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Asymptotic stability threshold of the 2D Couette flow in a finite channel

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Abstract. In this paper we study the asymptotic stability threshold of the Couette flow for the 2D Navier–Stokes equations in a finite channel $\Omega = \mathbb{T} \times [-1, 1]$ with Navier-slip boundary condition. It was proved that if the initial velocity v_0 satisfies $\|v_0 - (y, 0)\|_{H^4} \le c\nu^{1/3}$ for some small c independent of the viscosity coefficient ν, then the solution of the 2D Navier–Stokes equations rapidly converges to some shear flow close to Couette flow for $t \gg \nu^{-1/3}$. Moreover, we prove the optimal enhanced dissipation and inviscid damping estimates. To this end, we develop a new approach that does not rely on the construction of the Fourier multiplier. Therefore, our approach opens a way toward the asymptotic stability threshold problem for other laminar flows in a domain with a physical boundary.

1. Introduction

In this paper we study non-linear stability of the Couette flow $(y, 0)$ under the 2D Navier–Stokes equations in a finite channel $\Omega = \mathbb{T} \times [-1, 1]$ with Navier-slip boundary condition:

$$\begin{cases} \partial_t v - \nu\Delta v + v \cdot \nabla v + \nabla P = 0, \\ \nabla \cdot v = 0, \ \ v^2(t, x, \pm 1) = 0, \ \ \partial_y v^1(t, x, \pm 1) = 1, \\ v(0, x, y) = v_0(x, y), \end{cases}$$

where $v(t, x, y) = (v^1, v^2)$ is the velocity, $P(t, x, y)$ is the pressure, and ν is a small viscosity coefficient.

Let $u = v - (y, 0)$ be the perturbation of the velocity, which satisfies

$$\begin{cases} \partial_t u - \nu\Delta u + y\partial_x u + (u^2, 0) + u \cdot \nabla u + \nabla P = 0, \\ \nabla \cdot u = 0, \ \ u^2(t, x, \pm 1) = 0, \ \ \partial_y u^1(t, x, \pm 1) = 0, \\ u(0, x, y) = u^{(0)}(x, y). \end{cases}$$

Let $\omega = \partial_y u^1 - \partial_x u^2$ be the vorticity, which satisfies

$$\partial_t \omega - \nu\Delta\omega + y\partial_x\omega + u \cdot \nabla\omega = 0, \quad \omega(t, x, \pm 1) = 0. \tag{1.1}$$

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The velocity u is represented in terms of ω via the Biot–Savart law,

$$u = \nabla^\top(-\Delta)^{-1}\omega = (-\partial_y, \partial_x)(-\Delta)^{-1}\omega,$$

where $(-\Delta)^{-1}$ is taken with homogeneous Dirichlet condition on $y = \pm 1$.

In the case when $\nu = 0$, $\Omega = \mathbb{T} \times \mathbb{R}$, and the initial perturbation is in Gevrey class $2+$, Bedrossian and Masmoudi proved non-linear stability (inviscid damping) of the 2D Couette flow in a breakthrough work [8]; see the remarkable works [18, 23] for general stable monotone shear flows. In the case when $\nu > 0$ is small, we are concerned with the *transition threshold problem*, which was formulated by Bedrossian, Germain, and Masmoudi [2] as follows:

Given a norm $\|\cdot\|_X$, *determine a* $\beta = \beta(X)$ *so that*

$$\|u_0\|_X \ll \mathrm{Re}^{-\beta} \Rightarrow \text{stability},$$
$$\|u_0\|_X \gg \mathrm{Re}^{-\beta} \Rightarrow \text{instability}.$$

The exponent β is referred to as the transition threshold.

In the case when $\Omega = \mathbb{T} \times \mathbb{R}$, the following important results have been proved:

- if X is Gevrey class $2+$, then $\beta = 0$ [9];
- if X is Sobolev space, then $\beta \le \frac{1}{2}$ [10, 21];
- if X is Sobolev space, then $\beta \le \frac{1}{3}$ [22, 26];
- if X is Gevrey class $\frac{1}{s}$, $s \in [0, \frac{1}{2}]$, then $\beta \le \frac{1-2s}{3(1-s)}$ [19].

In a finite channel $\Omega = \mathbb{T} \times [-1, 1]$, Chen, Li, Wei, and Zhang proved $\beta \le \frac{1}{2}$ for the 2D Navier–Stokes equations with non-slip boundary condition [12] in Sobolev spaces. In a very recent work [7], the authors developed a new method to achieve the same threshold for shear flows near Couette under Navier-slip boundary condition in Sobolev spaces. See [13, 15–17, 27] for more related results. Due to the lift-up effect, the 3D transition threshold problem is quite different and more difficult, see [1, 3, 4, 14, 20, 25] for example.

The goal of this paper is to find the asymptotic stability threshold β for the perturbation in the Sobolev space. More precisely, we study the following question proposed in [22]:

Given a norm $\|\cdot\|_X$ $(X \subset L^2)$, *find a* $\beta = \beta(X)$ *as small as possible so that for the initial vorticity* $\|\omega^{(0)}\|_X \ll \nu^\beta$ *and for* $t > 0$,

$$\|\omega_{\neq}(t)\|_{L^2} \le Ce^{-c\nu^{1/3}t}\|\omega^{(0)}\|_X \quad \text{and} \quad \|u_{\neq}\|_{L^2_{t,x,y}} \le C\|\omega^{(0)}\|_X,$$

where

$$P_0 f = f_0(y) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x, y)\,dx, \quad P_{\neq} f = f_{\neq} = f - P_0 f.$$

We note that the first estimate corresponds to what is known as enhanced dissipation. The time integrability implied by the second estimate is consistent with the expected $1/t$ inviscid damping rate for Couette flow.

To achieve the asymptotic stability threshold, the proof in [22, 26] strongly relies on the Fourier multiplier method, which is based on Fourier analysis. Thus, it seems difficult to apply this method to the case with a physical boundary. To overcome this difficulty, we develop an approach that does not rely on the Fourier multiplier to achieve the same asymptotic stability threshold in a domain with a physical boundary.

Our main result is stated as follows.

Theorem 1.1. *There exist $\epsilon, \epsilon_1 \in (0, 1/4)$ such that for all $0 < \nu \le 1$, if the initial vorticity $\omega^{(0)}$ satisfies $\|\omega^{(0)}\|_{H^3} \le \epsilon_1 \nu^{1/3}$, $\omega^{(0)}(x, \pm 1) = 0$, then the solution $\omega(t)$ of (1.1) with initial data $\omega^{(0)}$ is global in time and satisfies the following stability estimates:*

$$\|\omega_{\neq}(t)\|_{L^2} \le C e^{-\epsilon \nu^{1/3} t} \|\omega^{(0)}\|_{H^3},$$
$$\|u_{\neq}(t)\|_{L^2} \le C(1+t)^{-1} e^{-\epsilon \nu^{1/3} t/2} \|\omega^{(0)}\|_{H^3}.$$

Let us give some remarks.

- According to the results in [22,26], the stability threshold $\beta = \frac{1}{3}$ should be optimal for perturbation in the Sobolev space. In very recent works [5, 6], the stability threshold $\beta = 0$ was established for perturbations in the Gevrey class. Under a non-slip boundary condition, recent work [11] indicates that the stability threshold $\beta \le 1/2$ achieved in [12] may be sharp for Sobolev perturbations.
- For simplicity, we require $\omega^{(0)} \in H^3(\Omega)$, which should not be optimal according to the result in [26], where $\omega^{(0)} \in H^2(\Omega)$ is required.
- We guess that similar results hold when the initial velocity is an $o(\nu^{1/3})$ perturbation of general stable monotone shear flows under the Euler equations. It has been proved that this kind of shear flow is also linearly stable under the Navier–Stokes equations even with non-slip boundary condition [13].
- Since our approach does not rely on constructing Fourier multipliers, it can be extended to the asymptotic stability threshold problem for other laminar flows, such as Poiseuille flow and Kolmogorov flow. For general flows, however, additional adaptations are required. These include deriving inviscid damping estimates for approximate solutions – particularly in non-monotone cases; establishing energy estimates over short timescales, and developing space-time estimates for the linearized Navier–Stokes equations, which in turn demand resolvent estimates.

Throughout this paper, we denote by C a constant independent of ν, t, which may be different from line to line, and $f \lesssim g$ stands for $f \le Cg$ for some absolute constant C. We will fix the constant $T_0 = \nu^{-1/6}$.

2. Sketch of the proof

We may write the initial data in the form

$$\omega^{(0)}(x, y) = W_{\text{in}}(y) + \omega^{(0)}_{\neq}(x, y), \quad W_{\text{in}} = P_0 \omega^{(0)}.$$

We define the heat extension with homogeneous Dirichlet boundary condition of W_{in} as

$$W(t,y)=e^{\nu t\partial_y^2}W_{\mathrm{in}}(y).$$

Let $U(t,y)$ be the resulting shear flow given by the corresponding Biot–Savart law:

$$U(t,y)=y-\partial_y(-\partial_y^2)^{-1}W(t,y).$$

Thanks to $\omega^{(0)}(x,\pm1)=0$ and $W_{\mathrm{in}}=P_0\omega^{(0)}$, we have $W_{\mathrm{in}}(\pm1)=0$ and $\partial_y^2W(t,y)=e^{\nu t\partial_y^2}\partial_y^2W_{\mathrm{in}}(y)$, $|\partial_y^jW(t,y)|\le\|\partial_y^jW_{\mathrm{in}}\|_{L^\infty}$ for $j=0,1,2$. Thus, for $t>0$,

$$\|U-y\|_{C^3}=\|\partial_y(-\partial_y^2)^{-1}W\|_{C^3}\lesssim\|W\|_{C^2}\le\|W_{\mathrm{in}}\|_{C^2}\lesssim\|W_{\mathrm{in}}\|_{H^3}\le\|\omega^{(0)}\|_{H^3}, \tag{2.1}$$

$$\|U'-1\|_{C^2}+\|U''\|_{C^1}+\|U'''\|_{L^\infty}\lesssim\|U-y\|_{C^3}\lesssim\|\omega^{(0)}\|_{H^3}. \tag{2.2}$$

Here $U'=\partial_yU$, $U''=\partial_y^2U$, and $U'''=\partial_y^3U$.

We denote

$$\omega_{\mathrm{ob}}(t,x,y)=\omega(t,x,y)-W(t,y),\quad u_{\mathrm{ob}}(t,x,y)=u(t,x,y)-(U(t,y)-y,0).$$

Then we have

$$\begin{cases}\partial_t\omega_{\mathrm{ob}}-\nu\Delta\omega_{\mathrm{ob}}+U\partial_x\omega_{\mathrm{ob}}-U''\partial_x\phi+u_{\mathrm{ob}}\cdot\nabla\omega_{\mathrm{ob}}=0,\\ u_{\mathrm{ob}}=(\partial_y,-\partial_x)\phi,\ \ \phi=\Delta^{-1}\omega_{\mathrm{ob}},\\ \omega_{\mathrm{ob}}(t,x,\pm1)=0,\ \ \omega_{\mathrm{ob}}|_{t=0}=\omega_{\neq}^{(0)}.\end{cases}$$

Next we construct an approximate solution ω_L of ω_{ob}. A natural idea is to solve ω_L by the following linearized equation:

$$\partial_t\omega_L-\nu\Delta\omega_L+U\partial_x\omega_L=0,\quad \omega_L|_{t=0}=\omega_{\neq}^{(0)}.$$

However, it is hard to give an explicit formula for the solution for the above equation. Instead, we solve ω_L by

$$\partial_t\omega_L-\nu t^2\partial_x^2\omega_L+U\partial_x\omega_L=0,\quad \omega_L|_{t=0}=\omega_{\neq}^{(0)}. \tag{2.3}$$

It turns out that ω_L is also a good approximate solution from Proposition 2.1. Indeed, let

$$Er_L=\partial_t\omega_L-\nu\Delta\omega_L+U\partial_x\omega_L,$$

and we write

$$\omega_L(t,x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}w_k^L(t,y)e^{ikx},\quad Er_L(t,x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}W_k(t,y)e^{ikx}.$$

Then we have

$$e^{ikyt}W_k=-\nu((\partial_y-ikt)^2-k^2+k^2t^2)(e^{ikyt}w_k^L)=-\nu(\partial_y^2-2ikt\partial_y-k^2)(e^{ikyt}w_k^L),$$

which along with Lemma 3.1 gives

$$\begin{aligned}\|W_k(t)\|_{L^2} &\leq C\nu(1+t)\|(\partial_y,k)^2(e^{ikyt}w_k^L)(t)\|_{L^2}\\ &\leq C\nu(1+t)e^{-2\nu^{1/3}t}E_k^L,\end{aligned}$$

which implies $\|Er_L(t)\|_{L^2} \leq C\nu(1+t)e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}$.

It is easy to see that $P_0\omega_L = 0$ and $\omega_L(t,x,\pm 1) = 0$. Let

$$\phi_L = \Delta^{-1}\omega_L, \quad u_L = (\partial_y, -\partial_x)\phi_L.$$

Now we denote

$$\omega_e = \omega_{\text{ob}} - \omega_L, \quad u_e = u_{\text{ob}} - u_L, \quad \phi_e = \phi - \phi_L,$$

which solve

$$\begin{cases}\partial_t\omega_e - \nu\Delta\omega_e + U\partial_x\omega_e - U''\partial_x\phi_e + u_{\text{ob}}\cdot\nabla\omega_e + u_e\cdot\nabla\omega_L + Er = 0,\\ u_e = (\partial_y, -\partial_x)\phi_e, \ \ \phi_e = \Delta^{-1}\omega_e,\\ \omega_e(t,x,\pm 1) = 0, \ \ \omega_e|_{t=0} = 0,\end{cases} \tag{2.4}$$

where

$$Er = Er_L - U''\partial_x\phi_L + u_L\cdot\nabla\omega_L.$$

2.1. Proof of Theorem 1.1

The proof is based on the following two key propositions.

Proposition 2.1. *It holds that for $t \geq 0$,*

$$\|\partial_x\omega_L(t)\|_{L^\infty} + \|(\partial_y + t\partial_x)\omega_L(t)\|_{L^\infty} \leq Ce^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}, \tag{2.5}$$

$$\|u_L^2(t)\|_{L^2} + \|(u_L^1 - tu_L^2)(t)\|_{L^2} \leq C(1+t)^{-2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}, \tag{2.6}$$

$$(1+t)\|u_L^2(t)\|_{L^\infty} + \|u_L^1(t)\|_{L^\infty} \leq C(1+t)^{-1}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}, \tag{2.7}$$

$$\|Er(t)\|_{L^2} \leq C\nu(1+t)e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3} + C(1+t)^{-2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}^2. \tag{2.8}$$

We denote

$$\begin{aligned} w_k^L(t,y) &= \frac{1}{2\pi}\int_{\mathbb{T}}\omega_L(t,x,y)e^{-ikx}\,dx,\\ w_{0,k}(y) &= \frac{1}{2\pi}\int_{\mathbb{T}}\omega^{(0)}(x,y)e^{-ikx}\,dx \quad \text{for } k\in\mathbb{Z}.\end{aligned}$$

Then (2.3) is reduced to

$$\partial_t w_k^L + \nu t^2k^2 w_k^L + ikUw_k^L = 0, \quad w_k^L|_{t=0} = w_{0,k}, \ k\neq 0, \ w_0^L = 0.$$

The solution is given by

$$w_k^L(t,y) = w_{0,k}(y)e^{-ikU_1(t,y)-\nu k^2t^3/3}, \quad U_1(t,y) = \int_0^t U(s,y)\,ds.$$

By (2.1), we have

$$\|U_1(t) - ty\|_{C^3} \lesssim t\|\omega^{(0)}\|_{H^3} \le \nu^{1/3}t. \tag{2.9}$$

Note that $e^{-\nu k^2t^3/3}$ contains the enhanced dissipation. Let ϕ_k^L solve

$$\Delta_k\phi_k^L = w_k^L, \quad \phi_k^L(\pm1) = 0, \quad \Delta_k := \partial_y^2 - k^2.$$

Then we have

$$\omega_L(t,x,y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} w_k^L(t,y)e^{ikx}, \quad \phi_L(t,x,y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \phi_k^L(t,y)e^{ikx}.$$

Thus, Proposition 2.1 reduces to the profile estimate of w_k^L given in Lemma 3.1 and the inviscid damping estimates of ϕ_k^L provided in Lemmas 3.4–3.6. Proposition 2.1 can then be proven by summing over each Fourier mode. See Section 3 for details.

Proposition 2.2. *There exist $\epsilon, \epsilon_1 \in (0, 1/4)$ such that for all $0 < \nu \le 1$, if the initial vorticity $\omega^{(0)}$ satisfies $\|\omega^{(0)}\|_{H^3} \le \epsilon_1\nu^{1/3}$, $\omega^{(0)}(x, \pm1) = 0$, then for $t \ge 0$,*

$$\|P_{\neq}\omega_e(t)\|_{L^2} \le Ce^{-\epsilon\nu^{1/3}t}\nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

Let us complete the proof of Theorem 1.1 by admitting the above two propositions.

Proof of Theorem 1.1. By (2.6) in Proposition 2.1, we have

$$\begin{aligned}\|u_L(t)\|_{L^2} &\le \|u_L^1(t)\|_{L^2} + \|u_L^2(t)\|_{L^2}\\ &\le (1+t)\|u_L^2(t)\|_{L^2} + \|(u_L^1 - tu_L^2)(t)\|_{L^2}\\ &\le C(1+t)^{-1}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}.\end{aligned}$$

By (2.5) in Proposition 2.1 and $P_0\omega_L = 0$, we have

$$\|\omega_L(t)\|_{L^2} \le \|\partial_x\omega_L(t)\|_{L^2} \le C\|\partial_x\omega_L(t)\|_{L^\infty} \le Ce^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}.$$

By Proposition 2.2, we have

$$\|P_{\neq}u_e(t)\|_{L^2} \le C\|P_{\neq}\omega_e(t)\|_{L^2} \le Ce^{-\epsilon\nu^{1/3}t}\nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

Thanks to $\omega_{\text{ob}}(t,x,y) = \omega(t,x,y) - W(t,y)$ and $u_{\text{ob}}(t,x,y) = u(t,x,y) - (U(t,y) - y, 0)$, we have $\omega_{\neq} = P_{\neq}\omega_{\text{ob}}$, $u_{\neq} = P_{\neq}u_{\text{ob}}$. Then we obtain

$$\begin{aligned}\|\omega_{\neq}(t)\|_{L^2} &= \|P_{\neq}\omega_{\text{ob}}(t)\|_{L^2} \le \|\omega_L(t)\|_{L^2} + \|P_{\neq}\omega_e(t)\|_{L^2}\\ &\le Ce^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3} + Ce^{-\epsilon\nu^{1/3}t}\nu^{1/3}\|\omega^{(0)}\|_{H^3} \le Ce^{-\epsilon\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}\end{aligned}$$

and

$$\begin{aligned}\|u_{\neq}(t)\|_{L^2} = \|P_{\neq}u_{\mathrm{ob}}(t)\|_{L^2} &\le \|u_L(t)\|_{L^2} + \|P_{\neq}u_e(t)\|_{L^2}\\ &\le C(1+t)^{-1}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3} + Ce^{-\epsilon\nu^{1/3}t}\nu^{1/3}\|\omega^{(0)}\|_{H^3}\\ &\le C(1+t)^{-1}e^{-\epsilon\nu^{1/3}t/2}\|\omega^{(0)}\|_{H^3}.\end{aligned}$$

This completes the proof of Theorem 1.1. ■

2.2. Proof of Proposition 2.2

The following proposition gives the uniform estimate of ω_e in a short timescale.

Proposition 2.3. *If* $0 \le t \le T_0 = \nu^{-1/6}$, *then*

$$\|\omega_e(t)\|_{L^2} \le C\nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

Proof. As $\omega_e|_{y=\pm1} = 0$, $\nabla\cdot u_{\mathrm{ob}} = 0$, $\int_\Omega (u_{\mathrm{ob}}\cdot\nabla\omega_e)\omega_e\,dx\,dy = 0$, we get by (2.4) that

$$\frac{1}{2}\frac{d}{dt}\|\omega_e\|_{L^2}^2 + \nu\|\nabla\omega_e\|_{L^2}^2 = \int_\Omega (U''\partial_x\phi_e - u_e\cdot\nabla\omega_L - Er)\omega_e\,dx\,dy,$$

which gives

$$\begin{aligned}\frac{d}{dt}\|\omega_e\|_{L^2} &\le \|U''\partial_x\phi_e - u_e\cdot\nabla\omega_L - Er\|_{L^2}\\ &\le \|\partial_x\phi_e\|_{L^2}\|U''\|_{L^\infty} + \|u_e\|_{L^2}\|\nabla\omega_L\|_{L^\infty} + \|Er\|_{L^2}.\end{aligned}$$

As $u_e = (\partial_y, -\partial_x)\phi_e$ and $\phi_e = \Delta^{-1}\omega_e$, we have $\|u_e\|_{L^2} = \|\nabla\phi_e\|_{L^2} \le C\|\omega_e\|_{L^2}$. Then by Grönwall's inequality and $\omega_e|_{t=0} = 0$, we get

$$\|\omega_e(t)\|_{L^2} \le \exp\Big(C\int_0^t (\|\nabla\omega_L(s)\|_{L^\infty} + \|U''\|_{L^\infty})\,ds\Big)\int_0^t \|Er(s)\|_{L^2}\,ds. \tag{2.10}$$

By (2.5) and (2.8) in Proposition 2.1 and (2.2), we have

$$\begin{aligned}&\|\nabla\omega_L(t)\|_{L^\infty} + \|U''\|_{L^\infty} \le C(1+t)\|\omega^{(0)}\|_{H^3} \le C(1+t)\nu^{\frac{1}{3}},\\ &\|Er(t)\|_{L^2} \lesssim \nu(1+t)\|\omega^{(0)}\|_{H^3} + (1+t)^{-2}\|\omega^{(0)}\|_{H^3}^2.\end{aligned}$$

Then (2.10) becomes

$$\|\omega_e(t)\|_{L^2} \le C\exp(C(1+t)^2\nu^{1/3})(\nu(1+t)^2\|\omega^{(0)}\|_{H^3} + \|\omega^{(0)}\|_{H^3}^2).$$

In particular, if $0 \le t \le T_0 = \nu^{-1/6}$, then we obtain

$$\|\omega_e(t)\|_{L^2} \le C(\nu^{2/3}\|\omega^{(0)}\|_{H^3} + \|\omega^{(0)}\|_{H^3}^2) \le C\nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

This completes the proof. ■

With the uniform estimates in a short timescale in hand, we only need to establish the uniform estimates of the solution in a long timescale $[T_0, T]$ for every $T > T_0$. From now on, all norms are taken over the interval $[T_0, T]$ unless stated otherwise, such as $\|f\|_{L^q L^p} = \|f\|_{L^q(T_0,T;L^p(\Omega))}$. The uniform estimate in a long timescale relies on the space-time estimates of the following linearized system:

$$\partial_t f - \nu\Delta f + U\partial_x f = g, \quad f(t,x,\pm 1) = 0, \quad \text{for } t \in [T_0, T].$$

We define the space-time norm $\|f\|_{X_T} = \|f\|_X$, where

$$\begin{aligned}\|f\|_X = \|f\|_{L^\infty L^2} + \nu^{1/2}\|\nabla f\|_{L^2L^2} + \|e^{\epsilon\nu^{1/3}t}P_{\neq} f\|_{L^\infty L^2} \\ + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\nabla P_{\neq} f\|_{L^2L^2} + \|e^{\epsilon\nu^{1/3}t}\partial_x\nabla\Delta^{-1} f\|_{L^2L^2} \\ + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f\|_{L^1L^2} + \nu^{1/3}\|e^{\epsilon\nu^{1/3}t}P_{\neq} f\|_{L^1L^2}.\end{aligned}$$

Proposition 2.4. *There exists* $0 < \epsilon < 1/4$ *such that*

$$\|f\|_X \le C\|f(T_0)\|_{L^2} + C\|e^{\epsilon\nu^{1/3}t}g\|_{L^1L^2}.$$

In Section 4, we will prove this proposition. The proof will use the hypocoercivity method introduced in [24] and follow some ideas introduced in the recent work [7].

To apply Proposition 2.4 for $f = \omega_e$, the main troublesome term is the reaction part of $u_e \cdot \nabla\omega_L$ due to the polynomial growth of $\nabla\omega_L$ for $t \le \nu^{-1/3}$. Notice that

$$u_e \cdot \nabla\omega_L + tu_e^2\partial_x\omega_L = u_e^1\partial_x\omega_L + u_e^2(\partial_y + t\partial_x)\omega_L,$$

which is smaller due to Proposition 2.1. This observation motivates us to introduce the reaction part ω_{re}, which solves (for $t \ge T_0$)

$$\partial_t\omega_{\text{re}} - \nu\Delta\omega_{\text{re}} + U\partial_x\omega_{\text{re}} - tu_e^2\partial_x\omega_L = 0, \quad \omega_{\text{re}}(t,x,\pm 1) = 0, \ \omega_{\text{re}}|_{t=T_0} = 0.$$

By (2.5), we have $\|\partial_x\omega_L\|_{L^\infty} \le C\|w^{(0)}\|_{H^3}$ for $t \lesssim \nu^{-1/3}$. Thus, $t\partial_x\omega_L$ is uniformly small with respect to ν if and only if $\|w^{(0)}\|_{H^3} \le \epsilon_1\nu^{1/3}$, as shown in Proposition 2.5. Otherwise, to relax the stability threshold, we must introduce the resonant toy model as in [19] to control the energy cascade across different modes, albeit at the cost of requiring Gevrey class regularity for the perturbations.

We have the following control for the reaction part ω_{re}, which will be proved in Section 5.

Proposition 2.5. *It holds that*

$$\|\omega_{\text{re}}\|_X \lesssim \nu^{-1/3}\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}\|\omega^{(0)}\|_{H^3}.$$

Let $\omega_* = \omega_e - \omega_{\text{re}}$, which solves

$$\partial_t\omega_* - \nu\Delta\omega_* + U\partial_x\omega_* + g = 0, \quad \omega_*(t,x,\pm 1) = 0, \ \omega_*|_{t=T_0} = \omega_e|_{t=T_0},$$

where

$$g = -U''\partial_x\phi_e + u_{\rm ob}\cdot\nabla\omega_e + u_e^1\partial_x\omega_L + u_e^2(\partial_y + t\partial_x)\omega_L + Er.$$

For the source g, we have the following uniform estimate.

Proposition 2.6. *It holds that*

$$\|e^{\epsilon\nu^{1/3}t}g\|_{L^1L^2} \lesssim \nu^{-2/3}\|\omega_e\|_X^2 + \nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

Proof. We estimate each term of g in several steps.

Step 1. Estimate of $U''\partial_x\phi_e$. As $\phi_e = \Delta^{-1}\omega_e$, we get by (2.2) that

$$\|U''\partial_x\phi_e\|_{L^2} \le \|U''\|_{L^\infty}\|\partial_x\phi_e\|_{L^2} \lesssim \|\omega^{(0)}\|_{H^3}\|P_{\neq}\omega_e\|_{L^2},$$

which gives

$$\begin{aligned}\|e^{\epsilon\nu^{1/3}t}U''\partial_x\phi_e\|_{L^1L^2} &\lesssim \|\omega^{(0)}\|_{H^3}\|e^{\epsilon\nu^{1/3}t}P_{\neq}\omega_e\|_{L^1L^2}\\ &\lesssim \nu^{-1/3}\|\omega^{(0)}\|_{H^3}\|\omega_e\|_X.\end{aligned}$$

Step 2. Estimate of $u_e^1\partial_x\omega_L + u_e^2(\partial_y + t\partial_x)\omega_L$. As $u_e = (\partial_y, -\partial_x)\phi_e = (\partial_y, -\partial_x)\Delta^{-1}\omega_e$, we get by (2.5) in Proposition 2.1 that

$$\begin{aligned}&\|u_e^2(\partial_y + t\partial_x)\omega_L\|_{L^2} \le \|u_e^2\|_{L^2}\|(\partial_y + t\partial_x)\omega_L\|_{L^\infty} \lesssim \|\omega_e\|_{L^2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3},\\ &\|u_e^1\partial_x\omega_L\|_{L^2} \le \|u_e^1\|_{L^2}\|\partial_x\omega_L\|_{L^\infty} \lesssim \|\omega_e\|_{L^2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3},\end{aligned}$$

which gives

$$\begin{aligned}&\|e^{\epsilon\nu^{1/3}t}(u_e^1\partial_x\omega_L + u_e^2(\partial_y + t\partial_x)\omega_L)\|_{L^1L^2}\\ &\quad\lesssim \|\omega_e\|_{L^\infty L^2}\|e^{-\nu^{1/3}t}\|_{L^1(T_0,T)}\|\omega^{(0)}\|_{H^3} \le \nu^{-1/3}\|\omega^{(0)}\|_{H^3}\|\omega_e\|_X.\end{aligned}$$

Step 3. Estimate of $u_{\rm ob}\cdot\nabla\omega_e$. As $u_e = u_{\rm ob} - u_L$, we have

$$u_{\rm ob}\cdot\nabla\omega_e = u_L\cdot\nabla\omega_e + u_e\cdot\nabla\omega_e.$$

For the first term, we have

$$u_L\cdot\nabla\omega_e = u_L^1\partial_x\omega_e + u_L^2\partial_y\omega_e.$$

By (2.7) in Proposition 2.1, we have

$$\begin{aligned}&\|u_L^1\|_{L^\infty L^\infty} \lesssim T_0^{-1}\|\omega^{(0)}\|_{H^3} = \nu^{1/6}\|\omega^{(0)}\|_{H^3},\\ &\|e^{\epsilon\nu^{1/3}t}u_L^2\|_{L^2L^\infty} \lesssim \|t^{-2}\|_{L^2(T_0,T)}\|\omega^{(0)}\|_{H^3} \lesssim T_0^{-1}\|\omega^{(0)}\|_{H^3} = \nu^{1/6}\|\omega^{(0)}\|_{H^3}.\end{aligned}$$

Then we infer that

$$\begin{aligned}\|e^{\epsilon\nu^{1/3}t}u_L\cdot\nabla\omega_e\|_{L^1L^2} &\lesssim \|u_L^1\|_{L^\infty L^\infty}\|e^{\epsilon\nu^{1/3}t}\partial_x\omega_e\|_{L^1L^2} + \|e^{\epsilon\nu^{1/3}t}u_L^2\|_{L^2L^\infty}\|\nabla\omega_e\|_{L^2L^2}\\ &\lesssim \nu^{1/6}\|\omega^{(0)}\|_{H^3}\nu^{-1/2}\|\omega_e\|_X = \nu^{-1/3}\|\omega^{(0)}\|_{H^3}\|\omega_e\|_X.\end{aligned}$$

For the second term, we have

$$u_e \cdot \nabla \omega_e = P_{\neq} u_e \cdot \nabla \omega_e + P_0 u_e^1 \partial_x \omega_e.$$

By Lemma A.1 and $u_e = (\partial_y, -\partial_x)\Delta^{-1}\omega_e$, we get

$$\begin{aligned}
&\|P_{\neq} u_e\|_{L^\infty} \lesssim \|\partial_x P_{\neq} u_e\|_{L^2}^{3/4} \|P_{\neq} u_e\|_{H^2}^{1/4} \lesssim \|\partial_x u_e\|_{L^2}^{3/4} \|\nabla \omega_e\|_{L^2}^{1/4},\\
&\|P_0 u_e\|_{L^\infty} \lesssim \|P_0 u_e\|_{H^1} \lesssim \|\omega_e\|_{L^2},
\end{aligned}$$

and then

$$\begin{aligned}
\|e^{\epsilon \nu^{1/3} t} P_{\neq} u_e\|_{L^2 L^\infty} &\lesssim \|e^{\epsilon \nu^{1/3} t} \partial_x u_e\|_{L^2 L^2}^{3/4} \|e^{\epsilon \nu^{1/3} t} \nabla \omega_e\|_{L^2 L^2}^{1/4}\\
&\lesssim \|\omega_e\|_X^{3/4} (\nu^{-1/2} \|\omega_e\|_X)^{1/4} = \nu^{-1/8} \|\omega_e\|_X.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\|e^{\epsilon \nu^{1/3} t} P_{\neq} u_e \cdot \nabla \omega_e\|_{L^1 L^2} &\le \|e^{\epsilon \nu^{1/3} t} P_{\neq} u_e\|_{L^2 L^\infty} \|\nabla \omega_e\|_{L^2 L^2}\\
&\lesssim \nu^{-1/8} \|\omega_e\|_X \nu^{-1/2} \|\omega_e\|_X,\\
\|e^{\epsilon \nu^{1/3} t} P_0 u_e^1 \partial_x \omega_e\|_{L^1 L^2} &\le \|\omega_e\|_{L^\infty L^2} \|e^{\epsilon \nu^{1/3} t} \partial_x \omega_e\|_{L^1 L^2}\\
&\lesssim \nu^{-1/2} \|\omega_e\|_X^2 \le \nu^{-2/3} \|\omega_e\|_X^2.
\end{aligned}$$

This shows that

$$\|e^{\epsilon \nu^{1/3} t} u_e \cdot \nabla \omega_e\|_{L^1 L^2} \lesssim \nu^{-2/3} \|\omega_e\|_X^2.$$

Step 4. Estimate of Er. By (2.8) in Proposition 2.1, we have

$$\begin{aligned}
\|e^{\epsilon \nu^{1/3} t} Er\|_{L^1 L^2} &\lesssim \|\nu(1+t) e^{-\nu^{1/3} t}\|_{L^1(T_0,T)} \|\omega^{(0)}\|_{H^3} + \|(1+t)^{-2}\|_{L^1(T_0,T)} \|\omega^{(0)}\|_{H^3}^2\\
&\lesssim \nu^{1/3} \|\omega^{(0)}\|_{H^3} + \|\omega^{(0)}\|_{H^3}^2 \lesssim \nu^{1/3} \|\omega^{(0)}\|_{H^3}.
\end{aligned}$$

Summing up, we arrive at

$$\begin{aligned}
\|e^{\epsilon \nu^{1/3} t} g\|_{L^1 L^2} &\lesssim \nu^{-1/3} \|\omega^{(0)}\|_{H^3} \|\omega_e\|_X + \nu^{-2/3} \|\omega_e\|_X^2 + \nu^{1/3} \|\omega^{(0)}\|_{H^3}\\
&\lesssim \|\omega^{(0)}\|_{H^3}^2 + \nu^{-2/3} \|\omega_e\|_X^2 + \nu^{1/3} \|\omega^{(0)}\|_{H^3}\\
&\lesssim \nu^{-2/3} \|\omega_e\|_X^2 + \nu^{1/3} \|\omega^{(0)}\|_{H^3}.
\end{aligned}$$

This completes the proof. ∎

Now we are in a position to prove the key Proposition 2.2.

Proof of Proposition 2.2. As $\omega_*(T_0) = \omega_e(T_0)$, we get by Proposition 2.3 that

$$\|\omega_*(T_0)\|_{L^2} = \|\omega_e(T_0)\|_{L^2} \le C \nu^{1/3} \|\omega^{(0)}\|_{H^3}.$$

Then by Proposition 2.4 and Proposition 2.6, we have

$$\|\omega_*\|_X \lesssim \|\omega_*(T_0)\|_{L^2} + \|e^{\epsilon\nu^{1/3}t}g\|_{L^1L^2} \lesssim \nu^{-2/3}\|\omega_e\|_X^2 + \nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

As $u_e^2 = -\partial_x\Delta^{-1}\omega_e$, we have

$$\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2} = \|e^{\epsilon\nu^{1/3}t}\partial_x\nabla\Delta^{-1}\omega_e\|_{L^2L^2} \le \|\omega_e\|_X,$$

and then by Proposition 2.5,

$$\|\omega_{\mathrm{re}}\|_X \lesssim \nu^{-1/3}\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}\|\omega^{(0)}\|_{H^3} \lesssim \nu^{-1/3}\|\omega_e\|_X\|\omega^{(0)}\|_{H^3}.$$

Thus, we obtain (recall that $\omega_* = \omega_e - \omega_{\mathrm{re}}$)

$$\begin{aligned}\|\omega_e\|_X &\le \|\omega_*\|_X + \|\omega_{\mathrm{re}}\|_X \lesssim \nu^{-2/3}\|\omega_e\|_X^2 + \nu^{1/3}\|\omega^{(0)}\|_{H^3} + \nu^{-1/3}\|\omega_e\|_X\|\omega^{(0)}\|_{H^3}\\ &\lesssim \nu^{-2/3}\|\omega_e\|_X^2 + \nu^{1/3}\|\omega^{(0)}\|_{H^3} + \|\omega^{(0)}\|_{H^3}^2\\ &\lesssim \nu^{-2/3}\|\omega_e\|_X^2 + \nu^{1/3}\|\omega^{(0)}\|_{H^3}. \end{aligned} \tag{2.11}$$

Now we use a bootstrap argument. Firstly, we assume that $\|\omega_e\|_{X_T} \le \varepsilon_0\nu^{2/3}$ for some small ε_0 determined later, which holds for some $T > T_0$ by Proposition 2.3 (for ϵ_1 small enough). Then (2.11) implies that

$$\|\omega_e\|_{X_T} \le C_1\varepsilon_0\|\omega_e\|_X + C_1\nu^{1/3}\|\omega^{(0)}\|_{H^3}.$$

Taking $\varepsilon_0 = (2C_1)^{-1}$, we can conclude that

$$\|\omega_e\|_{X_T} \le 2C_1\nu^{1/3}\|\omega^{(0)}\|_{H^3} \le 2C_1\epsilon_1\nu^{2/3}. \tag{2.12}$$

Taking $\epsilon_1 = \varepsilon_0/(4C_1)$, and using the continuity of $t \mapsto \|\omega_e\|_X$, we can conclude (2.12) for $T = +\infty$. Then the result follows from Proposition 2.3 and the definition of the X norm. ∎

3. Inviscid damping estimates for the linear part

In this section we prove Proposition 2.1. Recall that

$$w_k^L(t,y) = w_{0,k}(y)e^{-ikU_1(t,y)-\nu k^2t^3/3}$$

with U_1 satisfying (2.9).

Let $I = [-1,1]$ and $E_k^L = \|(\partial_y,k)^3 w_{0,k}\|_{L^2(I)}$ for $k \ne 0$. Then

$$\omega^{(0)}(x,y) = \sum_{k\in\mathbb{Z}} w_{0,k}(y)e^{ikx}, \qquad \sum_{k\in\mathbb{Z}\setminus\{0\}}(E_k^L)^2 \le C\|\omega^{(0)}\|_{H^3}^2. \tag{3.1}$$

We denote by $\|\cdot\|_{L^p}$ the $L^p(\Omega)$ norm or $L^p(I)$ norm, which is easy to distinguish from the context. We use the L^2 inner product $\langle f,g\rangle = \int_{-1}^1 f(y)\overline{g(y)}\,dy$, and use $\psi = \Delta_k^{-1}\omega$ to denote the unique solution of $\Delta_k\psi = \omega$, $\psi(\pm1) = 0$, here $\Delta_k = \partial_y^2 - k^2$, then $\psi_k^L = \Delta_k^{-1}w_k^L$.

Lemma 3.1. *For $k \in \mathbb{Z} \setminus \{0\}$, $t \ge 0$, $j = 0, 1, 2, 3$, we have*

$$\|(\partial_y, k)^j (e^{ikyt} w_k^L)(t)\|_{L^2} \le C|k|^{j-3} e^{-2(\nu k^2)^{1/3} t} E_k^L.$$

For $k \in \mathbb{Z} \setminus \{0\}$, $t \ge 0$, $j = 0, 1, 2$, we have

$$\|(\partial_y, k)^j (e^{ikyt} w_k^L)(t)\|_{L^\infty} \le C|k|^{j-\frac{5}{2}} e^{-2(\nu k^2)^{1/3} t} E_k^L.$$

Proof. Thanks to $w_k^L(t, y) = w_{0,k}(y) e^{-ikU_1(t,y) - \nu k^2 t^3/3}$, we have

$$e^{ikyt} w_k^L(t, y) = w_{0,k}(y) e^{-ik(U_1(t,y) - ty) - \nu k^2 t^3/3} = w_{0,k}(y) e^{i\gamma - \nu k^2 t^3/3},$$

where $\gamma = -k(U_1(t, y) - ty)$. By (2.9), we have $|\partial_y^j \gamma| \le C\nu^{1/3}|kt|$ for $j = 0, 1, 2, 3$ and then

$$\begin{aligned}
|\partial_y(e^{i\gamma})| &\le |\partial_y \gamma| \le C\nu^{1/3}|kt|,\\
|\partial_y^2(e^{i\gamma})| &\le |\partial_y^2 \gamma| + |\partial_y \gamma|^2 \le C\nu^{1/3}|kt| + C(\nu^{1/3}|kt|)^2,\\
|\partial_y^3(e^{i\gamma})| &\le |\partial_y^3 \gamma| + 3|\partial_y \gamma||\partial_y^2 \gamma| + |\partial_y \gamma|^3\\
&\le C\nu^{1/3}|kt| + C(\nu^{1/3}|kt|)^2 + C(\nu^{1/3}|kt|)^3,
\end{aligned}$$

which show that $|\partial_y^j(e^{i\gamma})| \le C(1 + \nu^{1/3}|kt|)^j$ for $j = 0, 1, 2, 3$, and

$$\begin{aligned}
|\partial_y^j(e^{i\gamma - \nu k^2 t^3/3})| &\lesssim (1 + \nu^{1/3}|kt|)^j e^{-\nu k^2 t^3/3} \lesssim |k|^{j/3}(1 + \nu k^2 t^3)^{j/3} e^{-\nu k^2 t^3/3}\\
&\lesssim |k|^{j/3} e^{-2(\nu k^2)^{1/3} t},\\
|(\partial_y, k)^j(e^{i\gamma - \nu k^2 t^3/3})| &\lesssim |k|^j e^{-2(\nu k^2)^{1/3} t}.
\end{aligned}$$

Thus, we infer that for $j = 0, 1, 2, 3$,

$$\begin{aligned}
\|(\partial_y, k)^j (e^{ikyt} w_k^L)(t)\|_{L^2} &= \|(\partial_y, k)^j (w_{0,k}(y) e^{i\gamma - \nu k^2 t^3/3})(t)\|_{L^2}\\
&\le C \sum_{l=0}^{j} \|(\partial_y, k)^l w_{0,k}(y)\|_{L^2} \|(\partial_y, k)^{j-l}(e^{i\gamma - \nu k^2 t^3/3})\|_{L^\infty}\\
&\le C \sum_{l=0}^{j} |k|^{l-3} E_k^L |k|^{j-l} e^{-2(\nu k^2)^{1/3} t}\\
&\le C|k|^{j-3} e^{-2(\nu k^2)^{1/3} t} E_k^L.
\end{aligned}$$

For $j = 0, 1, 2$, we get by the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
\|(\partial_y, k)^j (e^{ikyt} w_k^L)(t)\|_{L^\infty} &\le C\|(\partial_y, k)^{j+1}(e^{ikyt} w_k^L)(t)\|_{L^2}^{1/2} \|(\partial_y, k)^j (e^{ikyt} w_k^L)(t)\|_{L^2}^{1/2}\\
&\le C|k|^{j-\frac{5}{2}} e^{-2(\nu k^2)^{1/3} t} E_k^L.
\end{aligned}$$

This completes the proof. ∎

Remark 3.2. The estimate $\|(\partial_y, k)^j (e^{ikU_1} w_k^L)(t)\|_{L^2} \le C|k|^{j-3} e^{-\nu k^2 t^3/3} E_k^L$ is obvious. This lemma shows that a similar bound holds with U_1 replaced by yt. This ensures that there is no need to make a coordinate transform as in [22].

The following lemma will be used in Section 5.

Lemma 3.3. *It holds that*

$$t\|\partial_x^2 \omega_L(t)\|_{L^\infty} + t\|\partial_x \omega_L(t)\|_{L^\infty} \le C\nu^{-1/3} e^{-\nu^{1/3} t} \|\omega^{(0)}\|_{H^3}.$$

Proof. Notice that for $j = 1, 2$,

$$\omega_L(t,x,y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} w_k^L(t,y) e^{ikx}, \quad \partial_x^j \omega_L(t,x,y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (ik)^j w_k^L(t,y) e^{ikx}.$$

Then by Lemma 3.1, (3.1), and $(\nu k^2)^{1/3} t e^{-(\nu k^2)^{1/3} t} \le 1$, we have

$$\begin{aligned}
t\|\partial_x^2 \omega_L(t)\|_{L^\infty} + t\|\partial_x \omega_L(t)\|_{L^\infty} &\le Ct \sum_{k\in\mathbb{Z}\setminus\{0\}} (|k|^2 + |k|)\|(e^{ikyt} w_k^L)(t)\|_{L^\infty} \\
&\le Ct \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-\frac{1}{2}} E_k^L e^{-2(\nu k^2)^{1/3} t} \le C\nu^{-1/3} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-\frac{1}{2}-\frac{2}{3}} E_k^L e^{-(\nu k^2)^{1/3} t} \\
&\le C\nu^{-1/3} \Big(\sum_{k\in\mathbb{Z}\setminus\{0\}} (E_k^L)^2 \Big)^{1/2} e^{-\nu^{1/3} t} \le C\nu^{-1/3} e^{-\nu^{1/3} t} \|\omega^{(0)}\|_{H^3}.
\end{aligned}$$

This completes the proof. ∎

Lemma 3.4. *Assume that $k \in \mathbb{Z} \setminus \{0\}$, $t \ge 0$, $\omega \in H^1(-1,1)$, $\psi = -\Delta_k^{-1}\omega$. Then we have*

$$k^2 \|(\partial_y, k)\psi\|_{L^2} \le C(1+t)^{-1} \|(\partial_y, k)(e^{ikyt}\omega)\|_{L^2}.$$

Proof. Let $\omega_1 = \partial_y \omega + ikt\omega$. As $\psi = -\Delta_k^{-1}\omega$, we have

$$\begin{aligned}
|kt|(\|\partial_y \psi\|_{L^2}^2 + k^2 \|\psi\|_{L^2}^2) = |kt\langle \psi, \omega\rangle| &= |-i\langle \psi, \omega_1 - \partial_y \omega\rangle| = |\langle \psi, \omega_1\rangle + \langle \partial_y \psi, \omega\rangle| \\
&\le \|\psi\|_{L^2} \|\omega_1\|_{L^2} + \|\partial_y \psi\|_{L^2} \|\omega\|_{L^2} \\
&\le C(|k|\,\|\psi\|_{L^2} + \|\partial_y \psi\|_{L^2})(|k|^{-1} \|\omega_1\|_{L^2} + \|\omega\|_{L^2}),
\end{aligned}$$

from which, with $e^{ikyt}\omega_1 = \partial_y(e^{ikyt}\omega)$, we infer that

$$\begin{aligned}
|k|t(\|\partial_y \psi\|_{L^2} + |k|\,\|\psi\|_{L^2}) &\le C(|k|^{-1} \|\omega_1\|_{L^2} + \|\omega\|_{L^2}) \\
&\le C|k|^{-1} \|(\partial_y, k)(e^{ikyt}\omega)\|_{L^2}.
\end{aligned} \tag{3.2}$$

On the other hand, using $\|\partial_y \psi\|_{L^2}^2 + k^2 \|\psi\|_{L^2}^2 = \langle \psi, \omega\rangle \le \|\psi\|_{L^2} \|\omega\|_{L^2}$, we have

$$\|\partial_y \psi\|_{L^2} + |k|\,\|\psi\|_{L^2} \le C|k|^{-1} \|\omega\|_{L^2}. \tag{3.3}$$

Then the lemma is a direct consequence of (3.2) and (3.3). ∎

Lemma 3.5. *Assume that $k \in \mathbb{Z} \setminus \{0\}$, $t \geq 0$, $\omega \in H^2(-1,1)$, $\psi = -\Delta_k^{-1}\omega$. Then we have*

$$|k|^2 \|(\partial_y, k)(\partial_y \psi + ikt\psi)\|_{L^2} \leq C(1+t)^{-1} \|(\partial_y, k)^2 (e^{ikyt}\omega)\|_{L^2},$$
$$|k|^4 \|\psi\|_{L^2} \leq C(1+t)^{-2} \|(\partial_y, k)^2 (e^{ikyt}\omega)\|_{L^2}.$$

Proof. We define

$$\omega_1 = \partial_y \omega + ikt\omega, \quad \psi_1 = \partial_y \psi + ikt\psi, \quad \psi_2 = -\Delta_k^{-1}\omega_1, \quad \psi_3 = \psi_1 - \psi_2.$$

Then $-\Delta_k \psi = \omega$, $-\Delta_k \psi_1 = \omega_1$, and $\Delta_k \psi_3 = 0$. By Lemma 3.4, we have

$$\begin{aligned} k^2 \|(\partial_y, k)\psi_2\|_{L^2} &\leq C(1+t)^{-1} \|(\partial_y, k)(e^{ikyt}\omega_1)\|_{L^2} \\ &= C(1+t)^{-1} \|(\partial_y, k)\partial_y (e^{ikyt}\omega)\|_{L^2} \\ &\leq C(1+t)^{-1} \|(\partial_y, k)^2 (e^{ikyt}\omega)\|_{L^2}. \end{aligned} \tag{3.4}$$

To estimate ψ_3, we introduce $\gamma_j(y) = \frac{\sinh(k(y+j))}{j \sinh 2k}$, which satisfies

$$(\partial_y^2 - k^2)\gamma_j = 0, \ \ \gamma_j(j) = 1, \ \ \gamma_j(-j) = 0 \quad \text{for } j \in \{\pm 1\}.$$

Thus, we have

$$\psi_3 = \psi_3(-1)\gamma_{-1} + \psi_3(1)\gamma_1.$$

Thanks to $|\gamma_j'(j)| = |k \coth(2k)| \leq C|k|$ for $j \in \{\pm 1\}$, we infer that for $j = \pm 1$,

$$\begin{aligned} \|\partial_y \gamma_j\|_{L^2}^2 + k^2 \|\gamma_j\|_{L^2}^2 &= -\langle \gamma_j, (\partial_y^2 - k^2)\gamma_j \rangle + \gamma_j' \gamma_j |_{-1}^{1} \\ &= |\gamma_j' \gamma_j(j)| = |\gamma_j'(j)| \leq C|k|, \end{aligned} \tag{3.5}$$

which gives

$$\begin{aligned} \|(\partial_y, k)\psi_3\|_{L^2} &\leq |\psi_3(-1)| \, \|(\partial_y, k)\gamma_{-1}\|_{L^2} + |\psi_3(1)| \, \|(\partial_y, k)\gamma_1\|_{L^2} \\ &\leq C|k|^{\frac{1}{2}} (|\psi_3(-1)| + |\psi_3(1)|) \\ &= C|k|^{\frac{1}{2}} (|\partial_y \psi(-1)| + |\partial_y \psi(1)|). \end{aligned} \tag{3.6}$$

Here we used $\psi_3(j) = \psi_1(j) = \partial_y \psi(j)$ for $j = \pm 1$. Using the fact that

$$\langle \omega, \gamma_1 \rangle = -\langle (\partial_y^2 - k^2)\psi, \gamma_1 \rangle = -\langle \psi, (\partial_y^2 - k^2)\gamma_1 \rangle - (\partial_y \psi \gamma_1 - \psi \gamma_1')|_{-1}^{1} = -\partial_y \psi(1),$$

and $e^{ikyt}\omega_1 = \partial_y(e^{ikyt}\omega)$, we infer that for any $t \geq 0$,

$$\begin{aligned} |kt\partial_y \psi(1)| &= |kt\langle \omega, \gamma_1 \rangle| = |\langle \omega_1 - \partial_y \omega, \gamma_1 \rangle| = \left| \langle \omega_1, \gamma_1 \rangle + \langle \omega, \gamma_1' \rangle - \omega\gamma_1 |_{y=-1}^{y=1} \right| \\ &\leq \|\omega_1\|_{L^2} \|\gamma_1\|_{L^2} + \|\omega\|_{L^2} \|\gamma_1'\|_{L^2} + |\omega(1)| \\ &\leq C\|\omega_1\|_{L^2} |k|^{-\frac{1}{2}} + C\|\omega\|_{L^2} |k|^{\frac{1}{2}} + \|e^{ikyt}\omega\|_{L^\infty} \\ &\leq C|k|^{-\frac{1}{2}} (\|\omega_1\|_{L^2} + |k| \, \|\omega\|_{L^2}) + C\|e^{ikyt}\omega\|_{H^1}^{1/2} \|e^{ikyt}\omega\|_{L^2}^{1/2} \\ &\leq C|k|^{-\frac{1}{2}} \|(\partial_y, k)(e^{ikyt}\omega)\|_{L^2}. \end{aligned}$$

On the other hand, we have

$$|\partial_y\psi(1)| = |\langle\omega,\gamma_1\rangle| \le \|\omega\|_{L^2}\|\gamma_1\|_{L^2} \le C\|\omega\|_{L^2}|k|^{-\frac{1}{2}}.$$

This shows that

$$|\partial_y\psi(1)| \le C|k|^{-\frac{3}{2}}(1+t)^{-1}\|(\partial_y,k)(e^{ikyt}\omega)\|_{L^2}. \tag{3.7}$$

Similarly, we have

$$|\partial_y\psi(-1)| = |\langle\omega,\gamma_{-1}\rangle| \le C|k|^{-\frac{3}{2}}(1+t)^{-1}\|(\partial_y,k)(e^{ikyt}\omega)\|_{L^2}. \tag{3.8}$$

It follows from (3.6), (3.8), and (3.7) that

$$\begin{aligned}\|(\partial_y,k)\psi_3\|_{L^2} &\le C|k|^{-1}(1+t)^{-1}\|(\partial_y,k)(e^{ikyt}\omega)\|_{L^2}\\ &\le C|k|^{-2}(1+t)^{-1}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2},\end{aligned}$$

which along with (3.4) and $\psi_3 = \psi_1 - \psi_2$ implies

$$\begin{aligned}\|(\partial_y,k)\psi_1\|_{L^2} &\le \|(\partial_y,k)\psi_2\|_{L^2} + \|(\partial_y,k)\psi_3\|_{L^2}\\ &\le C|k|^{-2}(1+t)^{-1}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2}.\end{aligned}$$

This shows the first inequality by recalling $\psi_1 = \partial_y\psi + ikt\psi$.

Using Lemma 3.4 and $\psi_1 = \partial_y\psi + ikt\psi$, we have

$$|kt|\|\psi\|_{L^2} \le \|\partial_y\psi\|_{L^2} + \|\psi_1\|_{L^2} \le C|k|^{-3}(1+t)^{-1}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2},$$

which gives the second inequality (the case $0 \le t \le 1$ follows from Lemma 3.4). ■

Lemma 3.6. *Assume that $k \in \mathbb{Z}\setminus\{0\}$, $t \ge 0$, $\omega \in H^3(-1,1)$, $\omega(\pm1) = 0$, $\psi = -\Delta_k^{-1}\omega$. Then we have*

$$|k|^4\|\partial_y\psi + ikt\psi\|_{L^2} \le C(1+t)^{-2}\|(\partial_y,k)^3(e^{ikyt}\omega)\|_{L^2}.$$

Proof. Let ω_1, ψ_1, ψ_2, ψ_3, and γ_j for $j = \pm1$ be defined as in the proof of Lemma 3.5. Then (3.5) and (3.6) still hold.

By Lemma 3.5 and $e^{ikyt}\omega_1 = \partial_y(e^{ikyt}\omega)$, we have

$$\begin{aligned}|k|^4\|\psi_2\|_{L^2} &\le C(1+t)^{-2}\|(\partial_y,k)^2(e^{ikyt}\omega_1)\|_{L^2}\\ &= C(1+t)^{-2}\|(\partial_y,k)^2\partial_y(e^{ikyt}\omega)\|_{L^2}\\ &\le C(1+t)^{-2}\|(\partial_y,k)^3(e^{ikyt}\omega)\|_{L^2}.\end{aligned} \tag{3.9}$$

By (3.5) and $(\partial_y^2 - k^2)\gamma_j = 0$ we deduce that $\|\partial_y^m\gamma_j\|_{L^2} \le C|k|^{m-\frac{1}{2}}$ for $m \in \{0,1,2\}$, $j \in \{\pm1\}$. Then using the fact that $\langle\omega,\gamma_1\rangle = -\partial_y\psi(1)$, $\omega(\pm1) = 0$, we infer that for any $t \geq 0$,

$$\begin{aligned}|kt|^2|\partial_y\psi(1)| &= |kt|^2|\langle\omega,\gamma_1\rangle| = |kt|^2|\langle e^{ikyt}\omega\gamma_1, e^{ikyt}\rangle| = |\langle e^{ikyt}\omega\gamma_1, \partial_y^2(e^{ikyt})\rangle| \\ &= \big|\langle\partial_y^2(e^{ikyt}\omega\gamma_1), e^{ikyt}\rangle - e^{-ikyt}\partial_y(e^{ikyt}\omega\gamma_1)|_{y=-1}^{y=1}\big| \\ &\leq \|\partial_y^2(e^{ikyt}\omega\gamma_1)\|_{L^1} + \|\partial_y(e^{ikyt}\omega\gamma_1)\|_{L^\infty} \leq 2\|\partial_y^2(e^{ikyt}\omega\gamma_1)\|_{L^1} \\ &\leq C\sum_{j=0}^{2}\|\partial_y^j(e^{ikyt}\omega)\|_{L^2}\|\partial_y^{2-j}\gamma_1\|_{L^2} \leq C\sum_{j=0}^{2}\|\partial_y^j(e^{ikyt}\omega)\|_{L^2}|k|^{2-j-\frac{1}{2}} \\ &\leq C|k|^{-\frac{1}{2}}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2}.\end{aligned}$$

Here we also used $\omega(-1) = \gamma_1(-1) = 0$, $\partial_y(e^{ikyt}\omega\gamma_1)|_{y=-1} = 0$. On the other hand, we have $|\partial_y\psi(1)| = |\langle\omega,\gamma_1\rangle| \leq C\|\omega\|_{L^2}|k|^{-\frac{1}{2}}$. This shows that

$$|\partial_y\psi(1)| \leq C|k|^{-\frac{5}{2}}(1+t)^{-2}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2}. \tag{3.10}$$

Similarly, we have

$$|\partial_y\psi(-1)| = |\langle\omega,\gamma_{-1}\rangle| \leq C|k|^{-\frac{5}{2}}(1+t)^{-2}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2}. \tag{3.11}$$

It follows from (3.6), (3.10), and (3.11) that

$$\begin{aligned}\|(\partial_y,k)\psi_3\|_{L^2} &\leq C|k|^{-2}(1+t)^{-2}\|(\partial_y,k)^2(e^{ikyt}\omega)\|_{L^2} \\ &\leq C|k|^{-3}(1+t)^{-2}\|(\partial_y,k)^3(e^{ikyt}\omega)\|_{L^2},\end{aligned}$$

which along with (3.9) and $\psi_3 = \psi_1 - \psi_2$ implies

$$\|\psi_1\|_{L^2} \leq \|\psi_2\|_{L^2} + \|\psi_3\|_{L^2} \leq C|k|^{-4}(1+t)^{-2}\|(\partial_y,k)^3(e^{ikyt}\omega)\|_{L^2}.$$

This completes the proof by recalling $\psi_1 = \partial_y\psi + ikt\psi$. ∎

Next we prove Proposition 2.1.

Proof of Proposition 2.1. The proof is split into several steps.

Step 1. Estimate of ω_L. Notice that

$$\begin{aligned}\omega_L(t,x,y) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} w_k^L(t,y)e^{ikx}, \\ \partial_x\omega_L(t,x,y) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} ikw_k^L(t,y)e^{ikx}, \\ (\partial_y + t\partial_x)\omega_L(t,x,y) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} (\partial_y + ikt)w_k^L(t,y)e^{ikx} \\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \partial_y(e^{ikyt}w_k^L)(t,y)e^{ik(x-yt)}.\end{aligned}$$

Then by Lemma 3.1 and (3.1), we get

$$\begin{aligned}\|\partial_x\omega_L(t)\|_{L^\infty}+\|(\partial_y+t\partial_x)\omega_L(t)\|_{L^\infty}&\le C\sum_{k\in\mathbb{Z}\backslash\{0\}}\|(\partial_y,k)(e^{ikyt}w_k^L)(t)\|_{L^\infty}\\&\le C\sum_{k\in\mathbb{Z}\backslash\{0\}}|k|^{-\frac{3}{2}}E_k^Le^{-2(\nu k^2)^{1/3}t}\\&\le C\Big(\sum_{k\in\mathbb{Z}\backslash\{0\}}(E_k^L)^2\Big)^{1/2}e^{-2\nu^{1/3}t}\\&\le Ce^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3},\end{aligned}$$

which gives (2.5).

Step 2. Estimate of u_L. Notice that $u_L=(\partial_y,-\partial_x)\phi_L=(u_L^1,u_L^2)$, and

$$\begin{aligned}&\phi_L=\sum_{k\in\mathbb{Z}\backslash\{0\}}\phi_k^L(t,y)e^{ikx},\\&u_L^2(t,x,y)=-\partial_x\phi_L=-\sum_{k\in\mathbb{Z}\backslash\{0\}}ik\phi_k^L(t,y)e^{ikx},\end{aligned}\tag{3.12}$$

$$(u_L^1-tu_L^2)(t,x,y)=(\partial_y+t\partial_x)\phi_L=\sum_{k\in\mathbb{Z}\backslash\{0\}}(\partial_y+ikt)\phi_k^L(t,y)e^{ikx}.\tag{3.13}$$

As $\phi_k^L=\Delta_k^{-1}w_k^L$, by Lemmas 3.5, 3.6, and 3.1, we have

$$\begin{aligned}&|k|^4(1+t)^2(\|ik\phi_k^L(t)\|_{L^2}+\|(\partial_y+ikt)\phi_k^L(t)\|_{L^2})\\&\quad\le C|k|\,\|(\partial_y,k)^2(e^{ikyt}w_k^L)(t)\|_{L^2}+C\|(\partial_y,k)^3(e^{ikyt}w_k^L)(t)\|_{L^2}\\&\quad\le Ce^{-2(\nu k^2)^{1/3}t}E_k^L.\end{aligned}\tag{3.14}$$

Then by (3.12), (3.13), (3.14), and (3.1), we get

$$\begin{aligned}&\|u_L^2(t)\|_{L^2}^2+\|(u_L^1-tu_L^2)(t)\|_{L^2}^2\\&\quad\le C\sum_{k\in\mathbb{Z}\backslash\{0\}}(\|ik\phi_k^L(t)\|_{L^2}^2+\|(\partial_y+ikt)\phi_k^L(t)\|_{L^2}^2)\\&\quad\le C\sum_{k\in\mathbb{Z}\backslash\{0\}}(1+t)^{-4}e^{-4(\nu k^2)^{1/3}t}(E_k^L)^2\le C(1+t)^{-4}e^{-4\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}^2,\end{aligned}$$

which gives (2.6).

By Sobolev embedding and (3.14), we have

$$\begin{aligned}\|\psi_k^L(t)\|_{L^\infty}&=\|e^{ikyt}\phi_k^L(t)\|_{L^\infty}\le C\|e^{ikyt}\phi_k^L(t)\|_{H^1(I)}\\&\le C\|(\partial_y,k)(e^{ikyt}\phi_k^L)(t)\|_{L^2}=C\|(\partial_y+ikt,k)\phi_k^L(t)\|_{L^2}\\&\le C(1+t)^{-2}|k|^{-4}e^{-2(\nu k^2)^{1/3}t}E_k^L.\end{aligned}$$

Similarly, by Sobolev embedding and Lemmas 3.4, 3.5, and 3.1, we have

$$\begin{aligned}\|\partial_y\phi_k^L(t)\|_{L^\infty} &\le C\|e^{ikyt}\partial_y\phi_k^L(t)\|_{H^1(I)} \le C\|(\partial_y+ikt,k)\partial_y\phi_k^L(t)\|_{L^2}\\ &\le C(1+t)^{-1}|k|^{-2}(\|(\partial_y,k)^2(e^{ikyt}w_k^L)(t)\|_{L^2}\\ &\qquad\qquad + |k|\,\|(\partial_y,k)(e^{ikyt}w_k^L)(t)\|_{L^2})\\ &\le C(1+t)^{-1}|k|^{-3}e^{-2(\nu k^2)^{1/3}t}E_k^L.\end{aligned}$$

Then by (3.12), (3.1), and $u_L^1=\partial_y\phi_L$, we have

$$\begin{aligned}\|u_L^2(t)\|_{L^\infty} &\le C\sum_{k\in\mathbb{Z}\setminus\{0\}}\|ik\phi_k^L(t)\|_{L^\infty} \le C\sum_{k\in\mathbb{Z}\setminus\{0\}}(1+t)^{-2}|k|^{-3}e^{-2(\nu k^2)^{1/3}t}E_k^L\\ &\le C(1+t)^{-2}e^{-2\nu^{1/3}t}\Big(\sum_{k\in\mathbb{Z}\setminus\{0\}}(E_k^L)^2\Big)^{1/2} \le C(1+t)^{-2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3},\\ \|u_L^1(t)\|_{L^\infty} &\le C\sum_{k\in\mathbb{Z}\setminus\{0\}}\|\partial_y\phi_k^L(t)\|_{L^\infty} \le C\sum_{k\in\mathbb{Z}\setminus\{0\}}(1+t)^{-1}|k|^{-3}e^{-2(\nu k^2)^{1/3}t}E_k^L\\ &\le C(1+t)^{-1}e^{-2\nu^{1/3}t}\Big(\sum_{k\in\mathbb{Z}\setminus\{0\}}(E_k^L)^2\Big)^{1/2} \le C(1+t)^{-1}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3},\end{aligned}$$

which give (2.7).

Step 3. Estimate of Er. Recall $Er=Er_L-U''\partial_x\phi_L+u_L\cdot\nabla\omega_L$. As $\partial_x\phi_L=-u_L^2$, we get by (2.2) and (2.6) that

$$\|U''\partial_x\phi_L(t)\|_{L^2}\le\|U''\|_{L^\infty}\|u_L^2(t)\|_{L^2}\le C(1+t)^{-2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}^2.$$

Notice that

$$u_L\cdot\nabla\omega_L=u_L^1\partial_x\omega_L+u_L^2\partial_y\omega_L=(u_L^1-tu_L^2)\partial_x\omega_L+u_L^2(\partial_y+t\partial_x)\omega_L.$$

Then by (2.5) and (2.6) we get

$$\begin{aligned}\|u_L\cdot\nabla\omega_L(t)\|_{L^2} &\le \|(u_L^1-tu_L^2)(t)\|_{L^2}\|\partial_x\omega_L(t)\|_{L^\infty}+\|u_L^2(t)\|_{L^2}\|(\partial_y+t\partial_x)\omega_L(t)\|_{L^\infty}\\ &\le C(1+t)^{-2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}^2.\end{aligned}$$

Thanks to $(\partial_t-\nu t^2\partial_x^2+U\partial_x)\omega_L=0$ and $Er_L=(\partial_t-\nu\Delta+U\partial_x)\omega_L$, we get $Er_L=(\nu t^2\partial_x^2-\nu\Delta)\omega_L$ and

$$\omega_L(t,x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}w_k^L(t,y)e^{ikx},\quad Er_L(t,x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}W_k(t,y)e^{ikx},$$

where $W_k(t,y)=-\nu(\partial_y^2-k^2+k^2t^2)w_k^L(t,y)$. Then we have

$$\begin{aligned}e^{ikyt}W_k &= -\nu((\partial_y-ikt)^2-k^2+k^2t^2)(e^{ikyt}w_k^L)\\ &= -\nu(\partial_y^2-2ikt\partial_y-k^2)(e^{ikyt}w_k^L).\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}\|W_k(t)\|_{L^2} &= \|\nu(\partial_y^2 - 2ikt\partial_y - k^2)(e^{ikyt}w_k^L)(t)\|_{L^2}\\ &\le C\nu(1+t)\|(\partial_y, k)^2(e^{ikyt}w_k^L)(t)\|_{L^2}\\ &\le C\nu(1+t)|k|^{-1}e^{-2(\nu k^2)^{1/3}t}E_k^L \le C\nu(1+t)e^{-2\nu^{1/3}t}E_k^L.\end{aligned}$$

Then we get by (3.1) that

$$\begin{aligned}\|Er_L(t)\|_{L^2}^2 &\le C\sum_{k\in\mathbb{Z}\setminus\{0\}}\|W_k(t)\|_{L^2}^2 \le C\sum_{k\in\mathbb{Z}\setminus\{0\}}\nu^2(1+t)^2e^{-4\nu^{1/3}t}(E_k^L)^2\\ &\le C\nu^2(1+t)^2e^{-4\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}^2.\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}\|Er(t)\|_{L^2} &\le \|Er_L(t)\|_{L^2} + \|U''\partial_x\phi_L(t)\|_{L^2} + \|u_L\cdot\nabla\omega_L(t)\|_{L^2}\\ &\le C\nu(1+t)e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3} + C(1+t)^{-2}e^{-2\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}^2,\end{aligned}$$

which gives (2.8). ■

4. Space-time estimates for the linearized equation

This section is devoted to the proof of Proposition 2.4. Throughout this section, we assume that U satisfies (for some $\delta_0 \in (0, 1/4)$ small enough)

$$\|U(t,y) - y\|_{C^3} \le \delta_0, \quad \forall t \ge 0. \tag{4.1}$$

Since $\|\omega^{(0)}\|_{H^3} \le \epsilon_1\nu^{1/3} \le \epsilon_1$, for every fixed $\delta_0 > 0$, there exists $\epsilon_1 \in (0, 1)$ such that (2.1) implies (4.1).

4.1. Linear enhanced dissipation

Proposition 4.1. *There exists $\delta_0 \in (0, 1/4)$ such that if $U(t, y)$ satisfies (4.1) and f solves*

$$\partial_t f - \nu\Delta f + U\partial_x f = 0, \quad f(t, x, \pm 1) = 0, \ \forall t \ge 0, \tag{4.2}$$

then it holds that

$$\begin{aligned}&\|\partial_x f(t)\|_{L^2} \le C(\nu t)^{-1/2}(1+t)^{-1}\|f(0)\|_{L^2},\\ &\|P_{\neq} f(t)\|_{L^2} \le C(1+\nu t^3)^{-1}\|f(0)\|_{L^2}.\end{aligned}$$

We denote

$$f_k(t,y) := \frac{1}{2\pi}\int_{\mathbb{T}} f(t,x,y)e^{-ikx}\,dx, \quad \text{for } k \in \mathbb{Z}.$$

Then we have

$$\partial_t f_k - \nu\Delta_k f_k + ikUf_k = 0, \quad f_k(t, \pm 1) = 0.$$

Here, $\Delta_k = \partial_y^2 - k^2$. We need the following lemmas.

Lemma 4.2. *Let $\delta_0 \in (0, 1/4)$ and $U(t, y)$ satisfy (4.1). Then we have*

$$\frac{d}{dt}\|f_k\|_{L^2}^2 = -2\nu\|(\partial_y, k)f_k\|_{L^2}^2,$$
$$\frac{d}{dt}\|(\partial_y, k)f_k\|_{L^2}^2 + 2\nu\|\Delta_k f_k\|_{L^2}^2 \le 3|k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2},$$
$$\frac{d}{dt}\mathrm{Re}\langle ikf_k, \partial_y f_k\rangle + |k|^2\|\sqrt{U'}f_k\|_{L^2}^2 \le 2\nu|k|\,\|\partial_y f_k\|_{L^2}\|\Delta_k f_k\|_{L^2}.$$

Proof. As $f_k|_{y=\pm1} = 0$, by a direct calculation, we have

$$\frac{d}{dt}\|f_k\|_{L^2}^2 = 2\nu\,\mathrm{Re}\langle\Delta_k f_k, f_k\rangle = -2\nu\|(\partial_y, k)f_k\|_{L^2}^2,$$

which gives the first equality.

As $f_k|_{y=\pm1} = 0$, $\Delta_k = \partial_y^2 - k^2$, we have

$$\begin{aligned}
&\frac{d}{dt}\|(\partial_y, k)f_k\|_{L^2}^2 = -2\,\mathrm{Re}\langle\Delta_k f_k, \partial_t f_k\rangle = -2\nu\|\Delta_k f_k\|_{L^2}^2 + 2\,\mathrm{Re}\langle\Delta_k f_k, ikUf_k\rangle,\\
&\mathrm{Re}\langle\Delta_k f_k, ikUf_k\rangle = \mathrm{Re}\langle -k^2 f_k, ikUf_k\rangle - \mathrm{Re}\langle\partial_y f_k, \partial_y(ikUf_k)\rangle\\
&\qquad = -\,\mathrm{Re}\langle\partial_y f_k, ikU'f_k\rangle,\\
&|2\,\mathrm{Re}\langle\Delta_k f_k, ikUf_k\rangle| \le 2|k|\,\|U'\|_{L^\infty}\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2}.
\end{aligned}$$

This shows the second inequality by noticing that

$$\|U'\|_{L^\infty} \le 1 + \|U' - 1\|_{L^\infty} \le 1 + \delta_0 < 3/2.$$

Using integration by parts, we have

$$\begin{aligned}
\frac{d}{dt}\mathrm{Re}\langle ikf_k, \partial_y f_k\rangle &= \mathrm{Re}\langle ik(\nu\Delta_k f_k - ikUf_k), \partial_y f_k\rangle\\
&\quad + \mathrm{Re}\langle ikf_k, \partial_y(\nu\Delta_k f_k - ikUf_k)\rangle\\
&= -|k|^2\|\sqrt{U'}f_k\|_{L^2}^2 - \mathrm{Im}\langle k\nu\Delta_k f_k, \partial_y f_k\rangle - \mathrm{Im}\langle kf_k, \nu\partial_y\Delta_k f_k\rangle,\\
|\mathrm{Im}\langle k\nu\Delta_k f_k, \partial_y f_k\rangle + \mathrm{Im}\langle kf_k, \nu\partial_y\Delta_k f_k\rangle| &= 2|\mathrm{Im}\langle k\nu\Delta_k f_k, \partial_y f_k\rangle|\\
&\le 2\nu|k|\,\|\partial_y f_k\|_{L^2}\|\Delta_k f_k\|_{L^2},
\end{aligned}$$

which implies the third inequality. ∎

Lemma 4.3. *Let $\delta_0 \in (0, 1/4)$, $U(t, y)$ satisfy (4.1), $\gamma_0 = 1/900$, $\alpha_0 = 16\gamma_0$, $\beta_0 = 6\gamma_0$, and*

$$\Phi_k(t) = (1 + \gamma_0\nu|k|^2t^3)\|f_k\|_{L^2}^2 + \alpha_0\nu t\|(\partial_y, k)f_k\|_{L^2}^2 + \beta_0\nu t^2\,\mathrm{Re}\langle ikf_k, \partial_y f_k\rangle,$$

where f_k is $f_k(t, \cdot)$. Then it holds that for $t \ge 0$,

$$\Phi_k(t) \ge (1 + \gamma_0\nu|k|^2t^3/4)\|f_k\|_{L^2}^2 + \alpha_0\nu t\|(\partial_y, k)f_k\|_{L^2}^2/4, \quad \Phi_k'(t) \le 0.$$

Proof. As $\alpha_0 = 16\gamma_0 > 0$, $\beta_0 = 6\gamma_0$, we have

$$\begin{aligned}|\beta_0\nu t^2 \operatorname{Re}\langle ikf_k, \partial_y f_k\rangle| &\le \beta_0\nu t^2|k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2}\\ &\le \beta_0\nu(t^3|k|^2\|f_k\|_{L^2}^2 + 16t\|\partial_y f_k\|_{L^2}^2)/8\\ &= 3\nu(\gamma_0 t^3|k|^2\|f_k\|_{L^2}^2 + \alpha_0 t\|\partial_y f_k\|_{L^2}^2)/4,\end{aligned}$$

which implies the first inequality of the lemma.

Next we show $\Phi'(t) \le 0$. By Lemma 4.2 and $|\operatorname{Re}\langle ikf_k, \partial_y f_k\rangle| \le |k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2}$, we get

$$\begin{aligned}\Phi_k'(t) &\le 3\gamma_0\nu|k|^2t^2\|f_k\|_{L^2}^2 - 2\nu(1+\gamma_0\nu|k|^2t^3)\|(\partial_y,k)f_k\|_{L^2}^2 + \alpha_0\nu\|(\partial_y,k)f_k\|_{L^2}^2\\ &\quad - 2\alpha_0\nu^2 t\|\Delta_k f_k\|_{L^2}^2 + 3\alpha_0\nu t|k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2} + 2\beta_0\nu t|k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2}\\ &\quad - \beta_0\nu t^2|k|^2\|\sqrt{U'}f_k\|_{L^2}^2 + 2\beta_0\nu^2t^2|k|\,\|\partial_y f_k\|_{L^2}\|\Delta_k f_k\|_{L^2}\\ &= \nu|k|^2t^2\big(4\gamma_0\|f_k\|_{L^2}^2 - \beta_0\|\sqrt{U'}f_k\|_{L^2}^2\big) - \nu(1-\alpha_0)\|(\partial_y,k)f_k\|_{L^2}^2\\ &\quad - \big[\nu|k|^2t^2\gamma_0\|f_k\|_{L^2}^2 - (3\alpha_0+2\beta_0)\nu t|k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2} + \nu\|(\partial_y,k)f_k\|_{L^2}^2\big]\\ &\quad - \big[2\alpha_0\nu^2 t\|\Delta_k f_k\|_{L^2}^2 - 2\beta_0\nu^2t^2|k|\,\|\partial_y f_k\|_{L^2}\|\Delta_k f_k\|_{L^2}\\ &\qquad + 2\gamma_0\nu^2|k|^2t^3\|(\partial_y,k)f_k\|_{L^2}^2\big]\\ &= I_0 - I_1 - I_2 - I_3.\end{aligned}$$

By (4.1) and $\beta_0 = 6\gamma_0 > 0$, we have $\beta_0\cdot(3/4) > 4\gamma_0$ and

$$\begin{aligned}&\inf U' \ge 1 - \|U'-1\|_{L^\infty} \ge 1-\delta_0 > 3/4,\\ &\beta_0\|\sqrt{U'}f_k\|_{L^2}^2 \ge \beta_0\cdot(3/4)\|f_k\|_{L^2}^2 \ge 4\gamma_0\|f_k\|_{L^2}^2,\end{aligned}$$

which implies $I_0 \le 0$. As $\gamma_0 = 1/900$, $\alpha_0 = 16\gamma_0$, we have $\alpha_0 < 1$, which implies $I_1 \ge 0$.

As $\gamma_0 = 1/900 = 1/30^2$, $\alpha_0 = 16\gamma_0$, $\beta_0 = 6\gamma_0$, we have $3\alpha_0 + 2\beta_0 = 60\gamma_0$ and

$$\begin{aligned}I_2 &= \nu|k|^2t^2\gamma_0\|f_k\|_{L^2}^2 - (3\alpha_0+2\beta_0)\nu t|k|\,\|f_k\|_{L^2}\|\partial_y f_k\|_{L^2} + \nu\|(\partial_y,k)f_k\|_{L^2}^2\\ &= \nu\gamma_0(|k|t\|f_k\|_{L^2} - 30\|(\partial_y,k)f_k\|_{L^2})^2 \ge 0,\\ I_3 &= 2\alpha_0\nu^2t\|\Delta_k f_k\|_{L^2}^2 - 2\beta_0\nu^2t^2|k|\,\|\partial_y f_k\|_{L^2}\|\Delta_k f_k\|_{L^2} + 2\gamma_0\nu^2|k|^2t^3\|(\partial_y,k)f_k\|_{L^2}^2\\ &= 14\gamma_0\nu^2t\|\Delta_k f_k\|_{L^2}^2 + 2\gamma_0\nu^2t(3\|\Delta_k f_k\|_{L^2} - |k|t\|\partial_y f_k\|_{L^2})^2 + 2\gamma_0\nu^2|k|^4t^3\|f_k\|_{L^2}^2\\ &\ge 0.\end{aligned}$$

Summing up, we conclude that $\Phi_k'(t) \le I_0 - I_1 - I_2 - I_3 \le 0$. ∎

Now we prove Proposition 4.1.

Proof of Proposition 4.1. We write

$$f(t,x,y) = \sum_{k\in\mathbb{Z}} f_k(t,y)e^{ikx}.$$

Then we have

$$\|\partial_x f(t)\|^2_{L^2(\Omega)} = 2\pi \sum_{k\in\mathbb{Z}} \|ikf_k(t)\|^2_{L^2(I)}, \quad \|P_{\neq} f(t)\|^2_{L^2(\Omega)} = 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} \|f_k(t)\|^2_{L^2(I)}.$$

Thus, it is enough to prove that for every $k \in \mathbb{Z}\setminus\{0\}$, $t > 0$,

$$\begin{aligned} &|k|^2\|f_k(t)\|^2_{L^2} \leq C(\nu t)^{-1}(1+t)^{-2}\|f_k(0)\|^2_{L^2},\\ &\|f_k(t)\|^2_{L^2} \leq C(1+\nu t^3)^{-1}\|f_k(0)\|^2_{L^2}. \end{aligned} \tag{4.3}$$

By Lemma 4.3, we have

$$\begin{aligned} \|f_k(0)\|^2_{L^2} &= \Phi_k(0) \geq \Phi_k(t)\\ &\geq (1+\gamma_0\nu|k|^2t^3/4)\|f_k(t)\|^2_{L^2} + \alpha_0\nu t\|(\partial_y, k)f_k(t)\|^2_{L^2}/4\\ &\geq (1+\gamma_0\nu|k|^2t^3/4 + \alpha_0\nu|k|^2t/4)\|f_k(t)\|^2_{L^2}\\ &\geq \nu t(\gamma_0 t^2+\alpha_0)|k|^2\|f_k(t)\|^2_{L^2}/4, \end{aligned}$$

which gives the first inequality of (4.3). We also have (as $k \in \mathbb{Z}\setminus\{0\}$)

$$\|f_k(0)\|^2_{L^2} \geq (1+\gamma_0\nu|k|^2t^3/4)\|f_k(t)\|^2_{L^2} \geq (1+\gamma_0\nu t^3/4)\|f_k(t)\|^2_{L^2},$$

which gives the second inequality of (4.3). ■

4.2. Linear inviscid damping

Lemma 4.4. *There exists $\delta_0 \in (0, 1/4)$ such that if $U(t,y)$ satisfies (4.1) and f_k solves*

$$\partial_t f_k - \nu\Delta_k f_k + ikUf_k = \nu\Delta_k F_k, \quad \Delta_k\phi_k = f_k, \ \phi_k(t,\pm 1) = f_k(t,\pm 1) = 0,$$

where $k \in \mathbb{Z}$, then we have

$$\begin{aligned} &\|f_k(t)\|^2_{L^2} + \nu\int_0^t \|(\partial_y, k)f_k(s)\|^2_{L^2}\,ds + \int_0^t |k|^2\|(\partial_y,k)\phi_k(s)\|^2_{L^2}\,ds\\ &\quad\leq C\|f_k(0)\|^2_{L^2} + C\nu\int_0^t \|(\partial_y,k)F_k(s)\|^2_{L^2}\,ds. \end{aligned}$$

To prove Lemma 4.4, we need to use the operator $\mathfrak{J}_k$ on $L^2(I)$ for $k \in \mathbb{Z}\setminus\{0\}$ constructed in [7]. The operator $\mathfrak{J}_k$ is defined as

$$\mathfrak{J}_k[f](y) = |k|\text{p.v.}\frac{k}{|k|}\int_{-1}^1 \frac{1}{2i(y-y')}G_k(y,y')f(y')\,dy'.$$

Here, G_k is the Green's function for $\Delta_k := \partial_y^2 - |k|^2$ such that

$$G_k(y.y') = -\frac{1}{k\sinh 2k}\begin{cases} \sinh(k(1-y'))\sinh(k(1+y)), & y \leq y',\\ \sinh(k(1-y))\sinh(k(1+y')), & y \geq y'. \end{cases} \tag{4.4}$$

Then $\Delta_k^{-1}f(y) = \int_{-1}^1 G_k(y.y')f(y')\,dy'$. As $G_k|_{y=\pm 1} = 0$, we have $\mathfrak{J}_k[f]|_{y=\pm 1} = 0$.

The operator $\mathfrak{J}_k$ is primarily used to derive the inviscid damping estimate. Specifically, by Lemma B.2, we have

$$\mathrm{Re}\langle -ikUf_k, \mathfrak{J}_k f_k\rangle \le -\frac{|k|^2}{8}\|(\partial_y, k)\phi_k\|_{L^2}^2.$$

See the appendix for more properties of $\mathfrak{J}_k$.

Proof of Lemma 4.4. As $f_k|_{y=\pm 1} = 0$, by the energy estimate, we have

$$\begin{aligned}\frac{d}{dt}\|f_k\|_{L^2}^2 &= 2\,\mathrm{Re}\langle -ikUf_k + \nu\Delta_k f_k + \nu\Delta_k F_k, f_k\rangle = 2\nu\,\mathrm{Re}\langle \Delta_k f_k + \Delta_k F_k, f_k\rangle \\ &= -2\nu\|(\partial_y, k)f_k\|_{L^2}^2 - 2\nu\,\mathrm{Re}\langle \partial_y F_k, \partial_y f_k\rangle - 2\nu|k|^2\,\mathrm{Re}\langle F_k, f_k\rangle \\ &\le -2\nu\|(\partial_y, k)f_k\|_{L^2}^2 + 2\nu\|(\partial_y, k)f_k\|_{L^2}\|(\partial_y, k)F_k\|_{L^2} \\ &\le -\nu\|(\partial_y, k)f_k\|_{L^2}^2 + \nu\|(\partial_y, k)F_k\|_{L^2}^2,\end{aligned}$$

which implies

$$\begin{aligned}&\|f_k(t)\|_{L^2}^2 + \nu\int_0^t \|(\partial_y, k)f_k(s)\|_{L^2}^2\,ds \\ &\quad\le \|f_k(0)\|_{L^2}^2 + \nu\int_0^t \|(\partial_y, k)F_k(s)\|_{L^2}^2\,ds. \qquad (4.5)\end{aligned}$$

It remains to prove that

$$\int_0^t |k|^2\|(\partial_y, k)\phi_k(s)\|_{L^2}^2\,ds \le C\|f_k(0)\|_{L^2}^2 + C\nu\int_0^t \|(\partial_y, k)F_k(s)\|_{L^2}^2\,ds. \qquad (4.6)$$

For $k = 0$, (4.6) is clearly true. Now we assume $k \in \mathbb{Z}\setminus\{0\}$. Since $\mathfrak{J}_k$ is symmetric, we have

$$\frac{1}{2}\frac{d}{dt}\mathrm{Re}\langle f_k, \mathfrak{J}_k[f_k]\rangle = \mathrm{Re}\langle -ikUf_k + \nu\Delta_k f_k + \nu\Delta_k F_k, \mathfrak{J}_k[f_k]\rangle = T_1 + T_2 + T_3.$$

By Lemma B.2, we have

$$T_1 \le -\frac{|k|^2}{8}\|(\partial_y, k)\phi_k\|_{L^2}^2.$$

As $\mathfrak{J}_k[f_k]|_{y=\pm 1} = 0$, we get by (B.1) that

$$\begin{aligned}T_2 &= \nu\,\mathrm{Re}\langle \Delta_k f_k, \mathfrak{J}_k[f_k]\rangle = -\nu\,\mathrm{Re}\langle \partial_y f_k, \partial_y\mathfrak{J}_k[f_k]\rangle - \nu|k|^2\,\mathrm{Re}\langle f_k, \mathfrak{J}_k[f_k]\rangle \\ &\le \nu\|(\partial_y, k)f_k\|_{L^2}\|(\partial_y, k)\mathfrak{J}_k[f_k]\|_{L^2} \lesssim \nu\|(\partial_y, k)f_k\|_{L^2}^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}T_3 &\le \nu\|(\partial_y, k)F_k\|_{L^2}\|(\partial_y, k)\mathfrak{J}_k[f_k]\|_{L^2} \lesssim \nu\|(\partial_y, k)F_k\|_{L^2}\|(\partial_y, k)f_k\|_{L^2} \\ &\lesssim \nu\|(\partial_y, k)F_k\|_{L^2}^2 + \nu\|(\partial_y, k)f_k\|_{L^2}^2.\end{aligned}$$

Summing up, we obtain

$$\frac{1}{2}\frac{d}{dt}\mathrm{Re}\langle f_k,\mathfrak{J}_k[f_k]\rangle \le -\frac{|k|^2}{8}\|(\partial_y,k)\phi_k\|_{L^2}^2 + C\nu\|(\partial_y,k)f_k\|_{L^2}^2 + C\nu\|(\partial_y,k)F_k\|_{L^2}^2,$$

which implies

$$\begin{aligned}\int_0^t |k|^2\|(\partial_y,k)\phi_k(s)\|_{L^2}^2\,ds \le 4|\langle f_k(0),\mathfrak{J}_k[f_k(0)]\rangle| + 4|\langle f_k(t),\mathfrak{J}_k[f_k(t)]\rangle|\\ + C\nu\int_0^t \|(\partial_y,k)f_k(s)\|_{L^2}^2\,ds\\ + C\nu\int_0^t \|(\partial_y,k)F_k(s)\|_{L^2}^2\,ds.\end{aligned}$$

By Lemma B.1, we have

$$\begin{aligned}&|\langle f_k(t),\mathfrak{J}_k[f_k(t)]\rangle| \le \|f_k(t)\|_{L^2}\|\mathfrak{J}_k[f_k(t)]\|_{L^2} \lesssim \|f_k(t)\|_{L^2}^2,\\ &|\langle f_k(0),\mathfrak{J}_k[f_k(0)]\rangle| \lesssim \|f_k(0)\|_{L^2}^2.\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}\int_0^t |k|^2\|(\partial_y,k)\phi_k(s)\|_{L^2}^2\,ds \le C\|f_k(0)\|_{L^2}^2 + C\|f_k(t)\|_{L^2}^2\\ + C\nu\int_0^t \|(\partial_y,k)f_k(s)\|_{L^2}^2\,ds\\ + C\nu\int_0^t \|(\partial_y,k)F_k(s)\|_{L^2}^2\,ds,\end{aligned}$$

which along with (4.5) gives (4.6). ■

Summing up all the Fourier modes in Lemma 4.4, we have the following corollary.

Corollary 4.5. *There exists* $\delta_0 \in (0,1/4)$ *such that if* $U(t,y)$ *satisfies* (4.1) *and* f *solves*

$$\partial_t f - \nu\Delta f + U\partial_x f = \nu\Delta F, \quad f(t,x,\pm 1) = 0,\ \forall t \ge 0,$$

then we have

$$\begin{aligned}&\|f(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla f(s)\|_{L^2}^2\,ds + \int_0^t \|\partial_x\nabla\Delta^{-1}f(s)\|_{L^2}^2\,ds\\ &\le C\|f(0)\|_{L^2}^2 + C\nu\int_0^t \|\nabla F(s)\|_{L^2}^2\,ds.\end{aligned}$$

Restricting to the case $f = F = 0$ for $t \in [0,T_0]$, we have the following corollary.

Corollary 4.6. *There exists* $\delta_0 \in (0,1/4)$ *such that if* $U(t,y)$ *satisfies* (4.1) *and* f *solves*

$$\partial_t f - \nu\Delta f + U\partial_x f = \nu\Delta F, \quad f(t,x,\pm 1) = 0,\ \forall t \ge T_0,\quad f|_{t=T_0} = 0,$$

then we have

$$\|f\|_Y \lesssim \nu^{1/2}\|\nabla F\|_{L^2L^2} \le \|F\|_Y,$$

where

$$\|f\|_Y = \|f\|_{L^\infty L^2} + \nu^{1/2}\|\nabla f\|_{L^2 L^2} + \|\partial_x \nabla \Delta^{-1} f\|_{L^2 L^2}. \tag{4.7}$$

Corollary 4.6 will be used in the proof of Proposition 2.5.

4.3. Proof of Proposition 2.4

Let $S(t,s) = S(t,s,U)$ be the solution operator of (4.2). That is, $f(t,x,y) = S(t,s)g(x,y)$ solves (4.2) for $t \geq s$ with $f(s,x,y) = g(x,y)$. We infer from Corollary 4.5 (for $F = 0$) and Proposition 4.1 that

$$\begin{aligned}
&\|S(t,0)g\|_{L^2} \leq C\|g\|_{L^2}, \quad \forall t \geq 0, \quad \nu \int_0^\infty \|\nabla S(t,0)g\|_{L^2}^2\, dt \leq C\|g\|_{L^2}^2, \\
&\int_0^\infty \|\partial_x \nabla \Delta^{-1} S(t,0)g\|_{L^2}^2\, dt \leq C\|g\|_{L^2}^2, \\
&\int_0^\infty \|\partial_x S(t,0)g\|_{L^2}\, dt \leq C\int_0^\infty (\nu t)^{-1/2}(1+t)^{-1}\|g\|_{L^2}\, dt \leq C\nu^{-1/2}\|g\|_{L^2}, \\
&\int_0^\infty \|P_{\neq} S(t,0)g\|_{L^2}\, dt \leq C\int_0^\infty (1+\nu t^3)^{-1}\|g\|_{L^2}\, dt \leq C\nu^{-1/3}\|g\|_{L^2}.
\end{aligned}$$

Notice that $S(t,s,U) = S(t-s,0,\tau_s U)$ with $\tau_s U(t,y) = U(t+s,y)$, and $\tau_s U(t,y)$ also satisfies (4.1). Thus, there also holds for $t \geq s \geq 0$,

$$\|S(t,s)g\|_{L^2} \leq C\|g\|_{L^2}, \quad \nu \int_s^\infty \|\nabla S(t,s)g\|_{L^2}^2\, dt \leq C\|g\|_{L^2}^2, \tag{4.8}$$

$$\int_s^\infty \|\partial_x \nabla \Delta^{-1} S(t,s)g\|_{L^2}^2\, dt \leq C\|g\|_{L^2}^2, \tag{4.9}$$

$$\begin{aligned}
&\int_s^\infty \|\partial_x S(t,s)g\|_{L^2}\, dt \leq C\nu^{-1/2}\|g\|_{L^2}, \\
&\int_s^\infty \|P_{\neq} S(t,s)g\|_{L^2}\, dt \leq C\nu^{-1/3}\|g\|_{L^2}.
\end{aligned} \tag{4.10}$$

Let f solve $\partial_t f - \nu\Delta f + U\partial_x f = g$, $f(t,x,\pm 1) = 0$ for $t \in [T_0, T]$. Then we have

$$f(t) = S(t,T_0)f(T_0) + \int_{T_0}^t S(t,s)g(s)\, ds.$$

By using (4.8)–(4.10) and the Minkowski inequality, we deduce that for $t \in [T_0, T]$,

$$\begin{aligned}
&\|f(t)\|_{L^2} + \nu^{1/2}\left(\int_{T_0}^T \|\nabla f(t)\|_{L^2}^2\, dt\right)^{1/2} + \left(\int_{T_0}^T \|\partial_x \nabla \Delta^{-1} f(t)\|_{L^2}^2\, dt\right)^{1/2} \\
&\quad + \nu^{1/2}\int_{T_0}^T \|\partial_x f(t)\|_{L^2}\, dt + \nu^{1/3}\int_{T_0}^T \|P_{\neq} f(t)\|_{L^2}\, dt \\
&\leq C\|f(T_0)\|_{L^2} + C\int_{T_0}^T \|g(t)\|_{L^2}\, dt.
\end{aligned}$$

We introduce the following norm:

$$\|f\|_{\widetilde{X}} = \|f\|_{L^\infty L^2} + \nu^{1/2}\|\nabla f\|_{L^2L^2} + \|\partial_x \nabla \Delta^{-1} f\|_{L^2L^2} + \nu^{1/2}\|\partial_x f\|_{L^1L^2} + \nu^{1/3}\|P_{\neq} f\|_{L^1L^2}.$$

Then the result becomes

$$\|f\|_{\widetilde{X}} \le C\|f(T_0)\|_{L^2} + C\|g\|_{L^1L^2}. \tag{4.11}$$

Recall that

$$\|f\|_X = \|f\|_{L^\infty L^2} + \nu^{1/2}\|\nabla f\|_{L^2L^2} + \|e^{\epsilon\nu^{1/3}t} P_{\neq} f\|_{L^\infty L^2} + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\nabla P_{\neq} f\|_{L^2L^2} + \|e^{\epsilon\nu^{1/3}t}\partial_x \nabla \Delta^{-1} f\|_{L^2L^2} + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f\|_{L^1L^2} + \nu^{1/3}\|e^{\epsilon\nu^{1/3}t} P_{\neq} f\|_{L^1L^2}.$$

Then we have (as $\partial_x f = \partial_x P_{\neq} f$)

$$\|f\|_X \le \|f\|_{\widetilde{X}} + \|e^{\epsilon\nu^{1/3}t} P_{\neq} f\|_{\widetilde{X}}. \tag{4.12}$$

Now we can complete the proof of Proposition 2.4 by using (4.11) and (4.12). Since $\partial_t f - \nu\Delta f + U\partial_x f = g$, $f(t,x,\pm 1) = 0$, and U is independent of x we have

$$\partial_t P_{\neq} f - \nu\Delta P_{\neq} f + U\partial_x P_{\neq} f = P_{\neq} g, \quad P_{\neq} f(t,x,\pm 1) = 0.$$

Let $\tilde{f} = e^{\epsilon\nu^{1/3}t} P_{\neq} f$. Then

$$\partial_t \tilde{f} - \nu\Delta \tilde{f} + U\partial_x \tilde{f} = e^{\epsilon\nu^{1/3}t} P_{\neq} g + \epsilon\nu^{1/3}\tilde{f} =: \tilde{g}, \quad \tilde{f}(t,x,\pm 1) = 0.$$

Thus, (4.11) holds with (f, g) replaced by $(\tilde{f}, \tilde{g})$, i.e.,

$$\begin{aligned}\|\tilde{f}\|_{\widetilde{X}} &\le C\|\tilde{f}(T_0)\|_{L^2} + C\|e^{\epsilon\nu^{1/3}t} P_{\neq} g + \epsilon\nu^{1/3}\tilde{f}\|_{L^1L^2} \\ &\le Ce^{\epsilon\nu^{1/3}T_0}\|f(T_0)\|_{L^2} + C\|e^{\epsilon\nu^{1/3}t} P_{\neq} g\|_{L^1L^2} + C\epsilon\nu^{1/3}\|\tilde{f}\|_{L^1L^2} \\ &\le C\|f(T_0)\|_{L^2} + C\|e^{\epsilon\nu^{1/3}t} g\|_{L^1L^2} + C\epsilon\|\tilde{f}\|_{\widetilde{X}}.\end{aligned}$$

Here we used $T_0 = \nu^{-1/6}$, $\epsilon\nu^{1/3}T_0 = \epsilon\nu^{1/6} \le 1$. Taking $\epsilon \in (0, 1/4)$ so that $C\epsilon \le 1/2$, then

$$\|\tilde{f}\|_{\widetilde{X}} \le C\|f(T_0)\|_{L^2} + C\|e^{\epsilon\nu^{1/3}t} g\|_{L^1L^2}. \tag{4.13}$$

Finally, we conclude by (4.11)–(4.13) that

$$\begin{aligned}\|f\|_X &\le \|f\|_{\widetilde{X}} + \|e^{\epsilon\nu^{1/3}t} P_{\neq} f\|_{\widetilde{X}} = \|f\|_{\widetilde{X}} + \|\tilde{f}\|_{\widetilde{X}} \\ &\le C\|f(T_0)\|_{L^2} + C\|g\|_{L^1L^2} + C\|e^{\epsilon\nu^{1/3}t} g\|_{L^1L^2} \\ &\le C\|f(T_0)\|_{L^2} + C\|e^{\epsilon\nu^{1/3}t} g\|_{L^1L^2}.\end{aligned}$$

This completes the proof of Proposition 2.4.

5. Control of the reaction part

In this section we prove Proposition 2.5. Recall that

$$\partial_t \omega_{\mathrm{re}} - \nu\Delta\omega_{\mathrm{re}} + U\partial_x\omega_{\mathrm{re}} - tu_e^2\partial_x\omega_L = 0, \quad \omega_{\mathrm{re}}(t,x,\pm 1) = 0,\ \omega_{\mathrm{re}}|_{t=T_0} = 0. \quad (5.1)$$

Let f_1 solve

$$\partial_t f_1 + y\partial_x f_1 = te^{\nu^{1/3}t}u_e^2\partial_x\omega_L, \quad f_1|_{t=T_0} = 0.$$

As $\omega_L|_{y=\pm 1} = 0$, we have $f_1|_{y=\pm 1} = 0$. The motivation for f_1 is to extract the inviscid contribution free of y-derivatives, enabling it to be solved y-by-y. In particular, it requires no boundary conditions and can be solved via Fourier analysis as if it were defined on $\mathbb{T}\times\mathbb{R}$, as carried out in Lemma 5.6. To recover the y-derivatives in $\nu\Delta$, let f_2 solve

$$\partial_t f_2 - \nu\Delta f_2 + U\partial_x f_2 = \nu e^{-\nu^{1/3}t/2}\Delta f_1, \quad f_2(t,x,\pm 1) = 0,\ f_2|_{t=T_0} = 0.$$

By Corollary 4.6, we have

$$\|f_2\|_Y \lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y. \quad (5.2)$$

Let $f_a = e^{-\nu^{1/3}t/2} f_2 + e^{-\nu^{1/3}t} f_1$. Here, we employ the weight functions $e^{-\nu^{1/3}t/2}$, $e^{-\nu^{1/3}t}$ to derive the enhanced dissipation. Then $f_a|_{y=\pm 1} = 0$, $f_a|_{t=T_0} = 0$, and

$$\begin{aligned}&\partial_t f_a - \nu\Delta f_a + U\partial_x f_a + \nu^{1/3}e^{-\nu^{1/3}t/2} f_2/2 + \nu^{1/3}e^{-\nu^{1/3}t} f_1\\ &\quad = tu_e^2\partial_x\omega_L + e^{-\nu^{1/3}t}(U-y)\partial_x f_1.\end{aligned}$$

Thus, f_a is a good approximate solution and the error term $f_e = \omega_{\mathrm{re}} - f_a$ solves

$$\begin{cases}\partial_t f_e - \nu\Delta f_e + U\partial_x f_e = \nu^{1/3}e^{-\nu^{1/3}t/2} f_2/2 + \nu^{1/3}e^{-\nu^{1/3}t} f_1 - e^{-\nu^{1/3}t}(U-y)\partial_x f_1,\\ f_e(t,x,\pm 1) = 0, \quad f_e|_{t=T_0} = 0.\end{cases}$$

We need the following bound for the X norm.

Lemma 5.1. *For $\epsilon \in (0, 1/4)$ and Y defined in (4.7), we have*

$$\|f\|_X \lesssim \|e^{\nu^{1/3}t/2} f\|_Y + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f\|_{L^1L^2}.$$

Proof. Recall that

$$\begin{aligned}\|f\|_X &= \|f\|_{L^\infty L^2} + \nu^{1/2}\|\nabla f\|_{L^2L^2} + \|e^{\epsilon\nu^{1/3}t}P_{\neq} f\|_{L^\infty L^2}\\ &\quad + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\nabla P_{\neq} f\|_{L^2L^2} + \|e^{\epsilon\nu^{1/3}t}\partial_x\nabla\Delta^{-1} f\|_{L^2L^2}\\ &\quad + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f\|_{L^1L^2} + \nu^{1/3}\|e^{\epsilon\nu^{1/3}t}P_{\neq} f\|_{L^1L^2},\\ \|f\|_Y &= \|f\|_{L^\infty L^2} + \nu^{1/2}\|\nabla f\|_{L^2L^2} + \|\partial_x\nabla\Delta^{-1} f\|_{L^2L^2}.\end{aligned}$$

Then we have

$$\|f\|_X \le 2\|e^{\epsilon\nu^{1/3}t} f\|_Y + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f\|_{L^1L^2} + \nu^{1/3}\|e^{\epsilon\nu^{1/3}t} P_{\neq} f\|_{L^1L^2}.$$

As $\epsilon \in (0, 1/4)$, we have $\|e^{\epsilon\nu^{1/3}t} f\|_Y \le \|e^{\nu^{1/3}t/2} f\|_Y$ and

$$\begin{aligned}\nu^{1/3}\|e^{\epsilon\nu^{1/3}t} P_{\neq} f\|_{L^1L^2} &\le \nu^{1/3}\|e^{-\nu^{1/3}t/4}\|_{L^1(T_0,T)}\|e^{\nu^{1/3}t/2} f\|_{L^\infty L^2}\\ &\le 4\|e^{\nu^{1/3}t/2} f\|_{L^\infty L^2} \le 4\|e^{\nu^{1/3}t/2} f\|_Y.\end{aligned}$$

This completes the proof. ∎

5.1. Proof of Proposition 2.5

Let us first estimate f_e.

Lemma 5.2. *For $\epsilon \in (0, 1/4)$ as in Proposition 2.4 and Y defined in (4.7), we have*

$$\|f_e\|_X \lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y + \|\partial_x f_1\|_{L^\infty L^2}.$$

Proof. As $f_e(T_0) = 0$, we get by Proposition 2.4 that

$$\begin{aligned}\|f_e\|_X \lesssim\ & \nu^{1/3}\|e^{(\epsilon-1/2)\nu^{1/3}t} f_2\|_{L^1L^2} + \nu^{1/3}\|e^{(\epsilon-1)\nu^{1/3}t} f_1\|_{L^1L^2}\\ &+ \|e^{(\epsilon-1)\nu^{1/3}t}(U-y)\partial_x f_1\|_{L^1L^2}.\end{aligned}$$

As $\epsilon \in (0, 1/4)$, by (5.2) we have

$$\begin{aligned}\nu^{1/3}\|e^{(\epsilon-1/2)\nu^{1/3}t} f_2\|_{L^1L^2} &\le \nu^{1/3}\|e^{-\nu^{1/3}t/4}\|_{L^1(T_0,T)}\|f_2\|_{L^\infty L^2}\\ &\le 4\|f_2\|_{L^\infty L^2} \le 4\|f_2\|_Y \lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y,\\ \nu^{1/3}\|e^{(\epsilon-1)\nu^{1/3}t} f_1\|_{L^1L^2} &\le \nu^{1/3}\|e^{-\nu^{1/3}t/4}\|_{L^1(T_0,T)}\|e^{-\nu^{1/3}t/2} f_1\|_{L^\infty L^2}\\ &\le 4\|e^{-\nu^{1/3}t/2} f_1\|_{L^\infty L^2} \le 4\|e^{-\nu^{1/3}t/2} f_1\|_Y.\end{aligned}$$

As $0 < \epsilon < 1/4 < 1/2$, $\|\omega^{(0)}\|_{H^3} \le \epsilon_1\nu^{1/3} \le \nu^{1/3}$, by (2.1) we have

$$\begin{aligned}\|e^{(\epsilon-1)\nu^{1/3}t}(U-y)\partial_x f_1\|_{L^1L^2} &\le \|U-y\|_{L^\infty L^\infty}\|e^{-\nu^{1/3}t/2}\partial_x f_1\|_{L^1L^2}\\ &\lesssim \|\omega^{(0)}\|_{H^3}\|e^{-\nu^{1/3}t/4}\|_{L^1(T_0,T)}\|\partial_x f_1\|_{L^\infty L^2}\\ &\lesssim \|\partial_x f_1\|_{L^\infty L^2}.\end{aligned}$$

This completes the proof. ∎

Now we can give a bound for $\|\omega_{\mathrm{re}}\|_X$.

Lemma 5.3. *For $\epsilon \in (0, 1/4)$ as in Proposition 2.4 and Y defined in (4.7), we have*

$$\|\omega_{\mathrm{re}}\|_X \lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y + \|\partial_x f_1\|_{L^\infty L^2} + \nu^{1/6}\|\partial_x\omega_{\mathrm{re}}\|_X.$$

Proof. By Lemma 5.1 with $f = f_a$, we get

$$\|f_a\|_X \lesssim \|e^{\nu^{1/3}t/2} f_a\|_Y + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f_a\|_{L^1L^2}.$$

As $f_a = e^{-\nu^{1/3}t/2} f_2 + e^{-\nu^{1/3}t} f_1$, by (5.2) we have

$$\|e^{\nu^{1/3}t/2} f_a\|_Y \leq \|f_2\|_Y + \|e^{-\nu^{1/3}t/2} f_1\|_Y \lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y.$$

As $f_e = \omega_{\rm re} - f_a$ and $\partial_x\omega_{\rm re} = P_{\neq}\partial_x\omega_{\rm re}$, we have

$$\begin{aligned}\nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f_a\|_{L^1L^2} &\leq \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x f_e\|_{L^1L^2} + \nu^{1/2}\|e^{\epsilon\nu^{1/3}t}\partial_x\omega_{\rm re}\|_{L^1L^2}\\ &\leq \|f_e\|_X + \nu^{1/6}\|\partial_x\omega_{\rm re}\|_X.\end{aligned}$$

Thus, we obtain

$$\|f_a\|_X \lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y + \|f_e\|_X + \nu^{1/6}\|\partial_x\omega_{\rm re}\|_X.$$

As $f_e = \omega_{\rm re} - f_a$, we get by Lemma 5.2 that

$$\begin{aligned}\|\omega_{\rm re}\|_X \leq \|f_a\|_X + \|f_e\|_X &\lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y + \|f_e\|_X + \nu^{1/6}\|\partial_x\omega_{\rm re}\|_X\\ &\lesssim \|e^{-\nu^{1/3}t/2} f_1\|_Y + \|\partial_x f_1\|_{L^\infty L^2} + \nu^{1/6}\|\partial_x\omega_{\rm re}\|_X.\end{aligned}$$

This completes the proof. ∎

With the hand of Lemma 5.3, Proposition 2.5 is a direct consequence of the following Lemmas 5.4 and 5.5.

Lemma 5.4. *For* $\epsilon \in (0, 1/4)$ *as in Proposition 2.4, we have*

$$\|\partial_x\omega_{\rm re}\|_X \lesssim \nu^{-1/2}\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}\|\omega^{(0)}\|_{H^3}.$$

Proof. Applying ∂_x to (5.1), we get

$$\partial_t\partial_x\omega_{\rm re} - \nu\Delta\partial_x\omega_{\rm re} + U\partial_x^2\omega_{\rm re} - t\partial_x(u_e^2\partial_x\omega_L) = 0, \quad \partial_x\omega_{\rm re}|_{y=\pm1} = 0,\ \partial_x\omega_{\rm re}|_{t=T_0} = 0.$$

Then by Proposition 2.4, we have

$$\|\partial_x\omega_{\rm re}\|_X \lesssim \|e^{\epsilon\nu^{1/3}t} t\partial_x(u_e^2\partial_x\omega_L)\|_{L^1L^2}.$$

By Lemma 3.3, we have

$$\begin{aligned}\|t\partial_x(u_e^2\partial_x\omega_L)\|_{L^2} &\leq t\|\partial_x^2\omega_L(t)\|_{L^\infty}\|u_e^2\|_{L^2} + t\|\partial_x\omega_L(t)\|_{L^\infty}\|\partial_x u_e^2\|_{L^2}\\ &\lesssim \nu^{-1/3}e^{-\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}(\|u_e^2\|_{L^2} + \|\partial_x u_e^2\|_{L^2})\\ &\lesssim \nu^{-1/3}e^{-\nu^{1/3}t}\|\omega^{(0)}\|_{H^3}\|\nabla u_e^2\|_{L^2}.\end{aligned}$$

Here we used $u_e^2 = -\partial_x \phi_e$ to deduce that $P_0 u_e^2 = 0$, $\|u_e^2\|_{L^2} \le \|\partial_x u_e^2\|_{L^2} \le \|\nabla u_e^2\|_{L^2}$. Thus, we obtain

$$\begin{aligned}\|\partial_x \omega_{\mathrm{re}}\|_X &\lesssim \|e^{\epsilon\nu^{1/3}t} t\partial_x(u_e^2\partial_x\omega_L)\|_{L^1L^2} \lesssim \nu^{-1/3}\|\omega^{(0)}\|_{H^3}\|e^{(\epsilon-1)\nu^{1/3}t}\nabla u_e^2\|_{L^1L^2}\\ &\le \nu^{-1/3}\|\omega^{(0)}\|_{H^3}\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}\|e^{-\nu^{1/3}t}\|_{L^2(T_0,T)}\\ &\le \nu^{-1/2}\|\omega^{(0)}\|_{H^3}\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}.\end{aligned}$$

This completes the proof. ∎

Lemma 5.5. *For Y defined in (4.7), we have*

$$\|e^{-\nu^{1/3}t/2} f_1\|_Y + \|\partial_x f_1\|_{L^\infty L^2} \lesssim \nu^{-1/3}\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}\|\omega^{(0)}\|_{H^3}.$$

5.2. Proof of Lemma 5.5

We denote $\|f\|_{L^qL^p} = \|f\|_{L^q(T_0,T;L^p(D))}$ for $D = \Omega$ or $D = I$, which is easy to distinguish from the context. We need the following result.

Lemma 5.6. *Let $\omega(t,y)$ solve $\partial_t\omega + iky\omega + f = 0$ for $t \in [T_0,T]$, and $y \in I = [-1,1]$ and $\psi = -\Delta_k^{-1}\omega$, $\omega(T_0) = 0$. Assume that $k,l \in \mathbb{Z}$, $k \ne l$, $f \in L^2(T_0,T;H_0^1(-1,1))$. Then*

$$\begin{aligned}&\|\omega\|_{L^\infty L^2}^2 + |k|^2\|(\partial_y,k)\psi\|_{L^2L^2}^2 \le C|k-l|^{-2}\|(\partial_y,k-l)(e^{ilyt}f)\|_{L^2L^2}^2,\\ &\nu\|e^{-\nu^{1/3}t/2}(\partial_y,k)\omega\|_{L^2L^2}^2 \le C(1+|k|^2|k-l|^{-2})\|(\partial_y,k-l)(e^{ilyt}f)\|_{L^2L^2}^2.\end{aligned}$$

Proof. Step 1. Estimate of ω. We introduce the Fourier transform as

$$\begin{aligned}\widetilde{\omega}(t,\eta) &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\omega(t,y)e^{-iy\eta}\,dy,\\ \tilde{f}(t,\eta) &= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} f(t,y)e^{-iy\eta}\,dy.\end{aligned}$$

By Plancherel's formula, we have $\|\widetilde{\omega}(t)\|_{L^2(\mathbb{R})} = \|\omega(t)\|_{L^2(I)}$ and

$$\begin{aligned}\|(\partial_y,k-l)(e^{ilyt}f)\|_{L^2L^2}^2 &= \int_{T_0}^{T}\int_{\mathbb{R}}(|\eta+lt|^2+|k-l|^2)|\tilde{f}(t,\eta)|^2\,d\eta\,dt \qquad (5.3)\\ &= \int_{T_0}^{T}\int_{\mathbb{R}}(|\eta-kt+lt|^2+|k-l|^2)|\tilde{f}(t,\eta-kt)|^2\,d\eta\,dt.\end{aligned}$$

Notice that $\partial_t(e^{ikyt}\omega) + e^{ikyt}f = 0$ and $\omega(T_0) = 0$. Taking the Fourier transform, we get

$$\frac{d}{dt}\widetilde{\omega}(t,\eta-kt) = -\tilde{f}(t,\eta-kt), \quad \widetilde{\omega}(T_0,\eta-kT_0) = 0.$$

For fixed $\eta \in \mathbb{R}$, let $W(\eta) = \sup_{t\in[T_0,T]} |\widetilde{\omega}(t, \eta - kt)|$. Then we have

$$\begin{aligned} W(\eta) &\leq \int_{T_0}^{T} |\tilde{f}(t, \eta - kt)|\, dt \\ &\leq \Big(\int_{T_0}^{T} (|\eta - kt + lt|^2 + |k-l|^2)|\tilde{f}(t, \eta - kt)|^2\, dt \Big)^{1/2} \\ &\quad \times \Big(\int_{\mathbb{R}} \frac{dt}{|\eta - kt + lt|^2 + |k-l|^2} \Big)^{1/2}. \end{aligned}$$

Notice that ($s = t - \eta/(k-l)$)

$$\int_{\mathbb{R}} \frac{dt}{|\eta - kt + lt|^2 + |k-l|^2} = \frac{1}{|k-l|^2} \int_{\mathbb{R}} \frac{ds}{|s|^2 + 1} = \frac{\pi}{|k-l|^2},$$

and we have

$$|W(\eta)|^2 \leq \frac{\pi}{|k-l|^2} \int_{T_0}^{T} (|\eta - kt + lt|^2 + |k-l|^2)|\tilde{f}(t, \eta - kt)|^2\, dt,$$

which along with (5.3) implies

$$\int_{\mathbb{R}} |W(\eta)|^2\, d\eta \leq \frac{\pi}{|k-l|^2} \|(\partial_y, k-l)(e^{ilyt} f)\|_{L^2L^2}^2. \tag{5.4}$$

This gives an estimate of $\|\omega\|_{L^\infty L^2}$ by noting that for every $t \in [T_0, T]$,

$$\begin{aligned} \|\omega(t)\|_{L^2(I)}^2 = \|\widetilde{\omega}(t)\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} |\widetilde{\omega}(t, \eta)|^2\, d\eta \\ &= \int_{\mathbb{R}} |\widetilde{\omega}(t, \eta - kt)|^2\, d\eta \leq \int_{\mathbb{R}} |W(\eta)|^2\, d\eta. \end{aligned} \tag{5.5}$$

Step 2. Estimate of ψ. We first assume $k \neq 0$. Let

$$\psi_*(t, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\widetilde{\omega}(t, \eta)}{\eta^2 + k^2} e^{iy\eta}\, d\eta.$$

Then $\psi_*(t) \in H^2(\mathbb{R})$ satisfies $-\Delta_k \psi_* = \omega$ for $y \in [-1, 1]$. Thus,

$$\begin{aligned} \|(\partial_y, k)\psi(t)\|_{L^2}^2 &= \langle \psi(t), \omega(t)\rangle = \langle \psi(t), -\Delta_k \psi_*(t)\rangle \\ &= \langle \partial_y \psi(t), \partial_y \psi_*(t)\rangle + |k|^2 \langle \psi(t), \psi_*(t)\rangle \\ &\leq \|(\partial_y, k)\psi(t)\|_{L^2} \|(\partial_y, k)\psi_*(t)\|_{L^2(\mathbb{R})}, \end{aligned}$$

and (here $\|(\partial_y, k)\psi_*\|_{L^2L^2(\mathbb{R})} = \|(\partial_y, k)\psi_*\|_{L^2(T_0,T;L^2(\mathbb{R}))}$)

$$\begin{aligned} \|(\partial_y, k)\psi\|_{L^2L^2}^2 \leq \|(\partial_y, k)\psi_*\|_{L^2L^2(\mathbb{R})}^2 &= \int_{T_0}^{T} \int_{\mathbb{R}} \frac{|\widetilde{\omega}(t, \eta)|^2}{|\eta|^2 + |k|^2}\, d\eta\, dt \\ &= \int_{T_0}^{T} \int_{\mathbb{R}} \frac{|\widetilde{\omega}(t, \eta - kt)|^2}{|\eta - kt|^2 + |k|^2}\, d\eta\, dt. \end{aligned}$$

Then we have

$$
\begin{aligned}
|k|^2\|(\partial_y,k)\psi\|_{L^2L^2}^2 &\le \int_{T_0}^{T}\int_{\mathbb{R}} \frac{|k|^2|\widetilde{\omega}(t,\eta-kt)|^2}{|\eta-kt|^2+|k|^2}\,d\eta\,dt\\
&\le \int_{T_0}^{T}\int_{\mathbb{R}} \frac{|k|^2|W(\eta)|^2}{|\eta-kt|^2+|k|^2}\,d\eta\,dt\\
&\le \int_{\mathbb{R}}|W(\eta)|^2\int_{\mathbb{R}}\frac{|k|^2\,dt}{|\eta-kt|^2+|k|^2}\,d\eta=\pi\int_{\mathbb{R}}|W(\eta)|^2\,d\eta. \qquad (5.6)
\end{aligned}
$$

For $k=0$, (5.6) is clearly true. Now the first inequality follows from (5.4)–(5.6).

Step 3. Estimate of $\partial_y\omega$. As $\partial_t\omega+iky\omega+f=0$, we have

$$
\partial_t(e^{ilyt}\omega)+i(k-l)y(e^{ilyt}\omega)+e^{ilyt}f=0,
$$

and then

$$
(\partial_t+i(k-l)y)\partial_y(e^{ilyt}\omega)+i(k-l)(e^{ilyt}\omega)+\partial_y(e^{ilyt}f)=0.
$$

We also have $\omega(T_0)=0$, $\partial_y(e^{ilyt}\omega(T_0))=0$, thus, for $t\in[T_0,T]$,

$$
\begin{aligned}
\|\partial_y(e^{ilyt}\omega(t))\|_{L^2} &\le \int_{T_0}^{t}\|i(k-l)e^{ilys}\omega(s)+\partial_y(e^{ilys}f(s))\|_{L^2}\,ds\\
&\le \|k-l\|_{L^1(T_0,t)}\|\omega\|_{L^\infty L^2}+\|1\|_{L^2(T_0,t)}\|\partial_y(e^{ilyt}f)\|_{L^2L^2}\\
&\le |k-l|t\|\omega\|_{L^\infty L^2}+t^{\frac12}\|\partial_y(e^{ilyt}f)\|_{L^2L^2}.
\end{aligned}
$$

Thanks to $\partial_y\omega=e^{-ilyt}\partial_y(e^{ilyt}\omega)-ilt\omega$, we have (also for $t\in[T_0,T]$)

$$
\begin{aligned}
\|\partial_y\omega(t)\|_{L^2} &\le \|\partial_y(e^{ilyt}\omega(t))\|_{L^2}+|lt|\,\|\omega(t)\|_{L^2}\\
&\le (|k-l|+|l|)t\|\omega\|_{L^\infty L^2}+t^{\frac12}\|\partial_y(e^{ilyt}f)\|_{L^2L^2}.
\end{aligned}
$$

Therefore, (also using the first inequality of the lemma and $0<\nu\le 1$)

$$
\begin{aligned}
&\nu\|e^{-\nu^{1/3}t/2}\partial_y\omega\|_{L^2L^2}^2+\nu|k|^2\|e^{-\nu^{1/3}t/2}\omega\|_{L^2L^2}^2\\
&\le 2\nu(|k-l|+|l|)^2\|te^{-\nu^{1/3}t/2}\|_{L^2(T_0,T)}^2\|\omega\|_{L^\infty L^2}^2\\
&\quad+2\nu\|t^{\frac12}e^{-\nu^{1/3}t/2}\|_{L^2(T_0,T)}^2\|\partial_y(e^{ilyt}f)\|_{L^2L^2}^2\\
&\quad+\nu|k|^2\|e^{-\nu^{1/3}t/2}\|_{L^2(T_0,T)}^2\|\omega\|_{L^\infty L^2}^2\\
&\le C((|k-l|+|l|)^2+\nu^{2/3}|k|^2)\|\omega\|_{L^\infty L^2}^2+C\nu^{1/3}\|\partial_y(e^{ilyt}f)\|_{L^2L^2}^2\\
&\le C(|k-l|+|l|)^2|k-l|^{-2}\|(\partial_y,k-l)(e^{ilyt}f)\|_{L^2L^2}^2,
\end{aligned}
$$

which implies the second inequality of the lemma. ∎

Now we prove Lemma 5.5.

Proof of Lemma 5.5. Recall that

$$\partial_t f_1 + y\partial_x f_1 = te^{\nu^{1/3}t}u_e^2\partial_x\omega_L, \quad f_1|_{t=T_0}=0, \quad \omega_L(t,x,y)=\sum_{k\in\mathbb{Z}\setminus\{0\}} w_k^L(t,y)e^{ikx}.$$

Let $u_k^2(t,y)=\frac{1}{2\pi}\int_{\mathbb{T}} u_e^2(t,x,y)e^{-ikx}\,dx$. Then $u_e^2(t,x,y)=\sum_{k\in\mathbb{Z}} u_k^2(t,y)e^{ikx}$ and

$$u_e^2\partial_x\omega_L(t,x,y)=\sum_{k,l\in\mathbb{Z}}(u_k^2 ilw_l^L)(t,y)e^{i(k+l)x}=\sum_{k,l\in\mathbb{Z}}(u_{k-l}^2 ilw_l^L)(t,y)e^{ikx}.$$

Let $w_{k,l}$ solve

$$(\partial_t+iky)w_{k,l}(t,y)=e^{\nu^{1/3}t}tu_{k-l}^2(t,y)ilw_l^L(t,y)=F_{k,l}(t,y), \quad w_{k,l}(T_0,y)=0.$$

Then we have

$$f_1(t,x,y)=\sum_{k,l\in\mathbb{Z}} w_{k,l}(t,y)e^{ikx}=\sum_{k\in\mathbb{Z}}\widetilde{w}_k(t,y)e^{ikx}, \quad \widetilde{w}_k=\sum_{l\in\mathbb{Z}} w_{k,l}.$$

As $u_e^2=-\partial_x\phi_e$, we have $P_0u_e^2=0$, i.e., $u_0^2=0$; then $F_{k,l}=0$ and $w_{k,l}=0$ for $l=0$ or $k=l$. As $\omega_L|_{y=\pm1}=0$, we have $w_l^L|_{y=\pm1}=0$ and $F_{k,l}|_{y=\pm1}=0$. By Parseval's identity, we have

$$\|f_1(t)\|_{L^2(\Omega)}^2=2\pi\sum_{k\in\mathbb{Z}}\|\widetilde{w}_k(t)\|_{L^2(I)}^2, \quad \|\partial_x f_1(t)\|_{L^2(\Omega)}^2=2\pi\sum_{k\in\mathbb{Z}}|k|^2\|\widetilde{w}_k(t)\|_{L^2(I)}^2,$$
$$\|\partial_x\nabla\Delta^{-1}f_1(t)\|_{L^2(I)}^2=2\pi\sum_{k\in\mathbb{Z}}|k|^2\|(\partial_y,k)\Delta_k^{-1}\widetilde{w}_k(t)\|_{L^2(I)}^2,$$
$$\|\nabla f_1(t)\|_{L^2(\Omega)}^2=2\pi\sum_{k\in\mathbb{Z}}\|(\partial_y,k)\widetilde{w}_k(t)\|_{L^2(I)}^2,$$
$$\|\nabla u_e^2(t)\|_{L^2(\Omega)}^2=2\pi\sum_{k\in\mathbb{Z}}\|(\partial_y,k)u_k^2(t)\|_{L^2(I)}^2.$$

Recall that (see (4.7))

$$\|e^{-\nu^{1/3}t/2}f_1\|_Y=\|e^{-\nu^{1/3}t/2}f_1\|_{L^\infty L^2}+\nu^{1/2}\|e^{-\nu^{1/3}t/2}\nabla f_1\|_{L^2L^2} + \|e^{-\nu^{1/3}t/2}\partial_x\nabla\Delta^{-1}f_1\|_{L^2L^2},$$

where

$$\|e^{-\nu^{1/3}t/2}f_1\|_{L^\infty L^2}^2\le\|f_1\|_{L^\infty L^2}^2\le 2\pi\sum_{k\in\mathbb{Z}}\|\widetilde{w}_k\|_{L^\infty L^2}^2,$$
$$\nu\|e^{-\nu^{1/3}t/2}\nabla f_1\|_{L^2L^2}^2=2\pi\nu\sum_{k\in\mathbb{Z}}\|e^{-\nu^{1/3}t/2}(\partial_y,k)\widetilde{w}_k\|_{L^2L^2}^2,$$
$$\|e^{-\nu^{1/3}t/2}\partial_x\nabla\Delta^{-1}f_1\|_{L^2L^2}^2\le\|\partial_x\nabla\Delta^{-1}f_1\|_{L^2L^2}^2=2\pi\sum_{k\in\mathbb{Z}}|k|^2\|(\partial_y,k)\Delta_k^{-1}\widetilde{w}_k\|_{L^2L^2}^2,$$
$$\|\partial_x f_1\|_{L^\infty L^2}^2\le 2\pi\sum_{k\in\mathbb{Z}}|k|^2\|\widetilde{w}_k\|_{L^\infty L^2}^2.$$

Then we have

$$\begin{aligned}
&\|e^{-\nu^{1/3}t/2}f_1\|_Y^2+\|\partial_x f_1\|_{L^\infty L^2}^2\\
&\quad\lesssim \sum_{k\in\mathbb{Z}}\big[(1+|k|^2)\|\widetilde{w}_k\|_{L^\infty L^2}^2+|k|^2\|(\partial_y,k)\Delta_k^{-1}\widetilde{w}_k\|_{L^2L^2}^2\\
&\qquad\quad+\nu\|e^{-\nu^{1/3}t/2}(\partial_y,k)\widetilde{w}_k\|_{L^2L^2}^2\big],\\
&\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}^2=2\pi\sum_{k\in\mathbb{Z}}\|e^{\epsilon\nu^{1/3}t}(\partial_y,k)u_k^2\|_{L^2L^2}^2.
\end{aligned}$$

Let $E_k=\|e^{\epsilon\nu^{1/3}t}(\partial_y,k)u_k^2\|_{L^2L^2}$ and

$$\|f\|_{Y_k}^2=(1+|k|^2)\|f\|_{L^\infty L^2}^2+|k|^2\|(\partial_y,k)\Delta_k^{-1}f\|_{L^2L^2}^2+\nu\|e^{-\nu^{1/3}t/2}(\partial_y,k)f\|_{L^2L^2}^2.$$

Then (also using (3.1))

$$\begin{aligned}
&\|e^{-\nu^{1/3}t/2}f_1\|_Y^2+\|\partial_x f_1\|_{L^\infty L^2}^2\lesssim\sum_{k\in\mathbb{Z}}\|\widetilde{w}_k\|_{Y_k}^2,\\
&\sum_{k\in\mathbb{Z}}E_k^2\sum_{l\in\mathbb{Z}\setminus\{0\}}(E_l^L)^2\lesssim\|e^{\epsilon\nu^{1/3}t}\nabla u_e^2\|_{L^2L^2}^2\|\omega^{(0)}\|_{H^3}^2.
\end{aligned}$$

Thus, it is enough to show that

$$\sum_{k\in\mathbb{Z}}\|\widetilde{w}_k\|_{Y_k}^2\lesssim\nu^{-2/3}\sum_{k\in\mathbb{Z}}E_k^2\sum_{l\in\mathbb{Z}\setminus\{0\}}(E_l^L)^2=\nu^{-2/3}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}\setminus\{0\}}E_{k-l}^2(E_l^L)^2,$$

which is further reduced to the estimate

$$\|\widetilde{w}_k\|_{Y_k}^2\lesssim\nu^{-2/3}\sum_{l\in\mathbb{Z}\setminus\{0\}}E_{k-l}^2(E_l^L)^2,\quad\forall k\in\mathbb{Z}.\tag{5.7}$$

Thanks to $\widetilde{w}_k=\sum_{l\in\mathbb{Z}}w_{k,l}$, we only need to estimate $\|w_{k,l}\|_{Y_k}$. Recall that

$$(\partial_t+iky)w_{k,l}(t,y)=F_{k,l}(t,y),\quad w_{k,l}(T_0,y)=0,\ F_{k,l}|_{y=\pm1}=0.$$

Then by Lemma 5.6, we have (for $k\neq l$)

$$\begin{aligned}
&\|w_{k,l}\|_{L^\infty L^2}^2+|k|^2\|(\partial_y,k)\Delta_k^{-1}w_{k,l}\|_{L^2L^2}^2\leq C|k-l|^{-2}\|(\partial_y,k-l)(e^{ilyt}F_{k,l})\|_{L^2L^2}^2,\\
&\nu\|e^{-\nu^{1/3}t/2}(\partial_y,k)w_{k,l}\|_{L^2L^2}^2\leq C(1+|k|^2|k-l|^{-2})\|(\partial_y,k-l)(e^{ilyt}F_{k,l})\|_{L^2L^2}^2,
\end{aligned}$$

which show that

$$\begin{aligned}
\|w_{k,l}\|_{Y_k}^2&=(1+|k|^2)\|w_{k,l}\|_{L^\infty L^2}^2+|k|^2\|(\partial_y,k)\Delta_k^{-1}w_{k,l}\|_{L^2L^2}^2\\
&\quad+\nu\|e^{-\nu^{1/3}t/2}(\partial_y,k)w_{k,l}\|_{L^2L^2}^2\\
&\lesssim\big[(1+|k|^2)|k-l|^{-2}+1+|k|^2|k-l|^{-2}\big]\|(\partial_y,k-l)(e^{ilyt}F_{k,l})\|_{L^2L^2}^2\\
&\lesssim(1+|k|^2|k-l|^{-2})\|(\partial_y,k-l)(e^{ilyt}F_{k,l})\|_{L^2L^2}^2.
\end{aligned}$$

Recall that $F_{k,l} = e^{\nu^{1/3}t} t u_{k-l}^2 i l w_l^L$; we have

$$\begin{aligned}\|(\partial_y, k-l)(e^{ilyt} F_{k,l})\|_{L^2} &= e^{\nu^{1/3}t} t \|(\partial_y, k-l)(e^{ilyt} u_{k-l}^2 i l w_l^L)\|_{L^2} \\ &\lesssim e^{\nu^{1/3}t} t \|(\partial_y, k-l) u_{k-l}^2\|_{L^2} \|(\partial_y, 1)(e^{ilyt} l w_l^L)\|_{L^\infty}.\end{aligned}$$

For $l \in \mathbb{Z} \setminus \{0\}$, by Lemma 3.1, (3.1), and $(\nu l^2)^{1/3} t e^{-(\nu l^2)^{1/3} t} \leq 1$, we have

$$\begin{aligned}e^{\nu^{1/3}t} t \|(\partial_y, 1)(e^{ilyt} l w_l^L)\|_{L^\infty} &\lesssim e^{\nu^{1/3}t} t |l|^{-\frac{1}{2}} E_l^L e^{-2(\nu l^2)^{1/3} t} \leq t |l|^{-\frac{1}{2}} E_l^L e^{-(\nu l^2)^{1/3} t} \\ &\leq \nu^{-1/3} |l|^{-\frac{1}{2}-\frac{2}{3}} E_l^L \leq \nu^{-1/3} |l|^{-1} E_l^L.\end{aligned}$$

Then we obtain

$$\begin{aligned}\|(\partial_y, k-l)(e^{ilyt} F_{k,l})\|_{L^2 L^2} &\lesssim \nu^{-1/3} |l|^{-1} E_l^L \|(\partial_y, k-l) u_{k-l}^2\|_{L^2 L^2} \\ &\leq \nu^{-1/3} |l|^{-1} E_l^L E_{k-l},\end{aligned}$$

and (for $l \neq 0, k \neq l$)

$$\begin{aligned}\|w_{k,l}\|_{Y_k} &\lesssim (1 + |k||k-l|^{-1}) \|(\partial_y, k-l)(e^{ilyt} F_{k,l})\|_{L^2 L^2} \\ &\lesssim \nu^{-1/3} |l|^{-1} (1 + |k||k-l|^{-1}) E_l^L E_{k-l} \\ &\lesssim \nu^{-1/3} |l|^{-1} (1 + |l||k-l|^{-1}) E_l^L E_{k-l} = \nu^{-1/3} (|l|^{-1} + |k-l|^{-1}) E_l^L E_{k-l}.\end{aligned}$$

Thanks to $\widetilde{w}_k = \sum_{l \in \mathbb{Z}} w_{k,l}$, and $w_{k,l} = 0$ for $l = 0$ or $k = l$, we have

$$\begin{aligned}\|\widetilde{w}_k\|_{Y_k} &\leq \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \|w_{k,l}\|_{Y_k} \lesssim \nu^{-1/3} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} (|l|^{-1} + |k-l|^{-1}) E_l^L E_{k-l}, \\ \|\widetilde{w}_k\|_{Y_k}^2 &\lesssim \nu^{-2/3} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} (E_l^L E_{k-l})^2,\end{aligned}$$

which gives (5.7). Here we used

$$\sum_{l \in \mathbb{Z} \setminus \{0,k\}} (|l|^{-2} + |k-l|^{-2}) \leq 2 \sum_{l \in \mathbb{Z} \setminus \{0\}} |l|^{-2} \lesssim 1.$$

This completes the proof. ∎

A. Gagliardo–Nirenberg inequality

Lemma A.1. *If $P_0 f = 0$, then we have*

$$\|f\|_{L^\infty} \lesssim \|\partial_x f\|_{L^2}^{3/4} \|f\|_{H^2}^{1/4}.$$

Proof. Let $f_k(y) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x,y)e^{-ikx}\,dx$ for $k \in \mathbb{Z}$. As $P_0 f = 0$, $f(x,y) = \sum_{k\in\mathbb{Z}\setminus\{0\}} f_k(y)e^{ikx}$ and

$$\|\partial_x f\|_{L^2(\Omega)}^2 = 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} \|ikf_k\|_{L^2(I)}^2, \quad \|\nabla^2 f\|_{L^2(\Omega)}^2 = 2\pi \sum_{k\in\mathbb{Z}\setminus\{0\}} \|(\partial_y,k)^2 f_k\|_{L^2(I)}^2.$$

By the Gagliardo–Nirenberg inequality and the Hölder inequality, we have

$$\begin{aligned}
\|f\|_{L^\infty(\Omega)} &\le \sum_{k\in\mathbb{Z}\setminus\{0\}} \|f_k\|_{L^\infty(I)} \lesssim \sum_{k\in\mathbb{Z}\setminus\{0\}} \|f_k\|_{L^2(I)}^{3/4}\|f_k\|_{H^2(I)}^{1/4}\\
&\lesssim \Big(\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-3/2}\Big)^{1/2}\Big(\sum_{k\in\mathbb{Z}\setminus\{0\}} \|ikf_k\|_{L^2(I)}^2\Big)^{3/8}\\
&\quad\times\Big(\sum_{k\in\mathbb{Z}\setminus\{0\}} \|(\partial_y,k)^2 f_k\|_{L^2(I)}^2\Big)^{1/8}\\
&\lesssim \|\partial_x f\|_{L^2(\Omega)}^{3/4}\|f\|_{H^2(\Omega)}^{1/4}.
\end{aligned}$$

This completes the proof. ∎

B. Some properties of the operator $\mathfrak{J}_k$

The following result was proved in [7].

Lemma B.1. *Let $[\partial_y, \mathfrak{J}_k] = \mathfrak{H}_k$. It holds that*

$$\|\mathfrak{J}_k\|_{L^2\to L^2} \lesssim 1, \quad \|\mathfrak{H}_k\|_{L^2\to L^2} \lesssim |k|.$$

For $f, g \in L^2(I)$, we have

$$\overline{\mathfrak{J}_k[f]} = -\mathfrak{J}_k[\bar{f}], \quad \int_{-1}^1 \bar{f}\mathfrak{J}_k[g]\,dy = -\int_{-1}^1 \mathfrak{J}_k[\bar{f}]g\,dy = \int_{-1}^1 \overline{\mathfrak{J}_k[f]}g\,dy.$$

Thus, the operator $\mathfrak{J}_k$ is symmetric, i.e.,

$$\langle \mathfrak{J}_k[g], f\rangle = \langle g, \mathfrak{J}_k[f]\rangle.$$

For $k \in \mathbb{Z}\setminus\{0\}$, $f \in H^1(I)$, we have $\partial_y\mathfrak{J}_k f = \mathfrak{J}_k[\partial_y f] + \mathfrak{H}_k[f]$ and

$$\begin{aligned}
&\|\partial_y\mathfrak{J}_k[f]\|_{L^2} = \|\mathfrak{J}_k[\partial_y f]\|_{L^2} + \|\mathfrak{H}_k[f]\|_{L^2} \lesssim \|\partial_y f\|_{L^2} + |k|\|f\|_{L^2},\\
&\|(\partial_y,k)\mathfrak{J}_k[f]\|_{L^2} \lesssim \|(\partial_y,k)f\|_{L^2}.
\end{aligned} \tag{B.1}$$

Lemma B.2. *There exists $\delta_0 \in (0,1/4)$ such that if U satisfies (4.1), $\Delta_k\phi_k = f_k$, $\phi_k|_{y=\pm1} = 0$, then we have*

$$T_1 := \mathrm{Re}\langle -ikUf_k, \mathfrak{J}_k f_k\rangle \le -\frac{|k|^2}{8}\|(\partial_y,k)\phi_k\|_{L^2}^2.$$

This result was essentially proved in [7]. Recall that $W(t,y) = e^{\nu t\partial_y^2}W_{\rm in}(y)$ and $U(t,y) = y - \partial_y(-\partial_y^2)^{-1}W(t,y)$. We stress that the proof only requires the smallness of $\|U-y\|_{C^3}$ rather than $\|W\|_{H^4}$. For completeness, we will sketch the proof of Lemma B.2 as in [7], while focusing on where the smallness is used.

Proof of Lemma B.2. As in [7], by Taylor's expansion, we have

$$U(y') - U(y) = U'(y)(y'-y) + \frac{1}{2}U''(y)(y'-y)^2 + \frac{1}{2}\int_y^{y'} U'''(s)(y'-s)^2\,ds.$$

It is easy to see that the remainder satisfies

$$r(y,y') := \frac{1}{2(y'-y)}\int_y^{y'} U'''(s)(y'-s)^2\,ds \in L^\infty, \tag{B.2}$$

$$|r(y,y')| \lesssim \|U'''\|_{L^\infty}|y-y'|^2, \tag{B.3}$$

$$|\partial_y r(y,y')| + |\partial_{y'} r(y,y')| \lesssim \|U'''\|_{L^\infty}|y-y'|, \tag{B.4}$$

$$|\partial_{y'}\partial_y r(y,y')| \lesssim \|U'''\|_{L^\infty}. \tag{B.5}$$

Then we have (see [7] for more details)

$$\begin{aligned}
T_1 &= \frac{|k|^2}{4}\,\mathrm{Re}\int_{-1}^1\int_{-1}^1 \overline{f_k(y)}\frac{U(y)-U(y')}{y-y'}G_k(y,y')f_k(y')\,dy'\,dy\\
&= \frac{|k|^2}{4}\,\mathrm{Re}\int_{-1}^1\int_{-1}^1 \overline{f_k(y)}\frac{U'(y)(y-y')}{y-y'}G_k(y,y')f_k(y')\,dy'\,dy\\
&\quad + \frac{|k|^2}{8}\,\mathrm{Re}\int_{-1}^1\int_{-1}^1 \overline{f_k(y)}U''(y)(y'-y)G_k(y,y')f_k(y')\,dy'\,dy\\
&\quad + \frac{|k|^2}{4}\,\mathrm{Re}\int_{-1}^1\int_{-1}^1 \overline{f_k(y)}r(y,y')G_k(y,y')f_k(y')\,dy'\,dy\\
&= T_{11} + T_{12} + T_{13}.
\end{aligned}$$

For T_{11}, by using $\Delta_k\phi_k = f_k$, $\phi_k = \Delta_k^{-1}f_k$ and integration by parts, we obtain

$$\begin{aligned}
T_{11} &= \frac{|k|^2}{4}\,\mathrm{Re}\int_{-1}^1 \overline{\Delta_k\phi_k}U'\phi_k\,dy\\
&= -\frac{|k|^2}{4}\|\sqrt{U'}(\partial_y,k)\phi_k\|_{L^2}^2 - \frac{|k|^2}{4}\,\mathrm{Re}\int_{-1}^1 U''\partial_y\phi_k\overline{\phi_k}\,dy.
\end{aligned}$$

By (4.1), we have $\inf U' \geq 1 - \|U'-1\|_{L^\infty} \geq 1-\delta_0$ and $\|U''\|_{L^\infty} \leq \delta_0$, thus

$$\begin{aligned}
T_{11} &\leq -\frac{|k|^2}{4}(1-\delta_0)\|(\partial_y,k)\phi_k\|_{L^2}^2 + \frac{|k|^2}{4}\|U''\|_{L^\infty}\|\partial_y\phi_k\|_{L^2}\|\phi_k\|_{L^2}\\
&\leq -\frac{|k|^2}{4}(1-2\delta_0)\|(\partial_y,k)\phi_k\|_{L^2}^2.
\end{aligned}$$

For T_{12}, by using $\Delta_k \phi_k = f_k$, integration by parts, and the fact that the Green's function vanishes on the boundary in both variables, we have

$$\begin{aligned} T_{12} &= \frac{|k|^2}{8} \operatorname{Re} \int_{-1}^{1} \int_{-1}^{1} \overline{\Delta_k \phi_k(y)} U''(y)(y'-y) G_k(y,y') \Delta_k \phi_k(y') \, dy' \, dy \\ &= \frac{|k|^2}{8} \operatorname{Re} \int_{-1}^{1} \int_{-1}^{1} \overline{\nabla_{k,y} \phi_k(y)} \cdot \nabla_{k,y} (\nabla_{k,y'} \mathcal{N}_k(y,y') \cdot \nabla_{k,y'} \phi_k(y')) \, dy' \, dy. \end{aligned}$$

Here, $\mathcal{N}_k(y,y') = U''(y)(y'-y)G_k(y,y')$, $\nabla_{k,y} = (\partial_y, k)$, $\nabla_{k,y'} = (\partial_{y'}, k)$. Using (4.4), we can deduce that

$$\begin{aligned} &\|\mathcal{N}_k\|_{L^2_{y,y'}} \lesssim \|U''\|_{L^\infty} |k|^{-2}, \\ &\|\partial_y \mathcal{N}_k\|_{L^2_{y,y'}} + \|\partial_{y'} \mathcal{N}_k\|_{L^2_{y,y'}} \lesssim (\|U''\|_{L^\infty} + \|U'''\|_{L^\infty}) |k|^{-1}, \\ &\|\partial_y \partial_{y'} \mathcal{N}_k\|_{L^2_{y,y'}} \lesssim \|U''\|_{L^\infty} + \|U'''\|_{L^\infty}. \end{aligned}$$

Therefore, by Cauchy–Schwarz and (4.1), we have

$$|T_{12}| \lesssim (\|U''\|_{L^\infty} + \|U'''\|_{L^\infty}) |k|^2 \|(\partial_y, k)\phi_k\|_{L^2}^2 \lesssim \delta_0 |k|^2 \|(\partial_y, k)\phi_k\|_{L^2}^2.$$

Similarly, for T_{13}, we have

$$T_{13} = \frac{|k|^2}{4} \operatorname{Re} \int_{-1}^{1} \int_{-1}^{1} \overline{\nabla_{k,y} \phi_k(y)} \cdot \nabla_{k,y} (\nabla_{k,y'} (r G_k)(y,y') \cdot \nabla_{k,y'} \phi_k(y')) \, dy' \, dy.$$

By (4.4) and (B.3)–(B.5), we have (see [7] for more details)

$$\begin{aligned} \|r G_k\|_{L^2_{y,y'}} &\lesssim \|U'''\|_{L^\infty} |k|^{-2}, \\ \|\partial_y (r G_k)\|_{L^2_{y,y'}} + \|\partial_{y'} (r G_k)\|_{L^2_{y,y'}} &\lesssim \|U'''\|_{L^\infty} |k|^{-1}, \\ \|\partial_y \partial_{y'} (r G_k)\|_{L^2_{y,y'}} &\lesssim \|U'''\|_{L^\infty}. \end{aligned}$$

Therefore, by Cauchy–Schwarz and (4.1), we have

$$|T_{13}| \lesssim \|U'''\|_{L^\infty} |k|^2 \|(\partial_y, k)\phi_k\|_{L^2}^2 \lesssim \delta_0 |k|^2 \|(\partial_y, k)\phi_k\|_{L^2}^2.$$

Summing up, we arrive at

$$T_1 = T_{11} + T_{12} + T_{13} \le -\frac{|k|^2}{4} (1 - 2\delta_0) \|(\partial_y, k)\phi_k\|_{L^2}^2 + C\delta_0 |k|^2 \|(\partial_y, k)\phi_k\|_{L^2}^2,$$

which gives our result by taking δ_0 small enough. ■

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