

$L_{\kappa\omega}$ -equivalence of abelian groups with partial decomposition bases

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ABSTRACT – We consider the class of abelian groups possessing partial decomposition bases in the language $L_{\kappa\omega}$ for uncountable cardinals κ . Jacoby, Leistner, Loth and Strüngmann developed a numerical invariant deduced from the classical global Warfield invariant and proved that if two groups have identical modified Ulm invariants and Warfield invariants up to $\omega\delta$ for some ordinal δ , then they are equivalent in $L_{\infty\omega}^\delta$. Subsequently, Jacoby and Loth showed that the converse is true for appropriate δ . In this paper we prove that the modified Warfield invariant up to κ is expressible in $L_{\kappa\omega}$, thus a complete classification theorem in $L_{\kappa\omega}$ is obtained. This generalizes a result of Barwise and Eklof.

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Dedicated to László Fuchs on his 100th birthday

1. Introduction

The classical theorem of Ulm [21] presents invariants that classify countable abelian p -groups up to isomorphism. Hill [3] and Walker [22] were able to extend this result to the class of totally projective p -groups, the largest natural class of abelian p -groups in which the Ulm invariants distinguish between nonisomorphic p -groups. It was Szmielew [20] who first considered abelian groups from a model-theoretic point of

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view. Barwise and Eklof [1] took up Szmielew's approach and extended Ulm's theorem to the classification of arbitrary torsion abelian groups up to equivalence in some infinitary language, e.g., $L_{\infty\omega}$. Ulm's theorem was generalized to Warfield groups by Warfield [23] in the p -local case and by Hunter, Richman [7] and Stanton [19] in the global case. In [8] Jacoby developed the class of abelian groups with partial decomposition bases in order to extend Barwise and Eklof's [1] classification of torsion groups in $L_{\infty\omega}$ to Warfield groups. This class coincides with the class of k -groups defined by Hill and Megibben [4, 5] (see [12, 14]). Using modifications of the Ulm and Warfield invariants, groups with partial decomposition bases have been classified in $L_{\infty\omega}$, or equivalently, up to partial isomorphism ([13, 15]) as well as in $L_{\infty\omega}^\delta$ ([9–11]). It follows from [9, Theorem 6.10] (see the first part of Theorem 4.9) that given any cardinal $\kappa > \omega$, if two groups G and H with partial decomposition bases have identical modified Ulm and Warfield invariants up to κ , then G and H are equivalent in $L_{\kappa\omega}$. In this paper, the converse of this result is proved. More specifically, we show that the modified Warfield invariant up to κ is expressible in $L_{\kappa\omega}$ (Theorem 4.10). Therefore two $L_{\kappa\omega}$ -equivalent groups with partial decomposition bases have identical modified invariants up to κ . It follows that for groups with partial decomposition bases, their $L_{\kappa\omega}$ -equivalence classes are exactly their $L_{\infty\omega}^\kappa$ -equivalence classes (see Corollary 4.11).

For notation and terminology on abelian groups and on model theory, we may refer to [2, 6, 16, 18].

2. General and algebraic preliminaries

We accept the ZFC axioms of set theory. Throughout, we use $\alpha, \beta, \gamma, \dots$ to denote ordinals and $\kappa, \lambda, \dots$ to denote cardinals. The cardinality of a set X is written $|X|$. Recall that an infinite cardinal is *regular* if its cofinality is equal to itself, otherwise it is *singular*. For an ordinal α , let ω_α (or $\aleph_\alpha$) denote the α th infinite cardinal and set $\omega = \omega_0$. Then ω_α is regular if $\alpha = 0$ or α is a successor ordinal (see [17, p. 49]), thus α is a limit ordinal whenever ω_α is singular. If κ is an infinite cardinal, then $\kappa = \omega_\alpha$ for some α (cf. [17, pp. 29–30]).

All groups considered in this paper are abelian, written additively. Let G be a group and p a prime. The torsion part of G is denoted by tG and the p -torsion part is written $t_p G$. We write $G_p = G \otimes \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the group of integers localized at p . Notice that $t_p G \cong t(G_p)$. If S is a subset of G , then $\langle S \rangle$ denotes the subgroup of G generated by S . For all ordinals α , subgroups $p^\alpha G$ of G are defined inductively by $p^0 G = G$, $p^{\alpha+1} G = p(p^\alpha G)$, and $p^\alpha G = \bigcap_{\beta < \alpha} p^\beta G$ if α is a limit ordinal. Then the p -height $|x|_p$ (or $|x|_p^G$) of $x \in G$ is defined to be α if $x \in p^\alpha G \setminus p^{\alpha+1} G$ and $|x|_p = \infty$ if

$x \in p^\infty G := \bigcap_\alpha p^\alpha G$. There is a smallest ordinal τ such that $p^\tau G = p^\infty G$, called the p -length of G . It is called the *length* of G if G is a p -group. Let $G[p] = \{x \in G : px = 0\}$. The p -rank of G , written $r_p(G)$, is the cardinality of a maximal independent subset of $t_p G$. We have $r_p(G) = r_p(t_p G) = r_p(G[p]) = \dim_{\mathbb{Z}/p\mathbb{Z}}(G[p])$ (cf. [16, Corollary 1.4.3, p. 11]). The *torsion-free rank* of G , written $r_0(G)$, is the cardinality of a maximal independent set of elements of G of infinite order. We set $r_{p,0}(G) = r_p(G) + r_0(G)$ and define $\hat{r}_{p,0}(G) = \min\{\hat{r}_{p,0}(G), \omega\}$. A subgroup H of G is called p -isotype if $p^\alpha G \cap H = p^\alpha H$ for all ordinals α , and H is called *isotype* if it is p -isotype for all primes p . For example, tG is isotype and $t_p G$ is p -isotype in G .

For a prime p and an ordinal α , the *Ulm invariants* of G are defined by

$$u_p(\alpha, G) = \dim_{\mathbb{Z}/p\mathbb{Z}}((p^\alpha G)[p]/(p^{\alpha+1}G)[p])$$

with α an ordinal and

$$u_p(\infty, G) = \dim_{\mathbb{Z}/p\mathbb{Z}}(p^\infty G)[p].$$

Ulm [21] proved that these cardinal numbers are isomorphism invariants of countable torsion groups. Barwise and Eklof's [1] modified Ulm invariant is defined by

$$\hat{u}_p(\alpha, G) = \min\{u_p(\alpha, G), \omega\}.$$

By adding the invariant $\hat{u}_p(\infty, G) = \min\{u_p(\infty, G), \omega\}$, these invariants classify all torsion groups in $L_{\infty\omega}$ (see [1]).

By Ord we mean the class of all ordinals and define $\text{Ord}_\infty = \text{Ord} \cup \{\infty\}$. We adopt the convention that $\infty > \alpha$ whenever $\alpha \in \text{Ord}_\infty$ and set $\infty + 1 = \infty$. For simplicity, we may write $\alpha \wedge \beta$ instead of $\min\{\alpha, \beta\}$. An *Ulm sequence* is an increasing sequence $u = (\beta_i)_{i < \omega}$ where each β_i is an ordinal or the symbol ∞ . Two Ulm sequences $u = (\beta_i)$ and $v = (\gamma_i)$ are called *equivalent*, written $u \sim v$, if there exist $k, l < \omega$ such that $\beta_{i+k} = \gamma_{i+l}$ for all $i < \omega$. The p -Ulm sequence of $x \in G$ is the sequence $\|x\|_p = (|p^i x|_p)_{i < \omega}$.

An *Ulm matrix* is an $\omega \times \omega$ -matrix $M = [m_{p,i}]$ indexed over all primes and non-negative integers such that each row $M_p = (m_{p,i})_{i < \omega}$ is an Ulm sequence. For a group G , the *Ulm matrix of $x \in G$* (written $\|x\|$ or $\|x\|_G$) is defined to be the Ulm matrix $[|p^i x|_p]$ where the rows are the p -Ulm sequences of x with $p \in \mathbb{P}$. If n is a positive integer, we let nM be the Ulm matrix having $m_{p,i+|n|_p}$ as its (p, i) entry, where $|n|_p$ is the p -height of n in $\mathbb{Z}$. If $M = [m_{p,i}]$ and $N = [n_{p,i}]$, we write $M \geq N$ in the case $m_{p,i} \geq n_{p,i}$ for all (p, i) . Two Ulm matrices M and N are called *compatible* if there are positive integers m and n such that $mM \geq N$ and $nN \geq M$. This yields an equivalence relation, and the equivalence classes are called *compatibility classes*.

We recall the following definitions “up to α ” for a given ordinal α (see [16]): For $\beta, \gamma \in \text{Ord}_\infty$ we define $\beta \geq_\alpha \gamma$ to mean

$$\beta \wedge \alpha \geq \gamma \wedge \alpha.$$

If equality holds, then we write $\beta =_\alpha \gamma$. For Ulm matrices $M = [m_{p,i}]$ and $N = [n_{p,i}]$ define $M \geq_\alpha N$ if $m_{p,i} \geq_\alpha n_{p,i}$ for all p and i . Then M and N are called α -compatible, written $M \sim_\alpha N$, if and only if for some m and n , $mM \geq_\alpha N$ and $nN \geq_\alpha M$. Given compatibility classes c and c' of Ulm matrices, we write $c \sim_\alpha c'$ to mean $M \sim_\alpha N$ for some $M \in c$ and $N \in c'$.

For Ulm sequences $u = (\beta_i)$ and $v = (\gamma_i)$ we define $u \geq_\alpha v$ if and only if $\beta_i \geq_\alpha \gamma_i$ for all $i < \omega$. We call u and v α -equivalent, written $u \sim_\alpha v$, if for some $m, n < \omega$ we have $\alpha_{i+m} =_\alpha \beta_{i+n}$ for all $i < \omega$. Given two equivalence classes e and e' of Ulm sequences, we write $e \sim_\alpha e'$ if $u \sim_\alpha v$ for some $u \in e$ and $v \in e'$.

Let G be a group. A subset X of G is called a *decomposition set* if

- (1) all elements of X are independent and have infinite order;
- (2) for each $x = k_1x_1 + \cdots + k_nx_n$ ($k_1, \dots, k_n \in \mathbb{Z}$, $x_1, \dots, x_n \in X$) and prime p , we have $|x|_p = \min\{|k_i x_i|_p\}$.

Then X is called a *decomposition basis* for G if, in addition, $G/\langle X \rangle$ is torsion. A decomposition basis X for G is called *nice* if for every prime p , $\langle X \rangle_p$ is a p -nice subgroup of G_p , i.e., if every coset $x + \langle X \rangle_p$ ($x \in G_p$) has an element of maximal p -height. Groups possessing a nice decomposition basis X with simply presented quotient $G/\langle X \rangle$ are called *Warfield groups*. Stanton defined cardinal numbers that, together with the Ulm invariants, form a complete set of isomorphism invariants for Warfield groups (see Hunter–Richman [7], Stanton [19]).

Jacoby [8] generalized the concept of decomposition basis: Given a group G , a system $\mathcal{C}$ is called a *partial decomposition basis* for G if

- (1) $\mathcal{C}$ is a nonempty collection of finite subsets of G ;
- (2) if $X \in \mathcal{C}$, then X is a decomposition set;
- (3) if $X \in \mathcal{C}$ and $x \in G$, there is $Y \in \mathcal{C}$ such that $X \subseteq Y$ and $mx \in \langle Y \rangle$ for some positive integer m .

Now let G be a group with partial decomposition basis $\mathcal{C}$, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Jacoby [8] defined $\hat{w}(c, p, e, G)$ to be the largest integer n , if it exists, such that there are $X \in \mathcal{C}$ and $x_1, \dots, x_n \in X$ with $\|x_i\| \in c$ and $\|x_i\|_p \in e$ for all $i = 1, \dots, n$. If no such n exists, put $\hat{w}(c, p, e, G) = \omega$. Jacoby, Leistner, Loth and Strüngmann [9] generalized

this definition up to an ordinal α :

$$\hat{w}_\alpha(c, p, e, G) = \min\left\{\sum_{e' \sim_\alpha e, c' \sim_\alpha c} \hat{w}(c', p, e', G), \omega\right\}.$$

3. Model-theoretic preliminaries

The infinitary language $L_{\infty\omega}$ is an extension of a language of first-order logic that allows conjunctions and disjunctions over arbitrary sets of formulas and quantifications over finite sets of variables. The symbols of the language include a variable v_α for each ordinal α , and since we will be using this language to discuss abelian groups, we include the binary function symbol $+$, the unary function symbol $-$ and the constant 0 . We use these symbols to form atomic formulas in the usual way.

For each ordinal α we define, by induction on α , $L_{\infty\omega}^\alpha$ to be the smallest class Y that contains the atomic formulas and is closed under the following logical operations:

- (1) If $\varphi \in Y$ then $\neg\varphi \in Y$;
- (2) if Φ is a set, $\Phi \subseteq Y$, then $\bigwedge \Phi$ and $\bigvee \Phi \in Y$;
- (3) if $\varphi \in L_{\infty\omega}^\beta$ for some $\beta < \alpha$ and v is a variable, then $\forall v \varphi \in Y$ and $\exists v \varphi \in Y$.

In (2), $\bigwedge \Phi$ (resp. $\bigvee \Phi$) denotes the conjunction (resp. disjunction) of all elements of the set Φ . Notice that Φ can be of any cardinality and that the empty disjunction $\bigvee \emptyset$ is false everywhere (cf. [6, p. 28]). Then let $L_{\infty\omega} = \bigcup_{\alpha \in \text{Ord}} L_{\infty\omega}^\alpha$. An element φ of $L_{\infty\omega}$ is called a *formula* if φ has at most a finite number of free variables. When we write “ $\varphi \in L_{\infty\omega}$ ” it is understood that φ is a formula. *Sentences* are formulas with no free variables. For $\varphi \in L_{\infty\omega}$, the *quantifier rank* of φ is $\text{qr}(\varphi) = \min\{\alpha : \varphi \in L_{\infty\omega}^\alpha\}$.

The set of *subformulas* of φ , written $\text{sub}(\varphi)$, is defined inductively by the following:

- (1) If φ is atomic, then $\text{sub}(\varphi) = \{\varphi\}$;
- (2) if φ is $\neg\psi$, $\forall v \psi$ or $\exists v \psi$, then $\text{sub}(\varphi) = \text{sub}(\psi) \cup \{\varphi\}$;
- (3) if φ is $\bigwedge \Psi$ or $\bigvee \Psi$, then $\text{sub}(\varphi) = \bigcup_{\psi \in \Psi} \text{sub}(\psi) \cup \{\varphi\}$.

A subformula ψ of φ is called *proper* if $\psi \neq \varphi$. Hodges [6] defined the *complexity* of a formula $\varphi \in L_{\infty\omega}$ inductively to be the ordinal $\text{comp}(\varphi)$ such that

$$\text{comp}(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ is an atomic formula,} \\ \sup\{\text{comp}(\psi) + 1 : \psi \text{ is a proper subformula of } \varphi\} & \text{otherwise.} \end{cases}$$

For an infinite cardinal κ , the sublanguage $L_{\kappa\omega}$ of $L_{\infty\omega}$ is defined to be

$$L_{\kappa\omega} = \{\varphi \in L_{\infty\omega} : |\text{sub}(\varphi)| < \kappa\}$$

(see [1]). Two groups G and H are called $L_{\infty\omega}^\alpha$ -equivalent (resp. $L_{\kappa\omega}$ -equivalent), and we write

$$G \equiv_\alpha H \quad (\text{resp. } G \equiv_{\kappa\omega} H),$$

if G and H satisfy the same sentences of $L_{\infty\omega}^\alpha$ (resp. $L_{\kappa\omega}$).

The following result on $L_{\kappa\omega}$ is stated (without proof) in terms of cardinals $\kappa = \omega_\alpha$ in [1, p. 28]:

PROPOSITION 3.1. *Let κ be an infinite cardinal.*

- (1) *If κ is regular, then $L_{\kappa\omega}$ is the smallest class Y of formulas such that*
 - (a) *Y contains all atomic formulas;*
 - (b) *if φ is in Y , then $\neg\varphi$ is in Y ;*
 - (c) *if φ is in Y and v is a variable, then $\forall v\varphi$ and $\exists v\varphi$ are in Y ;*
 - (d) *if Ψ is a subset of Y and $|\Psi| < \kappa$, then $\bigwedge \Psi$ and $\bigvee \Psi$ are in Y .*
- (2) *If $\kappa = \omega_\alpha$, where α is a limit ordinal, then $L_{\kappa\omega} = \bigcup_{\omega \leq \lambda < \kappa} L_{\lambda\omega}$.*
- (3) *If $\varphi \in L_{\kappa\omega}$ then $\text{qr}(\varphi) < \kappa$, i.e., $L_{\kappa\omega} \subseteq \bigcup_{\gamma < \kappa} L_{\infty\omega}^\gamma$. Thus $L_{\infty\omega}^\kappa$ -equivalence implies $L_{\kappa\omega}$ -equivalence.*

PROOF. (1) It is clear that $L := L_{\kappa\omega}$ satisfies (a)–(c). Now assume that φ equals $\bigwedge \Phi$ or $\bigvee \Phi$, where $\Psi \subseteq L$ and $|\Psi| < \kappa$. Since κ is regular, we have $|\text{sub}(\varphi)| = |\bigcup_{\psi \in \Psi} \text{sub}(\psi) \cup \{\varphi\}| < \kappa$. Thus φ is in L .

To prove that L is the smallest such class, let M be any class of formulas satisfying (a)–(d). To show that $L \subseteq M$ we use induction on the complexity of formulas, so let $\varphi \in L$. If φ is atomic then $\varphi \in M$ by (a). Now assume that φ is not an atomic formula and let $\psi \in L$. If φ is $\neg\psi$, $\forall v\psi$ or $\exists v\psi$, then $\psi \in M$ by the induction hypothesis, thus $\varphi \in M$ by (b) and (c). Finally, assume that φ equals $\bigwedge \Psi$ or $\bigvee \Psi$ with $\Psi \subseteq L$, where $|\Psi| < \kappa$. By the induction hypothesis, $\Psi \subseteq M$, so by (d) we have $\varphi \in M$ as desired.

(2) If $\varphi \in L_{\kappa\omega}$ then $|\text{sub}(\varphi)| < \omega_\gamma$ for some $\gamma < \alpha$. Thus $\varphi \in L_{\omega_\gamma\omega} \subseteq \bigcup_{\beta < \alpha} L_{\omega_\beta\omega} = \bigcup_{\omega \leq \lambda < \kappa} L_{\lambda\omega}$.

(3) We write $\kappa = \omega_\alpha$ and prove the assertion by transfinite induction on α . First assume that ω_α is a regular cardinal; this includes the case $\alpha = 0$. We show that $M := \bigcup_{\gamma < \kappa} L_{\infty\omega}^\gamma$ satisfies (1) (a)–(d). Clearly, (a) and (b) are true. To see that M satisfies (c), let $\varphi \in M$. Then $\varphi \in L_{\infty\omega}^\gamma$ for some $\gamma < \kappa$, thus $\forall v\varphi$ and $\exists v\varphi$ are in $L_{\infty\omega}^{\gamma+1} \subseteq M$. To show that M satisfies (d), let $\Psi = \{\psi_i : i \in I\} \subseteq M$, where $|I| < \kappa$, say $\psi_i \in L_{\infty\omega}^{\gamma_i}$ with $\gamma_i < \kappa$ for all $i \in I$. Since κ is regular we have $\gamma' := \sup\{\gamma_i : i \in I\} < \kappa$, thus $\psi_i \in L_{\infty\omega}^{\gamma'}$ for all i . Hence both $\bigwedge \Psi$ and $\bigvee \Psi$ are in $L_{\infty\omega}^{\gamma'} \subseteq M$. By (1) we have $L_{\kappa\omega} \subseteq M$.

Now let $\alpha > 0$ and assume the claim holds for all $\beta < \alpha$. If ω_α is regular we are done by the paragraph above. If ω_α is singular then α is a limit ordinal, so by (2) and the induction hypothesis we have $L_{\omega_\alpha\omega} = \bigcup_{\beta < \alpha} L_{\omega_\beta\omega} \subseteq \bigcup_{\beta < \alpha} \bigcup_{\gamma < \omega_\beta} L_{\infty\omega}^\gamma = \bigcup_{\gamma < \omega_\alpha} L_{\infty\omega}^\gamma$, as desired. ■

4. $L_{\kappa\omega}$ -equivalence of groups with partial decomposition bases

Barwise and Eklof [1] examined torsion groups from the viewpoint of $L_{\infty\omega}$ and showed that key characteristics of p -groups are expressible in $L_{\infty\omega}$ and $L_{\kappa\omega}$, with some proof details omitted that are included here. The proofs apply to arbitrary groups as well.

LEMMA 4.1. *Let κ be an infinite cardinal, α an ordinal and p a prime.*

- (1) *There is a formula $\theta_\alpha(x)$ that has at most $|\alpha| + 1$ subformulas such that for any group G and any $a \in G$, $G \models \theta_\alpha[a]$ if and only if $a \in p^\alpha G$. More specifically, letting $s_\alpha = |\text{sub}(\theta_\alpha(x))|$, we have $s_0 = 1$, $s_\alpha = 2$ if $0 < \alpha < \omega$ and $s_\alpha = |\alpha|$ if $\alpha \geq \omega$.*
- (2) *If $\alpha < \kappa$ then $\theta_\alpha(x)$ is a formula of $L_{\kappa\omega}$.*

PROOF. (1) For $\theta_0(x)$, use $x = x$. If $0 < \alpha < \omega$ let $\theta_\alpha(x) = \exists y(x = p^\alpha y)$. To prove that $s_\alpha = |\alpha|$ for all $\alpha \geq \omega$ we use transfinite induction. It is clear that $s_\omega = \aleph_0$ since “ $x \in p^\omega G$ ” is expressible by $\theta_\omega(x) = \bigwedge_{\beta < \omega} \theta_\beta(x)$. Now let $\alpha > \omega$ and assume that $s_\beta = |\beta|$ for all $\omega \leq \beta < \alpha$. If $\alpha - 1$ exists, let $\theta_\alpha(x) = \exists y(\theta_{\alpha-1}(y) \wedge x = py)$ and if α is a limit ordinal, let $\theta_\alpha(x) = \bigwedge_{\beta < \alpha} \theta_\beta(x)$. In either case we conclude that $s_\alpha = |\alpha|$.

(2) follows from (1). ■

We will adopt the convention of using an algebraic formula in quotation marks to represent the corresponding expression in $L_{\infty\omega}$. For example, by Lemma 4.1 (1), “ $x \in p^\alpha G$ ” is expressible by a formula that has $\leq \lambda := |\alpha| + 1$ subformulas. Alternatively, we may write

“ $x \in p^\alpha G$ ” is in $L_{\infty\omega}$ and has $\leq \lambda$ subformulas.

Notice that “ $x \in tG$ ” is in $L_{\omega_1\omega}$ since it is expressible by the formula $\varphi(x) = \bigvee_{0 < n < \omega} (nx = 0)$. Likewise, “ $x \in t_p G$ ” and “ x has infinite order” are in $L_{\omega_1\omega}$.

LEMMA 4.2. *Suppose that H is a subgroup of a group G such that “ $x \in H$ ” is in $L_{\infty\omega}$ and has λ subformulas.*

- (1) If $a_1, \dots, a_m \in \mathbb{Z}$ then “ $a_1x_1 + \dots + a_mx_m \in H$ ” has $\lambda + 2$ subformulas.
- (2) “ $x_1, \dots, x_m$ are independent modulo H ” has $\aleph_0 \cdot \lambda$ subformulas.
- (3) “ x has infinite order modulo H ” and “ x has p -power order modulo H ” have $\aleph_0 \cdot \lambda$ subformulas, where p is a given prime.

PROOF. (1) Suppose that “ $x \in H$ ” is expressed by the formula $\psi(x)$. Then $a_1x_1 + \dots + a_mx_m \in H$ if and only if $\exists x(\psi(x) \wedge x = a_1x_1 + \dots + a_mx_m)$.

(2) $x_1, \dots, x_m$ are independent modulo H if and only if for all $a_1, \dots, a_m \in \mathbb{Z}$, $a_1x_1 + \dots + a_mx_m \in H$ implies $a_ix_i \in H$ for all $1 \leq i \leq m$. This may be written as $\bigwedge_{a_1 \in \mathbb{Z}} \dots \bigwedge_{a_m \in \mathbb{Z}} (“a_1x_1 + \dots + a_mx_m \in H” \rightarrow \bigwedge_{i=1}^m “a_ix_i \in H”)$. Now apply (1).

(3) “ x has infinite order modulo H ” is true if and only if $\bigwedge_{0 < n < \omega} \neg(“nx \in H”)$, and “ x has p -power order modulo H ” holds exactly if $\bigvee_{n < \omega} (“p^n x \in H”)$. ■

LEMMA 4.3. Let G be a group, p a prime, α an ordinal and $m < \omega$. Then “ $r_p(p^\alpha G) \geq m$ ” and “ $u_p(\alpha, G) \geq m$ ” are expressible by formulas that have at most $\aleph_0 \cdot |\alpha|$ subformulas.

PROOF. We have $r_p(p^\alpha G) \geq m$ if and only if there are m independent elements of p -height $\geq \alpha$, if and only if $G \models \exists x_1 \dots \exists x_m (\bigwedge_{i=1}^m “x_i \in p^\alpha G” \wedge “x_1, \dots, x_m$ are independent” $\wedge “x_1, \dots, x_m \in t_p G”)$.

Note that $u_p(\alpha, G) \geq m$ if and only if $G \models \exists x_1 \dots \exists x_m [\bigwedge_{i=1}^m (“x_i \in p^\alpha G” \wedge px_i = 0) \wedge “x_1, \dots, x_m$ are independent modulo $p^{\alpha+1} G”]$. ■

The following will be needed.

LEMMA 4.4. Let G be a group, p a prime, $\tau = \text{length}(t_p G)$ and $v \geq \tau$. Then $u_p(\infty, G) = r_p(p^v G)$.

PROOF. We have $u_p(\infty, G) = r_p(p^\infty G) = r_p(\bigcap_{\sigma \geq v} p^\sigma G \cap t_p G)$, so since $t_p G$ is p -isotype in G we obtain $u_p(\infty, G) = r_p(\bigcap_{\sigma \geq v} p^\sigma (t_p G)) = r_p(p^v (t_p G)) = r_p(p^v G \cap t_p G) = r_p(p^v G)$. ■

The next result was proved for p -groups in [1, Corollary 3.5 (1) $\Rightarrow$ (2)].

THEOREM 4.5. Let G and H be groups such that $G \equiv_{\kappa\omega} H$, where κ is a cardinal $> \omega$. Then we have

- (1) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all primes p and ordinals $\alpha < \kappa$;
- (2) if $\text{length}(t(G_p)) < \kappa$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.

PROOF. (1) Let p be a prime and $\alpha < \kappa$. If $\hat{u}_p(\alpha, G) = m < \omega$ then $G \models "u_p(\alpha, G) \geq m" \wedge \neg "u_p(\alpha, G) \geq m + 1"$, an expression with $\leq \aleph_0 \cdot |\alpha| < \kappa$ subformulas by Lemma 4.3. Thus H models the same expression, i.e., $\hat{u}_p(\alpha, H) = m$. Now assume that $\hat{u}_p(\alpha, G) = \omega$. Then $G \models \bigwedge_{m < \omega} "u_p(\alpha, G) \geq m"$, hence $\hat{u}_p(\alpha, H) = \omega$.

(2) Suppose $t(G_p) \cong t_p G$ has length $\tau < \kappa$. For $\sigma \geq \tau$ we have $(p^\sigma G)[p] = (p^\sigma G \cap t_p G)[p] = (p^\sigma(t_p G))[p] = (p^\tau(t_p G))[p]$, thus $u_p(\sigma, G) = 0$. Then (1) shows that $u_p(\sigma, H) = 0$ for all $\tau \leq \sigma < \kappa$. Assume that $\tau' := \text{length}(t(H_p)) \geq \kappa$. Since κ is a limit ordinal we have $u_p(\sigma, H) \neq 0$ for infinitely many $\tau < \sigma < \kappa$ by [16, Lemma 1.13.7 (2), p. 34], a contradiction. Therefore $\tau' < \kappa$. For $\nu = \max\{\tau, \tau'\}$ we have $u_p(\infty, G) = r_p(p^\nu G)$ and $u_p(\infty, H) = r_p(p^\nu H)$ (Lemma 4.4). Since " $r_p(p^\nu G) \geq m$ " has at most $\aleph_0 \cdot |\nu| < \kappa$ subformulas (Lemma 4.3), we conclude that $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$. ■

We wish to show that the modified Warfield invariant $\hat{w}_\alpha(c, p, e, G)$ is expressible in $L_{\kappa\omega}$ whenever $\alpha < \kappa$. First we turn to a modification of an invariant defined by Stanton [19], equivalent to $\hat{w}_\alpha(c, p, e, G)$, which we then prove is expressible in $L_{\kappa\omega}$.

For a group G , a prime p , an Ulm matrix M and an ordinal α , we define the following as in [16]:

$$\begin{aligned} G_\alpha(M) &= \{x \in G : \|x\| \geq_\alpha M\}, \\ G_\alpha(M^*)^\diamond &= \{x \in G_\alpha(M) : \|x\| \sim_\alpha M \text{ or } x \text{ is torsion}\}, \\ G_\alpha(M_p^*, p)^\diamond &= \{x \in G : \|x\|_p \geq_\alpha M_p \text{ and } |p^i x|_p \neq_\alpha m_{p,i} \text{ for infinitely many } i\}, \\ G_\alpha(M^*, p)^\diamond &= G_\alpha(M^*)^\diamond \cup G_\alpha(M_p^*, p)^\diamond, \\ G_\alpha(M^*, p) &= G_\alpha(M) \cap \langle G_\alpha(M^*, p)^\diamond \rangle. \end{aligned}$$

Then for a compatibility class c of Ulm matrices and an equivalence class e of Ulm sequences, the modified Stanton invariant is defined to be

$$\hat{s}_\alpha(c, p, e, G) = \sup\{\hat{r}_{p,0}(G_\alpha(M)/G_\alpha(M^*, p))\},$$

where the supremum is taken over all Ulm matrices $M \in c$ with $M_p \in e$ (cf. [16]). For groups with partial decomposition bases, this invariant can be expressed as follows.

THEOREM 4.6 (Jacoby–Loth [10, 16]). *Let G be a group with partial decomposition basis, α an ordinal, c a compatibility class of Ulm matrices, p a prime and e an equivalence class of Ulm sequences. Then we have*

- (1) $\hat{s}_\alpha(c, p, e, G) = \hat{w}_\alpha(c, p, e, G)$;
- (2) $\hat{s}_\alpha(c, p, e, G) = \hat{r}_{p,0}(G_\alpha(M)/G_\alpha(M^*, p))$ for all $M \in c$ with $M_p \in e$.

The next lemma will be useful.

LEMMA 4.7. *Let α be an ordinal and let $\beta, \gamma \in \text{Ord}_\infty$.*

- (1) $\beta \geq_\alpha \gamma$ if and only if $\beta \geq \gamma \wedge \alpha$.
- (2) $\beta \geq_\alpha \gamma$ and $\beta \neq_\alpha \gamma$ if and only if $\gamma \not\geq_\alpha \beta$.
- (3) $\gamma \not\geq_\alpha \beta$ if and only if $\beta \geq (\gamma + 1) \wedge \alpha$ and $\gamma < \alpha$.

PROOF. (1) If $\beta \geq_\alpha \gamma$ then $\beta \geq \beta \wedge \alpha \geq \gamma \wedge \alpha$ and if $\beta \geq \gamma \wedge \alpha$, then $\beta \wedge \alpha \geq \gamma \wedge \alpha$.

(2) $\beta \wedge \alpha \geq \gamma \wedge \alpha$ and $\beta \wedge \alpha \neq \gamma \wedge \alpha$ if and only if $\beta \wedge \alpha > \gamma \wedge \alpha$.

(3) By (1), $\gamma \not\geq_\alpha \beta$ if and only if $\gamma < \beta \wedge \alpha$. First, assume that $\gamma < \beta \wedge \alpha$. Then $\gamma < \alpha$ and since $\beta \geq \beta \wedge \alpha > \gamma$, we have $\beta \geq \gamma + 1 \geq (\gamma + 1) \wedge \alpha$.

Conversely, assume $\beta \geq (\gamma + 1) \wedge \alpha$ and $\gamma < \alpha$. Then $\gamma + 1 \leq \alpha$, hence $\beta \geq (\gamma + 1) \wedge \alpha = \gamma + 1$. Therefore $\beta \wedge \alpha > \gamma$. ■

LEMMA 4.8. *Let G be a group, p a prime and α an ordinal. Let $\lambda = \aleph_0 \cdot |\alpha|$.*

- (1) *If M is any Ulm matrix, then each of the expressions “ $x \in G_\alpha(M)$ ”, “ $x \in G_\alpha(M^*)^\diamond$ ”, “ $x \in G_\alpha(M_p^*, p)^\diamond$ ”, “ $x \in G_\alpha(M^*, p)^\diamond$ ” and “ $x \in G_\alpha(M^*, p)$ ” can be written as a formula that has $\leq \lambda$ subformulas.*
- (2) *If $m < \omega$, then “ $\hat{s}_\alpha(c, p, e, G) \geq m$ ” is expressible by a formula that has $\leq \lambda$ subformulas.*

PROOF. (1.1) We write $M = [m_{q,i}]$. Then Lemma 4.7 (1) shows that $x \in G_\alpha(M)$ if and only if $q^i x \in q^{m_{q,i} \wedge \alpha} G$ for all q and i . By Lemmas 4.1 (1) and 4.2 (1), the expression $\bigwedge_{(q,i) \in \mathbb{P} \times \omega} “q^i x \in q^{m_{q,i} \wedge \alpha} G”$ has $\leq \lambda$ subformulas.

(1.2) We have $x \in G_\alpha(M^*)^\diamond$ exactly if $x \in G_\alpha(M)$ and either x is torsion or $\|x\| \sim_\alpha M$. If $x \in G_\alpha(M)$, then $\|x\| \sim_\alpha M$ if and only if $jM \not\geq_\alpha \|x\|$ for every $j < \omega$. By Lemma 4.7 (3), this holds if and only if for every $j < \omega$ there are q and i , for which $|q^i x|_q \geq (m_{q,i+|j|_q} + 1) \wedge \alpha$ and $m_{q,i+|j|_q} < \alpha$. Therefore the statement may be written as “ $x \in G_\alpha(M)$ ” $\wedge$ (“ x is torsion” $\vee \bigwedge_{j < \omega} \bigvee_{(q,i) \in X_j} “q^i x \in q^{(m_{q,i+|j|_q} + 1) \wedge \alpha} G”$), where $X_j = \{(q, i) \in \mathbb{P} \times \omega : m_{q,i+|j|_q} < \alpha\}$ for $j < \omega$. Since the expressions in quotes have $\leq \lambda$ subformulas and $|X_j| \leq \aleph_0$ for all j , “ $x \in G_\alpha(M^*)^\diamond$ ” has $\leq \lambda$ subformulas.

(1.3) “ $x \in G_\alpha(M_p^*, p)^\diamond$ ” holds if and only if $\|x\|_p \geq_\alpha M_p$ and for every $j < \omega$ there exists an integer $i \geq j$ such that $|p^i x|_p \neq_\alpha m_{p,i}$. By Lemma 4.7 (2),(3) we have $|p^i x|_p \geq_\alpha m_{p,i}$ and $|p^i x|_p \neq_\alpha m_{p,i}$ if and only if $|p^i x|_p \geq (m_{p,i} + 1) \wedge \alpha$ and $m_{p,i} < \alpha$. Therefore the statement is expressible by $(\bigwedge_{i < \omega} “p^i x \in p^{m_{p,i} \wedge \alpha} G”) \wedge \bigwedge_{j < \omega} \bigvee_{i \in X_j} “p^i x \in p^{(m_{p,i} + 1) \wedge \alpha} G”$, where $X_j = \{i < \omega : i \geq j \text{ and } m_{p,i} < \alpha\}$. Again, the formula has $\leq \lambda$ subformulas.

(1.4) By (1.2) and (1.3), “ $x \in G_\alpha(M^*, p)^\diamond$ ” has $\leq \lambda$ subformulas.

(1.5) “ $x \in G_\alpha(M^*, p)$ ” is true if and only if $x \in G_\alpha(M)$ and there are $0 < n < \omega$ and $z_1, \dots, z_n \in G_\alpha(M^*, p)^\diamond$ such that $x = z_1 + \dots + z_n$. Thus “ $x \in G_\alpha(M^*, p)$ ” has $\leq \lambda$ subformulas.

(2) We have $\hat{s}_\alpha(c, p, e, G) \geq m$ if and only if for any $M \in c$ with $M_p \in e$, the following is true: $G \models \exists x_1 \dots \exists x_m [\bigwedge_{i=1}^m (“x_i \in G_\alpha(M)” \wedge “x_i \text{ has either infinite or } p\text{-power order modulo } G_\alpha(M^*, p)”)] \wedge “x_1, \dots, x_m \text{ are independent modulo } G_\alpha(M^*, p)”]$, an expression that has $\leq \lambda$ subformulas. ■

In $L_{\infty\omega}^\delta$, groups with partial decomposition bases are classified using modified Ulm and Warfield invariants.

THEOREM 4.9 (Jacoby–Leistner–Loth–Strüngmann [9], Jacoby–Loth [10]). *Let G and H be groups with partial decomposition bases, and let δ be an ordinal. Suppose*

- (1) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all primes p and $\alpha < \omega\delta$;
- (2) $\hat{w}_{\omega(v+1)}(c, p, e, G) = \hat{w}_{\omega(v+1)}(c, p, e, H)$ for every compatibility class c of Ulm matrices, prime p , equivalence class e of Ulm sequences, and $v < \delta$;
- (3) if $\text{length}(t(G_p)) < \omega\delta$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.

Then $G \equiv_\delta H$. If $\delta = \omega\gamma$ and γ is a limit ordinal, then the converse is also true.

If $\kappa > \omega$, then two $L_{\kappa\omega}$ -equivalent groups with partial decomposition bases have identical modified Warfield invariants up to κ .

THEOREM 4.10. *Let G and H be groups with partial decomposition bases and let κ be a cardinal $> \omega$ such that $G \equiv_{\kappa\omega} H$. If α is an ordinal $< \kappa$, then $\hat{w}_\alpha(c, p, e, G) = \hat{w}_\alpha(c, p, e, H)$ for all c, p, e .*

PROOF. Let $m < \omega$. Both “ $\hat{s}_\alpha(c, p, e, G) = m$ ” and “ $\hat{s}_\alpha(c, p, e, G) = \omega$ ” are in $L_{\kappa\omega}$ as they have $\leq \aleph_0 \cdot |\alpha| < \kappa$ subformulas by Lemma 4.8 (2), thus $\hat{s}_\alpha(c, p, e, G) = \hat{s}_\alpha(c, p, e, H)$. Therefore, $\hat{w}_\alpha(c, p, e, G) = \hat{w}_\alpha(c, p, e, H)$ by Theorem 4.6 (1). ■

COROLLARY 4.11. *Let G and H be groups with partial decomposition bases and let κ be a cardinal $> \omega$. The following conditions are equivalent:*

- (1) $G \equiv_{\kappa\omega} H$;
- (2) (a) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all primes p and ordinals $\alpha < \kappa$;
- (b) $\hat{w}_{\omega(v+1)}(c, p, e, G) = \hat{w}_{\omega(v+1)}(c, p, e, H)$ for every compatibility class c of Ulm matrices, prime p , equivalence class e of Ulm sequences, and ordinal $v < \kappa$;

(c) if $\text{length}(t(G_p)) < \kappa$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$;

(3) $G \equiv_\kappa H$.

PROOF. We have (2) $\Leftrightarrow$ (3) by Theorem 4.9 since $\omega\delta = \delta$ for $\delta = \kappa$. That (1) $\Rightarrow$ (2) follows from Theorems 4.5 and 4.10, and (3) $\Rightarrow$ (1) is immediate by Proposition 3.1 (3). ■

COROLLARY 4.12 (Barwise–Eklof [1]). *Let G and H be p -groups and let κ be a cardinal $> \omega$. The following conditions are equivalent:*

(1) $G \equiv_{\kappa\omega} H$;

(2) (a) $\hat{u}_p(\alpha, G) = \hat{u}_p(\alpha, H)$ for all ordinals $\alpha < \kappa$;

(b) if $\text{length}(G) < \kappa$, then $\hat{u}_p(\infty, G) = \hat{u}_p(\infty, H)$.

(3) $G \equiv_\kappa H$.

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