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GIT stable cubic threefolds and certain fourfolds of $K3^{[2]}$-type

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Abstract. We study the behaviour on some nodal hyperplanes of the isomorphism, described by Boissière–Camere–Sarti, between the moduli space of smooth cubic threefolds and the moduli space of hyperkähler fourfolds of $K3^{[2]}$-type with a non-symplectic automorphism of order three, whose invariant lattice has rank one and is generated by a class of square 6; along those hyperplanes, the automorphism degenerates by jumping to another family. We generalize their result to singular nodal cubic threefolds having one singularity of type A_i, for $i = 2, 3, 4$, providing birational maps between the loci of cubic threefolds where a generic element has an isolated singularity of the types A_i and some moduli spaces of hyperkähler fourfolds of $K3^{[2]}$-type with non-symplectic automorphism of order three belonging to different families. In order to treat the A_2 case, we introduce the notion of Kähler cone sections of K-type, generalizing the definition of K-general polarized hyperkähler manifolds.

1. Introduction

The relation between cubic hypersurfaces and irreducible holomorphic symplectic manifolds has attracted the interest of many experts during the last decades. Various examples of this interest can be found in literature. The first one is the classical result, due to Beauville and Donagi [8], stating that the Fano variety of lines on a cubic fourfold is deformation equivalent to the Hilbert square of a $K3$ surface. There are plenty of other examples, to cite a few [26, 31, 33], and the article of Boissière–Camere–Sarti [10] which is at the core of this article.

In [10], the authors prove the existence of an isomorphism ψ between the moduli space $\mathcal{C}_3^{\mathrm{sm}}$ of smooth cubic threefolds and the moduli space $\mathcal{N}_{\langle 6\rangle}^{\rho,\xi}$ of fourfolds of $K3^{[2]}$-type endowed with a special non-symplectic automorphism of order three. Moreover, they analyze the extension of the period map to singular cubics, given in [2], in order to give a geometric interpretation of the degenerations of the automorphism ψ along either the chordal or the singular nodal hyperplanes, where the cubic threefolds either acquire a nodal singularity or they are related to the chordal cubic. In particular, they find a birational

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morphism between the stable discriminant locus (corresponding to a generic nodal degeneration) $\Delta_3^{A_1}$ and the 9-dimensional moduli space of fourfolds of $K3^{[2]}$-type endowed with a non-symplectic automorphism of order three, having invariant lattice isometric to $U(3) \oplus \langle -2 \rangle$. In the exceptional locus of this birational morphism, there are some interesting subloci, e.g., cubic threefolds having an isolated singularity of type A_i for $i = 2, 3, 4$.

The aim of this article is to provide a similar result also for the closed subloci $\Delta_3^{A_2}$, $\Delta_3^{A_3}, \Delta_3^{A_4} \subset \Delta_3^{A_1}$, where $\Delta_3^{A_i}$ is the closure of the set of cubic threefolds having an isolated singularity of type A_i for $i = 1, \dots, 4$ taken in the moduli space of cubic threefolds. These threefolds are of our interest because Allcock proved in Theorem 1.1 of [1] that a singular cubic threefold is GIT stable if and only if all its singularities are of type A_i for $i = 2, 3, 4$. Therefore, they are in the strata at the boundary of the GIT compactification of the moduli space of smooth cubic threefolds. Different types of compactifications have been studied recently in many articles, e.g., [18, 19, 35, 48], and the already cited [1] and [2].

The main results of this article can be summarized as follows.

Theorem 1.1. *The $\Delta_3^{A_i}$ locus for $i = 1, \dots, 4$ is birational to a $(10 - i)$-dimensional moduli space of fourfolds of $K3^{[2]}$-type with Picard group of the generic member isometric to R_i endowed with a non-symplectic automorphism of order three, having invariant lattice isometric to T_i. These lattices are defined in the following table.*

i	T_i	R_i
1	$U(3) \oplus \langle -2 \rangle$	$U(3) \oplus \langle -2 \rangle$
2	$U \oplus A_2(-2) \oplus \langle -2 \rangle$	$U(3) \oplus \langle -2 \rangle$
3	$U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2 \rangle$	$U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2 \rangle$
4	$U \oplus E_6(-1) \oplus \langle -2 \rangle$	$U \oplus E_6(-1) \oplus \langle -2 \rangle$

In order to prove our results, we will proceed in the following way.

In Sections 2 and 3, we will set the notation and the background needed by introducing respectively basic facts about irreducible holomorphic symplectic manifolds (that will be the main object of this paper) and nodal cubics. Then, as noted in Section 4 of [10], in order to understand geometrically the degenerations of the automorphism along the nodal hyperplanes, one has to consider a moduli space of fourfolds of $K3^{[2]}$-type with an automorphism having an invariant lattice which is bigger than in the smooth case. So, we will start from a generic cubic threefold C in $\Delta_3^{A_i}$, then we will find a $K3$ surface $\hat{\Sigma}$ having the same period of C. The natural choice for $\hat{\Sigma}$ will be the one used in [2] to define the period of a nodal cubic. Indeed, as explained in Section 3, a cubic threefold having one isolated singularity can be seen as the zero locus of an homogeneous polynomial of the form

$$f(x_0 : x_1 : x_2 : x_3 : x_4) = x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4) = 0,$$

where the f_i are homogeneous polynomials of degree i. Then, the surface Σ given by the following intersection:

$$\begin{cases} f_2(x_1, x_2, x_3, x_4) = 0, \\ f_3(x_1, x_2, x_3, x_4) + x_5^3 = 0, \end{cases}$$

is the $K3$ surface with isolated singularities which we need to resolve.

After this, we will find some conditions on the Hilbert square $\hat{\Sigma}^{[2]}$ of $\hat{\Sigma}$ such that $\hat{\Sigma}^{[2]}$ is a "good candidate" for the relation we are looking for. Indeed, our aim is to define a birational map between $\Delta_3^{A_i}$ and some moduli space of fourfolds of $K3^{[2]}$-type. In order to do so, we will define an isomorphism between generic elements via the period map. Therefore a "good candidate" should be endowed with a marking, a non-symplectic automorphism and generic in a particular moduli space. Moreover, we want that the restriction of the period map to this moduli space is an isomorphism onto its image. Indeed, in general the period map for IHS manifolds is not an isomorphism because there may exist birational, non-isomorphic models in the fiber over a period. In order to ensure that this does not happen and avoid problems of separability, we will use, in a first instance, the notion of $K(T)$-generality for a fourfold of $K3^{[2]}$-type, introduced by Camere in Definition 3.10 of [17], recalled in Definition 2.14.

In Section 4, we will find a sufficient condition on the Picard group of $\hat{\Sigma}^{[2]}$ to make it a "good candidate", then we will use this result in Sections 5 and 6 to prove Theorem 1.1 for $i = 1$, 3 and 4. For these cases the result is proven, respectively, in Proposition 5.1, Proposition 6.1 and Proposition 6.2. Note that for $i = 1$ this coincides with the result stated in Proposition 4.6 of [10].

The case $i = 2$, where we start from a generic cubic in $\Delta_3^{A_2}$, behaves very differently respect to the other cases. This fact is somehow surprising at a first glance, but studying it we will find out many differences with the other cases; for example, in order to find $\hat{\Sigma}$, in this case we need to blow up three points which are permuted by a non-symplectic automorphism of order 3 instead of one fixed point as for $i = 3, 4$. Moreover, in the case with $i = 2$, we do not have the $K(T)$-generality property, thus, there are multiple birational non-isomorphic models. Therefore, in Section 7 we will discover that its associated $\hat{\Sigma}^{[2]}$ does not satisfy the sufficient condition in order to be a "good candidate" found in Section 4. So, in order to deal with the $\Delta_3^{A_2}$ locus, we will need to introduce in Section 8 the notion of Kähler cone sections of K-type. This is a technical condition which generalizes the notion of $K(T)$-generality and requires some work to do. It is particularly useful when one considers a subfamily polarized by a lattice bigger than the invariant lattice in the moduli space of IHS manifolds of a given type admitting a non-symplectic automorphism with a given action in cohomology. Finally, we will show in Section 9 that the notion just introduced leads us to the proof of Theorem 1.1 for $i = 2$.

2. Irreducible holomorphic symplectic manifols

In this section, we recall briefly a few facts about irreducible holomorphic symplectic manifolds. For any integer lattice L, we will denote by $L_{\mathbb{C}} := L \otimes \mathbb{C}$ its $\mathbb{C}$-linear extension.

Definition 2.1. An irreducible holomorphic symplectic (from now on, IHS) manifold X is a compact complex Kähler manifold which is simply connected and such that there exists an everywhere nondegenerate holomorphic 2-form ω_X such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega_X$.

The theory of IHS manifolds has been deeply studied over the last forty years. The first examples of IHS manifolds in higher dimensions were provided by Fujiki in dimension 4, and then extended to all even dimensions by Beauville [7]. Our interest will be predominantly in one family of deformations constructed as follows. Let Σ be a $K3$ surface; for our purposes, we can restrict to the case where Σ is projective.

Definition 2.2. We define the *Hilbert scheme of n points on Σ* as the variety that parameterizes zero-dimensional subschemes $(Z, \mathcal{O}_Z)$ of length n (i.e., $\dim \mathcal{O}_Z = n$) on the surface Σ, denoted by $\Sigma^{[n]}$. Oftentimes, we will denote it also by $\mathrm{Hilb}^n(\Sigma)$. We will refer to the Hilbert scheme $\Sigma^{[2]}$ of two points on Σ also as the *Hilbert square of Σ*.

This definition sometimes can bring, from our point of view, to a lack of geometric meaning, in the sense that it is not so clear how to generalize facts about $K3$ surfaces to Hilbert schemes of points on them. In order to deal with this fact, we can give another equivalent definition.

In line with Section 6 of [7], we use the following notations:

- $\Sigma^{(n)}$ stands for the variety of 0-cycles of degree n, defined as the quotient of

$$\Sigma^n := \overbrace{\Sigma \times \cdots \times \Sigma}^{n \text{ times}}$$

  by the symmetric group on n elements. We will refer to it also with $\mathrm{Sym}^n(\Sigma)$.
- We label the natural mapping associating each finite scheme with the corresponding 0-cycle (termed the *Hilbert–Chow morphism*) as $\varepsilon\colon \Sigma^{[n]} \to \Sigma^{(n)}$.
- We denote the locus of cycles in the form $p_1 + \cdots + p_n$ such that there exists $i \neq j$ with $p_i = p_j$, also called *diagonal*, as $D \subset \Sigma^{(n)}$.

Definition 2.3 (Alternative definition). Consider on Σ^n the action γ of the symmetric group S_n on $\mathbb{Z}/n\mathbb{Z}$ which permutes the factors. Then the quotient $\Sigma^{(n)} = (\Sigma^n)/S_n$ is singular on the diagonal. The blow-up of $\Sigma^{(n)}$ along the diagonal is the Hilbert scheme $\Sigma^{[n]}$ of n points on Σ, and the blow-up morphism is identified with the *Hilbert–Chow* morphism.

The other deformation types known at the state of art are those of the generalized Kummer manifolds, also due to Beauville [7], plus other two examples constructed by O'Grady in dimension 6 (see [43]) and 10 (see [42]).

The second cohomology group $H^2(X, \mathbb{Z})$ of an IHS manifold X is torsion-free, and it is equipped with a nondegenerate symmetric bilinear form (known as the Beauville–Bogomolov–Fujiki form), which gives it the structure of an integral lattice (see [24]). This lattice is a deformation invariant and in literature there can be found an explicit description for each example (see, e.g., [22]), therefore we will say that an IHS manifold X is of type L if $H^2(X, \mathbb{Z}) \simeq L$, implying that the deformation type is fixed.

Remark 2.4. There exists no general proof of the fact that if $H^2(X, \mathbb{Z}) \simeq H^2(Y, \mathbb{Z}) \simeq L$, for X and Y IHS manifolds and L a lattice, then $X \sim_{\mathrm{def}} Y$. Nevertheless, this fact is true for the known deformation families.

Consider an IHS manifold X of type L.

Definition 2.5. A *marking* on X is an isometry $\eta\colon H^2(X, \mathbb{Z}) \to L$. A *marked IHS manifold* is a pair (X, η), where X is an IHS manifold together with a marking $\eta\colon H^2(X, \mathbb{Z}) \to L$

on X. Two marked IHS manifolds (X_1, η_1) and (X_2, η_2) are isomorphic if there exists an isomorphism $f: X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^*$.

We define the coarse moduli space of marked IHS manifolds of type L as the set of marked manifolds (X, η), modulo the equivalence relation given by isomorphism of marked manifold. We denote it with $\mathcal{M}_L$.

On this space, one can define a *period map*

$$\begin{aligned}\mathcal{M}_L &\to \Omega_L\\ (X,\eta) &\mapsto \eta(H^{2,0}(X)),\end{aligned}$$

where $\Omega_L := \{\omega \in \mathbb{P}(L_{\mathbb{C}}) \mid (\omega,\omega) = 0,\ (\omega,\bar{\omega}) > 0\}$ is called the *period domain*. This map is a local homeomorphism (see Theorem 5 in [7]) and surjective when restricted to any connected component $\mathcal{M}_L^\circ \subset \mathcal{M}_L$ (see Theorem 9.1 in [29]). Finally, in $O(H^2(X,\mathbb{Z}))$ there exists an important subgroup $\mathrm{Mon}^2(X)$, called the *monodromy group*, consisting of the parallel transport operators of $H^2(X,\mathbb{Z})$ to itself (see Definition 1.1 in [36] for the details). This group is important due to the Hodge theoretic Torelli theorem.

Theorem 2.6 (Hodge theoretic Torelli theorem, Theorem 1.3 in [36]). *Let X and Y be IHS manifolds of the same deformation type. Then*:

- *X and Y are bimeromorphic if and only if there exists a parallel transport operator $f: H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ which is an isomorphism of integral Hodge structures.*
- *Let $f: H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be a parallel transport operator, which is an isomorphism of integral Hodge structures. There exists an isomorphism $\tilde{f}: X \to Y$ inducing f if and only if f maps some Kähler class on X to a Kähler class on Y.*

We define the *positive cone* of an IHS manifold X as the connected component $\mathcal{C}_X$ of the set $\{x \in H^{1,1}(X,\mathbb{R}) \mid x^2 > 0\}$ containing a Kähler class and its Kähler cone $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ as the cone consisting of Kähler classes.

Definition 2.7 (Theorem 6.2 in [3] and Proposition 1.5 in [38]). A monodromy birationally minimal (MBM) class is a rational class $\delta \in H^{1,1}(X) \cap H^2(X,\mathbb{Q})$ of negative square such that there exist a bimeromorphic map $f: X \dashrightarrow Y$ and a monodromy operator $h \in \mathrm{Mon}^2(X)$ such that the hyperplane $\delta^\perp \subset H^{1,1}(X) \cap H^2(X,\mathbb{R})$ contains a face of $h(f^*(\mathcal{K}_Y))$. We denote with $\Delta(X)$ the set of integral MBM classes which are also called *wall divisors*.

These classes are important as they give a wall and chamber structure to the positive cone, and moreover, the following theorem holds.

Theorem 2.8 (Theorem 6.2 in [3]). *Given an IHS manifold X, its Kähler cone $\mathcal{K}_X$ is a connected component of $\mathcal{C}_X \setminus \mathcal{H}_\Delta$, with $\mathcal{H}_\Delta$ defined as*

$$\mathcal{H}_\Delta := \bigcup_{\delta \in \Delta(X)} \delta^\perp \subset H^{1,1}(X) \cap H^2(X,\mathbb{R}).$$

Example 2.9 (Numerical characterization in the $K3^{[2]}$-case). We point out that by Theorem 22 in [27] and Theorem 1.2 in [37], there exists a numerical characterization of the elements in $\Delta(X)$ in the $K3^{[2]}$-type case. An effective class δ is a wall divisor on an IHS fourfold of $K3^{[2]}$-type X if and only if satisfies one of the following:

- $(\delta, \delta) = -2$,
- $(\delta, \delta) = -10$ and it has divisibility 2, i.e., $(\delta, H^2(X, \mathbb{Z})) \in 2\mathbb{Z}$.

Moreover, we recall that by Proposition 1.5 in [37], an effective class δ is monodromy reflective, that is, the reflection by δ is an integral monodromy operator, if and only if $(\delta, \delta) = -2$.

Moreover, there exists another important wall and chamber structure given by the subset $\mathcal{B}\Delta(X) \subset \Delta(X)$ of the *stably prime exceptional divisors* (see Section 6 of [36]). Here the *fundamental exceptional chamber* $\mathcal{F}\mathcal{E}_X$ is the connected component of

$$\mathcal{C}_X \setminus \bigcup_{\delta \in \mathcal{B}\Delta(X)} \delta^{\perp}$$

containing the Kähler cone, and by Proposition 5.6 in [36], the birational Kähler cone $\mathcal{B}\mathcal{K}_X$ satisfies the following: $\mathcal{B}\mathcal{K}_X \subset \mathcal{F}\mathcal{E}_X \subset \overline{\mathcal{B}\mathcal{K}_X}$. Given a marking η and fixing a connected component in $\mathcal{M}_L^{\circ} \subset \mathcal{M}_L$, we can translate these definitions to the lattice, e.g., $\Delta(L)$ will be the set consisting of elements $\eta(\delta)$ with δ a wall divisor for $(X, \eta) \in \mathcal{M}_L^{\circ}$, and analogously for $\mathcal{B}\Delta(L)$. Then we define $\mathcal{C}_L := \{x \in L \otimes \mathbb{R} \mid (x, x) > 0\}$ and, given again $(X, \eta) \in \mathcal{M}_L^{\circ}$, the monodromy group of L is $\mathrm{Mon}^2(L) := \eta \circ \mathrm{Mon}^2(X) \circ \eta^{-1}$. Now we continue our review talking about automorphisms on an IHS manifold X. A natural way to characterize them is to look at their action on the symplectic form ω_X; in fact, an automorphism σ is said *symplectic* if its action is trivial on ω_X, i.e., if $\sigma^*_{\mathbb{C}}(\omega_X) = \omega_X$, and non-symplectic otherwise. If no non-trivial power of a non-symplectic automorphism is symplectic, then the automorphism is called *purely non-symplectic*; this is always the case when the order is prime. In this last case, one can check with a simple computation that if an element is in the invariant lattice $H^2(X, \mathbb{Z})^{\sigma^*}$, then it is orthogonal to the symplectic form ω_X. Therefore, $H^2(X, \mathbb{Z})^{\sigma^*}$ is contained in the Néron–Severi lattice $\mathrm{NS}(X)$. Moreover, by Proposition 6 in [6], an IHS manifold admitting a non-symplectic automorphism is always projective.

2.1. Moduli spaces and period maps

In this section, we review the construction of the moduli spaces of lattice polarized IHS manifolds. These notions were first given and analyzed in [9] for the $K3^{[n]}$-type deformation; in [13], these results were generalized to the other deformation families.

Given an automorphism ρ of a lattice L and an embedding $j: T \hookrightarrow L$ of the invariant lattice $T \simeq L^{\rho}$, we give the following definition.

Definition 2.10. A (ρ, j)-polarization of an IHS manifold X of type L consists of the following data:

(i) a marking η;

(ii) a primitive embedding $\iota: T \hookrightarrow \mathrm{Pic}(X)$ such that $\eta \circ \iota = j$;

(iii) an automorphism $\sigma \in \mathrm{Aut}(X)$ such that $\sigma^*_{|H^{2,0}(X)} = \zeta \cdot \mathrm{id}$ (with ζ a primitive n-th root of unity) and η is a framing for σ, i.e., the following diagram commutes:

$$\begin{array}{ccc} H^2(X,\mathbb{Z}) & \xrightarrow{\sigma^*} & H^2(X,\mathbb{Z}) \\ \big\downarrow \eta & & \big\downarrow \eta \\ L & \xrightarrow{\quad\rho\quad} & L. \end{array}$$

The period domain is in this case (see Section 3.2 of [13])

$$\Omega_T^{\rho,\zeta} := \{x \in \mathbb{P}(S_\zeta) \mid h_S(x,x) > 0\},$$

where we denoted with S the orthogonal complement of T in L and with S_ζ the eigenspace relative to ζ inside $S_{\mathbb{C}}$. Given a sublattice $S \subset L$, we denote as $\Delta(S) := \Delta(L) \cap S$. Then, the period map on a connected component of the moduli space of (ρ, j)-polarized IHS manifolds X of type L is surjective on

$$\Omega_T^{\rho,\zeta} \setminus \bigcup_{\delta \in \Delta(S)} (\delta^\perp \cap \Omega_T^{\rho,\zeta})$$

by Proposition 3.12 in [13], but in order to give a bijective restriction we need to introduce another definition. Choose $K(T)$ as a connected component of

$$\mathcal{C}_T \setminus \bigcup_{\delta \in \Delta(S)} \delta^\perp \subset T_{\mathbb{R}}.$$

Definition 2.11. A (ρ, j)-polarized manifold (X, η) is $K(T)$-general if the cone generated by the invariant Kähler classes $\mathcal{K}_X^{\sigma^*}$ is identified with $K(T)$ under the polarization η, i.e., if $\eta(\mathcal{K}_X^{\sigma^*}) = K(T)$.

Now define the following:

$$\Gamma_T^{\rho,\zeta} := \{\gamma_{|S} \in O(S) \mid \gamma \in O(L),\ \gamma_{|T} = \mathrm{id},\ \gamma \circ \rho = \rho \circ \gamma\}$$

and

$$\Delta'(L) := \{\nu \in \Delta(L) \mid \nu = \nu_T + \nu_S,\ \nu_T \in T_{\mathbb{Q}},\ \nu_S \in S_{\mathbb{Q}},\ \nu_T^2, \nu_S^2 < 0\}.$$

Moreover, we set

$$\mathcal{H}_T := \bigcup_{\delta \in \Delta(S)} \delta^\perp \quad \text{and} \quad \mathcal{H}'_T := \bigcup_{\delta \in \Delta'(S)} \delta^\perp,$$

where again we use the convention that for a sublattice $S \subset L$, we denote $\Delta'(S) := \Delta'(L) \cap S$. Then Theorem 5.6 and Proposition 6.2 in [9] state that the restriction of the period map to the moduli space of $K(T)$-general IHS manifolds of type L

$$\mathcal{M}_{K(T)}^{\rho,\zeta} \to \Omega_T^{\rho,\zeta} \setminus (\mathcal{H}_T \cup \mathcal{H}'_T)$$

is an isomorphism and it induces an isomorphism on the quotients

$$\mathcal{P}_{K(T)}^{\rho,\zeta} : \mathcal{N}_{K(T)}^{\rho,\zeta} := \frac{\mathcal{M}_{K(T)}^{\rho,\zeta}}{\mathrm{Mon}^2(T,\rho)} \to \frac{\Omega_T^{\rho,\zeta} \setminus (\mathcal{H}_T \cup \mathcal{H}'_T)}{\Gamma_T^{\rho,\zeta}}, \tag{2.1}$$

where we denoted with $\mathrm{Mon}^2(T,\rho)$ the group of (ρ, T)-polarized monodromy operators:

$$\mathrm{Mon}^2(T,\rho) := \{g \in \mathrm{Mon}^2(L) \mid g_{|T} = \mathrm{id},\ g \circ \rho = \rho \circ g\}.$$

We now give a more general notion of polarization, introducing the (M, j)-polarization for a lattice M of signature $(1,t)$ with a primitive embedding $j\colon M \subset L$, given in Definition 3.1 of [16], as follows.

Definition 2.12. Given an IHS manifold X of type L, we say that it carries an (M, j)-polarization if it has

(1) a marking $\eta\colon H^2(X,\mathbb{Z}) \to L$;
(2) a primitive embedding $\iota\colon M \hookrightarrow \mathrm{Pic}(X)$ such that $\eta \circ \iota = j$.

Remark 2.13. The (ρ, j)-polarization is a special type of (M, j)-polarization in which $M = T$, the invariant lattice for the automorphism ρ, and on which we ask the existence of an automorphism satisfying item (iii) of Definition 2.10.

Moreover, it is not hard to see that the notion of $K(T)$-generality derives from the notion of $K(M)$-generality for an (M, j)-polarized IHS manifold given in Definition 3.10 of [17]. Indeed, let $\mathcal{C}_M$ be the connected component of the positive cone such that $\iota(\mathcal{C}_M)$ contains the Kähler cone $\mathcal{K}_X$ of an (M, j)-polarized IHS manifold X. We define also $K(M)$ as a connected component (also called *chamber*) of

$$\mathcal{C}_M \setminus \bigcup_{\delta \in \Delta(M)} \delta^{\perp} \subset M \otimes \mathbb{R}.$$

Definition 2.14. An (M, j)-polarized IHS manifold (X,η) is $K(M)$-general if $\iota(K(M)) = \mathcal{K}_X \cap \iota(\mathcal{C}_M)$.

In order to see that this definition is a generalization of the one given above, suppose that (X,η) is a (T, j)-polarized IHS manifold of type L, where T is the invariant lattice of an automorphism $\rho \in O(L)$. Suppose moreover that the (T, j)-polarization extends in a natural way to a (ρ, j)-polarization, i.e., that condition (iii) of Definition 2.10 is satisfied. Then, as T is the invariant sublattice of ρ, both definitions of $\mathcal{C}_T$ coincide. For the same reason, $\mathcal{K}_X^{\sigma^*} = \mathcal{K}_X \cap \iota(\mathcal{C}_T)$. Looking at Definition 2.10, we see that $\mathcal{K}_X \cap \iota(\mathcal{C}_T) = \mathcal{K}_X^{\sigma^*} = \iota(K(T))$ is equivalent to requiring that $\eta(\mathcal{K}_X \cap \iota(\mathcal{C}_T)) = \eta(\mathcal{K}_X^{\sigma^*}) = K(T)$. In [16], the author finds also a period map and its injective restriction with statements similar to the (ρ, j)-polarized case.

3. Cubic threefolds

In this section, we define the objects we are mainly interested in, i.e., the nodal cubic threefold C and the $K3$ surface whose Hilbert square will be our "good candidate", as said in the introduction. The construction highlighted here is standard, a good reference for it is [46].

Let $C \subset \mathbb{P}^4$ be a cubic threefold with an isolated singularity in $p_0 \in \mathbb{P}^4$, which we may assume to be $(1 : 0 : \cdots : 0)$. Thus its equation is of the form

$$f(x_0 : x_1 : x_2 : x_3 : x_4) = x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4) = 0,$$

where the f_i are homogeneous polynomials of degree i in $\mathbb{C}[x_1, x_2, x_3, x_4]$ sufficiently generic in $|\mathcal{O}_{\mathbb{P}^3}(3)|$. Let now $Y \subset \mathbb{P}^5$ be the triple cover of $\mathbb{P}^4$ branched over C, which is then described by the vanishing of the following polynomial:

$$F(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) = x_0 f_2(x_1, x_2, x_3, x_4) + f_3(x_1, x_2, x_3, x_4) + x_5^3,$$

using the same notation as above. This hypersurface has again an isolated singularity of type ADE at the point $p \in \mathbb{P}^5$ of coordinates $(1 : 0 : \ldots : 0)$.

Definition 3.1. We say that a cubic fourfold $Y \subset \mathbb{P}^5$ is *associated* to a cubic threefold $C \subset \mathbb{P}^4$ when $Y \to \mathbb{P}^4$ is a triple cover branched on C.

Let us consider now the hyperplane $H \subset \mathbb{P}^5$ given by $\{x_0 = 0\}$. In this hyperplane, which we will identify with $\mathbb{P}^4$ of coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$, we consider the surface Σ given by the following intersection:

$$\begin{cases} f_2(x_1, x_2, x_3, x_4) = 0, \\ f_3(x_1, x_2, x_3, x_4) + x_5^3 = 0. \end{cases} \tag{3.1}$$

This is the complete intersection of a quadric Q (defined by $f_2 = 0$) and a cubic K (defined by $f_3 + x_5^3 = 0$) in $\mathbb{P}^4$ when f_2 and f_3 are sufficiently generic. This surface is deeply linked to the fourfold Y, and a theorem by Wall [47] links the singularities of Σ to those of the blow-up $\mathrm{Bl}_p(Y)$ of Y at p.

Theorem 3.2 (Theorem 2.1 in [47]). *Let q be a singular point of Σ. If both Q and K have a singularity in q, then the whole line $\overline{pq}$ connecting p and q is singular in Y. If q is not a singularity of both Q and K and is an ADE singularity of type* $\mathbf{T}$ *for Q or K, then one of the followings holds*:

(i) *Q is smooth at q and the cubic fourfold Y has exactly two singularities, namely p and p', on the line $\overline{pq}$ and p' is of type* $\mathbf{T}$,

(ii) *Q is singular at q and the line $\overline{pq}$ meets Y only in p and the blow-up* $\mathrm{Bl}_p(Y)$ *of Y in p has a singularity of type* $\mathbf{T}$ *at q.*

As we asked p to be the only singularity on Y, the only possibility is the one described by item (ii) of Theorem 3.2. Thus the possibilities for the singularities of Σ are exactly those that can be found in the following table, based on Lemma 2.1 of [21].

$\mathbf{T}$	A_1	A_2	$A_{n\geq 3}$	D_4	$D_{n\geq 5}$	E_6	E_7	E_8
$\hat{\mathbf{T}}$	$\emptyset$	$\emptyset$	A_{n-2}	$3A_1$	$A_1 + D_{n-2}$	A_5	D_6	E_7

Here $\hat{T}$ is the type of the singularities that one can find on the exceptional divisor of the blow-up of a variety in a point p that has a singularity of type $\mathbf{T}$. Another interesting observation on Σ can be done following the argument of C. Lehn (see Lemma 3.3 and Theorem 3.6 in [32]) and Hassett (see Lemma 6.3.1 in [26]).

Theorem 3.3. *Let $Y \subset \mathbb{P}^5$ be a cubic fourfold with simple isolated singularities and suppose that it is neither reducible, nor a cone over a cubic threefold. Let $p \in Y$ be a singular point and assume that there exist no planes $\Pi \in Y$ such that $p \in \Pi$. Then the minimal resolution of $\Sigma := F(Y, p)$, the Fano variety of lines in Y passing through p, is a $K3$ surface. Moreover, $F(Y)$, the Fano variety of lines in Y, is birational to* $\mathrm{Hilb}^2(\Sigma)$.

Proof. We write here the explicit morphism, as it will be useful for the next sections.

For the first part, see the discussion above. Moreover, Σ is a (2,3)-complete intersection in $\mathbb{P}^4$ having only isolated ADE singularities, so it admits a minimal model which is a $K3$ surface.

For the rest of the proof, consider $W \subset Y$ the cone over Σ with vertex p. This is a Cartier divisor on Y cut out by the equation $f_2 = 0$. Hence a generic line $l \subset Y$ intersects W in exactly two points counted with multiplicity, thus defining a closed subscheme $\xi_{l\cap W}$ of length two on Σ. Therefore, we can define the birational map

$$\begin{aligned} \varphi^{-1} : F(Y) &\dashrightarrow \mathrm{Hilb}^2(\Sigma) \\ l &\mapsto \xi_{l\cap W}. \end{aligned}$$

The birational inverse of φ is given by the natural map

$$\begin{aligned} \varphi : \mathrm{Hilb}^2(\Sigma) &\to F(Y) \\ \xi &\mapsto l_\xi, \end{aligned}$$

where we define the residual line l_ξ as follows: the intersection between Y and $\langle \xi, p\rangle \simeq \mathbb{P}^2$ consists of a cone over ξ and a line l_ξ. ■

Remark 3.4. Note that φ has no indeterminacy points. Moreover, note that the indeterminacy locus of φ^{-1} is contained in $F(Y, p) \simeq \Sigma$. Indeed, looking at the definition of φ^{-1} in the proof above we can see that it is not defined when a line $l \subset Y$ is contained in W, the cone over Σ with vertex p. This means that either $l \subset \Sigma$ or $p \in l$; the former is impossible, for otherwise the plane $\Pi_{l,p} := \langle l, p\rangle$ would be contained in Y.

Remark 3.5. The condition of not having planes passing through the singular point of a cyclic cubic fourfold is a generic condition. For this reason, from now on we will suppose that this condition is satisfied by every cyclic cubic fourfold appearing also when not explicitly said.

3.1. Moduli space of cubic threefold as a ball quotient

In this section, we recall Allcock–Carlson–Toledo's construction of a period map for the moduli space of GIT stable cubic threefolds as done in [2]. This section is not to be intended as a complete overview of their work, but as a recollection of their results useful to understand the following sections.

We denote by $\mathcal{C}_3^s := |\mathcal{O}_{\mathbb{P}^4}(3)| /\!/ \mathrm{PGL}_5(\mathbb{C})$ the GIT moduli space of $\mathrm{PGL}_5(\mathbb{C})$-stable of stable cubic threefolds, and by $\mathcal{C}_3^{\mathrm{sm}}$ the sublocus of smooth cubic threefolds (this is the open set determined by the nonvanishing of the discriminant as shown in Chapter 5 of [39]). Now, given $C \in \mathcal{C}_3^{\mathrm{sm}}$, the idea outlined by Allcock–Carlson–Toledo is to use the associated cyclic cubic fourfold to induce a period map on $C \in \mathcal{C}_3^{\mathrm{sm}}$. Indeed, for any cubic fourfold Y, we can define a *marking*, i.e., an isometry

$$\eta : H^4_\circ(Y, \mathbb{Z}) \to S(-1) \simeq U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2$$

of the middle primitive cohomology. Moreover, the *period* of the marked pair (Y, η) is just $[\eta(H^{3,1}(Y))] \in \mathbb{P}(S(-1) \otimes \mathbb{C})$. Let σ be the covering automorphism of the associated cubic fourfold Y. Given a marking η for Y, we can define the abstract isometry induced by σ as $\rho := \eta \circ \sigma \circ \eta^{-1}$, and we define a *framing* as the equivalence class of markings $\tilde{\eta}$ compatible with ρ, i.e., $\tilde{\eta} \circ \rho = \rho \circ \tilde{\eta}$, up to action of $\mu_6 := \{\pm \mathrm{id}_{S(-1)}, \pm\rho, \pm\rho^2\}$.

Now, we denote by $\mathcal{F}_3^{\mathrm{sm}}$ the moduli space of framed smooth cubic threefolds and by $\Gamma := \{\gamma \in O(S(-1)) \mid \gamma \circ \rho = \rho \circ \gamma\}$. The latter acts on the former by composition with the framing, i.e., $(C, \eta) \mapsto (C, \gamma \circ \eta)$. As $\mu_6 \subset \Gamma$ acts trivially on $\mathcal{F}_3^{\mathrm{sm}}$, we consider $\mathbb{P}\Gamma := \Gamma/\mu_6$ and $\mathcal{C}_3^{\mathrm{sm}} \simeq \mathcal{F}_3^{\mathrm{sm}}/\mathbb{P}\Gamma$. So, any framing $\eta\colon H^4_\circ(Y, \mathbb{Z}) \to S(-1)$ induces an isomorphism $\eta\colon H^4_\circ(Y, \mathbb{Z}) \to S(-1)_\zeta$, where $S(-1)_\zeta$ is the eigenspace of $S(-1) \otimes \mathbb{C}$ for the eigenvalue ζ of the isometry ρ. Here, by ζ we denote a primitive third root of unity. Note that if we act on a marked cubic fourfold of period $\eta(H^{3,1}(Y))$ with an element of μ_6, we get that the period is multiplied by a non-zero scalar, therefore it remains well defined on the framed cubic threefolds. Therefore we have the following.

Theorem 3.6. *The period map sending a framed cubic threefold* (C, η) *to* $[\eta(H^{3,1}(Y))] \in \mathbb{P}(S(-1)_\zeta)$ *is an isomorphism onto the image equivariant with respect to the action of* $\mathbb{P}\Gamma$. *Moreover, the image is the complement of an hyperplane arrangement* $\mathcal{H}$, *where*

$$\mathcal{H} := \bigcup_{\delta \in S(-1),\, \delta^2 = 2} \delta^\perp.$$

Proof. See Theorem 1.9 in [2]. ∎

In [2], the authors study also an extension of the period map for the GIT stable cubic threefolds (see [1] for the details on GIT stability of cubic threefolds). In particular, in Chapter 6 of [2], the authors show, by studying the limit Hodge structure of the nodal degeneration of a cubic threefold, that the period map can be extended to $\Delta_3^{A_1}$ using the period of its associated $K3$ surface, i.e., the one defined in Section 3.

Theorem 3.7 (Theorem 6.1 in [2]). *The period map above defined can be holomorphically extended to an isomorphism between the GIT stable locus* $\mathcal{C}_3^s$ *and its image, mapping* $\Delta_3^{A_1}$ *to a divisor.*

3.2. Motivating example

Here we introduce the example which will be the core of our analysis. Given, as before, a ramified cyclic covering $Y \to \mathbb{P}^4$ branched along the cubic C, there exists a covering automorphism σ on Y acting by multiplication by a primitive third root of unity ζ. Any marking of the middle primitive cohomology $H^4_\circ(Y, \mathbb{Z}) \to S(-1)$ can be composed with the Abel–Jacobi map in order to induce a marking on the middle primitive cohomology of the Fano variety of lines $F(Y)$:

$$\begin{array}{ccc} H^4_\circ(Y, \mathbb{Z}) & \xrightarrow{\tilde{\eta}} & S(-1) \\ \big\downarrow A & & \big\downarrow -\mathrm{id} \\ H^2_\circ(F(Y), \mathbb{Z}) & \xrightarrow{\eta} & S. \end{array}$$

From [8], we know that S admits a unique, up to isometries, primitive embedding in L, and that this embedding has the lattice generated by an hyperplane section under the Plücker embedding $T \simeq \langle 6 \rangle$ as orthogonal complement. We are interested in giving a relation between cubic threefolds and IHS manifolds of $K3^{[2]}$-type. This is classically done (see, e.g., [2]) by looking at the cubic fourfolds which cover $\mathbb{P}^4$ and branch over a cubic threefold. Indeed, cubic fourfolds with an ordinary double point are described by Hassett in Section 4.2 of [26] as cubic fourfolds with discriminant six and they admit an associated $K3$ surface. Then, in Section 6 of *loc. cit.*, the author gives a nice description of the Plücker divisor on a fourfold of $K3^{[2]}$-type associated to a generic cubic fourfold of discriminant six: it is $\theta = 2\theta_{K3} - 3\varepsilon$, where θ_{K3} is a square six class on the associated $K3$ surface and ε is half of the exceptional class coming from the Hilbert–Chow morphism. In our case, even though we do not ask for genericity, we make the same choice; moreover, as they are all the same up to isometry, we choose $\theta_{K3} = 3u_1 + u_2$. So we have assigned an embedding $j: \langle 6 \rangle \to L$ and $j(\langle 6 \rangle)^{\perp} \simeq S$ as expected. Now using Corollary 1.5.2 in [41] (see also Proposition 4.5 below), we can extend the isometry

$$H^2_{\circ}(F(Y), \mathbb{Z}) \oplus \langle 6 \rangle \xrightarrow{\eta \oplus j} S(-1) \oplus j(\langle 6 \rangle)$$

to a marking $\bar{\eta}: H^2(F(Y), \mathbb{Z}) \to L$. Finally, $\sigma \in \mathrm{Aut}(Y)$ induces on $F(Y)$ an automorphism that will be also denoted with $\sigma \in \mathrm{Aut}(F(Y))$ in order to simplify the notation. Therefore, there exists a natural isometry on L, that is $\rho := \bar{\eta} \circ \sigma^* \circ \bar{\eta}^{-1}$. So, $F(Y)$ is an IHS manifold of $K3^{[2]}$-type and admits a (ρ, j)-polarization with the lattice $\langle 6 \rangle$ playing the role of T in Definition 2.10 and a primitive third root of unity as ζ. The period map $\mathcal{P}^{\rho,\zeta}_{\langle 6 \rangle}$ relative to this space has the period domain isomorphic to a 10-dimensional complex ball

$$\Omega^{\rho,\zeta}_{T} := \{x \in \mathbb{P}(S_{\zeta}) \mid h_S(x,x) > 0\} \simeq \mathbb{C}B^{10}.$$

4. Degeneracy lattices

In this section, we begin to study the degenerations of the automorphism ρ over the nodal hyperplane. In order to do so, we provide a general strategy which then will be used in each considered case.

We want to focus on the case described in Section 3.2. Take a period ω in $\mathcal{H}_{\Delta}$ and a point $(X, \eta) \in \mathcal{P}^{-1}_{\langle 6 \rangle}(\omega)$ in the fiber of the period map of $\langle 6 \rangle$-polarized IHS manifolds of $K3^{[2]}$-type. Remember that this is, by Theorem 3.7, the period of a nodal cubic threefold.

Remark 4.1. As explained in [10], the isometry $\rho \in O(L)$ is not represented by any automorphism of X. In fact, if ρ was represented by an automorphism of X, this one would be automatically non-symplectic. Consider now $l \in \mathrm{NS}(X)$ an ample class (it always exists as X is projective), and then consider $l + \rho^* l + (\rho^*)^2 l$, which is still ample and invariant. Therefore it is a multiple of the generator of the rank one invariant lattice; we call θ the primitive ample invariant class. Since $\delta_i \in S$, the divisor $\eta^{-1}(\delta_i)$ is orthogonal to θ, yielding a contradiction as $(\eta^{-1}(\delta_i), \theta) > 0$, by the ampleness of θ.

So, the main issue in finding an automorphism representing ρ is the fact that there exist some MBM classes in the Néron–Severi lattice which are orthogonal to the invariant lattice of the automorphism ρ. In order to deal with this, we define the following.

Definition 4.2. The *degeneracy lattice* of (X, η) is the sublattice of S generated by those MBM classes $\delta_i \in S$ which are orthogonal to ω.

This lattice is ρ-invariant and orthogonal to $j(\langle 6\rangle)$. So, in general, the degeneracy lattice will be $R_0 := \mathrm{Span}(\delta_1, \dots, \delta_n, \rho(\delta_1), \dots, \rho(\delta_n))$.

Remark 4.3. For each $i \in \{1, \dots, n\}$, the sublattice $\mathrm{Span}(\delta_i, \rho(\delta_i)) =: R_{\delta_i} \subset R_0$ is just the degeneracy lattice of a polarized IHS manifold (X, η) generic in $\mathcal{H}_\Delta$ and, with a simple computation, $R_{\delta_i} \simeq A_2(-1)$. For the sake of completeness, let us show this computation. By definition, $R_{\delta_i} \simeq \langle \delta_i, \rho(\delta_i)\rangle$, with $\delta_i^2 = (\rho(\delta_i))^2 = -2$. Moreover,

$$(\delta_i, \rho(\delta_i)) = (\rho(\delta_i), \rho^2(\delta_i)) = (\rho(\delta_i), -\delta_i - \rho(\delta_i)) = -(\rho(\delta_i), \delta_i) + 2,$$

which implies $(\delta_i, \rho(\delta_i)) = 1$.

Recall that in Section 3.2 we provided an embedding $j : \langle 6\rangle \to L$ such that $j(\langle 6\rangle)^\perp \simeq S$. With the help of j, we can induce an embedding of $\langle 6\rangle \oplus R_0$ which in general will not be primitive. So, we define $T_0 := \overline{\langle 6\rangle \oplus R_0}$ as the saturation of $\langle 6\rangle \oplus R_0$ in L. Note that $T_0 \hookrightarrow \mathrm{Pic}(X)$ by definition of degeneracy lattice, and that the equality will hold for a generic element, i.e., a $\langle 6\rangle$-polarized IHS fourfold of $K3^{[2]}$-type which has a generic period orthogonal to a fixed number of MBM classes $\delta_1, \dots, \delta_n$.

The strategy we will use in the following sections is similar to the one outlined in [10] and in Section 11 of [23]. We outline it here. First, we will prove the following claim for each family having generically the nodal period considered in each case.

Claim 4.4. *The Picard group of the Hilbert square $\hat{\Sigma}^{[2]}$ of the minimal resolution of the surface defined in Section 3 is generated by the square six polarization and $2i$ classes of square (-2) orthogonal to it, with $i = 1, \dots, 4$. Therefore it is generic in the above sense.*

Then we want to look for an isometry on L related to ρ which has a bigger invariant lattice. In particular, denoting with $S_0 := T_0^\perp$ in L, we look at

$$\mathrm{id}_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0).$$

If it can be lifted to an isometry $\rho_0 \in O(L)$, then as $T_0 \simeq \mathrm{Pic}(\hat{\Sigma}^{[2]})$, we can find an IHS manifold of $K3^{[2]}$-type with the same period (by definition of period of a nodal cubic given in [2]) generic in the space of the IHS manifolds of $K3^{[2]}$-type which are (ρ_0, j)-polarized. It is important to remark that by definition of the (ρ, j)-polarization, as the isometry $\rho \in O(L)$ comes from a non-symplectic automorphism σ on X, it can be restricted to an isometry $\rho_{S_0} \in O(S_0)$. Indeed, as $\omega \in \delta^\perp$, we deduce that $\zeta_3 \cdot (\rho(\delta), \omega) = (\rho(\delta), \rho(\omega)) = (\delta, \omega) = 0$, thus the isometry ρ can be restricted to an isometry of both the Picard lattice and its orthogonal complement. In order to lift the isometry, we will apply the following result.

Proposition 4.5 (Corollary 1.5.2 in [41]). *Let L be a finite index overlattice of $S \oplus T \subset L$ determined by the pair (H, γ), where $H < D_S$ is a subgroup and $\gamma\colon H \to D_T$ is a group monomorphism. Moreover, let $\rho_S \in O(S)$ and $\rho_T \in O(T)$ be two isometries such that the induced isometry on the discriminant $\rho_S^* \in O(D_S)$ restricts to an isometry of H. Then the isometry $\rho_S \oplus \rho_T \in O(S) \oplus O(T)$ lifts to an isometry $\rho \in O(L)$ of L if and only if $\rho_S|_H$ is conjugate to the induced isometry ρ_T^* via γ, or equivalently, $\gamma \circ \rho_S^*|_H = \rho_T^* \circ \gamma$.*

Proof. The first implication is obvious as $\rho_S = \rho|_S$ and $\rho_T = \rho|_T$. The second implication is done just by noting that in the diagram

$$\begin{array}{ccc} H & \xrightarrow{\rho_S^*} & H \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ \gamma(H) & \xrightarrow{\rho_T^*} & \gamma(H), \end{array}$$

the pair $(\rho_S^*(H) = H, \rho_T^* \circ \gamma \circ (\rho_S^*)^{-1} = \gamma)$ determines an overlattice isometric to L and we call this isometry ρ. ∎

Recall now that the Picard group of an IHS fourfold of $K3^{[2]}$-type can be written as

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle \varepsilon \rangle \simeq T_{K3} \oplus \langle -2 \rangle,$$

where ε is the (-2)-class given by half of the exceptional divisor introduced by the Hilbert–Chow morphism and $T_{K3} \subset L_{K3}$ is a sublattice of the $K3$ lattice $L_{K3} \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Thus we consider the isometry $\tilde{\rho_0} := \rho_0|_{T_{K3}}$. Then the following proposition proves that there exists an automorphism σ of the $K3$ surface $\hat{\Sigma}$ whose action on cohomology is conjugate to $\tilde{\rho_0}$.

Proposition 4.6. *Let Σ be a $K3$ surface and let $\rho \in O(L_{K3})$. Suppose that $\rho(\omega) = \lambda\omega$, where $\omega \in H^2(\Sigma, \mathbb{C})$ is the period of Σ and $1 \neq \lambda \in \mathbb{C}^*$. Then if $\mathrm{Pic}(\Sigma)$ is fixed by ρ, there exist $\sigma \in \mathrm{Aut}(\Sigma)$ and an element w in the Weyl subgroup of Σ such that $w\rho w^{-1} = \sigma$.*

This proposition is an immediate corollary of the following theorem by Namikawa.

Theorem 4.7 (Theorem 3.10 in [40]). *Let Σ be a $K3$ surface and G a finite subgroup of the group of isometries in $\Lambda = H^2(\Sigma, \mathbb{Z})$. Denote by ω the period of Σ, by Λ^G the sublattice of elements in Λ fixed by G, and set $S_{G,\Sigma} = (\Lambda^G)^\perp \cap \{\mathbb{C}\omega\}^\perp = (\Lambda^G)^\perp$ in $H^{1,1}_{\mathbb{Z}}(\Sigma)$. Then there exists an element t in the Weyl subgroup of Σ, $W(\Sigma)$, such that $tGt^{-1} \subset \mathrm{Aut}(\Sigma)$ if and only if*

- (i) $\mathbb{C}\omega$ *is G-invariant*;
- (ii) $S_{G,\Sigma}$ *contains no element of length -2*;
- (iii) *if $\omega \in \Lambda^G$, then $S_{G,\Sigma}$ is either 0 or nondegenerate and negative definite*;
- (iii') *if $\omega \notin \Lambda^G$, then Λ^G contains an element a with $(a, a) > 0$.*

Therefore, if we consider the natural automorphism on $\Sigma^{[2]}$ induced by σ, then we see that the former variety is equipped with an automorphism whose action in cohomology is conjugate to ρ_0.

Theorem 4.8. *Let C be a nodal cubic and $\hat{\Sigma}$, θ as in Section 3. Let $(\hat{\Sigma}^{[2]}, \eta)$ the Hilbert square of $(\hat{\Sigma}, \tilde{\eta})$ and suppose that $\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq j(\theta) \oplus W \simeq \langle 6 \rangle \oplus R_0 = T_0$. If the action induced by ρ on the discriminant group D_W is trivial, then the isometry $\mathrm{id}_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$ lifts to an isometry $\rho_0 \in O(L)$. Finally, if we define $K(T_0)$ as the chamber containing a Kähler class of $\Sigma^{[2]}$, then the latter is $K(T_0)$-general and defines a point in $\mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$.*

Proof. In order to prove the statement, it is sufficient to prove that the isometry $\mathrm{id}_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$ lifts to an isometry $\rho_0 \in O(L)$. In order to lift it, the condition stated in Proposition 4.5 is that the action induced by ρ on the discriminant group D_{S_0} is trivial. This discriminant group is isomorphic to a quotient of $D_W \oplus D_{S(-1)}$, but the action of ρ is trivial on D_W by hypothesis, therefore it is enough to check the triviality on $D_{S(-1)}$. Again $D_{S(-1)}$ is isomorphic to a quotient of $D_L \oplus D_{\langle 6\rangle}$, the action induced by ρ is trivial on the first factor because it is an order three isometry on $D_L \simeq \mathbb{Z}/2\mathbb{Z}$ and on the second by construction. The $K(T_0)$-generality of $\hat{\Sigma}^{[2]}$ follows from Lemma 5.2 in [9]. ■

Remark 4.9. In this section, we provided also an automorphism on $\hat{\Sigma}$ whose action on cohomology is conjugate to $\tilde{\rho_0}$. Therefore, we proved that, under the same hypotheses of Theorem 4.8, $(\hat{\Sigma}, \tilde{\eta}) \in \mathcal{K}^{\tilde{\rho_0},\zeta}_{T_{K3}}$, the moduli space of $(\tilde{\rho_0}, \tilde{j})$-polarized $K3$ surfaces.

In the next sections, we analyze case by case what happens. In order to perform this analysis, we define $\Delta_3^{A_i}$, for $i = 1, \dots, 4$, as the biggest sub-locus of the space of stable cubic threefolds where the generic element is a cubic with an isolated singularity of type A_i.

Remark 4.10. One can check (an explicit computation is done in Appendix B) that the dimension of the $\Delta_3^{A_i}$ locus is $10 - i$.

Notation. In the following sections, every time an IHS manifold endowed with a (ρ, j)-polarization will appear, we will use the following notations. The definition of the isometry playing the role of ρ will be clear in all cases. The embedding j will be constructed in the same way each time as follows. We will exhibit a square six class θ which has an embedding in L as in Section 2. This embedding will induce an embedding of the whole fixed lattice in L playing the role of j. Moreover, we will choose for any Hilbert square over a $K3$ surface the natural marking induced by a marking on a $K3$ surface, i.e., fixing a marking $\tilde{\eta}$ on a $K3$ surface $\hat{\Sigma}$, the natural marking η on its Hilbert square $\hat{\Sigma}^{[2]}$ induced by $\tilde{\eta}$ is the lifting of $\tilde{\eta} \oplus \mathrm{id}_{\langle -2\rangle}$ to the whole second cohomology group with integer coefficients. Then we will take the embedding ι in a way compatible with j, i.e., $\iota(\langle 6\rangle) = 2\theta_{K3} - 3\varepsilon$, with θ_{K3} fixed, and this embedding will induce also an embedding on its orthogonal. In order to lighten the notation and the exposition, we will often omit the markings where their presence is obvious.

5. Singularity of type A_1

In this section, we recall the results discussed in Section 4.3 of [10], where the authors provide a birationality result for the nodal hyperplane. In order to see the analogy with the loci we are interested in, we briefly review it giving a proof that fits our framework.

With the notation of Section 3, let p_0 be an isolated singularity of type A_1 for $C \subset \mathbb{P}^4$. This implies, for reasons of corank of the singularity (see Section 2 of [1] or directly do the computation), that f_2 is a rank 4 quadratic form and that $Y \subset \mathbb{P}^5$ has an isolated singularity of type A_2. Moreover, by genericity, we can assume that the surface Σ given by the following equations:

$$
(5.1)\qquad \begin{cases} f_2(x_1,x_2,x_3,x_4)=0,\\ f_3(x_1,x_2,x_3,x_4)+x_5^3=0, \end{cases}
$$

is a smooth $K3$ surface. Here $x_1,\dots,x_5$ are the homogeneous coordinates on the hyperplane $H\subset\mathbb{P}^5$ given by $\{x_0=0\}$. The covering automorphism σ on Y induces an automorphism τ of Σ. Explicitly, τ is given by $x_5\mapsto\zeta_3\cdot x_5$ and the identity on the other coordinates. The fixed locus of τ is a curve of genus 4. In [4], the authors show that the generic case (i.e., the one which we are considering) has $\mathrm{Pic}(\Sigma)\simeq U(3)$. Therefore, for the Hilbert square of Σ, it holds

$$
\mathrm{Pic}(\Sigma^{[2]})\simeq\mathrm{Pic}(\Sigma)\oplus\langle-2\rangle\simeq U(3)\oplus\langle-2\rangle\simeq\langle 6\rangle\oplus A_2(-1):=T_0.
$$

Note that the last isometry is given by

$$
U(3)\oplus\langle-2\rangle:=\langle u_1,u_2\rangle\oplus\langle\varepsilon\rangle\simeq\langle 2(u_1+u_2)-3\varepsilon\rangle\oplus\langle u_1-\varepsilon,\varepsilon-u_2\rangle\simeq j(\theta)\oplus W.
$$

This proves Claim 4.4. Moreover, τ induces an order three isometry $\rho\in O(L)$ with $j(\theta)$ as fixed sublattice. So, ρ restricts to an order three isometry without fixed points of $W\simeq A_2(-1)$. There exists only one isometry of $A_2(-1)$ of order three without fixed points modulo conjugation, and its action on the discriminant $D_{A_2(-1)}$ is trivial. Therefore, applying Theorem 4.8, there exists $\rho_0\in O(L)$ lifting $\mathrm{id}_{|T_0}\oplus\rho_{|S_0}$ and $(\Sigma^{[2]},\eta,\tau)\in\mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$.

Proposition 5.1 (Proposition 4.6 in [10]). *Let $R_0=\mathrm{Span}(\delta_1,\rho(\delta_1))\simeq A_2(-1)$ be the degeneracy lattice relative to a period ω of a generic cubic threefold having a single singularity of type A_1. Then the A_1 locus $\Delta_3^{A_1}$ is birational to the moduli space $\mathcal{N}^{\rho_0,\zeta}_{K(T_0)}$.*

Proof. The idea is to use the same structure of the proof of Proposition 4.6 in [10] within our theoretical approach. First we note that the extension of the period map $\mathcal{P}^3\colon\mathcal{C}_3^{\mathrm{sm}}\to[\mathbb{B}^{10}\setminus(\mathcal{H}_n\cup\mathcal{H}_c)]/\mathbb{P}\Gamma$ to the nodal locus is done (see Section 6 of [2]) by defining its period as the period of its associated $K3$ surface. In our case, the generic A_1 nodal cubic has the period

$$
\mathcal{P}^3(C):=\mathcal{P}^{\tilde\rho_0,\zeta}_{U(3)}((\Sigma,\eta)).
$$

The latter period map is defined by taking $\mathcal{K}^{\tilde\rho_0,\zeta}_{U(3)}$ as the moduli space of lattice polarized $K3$ surfaces with a non-symplectic automorphism of order three whose action on cohomology is conjugate to $\tilde\rho_0$. Following [23], this space comes with a period map

$$
\mathcal{P}^{\tilde\rho_0,\zeta}_{U(3)}\ :\ \mathcal{K}^{\tilde\rho_0,\zeta}_{U(3)}\to\Omega^{\tilde\rho_0,\zeta}_{U(3)}:=\{x\in\mathbb{P}((S_0)_\zeta)\mid h_{S_0}(x,x)>0\},
$$

which induces a bijection

$$
\mathcal{P}^{\tilde\rho_0,\zeta}_{U(3)}\ :\ \mathcal{K}^{\tilde\rho_0,\zeta}_{U(3)}\to\frac{\Omega^{\tilde\rho_0,\zeta}_{U(3)}\setminus\mathcal{H}_{U(3)}}{\Gamma^{\tilde\rho_0,\zeta}_{U(3)}},
$$

where

$$
\mathcal{H}_{U(3)}:=\bigcup_{\mu\in S_0,\ \mu^2=-2}\mu^\perp\cap\Omega^{\tilde\rho_0,\zeta}_{U(3)}\quad\text{and}\quad\Gamma^{\tilde\rho_0,\zeta}_{U(3)}:=\{\gamma\in O(L_{K3})\mid\gamma\circ\tilde\rho_0=\tilde\rho_0\circ\gamma\}.
$$

By definition, we find the following equality:

$$\frac{\Omega_{U(3)}^{\tilde{\rho}_0,\zeta}}{\Gamma_{U(3)}^{\tilde{\rho}_0,\zeta}} = \frac{\Omega_S^{\rho_0,\zeta} \cap \delta_1^{\perp}}{\Gamma_S^{\rho_0,\zeta}}.$$

As proven above, $(\Sigma^{[2]}, \eta, \tau)$ defines a point in $\mathcal{M}_{K(T_0)}^{\rho_0,\zeta}$. The period map in this space, following equation (2.1), descends to a bijection

$$\mathcal{P}_{T_0}^{\rho_0,\zeta} : \mathcal{N}_{K(T_0)}^{\rho_0,\zeta} = \frac{\mathcal{M}_{K(T_0)}^{\rho_0,\zeta}}{\mathrm{Mon}^2(T_0,\rho_0)} \to \frac{\Omega_{T_0}^{\rho_0,\zeta} \setminus (\mathcal{H}_{T_0} \cup \mathcal{H}'_{T_0})}{\Gamma_{T_0}^{\rho_0,\zeta}}.$$

By their definitions, we see that $\Omega_{T_0}^{\rho_0,\zeta} = \Omega_{U(3)}^{\tilde{\rho}_0,\zeta}$ and $\Gamma_{T_0}^{\rho_0,\zeta} = \Gamma_{U(3)}^{\tilde{\rho}_0,\zeta}$. Moreover, as $S_0 \subset L_{K3}$ is a sublattice of the unimodular $K3$ lattice, the following holds:

$$\mathcal{H}_{T_0} := \bigcup_{\mu \in S_0,\ \mu^2=-2} \mu^{\perp} \cap \Omega_{T_0}^{\rho_0,\zeta} = \mathcal{H}_{U(3)}.$$

Now it is straight-forward to see that $[\Omega_{T_0}^{\rho_0,\zeta} \setminus (\mathcal{H}_{T_0} \cup \mathcal{H}'_{T_0})]/\Gamma_{T_0}^{\rho_0,\zeta}$ is birational to $[\Omega_{U(3)}^{\tilde{\rho}_0,\zeta} \setminus \mathcal{H}_{U(3)}]/\Gamma_{U(3)}^{\tilde{\rho}_0,\zeta}$. So in order to conclude the proof, it is enough to show that $\Delta_3^{A_1}$ is birational to $\mathcal{K}_{U(3)}^{\tilde{\rho}_0,\zeta}$, as the claim of the proposition would follow through a composition of birational morphisms. But this is true as in [2] the authors show that the discriminant locus maps isomorphically to its image, the nodal hyperplane arrangement, through the period map, therefore a generic point in $\Delta_3^{A_1}$ is mapped isomorphically to the period of a generic $K3$ surface in $\mathcal{K}_{U(3)}^{\tilde{\rho}_0,\zeta}$, and by [23], this is an isomorphism. ■

6. Singularity of type A_3 and A_4

In this section, we prove Theorem 1.1 for $\Delta_3^{A_3}$ and $\Delta_3^{A_4}$. The cases treated in this section are very similar to the A_1 case; we will keep the same notation.

Let p_0 be a singularity of type A_k with $k = 3, 4$ for $C \subset \mathbb{P}^4$. Remember that if $k \geq 2$, the singularities have corank 1, therefore this translates in f_2 being a rank 3 quadratic form.

6.1. Singularity of type A_3

If C has an singularity of type A_3, then with a standard computation (just add a cube in the new variable), one shows that Y has an singularity of type E_6, thus, this time Σ has one isolated singularity of type A_5, call it $p \in \Sigma$. Let us consider $\hat{\Sigma}$, the minimal resolution of Σ which is a sequence of blow-ups of Σ at p. This is a $K3$ surface. The covering automorphism σ on Y descends to an automorphism τ on Σ. So, the singular point p is a fixed point for the automorphism τ. As the locus we are blowing up each time is fixed by the automorphism, there exists a unique lift $\hat{\tau}$ such that $\hat{\tau}$ is an automorphism of $\hat{\Sigma}$ and commutes with the resolution map β. The fixed locus for $\hat{\tau}$ consists of two curves and two points. Stegmann, in her PhD thesis [45], gives a detailed description of surfaces which are complete (2,3)-intersection in $\mathbb{P}^4$, and from Proposition 6.3.10 in [45], we find that the Picard lattice of $\hat{\Sigma}$ is just the span of $\langle C_1, E_1, E_2, E_3, E_4, E_5\rangle \subset \mathrm{Pic}(\hat{\Sigma})$ with the following Gram matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0\\ 0 & -2 & 1 & 0 & 0 & 0\\ 0 & 1 & -2 & 1 & 0 & 0\\ 1 & 0 & 1 & -2 & 1 & 0\\ 0 & 0 & 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Geometrically, the divisors $E_1, \dots, E_5$ are the (-2)-curves coming from the successive blow-ups of Σ. By genericity, the Picard lattice has rank 6 (see Appendix B for calculations), therefore

$$\langle C_1, E_1, E_2, E_3, E_4, E_5\rangle = \mathrm{Pic}(\hat{\Sigma}).$$

As the fixed locus of $\hat{\tau}$ consists of two curves and two isolated points, we deduce, using Table 2 in [4], that the Picard lattice admits an embedding of the invariant sublattice $U \oplus A_2(-1)^{\oplus 2}$. Moreover, this is an isomorphism of lattices. In fact,

$$\langle C_1 + E_3, C_1\rangle \oplus \langle E_1, E_2 - C_1\rangle \oplus \langle E_4 - C_1, E_5\rangle \simeq U \oplus A_2(-1)^{\oplus 2}$$

exhibits $U \oplus A_2(-1)^{\oplus 2}$ as a primitive sublattice of the Picard lattice. Now we can compute the determinants and notice that they are both 9, so the lattices are isomorphic. Moreover, the automorphism $\hat{\tau}$ is clearly non-symplectic (as it is the multiplication by a third root of unity ζ_3 of the last coordinate). Now we investigate some properties of $\hat{\Sigma}^{[2]}$. Its Picard lattice is

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \simeq U \oplus A_2(-1)^{\oplus 2} \oplus \langle -2\rangle \simeq E_6(-1) \oplus \langle 6\rangle := T_0,$$

where the last isomorphism is given by

$$\begin{aligned} &\mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \\ &\quad \simeq \langle E_1, E_2, E_3, E_4, E_5, C_1 - \varepsilon\rangle \oplus \langle 2(2C_1 + E_1 + 2E_2 + 3E_3 + 2E_4 + E_5) - 3\varepsilon\rangle, \end{aligned}$$

where ε is the (-2)-class which is half of the divisor introduced by the Hilbert–Chow morphism. It is useful to describe also the trascendental lattice $\mathrm{Tr}(\hat{\Sigma}^{[2]})$ of $\hat{\Sigma}^{[2]}$:

$$\mathrm{Tr}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Tr}(\hat{\Sigma}) \simeq U^{\oplus 2} \oplus E_8(-1) \oplus A_2(-1)^{\oplus 2}.$$

First we note that

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq E_6(-1) \oplus \langle 6\rangle \simeq W \oplus j(\theta),$$

with $\theta = 2\theta_{K3} - 3\varepsilon$, and again we assume $\theta_{K3} = 3u_1 + u_2$ with an isometry, so with another choice of the marking. This proves Claim 4.4. Again we can associate to $\hat{\tau}$ an order three isometry $\rho \in O(L)$ with $j(\theta)$ as fixed sublattice with Proposition 4.6. Therefore, it restricts to an order three isometry without fixed points on $W \simeq E_6(-1)$. As said in Example A.4, there exists only one, up to conjugation and sign, order three isometry without fixed points on E_6. We can write it explicitly. Let $E_6 = \langle e_1, \dots, e_6 \rangle$, with the standard Bourbaki numeration

$$\begin{array}{ccccccccc} e_1 & \text{——} & e_3 & \text{——} & e_4 & \text{——} & e_5 & \text{——} & e_6 \\ & & & & | & & & & \\ & & & & e_2. & & & & \end{array}$$

Then ρ is given by

$$\begin{aligned} e_1 &\mapsto -e_1 - e_3 - e_4 - e_5 - e_6, \\ e_2 &\mapsto e_3 + e_4 + e_5, \\ e_3 &\mapsto -e_2 - e_3 - e_4, \\ e_4 &\mapsto -e_4 - e_5, \\ e_5 &\mapsto e_4, \\ e_6 &\mapsto e_1 + e_2 + e_3 + e_4 + e_5. \end{aligned}$$

Now,

$$\mathbb{Z}/3\mathbb{Z} \simeq D_W = \left\langle -\frac{4}{3} e_1 - e_2 - \frac{5}{3} e_3 - 2e_4 - \frac{4}{3} e_5 - \frac{2}{3} e_6 \right\rangle,$$

and therefore, after substituting the expression of ρ on the generator of D_W, we note that the action induced by ρ is trivial on D_W. Finally, we can apply Theorem 4.8 to deduce also in this case that there exists $\rho_0 \in O(L)$ lifting $\mathrm{id}_{|T_0} \oplus \rho_{|S_0}$ and $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau}) \in \mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$.

Proposition 6.1. *Let $R_0 = \mathrm{Span}(\delta_1, \delta_2, \delta_3, \rho(\delta_1), \rho(\delta_2), \rho(\delta_3)) \simeq E_6(-1)$ be the degeneration lattice relative to a period ω of a generic cubic threefold having one singularity of type A_3. Then the A_3 locus $\Delta_3^{A_3}$ is birational to the moduli space $\mathcal{N}^{\rho_0,\zeta}_{K(T_0)}$.*

Proof. It is easy to see that one can adjust the same proof of Proposition 5.1 to this case and everything works. ∎

6.2. Singularity of type A_4

This section is analogous to the previous section, so we omit the details. If C has now a singularity of type A_4, then Y has a singularity of type E_8 and Σ an isolated singularity of type E_7, which we will call $p \in \Sigma$. Stegmann's work, see Proposition 6.3.12 in [45], and the computations in Appendix B, provide us once again the description of its Picard lattice $\mathrm{Pic}(\hat{\Sigma}) = \langle C_1, E_1, \dots, E_7 \rangle$ with the following Gram matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}.$$

The fixed locus of the automorphism $\hat{\tau}$, induced by the cover automorphism, is just the union of three isolated points and three curves. Therefore, we deduce from [4] that there exists an embedding of $U \oplus E_6(-1)$ in the Picard lattice which turns out to be an isometry. Indeed, it is immediate to see that

$$\mathrm{Pic}(\hat{\Sigma}) \simeq \langle C_1, C_1 + E_1\rangle \oplus \langle E_2 - C, E_3, \dots, E_7\rangle \simeq U \oplus E_6(-1),$$

and they both have determinant 3. Now $\hat{\Sigma}^{[2]}$ has Picard lattice

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \simeq U \oplus E_6(-1) \oplus \langle -2\rangle \simeq E_8(-1) \oplus \langle 6\rangle,$$

where the last isometry is given by

$$\begin{aligned}\mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2\rangle \simeq{}& \langle E_1, \dots, E_7, C_1 - \varepsilon\rangle \oplus \\ &\oplus \langle 2(-2C_1 + 3E_1 + 4E_2 + 5E_3 + 6E_4 + 4E_5 + 2E_6 + 3E_7) - 3\varepsilon\rangle,\end{aligned}$$

and ε is, as usual, the (-2)-class given by half of the divisor introduced by the Hilbert–Chow morphism. Once again, we note that with Proposition 4.6 we have the existence of an order 3 automorphism on $\hat{\Sigma}$ conjugate to $\tilde{\rho} = \rho_{|T_{K3}}$. This automorphism is, modulo conjugation, uniquely determined by its invariant lattice by [4], therefore it is $\hat{\tau}$. As in the previous section, we note that $\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq j(\theta) \oplus W$, proving Claim 4.4. Moreover, $W \simeq E_8(-1)$, which has trivial discriminant, therefore we can apply Theorem 4.8 and check that $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau}) \in \mathcal{M}^{\rho_0,\zeta}_{K(T_0)}$. Then we can deduce the following.

Proposition 6.2. *Let $R_0 = \mathrm{Span}(\delta_1, \dots, \delta_4, \rho(\delta_1), \dots, \rho(\delta_4)) \simeq E_8(-1)$ be the degeneracy lattice relative to a period ω of a generic cubic threefold having one singularity of type A_4. Then the A_4 locus $\Delta_3^{A_4}$ is birational to the moduli space $\mathcal{N}^{\rho_0,\zeta}_{K(T_0)}$.*

7. Singularity of type A_2

In this section, we begin the study of $\Delta_3^{A_2}$. As we will see in this section, this case is quite different from the other cases already treated; nevertheless, we will use the same notation in order to stress the similarities. Indeed, in this case the associated $K3$ has three isolated singularities instead of one.

With a standard computation (using, e.g., the recognition principle stated in [15]), we can see that when C has an isolated singularity of type A_2, Y has an isolated singularity of type D_4. The family of cubic threefolds having a singularity of type A_2 corresponds to the generic family of complete $(2,3)$-intersections in $\mathbb{P}^4$ where the quadratic part f_2 has rank 3 (a proof of this fact can be found in Appendix B). Thus, the singular locus of the quadric hypersurface defined by $f_2 = 0$ is a line $(l_1(t) : \cdots : l_4(t) : s)$; this line intersects the cubic hypersurface in three points p_1, p_2 and p_3 for f_3 sufficiently generic. Hence, Σ has exactly three singular points. Let σ be a covering automorphism on Y; it restricts to an order three automorphism τ on Σ, namely the one which maps $x_5 \mapsto \zeta_3 \cdot x_5$, where ζ_3 is just a primitive third root of unity. As we are interested in the minimal resolution $\hat{\Sigma}$ of Σ (which is just the blow-up of Σ at each p_i), we want to prove that the automorphism τ of Σ lifts to an automorphism $\hat{\tau}$ of $\hat{\Sigma}$, and to find its fixed locus.

Proposition 7.1. *With the above notation, there exists a unique lift* $\hat{\tau}$ *such that* $\hat{\tau}$ *is an automorphism of* $\hat{\Sigma}$ *and commutes with the map giving the minimal resolution of singularities* β. *Moreover,* $\mathrm{Fix}(\hat{\tau}) \simeq \mathrm{Fix}(\tau)$ *is a curve of genus* 4.

Proof. The singular locus $\mathrm{Sing}(\Sigma)$ is a proper orbit for the automorphism τ, where proper means that $\mathrm{Fix}(\tau) \cap \mathrm{Sing}(\Sigma) = \emptyset$. Indeed, the points in the singular locus are permuted by σ. Take $p_i \in \mathrm{Sing}(\Sigma)$ of coordinates $(\overline{l_1} : \cdots : \overline{l_4} : \bar{s})$. Then $(\overline{l_1} : \cdots : \overline{l_4} : \zeta_3 \bar{s})$ is again on the same line defined by the singular locus of $f_2 = 0$ and a different point of $\mathrm{Sing}(\Sigma)$. So $\mathrm{Sing}(\Sigma)$ is mapped to itself and therefore there exists a unique lift $\hat{\tau}$ such that $\hat{\tau}$ is an automorphism of $\hat{\Sigma}$ and commutes with β. By construction of $\hat{\tau}$, $\mathrm{Fix}(\hat{\tau}) \cap (\hat{\Sigma} \setminus \beta^{-1}(\mathrm{Sing}(\Sigma))) \simeq \mathrm{Fix}(\tau)$, as $\hat{\tau}$ acts in the same way of τ outside the exceptional divisors introduced by blowing up. Moreover, the diagram

$$\begin{array}{ccc} \hat{\Sigma} & \xrightarrow{\hat{\tau}} & \hat{\Sigma} \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\ \Sigma & \xrightarrow{\tau} & \Sigma \end{array}$$

commutes by the universal property of blow-ups, therefore

$$\hat{\tau}(\beta^{-1}(p_i)) = \beta^{-1}(\tau(p_i)) = \beta^{-1}(p_j) \quad \text{(where } j = i + 1 \text{ modulo 3).}$$

Therefore $\mathrm{Fix}(\hat{\tau}) \simeq \mathrm{Fix}(\tau)$ which is a curve of genus 4. In order to see that it is indeed a genus 4 curve, one can look at the explicit computation done in Proposition 4.7 of [4]. ■

It follows from the results in [4], since the automorphism $\hat{\tau}$ on the $K3$ surface $\hat{\Sigma}$ fixes exactly one curve, that the invariant lattice $T(\hat{\tau})$ in $H^2(\hat{\Sigma}, \mathbb{Z})$ is isometric to $U(3)$ and its orthogonal complement in $H^2(\hat{\Sigma}, \mathbb{Z})$ is isometric to $U \oplus U(3) \oplus E_8(-1)^{\oplus 2}$. As we are considering a generic Y with only one singularity of type D_4, and thus a generic K3 surface Σ with exactly three A_1 singularities, the Picard lattice T of $\hat{\Sigma}$ is of rank four. We use again the description given in Proposition 6.3.8 of [45] and we find a lattice $\langle C_1, E_1, E_2, E_3 \rangle \subset \mathrm{Pic}(\hat{\Sigma})$ with the following Gram matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}.$$

We consider C to be generic with a singularity of type A_2. Therefore, by genericity (using the computations of the rank of the Picard group in Appendix B), it holds

$$\langle C_1, E_1, E_2, E_3 \rangle = \mathrm{Pic}(\hat{\Sigma}).$$

Remark 7.2. Note that

$$\langle C_1, E_1, E_2, E_3 \rangle = \langle C_1, C_1 + E_1, -2C_1 - E_1 + E_2, -E_2 + E_3 \rangle \simeq U \oplus A_2(-2).$$

From the discussion above, we know that $T(\hat{\tau}) \simeq U(3)$ admits a primitive embedding $U(3) \hookrightarrow T$, so from this isomorphism we see that $U(3) \oplus A_2(-2) \subset U \oplus A_2(-2)$ can be written as a sublattice of finite index.

As in previous sections, we note the following isometry:

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Pic}(\hat{\Sigma}) \oplus \langle -2 \rangle \simeq U \oplus A_2(-2) \oplus \langle -2 \rangle \simeq \langle 6 \rangle \oplus D_4(-1).$$

This isometry can be described by

$$\mathrm{Pic}(\hat{\Sigma}^{[2]}) = \langle C_1, E_1, E_2, E_3, \varepsilon \rangle \simeq \langle -E_1, E_2, C_1 - \varepsilon, E_3, 2(2C_1 + E_1 + E_2 + E_3) - 3\varepsilon \rangle.$$

Here we are implicitly giving an embedding of the square six polarization class θ into the Picard lattice as $2\theta_{K3} - 3\varepsilon$, where θ_{K3} is a square six class on the $K3$ surface $\hat{\Sigma}$. As already remarked in Section 2, up to an isometry (so after a change of the marking), we can suppose θ_{K3} to be $3u_1 + u_2$ as in our setting. This proves Claim 4.4. It is useful to describe also the transcendental lattice $\mathrm{Tr}(\hat{\Sigma}^{[2]})$ of $\hat{\Sigma}^{[2]}$. So

$$\mathrm{Tr}(\hat{\Sigma}^{[2]}) \simeq \mathrm{Tr}(\hat{\Sigma}) \simeq U^{\oplus 2} \oplus E_8(-1) \oplus A_2(-1) \oplus D_4(-1).$$

In this case, we cannot proceed as described in Section 4. Indeed, let us denote

$$T_0 = \eta(\mathrm{Pic}(\hat{\Sigma}^{[2]})) \quad \text{and} \quad S_0 = T_0^{\perp} = \eta(\mathrm{Tr}(\hat{\Sigma}^{[2]})),$$

and prove the following proposition.

Proposition 7.3. *There exists no order three isometry $\rho_0 \in O(L)$ on the $K3^{[2]}$ lattice L which extends the isometry $id_{T_0} \oplus \rho_{|S_0} \in O(T_0) \oplus O(S_0)$.*

Proof. Suppose that it lifts to $\rho_0 \in O(L)$. We want to apply Proposition 4.5 to arrive to a contradiction. This means that there exists a subgroup $H < D_{S_0}$ which is glued to a subgroup $H' < D_{T_0}$. The only possibility is that $D_{D_4(-1)} < D_{S_0}$ is contained in the subgroup H. Moreover, when we restrict ρ_0 to T_0, it is the identity, so the order three isometry without fixed points $\rho_{|S_0}$ must be the identity on H. But we know from Example A.3 that there exists only one, up to conjugation, order three isometry without fixed points on $D_4(-1)$ and it is not trivial on $D_{D_4(-1)} \simeq (\mathbb{Z}/2\mathbb{Z})^2$. For the sake of completeness, we write explicitly the order three isometry ρ of $D_4(-1)$ without fixed points. We can express this lattice as $D_4(-1) = \langle d_1, d_2, d_3, d_4 \rangle$, with the following Gram matrix:

$$\begin{pmatrix} -2 & 0 & -1 & 0 \\ 0 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Then the isometry on the generators is given by:

$$\begin{aligned} d_1 &\mapsto -d_1 + d_3 - d_4, \\ d_2 &\mapsto -d_1 + d_2 - d_3, \\ d_3 &\mapsto -d_1 - d_2 + d_3 + d_4, \\ d_4 &\mapsto -d_2 + d_3 - d_4. \end{aligned}$$

After a standard computation, a basis of $D_{D_4(-1)}$ can be given by

$$\left\langle -d_1 + \frac{1}{2}d_2 + d_3 + \frac{1}{2}d_4, \frac{1}{2}d_1 - d_2 - d_3 - \frac{1}{2}d_4 \right\rangle = \langle a, b \rangle,$$

and we see that $\rho^*(a) = b$ and $\rho^*(b) = a + b$, where ρ^* is the map on $D_{D_4(-1)}$ induced by ρ. ■

We still have a non-symplectic automorphism of order three on the $K3$ surface $\hat{\Sigma}$ and, thus, on $\hat{\Sigma}^{[2]}$. Hence, as noted in Remark 4.3, we can see the latter as a point $(\hat{\Sigma}^{[2]}, \eta)$ having a degeneracy lattice $R_{\delta_1} \simeq R_{\delta_2} \simeq A_2(-1) \subset R_0$, "forgetting" one of the two roots orthogonal to its period (cf. with Definition 4.2). We now call

$$T_{\delta_1} = \langle 6 \rangle \oplus A_2(-1) \simeq \langle \theta, \delta_1 \rangle,$$

and we let S_{δ_1} be its orthogonal complement in L. Then, we can lift $\mathrm{id}_{|T_{\delta_1}} \oplus \rho_{|S_{\delta_1}}$ to a an isometry $\rho_{\delta_1} \in O(L)$. In order to see this, it is enough to note that in the proof of the lifting of the isometry in Theorem 4.8, the hypothesis on the Picard lattice is not used, the only thing used is the triviality of ρ on the discriminant, and in our case, this holds on R_{δ_1}. So, exactly as in the case A_1 done in Section 4.3 of [10], the isometry ρ_{δ_1} restricts to an isometry of $(\mathbb{Z}\varepsilon)^{\perp}$. Note that the fixed part of ρ_{δ_1} is by definition $\langle 6 \rangle \oplus A_2(-1) \simeq U(3) \oplus \langle -2 \rangle$, and if we consider a primitive embedding $T_{\delta_1} \subset \mathrm{Pic}(\hat{\Sigma}^{[2]})$, we get

$$T_{\delta_1} \oplus A_2(-2) \simeq \langle 6 \rangle \oplus A_2(-1) \oplus A_2(-2) \simeq U(3) \oplus A_2(-2) \oplus \langle -2 \rangle \subset \mathrm{Pic}(\hat{\Sigma}^{[2]}).$$

Lemma 7.4. *The action of $\hat{\tau}$ on $H^2(\hat{\Sigma}, \mathbb{Z})$ is conjugate to the isometry ρ_{δ_1} restricted to $(\mathbb{Z}\varepsilon)^{\perp}$.*

Proof. The proof is a straight-forward application of Theorem 4.7. Take $G = \langle \rho_{\delta_1} \rangle$, then condition (i) is simply verified. As $L^G \simeq U(3)$, also condition (iii') is easily verified. Lastly, $S_{G,X} \simeq A_2(-2)$, thus also condition (ii) is verified; therefore there exists an automorphism $\bar{\tau}$ on $\hat{\Sigma}$ whose action is conjugate to ρ_{δ_1}. If $\hat{\tau} = \bar{\tau}$, we are done. Otherwise, $\bar{\tau}$ is an order 3 automorphism whose invariant lattice is isomorphic to $U(3)$ and its fixed locus is a genus 4 curve $\bar{P}$. By Lemma 4.4 and Proposition 4.7 in [4], the embedding $\Phi_{|\bar{P}|}\colon \hat{\Sigma} \to \mathbb{P}^4$ is such that we can choose coordinates $(x_1 : \dots : x_5)$ of $\mathbb{P}^4$ such that the hyperplane H whose preimage $\Phi^{-1}(H) = \bar{P}$ is given by $x_5 = 0$. Moreover, the induced automorphism on the image is the automorphism of $\mathbb{P}^4$ that maps $x_5 \mapsto \zeta_3 \cdot x_5$. In other words, we can make a change of coordinates on $\hat{\Sigma}$ such that $\hat{\tau} = \bar{\tau}$. ∎

We now consider the natural automorphism $\hat{\tau}^{[2]}$ induced on $\hat{\Sigma}^{[2]}$ by $\hat{\tau}$. As its invariant lattice is T_{δ_1}, we see that $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau})$ defines a point in $\mathcal{N}_{T_{\delta_1}}^{\rho_{\delta_1}, \zeta}$. If $\hat{\Sigma}^{[2]}$ is moreover $K(T_{\delta_1})$-general, we can conclude as in Sections 5 and 6; but this turns out not to be the case, as proved by the following proposition.

Proposition 7.5. *$\hat{\Sigma}^{[2]}$ is not $K(T_{\delta_1})$-general.*

Proof. Note that by Theorem 6.18 in [36], the group of monodromies $\mathrm{Mon}^2(\hat{\Sigma})$ acts transitively on the set of exceptional chambers, therefore, up to taking a different birational model, in order to prove that $\hat{\Sigma}^{[2]}$ is not $K(T_{\delta_1})$-general, it is sufficient to show that there exists a wall divisor $\mu \in \Delta(\hat{\Sigma})$ not fixed by the action of $\hat{\tau}$ but such that $C^{\hat{\tau}} \cap \mu^{\perp} \neq 0$, where $C^{\hat{\tau}}$ denotes a connected component of the invariant positive cone. We will prove that there exist both -2 and -10 walls which are not fixed by $\hat{\tau}$ by exhibiting them. The wall divisor E_1 is not fixed by $\hat{\tau}$, but its orthogonal hyperplane intersects non-trivially $C^{\hat{\tau}}$, e.g., in $2C + E_1 + E_2 + E_3$. The same is true for the wall divisor $2E_1 + \varepsilon$. ∎

This proposition implies that $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau})$ is a non-separable point in $\mathcal{M}_{T_{\delta_1}}^{\rho_{\delta_1},\zeta}$. Moreover, this condition persists also if we consider $(\hat{\Sigma}^{[2]}, \eta, \hat{\tau})$ in the quotient $\mathcal{N}_{T_{\delta_1}}^{\rho_{\delta_1},\zeta}$. Indeed, in Section 4 of [9], the authors show that the points in the fiber over a period $\omega \in \mathcal{N}_{T_{\delta_1}}^{\rho_{\delta_1},\zeta}$ are in a one-to-one correspondence with the number of orbits of the monodromy action on the chambers. In the proof of Proposition 7.5, we found a (-10)-class defining a wall whose orbit cuts $K(T_{\delta_1})$, therefore there exist at least two chambers which are not in the same orbit of the monodromy action. This implies the following corollary.

Corollary 7.6. *There exist at least two non-biregular models in $\mathcal{N}_{T_{\delta_1}}^{\rho_{\delta_1},\zeta}$ which are over the same fiber of a period ω relative to a generic cubic threefold with a single singularity of type A_2.*

The pair $(\hat{\Sigma}^{[2]}, \eta)$ defines also a point in the moduli space of (M, j)-polarized IHS manifolds of $K3^{[2]}$-type, where $M := U \oplus A_2(-2) \oplus \langle -2 \rangle$. So we shift our interest in giving a formal characterization of marked IHS manifolds (X, ϕ) carrying the following commutative diagram in the data:

$$
\begin{array}{ccccccc}
T_{\delta_1} & \hookrightarrow & M & \overset{\iota}{\hookrightarrow} & \mathrm{Pic}(X) \subset H^2(X,\mathbb{Z}) & \xrightarrow{\sigma^*} & H^2(X,\mathbb{Z}) \\
 & \searrow^{j} & & & \downarrow \phi & & \downarrow \phi \\
 & & & & L & \xrightarrow{\rho_{\delta_1}} & L.
\end{array}
\tag{7.1}
$$

In order to arrive to a positive answer to our problem also for this case, we want to study a general situation when an IHS manifold admits, like in this case, two compatible polarizations.

8. Kähler cone sections of K-type

In this section, we introduce the notion of *Kähler cone sections of K-type* in order to avoid the problems of separability illustrated in the previous chapter.

Let (X, ϕ) be an (M, j)-polarized IHS manifold of type L, and let $T \subset M$ be a primitive sublattice. In this situation, the pair (X, ϕ) is both (M, j)-polarized and $(T, j_{|T})$-polarized, so we can compare the notions of $K(M)$ and $K(T)$ generality when the two chambers are chosen in such a way that

$$
\mathcal{K}_X \cap \iota(K(T)) \neq \emptyset \neq \mathcal{K}_X \cap \iota(K(M)),
$$

where ι denotes the $\mathbb{C}$-linear extension of the map ι of the diagram (7.1). Recall that $K(T)$ and $K(M)$ are, respectively, a connected component of $\mathcal{C}_T \setminus \bigcup_{\delta \in \Delta(S)} \delta^{\perp}$ and of $\mathcal{C}_M \setminus \bigcup_{\delta \in \Delta(M)} \delta^{\perp}$. By their definition, $\Delta(T) = \Delta(M) \cap T$ and $\mathcal{C}_T = \mathcal{C}_M \cap (T \otimes \mathbb{R})$, therefore our choice of the chambers $K(M)$ and $K(T)$ implies $K(M) \cap (T \otimes \mathbb{R}) \subset K(T)$, as the embedding of both via ι intersects $\mathcal{K}_X$.

Lemma 8.1. *If a pair (X, ϕ) as above is $K(T)$-general, then it is also $K(M)$-general.*

Proof. Suppose that (X,ϕ) is not $K(M)$-general; then, $\iota(\mathcal{C}_M)\cap\mathcal{K}_X$ is a proper subset of $\iota(K(M))$. By Theorem 2.8, there exists $\lambda\in\Delta(X)$ such that $\lambda^\perp\cap\iota(K(M))\neq\emptyset$ and $\phi(\lambda)\notin\Delta(M)$. But recall that $\Delta(M)\supset\Delta(T)$, and by our choice of the chambers, $K(M)\cap(T\otimes\mathbb{R})\subset K(T)$; then $\phi(\lambda)\notin\Delta(T)$ and $\lambda^\perp\cap\iota(K(T))\supset\lambda^\perp\cap\iota(K(M)\cap(T\otimes\mathbb{R}))\neq\emptyset$, implying that (X,ϕ) is not $K(T)$-general. ■

Remark 8.2. The converse to the previous statement is not true. A counterexample is given by the IHS fourfold $\hat{\Sigma}^{[2]}$ in Section 7. In fact, Proposition 7.5 shows that $(\hat{\Sigma}^{[2]},\eta)$ is not $K(T_{\delta_1})$-general, while it is clearly $K(M)$-general by Lemma 5.2 in [9].

So, we give the following more general definition.

Definition 8.3. Let K be a (connected and open) subset of a chamber $K(T)$ such that $K(T)\supset\phi(\mathcal{K}_X)$, with $\mathcal{K}_X$ denoting the Kähler cone of a (T,j)-polarized IHS manifold (X,ϕ). Then we say that X has a Kähler cone section of K-type if $K=\phi(\mathcal{K}_X)\cap\mathcal{C}_T$.

Remark 8.4. Note that if we choose a subset K which is not connected or not open, then no IHS manifold satisfies the above definition, as its Kähler cone is connected and open.

If the subset K chosen in Definition 8.3 is a proper subset of $K(T)$, any IHS manifold X with a Kähler section cone of K-type will not be $K(T)$-general. The following characterization gives us a link between the two definitions.

Proposition 8.5. *Let (X,ϕ) be a (T,j_T)-polarized IHS manifold of type L. If X has a Kähler cone section of K-type, then there exist*

(i) *a lattice M with an embedding $j_M\colon M\hookrightarrow L$;*

(ii) *a primitive embedding $\tilde{\iota}\colon T\hookrightarrow M$;*

(iii) *a chamber $K(M)$, i.e., a connected component of $\mathcal{C}_M\setminus\bigcup_{\delta\in\Delta(M)}\delta^\perp$, with $K(M)\cap\mathcal{C}_T=K$,*

such that (X,ϕ) is a $K(M)$-general, (M,j_M)-polarized IHS manifold. Conversely, if (X,ϕ) is a $K(M)$-general, (M,j_M)-polarized IHS manifold with $T\subset M$, then it has a Kähler cone section of K-type with $K=K(M)\cap\mathcal{C}_T$.

Proof. By Theorem 2.8, the Kähler cone of X is a connected component of $\mathcal{C}_X\setminus\mathcal{H}_\Delta$, so denote with $\Lambda\subset\Delta(X)$ the set of those MBM classes λ for which $\lambda^\perp$ is an extremal ray, i.e., if $\alpha,\beta\in\mathcal{C}_X$ are such that $\alpha+\beta\in\lambda^\perp$, then $\alpha,\beta\in\lambda^\perp$. Define M as a lattice of minimal rank containing T and $\phi(\Lambda)$ and such that every embedding in the chain of inclusions $T\subset M\subset M':=\phi(\mathrm{Pic}(X))$ is primitive and use the first to define $\tilde{\iota}$; fix an embedding $j_M\colon M\hookrightarrow L$ such that $(j_M)_{|T}=j_T$. Then by construction there exists a chamber $K(M)$ of $\mathcal{C}_M\setminus\mathcal{H}_{\Delta(M)}$ such that (X,ϕ) is (M,j_M)-polarized with $\iota_M:=j_M\circ\phi^{-1}$ and $K(M)$-general. Moreover, denoting with ι_T the embedding $T\hookrightarrow\mathrm{Pic}(X)$, we obtain that $\iota_M(K(M)\cap\mathcal{C}_T)=\mathcal{K}_X\cap\iota_T(\mathcal{C}_T)=\iota_M(K)$ by hypothesis and the injectivity of ι_M, therefore the first part of the statement is proved. The second one is obvious using the fact that ι_T and ι_M are injective and that, by construction, $\iota_T=(\iota_M)_{|T}$. Indeed, $\iota_T(K)=\iota_T(K(M)\cap\mathcal{C}_T)=\iota_M(K(M))\cap\iota_T(\mathcal{C}_T)=\mathcal{K}_X\cap\iota_T(\mathcal{C}_T)$. ■

Remark 8.6. Note that in the proof of the above proposition, we chose to exhibit a minimal $M\supset T$ for which the statement holds. In fact, the same holds for every lattice $M' \supset M \supset T$ for which (X, ϕ) admits an (M', j')-polarization by Lemma 8.1. In particular, it holds for $M' \simeq \mathrm{Pic}(X)$.

We now fix a chamber K and (X, η) a (T, j_T)-polarized IHS manifold with a Kähler cone section of K-type. According to Proposition 8.5, we can find a pair (M, j_M) such that (X, η) is (M, j_M)-polarized, $K(M)$-general IHS manifold, with $K(M) \cap \mathcal{C}_T = K$. We define $N := j_M(M)^{\perp}$ and $S := j_T(T)^{\perp}$. We can then consider the period map of the moduli space $\mathcal{M}_{K(M), j_M}$ of (M, j_M)-polarized IHS manifolds of type L which are $K(M)$-general:

$$\mathcal{P}_{K(M)} : \mathcal{M}_{K(M), j_M} \to \Omega^{M, j_M} \setminus (\mathcal{H}_M \cup \mathcal{H}'_{K(M)}),$$

where a marked pair (X, ϕ) is sent to $\phi(H^{2,0}(X))$, which is an isomorphism by Theorem 3.13 in [17]. We define the subfamily $\mathcal{F}^T_{j_M, K}$ of the (T, j_T)-polarized IHS manifold of type L with the Kähler cone section of type K and embedding j_M. Then we state the following theorem.

Theorem 8.7. *The period map $\mathcal{P}_{K(M)}$ restricted to $\mathcal{F}^T_{j_M, K}$ defines a bijection with*

$$\Omega^T_{j_M, K} := \{\omega \in \mathbb{P}(N_{\mathbb{C}}) \mid (\omega, \bar{\omega}) > 0,\ q(\omega) = 0\} \setminus ((\mathcal{H}_S \cap N_{\mathbb{C}}) \cup \mathcal{H}'_{K(M)}),$$

with

$$\mathcal{H}_{S_{\delta_1}} := \bigcup_{\nu \in \Delta(S)} H_\nu \quad \text{and} \quad \mathcal{H}'_{K(M)} := \bigcup_{\nu \in \Delta'(K(M))} H_\nu.$$

Proof. Consider the period map $\mathcal{P}_{K(M)}$. If we restrict this map to those IHS manifolds which are in $\mathcal{F}^T_{j_M, K}$, then the image cannot lie in $\mathcal{H}_S \cap N_{\mathbb{C}}$, because if there existed $\nu^{\perp} \in \mathcal{H}_S$ such that $\omega := \phi^{-1}(H^{2,0}(X)) \in H_\nu$, we would have $\phi^{-1}(\nu) \in \mathrm{NS}(X)$, by definition of $\mathrm{NS}(X)$ as the orthogonal complement of $H^{2,0}(X)$ in $H^2(X, \mathbb{Z})$. Therefore $\eta^{-1}(\nu)$ would be a wall divisor, yielding a contradiction using the same argument of Remark 4.1. Being $\mathcal{P}_{K(M)}$ an injective morphism, in order to get a bijection, we just need to show what is the image. Take a period $\omega \in \Omega^T_{j_M, K}$ and consider $(X, \phi) = \mathcal{P}^{-1}_{K(M)}(\omega)$; then it is immediate to see that (X, ϕ) is indeed an element of $\mathcal{F}^T_{j_M, K}$, as it is $K(M)$-general and by Proposition 8.5, it has a Kähler cone section of K-type. ■

9. A moduli space for the singularity of type A_2

In this section, we give the proof of Theorem 1.1 for $\Delta_3^{A_2}$ and continue the description started at the end of Section 7; therefore, we will use the same notation resumed in the diagram (7.1). Moreover, we recall that $T_{\delta_1} = U(3) \oplus \langle -2 \rangle$ and $M = U \oplus A_2(-2) \oplus \langle -2 \rangle$.

Let us define $K(M)$ as the connected component of $\mathcal{C}_M \setminus \bigcup_{\nu \in \Delta(M)} H_\nu$ which contains a Kähler class of $\hat{\Sigma}^{[2]}$, i.e., $\mathcal{K}_{\hat{\Sigma}^{[2]}} \subset \iota(K(M)) \otimes \mathbb{R}$. We choose in a compatible way also $K(T_{\delta_1})$, i.e., $\mathcal{K}_{\hat{\Sigma}^{[2]}} \cap \iota(K(T_{\delta_1})) \neq \emptyset \neq \mathcal{K}_{\hat{\Sigma}^{[2]}} \cap \iota(K(M))$; we also choose $K := K(M) \cap (T_{\delta_1} \otimes \mathbb{R}) \subset K(T_{\delta_1})$. Moreover, in the case of $\hat{\Sigma}^{[2]}$, the map $i : M \to \mathrm{Pic}(\hat{\Sigma}^{[2]})$ is an isomorphism, and thus $i(K(M)) = \mathcal{K}_{\hat{\Sigma}^{[2]}}$. So, we define the family $\mathcal{F}^{\rho_{\delta_1}, \zeta}_{K, T_{\delta_1}} \subset \mathcal{M}^{\rho_{\delta_1}, \zeta}_{T_{\delta_1}}$

of IHS manifolds in $\mathcal{M}_{T_{\delta_1}}^{\rho_{\delta_1},\zeta}$ which have a Kähler cone section of K-type and admit an embedding $T_{\delta_1} \hookrightarrow M \hookrightarrow \operatorname{Pic}(X)$ compatible with the polarization, i.e., for which the commutative diagram (7.1) is defined.

Remark 9.1. We do not want to choose the embedding of M in L in the data but only the embedding j of T_{δ_1}. It is an easy exercise, using Proposition 1.15.1 in [41], to see that there are only two possible non-isomorphic embeddings of T_{δ_1} in L, and that they correspond respectively to the two possible non-isomorphic embeddings of M in L (see Appendix C). Recall that we say that two embeddings j_1 and j_2 of M in L are isomorphic if there exists an automorphism $\varphi \in O(L)$ such that $j_2 = \varphi \circ j_1$. Therefore we choose the embedding j_k such that

$$j_k(T_{\delta_1})^{\perp} = j(T_{\delta_1})^{\perp} =: S_{\delta_1} \simeq U(3) \oplus U \oplus E_8(-1)^{\oplus 2} \supset \operatorname{Tr}(\hat{\Sigma}^{[2]}).$$

Note that doing so $(\hat{\Sigma}^{[2]}, \eta) \in \mathcal{F}_{K,T_{\delta_1}}^{\rho_{\delta_1},\zeta}$.

Proposition 9.2. *The period map $\mathcal{P}_{K(M)}$ restricted to $\mathcal{F}_{K,T_{\delta_1}}^{\rho_{\delta_1},\zeta}$ defines a bijection with*

$$\Omega_{K,T_{\delta_1}}^{\rho_{\delta_1},\zeta} := \{\omega \in \mathbb{P}(N_{\mathbb{C}}(\zeta)) \mid (\omega,\bar{\omega}) > 0\} \setminus \left((\mathcal{H}_{S_{\delta_1}} \cap N_{\mathbb{C}}) \cup \mathcal{H}'_{K(M)}\right),$$

with

$$\mathcal{H}_{S_{\delta_1}} := \bigcup_{\nu \in \Delta(S_{\delta_1})} H_\nu \quad \text{and} \quad \mathcal{H}'_{K(M)} := \bigcup_{\nu \in \Delta'(K(M))} H_\nu .$$

Proof. This result states that we can apply Theorem 8.7 also with the additional structure of the $(\rho_{\delta_1}, T_{\delta_1})$-polarization. The image of $\mathcal{P}_{K(M)}$ lies in the eigenspace $\mathbb{P}(N_{\mathbb{C}}(\zeta))$ by definition of Picard group. Take now a period $\omega \in \Omega_{M,K(M)}^{\rho_{\delta_1},\zeta}$ and consider $(X,\phi) = \mathcal{P}_{K(M)}^{-1}(\omega)$. We want to show that on X there exists an automorphism σ satisfying the properties required in diagram (7.1). Define

$$\sigma^* := \phi^{-1} \circ \rho_{\delta_1} \circ \phi.$$

It is an isomorphism of integral Hodge structures since

$$\sigma^*(\omega_X) := \sigma^*(\mathcal{P}_{K(M)}^{-1}(\omega)) = \phi^{-1}(\rho_{\delta_1}(\omega)) = \zeta\,\omega_X .$$

Moreover, it is a parallel transport operator as $\rho_{\delta_1} \in \operatorname{Mon}^2(L)$. If it preserves also a Kähler class, we can conclude with the Hodge theoretic Torelli Theorem 2.6. But X has a Kähler cone section of K-type, therefore

$$\mathcal{K}_X \cap i(T_{\delta_1}) \supset \mathcal{K}_X \cap i(T_{\delta_1}) \cap i(\mathcal{C}_M) = i(K(M)) \cap i(T_{\delta_1}) \neq \emptyset,$$

thus, ρ_{δ_1} fixes a Kähler class. ∎

Remark 9.3. Note that if (X_1,ϕ_1) and (X_2,ϕ_2) define the same point in $\mathcal{M}_{M,K(M)}$, i.e., if there exists a biregular morphism $f\colon X_1 \to X_2$ such that $\phi_1 = \phi_2 \circ f^*$ and $i_1 = i_2 \circ f^*$, then also $\sigma_1 = f^{-1} \circ \sigma_2 \circ f$ by Theorem 1.8 in [13].

Let

$$\mathrm{Mon}^2(M, j, \rho_{\delta_1}) := \{g \in \mathrm{Mon}^2(L) \mid g(M) = M,\ g(t) = t,\\ g \circ \rho_{\delta_1} = \rho_{\delta_1} \circ g,\ \forall t \in T_{\delta_1}\}$$

and denote its image in $O(N)$ via the restriction map with $\Gamma^{\rho_{\delta_1}}_{M,j}$. Note that $\mathrm{Mon}^2(M, j, \rho_{\delta_1})$ is the stabilizer of $\mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$ for the action of $\mathrm{Mon}^2(M, j)$ on $\mathcal{M}_{M,K(M,T_{\delta_1})}$. Moreover, by definition, the bijection defined in Proposition 9.2 is equivariant with respect to the action of $\mathrm{Mon}^2(M, j, \rho_{\delta_1})$ and of $\Gamma^{\rho_{\delta_1}}_{M,j}$. Denoting

$$\mathscr{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} := \mathscr{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} / \mathrm{Mon}^2(M, j, \rho_{\delta_1}),$$

we deduce the following corollary.

Corollary 9.4. *There exists a bijection between* $\mathscr{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$ *and* $\Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} / \Gamma^{\rho_{\delta_1}}_{M,j}$.

The whole formal construction made above is "natural" in the sense that it arises from the theory of $K3$ surfaces in a compatible way. Let us clarify this sentence. Consider a diagram similar to (7.1) where (X, ϕ) is, this time, a marked $K3$ surface such that the following commutative diagram is defined:

$$\begin{array}{ccccccc} U(3) & \hookrightarrow & U \oplus A_2(-2) & \overset{i}{\hookrightarrow} & \mathrm{Pic}(X) \subset H^2(X,\mathbb{Z}) & \xrightarrow{\sigma^*} & H^2(X,\mathbb{Z}) \\ & \searrow^{j} & & & \downarrow \phi & & \downarrow \phi \\ & & & & L_{K3} & \xrightarrow{\tilde{\rho}} & L_{K3}, \end{array}$$

where $\tilde{\rho}$ is ρ_{δ_1} restricted to $\langle -2 \rangle^{\perp}$. Define the subfamily of the $K3$ surfaces with an ample $U(3)$-polarization for which there exists the above diagram as $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ and note that $\hat{\Sigma} \in \mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$. Then with the same techniques of Proposition 9.2 we can find a generalized version of Theorem 11.3 in [23] which applies to our subfamily and obtain the following proposition. We denote with

$$\Gamma^{\tilde{\rho}}_{U(3),U\oplus A_2(-2)} := \{\gamma \in O(L_{K3}) \mid \gamma_{|U\oplus A_2(-2)} \in O(U \oplus A_2(-2)),\\ \gamma_{|U(3)} = \mathrm{id} \text{ and } \gamma \circ \tilde{\rho} = \tilde{\rho} \circ \gamma\}.$$

Proposition 9.5. *The period map defines a bijection between* $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$ *and*

$$\Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)} := \{\omega \in \mathbb{P}(N_{\mathbb{C}}(\zeta)) \mid (\omega, \bar{\omega}) > 0\} \setminus (\mathscr{H}_{S_{\delta_1}} \cap N_{\mathbb{C}}),$$

with

$$\mathscr{H}_{S_{\delta_1}} := \bigcup_{\nu \in \Delta(S_{\delta_1})} H_{\nu}.$$

Moreover, this bijection descends to bijection at the level of isomorphism classes, i.e.,

$$\Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)} / \Gamma^{\tilde{\rho}}_{U(3),U\oplus A_2(-2)}$$

parametrizes isomorphism classes of $K3$ surfaces in $\mathscr{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$.

Proof. Consider the period map for the $K3$ surfaces which are $(\tilde{\rho}, U(3))$-ample. By Theorem 11.2 in [23], it is a bijection with

$$\Omega^{\tilde{\rho}}_{U(3)} := \{\omega \in \mathbb{P}((S_{\delta_1} \otimes \mathbb{C})(\zeta)) \mid (\omega, \bar{\omega}) > 0\} \setminus \mathcal{H}_{S_{\delta_1}}.$$

If we moreover consider the restriction to our subfamily, then it is easy to see that Theorem 10.2 in [23] guarantees the assertion. ∎

We can now state the following proposition, which answers our question in a formal way.

Proposition 9.6. *Let* $R_0 = \mathrm{Span}(\delta_1, \delta_2, \rho(\delta_1), \rho(\delta_2)) \simeq D_4(-1)$ *be the degeneracy lattice relative to a period* ω *of a generic cubic threefold having a single singularity of type* A_2*. Then the* A_2 *locus* $\Delta_3^{A_2}$ *is birational to the moduli space* $\mathcal{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$.

Proof. The proof has again the same structure of the proof of Proposition 5.1; we review here the main points. Recall that the extension of the period map

$$\mathcal{P}^3 : \mathcal{C}_3^{\mathrm{sm}} \to \frac{\mathbb{B}^{10} \setminus (\mathcal{H}_n \cup \mathcal{H}_c)}{\mathbb{P}\Gamma}$$

to the nodal locus is done by [2] defining its period as the period of its associated $K3$ surface (see also Section 3.1). In our case, the generic A_2 nodal cubic has the period

$$\mathcal{P}^3(C) := \mathcal{P}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}((\Sigma, \eta)).$$

The latter period map is the same of Proposition 9.5. Following Proposition 9.5, this map yields an isomorphism between

$$\mathcal{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)} \quad \text{and} \quad \Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}.$$

As proven in Section 7, $(\hat{\Sigma}^{[2]}, \eta, \tau)$ defines a point in $\mathcal{F}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}}$. The period map in this space defines a bijection which, according to Corollary 9.4, descends to a bijection between

$$\mathcal{N}^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} \quad \text{and} \quad \Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} / \Gamma^{\rho_{\delta_1}}_{M,j}.$$

Note that

$$\Omega^{\rho_{\delta_1},\zeta}_{K,T_{\delta_1}} / \Gamma^{\rho_{\delta_1}}_{M,j} \quad \text{and} \quad \Omega^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)} / \Gamma^{\tilde{\rho}}_{U(3),U\oplus A_2(-2)}$$

are birational. So, in order to conclude the proof, it is enough to show that $\Delta_3^{A_2}$ is birational to the isomorphism classes of $K3$ surfaces in $\mathcal{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$, as the claim of the proposition would follow through a composition of birational morphisms. But this is true, as in Section 6 of [2] the authors show that the discriminant locus maps isomorphically to its image, the nodal hyperplane arrangement, through the period map, therefore a generic point in $\Delta_3^{A_2}$ is mapped isomorphically to the period of a generic $K3$ surface in $\mathcal{K}^{\tilde{\rho},\zeta}_{U(3),U\oplus A_2(-2)}$, and by Proposition 9.5, this is an isomorphism. ∎

Remark 9.7. Also Kondō notes in [30] this family as a codimension 1 family in the moduli space of curves of genus 4 which consists of smooth curves with a vanishing theta null.

A. Complex reflection groups and Springer theory

In this section, we will recall some notions about reflection groups and Springer theory. A deeper reference for this is [14], or Section 3 of [12].

Let V be a complex vector space of $\dim_{\mathbb{C}}(V) = n$ and let $W \subset \mathrm{GL}_n(V)$ be a finite subgroup of the complex general linear group of degree n. Given an element $g \in \mathrm{GL}_n(V)$, we define the subset $V^g := \{v \in V \mid g(v) = v\} \subset V$ of the elements fixed by g. Moreover, we define also the set

$$\mathrm{Ref}(W) := \{s \in W \mid \dim(V^s) = n - 1\}.$$

Definition A.1. A finite subgroup $W \subset \mathrm{GL}_n(V)$ is called a *complex reflection group* if $W = \langle \mathrm{Ref}(W) \rangle$.

A classic result, stated, e.g., in Theorem 4.1 of [14], is the following.

Theorem (Serre–Chevalley, Shepherd–Todd). *Given a complex reflection group W acting on a complex vector space V of $\dim_{\mathbb{C}}(V) = n$, there exist $f_1, \dots, f_n$ homogeneous polynomials of degree $d_1, \dots, d_n$ such that the invariant ring by the action of W is given by*

$$\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n].$$

The family $\{f_1, \dots, f_n\}$ is not unique (up to permutations), but the degrees $\{d_1, \dots, d_n\}$ are uniquely determined (up to permutations) by V and W. Moreover, a result by Solomon (Theorem 4.44 and Section 4.5.4 of [14]) implies that the graded $\mathbb{C}[V]^W$-module of W-invariant derivations of $\mathbb{C}[V]$ admits a homogeneous $\mathbb{C}[V]^W$-basis $(g_1, \dots, g_n)$ whose degrees $(d_1^*, \dots, d_n^*)$ are called co-degrees. Also the co-degrees are invariant up to permutation. Now, in order to state the results that we need from Springer theory, we need to define the following numbers for any $e \in \mathbb{N}$:

$$\lambda(e) := |\{1 \leq i \leq n \mid e \text{ divides } d_i\}|,$$
$$\lambda^*(e) := |\{1 \leq i \leq n \mid e \text{ divides } d_i^*\}|.$$

Moreover, if the primitive e-th root of unity ζ_e is a eigenvalue for the action of $w \in W$, we set $V(w, \zeta_e)$ to be the eigenspace of V with respect to the action of w relative to ζ_e. Then, putting together various results from Theorem C in [34] and from Theorems 3.4, 4.2 and 6.2 in [44], we state the following.

Theorem A.2 (Springer, Lehrer–Springer). *Let W be a complex reflection group acting on V. Then for every $e \in \mathbb{N}$, it holds $\lambda(e) = \max_{w \in W} \dim(V(w, \zeta_e))$. If, moreover, e is such that $\lambda(e) = \lambda^*(e)$, then the elements w_e which attain the maximum define a unique conjugacy class in W.*

Let us explain the results in this section with a couple of examples.

Example A.3 (Isometries of order 3 on D_4). From classical theory, that can be found, e.g., in Section 7 of Chapter 4 in [20], the group of lattice isometries of D_4 is $O(D_4) \simeq G_{28} \simeq W(F_4) =: W$, where $W(F_4)$ denotes the Coxeter group F_4 and G_{28} is the 28-th group in the Shepherd–Todd classification. If we consider its action on $V = \mathbb{C}^4$, it is a complex reflection group. In order to compute the degrees and co-degrees, one can use a computer algebra program like MAGMA or refer to [11]. In any case, we have that the degrees are

$$(d_1, d_2, d_3, d_4) = (2, 6, 8, 12),$$

and the co-degrees are

$$(d_1^*, d_2^*, d_3^*, d_4^*) = (0, 4, 6, 10).$$

So we can look at $\lambda(3) = \max_{w \in W} \dim(V(w, \zeta_3)) = 2 = \lambda^*(3)$. Suppose that this maximum is attained in $w_3 \in W = W(F_4) \subset \mathrm{GL}_4(V)$. Therefore w_3 is an integer matrix which admits ζ_3 as eigenvalue whose eigenspace is 2-dimensional, thus the same is true also for $\overline{\zeta_3}$. Moreover, as the triple $(W, V, 3)$ satisfies the hypothesis of Theorem A.2, w_3 is unique up to conjugation. Wrapping up everything, we proved that there exists only one isometry of D_4, up to conjugation, of order three and without fixed points (the last statement comes from the fact that the only eigenvalues are ζ_3 and $\overline{\zeta_3}$).

Example A.4 (Isometries of order 3 on E_6). Here we want to apply again the same idea. Lattice isometries of E_6 are just $O(E_6) \simeq \mathbb{Z}/2\mathbb{Z} \times G_{35} \simeq \mathbb{Z}/2\mathbb{Z} \times W(E_6)$ (again, this is a classical result which can be found in Section 8.3 of Chapter 4 in [20]), with the same convention as above. Now $W := W(E_6)$ acts as a complex reflection group on $V = \mathbb{C}^6$. The degrees are

$$(d_1, d_2, d_3, d_4, d_5, d_6) = (2, 5, 6, 8, 9, 12),$$

and the co-degrees are

$$(d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = (0, 3, 4, 6, 7, 10).$$

Now $\lambda(3) = \max_{w \in W} \dim(V(w, \zeta_3)) = 3 = \lambda^*(3)$, and we assume that w_3 attains the maximum. With the same argument as above, we obtain that w_3 has ζ_3 and $\overline{\zeta_3}$ as triple eigenvalues. Therefore we proved that, up to conjugation (and sign), there exists only one order three isometry without fixed points on E_6.

B. Dimension of moduli spaces

In this section, we compute the dimensions of the moduli spaces of families which are complete intersections in $\mathbb{P}^4$ of a quadric hypersurface of rank 3 or 4 and a cubic threefold.

Using the generalized Morse lemma and the recognition principle as done, e.g., in [28], one can arrive to a generic form for the families we are interested in and count the free parameters. But the computations are long, so we will use the following result, which is a direct application of the generalised Morse lemma (see Theorem 2.47 in Chapter I of [25] for a possible reference) and the recognition principle, see Lemma 1 in [15]. As this theorem appears on a PhD dissertation [28] which has not been published at the day we are writing this article, we include here its proof.

Theorem B.1 (Theorem 1.15 in [28]). *Let $Y \subset \mathbb{A}^n_{\mathbb{C}}$ be a hypersurface defined by a polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$, and assume that the origin is an isolated singular point of Y*

of corank one. Then, there exist polynomials $C_1, \ldots, C_{k+1}$ in the coefficients of P and depending on the choice of an analytic coordinate change such that the conditions

$$C_1 = \cdots = C_k = 0, \quad C_{k+1} \neq 0,$$

on the coefficients of P are equivalent to $(Y, 0)$ being of type A_k. Moreover, each C_i is homogeneous of degree $i-2$, and fixing the analytic coordinate change they depend on, there is an explicit algorithm computing them.

Proof. Let $k \in \mathbb{N}$. Using the generalized Morse lemma, we suppose that, after a suitable analytic coordinates change, P has the form

$$P(x) = x_1^2 + \cdots + x_{n-1}^2 + P_3(x_n) + \cdots + P_{k+1}(x_n) + \sum_{i=1}^{n-1} x_i\, Q_i(x_1, \ldots, x_n),$$

where each P_i is a polynomial of degree i and each Q_i of degree k. In order to apply the recognition principle, we take the weight

$$\alpha(A_k) = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{k+1}\right)$$

and note that the terms of degree $\alpha(A_k) < 1$ are $P_3(x_n) + \cdots + P_k(x_n)$, the terms of degree $\alpha(A_k) = 1$ are $x_1^2 + \cdots + x_{n-1}^2 + P_{k+1}(x_n)$, and the terms of degree $\alpha(A_k) > 1$ are $\sum_{i=1}^{n-1} x_i\, Q_i(x_1, \ldots, x_n)$. Therefore we write $C_i x_n = P_i(x_n)$ and conclude using the recognition principle. ■

This will lead us to prove the following proposition.

Proposition B.2. *The dimension of the family $\mathcal{K}_{A_i}$ associated to the cubic threefold having one A_i singularity (and thus of the subfamily of cubic threefolds having an isolated singularity of type A_i) for $i = 1, \ldots, 4$ is $10 - i$. A generic element Σ_{A_i} in $\mathcal{K}_{A_i}$ is such that $rk(\mathrm{Pic}(\Sigma_{A_i})) = 2i$.*

Proof. We start from the most general case, which is the A_1 case. Note that this is the only corank 0 case, so the theorem does not apply in this case. The equations are given by

$$\text{(B.1)} \qquad \begin{cases} f_2(x_1, x_2, x_3, x_4) = 0, \\ f_3(x_1, x_2, x_3, x_4) + a\, x_5^3 = 0. \end{cases}$$

So we have $\binom{3+2}{2} = 10$ parameters for the quadric and $\binom{3+3}{3} + 1 = 21$ for the cubic. Then we have to impose 4 conditions, because if two cubic hypersurfaces differ by a multiple of the quadric, they yield the same intersection. As every equation is defined up to a constant, the parameters are $10 + 21 - 4 - 1 - 1 = 25$. Then we have to consider the projective transformations which preserve the family, as projectivities are up to a constant are $4 \cdot 4 + 1 - 1 = 16$. Finally, the dimension of this family is $25 - 16 = 9$. Now, we consider the family $\mathcal{K}_{A_i}$ associated to the cubic threefold having one A_i singularity for $i \geq 2$. In these cases, we have the same parameters and projective transformations as before, but we need to add 1 condition for being a corank 1 singularity (this is equivalent to ask that $f_2 = 0$ has rank 3 as a quadric) and $i-2$ conditions coming from Theorem B.1. Therefore the dimension of the moduli space of the family of $(2,3)$-complete intersections in $\mathbb{P}^4$ associated to a cubic threefold having a singularity of type A_i is $10-i$. Then, if we take a generic element Σ_{A_i} in $\mathcal{K}_{A_i}$, by Section 9 of [5], we obtain $\mathrm{rk}(\mathrm{Pic}(\Sigma_{A_i})) = 22 - 2(10-i+1) = 2i$. ∎

C. An easy exercise

Here we outline the execution of the exercise mentioned in Remark 9.1.

Let L be the $K3^{[2]}$ lattice. Then $D_L \simeq \mathbb{Z}/2\mathbb{Z}$, with finite quadratic form $q_L = \langle 3/2 \rangle$. Moreover, let $T = U(3) \oplus \langle -2 \rangle$ and $M = U \oplus A_2(2) \oplus \langle -2 \rangle$. Clearly, $q_T \simeq q_L \oplus q_{U(3)}$ and $q_M \simeq q_L \oplus q_{A_2(2)}$, so given Proposition 1.15.1 in [41], recalled in Proposition 4.5, the only possibilities for the respective orthogonal complements for T in L are the following:

- the genus of the lattice with signature $(2,18)$ and discriminant form $q_T(-1) \oplus q_L$ is non-empty. Using the notation of Conway–Sloane ([20]), this is $\mathrm{II}_{(2,18)}\, 2_{\mathrm{I}}^{+2}\, 3^{-2}$. There exists only one class of isomorphism, represented by
$$U \oplus U(3) \oplus E_7 \oplus E_8 \oplus \langle -2 \rangle.$$
- The genus of the lattice with signature $(2,18)$ and discriminant form $q_{U(3)}(-1)$ is non-empty. Using the notation of Conway–Sloane ([20]), this is $\mathrm{II}_{(2,18)}\, 3^{-2}$. There exists only one class of isomorphism, represented by
$$U \oplus U(3) \oplus E_8^{\oplus 2}.$$

Analogously, for M,

- the genus of the lattice with signature $(2,16)$ and discriminant form $q_M(-1) \oplus q_L$ is non-empty. Using the notation of Conway–Sloane ([20]), this is $\mathrm{II}_{(2,16)}\, 2_{\mathrm{I}}^{-4}\, 3^{+1}$. There exists only one class of isomorphism, represented by
$$U^{\oplus 2} \oplus E_8 \oplus D_4 \oplus \langle -6 \rangle \oplus \langle -2 \rangle.$$
- The genus of the lattice with signature $(2,16)$ and discriminant form $q_{A_2(2)}(-1)$ is non-empty. Using the notation of Conway–Sloane ([20]), this is $\mathrm{II}_{(2,16)}\, 2_{\mathrm{II}}^{-2}\, 3^{+1}$. There exists only one class of isomorphism, represented by
$$U^{\oplus 2} \oplus E_8 \oplus D_4 \oplus A_2.$$

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