}(e_C e_D)\big)\frac{1}{[A:B]_0\, \mathrm{tr}(e_C e_D)} \\ &= \frac{2}{\delta}\eta\big(\delta\, \mathrm{tr}(e_C e_D)\big), \end{aligned}$$

which completes the proof. ■

Remark 3.13. It seems to be a difficult problem to generalize the formula in Theorem 3.12 to the non-irreducible case.

It turns out that $H(|\mathcal{F}(\Theta)|^2)$ is closely related to the Pimsner–Popa probabilistic index, at least in the case of irreducible subfactors. For a pair of subfactors P and Q of a II_1 factor M, let $\lambda(P, Q)$ be the probabilistic index (see [1, Definition 3.1], for instance). In general, $\lambda(P, Q)$ and $\lambda(Q, P)$ are not equal [1].

Corollary 3.14. *Let $(N \subset P, Q \subset M)$ be a quadruple of type II_1 factors with $[M : N] < \infty$ and $N' \cap M = \mathbb{C}$. Then,*

$$H\big(\left|\mathcal{F}(\Theta)\right|^2\big) = \frac{2}{\delta}\eta\big(\alpha_p \lambda(P, Q)\big) = \frac{2}{\delta}\eta\big(\alpha_q \lambda(Q, P)\big).$$

Proof. Use Theorem 3.12 and [1, Theorem 3.3]. ■

Theorem 3.15. *Let $(B \subset C, D \subset A)$ be a quadruple of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type. If the quadruple is a co-commuting square, then*

$$H\big(\left|\mathcal{F}(\Theta)\right|^2\big) = \frac{2}{\delta}\eta\left(\frac{\delta}{[A:C]_0[A:D]_0}\right).$$

Proof. First observe that the quadruple $(A \subset C_1, D_1 \subset A_1)$ is a commuting square. By Lemma 3.7 and Proposition 3.10, $q(C_1, D_1) = \delta\, e_{D_1} \star e_{C_1}$ is a projection. The following equations hold true:

$$\begin{aligned} H\big(\left|\mathcal{F}(\Theta)\right|^2\big) &= H\big(r^2(e_{D_1} \star e_{C_1})^2\big) \\ &= \mathrm{tr}\left(\eta\left(\frac{r^2}{\delta^2} q(C_1, D_1)\right)\right) \end{aligned}$$

$$= \operatorname{tr}\left(\eta\left(\frac{r^2}{\delta^2}\right)q(C_1, D_1)\right)$$
$$= \eta\left(\frac{r^2}{\delta^2}\right)\operatorname{tr}\left(q(C_1, D_1)\right)$$
$$= \frac{2r}{\delta}\eta\left(\frac{r}{\delta}\right)\frac{1}{r} \quad \text{(by Corollary 3.9)}$$
$$= \frac{2}{\delta}\eta\left(\frac{r}{\delta}\right).$$

Now, using the multiplicativity of the minimal index, it is easy to check that

$$\frac{r}{\delta} = \frac{\delta}{[A : C]_0[A : D]_0},$$

which concludes the proof. ■

We conclude the paper with the following question.

Problem. Let $B \subset A$ be an irreducible inclusion of simple unital C^*-algebras with a conditional expectation from A onto B of index-finite type and suppose C, D are two intermediate unital C^*-subalgebras. Can we provide a formula (in the spirit of Theorem 3.12) for $H(|\Theta|^2)$ instead of $H(|\mathcal{F}(\Theta)|^2)$?

Recall that a quadruple $(N \subset P, Q \subset M)$ of II_1 factors is called a quadrilateral if $N = P \vee Q$ and $M = P \wedge Q$ [16, Definition 3.5].

Corollary 3.16. *Suppose $(N \subset P, Q \subset M)$ is a quadrilateral of II_1 subfactors with $[M : N] < \infty$ and $N' \cap M = \mathbb{C}$ such that $[P : N], [Q : N] < 4$; then,*

$$H\left(|\Theta|^2\right) = \frac{2}{\delta^2}\eta\left(\frac{1}{[P : N] - 1}\right),$$
$$H\left(\left|\mathcal{F}(\Theta)\right|^2\right) = \frac{2}{\delta}\eta\left(\frac{\delta}{[M : N] - [M : P]}\right).$$

Proof. By [7], we know that $e_P e_Q e_P = e_N + \lambda(e_P - e_N)$, for $\lambda = \frac{\frac{[P:N]^2}{\dim_N L^2(PQ)} - 1}{[P:N]-1}$. Furthermore, by the proof of [7, Theorem 4.1.3], we have $\dim_N L^2(PQ) = [P : N]([P : N] - 1)$. Thus, $\lambda = \frac{1}{([P:N]-1)^2}$. Since $\eta(e_P e_Q e_P) = \eta(\lambda)(e_P - e_N)$, it follows that

$$H(|\Theta|^2) = \eta(\lambda)([M : P]^{-1} - [M : N]^{-1}) = \frac{2}{[M : N]}\eta(\frac{1}{[P : N] - 1}).$$

To prove the formula for $H(|\mathcal{F}(\Theta)|^2)$, we consider the dual factors $\bar{P}$ and $\bar{Q}$ from the basic constructions $P \subset M$ and $Q \subset M$, respectively (with the corresponding Jones projections $e_{\bar{P}}$ and $e_{\bar{Q}}$). Then, we have

$$\operatorname{tr}(e_{\bar{P}} e_{\bar{Q}}) = \frac{[M : P]}{[Q : N]}\operatorname{tr}(e_P e_Q).$$

Also, we know that (see [8]) $\mathrm{tr}(e_P e_Q) = (\dim_M L^2(\bar{P}\bar{Q}))^{-1}$ and therefore,

$$\mathrm{tr}(e_P e_Q) = \frac{1}{[M : P]\big([P : N] - 1\big)}.$$

Type II_1 factors are simple C^*-algebras and also there exists a trace preserving conditional expectation from M onto N. By [14, Proposition 1.3], there exists a quasi-basis for that conditional expectation. Hence, the assumptions of Theorem 3.12 are met and the proof is complete once we apply Theorem 3.12. ∎

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