

Free-boundary monotonicity for almost-minimizers of the relative perimeter

Gian Paolo Leonardi and Giacomo Vianello

Abstract. Let $E \subset \Omega$ be a local almost-minimizer of the relative perimeter in the open set $\Omega \subset \mathbb{R}^n$. We prove a free-boundary monotonicity inequality for E at a point $x \in \partial\Omega$, under a geometric property called “visibility”, that Ω is required to satisfy in a neighborhood of x . Incidentally, the visibility property is satisfied by a considerably large class of Lipschitz and possibly non-smooth domains. Then, we prove the existence of the density of the relative perimeter of E at x , as well as the fact that any blow-up of E at x is necessarily a perimeter-minimizing cone within the tangent cone to Ω at x .

1. Introduction

Consider an open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and fix $x_0 \in \partial\Omega$. The main goal of this work is to prove a free-boundary monotonicity inequality for a local almost-minimizer $E \subset \Omega$ of the relative perimeter at the boundary point x_0 , under a suitable geometric property of $\partial\Omega$ near x_0 , which we call “visibility”.

Monotonicity inequalities are key tools in the regularity theory for minimizers and almost-minimizers of the area functional. In the prototypical setting of E being a local perimeter minimizer near $0 \in \Omega$ (or even just a critical point of the perimeter functional) and given $0 < r_1 < r_2$ such that the ball B_{r_2} of radius r_2 and center 0 is contained in Ω , it is known that

$$\int_{(B_{r_2} \setminus B_{r_1}) \cap \partial^* E} \frac{\langle x, \nu_E \rangle^2}{|x|^{n+1}} d\mathcal{H}^{n-1}(x) \leq \frac{P(E; B_{r_2})}{r_2^{n-1}} - \frac{P(E; B_{r_1})}{r_1^{n-1}}, \quad (1.1)$$

where $P(E; B_r)$ is the perimeter of E in B_r , \mathcal{H}^{n-1} is the Hausdorff $(n-1)$ -dimensional measure in \mathbb{R}^n , $\partial^* E$ is the reduced boundary of E , and ν_E is the weak interior normal defined for \mathcal{H}^{n-1} -almost every $x \in \partial^* E$.

The first, fundamental consequences of an inequality like (1.1) are:

- (i) the monotonicity of the renormalized perimeter $r \mapsto \frac{P(E; B_r)}{r^{n-1}}$, which turns out to be a non-decreasing function admitting a finite limit as $r \rightarrow 0$ (denoted as $\theta_E(0)$, the perimeter density of E at 0);

- (ii) the fact that if a perimeter minimizer has a constant renormalized perimeter, then the left-hand side of (1.1) vanishes for all $0 < r_1 < r_2$, and consequently, E coincides up to null sets with a cone with vertex at the origin.

These two facts allow the application of the monotonicity formula in several steps of the proof of the $C^{1,\alpha}$ -regularity of the reduced boundary $\partial^* E$, as well as in the analysis of the singular set $\partial E \setminus \partial^* E$. After the pioneering work of De Giorgi [10] on the partial regularity of local perimeter minimizers, the internal regularity theory has been successfully extended to (rectifiable) sets of varying topological type [26], area-minimizing integral currents [4, 11–13, 17], and varifolds [1, 3]. In the codimension-1 case, in particular, the dimension of the singular set is sharply estimated as $\leq n - 8$, after the work of Bombieri–De Giorgi–Giusti [8] combined with a dimension-reduction argument due to Federer [29]. In these works, monotonicity formulas arise in specific but substantially equivalent forms, through proofs based either on comparison with cones or by testing the first variation of area with suitable radially symmetric vector fields.

A key feature of regularity theory is its stability under perturbations that, at small scales, have a higher order of infinitesimality than area. This has led to the extension of the regularity theory to wider classes of almost-minimizers including, for instance, minimizers of isoperimetric or prescribed mean curvature problems (see, e.g., [7, 19, 25, 28, 30]).

Coming to the boundary properties of almost-minimizers, we record regularity results proved for integral currents minimizing parametric elliptic integrals and k -varifolds with mean curvature in L^p for $p > k$, assuming $C^{1,\alpha}$ regularity of their boundary [2, 9, 14]. In the free-boundary case, regularity results have been proved for area-minimizing currents and varifolds when the domain Ω is either of class C^2 [20–23] or, more recently, a local C^2 -deformation of a wedge-type polyhedral cone [15]. In these cases, monotonicity properties of the renormalized area are shown by testing with locally constructed, almost-radial vector fields that are smooth and tangent to $\partial\Omega$.

The background motivation of this paper is the study of the free-boundary properties and regularity of perimeter almost-minimizers when the boundary of the domain Ω is Lipschitz but not necessarily smooth. We aim to consider Ω that might not be locally of class $C^{1,\alpha}$ or coincide with a smooth deformation of a polyhedral wedge (as assumed in [15]). Carrying over the full regularity program in such a general setting seems particularly challenging. For this reason, in this paper, we focus on the monotonicity property as a preliminary step towards the extension of the free-boundary regularity results mentioned so far.

1.1. Description of the main definitions and results

Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary and $E \subset \Omega$ be a measurable set. In what follows, E will be called a local almost-minimizer of the relative perimeter in Ω if for any $x \in \overline{\Omega}$, there exists $r_x > 0$ such that for any $0 < r < r_x$ and any measurable $F \subset \Omega$

with $F\Delta E \subset\subset B_r(x)$, one has

$$P(E; B_r(x) \cap \Omega) \leq P(F; B_r(x) \cap \Omega) + |F\Delta E|^{\frac{n-1}{n}} \psi_\Omega(E; x, r),$$

for a suitable function $\psi_\Omega(E; x, r)$ such that $\lim_{r \rightarrow 0^+} \psi_\Omega(E; x, r) = 0$. Of course, when the error term $|F\Delta E|^{\frac{n-1}{n}} \psi_\Omega(E; x, r)$ vanishes, we have a local perimeter minimizer. We conveniently introduce the following classical notation: given a function $f \in BV_{\text{loc}}(\Omega)$,¹ we define the *minimality gap* of f in A as

$$\Psi_\Omega(f; A) = |Df(\Omega \cap A)| - \inf \{ |Dg|(\Omega \cap A) : g \in BV_{\text{loc}}(\Omega), \text{spt}(g - f) \subset\subset A \}$$

and, when $f = \mathbf{1}_E$, we set $\Psi_\Omega(E; A) := \Psi_\Omega(\mathbf{1}_E; A)$. If E is an almost-minimizer and $x \in \overline{\Omega}$, it is immediate to check that $\Psi_\Omega(E; B_r(x)) \lesssim r^{n-1} \psi_\Omega(E; x, r)$. The two main applications of Theorem 5.2 (our general monotonicity inequality) are Corollary 5.3 and Theorem 6.1. For them, we will need to assume that the minimality gap on balls centered at $0 \in \partial\Omega$ decays to 0 suitably fast as $r \rightarrow 0^+$, which is expressed through the following summability property:

$$\int_0^R \frac{\Psi_\Omega(E; B_\rho)}{\rho^n} d\rho \simeq \int_0^R \frac{\psi_\Omega(E; 0, r)}{\rho} d\rho < +\infty. \quad (1.2)$$

Several notions of almost-minimality are available in the literature. Among those where the minimality error is in additive form, like ours, we mention the well-known Λ -minimality [5, 24], where the minimality error is estimated by $\Lambda|F\Delta E|$ for some $\Lambda > 0$, which is a special case of our definition. We also remark that the definition of almost-minimality given by Tamanini in [31]² is slightly more general than ours, even though we take direct inspiration from it. The reason why we are forced to consider a slightly stronger almost-minimality is that, in the boundary case, it seems impossible to use the monotonicity inequality to prove lower-density estimates for the perimeter and the volume of E at $x \in \partial\Omega$, as it happens for the interior case. On the other hand, a local almost-minimizer in our sense turns out to fulfill volume and perimeter density estimates directly (see Lemma 3.8). We point out that these estimates are crucially used in the proof of Theorem 6.1.

The key assumption needed for the proof of our monotonicity formula is the aforementioned visibility property, which we now briefly describe. Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary such that $0 \in \partial\Omega$ and in graphical form around 0 with respect to the variables $x' = (x_1, \dots, x_{n-1})$, so that Ω is locally the epigraph of some Lipschitz function of x' . We say that Ω satisfies the visibility property at 0 if there exist $R > 0$ and

¹Here we mean that $f \in L^1_{\text{loc}}(\Omega)$ and has bounded variation in $\Omega \cap A$, for any bounded set $A \subset \mathbb{R}^n$.

²In Tamanini's definition, the term $|F\Delta E|^{\frac{n-1}{n}}$ is replaced by r^{n-1} .

a function $v \in C^1([0, R])$ such that

$$(V1) \quad v(0) = v'(0) = 0 \text{ and } 0 \leq v' \leq 2^{-1};$$

(V2) The function

$$\gamma_v(r) := r^{-1} \sup_{0 < s \leq r} \sqrt{\frac{v(s)}{s} + v'(s)}$$

is summable on $(0, R)$;

(V3) for all $0 < r < R$, the segment joining the point $V_r = (0, \dots, 0, -v(r))$ with a point x belonging to $\partial\Omega \cap B_r$ does not intersect Ω .

The visibility property allows us to construct a quasi-conical competitor for a local almost-minimizer of the relative perimeter in Ω , which is a key step in the proof of our monotonicity inequality. It is worth observing that this property guarantees the existence of the tangent cone to Ω at 0 (Proposition 4.5). The visibility property at 0 is satisfied, for instance, by cones with vertex at 0 (with the trivial choice $v \equiv 0$), by $C^{1,\beta}$ open sets and also by convex sets suitably approximated by their tangent cone at 0; one can also easily construct examples of Lipschitz domains that do not satisfy the property (see Section 4.4 for more details). Another consequence of the visibility property is the existence of a foliation of a neighborhood of 0 by spheres with varying centers, which correspond to the level sets of a function ϕ of class C^1 , that is, such that $\phi^{-1}(r) = \partial B_r(V_r)$. Moreover, by (V1) and (V2) one can show that ϕ is C^1 -close to the function $|x|$ (see Lemma 4.9).

In our main result, Theorem 5.2, we establish the following monotonicity inequality for $f \in BV_{\text{loc}}(\Omega)$, under the visibility property:

$$\begin{aligned} & \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} |\langle v_f, \nabla \phi \rangle| d|Df| \right)^2 \\ & \leq 2 \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla \phi(x)|}{\phi(x)^{n-1}} d|Df| \right) \left[\mu_f(r_2) - \mu_f(r_1) + \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_\Omega(f; B_\rho(V_\rho)) d\rho \right. \\ & \quad \left. + G(f; r_1, r_2) \right]. \end{aligned} \quad (1.3)$$

Here, $0 < r_1 < r_2$ are sufficiently small, $\mathcal{A}_{r_1, r_2} = B_{r_2}(V_{r_2}) \setminus B_{r_1}(V_{r_1})$, v_f represents the Radon–Nikodym derivative of Df with respect to its total variation $|Df|$,

$$\mu_f(r) = \frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}},$$

and $G(f; r_1, r_2)$ is an error term whose precise expression is given in (5.1), which depends on the properties of the boundary of Ω , and goes to 0 when $r_2 \rightarrow 0$ as soon as

$$\mu_f(r) \leq C \quad \text{for all } r > 0 \text{ small enough and for some } C > 0. \quad (1.4)$$

Inequality (1.3) has important consequences when $f = \mathbf{1}_E$ and E is a local almost-minimizer of the relative perimeter in Ω satisfying the assumption in (1.2). Indeed, in this

case, the abovementioned density estimates guarantee that (1.4) holds. We then infer that the function

$$\int_0^r \frac{n-1}{\rho^n} \Psi_\Omega(f; B_\rho(V_\rho)) d\rho + G(f; 0, r)$$

is infinitesimal as $r \rightarrow 0^+$. Consequently, the density ratio $\mu_f(r)$ is almost-monotone, and hence it admits a finite limit $\theta_E(0)$ as $r \rightarrow 0^+$ (see Corollary 5.3).

Finally, the left-hand side of (1.3) can be interpreted as an approximate conical deviation term, since it is as small as E is close to being a cone with vertex at 0 (at least when $\phi(x) = |x|$, i.e., if the visibility property at 0 is satisfied with $v \equiv 0$). This observation is exploited in the proof of Theorem 6.1, where we show that any blow-up sequence of an almost-minimizer of the relative perimeter in Ω admits a subsequence that converges to a minimizing cone in the tangent cone to Ω at 0.

2. Preliminaries

Let $n \in \mathbb{N}$, $n \geq 2$. For $i = 1, \dots, n$, we denote by e_i the i -th vector of the standard basis of \mathbb{R}^n . Given $x, y \in \mathbb{R}^n$, their Euclidean product is $\langle x, y \rangle := \sum_{i=1}^n x_i y_i$, while the Euclidean norm of x is $|x| := \sqrt{\langle x, x \rangle}$. When needed, we will write $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We let $B_r(x)$ be the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$, and we set $B_r := B_r(0)$. Similarly, we let $B'_r(x')$ be the $(n-1)$ -dimensional open ball with center x' and radius r . We denote by either $\mathcal{L}^n(E)$ or $|E|$ the Lebesgue measure of $E \subset \mathbb{R}^n$, and we set $\omega_n := \mathcal{L}^n(B_1)$. Given a measurable set $A \subset \mathbb{R}^n$, we set

$$A^{(t)} := \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = t \right\} \quad \text{for all } 0 \leq t \leq 1.$$

Given two non-empty, bounded subsets $A, B \subset \mathbb{R}^n$, we denote by $\text{dist}_{\mathcal{H}}(A, B)$ the Hausdorff distance between A and B , that is,

$$\text{dist}_{\mathcal{H}}(A, B) = \max \left\{ \sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A) \right\}, \quad (2.1)$$

where $\text{dist}(x, Z) := \inf_{z \in Z} |x - z|$. Note that the Hausdorff distance becomes a proper distance function when restricted to compact sets (see [27]).

Given $\Omega \subset \mathbb{R}^n$ open and a vector-valued Radon measure $\mu = (\mu_1, \dots, \mu_p)$ on Ω , we denote by $|\mu|$ its total variation. Let $u = (u_1, \dots, u_p) : \Omega \rightarrow \mathbb{R}^p$ be summable with respect to $|\mu|$. Then $u \cdot \mu$ denotes the Radon measure defined by

$$u \cdot \mu(E) := \int_E u \cdot d\mu = \sum_{q=1}^p \int_E u_q d\mu_q.$$

It can be proved (see [6, Proposition 1.23]) that

$$|u \cdot \mu| = |u| \cdot \mu.$$

Given a Lipschitz map $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$, we denote by $\text{Lip}(g)$ its Lipschitz constant, and we denote by $\text{Lip}(\mathbb{R}^n)$ the space of Lipschitz real-valued functions. We record for future reference the following, elementary fact:

Lemma 2.1. *Let $\{f_j\}_{j \in \mathbb{N}} \subset \text{Lip}(\mathbb{R}^n)$ be such that $\sup_{j \geq 1} \text{Lip}(f_j) < \infty$. Assume further that $f_j(x) \rightarrow f(x)$ for all $x \in \mathbb{R}^n$. Then $f_j \rightarrow f$ locally uniformly on \mathbb{R}^n .*

Given $f \in L^1_{\text{loc}}(\Omega)$, we define

$$|Df|(\Omega) := \sup \left\{ \int_{\Omega} f(x) \operatorname{div} \phi(x) dx : \phi \in C^1_c(\Omega; \mathbb{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

We denote by $BV(\Omega)$ the space of functions $f \in L^1(\Omega)$ with the property that $|Df|(\Omega) < \infty$. In other words, a function $f \in L^1(\Omega)$ belongs to $BV(\Omega)$ if and only if its distributional gradient Df is represented by a \mathbb{R}^n -valued Radon measure with finite total variation.

When E is a measurable set, we define the perimeter of E in Ω as

$$P(E; \Omega) := |D\mathbf{1}_E|(\Omega),$$

where $\mathbf{1}_E$ is the characteristic function of E . We say that E has *finite perimeter* in Ω provided $P(E; \Omega) < +\infty$. In what follows, we will often use the following notation: given another open set $A \subset \mathbb{R}^n$, we define

$$P_\Omega(E; A) := P(E; A \cap \Omega),$$

with the short form $P_\Omega(E) := P_\Omega(E; A)$ whenever A contains Ω . We call P_Ω the relative perimeter in Ω , and $P_\Omega(E; A)$ the relative perimeter of E in Ω restricted to A .

For the specific purposes of this paper, it is convenient to define $BV_{\text{loc}}(\Omega)$ as the space of functions $f \in L^1_{\text{loc}}(\Omega)$ such that $f \in BV(\Omega \cap A)$ for all open, bounded sets $A \subset \mathbb{R}^n$. We remark that $f \in BV_{\text{loc}}(\Omega)$ implies that f has locally bounded variation in Ω , but the converse does not necessarily hold.

Let now $f \in BV_{\text{loc}}(\Omega)$. As a consequence of [24, Corollary 5.11] we can consider the density of Df with respect to $|Df|$ defined by

$$\nu_f(x) = \lim_{r \rightarrow 0^+} \frac{Df(B_r(x))}{|Df|(B_r(x))} \quad \text{for } |Df|\text{-a.e. } x \in \mathbb{R}^n,$$

and satisfying $Df = \nu_f \cdot |Df|$ and $|\nu_f(x)| = 1$ for $|Df|$ -almost every $x \in \Omega$. When E has locally finite perimeter in Ω , we set $\nu_E = \nu_{\mathbf{1}_E}$.

Next we recall some well-known facts concerning the total variation functional, BV functions, and sets of locally finite perimeter. For further details, the reader can consult, for example, [16, 18, 24].

Theorem 2.2 (Lower semicontinuity). *Let $f, f_j \in L^1_{\text{loc}}(\Omega)$ for $j \in \mathbb{N}$ be such that $f_j \rightarrow f$ in $L^1_{\text{loc}}(\Omega)$. Then*

$$|Df|(\Omega) \leq \liminf_j |Df_j|(\Omega).$$

BV functions can be approximated by smooth functions in a suitable sense.

Theorem 2.3 (Approximation). *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in BV(\Omega)$. Then there exists a sequence $\{f_j\}_{j \in \mathbb{N}} \in C^\infty(\Omega) \cap BV(\Omega)$ such that*

$$\|f_j - f\|_{L^1(\Omega)} \rightarrow 0, \quad |Df_j|(\Omega) \rightarrow |Df|(\Omega). \quad (2.2)$$

Moreover, provided (2.2) holds, we also have $Df_j \rightharpoonup^* Df$, that is, for all $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \phi \cdot dDf_j \rightarrow \int_{\mathbb{R}^n} \phi \cdot dDf, \quad \text{as } j \rightarrow \infty.$$

Remark 2.4. *If property (2.2) holds, we say that the sequence f_j strictly converges to f (see [6, Definition 3.14]). Thus Theorem 2.3 tells us in particular that $C^\infty(\Omega) \cap BV(\Omega)$ is dense in $BV(\Omega)$ with respect to the strict convergence. Moreover, the sequence $\{f_j\}_{j \in \mathbb{N}}$ can be chosen in such a way that the following extra property is satisfied (see [18, Remark 1.18]):*

$$\lim_{\rho \rightarrow 0^+} \rho^{-N} \int_{\Omega \cap B_\rho(x_0)} |f_j - f| dx = 0 \quad \text{for all } N > 0, x_0 \in \partial\Omega, j \in \mathbb{N}. \quad (2.3)$$

Theorem 2.5 (Compactness). *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary, let $C > 0$ be a fixed constant, and for $j \in \mathbb{N}$ let $f_j \in BV(\Omega)$ be such that*

$$\|f_j\|_{BV(\Omega)} := \|f_j\|_{L^1(\Omega)} + |Df_j|(\Omega) \leq C \quad \text{for every } j.$$

Then there exists $f \in BV(\Omega)$ and a subsequence f_{j_k} of f_j such that

$$\|f_{j_k} - f\|_{L^1(\Omega)} \rightarrow 0.$$

Theorem 2.6 (Coarea for BV functions). *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in L^1(\Omega)$. For $t \in \mathbb{R}$, let*

$$E_t = \{x \in \Omega : f(x) > t\}.$$

Then the following statements hold:

- (i) *if $f \in BV(\Omega)$, then E_t has finite perimeter for almost every $t \in \mathbb{R}$, $t \mapsto P(E_t; \Omega)$ is measurable, and*

$$|Df|(\Omega) = \int_{\mathbb{R}} P(E_t; \Omega) dt;$$

- (ii) *conversely, if $f \in L^1(\Omega)$, $t \mapsto P(E_t; \Omega)$, and*

$$\int_{\mathbb{R}} P(E_t; \Omega) dt < +\infty,$$

then $f \in BV(\Omega)$.

Theorem 2.7 (Coarea for Lipschitz functions). *If $\phi \in \text{Lip}(\mathbb{R}^n)$, $E \subset \mathbb{R}^n$ is a Borel set, and $f : \mathbb{R}^n \rightarrow [0, \infty)$ is a Borel function, then*

$$\int_E f |\nabla \phi| = \int_{\mathbb{R}} \int_{E \cap \{\phi=t\}} f d\mathcal{H}^{n-1} dt,$$

where \mathcal{H}^{n-1} denotes the $(n-1)$ -dimensional Hausdorff measure in \mathbb{R}^n .

We will also need the following well-known result, for which proof can be found in [24]:

Theorem 2.8 (Area Formula). *Let $M \subset \mathbb{R}^n$ be a locally \mathcal{H}^k -rectifiable set and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz continuous function, with $1 \leq k \leq m$. Then*

- (i) *for \mathcal{H}^k -almost all $x \in M$, the restriction of f to $x + T_x M$, where $T_x M$ is the approximated tangent space to M at x , is differentiable at x and we denote by $d^M f_x : T_x M \rightarrow \mathbb{R}^m$ its differential;*
- (ii) *the identity*

$$\int_{\mathbb{R}^m} \mathcal{H}^0(M \cap \{f = y\}) d\mathcal{H}^k(y) = \int_M J^M f d\mathcal{H}^k \quad (2.4)$$

holds, where $J^M f(x) := \sqrt{\det(d^M f_x^* \circ d^M f_x)}$ and $d^M f_x^*$ denotes the adjoint of $d^M f_x$.

We continue by introducing the notion of trace of a BV function on a Lipschitz boundary.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary and denote by ν_Ω the unit, inner normal vector defined \mathcal{H}^{n-1} -almost everywhere on $\partial\Omega$. Then there exists a unique linear and bounded inner trace operator*

$$\text{Tr}^+(\cdot, \partial\Omega) : BV(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1}),$$

such that, for all $f \in BV(\Omega)$,

$$\lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} |\text{Tr}^+(f, \partial\Omega)(x) - f(y)| dy = 0 \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Omega. \quad (2.5)$$

Moreover, for all $f \in BV(\Omega)$ and $\phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, the following Gauss–Green formula holds:

$$\int_\Omega f \operatorname{div} \phi dx = -(\phi \cdot Df) \Omega - \int_{\partial\Omega} \langle \phi, \nu_\Omega \rangle \text{Tr}^+(f, \partial\Omega) d\mathcal{H}^{n-1}.$$

The function $\text{Tr}^+(f, \partial\Omega)$ is called the *inner trace* of f on $\partial\Omega$. Formula (2.5) ensures in particular that

$$\text{Tr}^+(f, \partial\Omega)(x) = \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} f(y) dy \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega. \quad (2.6)$$

Remark 2.10. Under the assumptions of Theorem 2.9, the operator $\text{Tr}^+(\cdot, \partial\Omega)$ is also continuous with respect to the strict convergence in Ω (see [18, Theorem 2.11] or [6, Theorem 3.88]). Moreover, by (2.3) and (2.6), we have that the sequence $\{f_j\}_{j \in \mathbb{N}}$ of Theorem 2.3 satisfies

$$\begin{aligned} & |\text{Tr}^+(f_j, \partial\Omega)(x) - \text{Tr}^+(f, \partial\Omega)(x)| \\ &= \left| \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} f_j(y) dy - \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} f(y) dy \right| \\ &\leq \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} |f_j(y) - f(y)| dy = 0 \end{aligned}$$

for \mathcal{H}^{n-1} -almost every $x \in \partial\Omega$. In other words, $\text{Tr}^+(f_j, \partial\Omega) = \text{Tr}^+(f, \partial\Omega)$ \mathcal{H}^{n-1} -almost everywhere on $\partial\Omega$, for all j .

Let $\Omega, \Omega' \subset \mathbb{R}^n$ open sets with Lipschitz boundary, such that $\overline{\Omega} \subset \Omega'$. For any $f \in BV(\Omega')$, we can also consider a trace of f on $\partial\Omega$ computed with respect to $\Omega' \setminus \overline{\Omega}$. We thus set

$$\text{Tr}^-(f, \partial\Omega) := \text{Tr}^+(f, \partial(\Omega' \setminus \overline{\Omega})) \llcorner \partial\Omega.$$

The function $\text{Tr}^-(f, \partial\Omega) \in L^1(\partial\Omega; \mathcal{H}^{n-1})$ is called the *outer trace* of f on $\partial\Omega$.

Lemma 2.11. Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets. In addition, let Ω be bounded and have Lipschitz boundary, and assume that $\Omega \subset\subset \Omega'$. For all $f \in BV(\Omega')$, we have

$$|Df|(\partial\Omega) = \|\text{Tr}^+(f, \partial\Omega) - \text{Tr}^-(f, \partial\Omega)\|_{L^1(\partial\Omega; \mathcal{H}^{n-1})}. \quad (2.7)$$

Lemma 2.12. Fix an open set $A \subset \mathbb{R}^n$ and a Lipschitz function $\phi : A \rightarrow \mathbb{R}$ of class C^1 , such that $|\nabla\phi(x)| > 0$ for all $x \in A$. Set $A_r = \phi^{-1}(-\infty, r)$. Then for all $f \in BV(A)$, for \mathcal{L}^1 -almost every $r \in \mathbb{R}$, and \mathcal{H}^{n-1} -almost every $x \in \partial A_r \cap A$, we have

$$f(x) = \text{Tr}^+(f, \partial A_r)(x) = \text{Tr}^-(f, \partial A_r)(x). \quad (2.8)$$

Proof. We observe that, for almost every $r \in \mathbb{R}$,

$$\text{Tr}^+(f, \partial A_r) = \text{Tr}^-(f, \partial A_r), \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial A_r \cap A. \quad (2.9)$$

Indeed, for the proof of (2.9) we can combine (2.7) with $|Df|(\partial A_r \cap A) = 0$ for almost every $r \in \mathbb{R}$, which in turn comes from the fact that $|Df|$ is a finite measure and $\partial A_r \cap \partial A_s \cap A = \emptyset$ whenever $r \neq s$. Let $x \in A$ be a Lebesgue point for f . Then $x \in \partial A_r$ if and only if $r = \phi(x)$. Thanks to the smoothness of $\partial A_r \cap A$ (a consequence of the Implicit Function Theorem), we have

$$|A_r \cap B_\rho(x)| = \frac{1}{2} \rho^n + o(\rho^n) \quad \text{as } \rho \rightarrow 0^+,$$

and consequently,

$$\begin{aligned}
f(x) &= \lim_{\rho \rightarrow 0^+} \frac{1}{\omega_n \rho^n} \int_{B_\rho(x)} f(y) dy \\
&= \lim_{\rho \rightarrow 0^+} \frac{1}{\omega_n \rho^n} \left(\int_{A_r \cap B_\rho(x)} f(y) dy + \int_{B_\rho(x) \setminus A_r} f(y) dy \right) \\
&= \lim_{\rho \rightarrow 0^+} \frac{1}{2} \frac{1}{|A_r \cap B_\rho(x)|} \int_{A_r \cap B_\rho(x)} f(y) dy + \frac{1}{2} \frac{1}{|B_\rho(x) \setminus A_r|} \int_{B_\rho(x) \setminus A_r} f(y) dy \\
&= \frac{1}{2} \text{Tr}^+(f, \partial A_r)(x) + \frac{1}{2} \text{Tr}^-(f, \partial A_r)(x) = \text{Tr}^\pm(f, \partial A_r)(x).
\end{aligned}$$

Since the set of Lebesgue points for f coincides with A up to a \mathcal{L}^n -negligible set, by Theorem 2.7 we obtain that the first equality in (2.8) is verified for \mathcal{L}^1 -almost every $r \in \mathbb{R}$ and \mathcal{H}^{n-1} -almost every $x \in \partial A_r \cap A$, which together with (2.9) concludes the proof. ■

Proposition 2.13. *Under the assumptions of Lemma 2.12, we take $f, f_j \in BV(A)$ for $j \in \mathbb{N}$ such that*

$$\|f_j - f\|_{L^1(A)} \rightarrow 0, \quad |Df_j|(A) \rightarrow |Df|(A).$$

Then, for almost every $0 < r < 1$, we have

$$\text{Tr}(f, \partial A_r) := \text{Tr}^+(f, \partial A_r) = \text{Tr}^-(f, \partial A_r), \quad (2.10)$$

$$\text{Tr}(f_j, \partial A_r) := \text{Tr}^+(f_j, \partial A_r) = \text{Tr}^-(f_j, \partial A_r) \quad \text{for all } j \geq 1, \quad (2.11)$$

and

$$|Df_j|(A_r) \rightarrow |Df|(A_r), \quad \|\text{Tr}(f_j, \partial A_r) - \text{Tr}(f, \partial A_r)\|_{L^1(\partial A_r)} \rightarrow 0, \quad (2.12)$$

and hence, in particular, f_j strictly converges to f on A_r .

Proof. Thanks to Lemma 2.12, the two identities (2.10) and (2.11) hold for almost every $0 < r < 1$. In particular, for such r , we deduce that $|Df|(\partial A_r) = 0$, hence $|Df|(\overline{A_r}) = |Df|(A_r)$. Moreover, by Theorem 2.2, we have

$$\begin{aligned}
\liminf_{j \rightarrow \infty} |Df_j|(A_r) &\geq |Df|(A_r) = |Df|(\overline{A_r}) = |Df|(A) - |Df|(A \setminus \overline{A_r}) \\
&= \lim_{j \rightarrow \infty} |Df_j|(A) - |Df|(A \setminus \overline{A_r}) \\
&\geq \limsup_{j \rightarrow \infty} |Df_j|(A) - |Df|(A \setminus \overline{A_r}) \\
&= \limsup_{j \rightarrow \infty} |Df_j|(\overline{A_r}) \geq \limsup_{j \rightarrow \infty} |Df_j|(A_r),
\end{aligned}$$

which proves that

$$|Df|(A_r) = \lim_{j \rightarrow \infty} |Df_j|(A_r).$$

Since $\|f_j - f\|_{L^1(A)} \rightarrow 0$, we have in particular $\|f_j - f\|_{L^1(A_r)} \rightarrow 0$, and thus $\{f_j\}_{j \geq 1}$ strictly converges to f in A_r . Finally, (2.12) holds because, as observed in Remark 2.10, the inner trace operator is continuous with respect to the strict convergence. ■

Let $E \subset \Omega$ be a set of locally finite perimeter in Ω . We define the *reduced boundary* of E , denoted by $\partial^* E$, as the set of those points $x \in \Omega$ such that $|D\mathbf{1}_E(B_r(x))| > 0$, for all $r > 0$, and there exists a unit vector $\nu_E(x)$ such that

$$\nu_E(x) = \lim_{r \rightarrow 0^+} \frac{D\mathbf{1}_E(B_r(x))}{|D\mathbf{1}_E(B_r(x))|}.$$

Clearly, we have that $\nu_E(x) = \nu_{\mathbf{1}_E}(x)$ for $|D\mathbf{1}_E|$ -almost every $x \in \Omega$; therefore, the perimeter measure $|D\mathbf{1}_E|$ is concentrated on the reduced boundary $\partial^* E$.

Theorem 2.14 (De Giorgi–Federer’s Structure Theorem). *Let $E \subset \mathbb{R}^n$ be a set of locally finite perimeter. Then*

$$|D\mathbf{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \quad D\mathbf{1}_E = \nu_E \cdot \mathcal{H}^{n-1} \llcorner \partial^* E,$$

and the following Gauss–Green Formula holds:

$$\int_E \nabla \phi \, dx = \int_{\partial^* E} \phi \nu_E \, d\mathcal{H}^{n-1}, \quad \forall \phi \in C_c^1(\mathbb{R}^n).$$

Moreover, $\partial^* E$ is countably $(n-1)$ -rectifiable, that is, there exist countably many C^1 hypersurfaces $M_j \subset \mathbb{R}^n$ and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$ such that

$$\partial^* E \subset F \cup \bigcup_{j \geq 1} M_j.$$

Finally, $\partial^* E \subset E^{(1/2)}$ and $\mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)} \cup \partial^* E)) = 0$.

2.1. A local extension result

From now on, we assume that $\Omega \subset \mathbb{R}^n$ is an open set with Lipschitz boundary. We fix $\rho > 0$ such that, up to an isometry, there exists a Lipschitz function $\omega : B'_\rho \rightarrow \mathbb{R}$ and a constant $m > 0$, with $\omega(0) = 0$ and $m > \|\omega\|_{L^\infty(B'_\rho)}$, satisfying the following property: if we set $\mathcal{C}_{\rho,m} = B'_\rho \times (-m, m)$, we have

$$\Omega \cap \mathcal{C}_{\rho,3m} = \{x = (x', x_n) \in \mathbb{R}^n : x' \in B'_\rho, \omega(x') < x_n < 3m\}. \quad (2.13)$$

We aim to prove that, under this assumption, any measurable set $E \subset \Omega$ with $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$ ³ can be extended to a locally finite perimeter set \tilde{E} in $\Omega \cup \mathcal{C}_{\rho,m}$, in such a way that $\tilde{E} \cap \Omega = E \cap \Omega$, $P(\tilde{E}; \partial\Omega \cap \mathcal{C}_{\rho,m}) = 0$, and $P(\tilde{E}; S(B)) \leq C P(E; B)$, for all

³We recall that in this paper $f \in BV_{\text{loc}}(\Omega)$ means $f \in BV(A)$ for all $A \subset \Omega$ open and bounded.

Borel sets $B \subset \mathcal{C}_{\rho,m} \setminus \Omega$ and for some constant $C > 0$ depending on the dimension n and the function ω . In what follows, we will denote by $T_x E$ the approximate tangent space to $\partial^* E$ at x .⁴

Set $\mathcal{C}_\rho = B'_\rho \times \mathbb{R}$ and define the map $S : \mathcal{C}_\rho \rightarrow \mathcal{C}_\rho$ as

$$S(x) := (x', 2\omega(x') - x_n). \quad (2.14)$$

Note that S satisfies $S^2(x) = x$ for all x . Moreover, elementary computations show that

$$\text{Lip}(S) \leq \sqrt{3 + 6\text{Lip}(\omega)^2}. \quad (2.15)$$

Given $E \subset \Omega$ measurable with $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$, we define $\tilde{E} \subset \Omega \cup \mathcal{C}_\rho$ as

$$\tilde{E} = E \cup (S_\rho(E) \setminus \Omega), \quad (2.16)$$

where $S_\rho(E) = S(E \cap \mathcal{C}_\rho)$. Clearly, we have $\tilde{E} \cap \Omega = E \cap \Omega$. Further properties of \tilde{E} are stated in the next lemma.

Lemma 2.15. *Let $E \subset \Omega$ be a measurable set with $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$. Then, for almost all $x \in \partial^* E$, S restricted to $x + T_x E$ is differentiable at x , and we denote by $d^E S_x : T_x E \rightarrow \mathbb{R}^n$ its differential. Moreover, if \tilde{E} is the set defined in (2.16), we have*

$$P(\tilde{E}; \partial\Omega \cap \mathcal{C}_{\rho,m}) = 0 \quad (2.17)$$

and, for all Borel sets $B \subset S(\mathcal{C}_{\rho,m} \cap \Omega)$,

$$P(\tilde{E}; B) = P(S_\rho(E); B) = \int_{\partial^* E \cap S(B)} J^E S(x) d\mathcal{H}^{n-1}(x) \leq C P(E; S(B)), \quad (2.18)$$

where and $C = \text{Lip}(S)^{n-1}$.

Proof. Owing to Theorem 2.14, we know that $\partial^* E$ is countably $(n-1)$ -rectifiable. Then the fact that $S|_{T_x E}$ is differentiable at \mathcal{H}^{n-1} -almost every $x \in \partial^* E$ follows immediately from Theorem 2.8(i).

Now, we prove (2.17) in the following way: thanks to Lemma 2.11, we only need to check that, for \mathcal{H}^{n-1} -almost every $x \in \partial\Omega \cap \mathcal{C}_{\rho,m}$, we have

$$\text{Tr}^+(\mathbf{1}_{\tilde{E}}, \partial\Omega)(x) = \text{Tr}^-(\mathbf{1}_{\tilde{E}}, \partial\Omega)(x),$$

that is,

$$\text{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = \text{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x). \quad (2.19)$$

The proof of (2.19) goes as follows. We employ the characterization of the trace as a limit of averages: for \mathcal{H}^{n-1} -almost every $x \in \partial\Omega$, we have

$$\text{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = \lim_{r \rightarrow 0^+} \frac{|E \cap B_r(x) \cap \Omega|}{|B_r(x) \cap \Omega|}$$

⁴The approximate tangent space $T_x E$ is given by the orthogonal complement of $\nu_E(x)$.

and

$$\mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = \lim_{r \rightarrow 0^+} \frac{|S_\rho(E) \cap B_r(x) \setminus \Omega|}{|B_r(x) \setminus \Omega|}.$$

Then we combine this characterization with a consequence of (2.5)—namely, that the trace of a BV characteristic function coincides with a characteristic function \mathcal{H}^{n-1} -almost everywhere on $\partial\Omega$, to infer that we only need to show the equivalence

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = 0 \quad \Leftrightarrow \quad \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = 0.$$

One of the two required implications (the other can be discussed similarly) is

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = 0 \quad \Rightarrow \quad \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = 0.$$

This implication can be restated as

$$|E \cap B_r(x) \cap \Omega| = o(r^n) \quad \Rightarrow \quad |S_\rho(E) \cap B_r(x) \setminus \Omega| = o(r^n) \quad \text{as } r \rightarrow 0^+. \quad (2.20)$$

Up to taking $r > 0$ small enough, we have $B_r(x) \subset \mathcal{C}_{\rho,m}$, and hence, setting $L = \mathrm{Lip}(S)$ we get

$$\begin{aligned} |S_\rho(E) \cap B_r(x) \setminus \Omega| &\leq |S(E \cap S(B_r(x)) \cap \Omega)| \leq L^n |E \cap B_{Lr}(x) \cap \Omega| \\ &= L^n o(L^n r^n) = o(r^n) \quad \text{as } r \rightarrow 0^+, \end{aligned}$$

which proves the implication in (2.20) and concludes the proof of (2.17).

Finally, for the proof of (2.18), it is enough to show that

$$\mathcal{H}^{n-1}(\partial^* S(E) \Delta S(\partial^* E)) = 0. \quad (2.21)$$

Indeed, if (2.21) holds, Theorem 2.14 ensures that

$$P(S(E); B) = \mathcal{H}^{n-1}(\partial^* S(E) \cap B) = \mathcal{H}^{n-1}(S(\partial^* E) \cap B) = \mathcal{H}^{n-1}(S(\partial^* E \cap S(B))),$$

thus (2.18) is an immediate consequence of the Area Formula for rectifiable sets given in (2.4). Let us demonstrate (2.21). Again by Theorem 2.14, it suffices to prove that

$$S(E)^{(0)} = S(E^{(0)}), \quad S(E)^{(1)} = S(E^{(1)}).$$

Let us prove the first of the previous identities, as the proof of the other one is obtained by observing that $(\mathbb{R}^n \setminus E)^{(0)} = E^{(1)}$. Let $0 < \delta < \rho$, and $x \in \Omega \cap (B'_{\rho-\delta} \times \mathbb{R})$. Set $L = \mathrm{Lip}(S)$ as before, then by construction, for any $0 < r < \delta$, $S(B_r(S(x))) \subset B_{Lr}(x)$, and thus the Area Formula (see (2.4)) yields

$$|S(E) \cap B_r(S(x))| = \int_{E \cap B_r(S(x))} JS \, dx \leq L^n |B_{Lr}(x)|. \quad (2.22)$$

Since r is arbitrary and $S^{-1} = S$, it is easy to check that, thanks to (2.22), if $x \in E^{(0)}$ then $S(x) \in S(E)^{(0)}$, which proves the inclusion $S(E^{(0)}) \subset S(E)^{(0)}$. The reverse inclusion is proved in a completely analogous way. The proof of the lemma is then achieved thanks to (2.15). \blacksquare

3. Almost minimality

Here we introduce the almost-minimality for the relative perimeter, which is needed in the proofs of the free-boundary monotonicity results shown in Section 5. As a consequence of our definition, we show some properties of the *minimality gap* function, which will be needed later on, and (boundary) density estimates for the volume and the perimeter of an almost-minimizer.

Definition 3.1 (Almost-minimality). *Given $\Omega \subset \mathbb{R}^n$ open and $E \subset \Omega$ measurable, we say that E is a local almost-minimizer of P_Ω if, for any $x \in \overline{\Omega}$ there exists $r_x > 0$ such that, for any $0 < r < r_x$ and any measurable $F \subset \Omega$ with $F \Delta E \subset \subset B_r(x)$, one has*

$$P_\Omega(E; B_r(x)) \leq P_\Omega(F; B_r(x)) + |F \Delta E|^{\frac{n-1}{n}} \psi_\Omega(E; x, r), \quad (3.1)$$

for a suitable function $\psi_\Omega(E; x, r)$ such that $\lim_{r \rightarrow 0^+} \psi_\Omega(E; x, r) = 0$. If additionally

$$\psi_\Omega(E; r) := \sup_{x \in \overline{\Omega}} \psi_\Omega(E; x, r) \rightarrow 0 \quad \text{as } r \rightarrow 0^+,$$

then we say that E is an almost-minimizer of P_Ω .

We note that if $\psi_\Omega(E; x, r) = 0$, then E is a minimizer of P_Ω in $B_r(x)$. In particular, if $\psi_\Omega(E; x, r) = 0$ for all $x \in \overline{\Omega}$ and $r > 0$, then E is a (global) minimizer of P_Ω . We shall later discuss suitable summability properties of the function $r \rightarrow \psi_\Omega(E; x, r)$, which are required in the proof of the monotonicity formula.

It is worth recalling that, among the various notions of almost-minimality for sets of locally finite perimeter that can be found in the literature, the one expressed by (3.1) can be understood as a generalization of the well-known Λ -minimality property [5, 24]; however, it is slightly more restrictive than the notion considered by Tamanini in [31] (however, we point out that Tamanini's work is focused on internal regularity theory). For more completeness, we recall that a set E is a Λ -minimizer of P_Ω if, for any $x \in \overline{\Omega}$, there exists $r_x > 0$ such that, for any $0 < r < r_x$ and any measurable subset F of Ω with $F \Delta E \subset \subset B_r(x)$, one has

$$P_\Omega(E; B_r(x)) \leq P_\Omega(F; B_r(x)) + \Lambda |F \Delta E|.$$

It is then clear that a Λ -minimizer satisfies (3.1) with

$$\psi_\Omega(E; x, r) = \Lambda r \omega_n^{1/n}.$$

For better study of almost-minimizers, it is convenient to introduce the notion of *minimality gap*.

Definition 3.2. *Let $\Omega, A \subset \mathbb{R}^n$ be open sets and $f \in BV_{\text{loc}}(\Omega)$. The minimality gap of f in A relative to Ω is*

$$\Psi_\Omega(f; A) = |Df(\Omega \cap A)| - \inf \{ |Dg|(\Omega \cap A) : g \in BV_{\text{loc}}(\Omega), \text{spt}(g - f) \subset \subset A \}.$$

When $f = \mathbf{1}_E$, for some measurable subset $E \subset \mathbb{R}^n$, we write $\Psi_\Omega(E; A)$ in place of $\Psi_\Omega(\mathbf{1}_E; A)$.

3.1. Some estimates for the minimality gap

Lemma 3.3. *Let $\Omega, A \subset \mathbb{R}^n$ be open sets and $E \subset \Omega$ be such that $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$. Then*

$$\Psi_{\Omega}(E; A) = P_{\Omega}(E; A) - \inf \{P_{\Omega}(F; A) : \mathbf{1}_F \in BV_{\text{loc}}(\Omega), F \Delta E \subset\subset A\}.$$

Proof. Let us define

$$\begin{aligned} \mathcal{I}_1 &= \inf\{|Dg|(\Omega \cap A) : g \in BV_{\text{loc}}(\Omega), \text{spt}(g - \mathbf{1}_E) \subset\subset A\}, \\ \mathcal{I}_2 &= \inf\{P_{\Omega}(F; A) : \mathbf{1}_F \in BV_{\text{loc}}(\Omega), F \Delta E \subset\subset A\}. \end{aligned}$$

It suffices to show that $\mathcal{I}_1 = \mathcal{I}_2$. For sure, $\mathcal{I}_1 \leq \mathcal{I}_2$ because we can take $g = \mathbf{1}_F$ in the definition of \mathcal{I}_1 . Fix now $\varepsilon > 0$, and let $g \in BV_{\text{loc}}(\Omega)$ be such that $\text{spt}(g - \mathbf{1}_E) \subset\subset A$ and

$$|Dg|(\Omega \cap A) \leq \mathcal{I}_1 + \varepsilon. \quad (3.2)$$

For $t \in \mathbb{R}$, let us set

$$G_t = \{x \in \Omega : g(x) > t\}.$$

We observe that, for each $0 < t < 1$, $G_t \setminus \text{spt}(g - \mathbf{1}_E) = E \setminus \text{spt}(g - \mathbf{1}_E)$, and so

$$G_t \Delta E \subset \text{spt}(g - \mathbf{1}_E) \subset\subset A. \quad (3.3)$$

We can now exploit Theorem 2.6 and (3.3) to infer that

$$|Dg|(\Omega \cap A) = \int_{\mathbb{R}} P_{\Omega}(G_t; A) dt \geq \int_0^1 P_{\Omega}(G_t; A) dt \geq \mathcal{I}_2. \quad (3.4)$$

By (3.2) and (3.4), we deduce that $\mathcal{I}_1 \leq \mathcal{I}_2 \leq \mathcal{I}_1 + \varepsilon$ and then, since $\varepsilon > 0$ is arbitrary, we get $\mathcal{I}_1 = \mathcal{I}_2$ and complete the proof of the lemma. \blacksquare

Remark 3.4. *The minimality gap of a local almost-minimizer E satisfies the estimate*

$$\Psi_{\Omega}(E; B_r(x)) \leq \omega_n^{1-\frac{1}{n}} r^{n-1} \psi_{\Omega}(E; x, r)$$

for all $x \in \overline{\Omega}$, $0 < r < r_x$, where $\psi_{\Omega}(E; x, r)$ is the function appearing in (3.1). This means that $\Psi_{\Omega}(E; B_r(x)) = o(r^{n-1})$ as $r \rightarrow 0$. However, we anticipate here that the monotonicity formula proved in Section 5 will require a slightly stronger assumption on $\psi_{\Omega}(E; x, r)$, namely, that $r^{-1}\psi_{\Omega}(E; x, r)$ is summable on $(0, r_x)$. We notice that this kind of hypothesis is somehow well known in the context of regularity theory (see [31]).

Lemma 3.5. *Let $\Omega, A, A' \subset \mathbb{R}^n$ be open sets and $f \in BV_{\text{loc}}(\Omega)$. If $A \subset\subset A'$, then*

$$\Psi_{\Omega}(f; A) \leq \Psi_{\Omega}(f; A').$$

Proof. Let us fix $\varepsilon > 0$ and take $g \in BV_{\text{loc}}(\Omega)$ such that

$$\text{spt}(g - f) \subset\subset A \quad \text{and} \quad |Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \geq \Psi_{\Omega}(f; A) - \varepsilon.$$

Of course, g also satisfies $\text{spt}(g - f) \subset\subset A'$, hence we get

$$\begin{aligned} \Psi_{\Omega}(f; A') &\geq |Df|(\Omega \cap A') - |Dg|(\Omega \cap A') \\ &= |Df|(\Omega \cap A) + |Df|(\Omega \cap (A' \setminus A)) - |Dg|(\Omega \cap A) \\ &\quad - |Df|(\Omega \cap (A' \setminus A)) \\ &= |Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \geq \Psi_{\Omega}(f; A) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude the proof. \blacksquare

We now prove two key results. The first one is the lower semicontinuity property of the minimality gap for uniform sequences of local almost-minimizers. An extra, technical difficulty arising in the proof is that the tangent cone to the domain may not locally contain the dilations of the domain itself, since we are in the boundary case. This requires us to suitably extend a competitor from the tangent cone Ω_0 to the rescaled domains of the form $t^{-1}\Omega$, $t \rightarrow 0^+$ via Lemma 2.15. The second is an upper bound for the difference between the minimality gaps of two BV_{loc} functions.

In what follows, we will say that $\Omega_j \rightarrow \Omega$ locally in Hausdorff distance if there exist $r_0, m, L > 0$, with $r_0 < m$, and L -Lipschitz functions $\omega_j, \omega : B'_{r_0} \rightarrow (-m, m)$ providing local graphical representations of $\partial\Omega_j, \partial\Omega$, respectively, as in (2.13), such that $\omega_j \rightarrow \omega$ uniformly in B'_{r_0} .

Lemma 3.6. *For $j \in \mathbb{N}$, we let $\Omega_j, \Omega \subset \mathbb{R}^n$ be open sets with uniformly Lipschitz boundary, such that $0 \in \partial\Omega$ and $\Omega_j \rightarrow \Omega$ locally in Hausdorff distance. Let E_j, E be sets of locally finite perimeter, such that $E_j \subset \Omega_j$, $E \subset \Omega$, and $E_j \rightarrow E$ in $L^1(B_{r_0})$. Finally, we assume that E_j satisfies (3.1) for all $0 < r < r_0$ and $x = 0$, and that moreover we have*

$$\lim_{r \rightarrow 0^+} \sup_j \psi_{\Omega_j}(E_j; 0, r) = 0.$$

Then, E satisfies (3.1) for all $0 < r < r_0$, with $\psi_{\Omega}(E; 0, r) = \sup_j \psi_{\Omega_j}(E_j; 0, r)$, and

$$\liminf_j \Psi_{\Omega_j}(E_j; B_{r_0}) \geq \Psi_{\Omega}(E; B_{r_0}). \quad (3.5)$$

Moreover, if $\Psi_{\Omega_j}(E_j; B_{r_0}) \rightarrow 0$ as $j \rightarrow \infty$, then for almost all $0 < r < r_0$ we have

$$\lim_{j \rightarrow \infty} P_{\Omega_j}(E_j; B_r) = P_{\Omega}(E; B_r) \quad \text{for almost all } r > 0. \quad (3.6)$$

Proof. Let us fix $\varepsilon > 0$. Let $F \subset \Omega$ be such that $F \Delta E \subset\subset B_{r_0}$ and

$$\Psi_{\Omega}(E; B_{r_0}) \leq P_{\Omega}(E; B_{r_0}) - P_{\Omega}(F; B_{r_0}) + \varepsilon. \quad (3.7)$$

By Lemma 2.12, for all $j \geq 1$, for almost every $0 < r < r_0$ and \mathcal{H}^{n-1} -a.e. $x \in \partial B_r$, we have

$$\mathbf{1}_{E_j}(x) = \text{Tr}^{\pm}(E_j, \partial B_r)(x) \quad \text{and} \quad \mathbf{1}_E(x) = \text{Tr}^{\pm}(E, \partial B_r)(x), \quad (3.8)$$

where $\text{Tr}^\pm(A, \partial B_r) := \text{Tr}^\pm(\mathbf{1}_A, \partial B_r)$. By the L^1_{loc} -convergence of E_j to E , we can choose $r < r_0$ with the above property and the additional

$$P_\Omega(E; B_r) \geq P_\Omega(E; B_{r_0}) - \varepsilon, \quad (3.9)$$

then take j_ε large enough, such that

$$F \Delta E \subset\subset B_r, \quad (3.10)$$

$$\int_{\partial B_r} |\mathbf{1}_{E_j} - \mathbf{1}_E| d\mathcal{H}^{n-1} < \varepsilon \quad \text{for } j \geq j_\varepsilon. \quad (3.11)$$

Let us fix $\delta > 0$, define

$$U_\delta := \{x = (x', x_n) \in \mathcal{C}_{r,m} : \omega(x') - \delta < x_n \leq \omega(x')\},$$

and assume δ so small that

$$\mathcal{H}^{n-1}(\partial B_r \cap U_\delta) < \varepsilon \quad (3.12)$$

and

$$P_\Omega(E; B_r) \leq P(E; B_r \cap (\Omega + \delta e_n)) + \varepsilon. \quad (3.13)$$

Owing to the uniform convergence of ω_j to ω , we can select $j_\delta \geq 1$ such that $\|\omega_j - \omega\|_\infty < \delta$ for all $j \geq j_\delta$, then for those j we define

$$A_j := \Omega_j \cap \Omega \cap B_r, \quad B_j := (\Omega_j \setminus \Omega) \cap B_r,$$

and observe that $B_j \subset U_\delta$. Now we set

$$F_j := (\tilde{F} \cap \Omega_j \cap B_r) \cup (E_j \cap (B_{r_0} \setminus B_r)), \quad (3.14)$$

where $\tilde{F} = F \cup (S(F) \setminus \Omega)$ and S is the symmetry through $\partial\Omega$ defined in (2.14) (with $\rho = r_0$). Thanks to (3.10), we have $F_j \subset \Omega_j$ and $F_j \Delta E_j \subset\subset B_{r_0}$, which means that F_j is a competitor for E_j in the definition of $\Psi_{\Omega_j}(E_j, B_{r_0})$. Moreover, by (3.14), (3.11), and (3.12), we have

$$\begin{aligned} P_{\Omega_j}(F_j; B_{r_0}) &\leq P_{\Omega_j}(F_j; B_r) + P_{\Omega_j}(E_j; B_{r_0} \setminus \overline{B_r}) \\ &\quad + \int_{\partial B_r \cap \Omega_j \cap \Omega} |\mathbf{1}_{E_j} - \mathbf{1}_E| d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(\partial B_r \cap U_\delta) \\ &\leq P_{\Omega_j}(F_j; B_r) + P_{\Omega_j}(E_j; B_{r_0} \setminus \overline{B_r}) + 2\varepsilon. \end{aligned} \quad (3.15)$$

Let us compute $P(F_j; B_r \cap \Omega_j)$. By Lemma 2.15, $P(\tilde{F}; \partial\Omega \cap \mathcal{C}_{r_0,m}) = 0$, hence

$$P_{\Omega_j}(F_j; B_r) = P(F; A_j) + P(S(F); B_j). \quad (3.16)$$

Again by Lemma 2.15, up to possibly taking a smaller δ , we have

$$P(S(F); B_j) = P(\tilde{F}; B_j) \leq P(\tilde{F}; U_\delta) \leq \varepsilon. \quad (3.17)$$

Putting together (3.15), (3.16), and (3.17), and taking into account (3.8), we obtain

$$\begin{aligned}\Psi_{\Omega_j}(E_j; B_{r_0}) &\geq P_{\Omega_j}(E_j; B_{r_0}) - P_{\Omega_j}(F_j; B_{r_0}) \geq P_{\Omega_j}(E_j; B_r) - P(F; A_j) - 3\varepsilon \\ &\geq P_{\Omega_j}(E_j; B_r) - P_{\Omega}(F; B_{r_0}) - 3\varepsilon.\end{aligned}\quad (3.18)$$

Now, since for all $j \geq j_\varepsilon$ we have $\|\omega_j - \omega\|_\infty < \delta$, we infer that

$$B_r \cap (\Omega + \delta e_n) \subset B_r \cap \Omega_j.$$

This inclusion combined with (3.18) and (3.13) gives

$$\Psi_{\Omega_j}(E_j; B_{r_0}) \geq P(E_j; B_r \cap (\Omega + \delta e_n)) - P(F; B_{r_0} \cap \Omega) - 3\varepsilon, \quad (3.19)$$

so that taking the \liminf as $j \rightarrow \infty$ in (3.19), and using the lower semicontinuity of the perimeter, (3.13), (3.9), and (3.7), we find

$$\begin{aligned}\liminf_j \Psi_{\Omega_j}(E_j; B_{r_0}) &\geq \liminf_j P(E_j; B_r \cap (\Omega + \delta e_n)) - P_{\Omega}(F; B_{r_0}) - 3\varepsilon \\ &\geq P(E; B_r \cap (\Omega + \delta e_n)) - P_{\Omega}(F; B_{r_0}) - 3\varepsilon \\ &\geq P_{\Omega}(E; B_r) - P_{\Omega}(F; B_{r_0}) - 4\varepsilon \\ &\geq P_{\Omega}(E; B_{r_0}) - P_{\Omega}(F; B_{r_0}) - 5\varepsilon \geq \Psi_{\Omega}(E; B_{r_0}) - 6\varepsilon.\end{aligned}$$

Then, the arbitrary choice of ε implies (3.5). Now, the fact that E satisfies (3.1) with $\psi_{\Omega}(E; 0, r)$ as in the statement can be proved with the same argument used to show (3.5), also taking into account that $|E_j \Delta F_j| \rightarrow |E \Delta F|$ as $j \rightarrow \infty$. Finally, to prove (3.6) we consider the sequence F_j defined as before, but now with $F = E$. Choosing $\varepsilon > 0$ arbitrarily, for j large enough we obtain as before

$$\Psi_{\Omega_j}(E_j; B_{r_0}) \geq P_{\Omega_j}(E_j; B_{r_0}) - P_{\Omega_j}(F_j; B_{r_0}) \geq P_{\Omega_j}(E_j; B_r) - P_{\Omega}(E; B_r) - 3\varepsilon,$$

which gives the desired conclusion. \blacksquare

Lemma 3.7. *Let $\Omega, A \subset \mathbb{R}^n$ be open sets, with $\partial\Omega$ Lipschitz and A bounded, of class C^2 , and such that $\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) = 0$. Let $f, g \in BV_{\text{loc}}(\Omega)$, then*

$$\begin{aligned}|\Psi_{\Omega}(f, A) - \Psi_{\Omega}(g, A)| &\leq \left| |Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \right| \\ &\quad + \|\text{Tr}^+(f_0, \partial A) - \text{Tr}^+(g_0, \partial A)\|_{L^1(\partial A)},\end{aligned}\quad (3.20)$$

where f_0, g_0 denote the zero-extensions of f and g respectively on $A \setminus \Omega$.

Proof. Given $\varepsilon > 0$, there exists $h \in BV_{\text{loc}}(\Omega)$ with $\text{spt}(h - f) \subset\subset A$, such that

$$\begin{aligned}\Psi_{\Omega}(f, A) &\leq |Df|(A \cap \Omega) - |Dh|(A \cap \Omega) + \varepsilon \\ &\leq \left| |Df|(A \cap \Omega) - |Dg|(A \cap \Omega) \right| + \Psi_{\Omega}(g, A) + |D\tilde{h}|(A \cap \Omega) \\ &\quad - |Dh|(A \cap \Omega) + \varepsilon,\end{aligned}\quad (3.21)$$

where $\tilde{h} \in BV_{\text{loc}}(\Omega)$ will be suitably chosen, so that in particular $\text{spt}(\tilde{h} - g) \subset\subset A$. For the definition of \tilde{h} , we claim that it is possible to construct a sequence $A^{(k)}$ of inner parallel sets of A that converge to A , for which $|Df_0|(\partial A^{(k)}) = |Dg_0|(\partial A^{(k)}) = 0$ and, moreover,

$$\lim_k \int_{\partial A^{(k)}} |\text{Tr}^+(f_0 - g_0, \partial A^{(k)})| d\mathcal{H}^{n-1} = \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1}. \quad (3.22)$$

For the proof of (3.22) we argue as follows. Since A is of class C^2 , there exists $\delta > 0$ such that, for all $0 < t < \delta$, the map $\zeta_t(x) = x + t\nu_A(x)$ is a diffeomorphism of class C^1 between ∂A and the boundary ∂A_t of the inner parallel set $A_t = \{x \in A : \text{dist}(x, \partial A) > t\}$. Now, we consider two sequences $f_{0,j}, g_{0,j}$ of smooth approximations of f_0, g_0 on A , with traces $\text{Tr}^+(f_{0,j}, \partial A) = \text{Tr}^+(f_0, \partial A)$ and $\text{Tr}^+(g_{0,j}, \partial A) = \text{Tr}^+(g_0, \partial A)$, respectively (see Remark 2.10). By inspecting the proof of Anzellotti–Giaquinta’s approximation theorem, it is not restrictive to ask that the sequences $f_{0,j}, g_{0,j}$ also satisfy

$$\int_{A \setminus A_{1/k}} (|f_{0,j} - f_0| + |g_{0,j} - g_0|) dx \leq \frac{1}{k^2}, \quad (3.23)$$

for all j and for $k > \delta^{-1}$. We note that the tangential Jacobian of ζ_t satisfies $J\zeta_t(x) = 1 + O(t)$; hence, the area formula gives

$$\begin{aligned} & \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)| d\mathcal{H}^{n-1}(y) \\ &= (1 + O(t)) \int_{\partial A} |f_{0,j}(\zeta_t(x)) - g_{0,j}(\zeta_t(x))| d\mathcal{H}^{n-1}(x). \end{aligned} \quad (3.24)$$

As $t \rightarrow 0^+$ we have $\zeta_t(x) \rightarrow x$ uniformly. Therefore, by following the same Cauchy-sequence argument as in the classical construction of the trace (see, e.g., [16]), the compositions $f_{0,j}(\zeta_t(x))$ and $g_{0,j}(\zeta_t(x))$ can be shown to converge in $L^1(\partial A)$ to some limits $\hat{f}_{0,j}$ and $\hat{g}_{0,j}$, respectively. Hence, (3.24) implies

$$\lim_{t \rightarrow 0^+} \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)| d\mathcal{H}^{n-1}(y) = \int_{\partial A} |\hat{f}_{0,j}(x) - \hat{g}_{0,j}(x)| d\mathcal{H}^{n-1}(x). \quad (3.25)$$

At the same time, if we choose a vector field $\xi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ and set either $u_j = f_{0,j}$ or $u_j = g_{0,j}$, by the Gauss–Green Theorem we obtain

$$\begin{aligned} \int_{A_t} (u_j \text{div } \xi + \nabla u_j \cdot \xi) dx &= - \int_{\partial A_t} u_j \xi \cdot \nu_{A_t} d\mathcal{H}^{n-1} \\ &= -(1 + O(t)) \int_{\partial A} u_j(\zeta_t(x)) \xi(\zeta_t(x)) \cdot \nu_A(x) d\mathcal{H}^{n-1}(x); \end{aligned}$$

hence, taking the limit as $t \rightarrow 0^+$ gives

$$\int_A (u_j \text{div } \xi + \nabla u_j \cdot \xi) dx = - \int_{\partial A} \hat{u}_j \xi \cdot \nu_A d\mathcal{H}^{n-1},$$

which means that $\widehat{f}_{0,j}$ and $\widehat{g}_{0,j}$ coincide, respectively, with $\text{Tr}^+(f_0, \partial A)$ and $\text{Tr}^+(g_0, \partial A)$ up to \mathcal{H}^{n-1} -null sets and for all j , by the uniqueness of the trace. We can thus rewrite (3.25) as

$$\begin{aligned} & \lim_{t \rightarrow 0^+} \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)| d\mathcal{H}^{n-1}(y) \\ &= \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)(x)| d\mathcal{H}^{n-1}(x). \end{aligned} \quad (3.26)$$

To get (3.22) from (3.26), we must choose $A^{(k)}$ appropriately. To this aim, we apply the coarea formula to the integral in (3.23) and average the resulting inequality, deducing the existence of $0 < t_k < 1/k$ such that for all j ,

$$\int_{\partial A^{(k)}} (|f_{0,j} - \text{Tr}(f_0, \partial A^{(k)})| + |g_{0,j} - \text{Tr}(g_0, \partial A^{(k)})|) d\mathcal{H}^{n-1} \leq \frac{1}{k}, \quad (3.27)$$

where we have set $A^{(k)} = A_{t_k}$. By the triangle inequality and (3.27), we obtain

$$\int_{\partial A^{(k)}} |\text{Tr}^+(f_0 - g_0, \partial A^{(k)})| d\mathcal{H}^{n-1} \leq \int_{\partial A^{(k)}} |f_{0,j} - g_{0,j}| d\mathcal{H}^{n-1} + \frac{1}{k},$$

which gives (3.22) at once from (3.26).

Now we observe that $|Df_0|(\partial A^{(k)}) = |Dg_0|(\partial A^{(k)}) = 0$ because the inner and outer traces of f_0 and g_0 on $\partial A^{(k)}$ coincide; hence, we can define

$$\tilde{h} = h\mathbf{1}_{A^{(k)}} + g\mathbf{1}_{A \setminus A^{(k)}}.$$

Note that $\text{spt}(\tilde{h} - g) \subset\subset A$ and, if k is large enough, $\text{spt}(h - f) \subset\subset A^{(k)}$, so that

$$\begin{aligned} |D\tilde{h}|(A \cap \Omega) &\leq |Dh|(A^{(k)} \cap \Omega) + |Dg_0|(A \setminus \overline{A^{(k)}}) \\ &\quad + \int_{\partial A^{(k)}} |\text{Tr}(f_0 - g_0, \partial A^{(k)})| d\mathcal{H}^{n-1}. \end{aligned}$$

By choosing k large enough, we obtain $|Dg_0|(A \setminus A^{(k)}) < \varepsilon$, and thus

$$|D\tilde{h}|(A \cap \Omega) - |Dh|(A \cap \Omega) \leq \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1} + \varepsilon. \quad (3.28)$$

By combining (3.21) and (3.28), we get

$$\begin{aligned} \Psi_\Omega(f, A) &\leq \|Df|(A \cap \Omega) - |Dg|(A \cap \Omega)\| + \Psi_\Omega(g, A) \\ &\quad + \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1} + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we conclude

$$\begin{aligned} \Psi_\Omega(f, A) - \Psi_\Omega(g, A) &\leq \|Df|(A \cap \Omega) - |Dg|(A \cap \Omega)\| \\ &\quad + \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1} \end{aligned}$$

and, by exchanging the role of f and g in the argument above, we obtain (3.20). \blacksquare

3.2. Density estimates

In this subsection we establish perimeter and volume density estimates for almost-minimizers at a point either in Ω or on $\partial\Omega$.

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^n$ be an open set with a Lipschitz boundary. Let E be an almost-minimizer in Ω , and let $x \in \overline{\Omega}$. Assume that $P_\Omega(E; B_r(x)) > 0$ for all $r > 0$. Then, there exist constants $C \geq 1$, $r_0 > 0$, depending on Ω , E and x , such that*

$$C^{-1}r^{n-1} \leq P_\Omega(E; B_r(x)) \leq Cr^{n-1}, \quad (3.29)$$

$$\min(|E \cap B_r(x) \cap \Omega|, |(B_r(x) \cap \Omega) \setminus E|) \geq C^{-1}r^n, \quad (3.30)$$

for all $0 < r < r_0$.

Proof. Up to a translation, we assume that $x = 0$. We start proving (3.30). Given $r > 0$ we set

$$m(r) := |B_r \cap \Omega \cap E|, \quad \mu(r) := |B_r \cap \Omega \setminus E|.$$

Both m and μ are non-decreasing, and thus differentiable for almost all $r > 0$. By [24, Example 13.3], for almost all $r > 0$, we have

$$m'(r) = \mathcal{H}^{n-1}(E \cap \partial B_r \cap \Omega), \quad \mu'(r) = \mathcal{H}^{n-1}(\partial B_r \cap \Omega \setminus E).$$

Now, up to an isometry, and for r_0 sufficiently small and $L \geq 1$ sufficiently large, if we set $Q_{L,r_0} = B'_{r_0} \times (-Lr_0, Lr_0)$ and $\Omega_{L,r_0} := \Omega \cap Q_{L,r_0}$, we have that Ω_{L,r_0} is connected and has a Lipschitz boundary. Hence, it supports a relative isoperimetric inequality of the form

$$P(F; \Omega_{L,r_0}) \geq C_0 \min\{|F \cap \Omega_{L,r_0}|, |\Omega_{L,r_0} \setminus F|\}^{\frac{n-1}{n}}. \quad (3.31)$$

From (3.31) and the fact that $B_r \subset\subset Q_{L,r_0}$ for all $0 < r < r_0$, we infer that

$$m'(r) + P_\Omega(E; B_r) = P(E \cap B_r; \Omega_{L,r_0}) \geq C_0 \min\{m(r), \mu(r)\}^{\frac{n-1}{n}}. \quad (3.32)$$

and

$$\mu'(r) + P_\Omega(E; B_r) = P(B_r \setminus E; \Omega_{L,r_0}) \geq C_0 \min\{m(r), \mu(r)\}^{\frac{n-1}{n}} \quad (3.33)$$

for almost all $0 < r < r_0$. Set $0 < t < r$ and define the competitor

$$F_t = \begin{cases} E \cup B_t \cap \Omega & \text{if } m(t) > \mu(t), \\ E \setminus B_t \cap \Omega & \text{otherwise.} \end{cases}$$

We note that in the first case $F_t \Delta E = B_t \cap \Omega \setminus E$, while in the second case $F_t \Delta E = B_t \cap \Omega \cap E$. In any case, we have $F_t \Delta E \subset\subset B_r \cap \Omega$. Thus, by the almost-minimality of E in Ω , and for almost all $0 < t < r$, we infer that either

$$\begin{aligned} P_\Omega(E; B_r) &\leq P_\Omega(F_t; B_r) + \mu(t)^{\frac{n-1}{n}} \psi(r) \\ &\leq P_\Omega(E; B_r \setminus \overline{B_t}) + \mathcal{H}^{n-1}(\partial B_t \cap \Omega \setminus E) + \mu(r)^{\frac{n-1}{n}} \psi(r), \end{aligned} \quad (3.34)$$

or

$$\begin{aligned} P_\Omega(E; B_r) &\leq P_\Omega(F_t; B_r) + m(t)^{\frac{n-1}{n}} \psi(r) \\ &\leq P_\Omega(E; B_r \setminus \overline{B_t}) + \mathcal{H}^{n-1}(\partial B_t \cap \Omega \cap E) + m(r)^{\frac{n-1}{n}} \psi(r), \end{aligned} \quad (3.35)$$

where $\psi(r) := \psi_\Omega(E; 0, r)$. Taking the limit as $t \nearrow r$ in (3.34) and (3.35), and using (3.32), we deduce that, if $m(r) > \mu(r)$, then for almost all $0 < r < r_0$ we have

$$2\mu'(r) + \mu(r)^{\frac{n-1}{n}} \psi(r) \geq C_0 \mu(r)^{\frac{n-1}{n}},$$

while otherwise, by (3.33), we have

$$2m'(r) + m(r)^{\frac{n-1}{n}} \psi(r) \geq C_0 m(r)^{\frac{n-1}{n}}.$$

Therefore, calling $s(r) := \min\{m(r), \mu(r)\}$ and owing to the infinitesimality of $\psi(r)$ as $r \rightarrow 0$, we obtain

$$\frac{s'(r)}{s(r)^{\frac{n-1}{n}}} \geq C$$

for $0 < r < r_0$ and for some explicit constant $C > 0$ depending on C_0 . Integrating this inequality on the interval $(0, r)$, we obtain (3.30). Then, the first inequality in (3.29) follows from (3.30) and (3.32). Finally, the second inequality in (3.29) follows from the observation that, taking the limit as $t \nearrow r$ in (3.34) and possibly redefining \bar{r} and C , we have

$$P_\Omega(E; B_r) \leq \mathcal{H}(\partial B_r \cap \Omega \setminus E) + \mu(r)^{\frac{n-1}{n}} \psi(r) \leq C r^{n-1},$$

for every $0 < r < r_0$. ■

4. The visibility property

In this section, we introduce the visibility property and its main consequences. In what follows, $\Omega \subset \mathbb{R}^n$ denotes an open set with Lipschitz boundary such that $0 \in \partial\Omega$ and $\partial\Omega$ is a graph in a neighborhood of 0, as in (2.13). For notational convenience, we will only consider the visibility property at 0, but of course, we could equally define the property at a generic point of $\partial\Omega$.

Definition 4.1. *We say that Ω satisfies the visibility property provided there exists $T > 0$ and a function $u \in C^1([0, T])^5$ such that*

$$(V1) \quad u(0) = u'(0) = 0 \text{ and } 0 \leq u' \leq 2^{-1};$$

⁵By $u \in C^1([0, T])$ we mean that $u \in C^1(0, T)$ and there exist finite limits of $u(t)$ and $u'(t)$ as $t \rightarrow 0$.

(V2) The function

$$\gamma_u(t) := t^{-1} \sup_{0 < s \leq t} \sqrt{\frac{u(s)}{s} + u'(s)}$$

is summable on $(0, T)$;

(V3) for all $0 < t < T$, the segment joining the point $U_t = -u(t) e_n$ with a point x belonging to $\partial\Omega \cap B_t$ does not intersect Ω .

Remark 4.2. We note that (V1) and (V2) imply that $u(t) = o(t)$ and

$$t\gamma_u(t) \rightarrow 0 \tag{4.1}$$

as $t \rightarrow 0$. Moreover, since by (V1) we have $0 \leq u(t) \leq t$, we infer that

$$0 \leq \frac{u(t)}{t^2} = t^{-1} \frac{u(t)}{t} \leq t^{-1} \sqrt{\frac{u(t)}{t}} \leq \gamma_u(t).$$

Therefore, the summability of γ_u implies that of $t^{-2}u(t)$.

In the following proposition, we rewrite the assumption (V3) in the form of a property involving the functions $\omega(x')$ and $u(t)$. This will be particularly useful when checking the visibility property for relevant classes of domains (see the examples at the end of the section).

Proposition 4.3. The set Ω satisfies property (V3) in Definition 4.1 if and only if, for any $v \in \partial B'_1$ and for all $0 < t < T$, the slope $m_t(s)$ of the line connecting U_t with $(s, \omega(sv))$, given by

$$m_t(s) = \frac{\omega(sv) + u(t)}{s},$$

is non-increasing as a function of s , for $s > 0$ such that $s^2 + \omega(sv)^2 < t^2$.

Proof. Let us assume that (V3) holds, and set $\omega_v(s) = \omega(sv)$ for more simplicity. By contradiction, let $s_1 < s_2$ be such that $s_i^2 + \omega_v(s_i)^2 < t^2$, for $i = 1, 2$, and $m_t(s_1) < m_t(s_2)$. By the definition of m_t , we have

$$\frac{\omega_v(s_1) + u(t)}{s_1} < \frac{\omega_v(s_2) + u(t)}{s_2},$$

and so equivalently,

$$\omega_v(s_1) < \frac{s_1}{s_2}(\omega_v(s_2) + u(t)) - u(t).$$

This implies that the point

$$x = \left(s_1 v, \frac{s_1}{s_2}(\omega_v(s_2) + u(t)) - u(t) \right)$$

is internal to Ω and lies on the segment connecting $(s_2 v, \omega_v(s_2))$. This contradicts (V3).

Conversely, let us suppose that $m_t(s)$ is non-increasing in s , for $s > 0$ such that $s^2 + \omega_v(s)^2 < t^2$. Set $P(s) = (s\nu, \omega_v(s))$, then, arguing by contradiction, assume that $s_2 > 0$ is such that $s_2^2 + \omega_v(s_2)^2 < t^2$ and there exists $\lambda \in (0, 1)$ with the property

$$(1 - \lambda)U_t + \lambda P(s_2) = (\lambda s_2, \lambda(u(t) + \omega_v(s_2)) - u(t)) \in \Omega.$$

Then $\omega_v(\lambda s_2) < \lambda(u(t) + \omega_v(s_2)) - u(t)$. By continuity, for all $\delta > 0$ there exists $\lambda_\delta \in (\lambda, 1)$ such that

$$\lambda_\delta(u(t) + \omega_v(s_2)) - u(t) - \delta < \omega_v(\lambda_\delta s_2) < \lambda_\delta(u(t) + \omega_v(s_2)) - u(t). \quad (4.2)$$

Since the segment $[U_t, P(s_2)]$ is compactly contained in B_t , by (4.2), we can pick $\delta > 0$ small enough and a correspondent λ_δ such that

$$P(\lambda_\delta s_2) \in B_t. \quad (4.3)$$

Let $s_1 = \lambda_\delta s_2$. We observe that (4.3) and (4.2) imply

$$m_t(s_1) < m_t(s_2), \quad s_1^2 + \omega_v(s_1)^2 < t^2,$$

and this contradicts our hypothesis. This completes the proof of the proposition. \blacksquare

Corollary 4.4. *Assume that ω satisfies*

$$\langle x', \nabla \omega(x') \rangle \leq \omega(x') + u(|x'|) \quad \text{for a.e. } x' \in B'_\rho, \quad (4.4)$$

where $u : (0, T) \rightarrow \mathbb{R}$ is a non-decreasing function satisfying properties (V1) and (V2). Then Ω satisfies the visibility property.

Proof. Since ω is Lipschitz, the function m_t defined in the statement of Proposition 4.3 is almost everywhere differentiable, thus m_t is non-increasing if and only if $m'_t(s) \leq 0$ at almost every s . We observe that

$$m'_t(s) = \frac{\omega'_v(s)}{s} - \frac{\omega_v(s) + u(t)}{s^2},$$

thus $m'_t \leq 0$ if and only if

$$s \omega'_v(s) \leq \omega_v(s) + u(t). \quad (4.5)$$

The hypothesis in (4.4) implies that

$$s \omega'_v(s) \leq \omega_v(s) + u(s) \quad \text{for almost all } 0 < s < T. \quad (4.6)$$

Hence, if $s > 0$ is such that $s^2 + \omega_v(s)^2 < t^2$, then $s < t$, and by (4.6), since u is non-decreasing, we obtain

$$s \omega'_v(s) \leq \omega_v(s) + u(t),$$

which is precisely (4.5). Consequently, (V3) is verified thanks to Proposition 4.3. \blacksquare

4.1. Existence of the tangent cone

An important consequence of the visibility property is the existence of the tangent cone to Ω at 0.

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property. Then there exists the tangent cone to Ω at 0, denoted by Ω_0 . More precisely, if we set $\Omega_s := s^{-1}\Omega$ for $s > 0$, we obtain*

$$\lim_{s \rightarrow 0^+} \text{dist}_{\mathcal{H}}(\Omega_s \cap B_R, \Omega_0 \cap B_R) = 0 \quad \text{for all } R > 0. \quad (4.7)$$

Proof. Let us fix $v \in \partial B_1'$, and let

$$\omega_v(s) = \omega(sv), \quad s \geq 0,$$

where ω is the function realizing (2.13). Let $s > 0$ be a point where ω_v is differentiable and let $t = \sqrt{s^2 + \omega_v(s)^2}$. By (V3) in Definition 4.1, we deduce that the slope of the line connecting $(s, \omega_v(s))$ with U_t needs to be bounded below by $\omega'_v(s)$, that is,

$$\omega'_v(s) \leq \frac{\omega_v(s) + u(t)}{s}.$$

We set $L = \sqrt{1 + \text{Lip}(\omega)^2}$ and observe that $t \leq Ls$. Since u is non-decreasing by (V1), we infer

$$\omega'_v(s) \leq \frac{\omega_v(s) + u(Ls)}{s};$$

hence, we get

$$\left(\frac{\omega_v(s)}{s}\right)' = \frac{\omega'_v(s)}{s} - \frac{\omega_v(s)}{s^2} \leq \frac{u(Ls)}{s^2} \leq L^2 \frac{u(Ls)}{(Ls)^2}. \quad (4.8)$$

We integrate (4.8) between $s_1 < s_2$ thanks to (V2) (see Remark 4.2), and obtain

$$\frac{\omega_v(s_2)}{s_2} - \frac{\omega_v(s_1)}{s_1} \leq L^2 \int_{s_1}^{s_2} \frac{u(Ls)}{(Ls)^2} ds = L \int_{Ls_1}^{Ls_2} \frac{u(t)}{t^2} dt.$$

Thus, we conclude that the function

$$s \mapsto \frac{\omega_v(s)}{s} - L \int_0^{Ls} \frac{u(t)}{t^2} dt$$

is monotonically non-increasing in s and bounded by $\text{Lip}(\omega) + L \int_0^{Ls} t^{-2} u(t) dt$, for $0 < s < L^{-1}T$. Therefore, there exists

$$D_v^+ \omega(0) := \lim_{s \rightarrow 0^+} \frac{\omega_v(s)}{s} = \lim_{s \rightarrow 0^+} \frac{\omega_v(s)}{s} - L \int_0^{Ls} \frac{u(t)}{t^2} dt \in \mathbb{R}.$$

Let us define

$$\omega_0(x') = \begin{cases} |x'| D_{\frac{x'}{|x'|}}^+ \omega(0) & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0. \end{cases}$$

The function ω_0 is 1-homogeneous, and therefore the set

$$\Omega_0 = \{x \in \mathbb{R}^n : x_n > \omega_0(x')\}$$

is a cone with vertex at 0. Now, for all $s > 0$, we set

$$\omega_s(x') = \begin{cases} \frac{\omega(sx')}{s} & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0. \end{cases}$$

It is immediate to observe that $\omega_s(0) = 0$ and, setting $t = s|x'|$,

$$\omega_s(x') = \frac{\omega(t(x'/|x'|))}{t} |x'| \rightarrow |x'| D_{\frac{x'}{|x'|}}^+ \omega(0) = \omega_0(x') \quad \text{as } s \rightarrow 0^+.$$

Since $\{\omega_s\}_{s>0}$ is a one-parameter family of locally equibounded and equi-Lipschitz functions that pointwisely converge to ω_0 as $s \rightarrow 0^+$, we can apply Lemma 2.1 to conclude that this convergence is locally uniform. This easily implies the Hausdorff convergence stated in (4.7). \blacksquare

Remark 4.6. *We note that for the proof of Proposition 4.5, hypothesis (V2) of Definition 4.1 can be replaced by the weaker hypothesis of summability of $t^{-2}u(t)$ on $(0, T)$. We also observe that, if Ω is convex, then the existence of the tangent cone Ω_0 is always granted, even though (V2) may not be satisfied.*

4.2. An off-centric visibility property

The next lemma shows that the assumption (V3) in Definition 4.1 can be replaced by an equivalent assumption, where off-centric balls are taken instead of balls centered at 0. This off-centric visibility property will be useful later on.

Lemma 4.7. *The following properties are equivalent:*

- (i) Ω satisfies the visibility property;
- (ii) *There exist $R > 0$ and a function $v \in C^1(0, R)$ satisfying properties (V1), (V2) of Definition 4.1, and*
 - (V3') *For all $0 < r < R$, any segment joining the point $V_r = -v(r) e_n$ with a point x belonging to $\partial\Omega \cap B_r(V_r)$ does not intersect Ω .*

Proof. We prove that (i) implies (ii). Let $z(t) = t - u(t)$ where u is as in Definition 4.1. We can find $0 < T' < T$ such that $z(t)$ is an increasing C^1 diffeomorphism of the interval $(0, T')$ with the property

$$\frac{1}{2} t \leq z(t) \leq t.$$

Let $R = z(T')$. Then we can consider the inverse z^{-1} of z in $(0, R)$, which is an increasing diffeomorphism such that

$$r \leq z^{-1}(r) \leq 2r. \tag{4.9}$$

Setting $U_t = -u(t)e_n$, it follows that $B_{t-u(t)}(U_t) \subset B_t$ for all $0 < t < T$, and thus (V3) holds for all points $x \in \partial\Omega \cap B_{t-u(t)}(U_t)$. We then have that

$$B_r(U_{z^{-1}(r)}) \subset B_{z^{-1}(r)} \quad \text{for any } 0 < r < R.$$

Any line segment joining $x \in \partial\Omega \cap B_{z^{-1}(r)}$ with $U_{z^{-1}(r)}$ does not intersect Ω ; hence, the same property holds for any $x \in \partial\Omega \cap B_r(U_{z^{-1}(r)})$. Let

$$v(r) = u(z^{-1}(r)), \quad 0 < r < R.$$

It is clear that (V3') holds. By (4.9), up to possibly reducing the value of R , (V2) and $v' \leq 2^{-1}$ follow. Since both u and z^{-1} are non-decreasing, v is non-decreasing, and thus (V1) is also satisfied. A completely analogous argument shows that (ii) implies (i), and the proof is concluded. ■

For all $0 < r < R$, we define

$$\mathfrak{C}_r = \{V_r + t(z - V_r) : z \in \partial B_r(V_r) \cap \Omega, t > 0\}. \quad (4.10)$$

The set \mathfrak{C}_r is an open cone with vertex at V_r .

Lemma 4.8. *Assume that Ω satisfies the (off-centric) visibility property. Then, for all $0 < r < R$, the cone \mathfrak{C}_r contains $\Omega \cap B_r(V_r)$.*

Proof. By contradiction, let $x \in (\Omega \cap B_r(V_r)) \setminus \mathfrak{C}_r$. For all $0 < t \leq r$, let

$$x_t = V_r + t \frac{x - V_r}{|x - V_r|},$$

and note that $x = x_{t_0}$ for a suitable $0 < t_0 < r$. It is clear that $x_r \notin \Omega$, otherwise x_{t_0} would belong to \mathfrak{C}_r for all $t > 0$, which contradicts our assumption. We can then select a value $s \in (t_0, r]$ such that $x_s \in \partial\Omega$. This leads to a contradiction with the visibility because the segment joining V_r and $x_s \in B_r(V_r) \cap \partial\Omega$ contains $x_{t_0} = x \in \Omega$. ■

4.3. Foliation by off-centric spheres

Let us consider the family of off-centric balls $B_r(V_r)$, with $V_r = -v(r)e_n$, for $0 < r < R$. By the Implicit Function Theorem, we can easily show the existence of a smooth function ϕ , such that $\partial B_r(V_r)$ is the r -level set of ϕ . This means that the punctured ball $B_R(V_R) \setminus \{0\}$ is foliated by the spheres $\partial B_s(V_s) = \phi^{-1}(s)$ for $0 < s < R$.

Lemma 4.9. *There exists a function $\phi \in C^1(B_R(V_R) \setminus \{0\})$ such that $0 \leq \phi < R$ and*

$$\partial B_r(V_r) = \phi^{-1}(r) \quad \text{for any } 0 < r < R.$$

In particular, for any $x \in B_R(V_R) \setminus \{0\}$, we have

$$\frac{\nabla\phi(x)}{|\nabla\phi(x)|} = \frac{x - V_{\phi(x)}}{|x - V_{\phi(x)}|} \quad (4.11)$$

and

$$\left| \nabla \phi(x) - \frac{x}{|x|} \right| \leq 4 \sqrt{\frac{v(\phi(x))}{\phi(x)} + v'(\phi(x))}. \quad (4.12)$$

Proof. If $v(r)$ is identically 0, then there is nothing to prove because $\phi(x) = |x|$ in this case. We then suppose $v \neq 0$. Let us start by proving the existence of the function ϕ . We observe that, for any $x \in B_R(V_R)$, there exists a unique $r = r_x \in [0, R)$ such that $x \in \partial B_r(V_r)$. Indeed, if $x = 0$, we can take $r = 0$. Otherwise, let $F : (B_R(V_R) \setminus \{0\}) \times (0, R) \rightarrow \mathbb{R}$ be the function defined by

$$F(x, r) = |x - V_r|^2 - r^2. \quad (4.13)$$

It is immediate to observe that F is continuous. Moreover,

$$F(x, 0) = |x|^2 > 0 \quad \text{and} \quad F(x, R) < 0 \quad \text{for all } x \in B_R(V_R) \setminus \{0\}.$$

Hence, we can find $r \in (0, R)$ such that $F(x, r) = 0$, that is, such that $x \in \partial B_r(V_r)$. Let us show the uniqueness. Indeed, if $r, r' \in (0, R)$ have the property that

$$x \in \partial B_r(V_r) \cap \partial B_{r'}(V_{r'}),$$

then we get

$$|r - r'| = \| |x - V_r| - |x - V_{r'}| \| \leq |V_r - V_{r'}| = |v(r) - v(r')| \leq \frac{1}{2} |r - r'|,$$

and thus we must have $r = r'$. Now we can define $\phi(x) = r_x$. Let us show that $\phi \in \mathcal{C}^1(B_R(V_R) \setminus \{0\})$. To do so, we note that ϕ is implicitly defined by

$$F(x, \phi(x)) = 0,$$

where F is the function defined in (4.13). Easy computations give

$$\partial_r F(x, r) = -2r + 2v'(r)(x_n + v(r)).$$

Therefore, if we assume $F(x, r) = 0$ (that is, $x \in B_r(V_r)$) we obtain

$$\partial_r F(x, r) = -2r + 2v'(r)(x_n + v(r)) \leq 2r + |x_n + v(r)| \leq -r,$$

where the first inequality follows from the assumption $0 \leq v' \leq 2^{-1}$, while the second inequality follows from

$$|x_n + v(r)| = |\langle x - V_r, e_n \rangle| \leq |x - V_r| = r.$$

By the Implicit Function Theorem, we deduce that $\phi \in \mathcal{C}^1(B_R(V_R) \setminus \{0\})$. Identity (4.11) is a consequence of the fact that, if $\phi(x) = r$, then the vector $\nabla \phi(x)$ is orthogonal to the level set $\partial B_r(V_r) = \phi^{-1}(r)$ at x .

Let us now prove (4.12). We first observe that, if $\phi(x) = r > 0$, then

$$\nabla\phi(x) = -\frac{\partial_x F(x, r)}{\partial_r F(x, r)} = \frac{x + v(r)e_n}{r - v'(r)(x_n + v(r))}. \quad (4.14)$$

Then (4.14) yields

$$\begin{aligned} \left| \nabla\phi(x) - \frac{x}{|x|} \right|^2 &= \frac{|x + ve_n|^2}{(r - v'(x_n + v))^2} + 1 - 2 \left\langle \frac{x + ve_n}{r - v'(x_n + v)}, \frac{x}{|x|} \right\rangle \\ &= \frac{|x + ve_n|^2}{(r - v'(x_n + v))^2} + 1 - 2 \frac{|x + ve_n|^2 - v(x_n + v)}{(r - v'(x_n + v))|x|} \\ &= \frac{|x|r^2 + |x|(r - v'(x_n + v))^2 - 2(r - v'(x_n + v))[r^2 - v(x_n + v)]}{|x|(r - v'(x_n + v))^2} \\ &= \frac{\frac{|x|}{r} + \frac{|x|}{r} \left(1 - \frac{v'(x_n + v)}{r}\right)^2 - 2 \left(1 - \frac{v'(x_n + v)}{r}\right) \left(1 - \frac{v}{r} \frac{x_n + v}{r}\right)}{\frac{|x|}{r} \left(1 - \frac{v'(x_n + v)}{r}\right)^2} \\ &= \frac{1 + \left(1 - \frac{v'(x_n + v)}{r}\right)^2 - 2 \frac{r}{|x|} \left(1 - \frac{v'(x_n + v)}{r}\right) \left(1 - \frac{v}{r} \frac{x_n + v}{r}\right)}{\left(1 - \frac{v'(x_n + v)}{r}\right)^2}, \quad (4.15) \end{aligned}$$

where the short forms $v = v(r)$ and $v' = v'(r)$ have been used. Next we observe that $\|x\| - r = |x - |x - V_r||$, and hence

$$r - v(r) \leq |x| \leq r + v(r). \quad (4.16)$$

Exploiting (4.16) in (4.15) and recalling that $v'(r) \leq 1/2$ and $|x_n + v(r)| \leq r$, we get

$$\begin{aligned} \left| \nabla\phi(x) - \frac{x}{|x|} \right|^2 &\leq \frac{2 - 2 \left(1 - \frac{v(r)}{r + v(r)}\right) (1 - v'(r)) \left(1 - \frac{v(r)}{r}\right)}{(1 - v'(r))^2} \\ &\leq 8 \left(1 - \left(1 - \frac{v(r)}{r}\right)^2 (1 - v'(r))\right) \leq 16 \left(v'(r) + \frac{v(r)}{r}\right), \end{aligned}$$

and this concludes the proof. \blacksquare

4.4. Some examples

We note that there are examples of sets with Lipschitz boundary that do not satisfy the visibility property at 0, for instance, the epigraph of the function

$$\omega(x) = \begin{cases} x^2 \sin(|x|^{-1}) & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, one can easily check that at $x_k = \frac{1}{(2k+1)\pi}$ for $k \in \mathbb{N}$ we have $\omega(x_k) = 0$ and $\omega'(x_k) = 1$, and hence any visibility function u for which (V3) holds must satisfy $u(x_k) \geq x_k$, which contradicts both (V1) and (V2) (see Remark 4.2).

In the following, we exhibit some examples of domains for which the visibility holds. We recall that $\omega_\nu(s) = \omega(s\nu)$ for $s \geq 0$.

Example 4.10 (Lipschitz cones and outer star-shaped sets). *Let Ω be either a cone with respect to 0, or such that its complement $\mathbb{R}^n \setminus \Omega$ is locally star-shaped with respect to 0. It is immediate to check that Ω satisfies the visibility property with visibility function $u(t) \equiv 0$.*

Example 4.11 ($C^{1,\beta}$ -sets). *Let Ω have $C^{1,\beta}$ boundary and assume $0 \in \partial\Omega$. We show that Ω satisfies the visibility property. Up to a rotation we can assume that Ω admits a representation as in (2.13) with $\omega \in C^{1,\beta}(B'_\rho)$. By Corollary 4.4, it is enough to show that ω satisfies (4.4). Since $\nabla\omega$ is β -Hölder, we have*

$$\langle x', \nabla\omega(x') \rangle \leq \langle x', \nabla\omega(0) \rangle + u(|x'|), \quad (4.17)$$

where $u(t) = C t^{1+\beta}$ for some $C > 0$. Set $\bar{\omega}(x') := \omega(x') - \langle \nabla\omega(0), x' \rangle$ and note that $\bar{\omega}$ is $C^{1,\beta}$, $\bar{\omega}(0) = 0$, $\nabla\bar{\omega}(0) = 0$, and

$$\begin{aligned} |\omega(x') - \langle x', \nabla\omega(0) \rangle| &\leq \max_{|y'| \leq |x'|} |\nabla\bar{\omega}(y')| |x'| \leq (|\nabla\bar{\omega}(0)| + C|x|^\beta) |x'| \\ &= C|x'|^{1+\beta}. \end{aligned} \quad (4.18)$$

Putting together (4.17) and (4.18), we finally get

$$\langle x', \nabla\omega(x') \rangle \leq \omega(x') + u(|x'|) \quad \text{for a.e. } x' \in B'_\rho,$$

which is precisely (4.4). Since trivially u is non-decreasing and satisfies (V2), by Corollary 4.4 we infer that Ω satisfies the visibility property.

Example 4.12 (Convex sets satisfying (V2)). *Let Ω be a convex set with $0 \in \partial\Omega$. For $s > 0$, let $\Omega_s := s^{-1}\Omega$ and $\Omega_0 := \bigcup_{s>0} \Omega_s$. The set Ω_0 is the tangent cone to Ω at 0. Let*

$$u(r) := \text{dist}_{\mathcal{H}}(\Omega \cap B_r, \Omega_0 \cap B_r), \quad (4.19)$$

and assume that $u(r)$ satisfies (V2). We observe that u is non-decreasing in r . Let us prove that Ω satisfies the visibility property. As before, we assume that Ω admits a graphical representation as in (2.13) with the further property that $\omega : B'_\rho \rightarrow \mathbb{R}$ is convex. Using the notations introduced in the proof of Proposition 4.5, by the convexity of ω , we can define

$$\omega_0(x') = \begin{cases} |x'| D_{\frac{x'}{|x'|}}^+ \omega(0) & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0, \end{cases}$$

and deduce that

$$\Omega_0 = \{x \in \mathbb{R}^n : x_n > \omega_0(x')\}.$$

By the definition of u given in (4.19), we have

$$\|\omega - \omega_0\|_{L^\infty(B'_r)} \leq C u(r) \quad \text{for some } C > 0. \quad (4.20)$$

Owing to Corollary 4.4, the visibility property can be proved by showing that (4.4) holds. Thanks to the convexity of ω , for all $v \in \partial B'_1$, we have

$$D_v^+ \omega(0) \leq \omega'_v\left(\frac{s^+}{2}\right) := \lim_{\sigma \rightarrow 0^+} \frac{\omega_v(s/2 + \sigma) - \omega_v(s/2)}{\sigma} \quad \text{for all } 0 \leq s < \rho. \quad (4.21)$$

Moreover, for all $0 < \sigma < s/2$, by (4.20) and the convexity of ω_v , we have

$$\begin{aligned} \frac{\omega_v(s/2 + \sigma) - \omega_v(s/2)}{\sigma} &\leq \frac{\omega_v(s) - \omega_v(s/2)}{s/2} = 2 \frac{\omega_v(s)}{s} - \frac{\omega_v(s/2)}{s/2} \\ &= 2 \frac{\omega_v(s)}{s} - 2D_v^+ \omega(0) + D_v^+ \omega(0) - \frac{\omega_v(s/2)}{s/2} + D_v^+ \omega(0) \\ &= \frac{2}{s} (\omega_v(s) - \omega_0(sv) + \omega_0(sv/2) - \omega_v(s/2)) + D_v^+ \omega(0) \\ &\leq D_v^+ \omega(0) + \tilde{C} \frac{u(s)}{s} \quad \text{for some } \tilde{C} > 0. \end{aligned} \quad (4.22)$$

Putting together (4.21) and (4.22), we obtain

$$\left| D_v^+ \omega(0) - \omega'_v\left(\frac{s^+}{2}\right) \right| \leq \tilde{C} \frac{u(s)}{s}.$$

This suffices for us to conclude. In fact, if $x' \in B'_\rho \setminus \{0\}$ is such that ω is differentiable at x' , setting $s := |x'|$, $v := x'/|x'|$, for some $C > 0$ we achieve

$$\begin{aligned} \langle x', \nabla \omega(x') \rangle &= \langle sv, \nabla \omega(sv) \rangle = s \omega'_v(s^+) \leq s D_v^+ \omega(0) + C u(s) \\ &\leq \omega(sv) + C u(s) = \omega(x') + C u(|x'|). \end{aligned}$$

Since ω is convex, it is differentiable almost everywhere in B'_ρ , and so (4.4) is verified. Moreover, u is non-decreasing and satisfies (V2) by our assumption, and thus we conclude.

Example 4.13. For $i \in \mathbb{N}$, let $y_i = 2^{-i}$ and $\tilde{y}_i = 2^{-i} + 4^{-i}$. We observe that $y_{i+1} < \tilde{y}_{i+1} < y_i$. We define $P_i = (y_i, \tilde{y}_i)$, $Q_i = (\tilde{y}_i, \tilde{y}_i)$ and consider the polygonal curve formed by the segments $\overline{P_i Q_{i+1}}$, $\overline{Q_{i+1} P_{i+1}}$ for all $i \geq 0$. It is immediate to observe that this curve coincides with the graph of the function $\omega : [0, 1] \rightarrow \mathbb{R}$ defined by $\omega(0) = 0$ and

$$\omega(y) = \begin{cases} 2^{-(i+1)} + 4^{-(i+1)} & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ a_i y - b_i & \text{if } y \in (\tilde{y}_{i+1}, y_i], \end{cases} \quad (4.23)$$

where

$$a_i = \frac{1 + 3 \cdot 2^{-(i+1)}}{1 - 2^{-(i+1)}}, \quad b_i = \frac{4^{-i} + 2^{-(3i+1)}}{1 - 2^{-(i+1)}}.$$

Let $\Omega \subset \mathbb{R}^2$ be an open set such that

$$\Omega \cap ((-1, 1) \times \mathbb{R}) = \{x = (y, z) \in \mathbb{R}^2 : z > \omega(|y|)\}.$$

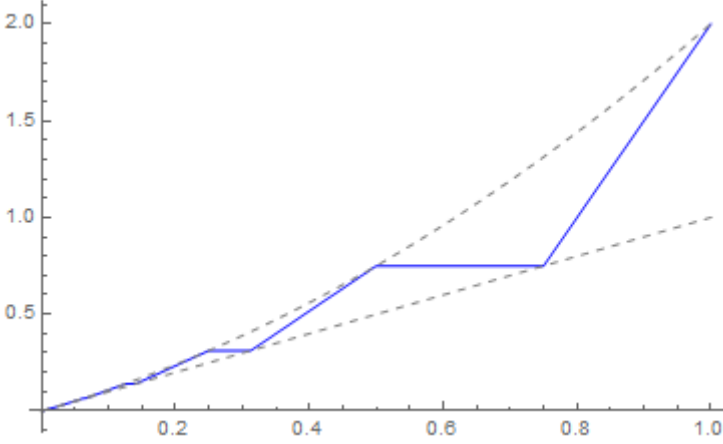


Figure 1. The graph of $\omega(y)$ “bounces” between the graphs of y and $y + y^2$.

Let us show that Ω satisfies the visibility condition at 0. Owing to Corollary 4.4, it suffices to show the existence of a non-decreasing function $u : (0, 1) \rightarrow \mathbb{R}$ satisfying (V1), (V2) and such that

$$y \omega'(y) \leq \omega(y) + u(y) \quad \text{for a.e. } y \in (0, 1). \quad (4.24)$$

By (4.23), for every $i \in \mathbb{N}$, we have

$$y \omega'(y) = \begin{cases} 0 & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ a_i y & \text{if } y \in (\tilde{y}_{i+1}, y_i]. \end{cases}$$

Thus, in order to verify (4.24), it suffices to choose a function $u = u(y)$ greater or equal than the function

$$\bar{u}(y) = \begin{cases} 0 & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ b_i & \text{if } y \in (\tilde{y}_{i+1}, y_i]. \end{cases}$$

We look for $\alpha > 0$ such that

$$u_\alpha(y) = \alpha y^2 \geq \bar{u}(y). \quad (4.25)$$

To obtain (4.25), it suffices to impose that

$$u_\alpha(\tilde{y}_{i+1}) = \alpha \tilde{y}_{i+1}^2 \geq b_i.$$

It is immediate to observe that

$$\alpha \tilde{y}_{n+1}^2 \geq \frac{\alpha}{4} 2^{-2i}, \quad b_i \leq 4 \cdot 2^{-2i}.$$

Hence, if, for instance, we take $\alpha = 16$, (4.25) holds. Since the function $u(y) = 16 y^2$ fulfills (V1) and (V2), we conclude that Ω satisfies the visibility condition at 0.

5. Free-boundary monotonicity

The present section is devoted to showing a free-boundary monotonicity property for local almost-minimizers of P_Ω at a point $x_0 \in \partial\Omega$ satisfying the visibility property up to an isometry (hence, from now on, we will directly assume $x_0 = 0$). Following the notation of the previous section, given $0 < r_1 < r_2 < R$, we recall that $V_r = -v(r)e_n$ and define

$$\mathcal{A}_{r_1, r_2} := B_{r_2}(V_{r_2}) \setminus \overline{B_{r_1}(V_{r_1})} = \phi^{-1}(r_1, r_2),$$

where $\phi(x)$ is the function defined in Lemma 4.9. We also conveniently introduce some further notation. Given $f \in BV_{\text{loc}}(\Omega)$ and $0 < r_1 < r_2$, we set

$$\mu_f(r) = \frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}}$$

and

$$\begin{aligned} G(f; r_1, r_2) &= \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla\phi| - 1) d|Df| d\rho \\ &\quad + \frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla\phi| - 1) d|Df| \\ &\quad - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla\phi| - 1) d|Df|. \end{aligned} \quad (5.1)$$

In the next proposition, we combine the visibility property with an upper bound on $\mu_f(r)$ and obtain the finiteness of $\lim_{\rho \rightarrow 0} G(f; \rho, r)$.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property as in Definition 4.1, and let $f \in BV_{\text{loc}}(\Omega)$. Assume that $\mu_f(r) \leq C$ for some constant $C > 0$ and for all $r \in (0, R)$. Then for $r \in (0, R)$, the limit*

$$\begin{aligned} G(f; r) := \lim_{\rho \rightarrow 0} G(f; \rho, r) &= \int_0^r \frac{n-1}{\rho^n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla\phi| - 1) d|Df| d\rho \\ &\quad + \frac{1}{r^{n-1}} \int_{\Omega \cap B_r(V_r)} (|\nabla\phi| - 1) d|Df| \end{aligned}$$

exists and is finite.

Proof. By (4.12), for all $x \in B_\rho(V_\rho)$, we have

$$\begin{aligned} \left| |\nabla\phi(x)| - 1 \right| &\leq \left| \nabla\phi(x) - \frac{x}{|x|} \right| \leq 4 \sqrt{\frac{v(\phi(x))}{\phi(x)} + v'(\phi(x))} \leq 4 \sup_{0 < r \leq \rho} \sqrt{\frac{v(r)}{r} + v'(r)} \\ &= 4\rho\gamma_v(\rho), \end{aligned}$$

where γ_v is the function defined in the visibility property (V2). Then, using the upper bound on μ_f , for $0 < \rho < r < R$ we obtain

$$\left| \rho^{1-n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla\phi| - 1) d|Df| \right| \leq 4C\rho\gamma_v(\rho).$$

Thanks to the summability of γ_v (see property (V2) of the visibility property) and to (4.1), from the last inequality we easily get the proof of the proposition. \blacksquare

Our monotonicity formula will then follow from the general inequality proved in the next theorem.

Theorem 5.2 (Monotonicity inequality). *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property, and let $f \in BV_{\text{loc}}(\Omega)$. Then, for $R > 0$ small enough, for almost every $0 < r_1 < r_2 < R$, we have*

$$\begin{aligned} & \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} |\langle v_f, \nabla\phi \rangle| d|Df| \right)^2 \\ & \leq 2 \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla\phi(x)|}{\phi(x)^{n-1}} d|Df| \right) \left[\mu_f(r_2) - \mu_f(r_1) \right. \\ & \quad \left. + \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_\Omega(f; B_\rho(V_\rho)) d\rho + G(f; r_1, r_2) \right], \end{aligned} \quad (5.2)$$

where v_f is such that $Df = v_f |Df|$.

Proof. We start by assuming $f \in BV(\Omega) \cap C^1(\Omega)$. For all $0 < r < R$ and $x \in \mathcal{C}_r \cap B_r(V_r)$, where \mathcal{C}_r is defined in (4.10), we let

$$Y_r(x) = V_r + r \frac{x - V_r}{|x - V_r|}.$$

Standard computations yield

$$\begin{aligned} DY_r(x) &= r \left[\frac{1}{|x - V_r|} D(x - V_r) + (x - V_r) \otimes \nabla |x - V_r|^{-1} \right] \\ &= \frac{r}{|x - V_r|} \left[\text{Id} - \frac{x - V_r}{|x - V_r|} \otimes \frac{x - V_r}{|x - V_r|} \right]. \end{aligned} \quad (5.3)$$

We define

$$g_r(x) = f(Y_r(x)) \quad \text{for all } x \in \mathcal{C}_r,$$

and then the ‘‘off-centric conical competitor’’ is

$$f_r(x) = \begin{cases} g_r(x) & \text{if } x \in \mathcal{C}_r \cap B_r(V_r), \\ f(x) & \text{if } x \in \Omega \setminus B_r(V_r). \end{cases}$$

By definition, f_r coincides with f in $\Omega \setminus B_r(V_r)$, and hence we infer

$$\Psi_\Omega(f; B_r(V_r)) \geq |Df|(\Omega \cap B_r(V_r)) - |Df_r|(\Omega \cap B_r(V_r)). \quad (5.4)$$

Then, by (5.4), we deduce that

$$\begin{aligned} |Df|(\Omega \cap B_r(V_r)) - \Psi_\Omega(f; B_r(V_r)) &\leq |Df_r|(\Omega \cap B_r(V_r)) \leq |Df_r|(\mathbb{C}_r \cap B_r(V_r)) \\ &= \int_{\mathbb{C}_r \cap B_r(V_r)} |\nabla g_r(x)| dx. \end{aligned} \quad (5.5)$$

Let us now compute the gradient of g_r . By (5.3), setting

$$v_r(x) = \frac{x - V_r}{|x - V_r|}, \quad Y_r = Y_r(x),$$

we obtain

$$\nabla g_r(x) = DY_r \cdot \nabla f(Y_r) = \frac{r}{|x - V_r|} \nabla f(Y_r)^{v_r(x)^\perp},$$

where $\nabla f(Y_r)^{v_r(x)^\perp}$ denotes the projection of $\nabla f(Y_r)$ onto the hyperplane

$$v_r(x)^\perp := \{y \in \mathbb{R}^n : \langle v_r(x), y \rangle = 0\}.$$

Going on with the computations, we obtain

$$\begin{aligned} |\nabla g_r(x)| &= \frac{r}{|x - V_r|} \sqrt{|\nabla f(Y_r)|^2 - \langle \nabla f(Y_r), v_r(x) \rangle^2} \\ &= \frac{r}{|x - V_r|} |\nabla f(Y_r)| \sqrt{1 - \frac{\langle \nabla f(Y_r), v_r(x) \rangle^2}{|\nabla f(Y_r)|^2}}. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \int_{\mathbb{C}_r \cap B_r(V_r)} |\nabla g_r| dx &= \int_0^r \int_{\mathbb{C}_r \cap \partial B_\rho(V_r)} \frac{r}{\rho} |\nabla f(Y_r)| \sqrt{1 - \frac{\langle \nabla f(Y_r), v_r \rangle^2}{|\nabla f(Y_r)|^2}} d\mathcal{H}^{n-1} d\rho \\ &= \int_0^r \left(\frac{\rho}{r}\right)^{n-2} d\rho \int_{\mathbb{C}_r \cap \partial B_r(V_r)} |\nabla f| \sqrt{1 - \frac{\langle \nabla f, v_r \rangle^2}{|\nabla f|^2}} d\mathcal{H}^{n-1} \\ &\leq \frac{r}{n-1} \left\{ \int_{\Omega \cap \partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} \right. \\ &\quad \left. - \frac{1}{2} \int_{\Omega \cap \partial B_r(V_r)} \frac{\langle \nabla f, v_r \rangle^2}{|\nabla f|} d\mathcal{H}^{n-1} \right\}. \end{aligned} \quad (5.6)$$

Combining (5.5) and (5.6), we get

$$\begin{aligned} &\frac{r}{2(n-1)} \int_{\Omega \cap \partial B_r(V_r)} \frac{\langle \nabla f, v_r \rangle^2}{|\nabla f|} d\mathcal{H}^{n-1} \\ &\leq \frac{r}{n-1} \int_{\Omega \cap \partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} - \int_{\Omega \cap B_r(V_r)} |\nabla f| dx + \Psi_\Omega(f; B_r(V_r)). \end{aligned} \quad (5.7)$$

Multiplying both sides of (5.7) by $(n-1)r^{-n}$ and observing that $r = \phi(y)$ for any $y \in \partial B_r(V_r)$, we get

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega \cap \partial B_r(V_r)} \left\langle \nabla f, \frac{\nabla \phi}{|\nabla \phi|} \right\rangle^2 \frac{1}{|\nabla f| \phi^{n-1}} d\mathcal{H}^{n-1} \\
& \leq \frac{1}{r^{n-1}} \int_{\Omega \cap \partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} + \frac{n-1}{r^n} \int_{\Omega \cap B_r(V_r)} |\nabla f| dx \\
& \quad + \frac{n-1}{r^n} \Psi_\Omega(f; B_r(V_r)) \\
& = \frac{d}{dr} \left(\frac{1}{r^{n-1}} \int_{\Omega \cap B_r(V_r)} |\nabla f| |\nabla \phi| dx \right) \\
& \quad + \frac{n-1}{r^n} \int_{\Omega \cap B_r(V_r)} |\nabla f| (|\nabla \phi| - 1) dx + \frac{n-1}{r^n} \Psi_\Omega(f; B_r(V_r)). \quad (5.8)
\end{aligned}$$

Let us now integrate (5.8) between $0 < r_1 < r_2 < R$. We then achieve

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \left\langle \nabla f(x), \frac{\nabla \phi(x)}{|\nabla \phi(x)|} \right\rangle^2 \frac{|\nabla \phi(x)|}{|\nabla f(x)| \phi(x)^{n-1}} dx \\
& \leq \frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} |\nabla f(x)| |\nabla \phi(x)| dx - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} |\nabla f(x)| |\nabla \phi(x)| dx \\
& \quad + \int_{r_1}^{r_2} \frac{n-1}{r^n} \int_{\Omega \cap B_r(V_r)} |\nabla f(x)| (|\nabla \phi(x)| - 1) dx dr \\
& \quad + \int_{r_1}^{r_2} \frac{n-1}{r^n} \Psi_\Omega(f; B_r(V_r)) dr. \quad (5.9)
\end{aligned}$$

By Hölder's Inequality, we get

$$\begin{aligned}
& \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} |\langle \nabla f(x), \nabla \phi(x) \rangle| \frac{dx}{\phi(x)^{n-1}} \right)^2 \\
& \leq \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla \phi(x)|}{\phi(x)^{n-1}} |\nabla f(x)| dx \right) \\
& \quad \cdot \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \langle \nabla f(x), \frac{\nabla \phi(x)}{|\nabla \phi(x)|} \rangle^2 \frac{|\nabla \phi(x)| dx}{|\nabla f(x)| \phi(x)^{n-1}} \right). \quad (5.10)
\end{aligned}$$

Putting together (5.9) and (5.10), we obtain

$$\begin{aligned}
& \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} |\langle \nabla f(x), \nabla \phi(x) \rangle| \frac{dx}{\phi(x)^{n-1}} \right)^2 \leq 2 \left(\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla \phi(x)|}{\phi(x)^{n-1}} |\nabla f(x)| dx \right) \\
& \quad \cdot \left(\frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} |\nabla f(x)| dx - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} |\nabla f(x)| dx \right.
\end{aligned}$$

$$+ \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_{\Omega}(f; B_{\rho}(V_{\rho})) d\rho + G(f; r_1, r_2), \quad (5.11)$$

where $G(f; r_1, r_2)$ is as in (5.1). This proves (5.2) for all $f \in BV(\Omega) \cap C^1(\Omega)$.

Let now $f \in BV(\Omega)$. We can select a sequence $f_j \in BV(\Omega) \cap C^1(\Omega)$ such that

$$\|f_j - f\|_{L^1(\Omega)} \rightarrow 0, \quad |Df_j|(\Omega) \rightarrow |Df|(\Omega), \quad Df_j \xrightarrow{*} Df \quad \text{in } \Omega. \quad (5.12)$$

In particular, by the continuity of the trace with respect to the strict convergence, we have

$$\|\text{Tr}^+(f_j, \partial\Omega) - \text{Tr}^+(f, \partial\Omega)\|_{L^1(\partial\Omega, \mathcal{H}^{n-1})} \rightarrow 0. \quad (5.13)$$

Let us consider the extensions

$$f_{0,j}(x) = \begin{cases} f_j(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \quad f_0(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

We observe that, by (5.12) and (5.13),

$$\|f_{0,j} - f_0\|_{L^1(\mathbb{R}^n)} \rightarrow 0, \quad |Df_{0,j}|(\mathbb{R}^n) \rightarrow |Df_0|(\mathbb{R}^n).$$

By Proposition 2.13, for almost all $0 < r < R$,

$$\begin{aligned} |Df_{0,j}|(B_r(V_r)) &\rightarrow |Df_0|(B_r(V_r)), \\ \|\text{Tr}^+(f_{0,j}, \partial B_r(V_r)) - \text{Tr}^+(f_0, \partial B_r(V_r))\|_{L^1(\partial B_r(V_r))} &\rightarrow 0, \end{aligned} \quad (5.14)$$

and in particular, owing to (5.13),

$$\begin{aligned} |Df_j|(\Omega \cap B_r(V_r)) &= |Df_{0,j}|(\Omega \cap B_r(V_r)) \rightarrow |Df_0|(\Omega \cap B_r(V_r)) \\ &= |Df|(\Omega \cap B_r(V_r)). \end{aligned} \quad (5.15)$$

Now (5.14) and (5.15) allow us to apply Lemma 3.7, and we deduce that

$$|\Psi_{\Omega}(f_j; B_r(V_r)) - \Psi_{\Omega}(f; B_r(V_r))| \rightarrow 0, \quad \text{as } j \rightarrow \infty$$

for almost all $0 < r < R$. This implies that

$$\int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_{\Omega}(f_j; B_{\rho}(V_{\rho})) d\rho \rightarrow \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_{\Omega}(f; B_{\rho}(V_{\rho})) d\rho.$$

Finally, to conclude that the right-hand side of (5.11) for $f = f_j$, passes to the limit as $j \rightarrow \infty$, giving precisely the right-hand side of (5.2), it suffices to show that the terms

$$\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} d|Df_j|, \quad \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla\phi| - 1) d|Df_j|, \quad \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla\phi| - 1) d|Df_j|$$

converge as $j \rightarrow \infty$ to

$$\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} d|Df|, \quad \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla \phi| - 1) d|Df|, \quad \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla \phi| - 1) d|Df|,$$

respectively. To see this, it suffices to construct a suitable partition of each domain, for instance, using portions of circular annuli whose boundaries are negligible for $|Df|$ and $|Df_j|$ for all $j \geq 1$, to uniformly approximate each integrand by simple functions (up to removing a small neighborhood of 0 in the case of the last two integrals). About the left-hand side of (5.11), we observe that (5.12) implies

$$Df_j \xrightarrow{*} Df \quad \text{in } \Omega \cap \mathcal{A}_{r_1, r_2}.$$

Now, for f smooth, the left-hand side of (5.2) and the left-hand side of (5.11) coincide. Moreover, we have

$$\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi(x)^{1-n} |\langle \nu_f(x), \nabla \phi(x) \rangle| d|Df|(x) = |\phi^{1-n} \nabla \phi \cdot Df_j|(\Omega \cap \mathcal{A}_{r_1, r_2}).$$

In particular, (5.12) implies that

$$\phi^{1-n} \nabla \phi \cdot Df_j \xrightarrow{*} \phi^{1-n} \nabla \phi \cdot Df,$$

and well-known properties of the weak-star convergence of Radon measures (see [24]) ensure that

$$|\phi^{1-n} \nabla \phi \cdot Df|(\Omega \cap \mathcal{A}_{r_1, r_2}) \leq \liminf_{j \rightarrow \infty} |\phi^{1-n} \nabla \phi \cdot Df_j|(\Omega \cap \mathcal{A}_{r_1, r_2}).$$

This implies (5.2) and concludes the proof of the theorem. \blacksquare

The next corollary, a first important consequence of Theorem 5.2, states the monotonicity of a suitable function of the radius r , which is defined by three terms: the renormalized perimeter $\mu_E(r)$, the integral of a renormalized minimality gap $\Psi_\Omega(E; B_r(V_r))$, and the visibility error $G(E, r)$. In particular, when E is an almost-minimizer, the infinitesimality of the second and third terms implies that $\mu_E(r)$ is “almost-increasing”, and hence that it admits a finite limit as $r \rightarrow 0$. This limit represents the perimeter density of E at 0; see Remark 5.4 below.

Corollary 5.3 (Boundary monotonicity for almost-minimizers). *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property. Let $E \subset \Omega$ be a local almost-minimizer such that $P_\Omega(E, B_r) > 0$ for all $r > 0$ and*

$$\int_0^R \rho^{-n} \Psi_\Omega(E; B_\rho(V_\rho)) d\rho < +\infty.$$

Then, using the same notation introduced at the beginning of the section, we can find $R' > 0$ such that the function

$$r \mapsto \mu_E(r) + (n-1) \int_0^r \rho^{-n} \Psi_\Omega(E; B_\rho(V_\rho)) d\rho + G(E; r)$$

is non-decreasing on $(0, R')$. Moreover, the terms $\int_0^r \rho^{-n} \Psi_\Omega(E; B_\rho(V_\rho)) d\rho$ and $G(E; r)$ are infinitesimal as $r \rightarrow 0$, and hence in particular, $\mu_E(r)$ is “almost-monotone” and the limit

$$\theta_E(0) := \lim_{r \rightarrow 0^+} \mu_E(r)$$

exists and is finite.

Proof of Corollary 5.3. By Lemma 3.8 and the fact that $B_r(V_r) \subset B_{r+v(r)}$, we can find constants $C, R > 0$ such that

$$\frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}} \leq \frac{|Df|(\Omega \cap B_{r+v(r)})}{(r+v(r))^{n-1}} \left(1 + \frac{v(r)}{r}\right)^{n-1} \leq C \quad \text{for all } 0 < r < R.$$

By combining Proposition 5.1 with (4.12) and the previous bound, up to redefining the constants $C, R > 0$, we obtain

$$|G(E; r)| \leq C \left(\int_0^r \gamma_v(\rho) d\rho + r\gamma_v(r) \right) \quad \text{for all } 0 < r < R. \quad (5.16)$$

Finally, the proof of the corollary follows directly from Theorem 5.2 and from the observation that the right-hand side of (5.16) is infinitesimal as $r \rightarrow 0^+$. ■

Remark 5.4. *It is easy to check that, under the assumptions of Corollary 5.3, one has*

$$\exists \lim_{r \rightarrow 0^+} \frac{P_\Omega(E; B_r)}{r^{n-1}} = \theta_E(0).$$

Indeed, this is an immediate consequence of the inclusions

$$B_{r-v(r)}(V_r) \subset B_r \subset B_{r+v(r)}(V_r)$$

combined with $v(r) = o(r)$ as $r \rightarrow 0$.

6. Blow-up limits of almost-minimizers are cones

We now apply Theorem 5.2 and prove that any blow-up limit of a local almost-minimizer E of P_Ω is a perimeter-minimizing cone.

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ satisfy the visibility property. Let $E \subset \Omega$ be a local almost-minimizer in Ω such that*

$$\int_0^R \frac{\Psi_\Omega(E; B_r)}{r^n} dr < \infty.$$

Fix a decreasing sequence $t_j \rightarrow 0$ and set $E_{t_j} = t_j^{-1}E$. Then, up to subsequences, E_{t_j} converges to $E_0 \subset \Omega_0$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. Moreover, E_0 is a non-trivial cone minimizing the relative perimeter in Ω_0 .

Proof. Set $E_j = E_{t_j}$ and $\Omega_j = \Omega_{t_j}$ for more simplicity. Then by the upper density estimate on the relative perimeter of E (Lemma 3.8) coupled with analogous estimates satisfied by Ω Lipschitz, we can find a constant $C > 0$ such that, for every fixed $R > 0$,

$$\begin{aligned} P(E_j; B_R) &\leq P(\Omega_j; B_R) + P_{\Omega_j}(E_j; B_R) \\ &= t_j^{1-n} (P(\Omega; B_{Rt_j}) + P_{\Omega}(E; B_{Rt_j})) \leq CR^{n-1}. \end{aligned}$$

By the compactness property of sequences of sets with uniformly bounded relative perimeter the ball B_R , we conclude that there exists a not relabeled subsequence E_j and a set E_0 of finite perimeter in B_R , such that $E_j \rightarrow E_0$ in $L^1(B_R)$ as $j \rightarrow \infty$. The fact that $E_0 \subset \Omega_0$ up to null sets is immediate, since $E_j \subset \Omega_j$ for all j , and the sequence Ω_j converges to the tangent cone Ω_0 locally in Hausdorff distance (and hence, in $L^1_{\text{loc}}(\mathbb{R}^n)$) thanks to Proposition 4.5. Up to a standard diagonal argument we can assume that the subsequence E_j converges to E_0 in $L^1_{\text{loc}}(\mathbb{R}^n)$. Moreover, by the lower-density estimates on the volume of E we also deduce that E_0 can be neither the empty set, nor the whole Ω_0 up to null sets (that is, E_0 is non-trivial).

By the scaling properties of the perimeter, for any fixed $R > 0$ we have

$$\Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = \frac{1}{t_j^{n-1}} \Psi_{\Omega}(E; B_{t_j R}) \leq \omega_n^{1-\frac{1}{n}} R^{n-1} \psi_{\Omega}(E; 0, t_j R) \rightarrow 0;$$

therefore, we can apply Lemma 3.6 and deduce that

$$\Psi_{\Omega_0}(E_0; B_R) \leq \liminf_j \Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = 0$$

for all $R > 0$ and, also owing to Corollary 5.3,

$$P_{\Omega_0}(E_0; B_R) = \lim_j P_{\Omega_{t_j}}(E_{t_j}; B_R) = \lim_j t_j^{1-n} P_{\Omega}(E; B_{t_j R} \cap \Omega) = R^{n-1} \theta_E(0).$$

Thus, E_0 is a minimizer for the relative perimeter in the cone Ω_0 such that

$$\frac{P(E_0; B_R \cap \Omega_0)}{R^{n-1}} = \theta_E(0) \quad \text{for all } R > 0.$$

Now, the monotonicity inequality (5.2) written for $f = \mathbf{1}_{E_0}$ and $\Omega = \Omega_0$ takes the form

$$\begin{aligned} &\left(\int_{\Omega_0 \cap (B_{r_2} \setminus B_{r_1})} \frac{|\langle \nu_{E_0}(x), x \rangle|}{|x|^n} d|D\mathbf{1}_{E_0}|(x) \right)^2 \\ &\leq \left(\int_{\Omega_0 \cap (B_{r_2} \setminus B_{r_1})} |x|^{1-n} d|D\mathbf{1}_{E_0}|(x) \right) \\ &\quad \cdot \left(\frac{P_{\Omega_0}(E_0; B_{r_2})}{r_2^{n-1}} - \frac{P_{\Omega_0}(E_0; B_{r_1})}{r_1^{n-1}} \right) = 0, \end{aligned}$$

for almost all $0 < r_1 < r_2$. The only possibility is then that $\langle \nu_{E_0}(x), x \rangle = 0$ at \mathcal{H}^{n-1} -almost every $x \in \partial^* E_0$. By [24, Proposition 28.8], we infer that E_0 is a cone with vertex at the origin, up to negligible sets, and the proof is concluded. \blacksquare

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Gian Paolo Leonardi

Department of Mathematics, University of Trento, via Sommarive 14, 38123 Povo, Italy;
gianpaolo.leonardi@unitn.it

Giacomo Vianello

Department of Mathematics “Tullio Levi-Civita”, University of Padua, via Trieste 63,
 35121 Padova, Italy; giacomo.vianello@unipd.it