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Algebraic dependence numbers and the cardinality of generating iterated function systems

Junda Zhang

Abstract. For a dust-like self-similar set (generated by an IFS of similarities with the strong separation condition), Elekes, Keleti and Máthé found an invariant, called algebraic dependence number, by considering its generating IFSs and isometry invariant self-similar measures. We find an intrinsic quantitative characterisation of this number: it is the dimension over $\mathbb{Q}$ of the vector space generated by the logarithms of all the common ratios of infinite geometric sequences in the gap length set, minus 1. Using this, we present a lower bound on the minimal cardinality of generating IFSs (with or without separation conditions) in terms of the gap lengths of a dust-like self-similar set. We also establish an analogous result for dust-like graph-directed attractors on complete metric spaces, and present a new proof of the logarithmic commensurability theorem for IFSs with the strong separation condition. These are new applications of the ratio analysis method and the gap sequence.

1. Introduction

We recall some basic concepts in fractal geometry. Given $m \geq 2$, a finite set of distinct contracting similarities $\{S_i\}_{i=1}^m$ on a complete metric space (M, d) is called a *standard* iterated function system (IFS), where $d(S_i(x), S_i(y)) = r_i d(x, y)$ for each $i = 1, 2, \dots, m$ ($0 < r_i < 1$ is called the *contraction ratio* of S_i). According to [29], the *attractor* of the IFS is the unique non-empty compact set $K \subset M$ such that

$$K = \bigcup_{i=1}^{m} S_i(K), \tag{1.1}$$

which is called a *self-similar set*. We call $\{S_i\}_{i=1}^m$ a *generating IFS* of K. We say that the *strong separation condition* (SSC) holds for the IFS when the union is disjoint in (1.1), and such K is called *dust-like*.

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This paper is motivated by the inverse fractal problem (see, for example, [11, 12, 22, 30]). Given a dust-like set, what can be said about its generating IFSs? This problem is also related to the tiling theory and image compression, see [22, Section 1] for details. Our results reveal the connection between the gap lengths of a dust-like self-similar set and its *algebraic dependence number* (Theorem 3.6), which is used to give lower bounds on the cardinality of generating IFSs (Corollary 3.8). The definition of gap lengths is intrinsic, and it is simply the lengths of the disjoint (non-empty bounded) complementary intervals for a dust-like set in $\mathbb{R}$.

In [15], the *algebraic dependence number* of an IFS is defined as the dimension over $\mathbb{Q}$ of the vector space generated by the logarithms of the contraction ratios of all similarities, minus 1. For a dust-like self-similar set in $\mathbb{R}^n$, [15, Theorem 5.7] shows that all its generating IFSs with the SSC have the same algebraic dependence number, so this number is an intrinsic characteristic for a dust-like self-similar set (independent of generating IFSs). The proof of [15, Theorem 5.7] uses an invariant measure characterisation of the algebraic dependence number. We generalise this theorem to compact metric spaces with a direct and intrinsic characterisation (without considering generating IFSs and measures). To be more specific, our Theorem 3.6 gives an intrinsic quantitative characterisation of this number: it is the dimension over $\mathbb{Q}$ of the vector space generated by the logarithms of all the common ratios of infinite geometric sequences in the gap length set (the collection of all gap lengths), minus 1.

We also consider graph-directed IFSs and graph-directed attractors based on a given directed graph, which generalize the concepts of IFSs and self-similar sets. They are introduced in [33] and have been studied intensively, including the geometric aspects (see, for example, [8, 9, 14, 20, 24, 35, 39]) and fractal analysis [4, 6, 26, 34]. A directed graph (V, E) consists of a finite set of vertices V and a finite set of directed edges E with loops and multiple edges allowed. Given a directed graph, we associate each vertex $v \in V$ with a complete metric space (M^v, d^v). Let $E_{uv} \subset E$ be the set of edges from vertex u to v. A graph-directed iterated function system (GD-IFS) $\{S_e : e \in E_{uv}\}$ consists of contracting similarities S_e from M^v to M^u, that is,

$$d^u\big(S_e(x), S_e(y)\big) = r_e d^v(x, y) \quad (0 < r_e < 1) \tag{1.2}$$

for all $x, y \in M^v$, where r_e is the contraction ratio of S_e.

We assume throughout that $d_u \geq 2$ for all $u \in V$ to avoid singletons, where d_u is the number of directed edges leaving u (see related discussion in [14]). For a GD-IFS $(V, E, (S_e)_{e\in E})$ based on such a directed graph, there exists a unique list of non-empty compact sets $(F_u \subset M^u)_{u\in V}$ such that, for all $u \in V$,

$$F_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(F_v), \tag{1.3}$$

see [13, Theorem 4.3.5 on p. 128]. We call the above $(F_u)_{u\in V}$ the *(list of) attractors* of the GD-IFS, and each F_u is called a *GD-attractor*. We say that the *strong separation condition* (SSC) holds for the GD-IFS if the union is disjoint in (1.3) for each $u \in V$, and such GD-attractors are called *dust-like*. Clearly, an IFS and its attractor can be viewed as a GD-IFS and its GD-attractor based on a directed graph with $\#V = 1$, where "#" denotes the cardinality of a finite set.

The graph-directed setting appears frequently in dynamical systems and the study of self-similar sets. Certain complex dynamical systems can be regarded as conformal GD-IFSs using a Markov partition [16, Section 5.5]. The orthogonal projection of certain self-similar sets are GD-attractors [19, Theorem 1.1]. When exact overlap occurs in an IFS, one can use a corresponding graph-directed system to study its attractor (see, for example, [10, 28]). Also in fractal analysis, GD-IFSs are used to determine whether the two walk-dimensions coincide for p.c.f. self-similar sets [25]. They are also related to tiling problems [37] and certain automaton [7].

Our idea is applying the ratio analysis method in [18] to the gap length set (formed by the gap sequence) of a dust-like GD-attractor (see [36] for applications of the gap sequence to the Lipschitz equivalence problem and box dimension estimates). Our results are useful for *inhomogeneous* (GD-)IFSs and (GD-)attractors, where the contraction ratios are not all equal. Separation conditions are often required in fractal geometry to obtain some precise structure information, and relaxing them is difficult in many problems.

In our paper, the SSC is (only) used for a precise formula of the gap length set [10, Theorem 2] for ratio analysis. We remark that the SSC is often required for inhomogeneous IFSs in related problems (see for example [1, 15, 23]). In Corollary 3.10 we use a geometric measure condition as in [22, Theorem 4.1] to remove the SSC condition.

This paper is organised as follows. In Section 2, we introduce the definition and expression of the gap length set of a dust-like GD-attractor, and necessary ratio analysis lemmas in [18]. In Section 3, we present our results. In Section 3.1, we use ratio analysis on the gap length set to obtain some lemmas and the logarithmic commensurability theorem for SSC IFSs (Theorem 3.1), which is used to generalize [15, Theorem 5.7] to metric spaces. In Section 3.2, we present Theorem 3.6, which characterizes the algebraic dependence number of (GD-)IFS in terms of (the geometric sequences in) the gap length set. In Section 3.3, we present the lower bound estimate on the cardinality of generating IFSs, with or without separation conditions (Corollaries 3.8 and 3.10).

2. Preliminaries

2.1. Gap lengths

Let us introduce some notions used in [10] concerning the gap sequence, a natural way to define the sizes and numbers of holes for disconnected sets. For a compact metric space (K, d) and $\delta > 0$, denote its diameter by $\mathrm{diam}(K)$. We say that $x, y \in K$ are δ-equivalent if there is a δ-chain connecting x and y, that is, there are points $x_i \in K$ $(i = 1, 2, \ldots, m)$ such that $x_1 = x$, $x_m = y$, $d(x_i, x_{i+1}) \leq \delta$ for all $i = 1, 2, \ldots, m-1$. According to this equivalence, we divide K into several δ-equivalence classes (intuitively, these small pieces are separated by distance δ, while any two points in each piece can be connected by some δ-chain). Let $\kappa(\delta)$ be the number of δ-equivalence classes of K, which is finite due to the compactness of K (since two points in an open $\delta/2$-metric ball are automatically δ-equivalent, $\kappa(\delta)$ does not exceed the covering number of K using open $\delta/2$-metric balls).

Definition 2.1 (Gap length set). The discontinuous points of the function $\kappa(\delta)$ $(0 < \delta < \mathrm{diam}(K))$ are called the *gap lengths* of K, and the collection of all the gap lengths of K is called the *gap length set* of K, denoted by $\mathrm{GL}(K)$ (that is, the set of discontinuous points of κ).

Remark 2.2. For compact sets on $\mathbb{R}$, this definition is intuitive and coincides with [18, Definition 2.1], see explanation and an example (middle-third Cantor set) in [36, Section 1]. The function $\kappa(\delta)$ is non-increasing in $\delta \in (0, \mathrm{diam}(K))$ since a δ-equivalence class is a δ'-equivalence class when $\delta < \delta'$. Thus, the gap lengths are at-most countably many, and arranging them with corresponding multiplicity (according to the value of κ) gives the *gap sequence* in [10, Definition 2], but we do not concern such multiplicity in our paper.

Definition 2.3 (Contraction ratio set). The *contraction ratio set* of a GD-IFS is defined as the set of the contraction ratios of the similarities, that is, $\{r_e : e \in E\}$ using the notation in (1.2).

The following lemma is a direct corollary of [10, Theorem 2]. We define the product of $A,\ B \subset \mathbb{R}$ as $AB = \{ab : a \in A,\ b \in B\}$. We regard a real number as a set when encountering the product with a set in $\mathbb{R}$, and $AB = \emptyset$ if $A = \emptyset$. Given a GD-IFS and a directed path $\mathbf{e} = e_1 e_2 \ldots e_k$ with edges e_i (for which the terminal vertex of e_i is the initial vertex of e_{i+1} when $i = 1, \ldots, k-1$), we write $r_{\mathbf{e}} := r_{e_1} r_{e_2} \ldots r_{e_k}$.

Lemma 2.4. *Let $G = (V, E)$ be a directed graph satisfying $d_w \geq 2$ for all $w \in V$, and let $(M^u)_{u \in V}$ be a list of complete metric spaces. Suppose that for some $u \in V$, $F_u \subset M^u$ is the GD-attractor of some GD-IFS (based on G) with the SSC. Then for*

each $u \in V$, there exist finite sets $\Lambda_u,\ \Gamma_u \subset (0,\infty)$ such that

$$\mathrm{GL}(F_u) = \Gamma_u \bigcup \Lambda_u \bigcup \Big(\bigcup_{v\in V} \Lambda_v R_{uv} \Big), \tag{2.1}$$

where $R_{uv} := \{r_{\mathbf{e}} : \mathbf{e}$ is a directed path from u to $v\}$.

Remark 2.5. When there is no directed path from u to v, then $R_{uv} = \emptyset$. We mention that [10, Theorem 2] gives the construction of Λ_u and Γ_u, although it is not used in our paper. Given a GD-IFS and its attractor list $(F_u \subset M^u)_{u\in V}$, we obtain $\mathrm{GL}(F_u)$ and write $\Gamma_u = \mathrm{GL}(F_u) \cap [\delta,\infty)$ for all $u \in V$, where

$$\delta := \inf\big\{ d^u(x_e, y_e) :\ x_e \in S_e(F_v),\ y_e \in F_u \setminus S_e(F_v),\ e \in E_{uv},\ u,\ v \in V \big\} > 0$$

due to the SSC. Then we write

$$\Lambda_u = \bigcup_{u\in V} \bigcup_{e\in E_{uv}} r_e \Big(\Gamma_v \bigcap [\delta, r_e^{-1}\delta) \Big)$$

for each $u \in V$, and it is clear that $\Lambda_u \subset (0,\delta)$.

We use the following notions in [18, Definition 2.5] and a corollary of the above lemma.

Definition 2.6. For a finite set $A = \{a_i\}_{i=1}^n \subset (0,\infty)$, define $A^{\mathbb{Z}_+^*}$ (resp. $A^{\mathbb{Q}_+^*}$) as the union of all products $\prod_{i=1}^n a_i^{m_i}$ where $(m_i)_{i=1}^n$ are non-zero vectors whose entries are nonnegative integers (resp. nonnegative rationals). Let $A^{\mathbb{Z}_+} = \{1\} \cup A^{\mathbb{Z}_+^*}$.

Corollary 2.7. *Let M be a complete metric space, and $K \subset M$ be the self-similar attractor of an IFS satisfying the SSC with contraction ratio set X. Then there exist two finite sets $\Lambda,\ \Gamma \subset (0,\infty)$ such that*

$$\mathrm{GL}(K) = \Gamma \bigcup \Lambda X^{\mathbb{Z}_+}. \tag{2.2}$$

2.2. Ratio analysis

In [18], the ratio analysis method is used to analyse sets Θ of positive real numbers, in terms of strictly decreasing geometric sequences $\{\theta' r^k\}_{k=0}^\infty$ that are contained in Θ.

Definition 2.8 ([18, Definition 2.4]). Let $\Theta \subset (0,\infty)$. For $\theta \in \Theta$, denote by

$$R_\Theta(\theta) = \big\{ r \in (0,1) :\ \text{there exists some } \theta' \in \Theta \text{ such that } \theta \in \{\theta' r^k\}_{k=0}^\infty \subset \Theta \big\}, \tag{2.3}$$

the set of common ratios of strictly decreasing infinite geometric sequences in Θ that contains θ (which may be empty).

This concept is quite natural since the gap length set of a dust-like attractor contains many geometric sequences. By definition, there is an obvious monotonicity: when $\theta \in \Theta_1 \subset \Theta_2 \subset (0, \infty)$,

$$R_{\Theta_1}(\theta) \subset R_{\Theta_2}(\theta), \tag{2.4}$$

since a geometric sequence in Θ_1 is also in Θ_2. We use the following two lemmas.

Lemma 2.9 ([18, Lemma 2.6 (i)]). *Let $A = \{a_i\}_{i=1}^n \subset (0, 1)$ be a finite set, and λ_j $(j = 1, \ldots, m)$ be positive real numbers (not necessarily distinct). Let $\Theta = \cup_{j=1}^m \lambda_j A_j$ where $A_j \subset A^{\mathbb{Z}_+}$ for $1 \le j \le m$. Then $R_\Theta(\theta) \subset A^{\mathbb{Q}_+^*}$ for all $\theta \in \Theta$.*

Lemma 2.10 ([18, Corollary 2.7]). *Let $X \subset (0, 1)$ and $\Lambda \subset (0, \infty)$ be two finite sets. Then*

$$X^{\mathbb{Z}_+^*} \subset R_{\Lambda X^{\mathbb{Z}_+}}(\theta) \subset X^{\mathbb{Q}_+^*}$$

for every $\theta \in \Lambda X^{\mathbb{Z}_+}$.

3. Main Results

3.1. Using ratio analysis to prove the logarithmic commensurability theorem

Logarithmic commensurability is central to the affine-embedding problem [1,2,15,21, 23, 38, 40] and the inverse fractal problem (generating IFSs). There are logarithmic commensurability results for affine maps (see, for example, [22, Theorem 1.1] and [15, Corollary 4.10]), and that for IFSs as the following states (see, for example, [38, Corollary 7.6] and references therein).

Conjecture 1 ([21, Conjecture 1.2]). *Suppose that K, F are two totally disconnected non-singleton self-similar sets in $\mathbb{R}^n$, which are the attractors of two IFSs with contraction ratio sets X, Y, respectively. Suppose that there exists an affine map f on $\mathbb{R}^n$ such that $f(F) \subset K$, then the* logarithmic commensurability *holds: $Y \subset X^{\mathbb{Q}_+^*}$.*

The following theorem verifies this in the case where $K = F$ is dust-like, which is a corollary of [17, Theorem 3.3] where K, F are Lipschitz equivalent and dust-like. The proof of [17, Theorem 3.3] uses martingale arguments on measures, while, for our special case, we use ratio analysis. The following theorem immediately implies [15, Theorem 5.7], thus we also obtain an intrinsic proof of [15, Theorem 5.7] on complete metric spaces (instead of $\mathbb{R}^n$, see Remark 3.5).

Theorem 3.1. *Let M be a complete metric space. Suppose that $K \subset M$ has two generating IFSs satisfying the SSC with contraction ratio sets A and X, respectively, then $X \subset A^{\mathbb{Q}_+^*}$, $A \subset X^{\mathbb{Q}_+^*}$.*

This theorem is a corollary of Lemma 3.3. To prove Lemma 3.3, we first prove a more general lemma for GD-attractors, which is also used in the next subsection.

Lemma 3.2. *Let $G = (V, E)$ be a directed graph satisfying $d_w \geq 2$ for all $w \in V$, and let $(M^u)_{u\in V}$ be a list of complete metric spaces. Suppose that for some $u \in V$, $F_u \subset M^u$ is the GD-attractor of some GD-IFS (based on G) satisfying the SSC with contraction ratio set A. Then $R_{\mathrm{GL}(F_u)}(\theta) \subset A^{\mathbb{Q}_+^*}$ for all $\theta \in \mathrm{GL}(F_u)$.*

Proof. We apply Lemma 2.9 to $\Theta = \mathrm{GL}(F_u)$ given by (2.1), by regarding the numbers in Γ_u, Λ_u and Λ_v $(v \in V)$ as λ_j, where the corresponding A_j are $\{1\}$ and R_{uv}, respectively. Since A is the contraction ratio set, $r_{\mathbf{e}} \in A^{\mathbb{Z}_+}$, the conditions in Lemma 2.9 are fulfilled, showing the desired. ∎

Lemma 3.3. *Let M be a complete metric space and $K \subset M$ be the self-similar attractor of an IFS satisfying the SSC with contraction ratio set X. Then for sufficiently small $\theta \in \mathrm{GL}(K)$,*

$$X \subset X^{\mathbb{Z}_+^*} \subset R_{\mathrm{GL}(K)}(\theta) \subset X^{\mathbb{Q}_+^*}. \tag{3.1}$$

Proof. For the left-hand side in (3.1), by Corollary 2.7 and our assumption, there exist two finite sets $\Lambda,\ \Gamma \subset (0, \infty)$ such that

$$\mathrm{GL}(K) = \Gamma \bigcup \Lambda X^{\mathbb{Z}_+}. \tag{3.2}$$

Thus, by the monotonicity (2.4) and Lemma 2.10, we know that for $\theta \in \Lambda X^{\mathbb{Z}_+} \subset \mathrm{GL}(K)$,

$$X^{\mathbb{Z}_+^*} \subset R_{\Lambda X^{\mathbb{Z}_+}}(\theta) \subset R_{\mathrm{GL}(K)}(\theta),$$

showing the desired for sufficiently small $\theta \in \mathrm{GL}(K)$ (smaller than the numbers in the finite set Γ in (3.2)).

The right-hand side in (3.1) directly follows from Lemma 3.2. The proof is complete. ∎

Proof of Theorem 3.1. By Lemma 3.3, for sufficiently small $\theta \in \mathrm{GL}(K)$,

$$A \subset R_{\mathrm{GL}(K)}(\theta) \subset A^{\mathbb{Q}_+^*}, X \subset R_{\mathrm{GL}(K)}(\theta) \subset X^{\mathbb{Q}_+^*}.$$

Thus

$$A \subset R_{\mathrm{GL}(K)}(\theta) \subset X^{\mathbb{Q}_+^*}, X \subset R_{\mathrm{GL}(K)}(\theta) \subset A^{\mathbb{Q}_+^*}.$$

The proof is complete. ∎

3.2. Intrinsic characterisation of the algebraic dependence number

We first define the algebraic dependence number of a GD-IFS. Denote by $\operatorname{span} B$ the vector space generated by $B \subset \mathbb{R}$ over $\mathbb{Q}$. Given a set $X \subset (0, \infty)$, we write $\log X := \{\log x : x \in X\}$.

Definition 3.4 (Algebraic (in)dependence number of GD-IFS). Given a GD-IFS with contraction ratio set A, define its *algebraic independence number* as the dimension of the vector space $\operatorname{span} \log A$, and its *algebraic dependence number* as its algebraic independence number minus 1.

Remark 3.5. For a dust-like self-similar set K in a complete metric space, all its generating IFSs with the SSC have the same algebraic dependence number by Theorem 3.1, thus we call this number the algebraic dependence number of K.

The following theorem shows the connection between the gap length set and the algebraic dependence number.

Theorem 3.6. *Suppose that F is a GD-IFS satisfying the SSC with GD-attractors $(F_u \subset M^u)_{u \in V}$ in complete metric spaces, based on a directed graph (V, E) with $d_w \geq 2$ for all $w \in V$. Then the algebraic independence number of F is no less than the dimension of*

$$\operatorname{span}\Big\{\log r : r \in \bigcup_{u \in V} \bigcup_{\theta \in \mathrm{GL}(F_u)} R_{\mathrm{GL}(F_u)}(\theta)\Big\}. \tag{3.3}$$

In particular, for a dust-like self-similar set K in a complete metric space, its algebraic dependence number plus 1 is equal to the dimension of

$$\operatorname{span}\Big\{\log r : r \in \bigcup_{\theta \in \mathrm{GL}(K)} R_{\mathrm{GL}(K)}(\theta)\Big\} = \operatorname{span}\big\{\log r : r \in R_{\mathrm{GL}(K)}(\theta)\big\} \tag{3.4}$$

for sufficiently small $\theta \in \mathrm{GL}(K)$.

Remark 3.7. As mentioned in the introduction, the second claim of this theorem gives an intrinsic quantitative characterisation of the algebraic dependence number for a dust-like self-similar set K in a complete metric space: it is the dimension over $\mathbb{Q}$ of the vector space generated by the logarithms of all the common ratios of infinite geometric sequences in the gap length set, minus 1. This is because every infinite geometric sequence in $\mathrm{GL}(K)$ has common ratio no greater than 1 since the set $\mathrm{GL}(K)$ has an upper bound $\operatorname{diam}(K)$. But $\log 1 = 0$ is useless to span vector spaces, thus only strictly decreasing infinite geometric sequences in $\mathrm{GL}(K)$ are essential.

In fact, we prove a stronger property (3.5) for dust-like self-similar sets: the vector space generated by the logarithms of the contraction ratios of any SSC generating IFS

(over $\mathbb{Q}$) is exactly the vector space generated by the logarithms of all the common ratios of geometric sequences in the gap length set (over $\mathbb{Q}$).

Proof. For the first inclusion, by using Lemma 3.2 and taking the logarithm,

$$B := \Big\{\log r : r \in \bigcup_{u\in V} \bigcup_{\theta\in \mathrm{GL}(F_u)} R_{\mathrm{GL}(F_u)}(\theta)\Big\} \subset \log A^{\mathbb{Q}_+^*},$$

where A is the contraction ratio set of F. It follows from Definition 2.6 that

$$\operatorname{span} B \subset \operatorname{span} \log A^{\mathbb{Q}_+^*} = \operatorname{span} \log A.$$

Then we know the dimension of $\operatorname{span} B$ is no greater than that of $\operatorname{span} \log A$, which is exactly the algebraic independence number of F.

For the second inclusion, let Φ be a generating IFS of K satisfying the SSC with contraction ratio set X. By using Lemma 3.3 and taking the logarithm,

$$\log X \subset B' \subset \log X^{\mathbb{Q}_+^*},$$

where B' can be either $\{\log r : r \in \cup_{\theta\in \mathrm{GL}(K)} R_{\mathrm{GL}(K)}(\theta)\}$ or $\{\log r : r \in R_{\mathrm{GL}(K)}(\theta)\}$ for sufficiently small $\theta \in \mathrm{GL}(K)$. Thus

$$\operatorname{span} \log X \subset \operatorname{span} B' \subset \operatorname{span} \log X^{\mathbb{Q}_+^*}.$$

By Definition 2.6, we have that $\operatorname{span} \log X = \operatorname{span} \log X^{\mathbb{Q}_+^*}$, and it follows that

$$\operatorname{span} \log X = \operatorname{span} B' = \operatorname{span} \log X^{\mathbb{Q}_+^*}, \tag{3.5}$$

which gives (3.4). Since the algebraic dependence number of K plus 1 is the algebraic independence number of Φ by Remark 3.5, that is, the dimension of $\operatorname{span} \log X$, the proof is complete. ∎

3.3. Lower bounds on the cardinality of generating IFSs

Algebraic independence numbers are natural lower bounds of the cardinality of IFSs, so for a given dust-like set, we are able to present an intrinsic estimate of the minimal cardinality of generating IFSs in terms of the gap lengths.

Corollary 3.8. *Let $G = (V, E)$ be a directed graph satisfying $d_w \geq 2$ for all $w \in V$, and let $(M^u)_{u\in V}$ be a list of complete metric spaces. Suppose that for some $u \in V$, $F_u \subset M^u$ is the GD-attractor of some GD-IFS F satisfying the SSC (based on G). Then the number of edges in G is no less than the dimension of*

$$\operatorname{span}\Big\{\log r : r \in \bigcup_{\theta\in \mathrm{GL}(F_u)} R_{\mathrm{GL}(F_u)}(\theta)\Big\}.$$

In particular, let K be the self-similar attractor of an SSC standard IFS Φ on a complete metric space, then the cardinality of Φ is no less than the dimension of

$$\operatorname{span}\Big\{\log r : r \in \bigcup_{\theta\in \mathrm{GL}(K)} R_{\mathrm{GL}(K)}(\theta)\Big\} = \operatorname{span}\big\{\log r : r \in R_{\mathrm{GL}(K)}(\theta)\big\}$$

for sufficiently small $\theta \in \mathrm{GL}(K)$.

Proof. For the first claim, just note that $\#E$, the number of edges in G, which is also the cardinality of contracting similarities in F, is no less than the cardinality of the contraction ratio set of F, and thus no less than the algebraic independence number of F. The first claim then follows from Theorem 3.6 and that

$$\begin{aligned}\operatorname{span}\Big\{\log r : r \in \bigcup_{\theta\in \mathrm{GL}(F_u)} R_{\mathrm{GL}(F_u)}(\theta)\Big\} \\ \subset \operatorname{span}\Big\{\log r : r \in \bigcup_{u\in V}\bigcup_{\theta\in \mathrm{GL}(F_u)} R_{\mathrm{GL}(F_u)}(\theta)\Big\}.\end{aligned}$$

The second claim immediately follows from the first claim and (3.4). ∎

Remark 3.9. We delete the union "$\cup_{u\in V}$" in (3.3) since we only have the information of one attractor at one vertex u. In addition, we do not give a lower bound on the number of vertices.

Furthermore, if the dust-like self-similar set K is in $\mathbb{R}$ and satisfies

$$\mathcal{H}^{\dim_{\mathrm{H}} K}(K) = \big(\operatorname{diam}(K)\big)^{\dim_{\mathrm{H}} K} \tag{3.6}$$

where $\dim_{\mathrm{H}} K$ is the Hausdorff dimension of K, we can remove the separation condition on generating IFSs.

Corollary 3.10. *Suppose that a self-similar set $K \subset \mathbb{R}$ is the attractor of an SSC standard IFS Φ and (3.6) is satisfied, then the cardinality of every generating IFS of K is no less than the dimension of*

$$\operatorname{span}\Big\{\log r : r \in \bigcup_{\theta\in \mathrm{GL}(K)} R_{\mathrm{GL}(K)}(\theta)\Big\}.$$

If further, the logarithm of the contraction ratio of each contracting similarity in Φ is different from each other and linearly independent over $\mathbb{Q}$, then Φ has the minimal cardinality among all generating IFSs of K.

Proof. For any generating IFS $\{S_i\}_{i=1}^m$ of K, there exists $I \subset \{1, 2, \ldots, m\}$ such that, the sub-IFS $\Psi := \{S_i\}_{i\in I}$ also has attractor K and satisfies the SSC, and

$$m \geq \#\Psi. \tag{3.7}$$

Indeed, whenever $S_i(K) \cap S_j(K) \neq \emptyset$, by the assumption (3.6) and [22, The claim in the proof of Theorem 4.1], $S_i(K) \subset S_j(K)$ (or $S_i(K) \supset S_j(K)$), so one can remove i (or j) from $\{1, 2, \ldots, m\}$, and repeat this process until the SSC is satisfied. The first inclusion then follows from using Corollary 3.8 for the lower bound of $\#\Psi$, which is also a lower bound for m by (3.7).

For the second inclusion, by assumption, the algebraic independence number of Φ is exactly its cardinality $\#\Phi$, thus the algebraic independence number of Ψ is also $\#\Phi$ by Remark 3.5. Since the cardinality of Ψ is no less than the algebraic independence number of Ψ, which is $\#\Phi$, we obtain $m \geq \#\Phi$ by (3.7). ∎

Condition (3.6) is easy to check by the similarities of generating IFSs in $\mathbb{R}$, see [22, Remark 4.1]. We end our paper by a brief overview on this condition. It is firstly discussed in Hausdorff's paper [27], then extended by Marion (called "perfect isotopic") [31, 32], and later by Ayer and Strichartz [3] for dust-like self-similar sets in $\mathbb{R}$: they present the same easily checkable equivalent condition. There is no known analogous equivalence in higher dimensional case to the best of our knowledge. The study of (3.6) for a GD-attractor K is more complicated than the self-similar case (see [5, Theorem 4.6]), and there is no known analogous equivalence even in $\mathbb{R}$ to the best of our knowledge.

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Junda Zhang
School of Mathematics, South China University of Technology, Wushan Road 381, Tianhe District, 510641 Guangzhou, P. R. China; summerfish@scut.edu.cn