



Panki Kim · Renming Song · Zoran Vondraček

Sharp two-sided Green function estimates for Dirichlet forms degenerate at the boundary

Received November 28, 2020; revised March 31, 2022

Abstract. The goal of this paper is to establish Green function estimates for a class of purely discontinuous symmetric Markov processes with jump kernels degenerate at the boundary and critical killing potentials. The jump kernel and the killing potential depend on several parameters. We establish sharp two-sided estimates on the Green functions of these processes for all admissible values of the parameters involved. Depending on the regions where the parameters belong, the estimates on the Green functions are different. In fact, the estimates have three different forms. As applications, we prove that the boundary Harnack principle holds in certain region of the parameters and fails in some other region of the parameters. Together with the main results of our previous paper [Potential Anal., online, 2021], we completely determine the region of the parameters where the boundary Harnack principle holds.

Keywords. Markov processes, Dirichlet forms, jump kernel, killing potential, Green function, Harnack inequality, Carleson estimate, boundary Harnack principle

1. Introduction and main results

In the last few decades, many important results have been obtained in the study of potential-theoretic properties for various types of jump processes in open subsets of \mathbb{R}^d . These include isotropic α -stable processes, more general symmetric Lévy and Lévy-type processes and their censored versions. The main results include the boundary Harnack principle [4, 5, 10, 14, 34, 38, 45], sharp two-sided Green function estimates [16, 19, 22, 23, 37, 42] and sharp two-sided Dirichlet heat kernel estimates [8, 9, 17–19, 33, 36]. In all these results, the jump kernel $J^D(x, y)$ of the process in the open set D is either the restriction of the jump kernel of the original process in \mathbb{R}^d or comparable to such a kernel and it does not tend to zero as x or y tends to the boundary of D . In this sense, one can say that the corresponding integro-differential operator is uniformly elliptic.

Panki Kim: Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 08826, Republic of Korea; pkim@snu.ac.kr

Renming Song: Department of Mathematics, University of Illinois, Urbana, IL 61801, USA; rsong@illinois.edu

Zoran Vondraček: Department of Mathematics, Faculty of Science, University of Zagreb, Zagreb, Croatia; vondra@math.hr

Mathematics Subject Classification (2020): Primary 60J45; Secondary 60J50, 60J76

Subordinate killed Brownian motions, and more generally, subordinate killed Lévy processes, form another important class of Markov processes. In the case of a stable subordinator, the generator of the subordinate killed Brownian motion is the spectral fractional Laplacian. The spectral fractional Laplacian and, more generally, fractional powers of elliptic differential operators in domains have been studied by many people in the PDE community [11–13, 15, 32, 46]. In contrast with killed Lévy processes and censored processes, the jump kernel of a subordinate killed Lévy process in an open subset $D \subset \mathbb{R}^d$ tends to zero near the boundary of D [39, 40, 44]. In this sense, the Dirichlet forms of subordinate killed Lévy processes are degenerate near the boundary. Partial differential equations degenerate at the boundary have been studied intensively in the PDE literature; see, for instance, [27, 29, 30, 35, 47] and the references therein.

In our recent paper [41], we introduced a class of symmetric Markov processes in open subsets $D \subset \mathbb{R}^d$ whose Dirichlet forms are degenerate at the boundary of D . This class of processes includes subordinate killed Lévy processes as special cases.

This paper is the second part of our investigation of the potential theory of Markov processes with jump kernels degenerate at the boundary. In [41] we studied Markov processes in open sets $D \subset \mathbb{R}^d$ defined via Dirichlet forms with jump kernels $J^D(x, y) = j(|x - y|)\mathcal{B}(x, y)$ (where $j(|x|)$ is the density of a pure jump isotropic Lévy process) and critical killing potentials κ . The function $\mathcal{B}(x, y)$ is assumed to satisfy certain conditions, and is allowed to decay at the boundary of the state space D . This is in contrast with all the works mentioned in the first paragraph where $\mathcal{B}(x, y)$ is assumed to be bounded between two positive constants, which can be viewed as a uniform ellipticity condition for non-local operators. In this sense, our paper [41] is the first systematic attempt to study the potential theory of general degenerate non-local operators defined in terms of Dirichlet forms. We proved in [41] that the Harnack inequality and Carleson's estimate are valid for non-negative harmonic functions with respect to these Markov processes.

When $D = \mathbb{R}_+^d := \{x = (\tilde{x}, x_d) : x_d > 0\}$, $j(|x - y|) = |x - y|^{-\alpha-d}$, $0 < \alpha < 2$, and $\kappa(x) := cx_d^{-\alpha}$, we showed in [41] that for certain values of the parameters involved in $\mathcal{B}(x, y)$ the boundary Harnack principle holds, while for some other values of the parameters the boundary Harnack principle fails (despite the fact that Carleson's estimate holds). The main goal of this paper is to establish sharp two-sided estimates on the Green functions of the corresponding processes for all admissible values of the parameters involved in $\mathcal{B}(x, y)$. These estimates imply anomalous boundary behavior for certain Green potentials (see Proposition 6.10), a feature recently studied both in the probabilistic and in the PDE literature [1, 11, 40]. As an application of these Green function estimates, we give a complete answer to the question for which values of the parameters the boundary Harnack principle holds true.

We first repeat the assumptions on \mathcal{B} that were introduced in [41]. Here and below, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

(A1) $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in \mathbb{R}_+^d$.

(A2) If $\alpha \geq 1$, then there exist $\theta > \alpha - 1$ and $C_1 > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \leq C_1 \left(\frac{|x - y|}{x_d \wedge y_d} \right)^\theta.$$

(A3) There exist $C_2 \geq 1$ and parameters $\beta_1, \beta_2, \beta_3, \beta_4 \geq 0$, with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C_2^{-1} \tilde{\mathcal{B}}(x, y) \leq \mathcal{B}(x, y) \leq C_2 \tilde{\mathcal{B}}(x, y), \quad x, y \in \mathbb{R}_+^d, \quad (1.1)$$

where

$$\begin{aligned} \tilde{\mathcal{B}}(x, y) := & \left(\frac{x_d \wedge y_d}{|x - y|} \wedge 1 \right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^{\beta_2} \left[\log \left(1 + \frac{(x_d \vee y_d) \wedge |x - y|}{x_d \wedge y_d \wedge |x - y|} \right) \right]^{\beta_3} \\ & \times \left[\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right]^{\beta_4}. \end{aligned} \quad (1.2)$$

(A4) For all $x, y \in \mathbb{R}_+^d$ and $a > 0$, $\mathcal{B}(ax, ay) = \mathcal{B}(x, y)$. In case $d \geq 2$, for all $x, y \in \mathbb{R}_+^d$ and $\tilde{z} \in \mathbb{R}^{d-1}$, $\mathcal{B}(x + (\tilde{z}, 0), y + (\tilde{z}, 0)) = \mathcal{B}(x, y)$.

Other than the requirements that $\beta_1 > 0$ if $\beta_3 > 0$ and $\beta_2 > 0$ if $\beta_4 > 0$, the parameters $\beta_1, \beta_2, \beta_3$ and β_4 are arbitrary. They control the rate at which \mathcal{B} goes to 0 at the boundary. Note that the term

$$\left(\frac{x_d \wedge y_d}{|x - y|} \wedge 1 \right)^{\beta_1} \left[\log \left(1 + \frac{(x_d \vee y_d) \wedge |x - y|}{x_d \wedge y_d \wedge |x - y|} \right) \right]^{\beta_3}$$

goes to 0 when one of x and y goes to the boundary, while the term

$$\left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^{\beta_2} \left[\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right]^{\beta_4}$$

goes to 0 when both x and y go to the boundary. Note that if $\mathcal{B}(x, y) \equiv c \tilde{\mathcal{B}}(x, y)$ for some positive constant c , then (A1)–(A4) trivially hold.

In the remainder of this paper, we always assume that

$$d > (\alpha + \beta_1 + \beta_2) \wedge 2, \quad p \in ((\alpha - 1)_+, \alpha + \beta_1),$$

$$J(x, y) = |x - y|^{-d-\alpha} \mathcal{B}(x, y) \text{ on } \mathbb{R}_+^d \times \mathbb{R}_+^d \text{ with } \mathcal{B} \text{ satisfying (A1)–(A4).}$$

To every parameter $p \in ((\alpha - 1)_+, \alpha + \beta_1)$, we associate a constant $C(\alpha, p, \mathcal{B}) \in (0, \infty)$ depending on α , p and \mathcal{B} defined as

$$\begin{aligned} C(\alpha, p, \mathcal{B}) = & \int_{\mathbb{R}^{d-1}} \frac{1}{(|\tilde{u}|^2 + 1)^{(d+\alpha)/2}} \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1 - s)^{1+\alpha}} \\ & \times \mathcal{B}((1 - s)\tilde{u}, 1, s\mathbf{e}_d) ds d\tilde{u}, \end{aligned} \quad (1.3)$$

where $\mathbf{e}_d = (\tilde{0}, 1)$. In case $d = 1$, $C(\alpha, p, \mathcal{B})$ is defined as

$$C(\alpha, p, \mathcal{B}) = \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha-p-1})}{(1-s)^{1+\alpha}} \mathcal{B}(1, s) ds.$$

Note that $\lim_{p \downarrow (\alpha-1)_+} C(\alpha, p, \mathcal{B}) = 0$, $\lim_{p \uparrow \alpha + \beta_1} C(\alpha, p, \mathcal{B}) = \infty$ and the function $p \mapsto C(\alpha, p, \mathcal{B})$ is strictly increasing (see [41, Lemma 5.4 and Remark 5.5]). Thus, the interval $((\alpha - 1)_+, \alpha + \beta_1)$ is the full admissible range for the parameter p .

Let

$$\kappa(x) = C(\alpha, p, \mathcal{B})x_d^{-\alpha}, \quad x \in \mathbb{R}_+^d, \quad (1.4)$$

be the killing potential. Note that κ depends on p , but we omit this dependence from the notation for simplicity. We denote by Y the Hunt process with jump kernel J and killing potential κ .

To be more precise, let us define

$$\mathcal{E}^{\mathbb{R}_+^d}(u, v) := \frac{1}{2} \int_{\mathbb{R}_+^d} \int_{\mathbb{R}_+^d} (u(x) - u(y))(v(x) - v(y))J(x, y) dy dx,$$

which is a symmetric form degenerate at the boundary due to **(A1)** and **(A3)**. By Fatou's lemma, $(\mathcal{E}^{\mathbb{R}_+^d}, C_c^\infty(\mathbb{R}_+^d))$ is closable in $L^2(\mathbb{R}_+^d, dx)$. Let $\mathcal{F}^{\mathbb{R}_+^d}$ be the closure of $C_c^\infty(\mathbb{R}_+^d)$ under $\mathcal{E}_1^{\mathbb{R}_+^d} := \mathcal{E}^{\mathbb{R}_+^d} + (\cdot, \cdot)_{L^2(\mathbb{R}_+^d, dx)}$. Then $(\mathcal{E}^{\mathbb{R}_+^d}, \mathcal{F}^{\mathbb{R}_+^d})$ is a regular Dirichlet form on $L^2(\mathbb{R}_+^d, dx)$. Set

$$\mathcal{E}(u, v) := \mathcal{E}^{\mathbb{R}_+^d}(u, v) + \int_{\mathbb{R}_+^d} u(x)v(x)\kappa(x) dx.$$

Since κ is locally bounded, the measure $\kappa(x)dx$ is a positive Radon measure charging no set of zero capacity. Let $\mathcal{F} := \widehat{\mathcal{F}^{\mathbb{R}_+^d}} \cap L^2(\mathbb{R}_+^d, \kappa(x) dx)$, where $\widehat{\mathcal{F}^{\mathbb{R}_+^d}}$ is the family of all quasi-continuous functions in $\mathcal{F}^{\mathbb{R}_+^d}$. By [31, Theorems 6.1.1 and 6.1.2], $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbb{R}_+^d, dx)$ with $C_c^\infty(\mathbb{R}_+^d)$ as a special standard core. Let $((Y_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}_+^d \setminus \mathcal{N}})$ be the associated Hunt process with lifetime ζ . By [41, Proposition 3.2], the exceptional set \mathcal{N} can be taken as the empty set. We add a cemetery point ∂ to the state space \mathbb{R}_+^d and define $Y_t = \partial$ for $t \geq \zeta$.

The process Y enjoys the following important scaling property shown in [41, Lemma 5.1]: For any $r > 0$ define the process $Y^{(r)}$ by $Y_t^{(r)} := rY_{r^{-1}t}$. Then under **(A1)**, the boundedness of \mathcal{B} and **(A4)**, $(Y^{(r)}, \mathbb{P}_{x/r})$ has the same law as (Y, \mathbb{P}_x) . The homogeneity property of \mathcal{B} from **(A4)** is crucial to establish this fact.

Recall that a Borel function $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ is said to be *harmonic* in an open set $V \subset \mathbb{R}_+^d$ with respect to Y if for every bounded open set $U \subset \bar{V} \subset V$,

$$f(x) = \mathbb{E}_x[f(Y_{\tau_U})] \quad \text{for all } x \in U, \quad (1.5)$$

where $\tau_U := \inf\{t > 0 : Y_t \notin U\}$ is the first exit time of Y from U . We say f is *regular harmonic* in V if (1.5) holds for V .

Let $G(x, y)$ denote the Green function of the process Y . The following theorem is our main result on Green function estimates. For two functions f and g , we use the notation $f \asymp g$ to denote that the quotient f/g stays bounded between two positive constants.

Theorem 1.1. *Assume that (A1)–(A4) and (1.4) hold true. Suppose that $d > (\alpha + \beta_1 + \beta_2) \wedge 2$ and $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. Then the process Y admits a Green function $G : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow [0, \infty]$ such that $G(x, \cdot)$ is continuous in $\mathbb{R}_+^d \setminus \{x\}$ and regular harmonic with respect to Y in $\mathbb{R}_+^d \setminus B(x, \varepsilon)$ for any $\varepsilon > 0$. Moreover, $G(x, y)$ has the following estimates:*

(1) *If $p \in ((\alpha - 1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$, then on $\mathbb{R}_+^d \times \mathbb{R}_+^d$,*

$$G(x, y) \asymp \frac{1}{|x - y|^{d-\alpha}} \left(\frac{x_d}{|x - y|} \wedge 1 \right)^p \left(\frac{y_d}{|x - y|} \wedge 1 \right)^p. \quad (1.6)$$

(2) *If $p = \alpha + \frac{\beta_1 + \beta_2}{2}$, then on $\mathbb{R}_+^d \times \mathbb{R}_+^d$,*

$$G(x, y) \asymp \frac{1}{|x - y|^{d-\alpha}} \left(\frac{x_d}{|x - y|} \wedge 1 \right)^p \left(\frac{y_d}{|x - y|} \wedge 1 \right)^p \times \left(\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right)^{\beta_4 + 1}.$$

(3) *If $p \in (\alpha + \frac{\beta_1 + \beta_2}{2}, \alpha + \beta_1)$, then on $\mathbb{R}_+^d \times \mathbb{R}_+^d$,*

$$\begin{aligned} & G(x, y) \\ & \asymp \frac{1}{|x - y|^{d-\alpha}} \left(\frac{x_d \wedge y_d}{|x - y|} \wedge 1 \right)^p \left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^{2\alpha - p + \beta_1 + \beta_2} \\ & \quad \times \left(\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right)^{\beta_4} \\ & = \frac{1}{|x - y|^{d-\alpha}} \left(\frac{x_d}{|x - y|} \wedge 1 \right)^p \left(\frac{y_d}{|x - y|} \wedge 1 \right)^p \left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^{-2(p - \alpha - (\beta_1 + \beta_2)/2)} \\ & \quad \times \left(\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right)^{\beta_4}. \end{aligned}$$

Note that when $\beta_1 \leq \beta_2$, case (1) covers all possible values of the parameter p , while when $\beta_2 < \beta_1$ the regimes of p in cases (1)–(3) are disjoint and exhaustive.

In fact, for lower bounds of Green functions, we have more general results (see Theorems 5.1 and 6.6). In these theorems, we establish lower bounds on the Green function $G^{B(w, R) \cap \mathbb{R}_+^d}(x, y)$ for Y killed upon exiting $B(w, R) \cap \mathbb{R}_+^d$ (where $w \in \partial \mathbb{R}_+^d$) in $B(w, (1 - \varepsilon)R) \cap \mathbb{R}_+^d$. The lower bounds on $G(x, y)$ in the theorem above are corollaries of these more general results.

Note that

$$p \mapsto 2\alpha - p + \beta_1 + \beta_2 = (\alpha + \beta_2) + (\alpha + \beta_1 - p)$$

is decreasing on $\alpha + \frac{\beta_1 + \beta_2}{2} \leq p < \alpha + \beta_1$, which has a somewhat strange and interesting consequence. Namely, the power of $\frac{x_d \wedge y_d}{|x - y|} \wedge 1$ is always p and we can increase

the exponent p of $\frac{x_d \wedge y_d}{|x-y|} \wedge 1$ all the way up to (just below) $\alpha + \beta_1$. But the exponent of $\frac{x_d \vee y_d}{|x-y|} \wedge 1$ is p only up to $\alpha + \frac{\beta_1 + [\beta_1 \wedge \beta_2]}{2}$ and one can increase the exponent only up to $\alpha + \frac{\beta_1 + [\beta_1 \wedge \beta_2]}{2}$. In the case $\beta_2 < \beta_1$, once p reaches $\alpha + \frac{\beta_1 + \beta_2}{2}$, the exponent of $\frac{x_d \vee y_d}{|x-y|} \wedge 1$ starts decreasing.

Estimates (1.6) can be equivalently stated as

$$G(x, y) \asymp \left(\frac{x_d y_d}{|x-y|^2} \wedge 1 \right)^p \frac{1}{|x-y|^{d-\alpha}} \quad \text{on } \mathbb{R}_+^d \times \mathbb{R}_+^d. \quad (1.7)$$

Note that, when $d \geq 3$ and $p = (d - \alpha)/(d - 2) \in (1, d/(d - 2))$, the estimates in (1.7) are those of a power of the Green function of killed Brownian motion in \mathbb{R}_+^d (see [28]).

Moreover, we can rewrite the estimates in Theorem 1.1 in a unified way: Let $a_p := 2(p - \alpha - \frac{\beta_1 + [\beta_1 \wedge \beta_2]}{2})$. Then on $\mathbb{R}_+^d \times \mathbb{R}_+^d$,

$$G(x, y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1 \right)^p \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1 \right)^{p-a_p+} \\ \times \log \left(2 + \mathbf{1}_{a_p \leq 0} \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|} \right)^{\beta_4 + \mathbf{1}_{a_p=0}}.$$

In [41, Theorem 1.3] we have proved that the boundary Harnack principle holds when either (a) $\beta_1 = \beta_2$ and $\beta_3 = \beta_4 = 0$, or (b) $p < \alpha$. In [41, Theorem 1.4] we have showed that when $\alpha + \beta_2 < p < \alpha + \beta_1$ the boundary Harnack principle fails. However, we were unable to determine what happens with the boundary Harnack principle in the remaining regions of the admissible parameters. As applications of our Green function estimates, we can completely resolve this issue and prove the following two results. In the remainder of this paper, we will only give the statements and proofs of the results for $d \geq 2$. The counterparts in the $d = 1$ case are similar and simpler.

For any $a, b > 0$ and $\tilde{w} \in \mathbb{R}^{d-1}$, we define a box

$$D_{\tilde{w}}(a, b) := \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : |\tilde{x} - \tilde{w}| < a, 0 < x_d < b\}.$$

Theorem 1.2. *Assume that (A1)–(A4) and (1.4) hold true. Suppose that $d > (\alpha + \beta_1 + \beta_2) \wedge 2$ and $p \in ((\alpha - 1)_+, \alpha + (\beta_1 \wedge \beta_2))$. Then there exists $C_3 \geq 1$ such that for all $r > 0$, $\tilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in \mathbb{R}_+^d which is harmonic in $D_{\tilde{w}}(2r, 2r)$ with respect to Y and vanishes continuously on $B((\tilde{w}, 0), 2r) \cap \partial\mathbb{R}_+^d$, we have*

$$\frac{f(x)}{x_d^p} \leq C_3 \frac{f(y)}{y_d^p}, \quad x, y \in D_{\tilde{w}}(r/2, r/2). \quad (1.8)$$

Theorem 1.2 implies that if two functions f, g in \mathbb{R}_+^d both satisfy the assumptions in Theorem 1.2, then

$$\frac{f(x)}{f(y)} \leq C_3^2 \frac{g(x)}{g(y)}, \quad x, y \in D_{\tilde{w}}(r/2, r/2).$$

We say that the non-scale-invariant boundary Harnack principle holds near the boundary of \mathbb{R}_+^d if there is a constant $\hat{R} \in (0, 1)$ such that for any $r \in (0, \hat{R}]$, there exists a constant $c = c(r) \geq 1$ such that for all $\tilde{w} \in \mathbb{R}^{d-1}$ and non-negative functions f, g in \mathbb{R}_+^d

which are harmonic in $\mathbb{R}_+^d \cap B((\tilde{w}, 0), r)$ with respect to Y and vanish continuously on $\partial\mathbb{R}_+^d \cap B((\tilde{w}, 0), r)$, we have

$$\frac{f(x)}{f(y)} \leq c \frac{g(x)}{g(y)} \quad \text{for all } x, y \in B((\tilde{w}, 0), r/2) \cap \mathbb{R}_+^d.$$

Theorem 1.3. *Suppose $d > \alpha + \beta_1 + \beta_2$ and $d \geq 2$. Assume that (A1)–(A4) and (1.4) hold true. If $\alpha + \beta_2 \leq p < \alpha + \beta_1$, then the non-scale-invariant boundary Harnack principle is not valid for Y .*

Thus, when $\alpha + \beta_2 \leq p < \alpha + (\beta_1 + \beta_2)/2$, the boundary Harnack principle is not valid for Y even though we have the standard form of the Green function estimates (1.7). This phenomenon has already been observed by the authors in [40] for subordinate killed Lévy processes.

The following two results proved in [41] will be fundamental for this paper. Note that, by the scaling property of Y (see [41, Lemma 5.1]), we can allow $r > 0$ instead of $r \in (0, 1]$.

Theorem 1.4 (Harnack inequality, [41, Theorem 1.1]). *Assume that (A1)–(A4) and (1.4) hold true and $p \in ((\alpha - 1)_+, \alpha + \beta_1)$.*

(a) *There exists a constant $C_4 > 0$ such that for any $r > 0$, any $B(x_0, r) \subset \mathbb{R}_+^d$ and any non-negative function f in \mathbb{R}_+^d which is harmonic in $B(x_0, r)$ with respect to Y , we have*

$$f(x) \leq C_4 f(y) \quad \text{for all } x, y \in B(x_0, r/2).$$

(b) *There exists a constant $C_5 > 0$ such that for any $L > 0$, any $r > 0$, any $x_1, x_2 \in \mathbb{R}_+^d$ with $|x_1 - x_2| < Lr$ and $B(x_1, r) \cup B(x_2, r) \subset \mathbb{R}_+^d$ and any non-negative function f in \mathbb{R}_+^d which is harmonic in $B(x_1, r) \cup B(x_2, r)$ with respect to Y , we have*

$$f(x_2) \leq C_5(L + 1)^{\beta_1 + \beta_2 + d + \alpha} f(x_1).$$

Since the half-space \mathbb{R}_+^d is κ -fat with characteristics $(R, 1/2)$ for any $R > 0$, we also have

Theorem 1.5 (Carleson's estimate, [41, Theorem 1.2]). *Assume that (A1)–(A4) and (1.4) hold true and $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. Then there exists a constant $C_6 > 0$ such that for any $w \in \partial\mathbb{R}_+^d$, $r > 0$, and any non-negative function f in \mathbb{R}_+^d that is harmonic in $\mathbb{R}_+^d \cap B(w, r)$ with respect to Y and vanishes continuously on $\partial\mathbb{R}_+^d \cap B(w, r)$, we have*

$$f(x) \leq C_6 f(\hat{x}) \quad \text{for all } x \in \mathbb{R}_+^d \cap B(w, r/2), \quad (1.9)$$

where $\hat{x} \in \mathbb{R}_+^d \cap B(w, r)$ with $\hat{x}_d \geq r/4$.

Assumptions (A1), (A2), (A3) and (A4) in this paper are assumptions (B1), (B4), (B7) and (B8) in [41], respectively. As a consequence of (A1)–(A4), $\mathcal{B}(x, y)$ also satisfies assumptions (B2), (B3), (B5) and (B6) in [41].

Now we explain the content of this paper and our strategy for proving the main results.

In Section 2 we first show that the process Y is transient and admits a symmetric Green function $G(x, y)$ (see Proposition 2.2). This is quite standard once we establish that the occupation measure $G(x, \cdot)$ of Y is absolutely continuous. We also show that $x \mapsto G(x, y)$ is harmonic away from y . As a consequence of the scaling property of Y and the invariance property of the half-space under scaling, one gets the following scaling property of the Green function: For all $x, y \in \mathbb{R}_+^d$,

$$G(x, y) = |x - y|^{\alpha-d} G\left(\frac{x}{|x - y|}, \frac{y}{|x - y|}\right).$$

In this paper, we use this property several times so that, to prove Theorem 1.1, we mainly deal with the case of $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \asymp 1$.

In Section 3, we show that the Green function $G(x, y)$ tends to 0 when x or y tends to the boundary. The proof of this result depends in a fundamental way on several lemmas from [41]. The decay of the Green function at the boundary allows us to apply Theorem 1.5 in later sections.

Section 4 is devoted to proving interior estimates on the Green function $G(x, y)$. Roughly, we show that if the points $x, y \in \mathbb{R}_+^d$ are closer to each other than to the boundary, then $G(x, y) \asymp |x - y|^{-d+\alpha}$. For the lower bound given in Proposition 4.1, we use a capacity argument. The upper bound is more difficult and relies on the Hardy inequality in [6, Corollary 3] and the heat kernel estimates of symmetric jump processes with large jumps of lower intensity in [2]. This is where the assumption $d > (\alpha + \beta_1 + \beta_2) \wedge 2$ is needed. The key to obtaining the interior upper estimate is to get a uniform estimate on the L^2 norm of $\int_{B(z,4)} G(x, y) dy$ on $B(z, 4)$ for all z sufficiently away from the boundary (see Proposition 4.5).

In Section 5, we give a lower bound for the Green function of the process Y killed upon exiting a half-ball centered at the boundary of \mathbb{R}_+^d and a preliminary upper bound for the Green function. The lower bound given in Theorem 5.1 is proved for $G^{B(w,R) \cap \mathbb{R}_+^d}(x, y)$, the Green function of the process Y killed upon exiting $B(w, R) \cap \mathbb{R}_+^d$, $w \in \partial \mathbb{R}_+^d$, for $x, y \in B(w, (1 - \varepsilon)R) \cap \mathbb{R}_+^d$. This gives the sharp lower bound of the Green function for $p \in ((\alpha - 1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$. A preliminary estimate of the upper bound is given in Lemma 5.5. Proofs of these estimates use the already mentioned fundamental lemmas from [41] and Theorem 1.5.

Section 6 is central to the paper. We first prove a technical Lemma 6.1 modeled after [1, Lemma 3.3] and its Corollary 6.3. They are both used throughout this section. In proving Theorem 1.1, one is led to double integrals involving the Green function (or the Green function of the killed process) twice and the jump kernel. The sharp bounds of these double integrals are essential in the proof of Theorem 1.1. To obtain the correct bound, we have to divide the region of integration into several parts and deal with them separately. These estimates are quite difficult and delicate; see Remark 6.8 below. By using the preliminary estimates of the Green function obtained in Section 5 and the explicit form of \tilde{B} , those integrals are successfully estimated by means of Lemma 6.1 and Corollary 6.3. As an application of the Green function estimates, we end the section with sharp two-sided

estimates on some killed potentials of the process Y , or in analytical language, with estimates of $\int_D G^D(x, y) y_d^\beta dy$ where D is a box of arbitrary size and $\beta > -p - 1$ (see Proposition 6.10 below), as well as estimates of $\int_{\mathbb{R}_+^d} G(x, y) y_d^\beta dy$. The latter estimates give precise information on the expected lifetime of the process Y .

In Section 7 we prove Theorems 1.2 and 1.3. Our Proposition 6.10 is powerful enough for us to cover the full range of the parameters.

We end this introduction by discussing some examples of explicit processes satisfying our assumptions, as well as a process which does not fall in the class considered here.

The first (motivating) example is a subordinate killed stable process Y whose infinitesimal generator is $-((-\Delta)_{\mathbb{R}_+^d}^{\delta/2})^{\gamma/2}$, where $\delta \in (0, 2]$ and $\gamma \in (0, 2)$. Its jump kernel is $J(x, y) = |x - y|^{-d-\alpha} \mathcal{B}(x, y)$ with $\alpha = \gamma\delta/2$ and $\mathcal{B}(x, y)$ satisfying (A3) with parameters as follows: If $\delta = 2$, then $\beta_1 = \beta_2 = 1$, $\beta_3 = \beta_4 = 0$. For $\delta \in (0, 2)$, (i) if $\gamma \in (1, 2)$, then $\beta_1 = \delta(1 - \gamma/2)$, $\beta_2 = \beta_3 = \beta_4 = 0$; (ii) if $\gamma = 1$, then $\beta_1 = \delta/2$, $\beta_3 = 1$, $\beta_2 = \beta_4 = 0$, (iii) if $\gamma \in (0, 1)$, then $\beta_1 = \delta/2$, $\beta_2 = (1 - \gamma)\delta/2$, $\beta_3 = \beta_4 = 0$. For more details see [41, (1.1), (1.2) and Section 2]. In all cases we have $p = \delta/2$, which can be deduced by comparing Green function estimates in Theorem 1.1 and [40, Theorem 6.4].

An example of a process with $\beta_4 > 0$ has been recently discovered in [25]. Let $\delta \in (0, 2)$, and let X be the reflected symmetric δ -stable process in $\bar{\mathbb{R}}_+^d := \{x = (\tilde{x}, x_d) : x_d \geq 0\}$ killed upon leaving \mathbb{R}_+^d , whose infinitesimal generator is the regional fractional Laplacian

$$\mathcal{L}f(x) = c(d, \delta) \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}_+^d, |y-x|>\varepsilon} (f(y) - f(x)) |x - y|^{-d-\delta} dy;$$

see [25, pp. 232–234] for details. Let $q \in [\delta - 1, \delta) \cap (0, \delta)$ and Z be the process corresponding to the the Feynman–Kac semigroup via the multiplicative functional

$$\exp\left(-C(d, \delta, q) \int_0^t (X_s^d)^{-\delta} ds\right),$$

where $C(d, \delta, q)$ is the positive constant (involving parameter q) defined in [24, p. 233]; see also [24, (3.5)]. Let S be an independent $\gamma/2$ -stable subordinator with $\gamma \in (0, 2)$ and set $\alpha = \delta\gamma/2$. Define a process Y by $Y_t = Z_{S_t}$ whose infinitesimal generator is $-(-\mathcal{L} + C(d, \delta, q)(x_d)^{-\delta})^{\gamma/2}$. The jump kernel of Y is of the form $J(x, y) = |x - y|^{-d-\alpha} \mathcal{B}(x, y)$ with $\mathcal{B}(x, y)$ satisfying (A1)–(A4). Moreover, the parameter β_4 in (A3) is equal to 1 for a certain value of q . For details, see [25, Example 7.3] and the paragraph above [41, Lemma 2.2].

The jump kernels of this paper are degenerate since \mathcal{B} approaches 0 at the boundary. There exist processes in \mathbb{R}_+^d whose jump kernels are of the form $|x - y|^{-d-\alpha} \mathcal{B}(x, y)$ with \mathcal{B} blowing up at the boundary. Here is an example. Let X be an isotopic α -stable process, and let Y be the process obtained from X by deleting the parts of the path outside \mathbb{R}_+^d . More precisely, let

$$A_t = \int_0^t 1_{\{X_s \in \mathbb{R}_+^d\}} ds,$$

be the occupation time in \mathbb{R}_+^d up to time t and let $\gamma_t = \inf\{s > 0 : A_s > t\}$. The process Y defined by $Y_t = X_{\gamma_t}$ is the trace of X in \mathbb{R}_+^d . It is called the path-censored α -stable process in [43]. Using [7, Theorem 6.1], one can show that the jump kernel of Y is of the form $|x - y|^{-d-\alpha} \mathcal{B}(x, y)$ with \mathcal{B} blowing up at the boundary.

Notation. Throughout this paper, the positive constants $\beta_1, \beta_2, \beta_3, \beta_4, \theta$ will remain the same. We will use the following convention: Capital letters $C, C_i, i = 1, 2, \dots$, will denote constants in the statements of results and assumptions. The labeling of these constants will remain the same. Lower case letters $c, c_i, i = 1, 2, \dots$, are used to denote constants in the proofs and the labeling of these constants starts anew in each proof. The notation $c_i = c_i(a, b, c, \dots)$, $i = 0, 1, 2, \dots$, indicates constants depending on a, b, c, \dots . We will use “:=” to mean “is defined to be”. For any $x \in \mathbb{R}^d$ and $r > 0$, we use $B(x, r)$ to denote the open ball of radius r centered at x . For a Borel subset V in \mathbb{R}^d , $|V|$ denotes the Lebesgue measure of V in \mathbb{R}^d , and $\delta_V := \text{dist}(V, \partial D)$. We use superscripts instead of subscripts for coordinates of processes, as in $Y = (Y^1, \dots, Y^d)$.

2. Existence of the Green function

Recall that ζ is the lifetime of Y . Let $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ be a Borel function and $\lambda \geq 0$. The λ -potential of f is defined by

$$G_\lambda f(x) := \mathbb{E}_x \int_0^\zeta e^{-\lambda t} f(Y_t) dt, \quad x \in \mathbb{R}_+^d.$$

When $\lambda = 0$, we write Gf instead of $G_0 f$ and call Gf the *Green potential* of f . If $g : \mathbb{R}_+^d \rightarrow [0, \infty)$ is another Borel function, then by the symmetry of Y we have

$$\int_{\mathbb{R}_+^d} G_\lambda f(x) g(x) dx = \int_{\mathbb{R}_+^d} f(x) G_\lambda g(x) dx. \quad (2.1)$$

For $A \in \mathcal{B}(\mathbb{R}_+^d)$, we let $G_\lambda(x, A) := G_\lambda \mathbf{1}_A(x)$ be the λ -occupation measure of A . In this section we show the existence of the Green function of the process Y , that is, the density of the 0-occupation measure. We start by recalling some of the results of [41, Section 3.1].

Let U be a relatively compact $C^{1,1}$ open subset of \mathbb{R}_+^d . For $\gamma > 0$ small enough, define a kernel $J_\gamma(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ by $J_\gamma(x, y) = J(x, y)$ for $x, y \in U$, and $J_\gamma(x, y) = \gamma|x - y|^{-d-\alpha}$ otherwise. Then there exist $c_1 > 0$ and $c_2 > 0$ such that (see the first display below [41, (3.3)])

$$c_1|x - y|^{-d-\alpha} \leq J_\gamma(x, y) \leq c_2|x - y|^{-d-\alpha}, \quad x, y \in \mathbb{R}^d.$$

For $u \in L^2(\mathbb{R}^d, dx)$, define

$$\mathcal{C}(u, u) := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))^2 J_\gamma(x, y) dx dy,$$

$$\mathcal{D}(\mathcal{C}) := \{u \in L^2(\mathbb{R}^d) : \mathcal{C}(u, u) < \infty\}.$$

Then there exists a conservative Feller and strongly Feller process Z associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ which has a continuous transition density (with respect to the Lebesgue measure) (see [21]). Let Z^U be the process Z killed upon exiting U and let $A_t := \int_0^t \tilde{\kappa}(Z_s^U) ds$ where $\tilde{\kappa}$ is a certain non-negative function defined in [41, Section 3.1] ($\tilde{\kappa}$ is non-negative when $\gamma > 0$ is small enough). Let Y^U be the process Y killed upon exiting U , and let $(Q_t^U)_{t \geq 0}$ denote its semigroup: For $f : U \rightarrow [0, \infty)$,

$$Q_t^U f(x) := \mathbb{E}_x[f(Y_t^U)] = \mathbb{E}_x[f(Y_t), t < \tau_U],$$

where $\tau_U := \inf\{t > 0 : Y_t \notin U\}$ is the first exit time from U . It is shown in [41, Section 3.1] that

$$Q_t^U f(x) = \mathbb{E}_x[\exp(-A_t)f(Z_t^U)], \quad t > 0, x \in U.$$

Moreover, Q_t^U has a transition density $q^U(t, x, y)$ (with respect to the Lebesgue measure) which is symmetric in x and y , and such that for all $y \in U$, $(t, x) \mapsto q^U(t, x, y)$ is continuous.

Let $G_\lambda^U f(x) := \int_0^\infty e^{-\lambda t} Q_t^U f(x) dt = \mathbb{E}_x \int_0^{\tau_U} e^{-\lambda t} f(Y_t) dt$ denote the λ -potential of Y^U and $G_\lambda^U(x, y) := \int_0^\infty e^{-\lambda t} q^U(t, x, y) dt$ the λ -potential density of Y^U . We will write G^U for G_0^U for simplicity. Then $G_\lambda^U(x, \cdot)$ is the density of the λ -occupation measure. In particular this shows that $G_\lambda^U(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure. Moreover, since $x \mapsto q^U(t, x, y)$ is continuous, we see that $x \mapsto G_\lambda^U(x, y)$ is lower semicontinuous. By Fatou's lemma this implies that $G_\lambda^U f$ is also lower semicontinuous.

Let $(U_n)_{n \geq 1}$ be a sequence of bounded $C^{1,1}$ open sets such that $U_n \subset \overline{U_n} \subset U_{n+1}$ and $\cup_{n \geq 1} U_n = \mathbb{R}_+^d$. For any Borel $f : \mathbb{R}_+^d \rightarrow [0, \infty)$,

$$G_\lambda f(x) = \mathbb{E}_x \int_0^\xi e^{-\lambda t} f(Y_t) dt = \uparrow \lim_{n \rightarrow \infty} \mathbb{E}_x \int_0^{\tau_{U_n}} e^{-\lambda t} f(Y_t) dt = \uparrow \lim_{n \rightarrow \infty} G_\lambda^{U_n} f(x), \quad (2.2)$$

where $\uparrow \lim$ denotes an increasing limit. In particular, if $A \in \mathcal{B}(\mathbb{R}_+^d)$ is of Lebesgue measure zero, then for every $x \in \mathbb{R}_+^d$,

$$G_\lambda(x, A) = \lim_{n \rightarrow \infty} G_\lambda^{U_n}(x, A) = \lim_{n \rightarrow \infty} G_\lambda^{U_n}(x, A \cap U_n) = 0.$$

Thus, $G_\lambda(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure for each $\lambda \geq 0$ and $x \in \mathbb{R}_+^d$. Together with (2.1) this shows that the conditions of [3, VI, Theorem (1.4)] are satisfied, which implies that the resolvent $(G_\lambda)_{\lambda > 0}$ is self-dual. In particular (see [3, pp. 256–257]), there exists a symmetric function $G(x, y)$ excessive in both variables such that

$$Gf(x) = \int_{\mathbb{R}_+^d} G(x, y) f(y) dy, \quad x \in \mathbb{R}_+^d.$$

We recall (see [3, II, Definition (2.1)]) that a measurable function $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ is λ -excessive, $\lambda \geq 0$, with respect to the process Y if for every $t \geq 0$ we have $\mathbb{E}_x[e^{-\lambda t} Y_t] \leq f(x)$ and $\lim_{t \rightarrow 0} \mathbb{E}_x[e^{-\lambda t} Y_t] = f(x)$, for every $x \in \mathbb{R}_+^d$; 0-excessive functions are simply called excessive.

We now show that Y is transient.

Lemma 2.1. *The process Y is transient in the sense that there exists $f : \mathbb{R}_+^d \rightarrow (0, \infty)$ such that $Gf < \infty$. More precisely, $G\kappa \leq 1$.*

Proof. Let $(Q_t)_{t \geq 0}$ denote the semigroup of Y . For any $A \in \mathcal{B}(\mathbb{R}_+^d)$, we use [31, (4.5.6)] with $h = \mathbf{1}_A$, $f = 1$, and let $t \rightarrow \infty$ to obtain

$$\mathbb{E}_{\mathbf{1}_A dx}(\zeta < \infty) \geq \mathbb{E}_{\mathbf{1}_A dx}(Y_{\zeta-} \in \mathbb{R}_+^d, \zeta < \infty) = \int_0^\infty \int_{\mathbb{R}_+^d} \kappa(x) Q_s \mathbf{1}_A(x) dx dt.$$

This can be rewritten as

$$\int_A \mathbb{P}_x(\zeta < \infty) dx \geq \int_{\mathbb{R}_+^d} \kappa(x) G \mathbf{1}_A(x) dx = \int_A G\kappa(x) dx.$$

Since this inequality holds for every $A \in \mathcal{B}(\mathbb{R}_+^d)$, we conclude that $\mathbb{P}_x(\zeta < \infty) \geq G\kappa(x)$ for a.e. $x \in \mathbb{R}_+^d$. Both functions $x \mapsto \mathbb{P}_x(\zeta < \infty)$ and $G\kappa$ are excessive. Since $G(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure (i.e., Hypothesis (L) holds, see [26, p. 112]), by [26, Proposition 9, p. 113] we conclude that $G\kappa(x) \leq \mathbb{P}_x(\zeta < \infty) \leq 1$ for all $x \in \mathbb{R}_+^d$. ■

As a consequence of Lemma 2.1, we have $G(x, y) < \infty$ for a.e. $y \in \mathbb{R}_+^d$. Another consequence is that, for every compact $K \subset \mathbb{R}_+^d$, $G \mathbf{1}_K$ is bounded. Indeed, by the definition of κ , we see that $\inf_K \kappa(x) =: c_K > 0$. Thus

$$G \mathbf{1}_K \leq c_K^{-1} G\kappa \leq c_K^{-1}. \quad (2.3)$$

Note that it follows from (2.2) that, for every non-negative Borel f , $G_\lambda f$ is lower semicontinuous, as an increasing limit of lower semicontinuous functions. Since every λ -excessive function is an increasing limit of λ -potentials (see [3, II, Proposition (2.6)]), we conclude that all λ -excessive functions of Y are lower semicontinuous. In particular, for every $y \in \mathbb{R}_+^d$, $G_\lambda(\cdot, y)$ is lower semicontinuous. Since $G(\cdot, y)$ is the increasing limit of $G_\lambda(\cdot, y)$ as $\lambda \rightarrow 0$, we see that $G(\cdot, y)$ is also lower semicontinuous.

Fix an open set B in \mathbb{R}_+^d and $x \in \mathbb{R}_+^d$ and let f be a non-negative Borel function on \mathbb{R}_+^d . By Hunt's switching identity [3, VI, Theorem (1.16)],

$$\mathbb{E}_x[Gf(Y_{\tau_B})] = \int_{\mathbb{R}_+^d} \mathbb{E}_x[G(Y_{\tau_B}, y)]f(y) dy = \int_{\mathbb{R}_+^d} \mathbb{E}_y[G(x, Y_{\tau_B})]f(y) dy.$$

Suppose further that $f = 0$ on B . Then by the strong Markov property [3, I, Definition (8.1)],

$$\begin{aligned} \int_{\mathbb{R}_+^d} G(x, y)f(y) dy &= \mathbb{E}_x \int_{\tau_B}^\infty f(Y_t) dt = \mathbb{E}_x[Gf(Y_{\tau_B})] \\ &= \int_{\mathbb{R}_+^d \setminus B} \mathbb{E}_y[G(x, Y_{\tau_B})]f(y) dy, \end{aligned}$$

and hence $G(x, y) = \mathbb{E}_y[G(x, Y_{\tau_B})]$ for a.e. $y \in \mathbb{R}_+^d \setminus B$. Since both sides are excessive (and thus excessive for the killed process $Y^{\mathbb{R}_+^d \setminus B}$), equality holds for every $y \in \mathbb{R}_+^d \setminus B$. By using Hunt's switching identity one more time, we arrive at

$$G(x, y) = \mathbb{E}_x[G(Y_{\tau_B}, y)] \quad \text{for all } x \in \mathbb{R}_+^d, y \in \mathbb{R}_+^d \setminus B.$$

In particular, if $y \in \mathbb{R}_+^d \setminus B$ is fixed, then the above equality says that $x \mapsto G(x, y)$ is regular harmonic in B with respect to Y . By symmetry, $y \mapsto G(x, y)$ is regular harmonic in B as well. By the Harnack inequality (Theorem 1.4), we conclude that $G(x, y) < \infty$ for all $y \in \mathbb{R}_+^d \setminus \{x\}$. This proves the following proposition on the existence of the Green function.

Proposition 2.2. *There exists a symmetric function $G : \mathbb{R}_+^d \times \mathbb{R}_+^d \rightarrow [0, \infty]$ which is lower semicontinuous in each variable and finite outside the diagonal such that for every non-negative Borel f ,*

$$Gf(x) = \int_{\mathbb{R}_+^d} G(x, y) f(y) dy.$$

Moreover, $G(x, \cdot)$ is harmonic with respect to Y in $\mathbb{R}_+^d \setminus \{x\}$ and regular harmonic with respect to Y in $\mathbb{R}_+^d \setminus B(x, \varepsilon)$ for any $\varepsilon > 0$.

Remark 2.3. We note in passing that all the results established above are valid, with the same proofs, for the process in any open set D (not necessarily \mathbb{R}_+^d), under conditions (1.3)–(1.6) and **(B1)**–**(B3)** from [41]. In particular, in the setup of [41], the process in any open set D studied there has a symmetric Green function.

For further use, we recall now the formula for the Green function of the process Y killed upon exiting an open set $B \subset \mathbb{R}_+^d$. Let $f : \mathbb{R}_+^d \rightarrow [0, \infty]$ be a measurable function vanishing on $\mathbb{R}_+^d \setminus B$. By the strong Markov property, for $x \in B$,

$$\begin{aligned} \int_{\mathbb{R}_+^d} G(x, y) f(y) dy &= \mathbb{E}_x \int_0^\infty f(Y_s) ds \\ &= \mathbb{E}_x \int_0^{\tau_B} f(Y_s) ds + \mathbb{E}_x \left(\mathbb{E}_{Y_{\tau_B}} \int_0^\infty f(Y_s) ds \right) \\ &= \mathbb{E}_x \int_0^\infty f(Y_s^B) ds + \mathbb{E}_x Gf(Y_{\tau_B}) \\ &= \mathbb{E}_x \int_0^\infty f(Y_s^B) ds + \int_{\mathbb{R}_+^d} \mathbb{E}_x[G(Y_{\tau_B}, y)] f(y) dy. \end{aligned}$$

By rearranging, we see that

$$G^B(x, y) := G(x, y) - \mathbb{E}_x[G(Y_{\tau_B}, y)] \tag{2.4}$$

is the Green function of Y^B .

We end this section with the scaling property of the Green function, which will be used several times in this paper.

Proposition 2.4. For all $x, y \in \mathbb{R}_+^d$, $x \neq y$,

$$G(x, y) = G\left(\frac{x}{|x-y|}, \frac{y}{|x-y|}\right) |x-y|^{\alpha-d}. \quad (2.5)$$

Proof. Let $r > 0$ and $Y_t^{(r)} := rY_{r^{-\alpha}t}$. Let $(\mathcal{E}^{(r)}, \mathcal{D}(\mathcal{E}^{(r)}))$ be the Dirichlet form of $Y^{(r)}$. It was shown in the proof of [41, Lemma 5.1] that $\mathcal{E}^{(r)}(f, g) = \mathcal{E}(f, g)$ for $f, g \in C_c^\infty(\mathbb{R}_+^d)$. Since $\mathcal{E}(Gf, g) = \int_{\mathbb{R}_+^d} f(x)g(x) dx$, we see that Gf is the 0-potential operator of $Y^{(r)}$.

In particular, $G^{(r)}(x, y) := G(x, y)$ is the Green function of $Y^{(r)}$.

Let (Q_t) be the semigroup of Y and $(Q_t^{(r)})$ the semigroup of $Y^{(r)}$. For $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ define $f^{(r)}(x) := f(rx)$. Then $Q_t^{(r)} f(x) = Q_{r^{-\alpha}t} f^{(r)}(x/r)$, implying that

$$\begin{aligned} G^{(r)} f(x) &= \int_0^\infty Q_t^{(r)} f(x) dt = \int_0^\infty Q_{r^{-\alpha}t} f^{(r)}(x/r) dt = r^\alpha \int_0^\infty Q_s f^{(r)}(x/r) ds \\ &= r^\alpha Gf^{(r)}(x/r). \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}_+^d} G(x, y) f(y) dy &= Gf(x) = r^\alpha Gf^{(r)}(x/r) = r^\alpha \int_{\mathbb{R}_+^d} G(x/r, y) f^{(r)}(y) dy \\ &= r^{\alpha-d} \int_{\mathbb{R}_+^d} G(x/r, z/r) f^{(r)}(z/r) dz = r^{\alpha-d} \int_{\mathbb{R}_+^d} G(x/r, y/r) f(y) dy. \end{aligned}$$

This implies that for every $x \in \mathbb{R}_+^d$, $G(x, y) = r^{\alpha-d} G(x/r, y/r)$ for a.e. y .

Note that since $(Y_t, \mathbb{P}_x) \stackrel{d}{=} (Y^{(r)}_t, \mathbb{P}_{x/r})$, the processes Y and $Y^{(r)}$ have the same excessive functions. Thus, if f is excessive for Y , it is also excessive for $Y^{(r)}$ and therefore $Q_{r^{-\alpha}t} f^{(r)} f(x/r) = Q_t^{(r)} f(x) \uparrow f(x)$ as $t \rightarrow 0$. Thus we also have $Q_t f^{(r)} f(y) \uparrow f(y) = f^{(r)}(y)$ as $t \rightarrow 0$, proving that $f^{(r)}$ is also excessive for Y . In particular, for every $x \in \mathbb{R}^d$, $y \mapsto r^{\alpha-d} G(x/r, y/r)$ is excessive for Y . Since this function is for a.e. y equal to the excessive function $y \mapsto G(x, y)$, it follows that they are equal everywhere. Thus for all $x, y \in \mathbb{R}_+^d$,

$$G(x, y) = r^{\alpha-d} G(x/r, y/r).$$

By taking $r = |x - y|$ we obtain (2.5). ■

3. Decay of the Green function

The goal of this section is to show that the Green function $G(x, y)$ vanishes at the boundary of \mathbb{R}_+^d . Recall that for $a, b > 0$ and $\tilde{w} \in \mathbb{R}^{d-1}$,

$$D_{\tilde{w}}(a, b) = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : |\tilde{x} - \tilde{w}| < a, 0 < x_d < b\}.$$

Due to (A4), without loss of generality, we mainly deal with the case $\tilde{w} = \tilde{0}$. We will write $D(a, b)$ for $D_{\tilde{0}}(a, b)$ and, for $r > 0$, $U(r) = D_{\tilde{0}}(r/2, r/2)$. Further we write U for $U(1)$.

In several places below we will need the following upper bound for $\mathcal{B}(x, y)$ proved in [41, Lemma 5.2 (a)]: There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \geq x_d$,

$$\mathcal{B}(x, y) \leq C x_d^{\beta_1} (|\log x_d|^{\beta_3} \vee 1) (1 + \mathbf{1}_{|y| \geq 1} (\log |y|)^{\beta_3}) |x - y|^{-\beta_1}. \quad (3.1)$$

We now recall three key lemmas from [41]. Recall that $Y_t = (Y_t^1, \dots, Y_t^d)$.

Lemma 3.1 ([41, Lemma 5.7]). *For all $x \in U$,*

$$\mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \leq x_d^p.$$

In the next two lemmas, we have used the scaling property of Y .

Lemma 3.2 ([41, Lemma 5.10]). *There exists $C_7 \in (0, 1)$ such that for all $r > 0$ and all $x = (\bar{0}, x_d) \in D(r/8, r/8)$,*

$$\mathbb{P}_x(Y_{\tau_{D(r/4, r/4)}} \in D(r/4, r) \setminus D(r/4, 3r/4)) \geq C_7(x_d/r)^p.$$

Lemma 3.3 ([41, Lemma 6.2]). *There exists $C_8 > 0$ such that for all $r > 0$ and all x in $D(2^{-5}r, 2^{-5}r)$,*

$$\mathbb{P}_x(Y_{\tau_{U(r)}} \in D(r, r)) \leq C_8(x_d/r)^p.$$

The Lévy system formula (see [31, Theorem 5.3.1] and the arguments in [20, p. 40]) states that for any non-negative Borel function F on $\mathbb{R}_+^d \times \mathbb{R}_+^d$ vanishing on the diagonal and any stopping time T ,

$$\mathbb{E}_x \sum_{s \leq T} F(Y_{s-}, Y_s) = \mathbb{E}_x \left(\int_0^T \int_{\mathbb{R}_+^d} F(Y_s, y) J(Y_s, y) dy ds \right), \quad x \in \mathbb{R}_+^d. \quad (3.2)$$

Here $Y_{s-} = \lim_{t \uparrow s} Y_t$ denotes the left limit of the process Y at time $s > 0$. We will use (3.2) in the following form: Let $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ be a Borel function, and let V, W be two Borel subsets of \mathbb{R}_+^d with disjoint closures. If $F(x, y) := \mathbf{1}_V(x) \mathbf{1}_W(y) f(y)$ and $T = \tau_V$, then (3.2) reads

$$\begin{aligned} \mathbb{E}_x[f(Y_{\tau_V}), Y_{\tau_V} \in W] &= \mathbb{E}_x \sum_{s \leq \tau_V} \mathbf{1}_V(Y_{s-}) \mathbf{1}_W(Y_s) f(Y_s) \\ &= \mathbb{E}_x \int_0^{\tau_V} \int_{\mathbb{R}_+^d} \mathbf{1}_V(Y_s) \mathbf{1}_W(y) f(y) J(Y_s, y) dy ds = \mathbb{E}_x \int_0^{\tau_V} \int_W f(y) J(Y_s, y) dy ds \end{aligned} \quad (3.3)$$

$$= \int_V G^V(x, z) \int_W f(y) J(z, y) dy dz. \quad (3.4)$$

The last line follows from the formula for the Green potential already described in Section 2.

The following lemma is an improvement of Lemma 3.3, since \mathbb{R}_+^d is a larger set than any $D(r, r)$.

Lemma 3.4. *There exists $C_9 > 0$ such that for all $r > 0$ and $x \in D(2^{-5}r, 2^{-5}r)$ we have*

$$\mathbb{P}_x(Y_{\tau_U(r)} \in \mathbb{R}_+^d) \leq C_9(x_d/r)^p. \quad (3.5)$$

Proof. By scaling, it suffices to prove (3.5) for $r = 1$. Let $U = U(1)$ and $D = D(1, 1)$. By Lemma 3.3 we only need to show that $\mathbb{P}_x(Y_{\tau_U} \in \mathbb{R}_+^d \setminus D) \leq c_1 x_d^p$ for some $c_1 > 0$. By using (3.3) (with $f \equiv 1$) in the second line and (3.1) in the third,

$$\begin{aligned} & \mathbb{P}_x(Y_{\tau_U} \in \mathbb{R}_+^d \setminus D) \\ &= \mathbb{E}_x \int_0^{\tau_U} \int_{\mathbb{R}_+^d \setminus D} J(w, Y_t) dw dt \\ &\leq c_2 \mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \int_{\mathbb{R}_+^d \setminus D} \frac{1 + \mathbf{1}_{|w|>1} (\log |w|)^{\beta_3}}{|w|^{d+\alpha+\beta_1}} dw. \end{aligned}$$

Since

$$\int_{\mathbb{R}_+^d \setminus D} \frac{1 + \mathbf{1}_{|w|>1} (\log |w|)^{\beta_3}}{|w|^{d+\alpha+\beta_1}} dw < \infty,$$

it follows from Lemma 3.1 that $\mathbb{P}_x(Y_{\tau_U} \in \mathbb{R}_+^d \setminus D) \leq c_3 x_d^p$. \blacksquare

The next result allows us to apply Theorem 1.5 to get Proposition 4.7, which is a key for us to get sharp two-sided Green function estimates.

Theorem 3.5. *For each $y \in \mathbb{R}_+^d$, $\lim_{x_d \rightarrow 0} G(x, y) = 0$.*

Proof. By translation invariance it suffices to show that $\lim_{|x| \rightarrow 0} G(x, y) = 0$. We fix $y \in \mathbb{R}_+^d$ and consider $x \in \mathbb{R}_+^d$ with $|x| < 2^{-10}y_d$. Let $B_1 = B(y, y_d/2)$ and $B_2 = B(y, y_d/4)$. For $z \in B_1$ we have $z_d \geq y_d/2$ so that $|z - y| \leq y_d/2 \leq z_d$. Moreover, $|z - x| \geq y_d/2 - x_d \geq (7/16)y_d$. Thus, by the regular harmonicity of $G(\cdot, y)$ (see Proposition 2.2),

$$\begin{aligned} G(x, y) &= \mathbb{E}_x[G(Y_{T_{B_1}}, y); Y_{T_{B_1}} \in B_1 \setminus B_2] + \mathbb{E}_x[G(Y_{T_{B_1}}, y); Y_{T_{B_1}} \in B_2] \\ &=: I_1 + I_2, \end{aligned} \quad (3.6)$$

where, for any $V \subset \mathbb{R}_+^d$, $T_V := \inf \{t > 0 : Y_t \in V\}$. By the Harnack inequality and Lemma 2.1,

$$\begin{aligned} \sup_{z \in B_1 \setminus B_2} G(z, y) &\leq \frac{c_1}{|B_1 \setminus B_2|} \int_{B_1 \setminus B_2} G(z, y) dz \leq c_2 \frac{y_d^\alpha}{y_d^d} \int_{B_1 \setminus B_2} G(y, z) \kappa(z) dz \\ &\leq c_2 y_d^{\alpha-d} G\kappa(y) \leq c_2 y_d^{\alpha-d}. \end{aligned}$$

In the second inequality we have used the definition of κ in (1.4), the fact that $z_d \asymp y_d$ in $B_1 \setminus B_2$, and that $|B_1 \setminus B_2| \asymp y_d^d$. Now we have

$$I_1 \leq \sup_{z \in B_1 \setminus B_2} G(z, y) \mathbb{P}_x(Y_{T_{B_1}} \in B_1 \setminus B_2) \leq \frac{c_2}{y^{d-\alpha}} \mathbb{P}_x(Y_{T_{B_1}} \in B_1 \setminus B_2).$$

Further, it is easy to check that $J(w, z) \asymp J(w, y)$ for all $w \in \mathbb{R}_+^d \setminus B_1$ and $z \in B_2$. Moreover, by Lemma 2.1,

$$\int_{B_2} G(y, z) dz \leq c_3 y_d^\alpha \int_{B_2} G(y, z) \kappa(z) dz \leq c_3 y_d^\alpha G \kappa(y) \leq c_3 y_d^\alpha.$$

Therefore, by (3.3) (with $f = G(\cdot, y)$) in the first line,

$$\begin{aligned} I_2 &= \mathbb{E}_x \int_0^{T_{B_1}} \int_{B_2} J(Y_t, z) G(z, y) dz dt \\ &\leq c_4 \mathbb{E}_x \int_0^{T_{B_1}} J(Y_t, y) y_d^\alpha dt \leq c_5 y_d^\alpha \mathbb{E}_x \int_0^{T_{B_1}} \left(\frac{1}{|B_2|} \int_{B_2} J(Y_t, z) dz \right) dt \\ &= \frac{c_6}{y_d^{d-\alpha}} \mathbb{P}_x(Y_{T_{B_1}} \in B_2). \end{aligned}$$

Inserting the estimates for I_1 and I_2 into (3.6) and using Lemma 3.4 we get

$$G(x, y) \leq \frac{c_7}{y_d^{d-\alpha}} \mathbb{P}_x(Y_{T_{B_1}} \in \mathbb{R}_+^d) \leq \frac{c_7}{y_d^{d-\alpha}} \mathbb{P}_x(Y_{\tau_U(y_d/4)} \in \mathbb{R}_+^d) \leq \frac{c_8}{y_d^{d-\alpha-p}} x_d^p,$$

which implies the claim. \blacksquare

4. Interior estimate of Green functions

4.1. Lower bound

We first use a capacity argument to show that there exists $c > 0$ such that $G(x, y) \geq c$ for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| = 1$ and $x_d \wedge y_d \geq 10$. For such x and y , let $U = B(x, 5)$, $V = B(x, 3)$ and $W_y = B(y, 1/2)$. Recall that, for any $W \subset \mathbb{R}_+^d$, $T_W = \inf\{t > 0 : Y_t \in W\}$. By the Krylov–Safonov type estimate [41, Lemma 3.12], there exists a constant $c_1 > 0$ such that

$$\mathbb{P}_x(T_{W_y} < \tau_U) \geq c_1 \frac{|W_y|}{|U|} = c_2 > 0. \quad (4.1)$$

Recall that Y^U is the process Y killed upon exiting U and $G^U(\cdot, \cdot)$ is the Green function of Y^U . The Dirichlet form of Y^U is $(\mathcal{E}, \mathcal{F}_U)$, where

$$\begin{aligned} \mathcal{E}(u, v) &:= \frac{1}{2} \int_U \int_U (u(x) - u(y))(v(x) - v(y)) J(x, y) dy dx + \int_U u(x)^2 \kappa_U(x) dx, \\ \kappa_U(x) &:= \int_{\mathbb{R}_+^d \setminus U} J(x, y) dy + \kappa(x), \quad x \in U, \end{aligned} \quad (4.2)$$

and $\mathcal{F}_U := \{u \in \mathcal{F} : u = 0 \text{ q.e. on } \mathbb{R}_+^d \setminus U\}$. Here q.e. means quasi-everywhere, that is, except on a set of capacity zero with respect to Y . Let μ be the capacitary measure of W_y with respect to Y^U (i.e., with respect to the corresponding Dirichlet form). Then μ is

concentrated on $\overline{W_y}$, $\mu(U) = \text{Cap}^{Y^U}(W_y)$ and $\mathbb{P}_x(T_{W_y} < \tau_U) = G^U \mu(x)$. By (4.1) and applying Theorem 1.4 (Harnack inequality) to the function $G(x, \cdot)$, we get

$$\begin{aligned} c_2 &\leq \mathbb{P}_x(T_{W_y} < \tau_U) = G^U \mu(x) = \int_U G^U(x, z) \mu(dz) \leq \int_U G(x, z) \mu(dz) \\ &\leq c_3 G(x, y) \mu(U) = c_3 G(x, y) \text{Cap}^{Y^U}(W_y). \end{aligned} \quad (4.3)$$

Let X be the isotropic α -stable process in \mathbb{R}^d with the jump kernel $j(x, y) := |x - y|^{-d-\alpha}$. For $u, v : \mathbb{R}^d \rightarrow \mathbb{R}$, let

$$\begin{aligned} \mathcal{Q}(u, v) &:= \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) j(|x - y|) dy dx, \\ \mathcal{D}(\mathcal{Q}) &:= \{u \in L^2(\mathbb{R}^d, dx) : \mathcal{Q}(u, u) < \infty\}. \end{aligned}$$

Then $(\mathcal{Q}, \mathcal{D}(\mathcal{Q}))$ is the regular Dirichlet form corresponding to X . Let X^U denote the part of the process X in U . The Dirichlet form of X^U is $(\mathcal{Q}, \mathcal{D}_U(\mathcal{Q}))$, where

$$\begin{aligned} \mathcal{Q}^U(u, v) &:= \frac{1}{2} \int_U \int_U (u(x) - u(y))(v(x) - v(y)) j(|x - y|) dy dx \\ &\quad + \int_U u(x)^2 \kappa_U^X(x) dx, \\ \kappa_U^X(x) &:= \int_{\mathbb{R}^d \setminus U} j(|x - y|) dy, \quad x \in U, \end{aligned}$$

and $\mathcal{D}_U(\mathcal{Q}) := \{u \in \mathcal{D}(\mathcal{Q}) : u = 0 \text{ q.e. on } \mathbb{R}^d \setminus U\}$. Using calculations similar to those in [41, p. 13], one can show that $\kappa_U(x) \asymp \kappa_U^X(x)$ for $x \in U$. Thus, there exists $c_4 > 0$ such that $\mathcal{E}(u, u) \leq c_4 \mathcal{Q}^U(u, u)$ for all $u \in C_c^\infty(U)$ which is a core for both $(\mathcal{Q}, \mathcal{D}_U(\mathcal{Q}))$ and $(\mathcal{E}, \mathcal{F}_U)$. This implies that

$$\text{Cap}^{Y^U}(W_y) \leq c_4 \text{Cap}^{X^U}(W_y) \leq c_4 \text{Cap}^{X^U}(V).$$

The last term, $\text{Cap}^{X^U}(V)$, the capacity of V with respect to X^U , is just a number, say c_5 , depending only on the radii of V and U . Hence, $\text{Cap}^{Y^U}(W_y) \leq c_4 c_5$. Inserting this in (4.3), we get

$$G(x, y) \geq c_2 c_3^{-1} c_4^{-1} c_5^{-1}.$$

Combining this with the Harnack inequality (Theorem 1.4) and (2.5), we immediately get

Proposition 4.1. *For any $C_{10} > 0$, there exists a constant $C_{11} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \leq C_{10}(x_d \wedge y_d)$,*

$$G(x, y) \geq C_{11} |x - y|^{-d+\alpha}.$$

Proof. We have shown above that there is $c_1 > 0$ such that $G(z, w) \geq c_1$ for all $z, w \in \mathbb{R}_+^d$ with $|z - w| = 1$ and $z_d \wedge w_d \geq 10$. By the Harnack inequality (Theorem 1.4), there exists $c_2 > 0$ such that $G(z, w) \geq c_2$ for all $z, w \in \mathbb{R}_+^d$ with $|z - w| = 1$ and $z_d \wedge w_d > C_{10}^{-1}$.

Now let $x, y \in \mathbb{R}_+^d$ satisfy $|x - y| \leq C_{10}(x_d \wedge y_d)$ and set

$$x^{(0)} = \frac{x}{|x - y|}, \quad y^{(0)} = \frac{y}{|x - y|}.$$

Then $|x^{(0)} - y^{(0)}| = 1$ and $x_d^{(0)} \wedge y_d^{(0)} > C_{10}^{-1}$ so that $G(x^{(0)}, y^{(0)}) \geq c_2$. By scaling (Proposition 2.4),

$$G(x, y) = G(x^{(0)}, y^{(0)})|x - y|^{\alpha-d} \geq \frac{c_2}{|x - y|^{d-\alpha}}. \quad \blacksquare$$

As a corollary of the lower bound above we see that for every $x \in \mathbb{R}_+^d$,

$$\lim_{y \rightarrow x} G(x, y) = +\infty.$$

4.2. Upper bound

The purpose of this subsection is to establish the interior upper bound on the Green function G (Proposition 4.6). By (2.5) and the Harnack inequality (Theorem 1.4), it suffices to deal with $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$ and $x_d = y_d > 10$.

We fix now two points $x^{(0)}$ and $y^{(0)}$ in \mathbb{R}_+^d such that $|x^{(0)} - y^{(0)}| = 1$, $x_d^{(0)} = y_d^{(0)} > 10$ and $\widetilde{x^{(0)}} = \widetilde{0}$. Let $E = B(x^{(0)}, 1/4)$, $F = B(y^{(0)}, 1/4)$ and $D = B(x^{(0)}, 4)$. Let $f = G\mathbf{1}_E$ and $u = G\mathbf{1}_D$. Since $z \mapsto G(y^{(0)}, z)$ is harmonic in $B(x^{(0)}, 1/2)$ with respect to Y and f is harmonic in $B(y^{(0)}, 1/2)$ with respect to Y , by applying the Harnack inequality (Theorem 1.4) to f and $z \mapsto G(y^{(0)}, z)$, we get

$$f(y^{(0)}) = \int_E G(y^{(0)}, z) dz \geq c|E|G(y^{(0)}, x^{(0)}) \quad \text{and} \quad \int_F f(y)^2 dy \geq c|F|f(y^{(0)})^2.$$

Thus, using the symmetry of G , we obtain

$$G(x^{(0)}, y^{(0)}) \leq \frac{c}{|E|} f(y^{(0)}) \leq \frac{c}{|E|} \left(\frac{c}{|F|} \int_F f(y)^2 dy \right)^{1/2} \leq \frac{c^{3/2}}{|E|^{3/2}} \|u\|_{L^2(D)}, \quad (4.4)$$

for some constant $c > 0$. The key is to get uniform estimate on the L^2 norm of $u = G\mathbf{1}_D$ (see Proposition 4.5). To get the desired uniform estimate, we will use the Hardy inequality in [6, Corollary 3] and the heat kernel estimates of symmetric jump processes with large jumps of lower intensity in [2].

By (A3), we have

$$\mathcal{B}(x, y) \geq c_1 \begin{cases} |x - y|^{-\beta_1 - \beta_2} & \text{if } |x - y| \geq 1 \text{ and } x_d \wedge y_d \geq 1, \\ 1 & \text{if } |x - y| < 1 \text{ and } x_d \wedge y_d \geq 1. \end{cases} \quad (4.5)$$

Define

$$\phi(r) := r^\alpha \mathbf{1}_{r < 1} + r^{\alpha + \beta_1 + \beta_2} \mathbf{1}_{r \geq 1} \quad \text{and} \quad \Phi(r) := \frac{r^2}{\int_0^r \frac{s}{\phi(s)} ds}.$$

Let $\bar{\beta} := (\alpha + \beta_1 + \beta_2) \wedge 2$. Then

$$\Phi(r) \asymp \begin{cases} r^\alpha & \text{if } r \leq 1, \\ r^{\bar{\beta}} & \text{if } r > 1 \text{ and } \alpha + \beta_1 + \beta_2 \neq 2, \\ r^2/\log(1+r) & \text{if } r > 1 \text{ and } \alpha + \beta_1 + \beta_2 = 2, \end{cases}$$

which implies that

$$c_2 \left(\frac{R}{r}\right)^\alpha \leq \frac{\Phi(R)}{\Phi(r)} \leq c_3 \left(\frac{R}{r}\right)^{\bar{\beta}}, \quad 0 < r \leq R < \infty. \quad (4.6)$$

For $a > 0$, let $\mathbb{R}_{a+}^d := \{x \in \mathbb{R}_+^d : x_d \geq a\}$. Define

$$K(r) := \begin{cases} r^{-d-\alpha} & \text{if } r \leq 1, \\ r^{-d-\alpha-\beta_1-\beta_2} & \text{if } r > 1, \end{cases} \quad (4.7)$$

$$Q(u, u) := \int_{\mathbb{R}_{1+}^d} \int_{\mathbb{R}_{1+}^d} (u(x) - u(y))^2 K(|x - y|) dx dy. \quad (4.8)$$

Note that, by (4.5),

$$K(|x - y|) \leq c_4 J(x, y) \leq c_5 j(|x - y|), \quad (x, y) \in \mathbb{R}_{1+}^d \times \mathbb{R}_{1+}^d, \quad (4.9)$$

for some positive constants c_4 and c_5 . Consider the Dirichlet form $(Q, \mathcal{D}(Q))$ on \mathbb{R}_{1+}^d , where

$$\mathcal{D}(Q) := \{u \in L^2(\mathbb{R}_{1+}^d) : Q(u, u) < \infty\}. \quad (4.10)$$

Let

$$\begin{aligned} \tilde{Q}(u, u) &:= \int_{\mathbb{R}_{1+}^d} \int_{\mathbb{R}_{1+}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy \\ \mathcal{D}(\tilde{Q}) &:= \{u \in L^2(\mathbb{R}_{1+}^d) : \tilde{Q}(u, u) < \infty\}. \end{aligned}$$

It follows from [5, Remark 2.1 (1)] (more precisely the first sentence on [5, p. 98]) that $(\tilde{Q}, \mathcal{D}(\tilde{Q}))$ is a regular Dirichlet form. Moreover, we have

$$\begin{aligned} \tilde{Q}(u, u) &= \int_{\mathbb{R}_{1+}^d \times \mathbb{R}_{1+}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy \\ &= \int_{\mathbb{R}_{1+}^d \times \mathbb{R}_{1+}^d} \mathbf{1}_{|x-y| \leq 1} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy \\ &\quad + \int_{\mathbb{R}_{1+}^d \times \mathbb{R}_{1+}^d} \mathbf{1}_{|x-y| > 1} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx dy \\ &\leq Q(u, u) + 4\|u\|_{L^2(\mathbb{R}_{1+}^d)}^2 \sup_{y \in \mathbb{R}_{1+}^d} \int_{\mathbb{R}_{1+}^d} \mathbf{1}_{|x-y| > 1} |x - y|^{-d-\alpha} dx \\ &\leq Q(u, u) + 4\|u\|_{L^2(\mathbb{R}_{1+}^d)}^2 \int_{\mathbb{R}^d} \mathbf{1}_{|z| > 1} |z|^{-d-\alpha} dz = Q(u, u) + c_6 \|u\|_{L^2(\mathbb{R}_{1+}^d)}^2. \end{aligned}$$

This implies that the Dirichlet form $(Q, \mathcal{D}(Q))$ is also regular on $L^2(\mathbb{R}_{1+}^d, dx)$.

Let $X^{(1)} = (X_t^{(1)})_{t \geq 0}$ be the symmetric Hunt process associated with $(Q, \mathcal{D}(Q))$ and denote by $p^{(1)}(t, x, y)$ the transition density of $X^{(1)}$. Then [2, Theorem 4.6] says that there exists $c_7 > 0$ such that

$$p^{(1)}(t, x, y) \leq c_7 \left(\frac{1}{\Phi^{-1}(t)^d} \wedge \frac{t}{|x - y|^d \Phi(|x - y|)} \right), \quad t > 0, x, y \in \mathbb{R}_{1+}^d, \quad (4.11)$$

and [2, Theorem 2.19 (i)] says that there exists $c_8 > 0$ such that

$$p^{(1)}(t, x, y) \geq \frac{c_8}{\Phi^{-1}(t)^d}, \quad t > 0, x, y \in \mathbb{R}_{1+}^d \text{ with } |x - y| \leq \Phi^{-1}(t). \quad (4.12)$$

Recall that we have assumed $d > \bar{\beta}$. By using (4.6), (4.11) and (4.12), we can compute (see [6, p. 241]) that for every $\gamma \in (0, (d/\bar{\beta} - 1) \wedge 2)$,

$$\begin{aligned} h(x, y) &:= \int_0^\infty t^\gamma p^{(1)}(t, x, y) dt \asymp \frac{\Phi(|x - y|)^{\gamma+1}}{|x - y|^d}, \quad x, y \in \mathbb{R}_{1+}^d, \\ \bar{h}(x, y) &:= \int_0^\infty t^{\gamma-1} p^{(1)}(t, x, y) dt \asymp \frac{\Phi(|x - y|)^\gamma}{|x - y|^d}, \quad x, y \in \mathbb{R}_{1+}^d. \end{aligned}$$

This is the only place where the assumption $d > \bar{\beta}$ is used. Set $x^* = (\tilde{0}, 1)$ and let

$$q(x) := \frac{\bar{h}(x, x^*)}{h(x, x^*)} \asymp \frac{1}{\Phi(|x - x^*|)}.$$

It follows from the Hardy inequality in [6, Theorem 2 and Corollary 3] that there exists $c_9 > 0$ such that

$$Q(u, u) \geq c_9 \int_{\mathbb{R}_{1+}^d} u(x)^2 \frac{dx}{\Phi(|x - x^*|)} \quad \text{for all } u \in L^2(\mathbb{R}_{1+}^d). \quad (4.13)$$

This estimate can be improved to obtain the following result.

Proposition 4.2. *There exists a constant $C_{12} > 0$ such that for all $u \in \mathcal{D}(Q)$ and all $z_a = (\tilde{0}, a)$ with $a \geq 0$,*

$$Q(u, u) \geq C_{12} \int_{\mathbb{R}_{1+}^d} u(x + z_a)^2 \frac{dx}{\Phi(|x - x^*|)}.$$

Proof. Let $z_a = (\tilde{0}, a)$, $a \geq 0$. Then

$$\begin{aligned} & \int_{\mathbb{R}_{1+}^d} \int_{\mathbb{R}_{1+}^d} (u(x + z_a) - u(y + z_a))^2 K(|x - y|) dx dy \\ &= \int_{\mathbb{R}_{(1+a)+}^d} \int_{\mathbb{R}_{(1+a)+}^d} (u(x) - u(y))^2 K(|x - y|) dx dy \leq Q(u, u) < \infty. \end{aligned}$$

Thus, $u(\cdot + z_a) \in \mathcal{D}(Q)$ by (4.10) and

$$\begin{aligned} Q(u(\cdot + z_a), u(\cdot + z_a)) &= \int_{\mathbb{R}_{1+}^d} \int_{\mathbb{R}_{1+}^d} (u(x + z_a) - u(y + z_a))^2 K(|x - y|) dx dy \\ &\leq Q(u, u). \end{aligned}$$

Since clearly $u(\cdot + z_a) \in L^2(\mathbb{R}_{1+}^d)$, the claim follows from (4.13). \blacksquare

We have shown in Lemma 2.1 that $(\mathcal{E}, \mathcal{F})$ is transient. Let $(\mathcal{E}, \mathcal{F}_e)$ be its extended Dirichlet space.

Lemma 4.3. *There exists $C_{13} > 0$ such that for any $h \in \mathcal{F}_e$ and any $z_a = (\tilde{0}, a)$ with $a \geq 0$,*

$$\int_{\mathbb{R}_{1+}^d} \frac{|h(x + z_a)|^2}{\Phi(|x - x^*|)} dx \leq C_{13} \mathcal{E}(h, h).$$

Proof. Let $h \in \mathcal{F}_e$. There exists an approximating sequence $(g_n)_{n \geq 1}$ in \mathcal{F} such that $\mathcal{E}(h, h) = \lim_{n \rightarrow \infty} \mathcal{E}(g_n, g_n)$ and $h = \lim_{n \rightarrow \infty} g_n$ a.e. Since $g_n \in L^2(\mathbb{R}_{1+}^d, dx)$, we have $g_n \mathbf{1}_{\mathbb{R}_{1+}^d} \in L^2(\mathbb{R}_{1+}^d, dx)$. Further, by (4.9),

$$Q(g_n \mathbf{1}_{\mathbb{R}_{1+}^d}, g_n \mathbf{1}_{\mathbb{R}_{1+}^d}) \leq c_1 \mathcal{E}(g_n, g_n) < \infty,$$

so that $g_n \mathbf{1}_{\mathbb{R}_{1+}^d} \in \mathcal{D}(Q)$ by (4.10).

Now, using Proposition 4.2 and the above inequality, we have

$$\mathcal{E}(g_n, g_n) \geq c_1^{-1} Q(g_n \mathbf{1}_{\mathbb{R}_{1+}^d}, g_n \mathbf{1}_{\mathbb{R}_{1+}^d}) \geq c_2 \int_{\mathbb{R}_{1+}^d} g_n(x + z_a)^2 \frac{dx}{\Phi(|x - x^*|)}$$

for some constant $c_2 > 0$. By Fatou's lemma,

$$\begin{aligned} \mathcal{E}(h, h) &= \lim_{n \rightarrow \infty} \mathcal{E}(g_n, g_n) \geq c_2 \int_{\mathbb{R}_{1+}^d} \liminf_{n \rightarrow \infty} g_n(x + z_a)^2 \frac{dx}{\Phi(|x - x^*|)} \\ &= c_2 \int_{\mathbb{R}_{1+}^d} h(x + z_a)^2 \frac{dx}{\Phi(|x - x^*|)}. \quad \blacksquare \end{aligned}$$

By [31, Theorem 1.5.4], for any non-negative Borel function f satisfying the condition $\int_{\mathbb{R}_{1+}^d} f(x) Gf(x) dx < \infty$, we have $Gf \in \mathcal{F}_e$ and $\mathcal{E}(Gf, Gf) = \int_{\mathbb{R}_{1+}^d} f(x) Gf(x) dx$. Thus Lemma 4.3 yields

Corollary 4.4. *There exists $C_{14} > 0$ such that for every non-negative Borel function f satisfying $\int_{\mathbb{R}_{1+}^d} f(x) Gf(x) dx < \infty$ and every $z_a = (\tilde{0}, a)$ with $a \geq 0$,*

$$\int_{\mathbb{R}_{1+}^d} \frac{|Gf(x + z_a)|^2}{\Phi(|x - x^*|)} dx \leq C_{14} \int_{\mathbb{R}_{1+}^d} f(x) Gf(x) dx.$$

Proposition 4.5. *There exists $C_{15} > 0$ such that for every $x^{(0)} \in \mathbb{R}_{1+}^d$ with $x_d^{(0)} > 6$,*

$$\int_{B(x^{(0)}, 4)} (G\mathbf{1}_{B(x^{(0)}, 4)}(x))^2 dx \leq C_{15}.$$

Proof. Without loss of generality we assume that $x^{(0)} = (\tilde{0}, x_d^{(0)})$. Set $B = B(x^{(0)}, 4)$ and let $u = G\mathbf{1}_B$. We first note that, by (2.3), $G\mathbf{1}_B \leq c_B^{-1}$, and therefore $\|u\|_{L^2(B)} < \infty$.

Let $z = (\tilde{0}, x_d^{(0)} - 6)$ and $\tilde{B} = B((\tilde{0}, 6), 4) \subset \mathbb{R}_{2+}^d$. By using the change of variables $w = x - z$ and the fact that $\Phi(|w - x^*|) \asymp 1$ for $w \in \tilde{B}$ in the first line, and then Corollary 4.4 and the Cauchy inequality in the third line below, we have

$$\begin{aligned} \|u\|_{L^2(B)}^2 &= \int_{\tilde{B}} |u(w+z)|^2 dw \leq c_1 \int_{\tilde{B}} |u(w+z)|^2 \frac{dw}{\Phi(|w-x^*|)} \\ &\leq c_1 \int_{\mathbb{R}_{1+}^d} |u(w+z)|^2 \frac{dw}{\Phi(|w-x^*|)} = c_1 \int_{\mathbb{R}_{1+}^d} |G\mathbf{1}_B(w+z)|^2 \frac{dw}{\Phi(|w-x^*|)} \\ &\leq c_2 \int_{\mathbb{R}_+^d} \mathbf{1}_B(x) G\mathbf{1}_B(x) dx \leq c_2 |B|^{1/2} \|u\|_{L^2(B)}. \end{aligned}$$

Since $\|u\|_{L^2(B)} < \infty$, it follows that $\|u\|_{L^2(B)} \leq c_2 |B|^{1/2}$. This completes the proof. ■

Coming back to (4.4), by Proposition 4.5, we see that the right-hand side is bounded above by a constant, and therefore $G(x^{(0)}, y^{(0)}) \leq c$.

Proposition 4.6. *There exists a constant $C_{16} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ satisfying $|x - y| \leq 8(x_d \wedge y_d)$,*

$$G(x, y) \leq C_{16} |x - y|^{-d+\alpha}.$$

Proof. This is analogous to the proof of Proposition 4.1. We omit the details. ■

Using Theorem 3.5, we can combine Proposition 4.6 with Theorem 1.5 to get the following result, which is key for us to get sharp two-sided Green functions estimates.

Proposition 4.7. *There exists a constant $C_{17} > 0$ such that for all $x, y \in \mathbb{R}_+^d$,*

$$G(x, y) \leq C_{17} |x - y|^{-d+\alpha}. \quad (4.14)$$

Proof. By Proposition 4.6, there exists $c_1 > 0$ such that $G(x, y) \leq c_1$ for all $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$ and $x_d \wedge y_d \geq 1/8$.

Suppose that $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$ and $x_d \leq y_d$ and $x_d < 1/8 < y_d$. Since $z \mapsto G(z, y)$ is harmonic in $B((\tilde{x}, 0), 1/4)$ with respect to Y and vanishes on the boundary of \mathbb{R}_+^d by Theorem 3.5, we can use Theorem 1.5 to find that there exists $c_2 > 0$ such that

$$G(x, y) \leq c_2 G(x + (\tilde{0}, 1/8), y) \leq c_2 c_1. \quad (4.15)$$

Suppose that $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$ and $x_d \leq y_d$ and $y_d \leq 1/8$. Then, since $z \mapsto G(z, y)$ is harmonic in $B((\tilde{x}, 0), 1/4)$ with respect to Y and vanishes on the boundary of \mathbb{R}_+^d , by (4.15) and Theorem 1.5 we see that $G(x, y) \leq c_2 G(x + (\tilde{0}, 1/8), y) \leq c_2^2 c_1$. Thus for all $x, y \in \mathbb{R}_+^d$ with $|x - y| = 1$, we have $G(x, y) \leq C$. Therefore, by (2.5),

$$G(x, y) \leq C |x - y|^{-d+\alpha}, \quad x, y \in \mathbb{R}_+^d. \quad \blacksquare$$

5. Preliminary Green function estimates

The results of this section are valid for all $p \in ((\alpha - 1)_+, \alpha + \beta_1)$.

5.1. Lower bound

The goal of this subsection is to prove the following result, which is used later to prove the sharp lower bound of the Green function for $p \in ((\alpha - 1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$.

Theorem 5.1. *Suppose $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. For any $\varepsilon \in (0, 1/4)$, there exists a constant $C_{18} > 0$ such that for all $w \in \partial\mathbb{R}_+^d$, $R > 0$ and $x, y \in B(w, (1 - \varepsilon)R) \cap \mathbb{R}_+^d$,*

$$G^{B(w,R) \cap \mathbb{R}_+^d}(x, y) \geq C_{18} \left(\frac{x_d}{|x - y|} \wedge 1 \right)^p \left(\frac{y_d}{|x - y|} \wedge 1 \right)^p \frac{1}{|x - y|^{d-\alpha}}.$$

The theorem will be proved through three lemmas. For any $a > 0$, let $B_a^+ := B(0, a) \cap \mathbb{R}_+^d$. Recall that $\mathbb{R}_{a+}^d = \{x \in \mathbb{R}_+^d : x_d \geq a\}$.

Lemma 5.2. *For any $\varepsilon \in (0, 1)$ and $M > 1$, there exists a constant $C_{19} > 0$ such that for all $y, z \in B_{1-\varepsilon}^+$ with $|y - z| \leq M(y_d \wedge z_d)$,*

$$G^{B_1^+}(y, z) \geq C_{19}|y - z|^{-d+\alpha}.$$

Proof. By using (2.4) in the first equality below, it follows from Propositions 4.7 and 4.1 that there exists $c_1 > 1$ such that for all $y, z \in B_{1-\varepsilon}^+$ with $|y - z| \leq M(y_d \wedge z_d)$,

$$G^{B_1^+}(y, z) = G(y, z) - \mathbb{E}_y[G(Y_{\tau_{B_1^+}}, z)] \geq c_1^{-1}|y - z|^{-d+\alpha} - c_1\varepsilon^{-d+\alpha}.$$

Now, choose $\delta = (2c_1^2)^{-\frac{1}{d-\alpha}}$. Then for all $y, z \in B_{1-\varepsilon}^+$ with $|y - z| \leq (\delta\varepsilon) \wedge M(y_d \wedge z_d)$,

$$\begin{aligned} G^{B_1^+}(y, z) &\geq c_1^{-1}|y - z|^{-d+\alpha} - c_1(\delta^{-1}|y - z|)^{-d+\alpha} \\ &\geq (c_1^{-1} - c_1\delta^{d-\alpha})|y - z|^{-d+\alpha} = (2c_1)^{-1}|y - z|^{-d+\alpha}. \end{aligned} \quad (5.1)$$

Assume that $y, z \in B_{1-\varepsilon}^+$ with $|y - z| \leq M(y_d \wedge z_d)$ are such that also $|y - z| \leq \delta\varepsilon$. Then clearly $|y - z| \leq (\delta\varepsilon) \wedge M(y_d \wedge z_d)$, and (5.1) proves the lemma.

Now, we assume that $y, z \in B_{1-\varepsilon}^+$ with $|y - z| \leq M(y_d \wedge z_d)$, but $|y - z| > \delta\varepsilon$; see Figure 1. Since $y_d \wedge z_d > \delta\varepsilon/M$, we have

$$y, z \in B_{1-\varepsilon}^+ \cap \mathbb{R}_{(\delta\varepsilon/M)+}^d. \quad (5.2)$$

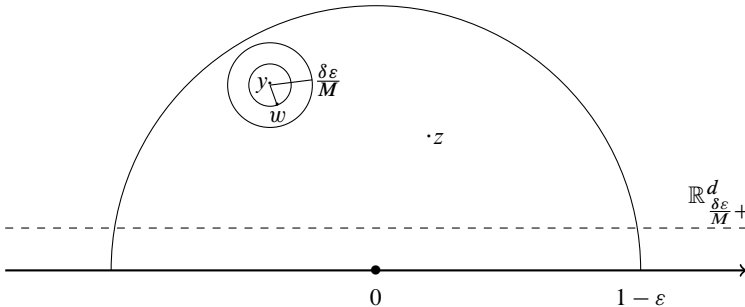


Fig. 1

Therefore we can choose a point $w \in B(y, \delta\varepsilon/M)$ such that $|y - w| = \delta\varepsilon/(2M)$ and $w \in B_{1-\varepsilon}^+ \cap \mathbb{R}_{(\delta\varepsilon/M)_+}^d$. Since $M(y_d \wedge w_d) > \delta\varepsilon > |y - w|$, we can use (5.1) for the points y and w to conclude that

$$G^{B_1^+}(y, w) \geq (2c_1)^{-1}|y - w|^{-d+\alpha} = (2c_1)^{-1}(\delta\varepsilon/(2M))^{-d+\alpha} =: c_2.$$

Since $G^{B_1^+}(y, \cdot)$ is harmonic in $B(w, \delta\varepsilon/(4M)) \cup B(z, \delta\varepsilon/(4M))$ by (5.2), we can use Theorem 1.4 (b) and the fact that $|y - z| < \delta\varepsilon$ to get

$$G^{B_1^+}(y, z) \geq c_3 G^{B_1^+}(y, w) \geq c_4 \geq c_5 |y - z|^{-d+\alpha}. \quad \blacksquare$$

Lemma 5.3. *Suppose $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. For every $\varepsilon \in (0, 1/4)$ and $M, N > 1$, there exists a constant $C_{20} > 0$ such that for all $x, z \in B_{1-\varepsilon}^+$ with $x_d \leq z_d$ satisfying $x_d/N \leq |x - z| \leq Mz_d$,*

$$G^{B_1^+}(x, z) \geq C_{20} x_d^p |x - z|^{-d+\alpha-p}.$$

Proof. Without loss of generality, we assume $M > 4/\varepsilon$. If $|x - z| \leq Mz_d$ and $|x - z| \geq 20Mx_d$, let $r = \frac{|x-z|}{10M} \leq \frac{1}{5M} \leq \frac{\varepsilon}{20}$. Since $x \mapsto G^{B_1^+}(x, z)$ is regular harmonic in $D_{\bar{x}}(r, r)$, and $D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r) \subset B_{1-\varepsilon/4}^+$, by Lemmas 5.2 and 3.2 we have

$$\begin{aligned} G^{B_1^+}(x, z) &\geq \mathbb{E}_x[G^{B_1^+}(Y_{\tau_{D_{\bar{x}}(r,r)}}(z) : Y_{\tau_{D_{\bar{x}}(r,r)}} \in D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r)] \\ &\geq c_1 |x - z|^{-d+\alpha} \mathbb{P}_x(Y_{\tau_{D_{\bar{x}}(r,r)}} \in D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r)) \geq c_2 x_d^p |x - z|^{-d+\alpha-p}, \end{aligned}$$

since, for $y \in D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r)$, $|y - z| \leq |x - z| + |x - y| \leq 5(2M + 1)r \leq 2(2M + 1)(y_d \wedge z_d)$.

If $|x - z| \leq Mz_d$ and $x_d/N < |x - z| < 20Mx_d$, we simply use Lemma 5.2 (since $|x - z| < 12M(x_d \wedge z_d)$) to get

$$G^{B_1^+}(x, z) \geq c_3 |x - z|^{-d+\alpha} \geq c_3 N^{-p} x_d^p |x - z|^{-d+\alpha-p}. \quad \blacksquare$$

Lemma 5.4. *Suppose $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. For every $\varepsilon \in (0, 1/4)$ and $M \geq 40/\varepsilon$, there exists a constant $C_{21} > 0$ such that for all $x, z \in B_{1-\varepsilon}^+$ with $x_d \leq z_d$ satisfying $|x - z| \geq Mz_d$,*

$$G^{B_1^+}(x, z) \geq C_{21} x_d^p z_d^p |x - z|^{-d+\alpha-2p}.$$

Proof. Let $r = \frac{2|x-z|}{M} \leq \frac{4}{M} \leq \frac{\varepsilon}{10}$. Since $x \mapsto G^{B_1^+}(x, z)$ is regular harmonic in $D_{\bar{x}}(r, r)$, and $D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r) \subset B_{1-\varepsilon/4}^+$, by Lemmas 5.3 and 3.2 we have

$$\begin{aligned} G^{B_1^+}(x, z) &\geq \mathbb{E}_x[G^{B_1^+}(Y_{\tau_{D_{\bar{x}}(r,r)}}(z) : Y_{\tau_{D_{\bar{x}}(r,r)}} \in D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r)] \\ &\geq c_1 z_d^p |x - z|^{-d+\alpha-2p} \mathbb{P}_x(Y_{\tau_{D_{\bar{x}}(r,r)}} \in D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r)) \geq c_2 x_d^p z_d^p |x - z|^{-d+\alpha-2p} \end{aligned}$$

since, for $y \in D_{\bar{x}}(r, 4r) \setminus D_{\bar{x}}(r, 3r)$, $|y - z| \leq |x - z| + |x - y| \leq (M/2 + 5)r \leq (M/2 + 5)y_d$ and $|y - z| \geq |x - z| - |x - y| \geq 75r \geq 150z_d$. \blacksquare

Combining the above result with scaling, we get the result of Theorem 5.1.

5.2. Upper bound

The goal of this subsection is to get the following preliminary upper bound on the Green function.

Lemma 5.5. *Suppose $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. There exists $C_{22} > 0$ such that*

$$G(x, y) \leq C_{22} \left(\frac{x_d \wedge y_d}{|x - y|} \wedge 1 \right)^p \frac{1}{|x - y|^{d-\alpha}}, \quad x, y \in \mathbb{R}_+^d. \quad (5.3)$$

Proof. Suppose $x, y \in \mathbb{R}_+^d$ satisfy $\tilde{x} = \tilde{0}$, $x_d \leq 2^{-9}$ and $|x - y| = 1$. Let $r = 2^{-8}$. For $z \in U(r)$ and $w \in \mathbb{R}_+^d \setminus D(r, r)$, we have $|w - z| \asymp |w|$. Thus, by using (3.1) and Proposition 4.7,

$$\begin{aligned} & \int_{\mathbb{R}_+^d \setminus D(r, r)} G(w, y) \mathcal{B}(z, w) |z - w|^{-d-\alpha} dw \\ & \leq c_1 z_d^{\beta_1} (|\log z_d|^{\beta_3} \vee 1) \int_{\mathbb{R}_+^d \setminus D(r, r)} \frac{G(w, y)}{|w|^{d+\alpha+\beta_1}} (1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}) dw \\ & \leq c_2 z_d^{\beta_1} |\log z_d|^{\beta_3} \int_{\mathbb{R}_+^d \setminus D(r, r)} \frac{1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}}{|w - y|^{d-\alpha} |w|^{d+\alpha+\beta_1}} dw. \end{aligned} \quad (5.4)$$

Hence, by using (3.4) and (3.1) in the second line, and Lemma 3.1 in the third,

$$\begin{aligned} & \mathbb{E}_x[G(Y_{\tau_{U(r)}}, y); Y_{\tau_{U(r)}} \notin D(r, r)] \\ & \leq c_3 \mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} |\log(Y_t^d)|^{\beta_3} dt \int_{\mathbb{R}_+^d \setminus D(r, r)} \frac{1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}}{|w - y|^{d-\alpha} |w|^{d+\alpha+\beta_1}} dw \\ & \leq c_4 x_d^p \int_{\mathbb{R}_+^d \setminus D(r, r)} \frac{1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}}{|w - y|^{d-\alpha} |w|^{d+\alpha+\beta_1}} dw. \end{aligned}$$

Let

$$\begin{aligned} & \int_{\mathbb{R}_+^d \setminus D(r, r)} \frac{1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}}{|w - y|^{d-\alpha} |w|^{d+\alpha+\beta_1}} dw = \int_{\mathbb{R}_+^d \cap B(y, r)} + \int_{\mathbb{R}_+^d \setminus (D(r, r) \cup B(y, r))} \\ & =: I + II. \end{aligned} \quad (5.5)$$

It is easy to see

$$II \leq r^{-d+\alpha} \int_{\mathbb{R}_+^d \setminus (D(r, r) \cup B(y, r))} \frac{1 + \mathbf{1}_{|w| \geq 1} (\log |w|)^{\beta_3}}{|w|^{d+\alpha+\beta_1}} dw < \infty, \quad (5.6)$$

$$I \leq c_5 \int_{\mathbb{R}_+^d \cap B(y, r)} \frac{1}{|w - y|^{d-\alpha}} dw < \infty. \quad (5.7)$$

Thus,

$$\mathbb{E}_x[G(Y_{\tau_{U(r)}}, y); Y_{\tau_{U(r)}} \notin D(r, r)] \leq c_6 x_d^p. \quad (5.8)$$

Let $x_0 := (\tilde{0}, r)$. By Theorem 1.5, Proposition 4.7 and Lemma 3.3, we have

$$\mathbb{E}_x[G(Y_{\tau_{U(r)}}, y); Y_{\tau_{U(r)}} \in D(r, r)] \leq c_7 G(x_0, y) \mathbb{P}_x(Y_{\tau_{U(r)}} \in D(r, r)) \leq c_8 x_d^p. \quad (5.9)$$

Combining (5.8) and (5.9), we see that for $x, y \in \mathbb{R}_+^d$ satisfying $x_d \leq 2^{-9}$ and $|x - y| = 1$,

$$G(x, y) = \mathbb{E}_x[G(Y_{\tau_{U(r)}}, y); Y_{\tau_{U(r)}} \notin D(r, r)] + \mathbb{E}_x[G(Y_{\tau_{U(r)}}, y); Y_{\tau_{U(r)}} \in D(r, r)] \leq c_9 x_d^p.$$

Combining this with Proposition 4.7, (2.5) and symmetry, we immediately get the desired conclusion. \blacksquare

6. Proof of Theorem 1.1

We begin this section by introducing an auxiliary function that will be needed later. For $\gamma \in \mathbb{R}$ and $\beta \geq 0$, we define a function on $(0, 1]$ by

$$F(x; \gamma, \beta) := \int_x^1 h^\gamma \left(\log \frac{2}{h} \right)^\beta dh.$$

Note that $F(\cdot; \gamma, \beta)$ is a decreasing function on $(0, 1]$ and that, when $\gamma > -1$, $F(0+; \gamma, \beta)$ is finite. It is obvious that

$$F(x; \gamma, 0) = \begin{cases} \frac{1}{\gamma+1}(1 - x^{\gamma+1}), & \gamma \neq -1, \\ -\log x, & \gamma = -1 \end{cases}$$

and

$$F(x; -1, \beta) = \frac{1}{1+\beta} \left(\left(\log \frac{2}{x} \right)^{1+\beta} - (\log 2)^{1+\beta} \right). \quad (6.1)$$

Note also that for any $b \in (0, 1)$, on $(0, b]$, when $\gamma > -1$,

$$F(0; \gamma, \beta) - F(x; \gamma, \beta) \asymp x^{\gamma+1} \left(\log \frac{2}{x} \right)^\beta, \quad (6.2)$$

and when $\gamma < -1$,

$$F(x; \gamma, \beta) \asymp x^{\gamma+1} \left(\log \frac{2}{x} \right)^\beta, \quad (6.3)$$

with comparison constants depending on $\beta \geq 0$ and $\gamma < -1$.

We first present a technical lemma inspired by [1, Lemma 3.3]. This lemma will be used several times in this section. For $x = (\tilde{0}, x_d) \in \mathbb{R}_+^d$ and $\gamma, q, \delta \in \mathbb{R}$, $R > 0$, $\beta \geq 0$ and $y \in \mathbb{R}_+^d$ with $y_d \in (0, R)$, we define

$$\begin{aligned} f(y; \gamma, \beta, q, \delta, x) \\ := y_d^\gamma |x - y|^{-d+\alpha-q} \left(\log \left(1 + \frac{2R}{y_d} \right) \right)^\beta \left(\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right)^\delta \end{aligned}$$

and

$g(y; \beta, q, \delta, x)$

$$:= \left(\frac{x_d}{|x-y|} \wedge 1 \right)^q |x-y|^{-d+\alpha} \left(\log \left(1 + \frac{2R}{y_d} \right) \right)^\beta \left(\log \left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|} \right) \right)^\delta.$$

Note that for $0 < y_d < R$ we have $\log(1 + 2R/y_d) \asymp \log(2R/y_d)$. In almost all our applications of Lemma 6.1 and Corollary 6.3 below, the parameter δ will be 0. The only exception is Proposition 6.10 where we will have δ equal to 0, β_4 or $\beta_4 + 1$.

Lemma 6.1. *Let $R \in (0, \infty)$ and $x = (\tilde{0}, x_d)$ with $x_d \leq 2R/3$. Fix $0 < a_1 \leq x_d/2$ and $3x_d/2 \leq a_3 \leq a_2 \leq R$. We have the following comparison relations, with comparison constants independent of R, a_1, a_2, a_3 and $x_d \in (0, 2R/3)$:*

(i) *If $\gamma > -1$ and $q > \alpha - 1$, then*

$$I_1 := \int_{D(R, a_1)} f(y; \gamma, \beta, q, \delta, x) dy \asymp x_d^{\alpha-q-1} a_1^{\gamma+1} \left(\log \frac{2R}{a_1} \right)^\beta.$$

(ii) *If $q > \alpha - 1$, then*

$$I_2 := \int_{D(R, a_2) \setminus D(R, a_3)} f(y; \gamma, \beta, q, \delta, x) dy \\ \asymp R^{\gamma+\alpha-q} \left(F \left(\frac{a_3}{R}; \gamma + \alpha - q - 1, \beta \right) - F \left(\frac{a_2}{R}; \gamma + \alpha - q - 1, \beta \right) \right).$$

(iii) *If $q > \alpha - 1$, then*

$$I_3 := \int_{D(R, 3x_d/2) \setminus D(R, x_d/2)} g(y; \beta, q, \delta, x) dy \asymp x_d^\alpha \left(\log \frac{2R}{x_d} \right)^\beta.$$

Proof. (i) In $D(R, a_1)$, $y_d < x_d$. Without loss of generality, we replace $\log(1 + 2R/y_d)$ with $\log(2R/y_d)$. Thus, using the change of variables $y_d = x_d h$ and $r = x_d s$ in the third line below, we get

$$I_1 \asymp \int_0^R r^{d-2} \int_0^{a_1} \frac{y_d^\gamma}{((x_d - y_d) + r)^{d-\alpha+q}} \left(\log \frac{2R}{y_d} \right)^\beta \\ \times \left(\log \left(1 + \frac{(x_d - y_d) + r}{x_d} \right) \right)^\delta dy_d dr \\ = x_d^{\alpha-q+\gamma} \int_0^{R/x_d} s^{d-2} \int_0^{a_1/x_d} \frac{h^\gamma}{[(1-h) + s]^{d-\alpha+q}} \\ \times \left(\log \frac{2R/x_d}{h} \right)^\beta (\log(2-h+s))^\delta dh ds,$$

which, using $1-h \asymp 1$ (because $0 < a_1 \leq x_d/2$), is comparable to

$$x_d^{\alpha-q+\gamma} \int_0^{R/x_d} \frac{s^{d-2} (\log(2+s))^\delta}{(1+s)^{d-\alpha+q}} ds \left(\int_0^{a_1/x_d} h^\gamma \left(\log \frac{2R/x_d}{h} \right)^\beta dh \right).$$

Note that, since $q > \alpha - 1$,

$$\int_1^{3/2} \frac{(\log(2+s))^\delta}{s^{2-\alpha+q}} ds \leq \int_1^{R/x_d} \frac{(\log(2+s))^\delta}{s^{2-\alpha+q}} ds \leq \int_1^\infty \frac{(\log(2+s))^\delta}{s^{2-\alpha+q}} ds < \infty.$$

Therefore, using this inequality and (6.2), after a change of variables we get

$$\begin{aligned} I_1 &\asymp x_d^{\alpha-q+\gamma} \left(\int_0^1 \frac{s^{d-2} (\log(2+s))^\delta}{(1+s)^{d-\alpha+q}} ds + \int_1^{R/x_d} \frac{(\log(2+s))^\delta}{s^{2-\alpha+q}} ds \right) \\ &\quad \times \left(\frac{R}{x_d} \right)^{\gamma+1} \left(F(0; \gamma, \beta) - F\left(\frac{a_1}{R}; \gamma, \beta\right) \right) \\ &\asymp x_d^{\alpha-q-1} a_1^{\gamma+1} \left(\log \frac{2R}{a_1} \right)^\beta. \end{aligned}$$

(ii) In $D(R, a_2) \setminus D(R, a_3)$, $y_d > x_d$. Thus, using the change of variables $y_d = x_d h$ and $r = x_d s$ in the third line below, we get

$$\begin{aligned} I_2 &\asymp \int_0^R r^{d-2} \int_{a_3}^{a_2} \frac{y_d^\gamma}{((y_d - x_d) + r)^{d-\alpha+q}} \left(\log \frac{2R}{y_d} \right)^\beta \\ &\quad \times \left(\log \left(1 + \frac{(y_d - x_d) + r}{y_d} \right) \right)^\delta dy_d dr \\ &= x_d^{\alpha-q+\gamma} \int_{a_3/x_d}^{a_2/x_d} \int_0^{R/x_d} \frac{s^{d-2} h^\gamma}{[(h-1) + s]^{d-\alpha+q}} \left(\log \frac{2R/x_d}{h} \right)^\beta \\ &\quad \times \left(\log \left(1 + \frac{h-1+s}{h} \right) \right)^\delta ds dh, \end{aligned}$$

which, by the change of variables $s = (h-1)t$, is equal to

$$\begin{aligned} x_d^{\alpha-q+\gamma} \int_{a_3/x_d}^{a_2/x_d} \int_0^{\frac{R}{(h-1)x_d}} \frac{h^\gamma t^{d-2}}{(h-1)^{1-\alpha+q} (1+t)^{d-\alpha+q}} \left(\log \frac{2R/x_d}{h} \right)^\beta \\ \times \left(\log \left(1 + \frac{(h-1)(1+t)}{h} \right) \right)^\delta dt dh. \quad (6.4) \end{aligned}$$

Note that since $3x_d/2 \leq a_3 \leq hx_d \leq a_2 \leq R$ we have

$$\frac{R}{(h-1)x_d} \geq \frac{R}{a_2 - x_d} \geq 1, \quad a_3/x_d \leq h \leq a_2/x_d.$$

Thus, using $q > \alpha - 1$, we find that for $a_3/x_d \leq h \leq a_2/x_d$,

$$\begin{aligned} \int_{1/2}^1 \frac{(\log(2+t))^\delta}{(1+t)^{2-\alpha+q}} dt &\leq \int_{1/2}^{\frac{R}{(h-1)x_d}} \frac{(\log(2+t))^\delta}{(1+t)^{2-\alpha+q}} dt \\ &\leq \int_{1/2}^\infty \frac{(\log(2+t))^\delta}{(1+t)^{2-\alpha+q}} dt < \infty. \end{aligned}$$

Therefore, using $(h-1)/h \asymp 1$ and the display above, (6.4) is comparable to

$$\begin{aligned}
& x_d^{\alpha-q+\gamma} \int_{a_3/x_d}^{a_2/x_d} h^{\gamma+\alpha-q-1} \left(\log \frac{2R/x_d}{h} \right)^\beta \int_0^{\frac{R}{(h-1)x_d}} \frac{t^{d-2}}{(1+t)^{d-\alpha+q}} (\log(2+t))^\delta dt dh \\
& \asymp x_d^{\alpha-q+\gamma} \int_{a_3/x_d}^{a_2/x_d} h^{\gamma+\alpha-q-1} \left(\log \frac{2R/x_d}{h} \right)^\beta \\
& \quad \times \left(\int_0^{1/2} t^{d-2} dt + \int_{1/2}^{\frac{R}{(h-1)x_d}} \frac{(\log(2+t))^\delta}{(1+t)^{2-\alpha+q}} dt \right) dh \\
& \asymp x_d^{\alpha-q+\gamma} \int_{a_3/x_d}^{a_2/x_d} h^{\gamma+\alpha-q-1} \left(\log \frac{2R/x_d}{h} \right)^\beta dh \\
& \asymp R^{\gamma+\alpha-q} \left(F\left(\frac{a_3}{R}; \gamma + \alpha - q - 1, \beta\right) - F\left(\frac{a_2}{R}; \gamma + \alpha - q - 1, \beta\right) \right).
\end{aligned}$$

(iii) Let $B(x) = \{(\tilde{y}, y_d) : |\tilde{y}| < x_d/2, |y_d - x_d| < x_d/2\}$. Note that

$$\begin{aligned}
I_3 &= \int_{B(x)} g(y; \beta, q, \delta, x) dy \\
& \quad + \int_{(D(R, 3x_d/2) \setminus D(R, x_d/2)) \setminus B(x)} g(y; \beta, q, \delta, x) dy \\
& =: I_{31} + I_{32}.
\end{aligned}$$

Note that in both I_{31} and I_{32} we have $\log(2R/y_d) \asymp \log(2R/x_d)$ (since $y_d \asymp x_d$), and therefore this term comes out of the integral. When $y \in B(x)$, $x_d \asymp y_d \geq |x - y|$ so that $(\log(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}))^\delta \asymp 1$. Therefore

$$I_{31} \asymp \left(\log \frac{2R}{x_d} \right)^\beta \int_{B(x)} |x - y|^{-d+\alpha} dy \asymp x_d^\alpha \left(\log \frac{2R}{x_d} \right)^\beta.$$

In $(D(R, 3x_d/2) \setminus D(R, x_d/2)) \setminus B(x)$, we have $y_d \asymp x_d$ and $x_d \leq 2|x - y|$. Thus, using the change of variables $y_d = rt + x_d$ in the last line below, we get

$$\begin{aligned}
I_{32} & \asymp x_d^q \left(\log \frac{2R}{x_d} \right)^\beta \\
& \quad \times \int_{(D(R, 3x_d/2) \setminus D(R, x_d/2)) \setminus B(x)} |x - y|^{-d+\alpha-q} \left(\log \left(1 + \frac{|x-y|}{x_d} \right) \right)^\delta dy \\
& \asymp x_d^q \left(\log \frac{2R}{x_d} \right)^\beta \\
& \quad \times \int_{x_d/2}^R r^{d-2} \int_{x_d/2}^{3x_d/2} (|x_d - y_d| + r)^{-d+\alpha-q} \left(\log \left(1 + \frac{|x_d - y_d| + r}{x_d} \right) \right)^\delta dy_d dr \\
& = x_d^q \left(\log \frac{2R}{x_d} \right)^\beta \int_{x_d/2}^R r^{\alpha-q-1} \int_{-\frac{x_d}{2r}}^{\frac{x_d}{2r}} (|t|+1)^{-d+\alpha-q} \left(\log \left(1 + \frac{r(|t|+1)}{x_d} \right) \right)^\delta dt dr,
\end{aligned}$$

which, by the change of variables $r = x_d s$, is comparable to

$$x_d^\alpha \left(\log \frac{2R}{x_d} \right)^\beta \int_{1/2}^{R/x_d} s^{\alpha-q-1} \int_0^{1/s} \frac{(\log(1+s(t+1)))^\delta}{(t+1)^{d-\alpha+q}} dt ds. \quad (6.5)$$

Note that, since $q > \alpha - 1$,

$$\begin{aligned} \int_0^{1/s} \frac{(\log(1+s(t+1)))^\delta}{(t+1)^{d-\alpha+q}} dt &\asymp (\log(1+s))^\delta \int_0^{1/s} \frac{dt}{(t+1)^{d-\alpha+q}} \\ &\asymp \frac{(\log(1+s))^\delta}{s}, \quad s > 1/2, \end{aligned}$$

and

$$\int_{1/2}^{3/2} \frac{(\log(1+s))^\delta}{s^{q+2-\alpha}} ds \leq \int_{1/2}^{R/x_d} \frac{(\log(1+s))^\delta}{s^{q+2-\alpha}} ds \leq \int_{1/2}^\infty \frac{(\log(1+s))^\delta}{s^{q+2-\alpha}} ds < \infty.$$

Therefore, by the above inequalities, (6.5) is comparable to

$$x_d^\alpha \left(\log \frac{2R}{x_d} \right)^\beta \int_{1/2}^{R/x_d} \frac{(\log(1+s))^\delta}{s^{q+2-\alpha}} ds \asymp x_d^\alpha \left(\log \frac{2R}{x_d} \right)^\beta. \quad \blacksquare$$

Remark 6.2. Note that it follows from the proof of Lemma 6.1 (i) that $I_1 = \infty$ for $\gamma \leq -1$.

Corollary 6.3. Let $R > 0$, $q > \alpha - 1$, $\delta \in \mathbb{R}$, $\gamma > -1$, $\beta \geq 0$, and $x = (\tilde{0}, x_d)$.

(i) We have the following comparison result, with the comparison constant independent of R and $x_d \in (0, R/2)$:

$$\begin{aligned} \int_{D(R,R)} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^q f(y; \gamma, \beta, 0, \delta, x) dy \\ \asymp \begin{cases} R^{\alpha+\gamma-q} x_d^q & \text{if } \alpha - 1 < q < \alpha + \gamma, \\ x_d^q \left(\log \frac{2R}{x_d} \right)^{\beta+1} & \text{if } q = \alpha + \gamma, \\ x_d^{\alpha+\gamma} \left(\log \frac{2R}{x_d} \right)^\beta & \text{if } q > \alpha + \gamma. \end{cases} \end{aligned}$$

(ii) Let $a \in (0, R]$ and $\alpha - 1 < q < \alpha + \gamma$. Then there is a constant C_{23} independent of R , a and $x_d \in (0, R/2)$ such that

$$\int_{D(R,a)} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^q f(y; \gamma, \beta, 0, \delta, x) dy \leq C_{23} x_d^q a^{\alpha+\gamma-q} \left(\log \frac{2R}{a} \right)^\beta. \quad (6.6)$$

Proof. (i) Set $a_1 = x_d/2$, $a_2 = R$ and $a_3 = 3x_d/2$ in Lemma 6.1. In $D(R, x_d/2)$ and $D(R, R) \setminus D(R, 3x_d/2)$, we have $x_d \leq c|x-y|$. Therefore,

$$\begin{aligned} \int_{D(R, x_d/2)} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^q f(y; \gamma, \beta, 0, \delta, x) dy \\ \asymp x_d^q \int_{D(R, x_d/2)} f(y; \gamma, \beta, q, \delta, x) dy \asymp x_d^{\alpha+\gamma} \left(\log \frac{2R}{x_d} \right)^\beta. \end{aligned}$$

Using $3x_d/2 < 3R/4$ (so that $3x_d/2R \leq 3/4$), (6.1) and (6.3), we get

$$\begin{aligned}
& \int_{D(R,R) \setminus D(R,3x_d/2)} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^q f(y; \gamma, \beta, 0, \delta, x) dy \\
& \asymp x_d^q \int_{D(R,R) \setminus D(R,3x_d/2)} f(y; \gamma, \beta, q, \delta, x) dy \\
& \asymp x_d^q R^{\gamma+\alpha-q} F\left(\frac{3x_d}{2R}; \gamma + \alpha - q - 1, \beta\right) \\
& \asymp \begin{cases} x_d^q R^{\alpha+\gamma-q} & \text{if } \alpha - 1 < q < \alpha + \gamma, \\ x_d^q \left(\log \frac{2R}{x_d}\right)^{\beta+1} & \text{if } q = \alpha + \gamma, \\ x_d^{\alpha+\gamma} \left(\log \frac{2R}{x_d}\right)^\beta & \text{if } q > \alpha + \gamma. \end{cases}
\end{aligned}$$

In $D(R, 3x_d/2) \setminus D(R, x_d/2)$ we have $y_d \asymp x_d$, so

$$\begin{aligned}
& \int_{D(R,3x_d/2) \setminus D(R,x_d/2)} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^q f(y; \gamma, \beta, 0, \delta, x) dy \\
& \asymp x_d^\gamma \left(\log \frac{2R}{x_d}\right)^\beta \int_{D(R,3x_d/2) \setminus D(R,x_d/2)} g(y; q, \delta, x) dy \asymp x_d^{\alpha+\gamma} \left(\log \frac{2R}{x_d}\right)^\beta.
\end{aligned}$$

By adding up these three displays we get the claim.

(ii) If $a \leq x_d/2$, then by Lemma 6.1 (i) (with $a_1 = a$) and the assumption $\alpha - q - 1 < 0$, we find that the integral in (6.6) is less than $c x_d^q (x_d^{\alpha-q-1} a^{\gamma+1} (\log(2R/a))^\beta) \leq x_d^q a^{\alpha+\gamma-q} (\log(2R/a))^\beta$. If $x_d/2 \leq a \leq 3x_d/2$, we split the integral into two parts – over $D(R, x_d/2)$ and $D(R, a) \setminus D(R, x_d/2)$. The first one is by Lemma 6.1 (i) comparable with $x_d^q x_d^{\alpha-q+\gamma} (\log(4R/x_d))^\beta \asymp x_d^q a^{\alpha+\gamma-q} (\log(2R/a))^\beta$, while the second one is by Lemma 6.1 (iii) smaller than $x_d^\gamma x_d^\alpha (\log(2R/x_d))^\beta = x_d^q x_d^{\alpha+\gamma-q} (\log(2R/x_d))^\beta \asymp x_d^q a^{\alpha+\gamma-q} (\log(2R/a))^\beta$. Finally, if $a \in (3x_d/2, R]$, then by using Lemma 6.1 (ii) (with $a_2 = a$, $a_3 = 3x_d/2$) and the assumption $q < \alpha + \gamma$ we deduce that the integral over $D(R, a) \setminus D(R, 3x_d/2)$ is bounded by above by $c x_d^q a^{\alpha+\gamma-q} (\log(2R/a))^\beta$. ■

6.1. Green function upper bound for $p \in ((\alpha - 1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$

In this subsection we deal with the case

$$p \in ((\alpha - 1)_+, \alpha + 2^{-1}[\beta_1 + (\beta_1 \wedge \beta_2)]). \quad (6.7)$$

If $\beta_2 > 0$, then there exists $0 < \tilde{\beta}_2 < \beta_2$ such that

$$p \in ((\alpha - 1)_+, \alpha + 2^{-1}[\beta_1 + (\beta_1 \wedge \tilde{\beta}_2)]). \quad (6.8)$$

Further, if $\beta_4 > 0$, there is $c > 0$ such that for all $s \in (0, 1)$,

$$s^{\beta_2} (\log(1 + 8/s))^{\beta_4} \leq c s^{\tilde{\beta}_2}. \quad (6.9)$$

Let

$$\varepsilon_0 = \begin{cases} 0 & \text{if } \beta_3 = 0, \\ 2^{-1}(\alpha + \beta_1 - p) & \text{if } \beta_3 > 0. \end{cases}$$

Note that

$$[\log(1+s)]^{\beta_3} \leq cs^{\varepsilon_0}, \quad s \geq 1. \quad (6.10)$$

Recall $D_{\tilde{w}}(a, b) = \{x = (\tilde{x}, x_d) \in \mathbb{R}^d : |\tilde{x} - \tilde{w}| < a, 0 < x_d < b\}$.

Lemma 6.4. *Suppose that (6.7) holds. There exists $C_{24} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ with $|\tilde{x} - \tilde{y}| > 3$ and $0 < x_d, y_d < 1/4$,*

$$\begin{aligned} & \int_{D_{\tilde{x}}(1,1)} \int_{D_{\tilde{y}}(1,1)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2}}{|x-w|^{d-\alpha} |y-z|^{d-\alpha}} \\ & \quad \times \left(\log \left(1 + \frac{w_d \vee z_d}{w_d \wedge z_d} \right) \right)^{\beta_3} \left(\log \left(1 + \frac{8}{w_d \vee z_d} \right) \right)^{\beta_4} dz dw \leq C_{24} x_d^p y_d^p. \end{aligned} \quad (6.11)$$

Proof. Define $\hat{\beta}_1 = \beta_1 - \varepsilon_0$, $\hat{\beta}_2 = \tilde{\beta}_2 + \varepsilon_0$. Note that by the definition of ε_0 , $p < \alpha + \hat{\beta}_1$. Note also that by (6.9) we can estimate $(w_d \vee z_d)^{\beta_2} (\log(1 + 8/(w_d \vee z_d)))^{\beta_4}$ by a constant times $(w_d \vee z_d)^{\hat{\beta}_2}$. By (6.10) and Tonelli's theorem, the left-hand side of (6.11) is less than or equal to

$$\begin{aligned} & c_1 \int_{D_{\tilde{x}}(1,1)} \int_{D_{\tilde{y}}(1,1)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\hat{\beta}_1} (w_d \vee z_d)^{\hat{\beta}_2}}{|x-w|^{d-\alpha} |y-z|^{d-\alpha}} dz dw \\ & = c_1 \left(\int_{\{(z,w) \in D_{\tilde{x}}(1,1) \times D_{\tilde{y}}(1,1) : z_d < w_d\}} + \int_{\{(z,w) \in D_{\tilde{x}}(1,1) \times D_{\tilde{y}}(1,1) : z_d \geq w_d\}} \right) \\ & \quad \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\hat{\beta}_1} (w_d \vee z_d)^{\hat{\beta}_2}}{|x-w|^{d-\alpha} |y-z|^{d-\alpha}} dz dw \\ & = c_1 \int_{D_{\tilde{x}}(1,1)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{w_d^{\hat{\beta}_2}}{|x-w|^{d-\alpha}} \left(\int_{D_{\tilde{y}}(1,w_d)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_1}}{|y-z|^{d-\alpha}} dz \right) dw \\ & \quad + c_1 \int_{D_{\tilde{y}}(1,1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_2}}{|y-z|^{d-\alpha}} \left(\int_{D_{\tilde{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{w_d^{\hat{\beta}_1}}{|x-w|^{d-\alpha}} dw \right) dz. \end{aligned}$$

By symmetry, we only need to bound the last term above.

Since $\hat{\beta}_1 + \alpha > p > \alpha - 1$, we can apply Corollary 6.3 (ii) (with $R = 1$, $a = z_d$, $q = p$, $\gamma = \hat{\beta}_1$ and $\beta = \delta = 0$) to get

$$\begin{aligned} & \int_{D_{\tilde{y}}(1,1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_2}}{|y-z|^{d-\alpha}} \left(\int_{D_{\tilde{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{w_d^{\hat{\beta}_1}}{|x-w|^{d-\alpha}} dw \right) dz \\ & \leq c_4 x_d^p \int_{D_{\tilde{y}}(1,1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_2 + \alpha + \beta_1 - p}}{|y-z|^{d-\alpha}} dz. \end{aligned}$$

By (6.7) we have

$$(\tilde{\beta}_2 + \alpha + \beta_1 - p) + \alpha > p.$$

Thus, we can apply Corollary 6.3 (ii) again (with $R = 1$, $a = 1$, $q = p$, $\gamma = \tilde{\beta}_2 + \alpha + \beta_1 - p$ and $\beta = \delta = 0$) and conclude that

$$\int_{D_{\tilde{y}}(1,1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_2}}{|y-z|^{d-\alpha}} \left(\int_{D_{\tilde{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{w_d^{\hat{\beta}_1} dw}{|x-w|^{d-\alpha}} \right) dz \leq c_5 x_d^p y_d^p. \quad \blacksquare$$

Lemma 6.5. *Suppose (6.7) holds. There exists $C_{25} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ with $|\tilde{x} - \tilde{y}| > 4$ and $0 < x_d, y_d < 1/4$,*

$$G(x, y) \leq C_{25} x_d^p y_d^p.$$

Proof. Assume $x = (\tilde{0}, x_d)$ with $0 < x_d < 1/4$, and let $D = D(1, 1)$ and $V = D_{\tilde{y}}(1, 1)$. By Lemma 5.5,

$$G(w, y) \leq c_1 \left(\frac{y_d}{|w-y|} \wedge 1 \right)^p \leq c_2 y_d^p, \quad w \in \mathbb{R}^d \setminus V.$$

Thus by Lemma 3.4,

$$\mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \notin V] \leq c_3 y_d^p \mathbb{P}_x(Y_{\tau_D} \in \mathbb{R}_+^d) \leq c_4 y_d^p x_d^p.$$

On the other hand, since $2 < |z-w| < 8$ for $(w, z) \in D \times V$, it follows that $\log\left(1 + \frac{|z-w|}{(w_d \vee z_d) \wedge |z-w|}\right) \leq \log\left(1 + \frac{8}{w_d \vee z_d}\right)$, and thus

$$J(w, z) \leq c_5 (w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2} \left(\log\left(1 + \frac{w_d \vee z_d}{w_d \wedge z_d}\right) \right)^{\beta_3} \left(\log\left(1 + \frac{8}{w_d \vee z_d}\right) \right)^{\beta_4}$$

for $(w, z) \in D \times V$. By using the Lévy system formula (3.4) (with $f = G(\cdot, y)$) in the first equality, and (5.3) in the third line, we conclude that

$$\begin{aligned} & \mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \in V] \\ &= \int_D G^D(x, w) \int_V J(w, z) G(z, y) dz dw \leq \int_D G(x, w) \int_V J(w, z) G(z, y) dz dw \\ &\leq c_8 \int_D \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{1}{|x-w|^{d-\alpha}} \\ &\quad \times \int_V (w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2} \left(\log\left(1 + \frac{w_d \vee z_d}{w_d \wedge z_d}\right) \right)^{\beta_3} \\ &\quad \times \left(\log\left(1 + \frac{8}{w_d \vee z_d}\right) \right)^{\beta_4} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{dz}{|y-z|^{d-\alpha}} dw, \end{aligned}$$

which is less than or equal to $c_6 x_d^p y_d^p$ by Lemma 6.4. Therefore

$$G(x, y) = \mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \notin V] + \mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \in V] \leq c_7 x_d^p y_d^p. \quad \blacksquare$$

6.2. Green function estimates for $p \in [\alpha + \frac{\beta_1 + \beta_2}{2}, \alpha + \beta_1)$

In this subsection we deal with the case

$$\alpha + \frac{\beta_1 + \beta_2}{2} \leq p < \alpha + \beta_1. \quad (6.12)$$

Note that (6.12) implies $\beta_2 < \beta_1$ and

$$\alpha + \beta_2 < p, \quad (6.13)$$

$$2\alpha - 2p + \beta_1 + \beta_2 \leq 0. \quad (6.14)$$

Recall that $B_a^+ := B(0, a) \cap \mathbb{R}_+^d$, $a > 0$. The lower bound in the following theorem sharpens the lower bound in Lemma 5.4 under the assumption (6.12).

Theorem 6.6. *Suppose (6.12) holds. For every $\varepsilon \in (0, 1/4)$, there exists a constant $C_{26} > 0$ such that for all $w \in \partial\mathbb{R}_+^d$, $R > 0$ and $x, y \in B(w, (1 - \varepsilon)R) \cap \mathbb{R}_+^d$,*

$$G^{B(w, R) \cap \mathbb{R}_+^d}(x, y) \geq \frac{C_{26}}{|x - y|^{d - \alpha}} \left(\frac{x_d \wedge y_d}{|x - y|} \wedge 1 \right)^p \\ \times \begin{cases} \left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^{2\alpha - p + \beta_1 + \beta_2} \left(\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right)^{\beta_4} & \text{if } \alpha + \frac{\beta_1 + \beta_2}{2} < p < \alpha + \beta_1, \\ \left(\frac{x_d \vee y_d}{|x - y|} \wedge 1 \right)^p \left(\log \left(1 + \frac{|x - y|}{(x_d \vee y_d) \wedge |x - y|} \right) \right)^{\beta_4 + 1} & \text{if } p = \alpha + \frac{\beta_1 + \beta_2}{2}. \end{cases}$$

Proof. By scaling, translation and symmetry, without loss of generality, we assume that $w = 0$, $R = 1$ and $x_d \leq y_d$. Moreover, by Theorem 5.1, we only need to show that there exists a constant $c_1 > 0$ such that for all $x, y \in B_{1 - \varepsilon}^+$ with $x_d \leq y_d$ satisfying $|x - y| \geq (40/\varepsilon)y_d$,

$$G^{B_1^+}(x, y) \geq \frac{c_1 x_d^p}{|x - y|^{d + \alpha + \beta_1 + \beta_2}} \begin{cases} y_d^{2\alpha - p + \beta_1 + \beta_2} (\log(|x - y|/y_d))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ y_d^p (\log(|x - y|/y_d))^{\beta_4 + 1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \quad (6.15)$$

We assume that $x, y \in B_{1 - \varepsilon}^+$ with $x_d \leq y_d$ satisfying $|x - y| \geq (40/\varepsilon)y_d$. By the Harnack inequality (Theorem 1.4), we can further assume that $4x_d \leq y_d$. Let $M = 40/\varepsilon$ and $r = 4|x - y|/M$.

By the Lévy system formula (3.4) (with $f = G^{B_1^+}(\cdot, y)$) and regular harmonicity of $w \mapsto G^{B_1^+}(w, y)$ on $D_{\bar{x}}(2r, 2r)$,

$$G^{B_1^+}(x, y) \geq \mathbb{E}_x[G^{B_1^+}(Y_{\tau_{D_{\bar{x}}}(2r, 2r)}, y); Y_{\tau_{D_{\bar{x}}}(2, 2)} \in D_{\bar{y}}(r, r)] \\ = \int_{D_{\bar{x}}(2r, 2r)} G^{D_{\bar{x}}(2r, 2r)}(x, w) \int_{D_{\bar{y}}(r, r)} J(w, z) G^{B_1^+}(z, y) dz dw \\ \geq \int_{D_{\bar{x}}(r, r)} G^{D_{\bar{x}}(2r, 2r)}(x, w) \int_{D_{\bar{y}}(r, r)} J(w, z) G^{B_1^+}(z, y) dz dw \\ \geq \int_{D_{\bar{x}}(r, r)} G^{B((\bar{x}, 0), 2r) \cap \mathbb{R}_+^d}(x, w) \int_{D_{\bar{y}}(r, r)} J(w, z) G^{B_1^+}(z, y) dz dw. \quad (6.16)$$

Since $D_{\tilde{x}}(r, r) \subset B((\tilde{x}, 0), \sqrt{2}r) \cap \mathbb{R}_+^d$ and $D_{\tilde{y}}(r, r) \subset B_{1-\varepsilon/4}^+$, by Theorem 5.1 we have

$$\begin{aligned} & G^{B((\tilde{x}, 0), 2r) \cap \mathbb{R}_+^d}(x, w) \\ & \geq c_2 \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \left(\frac{w_d}{|w-x|} \wedge 1 \right)^p \frac{1}{|x-w|^{d-\alpha}}, \quad w \in D_{\tilde{x}}(r, r), \end{aligned} \quad (6.17)$$

and

$$G^{B_1^+}(z, y) \geq c_3 \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \left(\frac{z_d}{|z-y|} \wedge 1 \right)^p \frac{1}{|y-z|^{d-\alpha}}, \quad z \in D_{\tilde{y}}(r, r). \quad (6.18)$$

Moreover, since $(w_d \vee z_d) \leq |z-w| \asymp r$ for $(w, z) \in D_{\tilde{x}}(r, r) \times D_{\tilde{y}}(r, r)$, we have

$$\begin{aligned} J(w, z) & \geq c_4 |w-z|^{-d-\alpha} \left(\frac{w_d \wedge z_d}{|w-z|} \wedge 1 \right)^{\beta_1} \left(\frac{w_d \vee z_d}{|w-z|} \wedge 1 \right)^{\beta_2} \\ & \quad \times \left(\log \left(1 + \frac{|w-z|}{(w_d \vee z_d) \wedge |w-z|} \right) \right)^{\beta_4} \\ & \geq c_5 \frac{(w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2}}{r^{d+\alpha+\beta_1+\beta_2}} \left(\log \frac{2r}{w_d \vee z_d} \right)^{\beta_4}, \quad (w, z) \in D_{\tilde{x}}(r, r) \times D_{\tilde{y}}(r, r). \end{aligned} \quad (6.19)$$

Using (6.17)–(6.19), we obtain

$$\begin{aligned} & \int_{D_{\tilde{x}}(r, r)} G^{B((\tilde{x}, 0), 2r) \cap \mathbb{R}_+^d}(x, w) \int_{D_{\tilde{y}}(r, r)} J(w, z) G^{B_1^+}(z, y) dz dw \\ & \geq \frac{c_6}{r^{d+\alpha+\beta_1+\beta_2}} \int_{D_{\tilde{x}}(r, r)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \left(\frac{w_d}{|w-x|} \wedge 1 \right)^p \frac{1}{|x-w|^{d-\alpha}} \\ & \quad \times \int_{D_{\tilde{y}}(r, r)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \left(\frac{z_d}{|z-y|} \wedge 1 \right)^p \\ & \quad \times \frac{(w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2}}{|y-z|^{d-\alpha}} \left(\log \frac{2r}{w_d \vee z_d} \right)^{\beta_4} dz dw \\ & \geq \frac{c_7}{r^{d+\alpha+\beta_1+\beta_2}} \int_{D_{\tilde{y}}(r, r) \setminus D_{\tilde{y}}(r, 3y_d/2)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \left(\frac{z_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\beta_2}}{|y-z|^{d-\alpha}} \\ & \quad \times \left(\log \frac{2r}{z_d} \right)^{\beta_4} \left(\int_{D_{\tilde{x}}(r, z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \left(\frac{w_d}{|w-x|} \wedge 1 \right)^p \frac{w_d^{\beta_1} dw}{|x-w|^{d-\alpha}} \right) dz \\ & \geq \frac{c_8 x_d^p y_d^p}{r^{d+\alpha+\beta_1+\beta_2}} \int_{D_{\tilde{y}}(r, r) \setminus D_{\tilde{y}}(r, 3y_d/2)} \frac{z_d^{p+\beta_2}}{|y-z|^{d+2p-\alpha}} \left(\log \frac{2r}{z_d} \right)^{\beta_4} \\ & \quad \times \left(\int_{D_{\tilde{x}}(r, z_d) \setminus D_{\tilde{x}}(r, 3x_d/2)} \frac{w_d^{p+\beta_1} dw}{|x-w|^{d+2p-\alpha}} \right) dz. \end{aligned}$$

Now by applying Lemma 6.1 (ii) with $R = r$, $a_2 = z_d$, $a_3 = 3x_d/2$, $\gamma = p + \beta_1$, $q = 2p$ and $\beta = \delta = 0$ in the inner integral, we find that for $z_d \geq 3y_d/2$,

$$\int_{D_{\tilde{x}}(r, z_d) \setminus D_{\tilde{x}}(r, 3x_d/2)} \frac{w_d^{p+\beta_1} dw}{|x-w|^{d+2p-\alpha}} \geq c_9 (z_d^{\alpha-p+\beta_1} - (3x_d/2)^{\alpha-p+\beta_1}) \geq c_{10} z_d^{\alpha-p+\beta_1}.$$

In the last inequality above, we have used the the assumption $4x_d \leq y_d$ so that for all $z_d \geq 3y_d/2$ we have $z_d/4 \geq 3x_d/2$. Thus,

$$\begin{aligned} & \int_{D_{\tilde{x}}(r, r)} G^{B((\tilde{x}, 0), 2r) \cap \mathbb{R}_+^d}(x, w) \int_{D_{\tilde{y}}(r, r)} J(w, z) G^{B_1^+}(z, y) dz dw \\ & \geq \frac{c_{11} x_d^p y_d^p}{r^{d+\alpha+\beta_1+\beta_2}} \int_{D_{\tilde{y}}(r, r) \setminus D_{\tilde{y}}(r, 3y_d/2)} \frac{z_d^{\beta_1+\beta_2+\alpha}}{|y-z|^{d+2p-\alpha}} \left(\log \frac{2r}{z_d} \right)^{\beta_4} dz. \end{aligned} \quad (6.20)$$

Finally, applying Lemma 6.1 (ii) with $R = r$, $a_2 = r$, $a_3 = 3y_d/2$, $\gamma = \alpha + \beta_1 + \beta_2$, $q = 2p$, $\beta = \beta_4$ and $\delta = 0$ and using the fact that $y_d < r/4$, we conclude that the above is greater than or equal to

$$\frac{c_{12} x_d^p y_d^p}{r^{d+\alpha+\beta_1+\beta_2}} \begin{cases} y_d^{2\alpha-2p+\beta_1+\beta_2} \left(\log \frac{r}{y_d} \right)^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ \left(\log \frac{r}{y_d} \right)^{\beta_4+1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \quad (6.21)$$

Recalling that $r = 4|x-y|/M$ and combining (6.16), (6.20) and (6.21), we have proved that (6.15) holds. \blacksquare

We now consider the upper bound of $G(x, y)$.

Lemma 6.7. *Suppose (6.12) holds. There exists $C_{27} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ with $|\tilde{x} - \tilde{y}| > 3$, and $0 < 4x_d \leq y_d < \frac{1}{4}$ or $0 < 4y_d \leq x_d < \frac{1}{4}$,*

$$\begin{aligned} & \int_{D_{\tilde{x}}(1, 1)} dw \int_{D_{\tilde{y}}(1, 1)} dz \left(\frac{x_d \wedge w_d}{|w-x|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2}}{|x-w|^{d-\alpha} |y-z|^{d-\alpha}} \\ & \quad \times \left(\log \left(1 + \frac{w_d \vee z_d}{w_d \wedge z_d} \right) \right)^{\beta_3} \left(\log \left(1 + \frac{2}{w_d \vee z_d} \right) \right)^{\beta_4} \left(\frac{y_d \wedge z_d}{|z-y|} \wedge 1 \right)^p \\ & \leq C_{27} (x_d \wedge y_d)^p \begin{cases} (x_d \vee y_d)^{2\alpha-p+\beta_1+\beta_2} (\log(1/(x_d \vee y_d)))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ (x_d \vee y_d)^p (\log(1/(x_d \vee y_d)))^{\beta_4+1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \end{aligned} \quad (6.22)$$

Proof. By symmetry, we only need to consider the case $0 \leq 4x_d \leq y_d \leq 1/4$. Define

$$\varepsilon_0 := 2^{-1} \mathbf{1}_{\beta_3 > 0} [(\alpha + \beta_1 - p) \wedge (p - \alpha - \beta_2)], \quad \hat{\beta}_1 := \beta_1 - \varepsilon_0, \quad \hat{\beta}_2 := \beta_2 + \varepsilon_0.$$

Note that $p < \alpha + \hat{\beta}_1$ and $p > \alpha + \hat{\beta}_2$ by (6.13).

By (6.10),

$$\begin{aligned}
& \int_{D_{\bar{x}}(1,1)} dw \int_{D_{\bar{y}}(1,1)} dz \left(\frac{x_d \wedge w_d}{|w-x|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2}}{|x-w|^{d-\alpha} |y-z|^{d-\alpha}} \\
& \quad \times \left(\log \left(1 + \frac{w_d \vee z_d}{w_d \wedge z_d} \right) \right)^{\beta_3} \left(\log \frac{2}{w_d \vee z_d} \right)^{\beta_4} \left(\frac{y_d \wedge z_d}{|z-y|} \wedge 1 \right)^p \\
& \leq c_1 \left(\int_{\{(z,w) \in D_{\bar{x}}(1,1) \times D_{\bar{y}}(1,1) : z_d < w_d\}} + \int_{\{(z,w) \in D_{\bar{x}}(1,1) \times D_{\bar{y}}(1,1) : z_d \geq w_d\}} \right) \\
& \quad \times \left(\frac{x_d \wedge w_d}{|w-x|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\hat{\beta}_1} (w_d \vee z_d)^{\hat{\beta}_2}}{|x-w|^{d-\alpha} |y-z|^{d-\alpha}} \left(\log \frac{2}{w_d \vee z_d} \right)^{\beta_4} \left(\frac{y_d \wedge z_d}{|z-y|} \wedge 1 \right)^p dz dw \\
& \leq c_1 \int_{D_{\bar{y}}(1,1)} \left(\frac{y_d \wedge z_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_1}}{|y-z|^{d-\alpha}} \\
& \quad \times \int_{D_{\bar{x}}(1,1) \setminus D_{\bar{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2} dw}{|x-w|^{d-\alpha}} dz \\
& + c_1 \int_{D_{\bar{y}}(1,1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{(\log(2/z_d))^{\beta_4} z_d^{\hat{\beta}_2}}{|y-z|^{d-\alpha}} \left(\int_{D_{\bar{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{w_d^{\hat{\beta}_1} dw}{|x-w|^{d-\alpha}} \right) dz \\
& =: I_1 + I_2.
\end{aligned}$$

Since $\hat{\beta}_1 > p - \alpha > \beta_2 \geq 0$, we can apply Corollary 6.3 (ii) to estimate the inner integral in I_2 to get

$$I_2 \leq c_2 x_d^p \int_{D_{\bar{y}}(1,1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\beta_2 + \alpha + \beta_1 - p}}{|y-z|^{d-\alpha}} \left(\log \frac{2}{z_d} \right)^{\beta_4} dz. \quad (6.23)$$

By (6.14),

$$0 < \beta_2 + \alpha + \beta_1 - p \leq p - \alpha.$$

Thus we can apply Corollary 6.3 (i) to find that (and by using $y_d < 1/4$ we may replace 2 with 1)

$$I_2 \leq c_3 x_d^p \begin{cases} y_d^{2\alpha - p + \beta_1 + \beta_2} (\log(1/y_d))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ y_d^p (\log(1/y_d))^{\beta_4 + 1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \quad (6.24)$$

We now consider

$$\begin{aligned}
I_1 & \leq c_1 \int_{D_{\bar{y}}(1,2x_d)} \left(\frac{z_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_1}}{|y-z|^{d-\alpha}} \\
& \quad \times \int_{D_{\bar{x}}(1,1) \setminus D_{\bar{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2} dw}{|x-w|^{d-\alpha}} dz \\
& + c_1 \int_{D_{\bar{y}}(1,1) \setminus D_{\bar{y}}(1,2x_d)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_1}}{|y-z|^{d-\alpha}} \\
& \quad \times \int_{D_{\bar{x}}(1,1) \setminus D_{\bar{x}}(1,z_d)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2} dw}{|x-w|^{d-\alpha}} dz
\end{aligned}$$

$$\begin{aligned}
&\leq c_1 \int_{D_{\bar{y}}(1, 2x_d)} \frac{z_d^{\hat{\beta}_1+p}}{|y-z|^{d-\alpha+p}} dz \int_{D_{\bar{x}}(1, 1)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2}}{|x-w|^{d-\alpha}} dw \\
&+ c_1 x_d^p \int_{D_{\bar{y}}(1, 1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\hat{\beta}_1}}{|y-z|^{d-\alpha}} \int_{D_{\bar{x}}(1, 1) \setminus D_{\bar{x}}(1, z_d)} \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2}}{|x-w|^{d-\alpha+p}} dw dz \\
&=: I_{11} + x_d^p I_{12}.
\end{aligned}$$

Since $p \geq \alpha$ and $4x_d \leq y_d$, we can apply Lemma 6.1 (i) (with $a_1 = 2x_d$, $\gamma = p + \hat{\beta}_1$, $q = p$, $\beta = \delta = 0$) to get

$$\int_{D_{\bar{y}}(1, 2x_d)} \frac{z_d^{\hat{\beta}_1+p}}{|y-z|^{d-\alpha+p}} dz \leq c_4 y_d^{\alpha-p-1} x_d^{p+\hat{\beta}_1+1}.$$

Since $\alpha + \hat{\beta}_2 < p$, by Corollary 6.3 (i) it follows that

$$\int_{D_{\bar{x}}(1, 1)} \left(\frac{x_d}{|w-x|} \wedge 1 \right)^p \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2}}{|x-w|^{d-\alpha}} dw \leq c_5 x_d^{\alpha+\hat{\beta}_2} \left(\log \frac{2}{x_d} \right)^{\beta_4}.$$

Thus, we have

$$\begin{aligned}
I_{11} &\leq c_6 y_d^{\alpha-p-1} x_d^{p+\hat{\beta}_1+1} x_d^{\alpha+\hat{\beta}_2} (\log(2/x_d))^{\beta_4} \\
&= c_6 x_d^p x_d^{\alpha+\beta_1+\beta_2+1} (\log(2/x_d))^{\beta_4} y_d^{\alpha-p-1} \\
&\leq c_6 x_d^p y_d^{\alpha+\beta_1+\beta_2+1} (\log(2/y_d))^{\beta_4} y_d^{\alpha-p-1} \\
&\leq \tilde{c}_6 x_d^p y_d^{2\alpha-p+\beta_1+\beta_2} (\log(1/y_d))^{\beta_4}.
\end{aligned} \tag{6.25}$$

Here we have used the fact that $t \mapsto t^{\alpha+\beta_1+\beta_2+1} (\log(2/t))^{\beta_4}$ is almost increasing on $(0, 1/4]$.

Finally, we take care of I_{12} . Note that for every $z \in D_{\bar{y}}(1, 1) \setminus D_{\bar{y}}(1, 2x_d)$, we have $z_d > 2x_d$ and so, since $\alpha + \hat{\beta}_2 < p$, by Lemma 6.1 (ii) with $R = a_2 = 1$, $a_3 = z_d$, $\gamma = \hat{\beta}_2$, $q = p$, $\beta = \beta_4$, $\delta = 0$,

$$\int_{D_{\bar{x}}(1, 1) \setminus D_{\bar{x}}(1, z_d)} \frac{(\log(2/w_d))^{\beta_4} w_d^{\hat{\beta}_2}}{|x-w|^{d-\alpha+p}} dw \leq c z_d^{\alpha+\hat{\beta}_2-p} \left(\log \frac{2}{z_d} \right)^{\beta_4}.$$

Thus,

$$I_{12} \leq c_8 \int_{D_{\bar{y}}(1, 1)} \left(\frac{y_d}{|z-y|} \wedge 1 \right)^p \frac{z_d^{\beta_2+\alpha+\beta_1-p}}{|y-z|^{d-\alpha}} \left(\log \frac{2}{z_d} \right)^{\beta_4} dz.$$

By the same argument as that in in (6.23) and (6.24), we now have

$$I_{12} \leq c_9 \begin{cases} y_d^{2\alpha-p+\beta_1+\beta_2} (\log(1/y_d))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ y_d^p (\log(1/y_d))^{\beta_4+1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \tag{6.26}$$

By combining (6.24)–(6.26) and symmetry, we have proved the lemma. \blacksquare

Remark 6.8. In the proof of Lemma 6.7, if we used Tonelli's theorem on I_1 and estimated it as I_2 (instead of using the argument to bound I_{11} and I_{12} separately), we would not obtain the sharp upper bound.

Proposition 6.9. *Suppose (6.12) holds. There exists $C_{28} > 0$ such that for all $x, y \in \mathbb{R}_+^d$ with $0 < x_d, y_d < 1/4$ with $|\tilde{x} - \tilde{y}| > 4$,*

$$G(x, y) \leq C_{28}(x_d \wedge y_d)^p \begin{cases} (x_d \vee y_d)^{2\alpha-p+\beta_1+\beta_2} (\log(1/(x_d \vee y_d)))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ (x_d \vee y_d)^p (\log(1/(x_d \vee y_d)))^{\beta_4+1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases}$$

Proof. Without loss of generality, we assume $\tilde{x} = \tilde{0}$. By symmetry, we consider the case $0 < x_d \leq y_d < 1/4$ only. By the Harnack inequality (Theorem 1.4), it suffices to deal with the case $0 < 4x_d \leq y_d < 1/4$. Let $D = D(1, 1)$ and $V = D_{\tilde{y}}(1, 1)$. By the Lévy system formula (3.4) (with $f = G(\cdot, y)$), (5.3), Lemma 6.7, and the fact that $2 < |z - w| < 8$ below (so that $|z - w| \asymp 2$)

$$\begin{aligned} & \mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \in V] \\ &= \int_D G^D(x, w) \int_V J(w, z) G(z, y) dz dw \leq \int_D G(x, w) \int_V J(w, z) G(z, y) dz dw \\ &\leq c_1 \int_{D_{\tilde{x}}(1, 1)} dw \int_{D_{\tilde{y}}(1, 1)} dz \left(\frac{x_d \wedge w_d}{|w - x|} \wedge 1 \right)^p \frac{(w_d \wedge z_d)^{\beta_1} (w_d \vee z_d)^{\beta_2}}{|x - w|^{d-\alpha} |y - z|^{d-\alpha}} \\ &\quad \times \left(\log \left(1 + \frac{w_d \vee z_d}{w_d \wedge z_d} \right) \right)^{\beta_3} \left(\log \left(1 + \frac{2}{w_d \vee z_d} \right) \right)^{\beta_4} \left(\frac{y_d \wedge z_d}{|z - y|} \wedge 1 \right)^p \\ &\leq c_2 (x_d \wedge y_d)^p \begin{cases} (x_d \vee y_d)^{2\alpha-p+\beta_1+\beta_2} (\log(1/(x_d \vee y_d)))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ (x_d \vee y_d)^p \log(1/(x_d \vee y_d))^{\beta_4+1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \end{aligned}$$

Moreover, by the same argument as in the proof of Lemma 6.5, we also have

$$\mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \notin V] \leq c_3 y_d^p \mathbb{P}_x(Y_{\tau_D} \in \mathbb{R}_+^d) \leq c_4 y_d^p x_d^p.$$

Therefore

$$\begin{aligned} G(x, y) &= \mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \notin V] + \mathbb{E}_x[G(Y_{\tau_D}, y); Y_{\tau_D} \in V] \\ &\leq c_5 (x_d \wedge y_d)^p \begin{cases} (x_d \vee y_d)^{2\alpha-p+\beta_1+\beta_2} (\log(1/(x_d \vee y_d)))^{\beta_4} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 < 0, \\ (x_d \vee y_d)^p (\log(1/(x_d \vee y_d)))^{\beta_4+1} & \text{if } 2\alpha - 2p + \beta_1 + \beta_2 = 0. \end{cases} \quad \blacksquare \end{aligned}$$

6.3. Proof of Theorem 1.1 and estimates of potentials

With the preparations in the previous two subsections, we are now ready to prove Theorem 1.1. We recall [41, Theorem 3.14] on the Hölder continuity of bounded harmonic functions: There exist constants $c > 0$ and $\gamma \in (0, 1)$ such that for every $x_0 \in \mathbb{R}_+^d$, $r \in (0, 1]$ such that $B(x_0, 2r) \subset \mathbb{R}_+^d$ and every bounded $f : \mathbb{R}_+^d \rightarrow [0, \infty)$ which is harmonic in $B(x_0, 2r)$, we have

$$|f(x) - f(y)| \leq c \|f\|_\infty (|x - y|/r)^\gamma \quad \text{for all } x, y \in B(x_0, r). \quad (6.27)$$

Proof of Theorem 1.1. The existence and regular harmonicity of the Green function D were shown in Proposition 2.2. We now prove the continuity of G . We fix $x_0, y_0 \in \mathbb{R}_+^d$ and choose a positive a small enough so that $B(x_0, 4a) \cap B(y_0, 4a) = \emptyset$ and $B(x_0, 4a) \cup B(y_0, 4a) \subset \mathbb{R}_+^d$.

We recall that by [41, Proposition 3.11(b)], $\mathbb{E}_y \tau_{B(x_0, 2a)} \leq c_1 a^\alpha$ for all $y \in B(x_0, a)$. Let $N \geq 1/a$. By using (3.4) in the second line and Proposition 4.7 in the fourth, we have, for every $y \in B(x_0, a)$,

$$\begin{aligned} & \mathbb{E}_y [G(Y_{\tau_{B(x_0, 2a)}}, y_0); Y_{\tau_{B(x_0, 2a)}} \in B(y_0, 1/N)] \\ &= \mathbb{E}_y \left(\int_0^{\tau_{B(x_0, 2a)}} \int_{B(y_0, 1/N)} G(w, y_0) J(Y_s, w) dw ds \right) \\ &\leq \left(\sup_{y \in B(x_0, a)} \mathbb{E}_y \tau_{B(x_0, 2a)} \right) \left(\sup_{z \in B(x_0, 2a)} \int_{B(y_0, 1/N)} J(z, w) G(w, y_0) dw \right) \\ &\leq c_1 a^\alpha (8a)^{-d-\alpha} \int_{B(y_0, 1/N)} |w - y_0|^{-d+\alpha} dw = c_2 a^{-d} (1/N)^\alpha. \end{aligned}$$

Now choose N large enough so that $c_2 a^{-d} (1/N)^\alpha < \varepsilon/4$. Then

$$\sup_{y \in B(x_0, a)} \mathbb{E}_y [G(Y_{\tau_{B(x_0, 2a)}}, y_0); Y_{\tau_{B(x_0, 2a)}} \in B(y_0, 1/N)] < \varepsilon/4.$$

Since by Proposition 4.7,

$$x \mapsto h(x) := \mathbb{E}_x [G(Y_{\tau_{B(x_0, 2a)}}, y_0); Y_{\tau_{B(x_0, 2a)}} \in \mathbb{R}_+^d \setminus B(y_0, 1/N)]$$

is a bounded function which is harmonic on $B(x_0, a)$, it is continuous by (6.27) so we can choose a $\delta \in (0, a)$ such that $|h(x) - h(x_0)| < \varepsilon/2$ for all $x \in B(x_0, \delta)$. Therefore, for all $x \in B(x_0, \delta)$,

$$\begin{aligned} & |G(x, y_0) - G(x_0, y_0)| \\ &\leq |h(x) - h(x_0)| + 2 \sup_{y \in B(x_0, a)} \mathbb{E}_y [G(Y_{\tau_{B(x_0, 2a)}}, y_0); Y_{\tau_{B(x_0, 2a)}} \in B(y_0, 1/N)] < \varepsilon. \end{aligned}$$

Now combining Theorem 5.1 and Lemma 6.5 with (2.5), we arrive at Theorem 1.1 (1); and combining Theorem 6.6, Proposition 6.9 and (2.5), we arrive at Theorem 1.1 (2, 3). \blacksquare

As an application of Theorem 1.1, we get the following estimates on killed potentials of Y .

Proposition 6.10. *Suppose that $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. Then for any $\tilde{w} \in \mathbb{R}^{d-1}$, any Borel set D satisfying $D_{\tilde{w}}(R/2, R/2) \subset D \subset D_{\tilde{w}}(R, R)$ and any $x = (\tilde{w}, x_d)$ with $0 < x_d \leq R/10$,*

$$\mathbb{E}_x \int_0^{\tau_D} (Y_t^d)^\gamma dt = \int_D G^D(x, y) y_d^\gamma dy \asymp \begin{cases} R^{\alpha+\gamma-p} x_d^p, & \gamma > p - \alpha, \\ x_d^p \log(R/x_d), & \gamma = p - \alpha, \\ x_d^{\alpha+\gamma}, & -p - 1 < \gamma < p - \alpha, \end{cases} \quad (6.28)$$

where the comparison constant is independent of $\tilde{w} \in \mathbb{R}^{d-1}$, D , R and x .

Proof. Without loss of generality, we assume $\tilde{w} = \tilde{x} = \tilde{0}$.

(i) *Upper bound:* Note that, by Lemma 5.5,

$$\begin{aligned} \int_D G^D(x, y) y_d^\gamma dy &\leq \int_{D(R, R)} G(x, y) y_d^\gamma dy \\ &\leq c_0 \left(\int_{D(R, x_d/2)} y_d^{\gamma+p} |x - y|^{\alpha-d-p} dy \right. \\ &\quad \left. + \int_{D(R, R) \setminus D(R, x_d/2)} y_d^\gamma \left(\frac{x_d}{|x - y|} \wedge 1 \right)^p |x - y|^{\alpha-d} dy \right) \\ &= c_0 \left(\int_{D(R, x_d/2)} f(y; \gamma + p, 0, p, 0, x) dy \right. \\ &\quad \left. + x_d^\gamma \int_{D(R, 3x_d/2) \setminus D(R, x_d/2)} g(y; 0, p, 0, x) dy \right. \\ &\quad \left. + x_d^p \int_{D(R, R) \setminus D(R, 3x_d/2)} f(y; \gamma, 0, p, 0, x) dy \right) \\ &=: c_0(I_1 + I_2 + I_3). \end{aligned}$$

Suppose first $-p - 1 < \gamma < p - \alpha$. We apply Lemma 6.1 (i) to I_1 (which is allowed since $\gamma + p > -1$) and Lemma 6.1 (iii) to I_2 . Then

$$I_1 \asymp x_d^{\alpha-p-1} (x_d/2)^{\gamma+p+1} \asymp x_d^{\alpha+\gamma} \quad \text{and} \quad I_2 \asymp x_d^\gamma x_d^\alpha = x_d^{\alpha+\gamma}.$$

Finally,

$$I_3 \asymp x_d^p R^{\gamma+\alpha-p} F\left(\frac{3x_d}{2R}; \gamma + \alpha - p - 1, 0\right) \asymp x_d^p R^{\gamma+\alpha-p} \left(\frac{3x_d}{2R}\right)^{\gamma+\alpha-p} \asymp x_d^{\alpha+\gamma}.$$

Here the first asymptotic equality follows from Lemma 6.1 (ii) (with $a_2 = R$ and $a_3 = 3x_d/2$) and the second from the definition of $F(\cdot; \cdot, 0)$.

This completes the proof of the upper bound in the case $-p - 1 < \gamma < p - \alpha$. The other two cases are similar, but simpler, since one can directly use Corollary 6.3 (i) with Lemma 5.5. We omit the details.

(ii) *Lower bound:* We first note that by Theorem 5.1,

$$\begin{aligned} \mathbb{E}_x \int_0^{\tau_D} (Y_t^d)^\gamma dt &\geq \int_{B_{R/2}^+} y_d^\gamma G^{B_{R/2}^+}(x, y) dy \geq \int_{D(R/5, R/5)} y_d^\gamma G^{B_{R/2}^+}(x, y) dy \\ &\geq c x_d^p \int_{D(R/5, R/5) \setminus D(R/5, 3x_d/2)} \frac{y_d^{p+\gamma} dy}{|x-y|^{d-\alpha+2p}} \\ &= c x_d^p \int_{D(R/5, R/5) \setminus D(R/5, 3x_d/2)} f(y; \gamma + p, 0, 2p, 0, x) dy. \end{aligned}$$

Since $3x_d/2 < 3R/20$ (so that $3x_d/(2R/5) \leq 3/4$), using Lemma 6.1 (ii) (with $a_2 = R/5$ and $a_3 = 3x_d/2$) and applying (6.1) and (6.3), we immediately get the lower bound. ■

Remark 6.11. (a) It follows from the proof of Proposition 6.10 and Remark 6.2 that

$$\int_D G^D(x, y) y_d^\gamma dy = \infty \quad \text{if } \gamma \leq -p - 1.$$

(b) By Proposition 6.10, for any $\beta_1 \geq 0$, and all $r \in (0, 1]$ and $x \in U(r)$,

$$\begin{aligned} r^{\alpha+\beta_1-p} x_d^p &\asymp \mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} dt \leq \mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_1} |\log Y_t^d|^{\beta_3} dt \\ &\leq c \mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{(p-\alpha+\beta_1)/2} dt \asymp r^{\alpha+(p-\alpha+\beta_1)/2-p} x_d^p = r^{(\alpha+\beta_1-p)/2} x_d^p \leq x_d^p. \end{aligned}$$

Thus, Proposition 6.10 is a significant generalization of Lemma 3.1.

We end this section with the following corollary, which follows from Proposition 6.10 and Remark 6.11 by letting $R \rightarrow \infty$. Recall that $Y_{\zeta-} = \lim_{t \uparrow \zeta} Y_t$ denotes the left limit of the process Y at its lifetime.

Corollary 6.12. *Suppose that $p \in ((\alpha - 1)_+, \alpha + \beta_1)$. Then for all $x \in \mathbb{R}_+^d$,*

$$\mathbb{E}_x \int_0^\zeta (Y_t^d)^\gamma dt = \int_{\mathbb{R}_+^d} G(x, y) y_d^\gamma dy \asymp \begin{cases} \infty, & \gamma \geq p - \alpha \text{ or } \gamma \leq -p - 1, \\ x_d^{\alpha+\gamma}, & -p - 1 < \gamma < p - \alpha. \end{cases}$$

In particular, for all $x \in \mathbb{R}_+^d$, we have $\mathbb{P}_x(Y_{\zeta-} \in \mathbb{R}_+^d, \zeta < \infty) = G\kappa(x) \asymp c > 0$ and

$$\mathbb{E}_x[\zeta] \asymp \begin{cases} \infty, & p \leq \alpha, \\ x_d^\alpha, & p > \alpha. \end{cases}$$

7. Boundary Harnack principle

In this section we give a proof of Theorem 1.2. We start with a lemma providing important estimates of the jump kernel J needed in the proof. Recall that $U = D(1/2, 1/2)$.

Note the exponent $\beta_1 - \varepsilon$ in (7.3) below is not necessarily positive, but is always strictly larger than -1 .

Lemma 7.1. *Suppose $p \in ((\alpha - 1)_+, \alpha + (\beta_1 \wedge \beta_2))$ and let*

$$k(y) = \frac{(y_d \wedge 1)^{\beta_1} (y_d \vee 1)^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} (1 + |\log y_d|)^{\beta_3} \left(\log \left(1 + \frac{|y|}{y_d \vee 1} \right) \right)^{\beta_4}. \quad (7.1)$$

(a) *Let $z^{(0)} = (\tilde{0}, 2^{-2})$. Then for any $z \in B(z^{(0)}, 2^{-3})$ and $y \in \mathbb{R}_+^d \setminus D(1, 1)$,*

$$J(z, y) \geq ck(y). \quad (7.2)$$

(b) *Let*

$$\varepsilon = \beta_1 + \alpha - p - \frac{\beta_2 + \alpha - p}{M}, \quad \text{where } M = 1 + \left(\frac{\beta_2 + \alpha - p}{\beta_1 + \alpha - p} \vee 1 \right).$$

Then for any $z \in U$ and $y \in \mathbb{R}_+^d \setminus D(1, 1)$,

$$J(z, y) \leq cz_d^{\beta_1 - \varepsilon} k(y). \quad (7.3)$$

Proof. (a) For $z \in B(z^{(0)}, 2^{-3})$ and $y \in \mathbb{R}_+^d \setminus D(1, 1)$, $z_d \asymp z_d^{(0)} = 2^{-2}$ and $|z - y| \asymp |z^{(0)} - y| \asymp |y| > c$, which immediately implies (7.2).

(b) Let $\delta = (1 - 1/M)(\beta_2 + \alpha - p) > 0$. We first note that by the definitions of M , δ and ε , we have

$$\varepsilon > \beta_1 + \alpha - p - \frac{\beta_2 + \alpha - p}{\frac{\beta_2 + \alpha - p}{\beta_1 + \alpha - p} \vee 1} = \beta_1 + \alpha - p - (\beta_1 + \alpha - p) \wedge (\beta_2 + \alpha - p) \geq 0 \quad (7.4)$$

and

$$\begin{aligned} \beta_2 + \varepsilon &= \beta_2 + \beta_1 + \alpha - p - \frac{\beta_2 + \alpha - p}{M} = \beta_1 + \left(1 - \frac{1}{M} \right) (\beta_2 + \alpha - p) \\ &= \beta_1 + \delta > \beta_1. \end{aligned} \quad (7.5)$$

Assume that $z \in U$ and $y \in \mathbb{R}_+^d \setminus D(1, 1)$. Since $|z - y| \asymp |y| \geq c(z_d \vee y_d)$,

$$\begin{aligned} J(z, y) &\asymp \frac{(z_d \wedge y_d)^{\beta_1} (z_d \vee y_d)^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(\log \left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d} \right) \right)^{\beta_3} \\ &\quad \times \left(\log \left(1 + \frac{|y|}{(y_d \vee z_d) \wedge |y|} \right) \right)^{\beta_4}. \end{aligned} \quad (7.6)$$

Clearly, if $y_d \geq 3/4 > 1/2 \geq z_d$, then

$$\frac{|y|}{(y_d \vee z_d) \wedge |y|} \asymp \frac{|y|}{(y_d \vee 1) \wedge |y|} \asymp \frac{|y|}{y_d \vee 1}$$

and

$$\begin{aligned} \log \left(1 + \frac{y_d}{z_d} \right) &\leq 3 \log \frac{y_d}{z_d} \leq 3 \left(|\log y_d| + \log \frac{1}{z_d} \right) \\ &\leq 6 |\log y_d| \log \frac{1}{z_d} + 3 \log \frac{1}{z_d} \leq 6 \left(\log \frac{1}{z_d} \right) (1 + |\log y_d|). \end{aligned}$$

Thus, for $z \in U$ and $y \in \mathbb{R}_+^d \setminus D(1, 1)$ with $y_d \geq 3/4$,

$$\begin{aligned} J(z, y) &\asymp \frac{z_d^{\beta_1} y_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(\log \frac{y_d}{z_d} \right)^{\beta_3} \left(\log \left(1 + \frac{|y|}{y_d \vee 1} \right) \right)^{\beta_4} \\ &\leq c z_d^{\beta_1} \left(\log \frac{1}{z_d} \right)^{\beta_3} k(y). \end{aligned} \quad (7.7)$$

It is easy to see from (7.6) that for $(z, y) \in U \times (\mathbb{R}_+^d \setminus D(1, 1))$ with $y_d < 3/4$ and $z_d > y_d$,

$$J(z, y) \leq c \frac{y_d^{\beta_1} z_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(\log \frac{1}{y_d} \right)^{\beta_3} \left(\log \frac{|y|}{z_d} \right)^{\beta_4}.$$

Since $\delta > 0$, we have

$$\begin{aligned} z_d^\delta \left(\log \frac{|y|}{z_d} \right)^{\beta_4} &= |y|^\delta \left(\frac{z_d}{|y|} \right)^\delta \left(\log \frac{|y|}{z_d} \right)^{\beta_4} \\ &\leq c |y|^\delta \left(\frac{2^{-1}}{|y|} \right)^\delta \left(\log \frac{|y|}{2^{-1}} \right)^{\beta_4} \leq c (\log(2|y|))^{\beta_4}, \quad 0 < z_d \leq 1/2 < 1 < |y|. \end{aligned} \quad (7.8)$$

Thus, using (7.5),

$$J(z, y) \leq c \frac{y_d^{\beta_1} z_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(\log \frac{1}{y_d} \right)^{\beta_3} \left(\log \frac{|y|}{z_d} \right)^{\beta_4} \leq c z_d^{\beta_2-\delta} k(y) = c z_d^{\beta_1-\varepsilon} k(y). \quad (7.9)$$

Since $\varepsilon > 0$ by (7.4), we have

$$z_d^\varepsilon \left(\log \frac{1}{z_d} \right)^{\beta_3} \leq c y_d^\varepsilon \left(\log \frac{1}{y_d} \right)^{\beta_3}, \quad 0 < z_d \leq y_d < 3/2,$$

so that by using the same argument as in (7.8),

$$\begin{aligned} z_d^\varepsilon \left(\log \frac{1}{z_d} \right)^{\beta_3} \left(\log \frac{|y|}{y_d} \right)^{\beta_4} &\leq c y_d^{\varepsilon-\delta} \left(\log \frac{1}{y_d} \right)^{\beta_3} y_d^\delta \left(\log \frac{|y|}{y_d} \right)^{\beta_4} \\ &\leq c y_d^{\varepsilon-\delta} \left(\log \frac{1}{y_d} \right)^{\beta_3} (\log(2|y|))^{\beta_4}, \quad 0 < z_d \leq y_d < 3/2 < 1 < |y|. \end{aligned}$$

Thus using (7.5) in the last inequality below, we find that, for $(z, y) \in U \times (\mathbb{R}_+^d \setminus D(1, 1))$ with $y_d < 3/4$ and $z_d \leq y_d$,

$$\begin{aligned} J(z, y) &\leq c \frac{z_d^{\beta_1} y_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(\log \frac{1}{z_d} \right)^{\beta_3} \left(\log \frac{|y|}{y_d} \right)^{\beta_4} \\ &= c z_d^{\beta_1-\varepsilon} \frac{y_d^{\beta_2}}{|y|^{d+\alpha+\beta_1+\beta_2}} z_d^\varepsilon \left(\log \frac{1}{z_d} \right)^{\beta_3} \left(\log \frac{|y|}{y_d} \right)^{\beta_4} \\ &\leq c z_d^{\beta_1-\varepsilon} \frac{y_d^{\beta_2+\varepsilon-\delta}}{|y|^{d+\alpha+\beta_1+\beta_2}} \left(\log \frac{1}{y_d} \right)^{\beta_3} (\log(2|y|))^{\beta_4} \leq c z_d^{\beta_1-\varepsilon} k(y). \end{aligned} \quad (7.10)$$

Combining (7.7), (7.9) and (7.10), and using the inequality

$$z^{\beta_1 - \varepsilon} \vee (z^{\beta_1} (\log(1/z_d))^{\beta_3}) \leq c z_d^{\beta_1 - \varepsilon}, \quad z \in U,$$

we get the upper bound (7.3) for $J(z, y)$. \blacksquare

Proof of Theorem 1.2. By scaling, we just need to consider the case $r = 1$. Moreover, by Theorem 1.4 (b), it suffices to prove (1.8) for $x, y \in D_{\tilde{w}}(2^{-8}, 2^{-8})$.

Since f is harmonic in $D_{\tilde{w}}(2, 2)$ and vanishes continuously on $B((\tilde{w}, 0), 2) \cap \partial\mathbb{R}_+^d$, it is regular harmonic in $D_{\tilde{w}}(7/4, 7/4)$ and vanishes continuously on $B((\tilde{w}, 0), 7/4) \cap \partial\mathbb{R}_+^d$. Throughout the remainder of this proof, we assume that $x \in D_{\tilde{w}}(2^{-8}, 2^{-8})$. Without loss of generality we take $\tilde{w} = 0$.

Define $z^{(0)} = (\tilde{0}, 2^{-2})$. By Theorem 1.4 (b) and Lemma 3.2, we have

$$\begin{aligned} f(x) &= \mathbb{E}_x[f(Y_{\tau_U})] \geq \mathbb{E}_x[f(Y_{\tau_U}); Y_{\tau_U} \in D(1/2, 1) \setminus D(1/2, 3/4)] \\ &\geq c_1 f(z^{(0)}) \mathbb{P}_x(Y_{\tau_{D_{\tilde{x}}}(1/4, 1/4)}} \in D_{\tilde{x}}(1/4, 1) \setminus D_{\tilde{x}}(1/4, 3/4)) \geq c_2 f(z^{(0)}) x_d^p. \end{aligned} \quad (7.11)$$

Let k be the function defined in (7.1). Using (7.2), the harmonicity of f , the Lévy system formula and [41, Proposition 3.11 (a)],

$$\begin{aligned} f(z^{(0)}) &\geq \mathbb{E}_{z^{(0)}}[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1, 1)] \\ &\geq \mathbb{E}_{z^{(0)}} \int_0^{\tau_{B(z^{(0)}, 2^{-3})}} \int_{\mathbb{R}_+^d \setminus D(1, 1)} J(Y_t, y) f(y) dy dt \\ &\geq c_{10} \mathbb{E}_{z^{(0)}} \tau_{B(z^{(0)}, 2^{-3})} \int_{\mathbb{R}_+^d \setminus D(1, 1)} k(y) f(y) dy \geq c_{11} \int_{\mathbb{R}_+^d \setminus D(1, 1)} k(y) f(y) dy. \end{aligned} \quad (7.12)$$

Now we assume that $z \in U$ and $y \in \mathbb{R}_+^d \setminus D(1, 1)$. Let ε be defined as in Lemma 7.1. Since $\beta_1 - \varepsilon > \beta_1 - (\alpha + \beta_1 - p) = p - \alpha$, by Proposition 6.10 and (7.3) we have

$$\begin{aligned} \mathbb{E}_x[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1, 1)] &= \mathbb{E}_x \int_0^{\tau_U} \int_{\mathbb{R}_+^d \setminus D(1, 1)} J(Y_t, y) f(y) dy dt \\ &\leq c \mathbb{E}_x \int_0^{\tau_U} (Y_t^d)^{\beta_1 - \varepsilon} dt \int_{\mathbb{R}_+^d \setminus D(1, 1)} k(y) f(y) dy \leq c x_d^p \int_{\mathbb{R}_+^d \setminus D(1, 1)} k(y) f(y) dy. \end{aligned} \quad (7.13)$$

Combining this with (7.12), we now have

$$\mathbb{E}_x[f(Y_{\tau_U}); Y_{\tau_U} \notin D(1, 1)] \leq c x_d^p f(z^{(0)}). \quad (7.14)$$

On the other hand, since f is a non-negative function in \mathbb{R}_+^d which is harmonic in $D_{\tilde{w}}(2, 2)$ with respect to Y and vanishes continuously on $B((\tilde{w}, 0), 2) \cap \partial\mathbb{R}_+^d$, by Theorem 1.4 (b) and Carleson's estimate (Theorem 1.5) we conclude that $f(v) \leq c_{16} f(z^{(0)})$

for all $v \in D(1, 1)$. Therefore, by Lemma 3.3,

$$\mathbb{E}_x[f(Y_{\tau_U}); Y_{\tau_U} \in D(1, 1)] \leq c_{16} f(z^{(0)}) \mathbb{P}_x(Y_{\tau_U} \in D(1, 1)) \leq c_{17} f(z^{(0)}) x_d^p. \quad (7.15)$$

Combining (7.14), (7.15) and (7.11) we get $f(x) \asymp x_d^p f(z^{(0)})$ for all $x \in D(2^{-8}, 2^{-8})$, which implies that for all $x, y \in D(2^{-8}, 2^{-8})$,

$$\frac{f(x)}{f(y)} \leq c_7 \frac{x_d^p}{y_d^p},$$

which is the conclusion of the theorem. \blacksquare

Proof of Theorem 1.3. The case $\alpha + \beta_2 < p < \alpha + \beta_1$ has been dealt with in [41, Theorem 1.4.], so we only need to deal with $p = \alpha + \beta_2$. The proof is the same as that of [41, Theorem 1.4.], except that we can now use Proposition 6.10 to get, for all $r > 0$ and $x = (\tilde{0}, x_d)$ with $0 < x_d \leq r/10$,

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_2} dt \asymp x_d^{\beta_2 + \alpha} \log(r/x_d) = x_d^p \log(r/x_d). \quad (7.16)$$

Moreover, using (7.16), we also get, for every $r > 0$ and $x \in U(r)$,

$$\mathbb{E}_x \int_0^{\tau_{U(r)}} (Y_t^d)^{\beta_2} dt \leq \mathbb{E}_x \int_0^{\tau_{D_x(5r, 5r)}} (Y_t^d)^{\beta_2} dt \leq c_0 x_d^p \log(r/x_d). \quad (7.17)$$

The displays (7.16) and (7.17) will be used to replace [41, Lemmas 5.11 and 5.12]. We provide the full proof for the convenience of the reader.

Suppose that the non-scale-invariant BHP holds near the boundary of \mathbb{R}_+^d (see the paragraph before Theorem 1.3).

Note that by taking $g(x) = \mathbb{P}_x(Y_{\tau_U} \in D(1/2, 1) \setminus D(1/2, 3/4))$, we see from Lemma 3.2 that there exists $\hat{R} \in (0, 1)$ such that for any $r \in (0, \hat{R}]$ there exists $c_1 = c_1(r) > 0$ such that for any non-negative function f in \mathbb{R}_+^d which is harmonic in $\mathbb{R}_+^d \cap B(0, r)$ with respect to Y and vanishes continuously on $\partial\mathbb{R}_+^d \cap B(0, r)$,

$$\frac{f(x)}{f(y)} \leq c_1 \frac{x_d^p}{y_d^p} \quad \text{for all } x, y \in \mathbb{R}_+^d \cap B(0, r/2). \quad (7.18)$$

Let $r_0 = \hat{R}/4$ and choose a point $z_0 \in \partial\mathbb{R}_+^d$ with $|z_0| = 4$. For $n \in \mathbb{N}$, $B(z_0, 1/n)$ does not intersect $B(0, 2r_0)$. We define

$$K_n := \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} |\log y_d|^{\beta_3 + \beta_4} dy, \quad f_n(y) := K_n^{-1} y_d^{-\beta_1} \mathbf{1}_{\mathbb{R}_+^d \cap B(z_0, 1/n)}(y),$$

and

$$\begin{aligned} g_n(x) &:= \mathbb{E}_x[f_n(Y_{\tau_{U(r_0)}})] \\ &= \mathbb{E}_x \int_0^{\tau_{U(r_0)}} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} J(Y_t, y) f_n(y) dy dt, \quad x \in U(r_0). \end{aligned}$$

We claim that there exists $c_2 > 0$ such that

$$\liminf_{n \rightarrow \infty} g_n(x) \geq c_2 x_d^{\beta_2 + \alpha} \log(r_0/x_d) = c_2 x_d^p \log(r_0/x_d) \quad (7.19)$$

for all $x = x^{(s)} = (\tilde{0}, s) \in \mathbb{R}_+^d$ with $s \in (0, r_0/10)$.

Here is a proof of the claim above. Since

$$6 > |z - y| > 2 > y_d \wedge z_d \quad \text{for } (y, z) \in (\mathbb{R}_+^d \cap B(z_0, 1/n)) \times U(r_0),$$

using **(A3)** we have for $(y, z) \in (\mathbb{R}_+^d \cap B(z_0, 1/n)) \times U(r_0)$,

$$\begin{aligned} J(z, y) &\asymp (z_d \wedge y_d)^{\beta_1} (z_d \vee y_d)^{\beta_2} \left(\log \left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{z_d \vee y_d} \right)^{\beta_4} \\ &\asymp \frac{z_d^{\beta_1} y_d^{\beta_1}}{(z_d \vee y_d)^{\beta_1 - \beta_2}} \left(\log \left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{z_d \vee y_d} \right)^{\beta_4}. \end{aligned}$$

Therefore, for $x \in U(r_0)$,

$$\begin{aligned} g_n(x) &\asymp K_n^{-1} \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (Y_t^d \vee y_d)^{-(\beta_1 - \beta_2)} \\ &\quad \times \left(\log \left(1 + \frac{Y_t^d \vee y_d}{Y_t^d \wedge y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{Y_t^d \vee y_d} \right)^{\beta_4} dy dt. \quad (7.20) \end{aligned}$$

Note that, using $\sup_{t \geq 1} t^{-(\beta_1 - \beta_2)} (\log(1+t))^{\beta_3} < \infty$, for $z \in U(r_0)$,

$$\begin{aligned} &K_n^{-1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (z_d \vee y_d)^{-(\beta_1 - \beta_2)} \left(\log \left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d} \right) \right)^{\beta_3} \left(\log \left(\frac{1}{z_d \vee y_d} \right) \right)^{\beta_4} dy \\ &\leq \frac{K_n^{-1}}{z_d^{\beta_1 - \beta_2}} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n) \cap \{z_d \leq y_d\}} (z_d/y_d)^{\beta_1 - \beta_2} \left(\log \left(1 + \frac{y_d}{z_d} \right) \right)^{\beta_3} \left(\log \frac{1}{y_d} \right)^{\beta_4} dy \\ &\quad + \frac{K_n^{-1}}{z_d^{\beta_1 - \beta_2}} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n) \cap \{z_d > y_d\}} \left(\log \left(1 + \frac{z_d}{y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{z_d} \right)^{\beta_4} dy \\ &\leq c_3 \frac{K_n^{-1}}{z_d^{\beta_1 - \beta_2}} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} \left(\log \frac{1}{y_d} \right)^{\beta_3 + \beta_4} dy \leq c_4 z_d^{-(\beta_1 - \beta_2)} \quad (7.21) \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} K_n^{-1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (z_d \vee y_d)^{-(\beta_1 - \beta_2)} \left(\log \left(1 + \frac{z_d \vee y_d}{z_d \wedge y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{z_d \vee y_d} \right)^{\beta_4} dy \\ = z_d^{-(\beta_1 - \beta_2)}. \end{aligned}$$

Moreover, by (7.16), $\mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_2} dt < \infty$ for all $x \in U(r_0)$. Thus we can use the dominated convergence theorem to deduce that for all $x \in U(r_0)$,

$$\begin{aligned}
\lim_{n \rightarrow \infty} K_n^{-1} \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (Y_t^d \vee z_d)^{-(\beta_1 - \beta_2)} \\
\times \left(\log \left(1 + \frac{Y_t^d \vee y_d}{Y_t^d \wedge y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{Y_t^d \vee y_d} \right)^{\beta_4} dy dt \\
= \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} (Y_t^d)^{-(\beta_1 - \beta_2)} dt = \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_2} dt. \quad (7.22)
\end{aligned}$$

Combining (7.22) with (7.16) we conclude that (7.19) holds true.

From (7.20), (7.21) and (7.17) we see that for all $x \in U(r_0)$,

$$\begin{aligned}
g_n(x) &\leq c_5 K_n^{-1} \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_1} \int_{\mathbb{R}_+^d \cap B(z_0, 1/n)} (Y_t^d)^{-(\beta_1 - \beta_2)} \\
&\quad \times \left(\log \left(1 + \frac{Y_t^d \vee y_d}{Y_t^d \wedge y_d} \right) \right)^{\beta_3} \left(\log \frac{1}{Y_t^d \vee y_d} \right)^{\beta_4} dy dt \\
&\leq c_6 \mathbb{E}_x \int_0^{\tau_{U(r_0)}} (Y_t^d)^{\beta_2} dt \leq c_7 x_d^p \log(r_0/x_d). \quad (7.23)
\end{aligned}$$

Thus the g_n 's are non-negative functions in \mathbb{R}_+^d which are harmonic in $\mathbb{R}_+^d \cap B(0, 2^{-2}r_0)$ with respect to Y and vanish continuously on $\partial\mathbb{R}_+^d \cap B(0, 2^{-2}r_0)$. Therefore, by (7.18),

$$\frac{g_n(y)}{g_n(w)} \leq c_1 \frac{y_d^p}{w_d^p} \quad \text{for all } y \in D \cap B(0, 2^{-3}r_0),$$

where $w = (\tilde{0}, 2^{-3}r_0)$ and $c_1 = c_1(2^{-2}r_0)$. Thus by (7.23), for all $y \in \mathbb{R}_+^d \cap B(0, 2^{-3}r_0)$,

$$\limsup_{n \rightarrow \infty} g_n(y) \leq c_1 \limsup_{n \rightarrow \infty} g_n(w) \frac{y_d^p}{w_d^p} \leq c_8 y_d^p.$$

This and (7.19) imply that for all $x = x^{(s)} = (\tilde{0}, s) \in \mathbb{R}_+^d$ with $s \in (0, r_0/10)$, we have $x_d^p \log(r_0/x_d) \leq c_9 x_d^p$, which gives a contradiction. \blacksquare

Acknowledgments. We thank the anonymous referees for very helpful comments which led to improvements of this paper.

Funding. Panki Kim's work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (NRF-2021R1A4A1027378).

Part of the research for this paper was done while Renming Song was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004) and by the Priority Academic Program Development of Jiangsu Higher Education Institutions. His research was also supported in part by a grant from the Simons Foundation (#429343, Renming Song).

Zoran Vondraček's research was supported in part by the Croatian Science Foundation under the project 4197.

References

- [1] Abatangelo, N. Gomez-Castro, D., Vázquez, J. L.: Singular boundary behaviour and large solutions for fractional elliptic equations. J. London Math. Soc., to appear; arXiv:1910.00366

- [2] Bae, J., Kang, J., Kim, P., Lee, J.: Heat kernel estimates and their stabilities for symmetric jump processes with general mixed polynomial growths on metric measure spaces. arXiv:1904.10189v2 (2019)
- [3] Blumenthal, R. M., Gettoor, R. K.: *Markov Processes and Potential Theory*. Pure Appl. Math. 29, Academic Press, New York (1968) Zbl 0169.49204 MR 0264757
- [4] Bogdan, K.: The boundary Harnack principle for the fractional Laplacian. *Studia Math.* **123**, 43–80 (1997) Zbl 0870.31009 MR 1438304
- [5] Bogdan, K., Burdzy, K., Chen, Z.-Q.: Censored stable processes. *Probab. Theory Related Fields* **127**, 89–152 (2003) Zbl 1032.60047 MR 2006232
- [6] Bogdan, K., Dyda, B., Kim, P.: Hardy inequalities and non-explosion results for semigroups. *Potential Anal.* **44**, 229–247 (2016) Zbl 1335.31007 MR 3460023
- [7] Bogdan, K., Grzywny, T., Pietruska-Pałuba, K., Rutkowski, A.: Extension and trace for non-local operators. *J. Math. Pures Appl.* (9) **137**, 33–69 (2020) Zbl 1452.46023 MR 4088505
- [8] Bogdan, K., Grzywny, T., Ryznar, M.: Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.* **38**, 1901–1923 (2010) Zbl 1204.60074 MR 2722789
- [9] Bogdan, K., Grzywny, T., Ryznar, M.: Dirichlet heat kernel for unimodal Lévy processes. *Stochastic Process. Appl.* **124**, 3612–3650 (2014) Zbl 1320.60112 MR 3249349
- [10] Bogdan, K., Kulczycki, T., Kwaśnicki, M.: Estimates and structure of α -harmonic functions. *Probab. Theory Related Fields* **140**, 345–381 (2008) Zbl 1146.31004 MR 2365478
- [11] Bonforte, M., Figalli, A., Vázquez, J. L.: Sharp global estimates for local and nonlocal porous medium-type equations in bounded domains. *Anal. PDE* **11**, 945–982 (2018) Zbl 1443.35067 MR 3749373
- [12] Bonforte, M., Sire, Y., Vázquez, J. L.: Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discrete Contin. Dynam. Systems* **35**, 5725–5767 (2015) Zbl 1347.35129 MR 3393253
- [13] Bonforte, M., Vázquez, J. L.: Fractional nonlinear degenerate diffusion equations on bounded domains part I. Existence, uniqueness and upper bounds. *Nonlinear Anal.* **131**, 363–398 (2016) Zbl 1328.35009 MR 3427986
- [14] Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32**, 1245–1260 (2007) Zbl 1143.26002 MR 2354493
- [15] Caffarelli, L. A., Stinga, P. R.: Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **33**, 767–807 (2016) Zbl 1381.35211 MR 3489634
- [16] Chen, Z.-Q., Kim, P.: Green function estimate for censored stable processes. *Probab. Theory Related Fields* **124**, 595–610 (2002) Zbl 1016.60070 MR 1942325
- [17] Chen, Z.-Q., Kim, P., Song, R.: Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc.* **12**, 1307–1329 (2010) Zbl 1203.60114 MR 2677618
- [18] Chen, Z.-Q., Kim, P., Song, R.: Two-sided heat kernel estimates for censored stable-like processes. *Probab. Theory Related Fields* **146**, 361–399 (2010) Zbl 1190.60068 MR 2574732
- [19] Chen, Z.-Q., Kim, P., Song, R.: Dirichlet heat kernel estimates for rotationally symmetric Lévy processes. *Proc. London Math. Soc.* (3) **109**, 90–120 (2014) Zbl 1304.60080 MR 3237737
- [20] Chen, Z.-Q., Kumagai, T.: Heat kernel estimates for stable-like processes on d -sets. *Stochastic Process. Appl.* **108**, 27–62 (2003) Zbl 1075.60556 MR 2008600
- [21] Chen, Z.-Q., Kumagai, T.: Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Related Fields* **140**, 277–317 (2008) Zbl 1131.60076 MR 2357678
- [22] Chen, Z.-Q., Song, R.: Estimates on Green functions and Poisson kernels for symmetric stable processes. *Math. Ann.* **312**, 465–501 (1998) Zbl 0918.60068 MR 1654824
- [23] Chen, Z.-Q., Tokle, J.: Global heat kernel estimates for fractional Laplacians in unbounded open sets. *Probab. Theory Related Fields* **149**, 373–395 (2011) Zbl 1231.60078 MR 2776619

- [24] Cho, S., Kim, P., Song, R., Vondraček, Z.: Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings. *J. Math. Pures Appl. (9)* **143**, 208–256 (2020) Zbl [1460.60087](#) MR [4163128](#)
- [25] Cho, S., Kim, P., Song, R., Vondraček, Z.: Heat kernel estimates for subordinate Markov processes and their applications. *J. Differential Equations* **316**, 28–93 (2022) Zbl [1491.60135](#) MR [4374849](#)
- [26] Chung, K. L., Walsh, J. B.: *Markov Processes, Brownian Motion, and Time Symmetry*. 2nd ed., Grundlehren Math. Wiss. 249, Springer, New York (2005) Zbl [1082.60001](#) MR [2152573](#)
- [27] Daskalopoulos, P., Lee, K.-A.: Hölder regularity of solutions of degenerate elliptic and parabolic equations. *J. Funct. Anal.* **201**, 341–379 (2003) Zbl [1107.35033](#) MR [1986693](#)
- [28] Dellacherie, C., Duarte, M., Martínez, S., San Martín, J., Vandaele, P.: Powers of Brownian Green potentials. *Potential Anal.* **56**, 227–265 (2022) Zbl [1482.60103](#) MR [4367940](#)
- [29] Feehan, P. M. N., Pop, C. A.: Schauder a priori estimates and regularity of solutions to boundary-degenerate elliptic linear second-order partial differential equations. *J. Differential Equations* **256**, 895–956 (2014) Zbl [1286.35116](#) MR [3128929](#)
- [30] Fornaro, S., Metafune, G., Pallara, D., Schnaubelt, R.: Degenerate operators of Tricomi type in L^p -spaces and in spaces of continuous functions. *J. Differential Equations* **252**, 1182–1212 (2012) Zbl [1235.35139](#) MR [2853535](#)
- [31] Fukushima, M., Oshima, Y., Takeda, M.: *Dirichlet Forms and Symmetric Markov Processes*. De Gruyter Stud. Math. 19, de Gruyter, Berlin (2011) Zbl [1227.31001](#) MR [2778606](#)
- [32] Grubb, G.: Regularity of spectral fractional Dirichlet and Neumann problems. *Math. Nachr.* **289**, 831–844 (2016) Zbl [1448.47062](#) MR [3503820](#)
- [33] Grzywny, T., Kim, K.-Y., Kim, P.: Estimates of Dirichlet heat kernel for symmetric Markov processes. *Stochastic Process. Appl.* **130**, 431–470 (2020) Zbl [1468.60097](#) MR [4035035](#)
- [34] Grzywny, T., Kwaśnicki, M.: Potential kernels, probabilities of hitting a ball, harmonic functions and the boundary Harnack inequality for unimodal Lévy processes. *Stochastic Process. Appl.* **128**, 1–38 (2018) Zbl [1386.60265](#) MR [3729529](#)
- [35] Kim, K.-H.: Sobolev space theory of parabolic equations degenerating on the boundary of C^1 domains. *Comm. Partial Differential Equations* **32**, 1261–1280 (2007) Zbl [1124.35031](#) MR [2354494](#)
- [36] Kim, P., Mimica, A.: Estimates of Dirichlet heat kernels for subordinate Brownian motions. *Electron. J. Probab.* **23**, art. 64, 45 pp. (2018) Zbl [1410.60082](#) MR [3835470](#)
- [37] Kim, P., Song, R., Vondraček, Z.: Two-sided Green function estimates for killed subordinate Brownian motions. *Proc. London Math. Soc. (3)* **104**, 927–958 (2012) Zbl [1248.60091](#) MR [2928332](#)
- [38] Kim, P., Song, R., Vondraček, Z.: Global uniform boundary Harnack principle with explicit decay rate and its application. *Stochastic Process. Appl.* **124**, 235–267 (2014) Zbl [1296.60198](#) MR [3131293](#)
- [39] Kim, P., Song, R., Vondraček, Z.: Potential theory of subordinate killed Brownian motion. *Trans. Amer. Math. Soc.* **371**, 3917–3969 (2019) Zbl [1450.60040](#) MR [3917213](#)
- [40] Kim, P., Song, R., Vondraček, Z.: On the boundary theory of subordinate killed Lévy processes. *Potential Anal.* **53**, 131–181 (2020) Zbl [1451.60090](#) MR [4117984](#)
- [41] Kim, P., Song, R., Vondraček, Z.: On potential theory of Markov processes with jump kernels decaying at the boundary. *Potential Anal.* (online, 2021)
- [42] Kulczycki, T.: Properties of Green function of symmetric stable processes. *Probab. Math. Statist.* **17**, 339–364 (1997) Zbl [0903.60063](#) MR [1490808](#)
- [43] Kyprianou, A. E., Pardo, J. C., Watson, A. R.: Hitting distributions of α -stable processes via path censoring and self-similarity. *Ann. Probab.* **42**, 398–430 (2014) Zbl [1306.60051](#) MR [3161489](#)

-
- [44] Song, R., Vondraček, Z.: Potential theory of subordinate killed Brownian motion in a domain. *Probab. Theory Related Fields* **125**, 578–592 (2003) Zbl [1022.60078](#) MR [1974415](#)
- [45] Song, R., Wu, J.-M.: Boundary Harnack principle for symmetric stable processes. *J. Funct. Anal.* **168**, 403–427 (1999) Zbl [0945.31006](#) MR [1719233](#)
- [46] Stinga, P. R., Torrea, J. L.: Extension problem and Harnack’s inequality for some fractional operators. *Comm. Partial Differential Equations* **35**, 2092–2122 (2010) Zbl [1209.26013](#) MR [2754080](#)
- [47] Wang, C., Xin, Z.: Optimal Hölder continuity for a class of degenerate elliptic problems with an application to subsonic-sonic flows. *Comm. Partial Differential Equations* **36**, 873–924 (2011) Zbl [1245.76132](#) MR [2769112](#)