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PBW theory for quantum affine algebras

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Abstract. Let $U'_q(\mathfrak{g})$ be a quantum affine algebra of *arbitrary* type and let $\mathscr{C}^0_\mathfrak{g}$ be Hernandez-Leclerc's category. We can associate the quantum affine Schur–Weyl duality functor $\mathcal{F}_\mathcal{D}$ to a duality datum $\mathcal{D}$ in $\mathscr{C}^0_\mathfrak{g}$. In this paper, we introduce the notion of a strong (complete) duality datum $\mathcal{D}$ and prove that, when $\mathcal{D}$ is strong, the induced duality functor $\mathcal{F}_\mathcal{D}$ sends simple modules to simple modules and preserves the invariants Λ, $\widetilde{\Lambda}$ and Λ^∞ introduced by the authors. We next define the reflections $\mathscr{S}_k$ and $\mathscr{S}_k^{-1}$ acting on strong duality data $\mathcal{D}$. We prove that if $\mathcal{D}$ is a strong (resp. complete) duality datum, then $\mathscr{S}_k(\mathcal{D})$ and $\mathscr{S}_k^{-1}(\mathcal{D})$ are also strong (resp. complete) duality data. This allows us to make new strong (resp. complete) duality data by applying the reflections $\mathscr{S}_k$ and $\mathscr{S}_k^{-1}$ from known strong (resp. complete) duality data. We finally introduce the notion of affine cuspidal modules in $\mathscr{C}^0_\mathfrak{g}$ by using the duality functor $\mathcal{F}_\mathcal{D}$, and develop the cuspidal module theory for quantum affine algebras similar to the quiver Hecke algebra case. When $\mathcal{D}$ is complete, we show that all simple modules in $\mathscr{C}^0_\mathfrak{g}$ can be constructed as the heads of ordered tensor products of affine cuspidal modules. We further prove that the ordered tensor products of affine cuspidal modules have the unitriangularity property. This generalizes the classical simple module construction using ordered tensor products of fundamental modules.

Keywords. Affine cuspidal modules, quantum affine Schur–Weyl duality, Hernandez–Leclerc category, quantum affine algebra, quiver Hecke algebra, PBW theory

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1. Introduction

Let q be an indeterminate and let $\mathscr{C}_{\mathfrak{g}}$ be the category of finite-dimensional integrable modules over a quantum affine algebra $U_q'(\mathfrak{g})$. The category $\mathscr{C}_{\mathfrak{g}}$ occupies an important position in the study of quantum affine algebras because of its rich structure. The simple modules in $\mathscr{C}_{\mathfrak{g}}$ are indexed by using n-tuples of polynomials with constant term 1 (called *Drinfeld polynomials*) ([5–7] for the untwisted cases and [8] for the twisted cases). The simple modules can be obtained as the head of ordered tensor product of *fundamental representations* [1, 25, 53], and a geometric approach to simple modules was also studied in [45, 46, 53].

Let $\mathfrak{g}_0$ be a finite-dimensional simple Lie algebra of ADE type and let $U_q'(\mathfrak{g})$ be a quantum affine algebra of untwisted affine ADE type. Hernandez and Leclerc [15] introduced the monoidal full subcategory $\mathscr{C}_{\mathfrak{g}}^0$ of $\mathscr{C}_{\mathfrak{g}}$, which consists of objects all of whose simple subquotients are obtained from the heads of tensor products of certain fundamental representations. Any simple module in $\mathscr{C}_{\mathfrak{g}}$ can be obtained as a tensor product of suitable parameter shifts of simple modules in $\mathscr{C}_{\mathfrak{g}}^0$. For each Dynkin quiver Q of $\mathfrak{g}_0$ with a height function, Hernandez and Leclerc [16] introduced a monoidal subcategory $\mathscr{C}_Q$ of $\mathscr{C}_{\mathfrak{g}}^0$. The category $\mathscr{C}_Q$ is defined by using certain fundamental representations parameterized by vertices of the Auslander–Reiten quiver of Q. It turns out that the complexified Grothendieck ring $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathscr{C}_Q)$ is isomorphic to the coordinate ring $\mathbb{C}[N]$ of the unipotent group N associated with $\mathfrak{g}_0$ and, under this isomorphism, the set of isomorphism classes of simple modules in $\mathscr{C}_Q$ corresponds to the *upper global basis* (or *dual canonical basis*) of $\mathbb{C}[N]$ [16].

In [23,34,47,51], the notion of the categories $\mathscr{C}_{\mathfrak{g}}^0$ and $\mathscr{C}_Q$ is extended to all untwisted and twisted quantum affine algebras. Suppose that $U_q'(\mathfrak{g})$ is of an arbitrary affine type. We consider the set $\sigma(\mathfrak{g}) := I_0 \times \mathbf{k}^\times/\sim$, where the equivalence relation is given by (2.7), with the arrows determined by the pole of R-matrices between tensor products of fundamental representations $V(\varpi_i)_x$ $((i,x) \in \sigma(\mathfrak{g}))$. Let $\sigma_0(\mathfrak{g})$ be a connected component of $\sigma(\mathfrak{g})$. The category $\mathscr{C}_{\mathfrak{g}}^0$ is defined to be the full subcategory of $\mathscr{C}_{\mathfrak{g}}$ determined by $\sigma_0(\mathfrak{g})$ (see Section 2.5). Let $\mathfrak{g}_{\mathrm{fin}}$ be the simple Lie algebra of type $X_{\mathfrak{g}}$ defined in (6.1). Note that, when $\mathfrak{g}$ is of untwisted affine type ADE, $\mathfrak{g}_{\mathrm{fin}}$ coincides with $\mathfrak{g}_0$. A Q-datum is a triple $\mathscr{Q} := (\triangle, \sigma, \xi)$ consisting of the Dynkin diagram $\triangle$ of $\mathfrak{g}_{\mathrm{fin}}$, an automorphism σ on $\triangle$ and a height function ξ (see Section 6.2). When $\mathfrak{g}$ is of untwisted affine type ADE, σ is the identity and $\mathscr{Q}$ is equal to a Dynkin quiver with a height function. For a Q-datum $\mathscr{Q}$, the monoidal subcategory $\mathscr{C}_{\mathscr{Q}}$ of $\mathscr{C}_{\mathfrak{g}}^0$ was introduced in [16] for untwisted affine type ADE, in [23] for twisted affine types $A^{(2)}$ and $D^{(2)}$, in [34,51] for untwisted affine types $B^{(1)}$ and $C^{(1)}$, and in [47] for exceptional affine type. Similarly to the untwisted affine ADE case, the category $\mathscr{C}_{\mathscr{Q}}$ categorifies the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group N associated with $\mathfrak{g}_{\mathrm{fin}}$. The simple Lie algebra $\mathfrak{g}_{\mathrm{fin}}$ is more deeply related to the structure of the category $\mathscr{C}_{\mathfrak{g}}$. It is proved in [31] that the simply-laced root system $\Upsilon_{\mathfrak{g}}$ of $\mathfrak{g}_{\mathrm{fin}}$ arises from $\mathscr{C}_{\mathfrak{g}}$ in a natural way and the block decompositions of $\mathscr{C}_{\mathfrak{g}}$ and $\mathscr{C}_{\mathfrak{g}}^0$ are parameterized by the lattice associated with the root system $\Upsilon_{\mathfrak{g}}$. In the course of the proof, the new invariants Λ and Λ^∞ for $\mathscr{C}_{\mathfrak{g}}$ introduced in [27] are used in a crucial way. These invariants are quantum affine algebra analogues of the invariants (with the same notations) for the quiver Hecke algebras [21,24].

Let R_{C} be a *quiver Hecke algebra* (or *Khovanov–Lauda–Rouquier algebras*) corresponding to a generalized Cartan matrix C and denote by R_{C}-gmod its finite-dimensional graded module category. The algebra R_{C} categorifies the half of the quantum group $U_q(\mathfrak{g})$ associated with C [38, 39, 49]. The simple R_{C}-modules were studied and classified by using the structure of $U_q^-(\mathfrak{g})$ via categorification [2,17,37,41–43,52]. When R_{C} is symmetric and the base field is of characteristic 0, the set of isomorphism classes of simple R_{C}-modules corresponds to the upper global basis of $U_q^-(\mathfrak{g})$ [48,54]. Suppose that C is of finite type. One of the most successful constructions for simple R_{C}-modules is the construction using *cuspidal modules* via the (dual) PBW theory for $U_q^-(\mathfrak{g})$. For a reduced expression $\underline{w_0}$ of the longest element w_0 of the Weyl group W_{C}, one can define the associated cuspidal modules $\{\mathsf{V}_k\}_{k=1}^{\ell}$, which correspond to the *dual PBW vectors*, and all simple R_{C}-modules are obtained as the simple heads of ordered tensor products of cuspidal modules. The construction using Lyndon words was introduced in [41] (see also [17]) and the construction in a general setting with a convex order was studied in [37,43]. It was also studied in [40,44] for an affine case, and in [52] for a symmetrizable case from the viewpoint of MV polytopes.

The *quantum affine Schur–Weyl duality* [21] gives a connection between quiver Hecke algebras and quantum affine algebras. The quantum affine Schur–Weyl duality says that, for each *duality datum* $\mathcal{D} = \{\mathsf{L}_i\}_{i \in J} \subset \mathscr{C}_{\mathfrak{g}}$ associated with a generalized symmetric Cartan matrix C, there exists a monoidal functor $\mathcal{F}_{\mathcal{D}}$, called briefly a *duality functor*, from the category R_{C}-gmod to the category $\mathscr{C}_{\mathfrak{g}}$. The duality functor is very interesting and useful, but it is difficult to handle it because the functor does not enjoy good properties in general. When $\mathfrak{D}$ arises from a Q-datum, the duality functor $\mathscr{F}_{\mathfrak{D}}$ enjoys good properties. It was shown in [11, 20, 23, 34, 47] that, for each choice of Q-data $\mathscr{Q}$, the quantum affine Schur–Weyl duality functor

$$\mathscr{F}_{\mathscr{Q}}\colon R^{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod} \to \mathscr{C}_{\mathscr{Q}} \subset \mathscr{C}^0_{\mathfrak{g}}$$

is exact and sends simple modules to simple modules, thus it induces an isomorphism at the Grothendieck ring level. Here $R^{\mathfrak{g}_{\mathrm{fin}}}$ is the symmetric quiver Hecke algebra associated with $\mathfrak{g}_{\mathrm{fin}}$. In this viewpoint, it is natural and important to ask which conditions for $\mathfrak{D}$ provide the duality functor $\mathscr{F}_{\mathfrak{D}}$ with such good properties, and what properties are preserved from R_{C}-gmod to $\mathscr{C}_{\mathfrak{g}}$ under the duality functor $\mathscr{F}_{\mathfrak{D}}$.

This paper is a complete version of the announcement [29]. The main results of this paper can be summarized as follows:

(i) Let $U_q'(\mathfrak{g})$ be a quantum affine algebra of *arbitrary type*. We find a sufficient condition for a duality datum $\mathfrak{D} = \{\mathsf{L}_i\}_{i\in J}$ to provide the functor $\mathscr{F}_{\mathfrak{D}}$ with good properties. We introduce the notion of *strong duality datum* by investigating *root modules*. We prove that the associated duality functor $\mathscr{F}_{\mathfrak{D}}$ sends simple modules to simple modules and preserves the invariants Λ, $\widetilde{\Lambda}$ and Λ^{∞}. We also introduce the notion of *complete duality datum*, which can be understood as a generalization of the duality datum arising from a Q-datum. It turns out that the Cartan matrix C associated with a compete duality datum $\mathfrak{D}$ is equal to the one of $\mathfrak{g}_{\mathrm{fin}}$.

(ii) We introduce the *reflections* $\mathscr{S}_i$ and $\mathscr{S}_i^{-1}$ $(i \in J)$ acting on strong duality data $\mathfrak{D}$. We prove that if $\mathfrak{D}$ is a strong (resp. complete) duality datum, then $\mathscr{S}_i(\mathfrak{D})$ and $\mathscr{S}_i^{-1}(\mathfrak{D})$ are also strong (resp. complete) duality data. This allows us to create new strong (resp. complete) duality data from known strong (resp. complete) duality data by applying a finite sequence of the reflections $\mathscr{S}_i$ and $\mathscr{S}_i^{-1}$. Indeed, the family $\{\mathscr{S}_i\}_{i\in J}$ satisfies the *braid relations*, etc. [28]. It will be discussed in a forthcoming paper.

(iii) We introduce the notion of *affine cuspidal modules* for the category $\mathscr{C}^0_{\mathfrak{g}}$. Let $\mathfrak{D}$ be a complete duality datum associated with a Cartan matrix C. For a reduced expression $\underline{w_0}$ of the longest element of the Weyl group W_{C}, we define the affine cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ for $\mathscr{C}^0_{\mathfrak{g}}$ by using the duality functor $\mathscr{F}_{\mathfrak{D}}$, the right and left duals $\mathscr{D}$, $\mathscr{D}^{-1}$, and the cuspidal modules $\{\mathsf{V}_k\}_{k=1}^{\ell}$ of the quiver Hecke algebra R_{C} associated with $\underline{w_0}$. If $\mathfrak{D}$ arises from a Q-datum, then the affine cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ consist of fundamental modules. But, in general, affine cuspidal modules are not fundamental. We prove that all simple modules in $\mathscr{C}^0_{\mathfrak{g}}$ can be obtained uniquely as the simple heads of the ordered tensor products $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$, called *standard modules*, of cuspidal modules. We then show that the standard module $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ has the *unitriangularity property*. This generalizes the classical simple module construction taking the head of ordered tensor products of fundamental representations [1, 25, 45, 46, 53]. The unitriangularity property allows us to define a monoidal subcategory $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$

of $\mathscr{C}_{\mathfrak{g}}^0$ for an interval $[a, b]$, which is a generalization of the subcategory $\mathscr{C}_l$ ($l \in \mathbb{Z}_{>0}$) introduced in [15]. This approach can be understood as a counterpart of the PBW theory for quiver Hecke algebras via the duality functor $\mathcal{F}_{\mathfrak{D}}$. Hence we establish a base to answer the monoidal categorification conjecture for various monoidal subcategories of $\mathscr{C}_{\mathfrak{g}}^0$ in the same spirit of [24, 27]. The monoidal categorification conjecture will be discussed in a forthcoming paper [32, 33].

We remark that when $U_q'(\mathfrak{g})$ is of untwisted affine ADE type, it has been established by Hernandez–Leclerc that the complexified Grothendieck ring $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathscr{C}_{\mathfrak{g}}^0)$ can be written as a product of copies of $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathscr{C}_Q) \simeq \mathbb{C}[N]$, where N is the unipotent group associated with $\mathfrak{g}_0$ (see the proof of [16, Theorem 7.3]). When the orientation of the quiver Q varies, one gets various copies of $\mathbb{C}[N]$ in $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathscr{C}_{\mathfrak{g}}^0)$ and the basis of standard modules correspond to various PBW basis. The PBW theory developed in this paper explains this story transparently at the level of the module category.

Let us explain our results more precisely. Let $U_q'(\mathfrak{g})$ be a quantum affine algebra of an arbitrary type. We first investigate several properties of root modules about the new invariants Λ, $\mathfrak{d}$, etc in Section 3. A root module is a real simple module L such that

$$\mathfrak{d}(L, \mathscr{D}^k(L)) = \delta(k = \pm 1) \quad \text{for any } k \in \mathbb{Z}.$$

Note that the name "root module" comes from Lemma 4.15. We prove several lemmas and propositions on root modules, which are used crucially in the proofs of the main results.

We next deal with the quantum affine Schur–Weyl duality. Let $\mathfrak{D}$ be a duality datum associated with a generalized Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$ of symmetric type. We study the *affinizations* of modules appearing in both the categories R_{C}-gmod and $\mathscr{C}_{\mathfrak{g}}$ as pro-objects and slightly modify the definition of quantum affine Schur–Weyl duality in order that the duality functor $\mathcal{F}_{\mathfrak{D}}$ preserves the affinizations (Theorem 4.2). This allows us to compare the invariants Λ, $\mathfrak{d}$, etc. between quiver Hecke algebras and quantum affine algebras via the duality functor $\mathcal{F}_{\mathfrak{D}}$. When $\mathfrak{D} = \{\mathsf{L}_i\}_{i \in J}$ is a strong duality datum of a Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$ of simply-laced finite type (Definition 4.7), we prove that $\mathcal{F}_{\mathfrak{D}}$ sends simple modules to simple modules (Theorem 4.10), i.e., $\mathcal{F}_{\mathfrak{D}}$ is faithful (Corollary 4.11), and $\mathcal{F}_{\mathfrak{D}}$ preserves the invariants: for any simple modules M, N in R_{C}-gmod,

(i) $\Lambda(M, N) = \Lambda(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N))$,
(ii) $\mathfrak{d}(M, N) = \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N))$,
(iii) $(\mathrm{wt}\, M, \mathrm{wt}\, N) = -\Lambda^\infty(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N))$,
(iv) $\mathfrak{d}(\mathscr{D}^k \mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) = 0$ for any $k \neq 0, \pm 1$,
(v) $\widetilde{\Lambda}(M, N) = \mathfrak{d}(\mathscr{D}\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) = \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}^{-1}\mathcal{F}_{\mathfrak{D}}(N))$

(see Theorem 4.12). The key part of the proof is to show that the invariants for *determinantial modules* $\mathsf{D}(w\Lambda, \Lambda)$ (see Section 2.2) are preserved under $\mathcal{F}_{\mathfrak{D}}$ (Theorem 4.9). Corollary 4.14 says that the duality functor $\mathcal{F}_{\mathfrak{D}}$ induces an injective ring homomorphism

$$K_{q=1}(R_{\mathsf{C}}\text{-gmod}) \rightarrowtail K(\mathscr{C}_{\mathfrak{g}}),$$

where $K_{q=1}(R_{\mathsf{C}}\text{-gmod})$ is the specialization of $K(R_{\mathsf{C}}\text{-gmod})$ at $q = 1$. Interestingly, the ε_i and ε_i^* in the crystal theory for R_{C}-gmod can be interpreted in terms of the invariants $\mathfrak{d}$ for $\mathscr{C}_{\mathfrak{g}}$ (Corollary 4.13).

Let $\mathcal{D} = \{\mathsf{L}_i\}_{i\in J}$ be a strong duality datum associated with a Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j\in J}$ of simply-laced finite type, and define $\mathscr{C}_{\mathcal{D}}$ to be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}^0$ that

(a) contains $\mathcal{F}_{\mathcal{D}}(L)$ for any simple R_{C}-module L,

(b) is stable by taking subquotients, extensions, and tensor products.

The induced map $[\mathcal{F}_{\mathcal{D}}]$ gives an isomorphism between $K(\mathscr{C}_{\mathcal{D}})$ and $K_{q=1}(R_{\mathsf{C}}\text{-gmod})$ as a ring. We introduce the notion of *unmixed pairs* of modules in $\mathscr{C}_{\mathfrak{g}}$ (Definition 5.1) and investigate several properties. Lemma 5.5 says that if (M, N) is an unmixed pair of simple modules in R_{C}-gmod, then $(\mathcal{F}_{\mathcal{D}}(M), \mathcal{F}_{\mathcal{D}}(N))$ is strongly unmixed. Let w_0 be the longest element of the Weyl group W_{C} of $\mathfrak{g}_{\mathsf{C}}$, and ℓ the length of w_0. We define the affine cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbb{Z}} \subset \mathscr{C}_{\mathfrak{g}}^0$ to be the simple $U_q'(\mathfrak{g})$-modules given by

(a) $\mathsf{S}_k = \mathcal{F}_{\mathcal{D}}(\mathsf{V}_k)$ for any $k = 1, \dots, \ell$,

(b) $\mathsf{S}_{k+\ell} = \mathscr{D}(\mathsf{S}_k)$ for any $k \in \mathbb{Z}$,

where $\{\mathsf{V}_k\}_{k=1}^{\ell} \subset R_{\mathsf{C}}$-gmod are the cuspidal modules associated with $\underline{w_0}$. Note that the cuspidal module V_k corresponds to the dual PBW vectors associated with $\underline{w_0}$ under the categorification using quiver Hecke algebras. We then prove that S_a is a root module for any $a \in \mathbb{Z}$, and $(\mathsf{S}_a, \mathsf{S}_b)$ is strongly unmixed for any $a > b$, which tells us that the ordered tensor product $\mathsf{S}_{k_1}^{\otimes a_1} \otimes \cdots \otimes \mathsf{S}_{k_t}^{\otimes a_t}$ has a simple head for any decreasing integers $k_1 > \cdots > k_t$ and $a_1, \dots, a_t \in \mathbb{Z}_{\ge 0}$ (Proposition 5.7). We next define the reflections $\mathscr{S}_k$ and $\mathscr{S}_k^{-1}$ on duality data (see (5.3)) and prove that the reflections preserve strong duality data with the same Cartan matrix (Proposition 5.9). Furthermore, we characterize simple modules in the intersections $\mathscr{C}_{\mathscr{S}_i(\mathcal{D})} \cap \mathscr{C}_{\mathcal{D}}$ and $\mathscr{C}_{\mathscr{S}_i^{-1}(\mathcal{D})} \cap \mathscr{C}_{\mathcal{D}}$ by using the cuspidal modules $\{\mathsf{V}_k\}_{k=1}^{\ell}$ (Proposition 5.11).

We finally introduce the notion of a complete duality datum (Definition 6.1). We prove that if $\mathcal{D} = \{\mathsf{L}_i\}_{i\in J}$ is a complete duality datum, then the associated Cartan matrix C has the same type as that of $\mathfrak{g}_{\mathrm{fin}}$. Note that the root system $\Upsilon_{\mathfrak{g}}$ of $\mathfrak{g}_{\mathrm{fin}}$ provides the block decomposition of $\mathscr{C}_{\mathfrak{g}}$ [31]. The reflections $\mathscr{S}_k$ and $\mathscr{S}_k^{-1}$ preserve complete duality data with the same Cartan matrix (Theorem 6.3), and the duality datum $\mathcal{D}_{\mathcal{Q}}$ arising from a Q-datum $\mathcal{Q} = (\triangle, \sigma, \xi)$ is complete (Proposition 6.5). By the definition, $\mathscr{C}_{\mathcal{D}_{\mathcal{Q}}}$ is equal to $\mathscr{C}_{\mathcal{Q}}$. Since a new complete duality datum can be constructed by applying the reflections to $\mathcal{D}_{\mathcal{Q}}$, when $\mathcal{D}$ is complete, the category $\mathscr{C}_{\mathcal{D}}$ can be viewed as a generalization of $\mathscr{C}_{\mathcal{Q}}$. We now assume that $\mathcal{D}$ is complete. Let $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ be the affine cuspidal modules corresponding to $\mathcal{D}$ and a reduced expression $\underline{w_0}$, and set $\mathbf{Z} := \mathbb{Z}_{\ge 0}^{\oplus\mathbb{Z}}$. We denote by $\prec$ the bi-lexicographic order on $\mathbf{Z}$. For any $\mathbf{a} = (a_k)_{k\in\mathbb{Z}} \in \mathbf{Z}$, we define the standard module by

$$\mathsf{P}_{\mathcal{Q},\underline{w_0}}(\mathbf{a}) := \cdots \otimes \mathsf{S}_1^{\otimes a_1} \otimes \mathsf{S}_0^{\otimes a_0} \otimes \mathsf{S}_{-1}^{\otimes a_{-1}} \otimes \cdots,$$

and set $\mathsf{V}_{\mathcal{Q},\underline{w_0}}(\mathbf{a}) := \mathrm{hd}(\mathsf{P}_{\mathcal{Q},\underline{w_0}}(\mathbf{a}))$. We prove that $\mathsf{V}_{\mathcal{Q},\underline{w_0}}(\mathbf{a})$ is simple for any $\mathbf{a} \in \mathbf{Z}$ and the set $\{\mathsf{V}_{\mathcal{Q},\underline{w_0}}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Z}\}$ is a complete and irredundant set of simple modules of $\mathscr{C}_{\mathfrak{g}}^0$ up to isomorphism (Theorem 6.10). Furthermore, Theorem 6.12 says that if V is a simple subquotient of $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ which is not isomorphic to $\mathsf{V}_{Q,\underline{w_0}}(\mathbf{a})$, then

$$\mathbf{a}_{Q,\underline{w_0}}(V) \prec \mathbf{a},$$

which means that the module $\mathsf{P}_{Q,\underline{w_0}}(\mathbf{a})$ has the unitriangularity property with respect to $\prec$. For an interval $[a,b]$, we define $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w_0}}$ to be the full subcategory of $\mathscr{C}_{\mathfrak{g}}$ whose objects have all their composition factors V satisfying

$$b \geq l(\mathbf{a}_{\mathcal{D},\underline{w_0}}(V)) \quad \text{and} \quad r(\mathbf{a}_{\mathcal{D},\underline{w_0}}(V)) \geq a,$$

where l and r are defined in (6.8). By unitriangularity, the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathcal{D},\underline{w_0}}$ is stable by taking tensor products, and it also enjoys the same properties (Theorem 6.16).

This paper is organized as follows. In Section 2, we give the necessary background on quiver Hecke algebras, quantum affine algebras, and the invariants related to R-matrices. In Section 3, we introduce the notion of root modules and investigate several properties. In Section 4, we study affinizations and the duality functor $\mathscr{F}_{\mathcal{D}}$, and prove that when $\mathcal{D}$ is strong, $\mathscr{F}_{\mathcal{D}}$ sends simple modules to simple modules and preserves the new invariants. In Section 5, we introduce the notions of affine cuspidal modules and reflections, and prove that the reflections preserve the strong duality data. In Section 6, we study the PBW-theoretic approach to $\mathscr{C}_{\mathfrak{g}}^{0}$ using a complete duality datum and affine cuspidal modules.

2. Preliminaries

Convention. (i) For a statement P, $\delta(P)$ is 1 or 0 according as P is true or not.

(ii) For a field $\mathbf{k}$, $a \in \mathbf{k}$ and $f(z) \in \mathbf{k}(z)$, we denote by $\mathrm{zero}_{z=a}\, f(z)$ the order of the zero of $f(z)$ at $z = a$.

(iii) For a ring A, $A^{\times}$ is the set of invertible elements of A.

2.1. *Quantum groups*

Let I be an index set. A quintuple $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^{\vee}, \Pi^{\vee})$ called a (symmetrizable) *Cartan datum* consists of

(a) a generalized *Cartan matrix* $\mathsf{A} = (a_{ij})_{i,j\in I}$,

(b) a free abelian group P, called the *weight lattice*,

(c) $\Pi = \{\alpha_i \mid i \in I\} \subset \mathsf{P}$, called the set of *simple roots*,

(d) $\mathsf{P}^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}, \mathbb{Z})$, called the *coweight lattice*,

(e) $\Pi^{\vee} = \{h_i \in \mathsf{P}^{\vee} \mid i \in I\}$, called the set of *simple coroots*

satisfying the following:

(i) $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$,

(ii) Π is linearly independent over $\mathbb{Q}$,

(iii) for each $i \in I$, there exists $\Lambda_i \in \mathsf{P}$, called the *fundamental weight*, such that $\langle h_j, \Lambda_i\rangle = \delta_{j,i}$ for all $j \in I$.

(iv) there is a symmetric bilinear form $(\cdot\,,\cdot)$ on P satisfying

$$(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0} \quad \text{and} \quad \langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}.$$

We set $\mathsf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ and $\mathsf{Q}^+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and define $\mathrm{ht}(\beta) = \sum_{i\in I} k_i$ for $\beta = \sum_{i\in I} k_i\alpha_i \in \mathsf{Q}^+$. We define $\mathsf{P}^+ := \{\Lambda \in \mathsf{P} \mid \langle h_i, \Lambda\rangle \in \mathbb{Z}_{\geq 0} \text{ for any } i \in I\}$.

We write Φ^+ for the set of positive roots associated with A and set $\Phi^- := -\Phi^+$. Denote by W the *Weyl group*, which is the subgroup of $\mathrm{Aut}(\mathsf{P})$ generated by $s_i(\lambda) := \lambda - \langle h_i, \lambda\rangle\alpha_i$ for $i \in I$.

We denote by $U_q(\mathfrak{g})$ the *quantum group* associated with $(\mathsf{A}, \mathsf{P}, \mathsf{P}^\vee, \Pi, \Pi^\vee)$, which is a $\mathbb{Q}(q)$-algebra generated by f_i, e_i $(i \in I)$ and q^h $(h \in \mathsf{P}^\vee)$ with certain defining relations (see [18, Chapter 3] for details). We denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by e_i's (resp. f_i's). Set $\mathbf{A} := \mathbb{Z}[q, q^{-1}]$ and write $U_{\mathbf{A}}^{\pm}(\mathfrak{g})$ for the $\mathbf{A}$-lattice of $U_q^{\pm}(\mathfrak{g})$, which is the $\mathbf{A}$-subalgebra generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$) for $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$. We define the *unipotent quantum coordinate ring*

$$A_q(\mathfrak{n}) := \bigoplus_{\beta\in\mathsf{Q}^-} A_q(\mathfrak{n})_\beta \quad \text{where} \quad A_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q)),$$

and denote by $A_q(\mathfrak{n})_{\mathbf{A}}$ the $\mathbf{A}$-lattice of $A_q(\mathfrak{n})$. Note that $A_q(\mathfrak{n})$ is isomorphic to $U_q^-(\mathfrak{g})$ as a $\mathbb{Q}(q)$-algebra [24, Lemma 8.2.2].

2.2. *Quiver Hecke algebras*

Let $\mathbf{k}$ be a field and let $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ be a Cartan datum. Choose polynomials

$$Q_{i,j}(u, v) = \delta(i \neq j) \sum_{\substack{(p,q)\in\mathbb{Z}^2_{\geq 0} \\ (\alpha_i,\alpha_i)p+(\alpha_j,\alpha_j)q=-2(\alpha_i,\alpha_j)}} t_{i,j;p,q}u^p v^q \in \mathbf{k}[u, v]$$

with $t_{i,j;p,q} \in \mathbf{k}$, $t_{i,j;p,q} = t_{j,i;q,p}$ and $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$. Note that $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ for $i, j \in I$. Let $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1}\rangle$ be the symmetric group on n letters with the action of $\mathfrak{S}_n$ on I^n by place permutation. For $\beta \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = n$, we set

$$I^\beta := \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Definition 2.1. Let $\beta \in \mathsf{Q}^+$ with $\mathrm{ht}(\beta) = n$. The *quiver Hecke algebra $R(\beta)$* associated with the parameters $\{Q_{i,j}\}_{i,j\in I}$ is the $\mathbf{k}$-algebra generated by $\{e(\nu)\}_{\nu\in I^\beta}$, $\{x_k\}_{k=1}^n$, $\{\tau_m\}_{m=1}^{n-1}$ satisfying the following defining relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1,$$

$$x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu)x_k,$$

$$
\begin{aligned}
&\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k\tau_m = \tau_m\tau_k \ \text{ if } |k-m|>1,\\
&\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1})e(\nu),\\
&(\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } m=k,\ \nu_k=\nu_{k+1},\\ e(\nu) & \text{if } m=k+1,\ \nu_k=\nu_{k+1},\\ 0 & \text{otherwise,}\end{cases}\\
&(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu)\\
&\qquad = \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}}e(\nu) & \text{if } \nu_k = \nu_{k+2},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

The algebra $R(\beta)$ has the $\mathbb{Z}$-grading defined by

$$\deg e(\nu) = 0, \quad \deg\, x_k e(\nu) = (\alpha_{\nu_k},\alpha_{\nu_k}), \quad \deg\, \tau_l e(\nu) = -(\alpha_{\nu_l},\alpha_{\nu_{l+1}}).$$

For a $\mathbb{Z}$-graded **k**-algebra A, we denote by A-gMod the category of graded left A-modules, and write A-gproj (resp. A-gmod) for the full subcategory of A-gMod consisting of finitely generated projective (resp. finite-dimensional) graded A-modules. We set R-gproj $:= \bigoplus_{\beta\in \mathsf{Q}^+} R(\beta)$-gproj and R-gmod $:= \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)$-gmod.

For $M \in R(\beta)$-gMod and $N \in R(\gamma)$-gMod, we define their convolution product by

$$M \circ N := R(\beta+\gamma)e(\beta,\gamma)\otimes_{R(\beta)\otimes R(\gamma)} (M\otimes N),$$

where $e(\beta,\gamma) = \sum_{\nu_1\in I^\beta,\,\nu_2\in I^\gamma} e(\nu_1 * \nu_2)$. Here $\nu_1 * \nu_2$ is the concatenation of ν_1 and ν_2. We denote by $M \nabla N$ the head of $M\circ N$ and by $M \Delta N$ the socle of $M \circ N$. We say that simple R-modules M and N *strongly commute* if $M \circ N$ is simple. A simple R-module L is *real* if $L \circ L$ is simple. For $i \in I$ and an $R(\beta)$-module M, we define

$$E_i(M) := e(\alpha_i, \beta-\alpha_i)M, \quad F_i(M) := R(\alpha_i)\circ M,$$

and

$$
\begin{aligned}
&\mathrm{wt}(M) := -\beta,\\
&\varepsilon_i(M) := \max\{k \ge 0 \mid E_i^k(M) \ne 0\},\\
&\varphi_i(M) := \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle.
\end{aligned}
$$

For $i \in I$, we denote by $L(i)$ the self-dual 1-dimensional simple $R(\alpha_i)$-module. For a simple module M, $\tilde{f}_i(M)$ (resp. $\tilde{e}_i(M)$) is the self-dual simple R-module isomorphic to $L(i)\nabla M$ (resp. $\mathrm{soc}(E_i M)$). One also defines E_i^*, F_i^*, ε_i^*, etc. in the same manner, replacing $e(\alpha_i, \beta-\alpha_i)$ and $R(\alpha_i)\circ -$ by $e(\beta-\alpha_i,\alpha_i)$ and $-\circ R(\alpha_i)$.

Theorem 2.2 ([38, 39, 49]). *There exist* **A**-*bialgebra isomorphisms*

$$U_{\mathbf{A}}^-(\mathfrak{g}) \xrightarrow{\sim} K(R\text{-gproj}) \quad \textit{and} \quad A_q(\mathfrak{n})_{\mathbf{A}} \xrightarrow{\sim} K(R\text{-gmod}),$$

where $K(R$-gproj) *and* $K(R$-gmod) *are the Grothendieck groups of* R-gproj *and* R-gmod.

Definition 2.3. The quiver Hecke algebra $R(\beta)$ is said to be *symmetric* if $Q_{i,j}(u,v)$ is a polynomial in $u-v$ for any $i,j \in I$.

When R is symmetric, the Cartan matrix A is of symmetric type. In this case we assume that $(\alpha_i,\alpha_i)=2$ for all $i \in I$.

In what follows, we assume that R is symmetric.

Let z be an indeterminate with homogeneous degree 2. For an $R(\beta)$-module M, we denote by M^{aff} the *affinization* of M [21, 35]. If $R(\beta)$ is symmetric, then $M^{\mathrm{aff}} = \mathbf{k}[z] \otimes_{\mathbf{k}} M$ and the $R(\beta)$-module structure of M^{aff} is defined by

$$\begin{aligned} e(\nu)(f \otimes m) &= f \otimes e(\nu)m, \\ x_j(f \otimes m) &= (zf) \otimes m + f \otimes x_j m, \\ \tau_k(f \otimes m) &= f \otimes (\tau_k m) \end{aligned}$$

for $f \in \mathbf{k}[z]$, $m \in M$, $\nu \in I^\beta$ and admissible j,k. We sometimes write M_z instead of M^{aff} to emphasize z.

Let $\beta \in \mathsf{Q}^+$ and $m = \mathrm{ht}(\beta)$. For $k = 1, \ldots, m-1$ and $\nu \in I^\beta$, the *intertwiner* $\varphi_k \in R(\beta)$ is defined by

$$\varphi_k e(\nu) := \begin{cases} (\tau_k x_k - x_k \tau_k)e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tau_k e(\nu) & \text{otherwise.} \end{cases}$$

Note that $\{\varphi_k\}_{k=1}^{m-1}$ satisfies the braid relation. Hence, we can define φ_w for any $w \in \mathfrak{S}_m$. Let M be an $R(\beta)$-module with $\mathrm{ht}(\beta)=m$ and N an $R(\beta')$-module with $\mathrm{ht}(\beta')=n$. Let $w[n,m]$ be the element of $\mathfrak{S}_{m+n}$ which sends $k \mapsto k+m$ for $1 \le k \le n$ and $k \mapsto k-n$ if $n < k \le m+n$. Then the $R(\beta)\otimes R(\beta')$-linear map $M \otimes N \to N \circ M$ defined by $u \otimes v \mapsto \varphi_{w[n,m]}(v \otimes u)$ can be extended to the $R(\beta+\beta')$-module homomorphism (up to a grading shift)

$$R_{M,N} \colon M \circ N \to N \circ M.$$

For non-zero R-modules M and N, we set

$$R^{\mathrm{ren}}_{M_z,N_{z'}} := (z'-z)^{-s} R_{M_z,N_{z'}} \colon M_z \circ N_{z'} \to N_{z'} \circ M_z,$$

where s is the largest integer such that $R_{M_z,N_{z'}}(M_z \circ N_{z'}) \subset (z'-z)^s N_{z'} \circ M_z$. We call it the *renormalized R-matrix*. Then, we define

$$\mathbf{r}_{M,N} \colon M \circ N \to N \circ M$$

as the specialization of $R^{\mathrm{ren}}_{M_z,N_{z'}}$ at $z = z' = 0$ (up to a constant multiple), which never vanishes by the definition (see [21, Section 1] and [35, Section 2] for details).

Definition 2.4. Let M and N be simple R-modules. We set

$$\begin{aligned} \Lambda(M,N) &:= \deg(\mathbf{r}_{M,N}), \\ \widetilde{\Lambda}(M,N) &:= \tfrac{1}{2}\big(\Lambda(M,N) + (\mathrm{wt}(M),\mathrm{wt}(N))\big), \\ \mathfrak{d}(M,N) &:= \tfrac{1}{2}\big(\Lambda(M,N) + \Lambda(N,M)\big). \end{aligned}$$

Many properties of Λ, $\widetilde{\Lambda}$, and $\mathfrak{d}$ were obtained in [24, 26, 30].

We now define the monoidal subcategories $\mathscr{C}_w$, $\mathscr{C}_{*,v}$ and $\mathscr{C}_{w,v}$ of R-gmod for $w, v \in \mathsf{W}$. For $M \in R(\beta)$-gMod, we define

$$\mathsf{W}(M) := \{\gamma \in \mathsf{Q}^+ \cap (\beta - \mathsf{Q}^+) \mid e(\gamma, \beta - \gamma)M \neq 0\},$$
$$\mathsf{W}^*(M) := \{\gamma \in \mathsf{Q}^+ \cap (\beta - \mathsf{Q}^+) \mid e(\beta - \gamma, \gamma)M \neq 0\}.$$

For $w \in \mathsf{W}$, we denote by $\mathscr{C}_w$ the full subcategory of R-gmod whose objects M satisfy

$$\mathsf{W}(M) \subset \mathrm{span}_{\mathbb{R}_{\geq 0}}(\Phi^+ \cap w\Phi^-).$$

Similarly, for $v \in \mathsf{W}$, we define $\mathscr{C}_{*,v}$ to be the full subcategory of R-gmod whose objects N satisfy

$$\mathsf{W}^*(N) \subset \mathrm{span}_{\mathbb{R}_{\geq 0}}(\Phi^+ \cap v\Phi^+).$$

Finally, we define $\mathscr{C}_{w,v} := \mathscr{C}_w \cap \mathscr{C}_{*,v}$.

When $\mathfrak{g}$ is of finite type, we have $\mathscr{C}_{w_0} = R$-gmod and

$$M \in \mathscr{C}_{s_i w_0} \quad \text{if and only if} \quad \varepsilon_i(M) = 0,$$
$$M \in \mathscr{C}_{*,s_i} \quad \text{if and only if} \quad \varepsilon_i^*(M) = 0,$$

for any R-module M in R-gmod and $i \in I$. Here w_0 denotes the longest element of W (see [26] for details).

Let $\underline{w} := s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in \mathsf{W}$ and define

$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad \text{for } k = 1, \ldots, l. \tag{2.1}$$

Then we equip $\Phi^+ \cap w\Phi^- = \{\beta_1, \ldots, \beta_l\}$ with the *convex order* $\prec$ on $\Phi^+ \cap w\Phi^-$, i.e., $\beta_a \prec \beta_b$ for any $a < b$. For $\beta \in \Phi^+ \cap w\Phi^-$, a pair (α, γ) is called a *minimal pair* of β if $\beta = \alpha + \gamma$, $\alpha \prec \gamma$ and there exists no pair (α', γ') such that $\beta = \alpha' + \gamma'$ and $\alpha \prec \alpha' \prec \gamma' \prec \gamma$. The convex order provides the *PBW vectors* $\{E(\beta_k)\}_{k=1}^{l}$ in $U_{\mathbf{A}}^-(\mathfrak{g})$ and the *dual PBW vectors* $\{E^*(\beta_k)\}_{k=1}^{l}$ in $A_q(\mathfrak{n})_{\mathbf{A}}$. We set $A_q(\mathfrak{n}(w))$ to be the subalgebra of $A_q(\mathfrak{n})$ generated by $E^*(\beta_k)$ for $k = 1, \ldots, l$. The category $\mathscr{C}_w$ categorifies the algebra $A_q(\mathfrak{n}(w))$ [24, 26].

For $k = 1, \ldots, l$, let V_k be the *cuspidal module* corresponding to β_k with respect to $\underline{w}$ (see [26, Section 2] for a precise definition). Under the categorification, the cuspidal module V_k corresponds to the dual PBW vector $E^*(\beta_k)$. It is known that the set

$$\{\mathrm{hd}(\mathsf{V}_l^{\circ a_l} \circ \cdots \circ \mathsf{V}_1^{\circ a_1}) \mid (a_1, \ldots, a_l) \in \mathbb{Z}_{\geq 0}^l\}$$

gives a complete set of pairwise non-isomorphic simple graded modules in $\mathscr{C}_w$, up to a grading shift [3, 37, 43, 52]. Note that, for a minimal pair (β_a, β_b) of β_k, there exists an isomorphism

$$\mathsf{V}_a \nabla \mathsf{V}_b \simeq \mathsf{V}_k \tag{2.2}$$

(see [43, Lemma 4.2] and [3, Section 4.3]).

For $\Lambda \in \mathsf{P}^+$ and $w, v \in \mathsf{W}$ with $w \geq v$, we denote by $\mathsf{D}(w\Lambda, v\Lambda)$ the *determinantial module* in R-gmod corresponding to the pair $(w\Lambda, v\Lambda)$ (see [24, Section 10.2] and [26, Section 4] for precise definitions). Under the categorification, the determinantial module $\mathsf{D}(w\Lambda, v\Lambda)$ corresponds to the *unipotent quantum minor* $D(w\Lambda, v\Lambda)$ in $A_q(\mathfrak{n})$ [26, Proposition 4.1].

From now on, *we assume that* **k** *is a field of characteristic* 0 *and that* R *is a symmetric quiver Hecke algebra of finite* ADE *type.*

Note that, under the categorification by R-gmod, the *upper global basis* (or *dual canonical basis*) of $A_q(\mathfrak{n})$ corresponds to the set of isomorphism classes of simple R-modules [48, 54]. Then the reflection functor $\mathcal{T}_i$ constructed in [37] gives an equivalence of categories

$$\mathcal{T}_i \colon \mathcal{C}_{s_i w_0} \xrightarrow{\sim} \mathcal{C}_{*,s_i}.$$

Note that $\mathcal{T}_i$ is denoted by $\mathcal{T}_i^*$ in [26]. Since, at the crystal level, this functor corresponds to the Saito crystal reflection [50], we have

$$\mathcal{T}_i(M) \simeq \tilde{f}_i^{\varphi_i^*(M)} \tilde{e}_i^{*\,\varepsilon_i^*(M)}(M) \tag{2.3}$$

for a simple module M with $\varepsilon_i(M) = 0$. For a reduced expression $\underline{w} := s_{i_1} \cdots s_{i_l}$, the cuspidal module V_k can be computed as follows (see [26, Section 5]):

$$\mathsf{V}_k \simeq \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{k-1}}(L(i_k)) \quad \text{for } k = 1, \ldots, l. \tag{2.4}$$

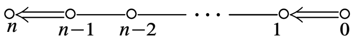
2.3. *Quantum affine algebras*

We assume that $\mathsf{A} = (a_{i,j})_{i,j \in I}$ is an affine Cartan matrix. Note that the rank of P is $|I| + 1$. We denote by $\delta \in \mathsf{Q}$ the *imaginary root* and by c the *central element* in $\mathsf{P}^\vee$. Note that the positive imaginary root Δ_+^{im} is equal to $\mathbb{Z}_{>0}\delta$ and the center of $\mathfrak{g}$ is generated by c. We write $\mathsf{P}_{\mathrm{cl}} := \mathsf{P}/(\mathsf{P} \cap \mathbb{Q}\delta)$, called the *classical weight lattice*, and take $\rho \in \mathsf{P}$ (resp. $\rho^\vee \in \mathsf{P}^\vee$) such that $\langle h_i, \rho \rangle = 1$ (resp. $\langle \rho^\vee, \alpha_i \rangle = 1$) for any $i \in I$. We choose a $\mathbb{Q}$-valued non-degenerate symmetric bilinear form $(\ ,\ )$ on P satisfying

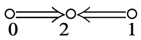
$$\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad \langle c, \lambda \rangle = (\delta, \lambda)$$

for any $i \in I$ and $\lambda \in \mathsf{P}$. We define $\mathfrak{g}$ to be the *affine Kac–Moody algebra* associated with A. We shall use the standard convention in [19] to choose $0 \in I$ except for $A_{2n}^{(2)}$ type, in which we take the longest simple root as α_0, and $B_2^{(1)}$ and $A_3^{(2)}$ types, in which we take the following Dynkin diagrams:

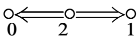
$A_{2n}^{(2)}$:



$B_2^{(1)}$:



$A_3^{(2)}$:



Note that $B_2^{(1)}$ and $A_3^{(2)}$ in the above diagram are denoted by $C_2^{(1)}$ and $D_3^{(2)}$ respectively in [19].

Set $I_0 := I \setminus \{0\}$.

Let q be an indeterminate and $\mathbf{k}$ the algebraic closure of the subfield $\mathbb{C}(q)$ in the algebraically closed field $\widehat{\mathbf{k}} := \bigcup_{m>0} \mathbb{C}((q^{1/m}))$. For $m, n \in \mathbb{Z}_{\geq 0}$ and $i \in I$, we define $q_i = q^{(\alpha_i,\alpha_i)/2}$ and

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i![n]_i!}.$$

Let d be the smallest positive integer such that $d(\alpha_i, \alpha_i)/2 \in \mathbb{Z}$ for all $i \in I$.

Definition 2.5. The *quantum affine algebra* $U_q(\mathfrak{g})$ associated with an affine Cartan datum $(\mathsf{A}, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ is the associative algebra over $\mathbf{k}$ with 1 generated by e_i, f_i $(i \in I)$ and q^h $(h \in d^{-1}\mathsf{P}^\vee)$ satisfying the following relations:

(i) $q^0 = 1, \quad q^h q^{h'} = q^{h+h'}$ for $h, h' \in d^{-1}\mathsf{P}^\vee$,

(ii) $q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i$ for $h \in d^{-1}\mathsf{P}^\vee, i \in I$,

(iii) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q_i^{h_i}$,

(iv) $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$ for $i \neq j$,

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$.

Let us denote by $U_q'(\mathfrak{g})$ the $\mathbf{k}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ $(i \in I)$. Let $\mathscr{C}_\mathfrak{g}$ be the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules, i.e., finite-dimensional modules M with a weight decomposition

$$M = \bigoplus_{\lambda \in \mathsf{P}_{\mathrm{cl}}} M_\lambda \quad \text{where } M_\lambda = \{u \in M \mid K_i u = q_i^{\langle h_i,\lambda\rangle} u\}.$$

Note that the trivial module $\mathbf{1}$ is contained in $\mathscr{C}_\mathfrak{g}$ and the tensor product $\otimes$ gives a monoidal category structure on $\mathscr{C}_\mathfrak{g}$. The monoidal category $\mathscr{C}_\mathfrak{g}$ is rigid. For $M \in \mathscr{C}_\mathfrak{g}$, we denote by $\mathscr{D}M$ and $\mathscr{D}^{-1}M$ the right dual and the left dual of M, respectively. Hence we have the evaluation morphisms

$$M \otimes \mathscr{D}M \to \mathbf{1} \quad \text{and} \quad \mathscr{D}^{-1}M \otimes M \to \mathbf{1}.$$

We extend this to $\mathscr{D}^k$ for $k \in \mathbb{Z}$. We set $M^{\otimes k} := M \otimes \cdots \otimes M$ (k factors) for $k \in \mathbb{Z}_{\geq 0}$. For $M, N \in \mathscr{C}_\mathfrak{g}$, we denote by $M \nabla N$ the head of $M \otimes N$ and by $M \Delta N$ the socle of $M \otimes N$. We say that M and N *strongly commute* if $M \otimes N$ is simple. A simple $U_q'(\mathfrak{g})$-module L is *real* if $L \otimes L$ is simple.

A simple module L in $\mathscr{C}_\mathfrak{g}$ contains a non-zero vector $u \in L$ of weight $\lambda \in \mathsf{P}_{\mathrm{cl}}$ such that (i) $\langle h_i, \lambda\rangle \geq 0$ for all $i \in I_0$, (ii) all the weights of L are contained in $\lambda - \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\,\mathrm{cl}(\alpha_i)$, where $\mathrm{cl}\colon \mathsf{P} \to \mathsf{P}_{\mathrm{cl}}$ is the canonical projection. Such a λ is unique and u is unique up to a constant multiple. We call λ the *dominant extremal weight* of L and u a *dominant extremal weight vector* of L. For each $i \in I_0$, we set

$$\varpi_i := \gcd(\mathsf{c}_0, \mathsf{c}_i)^{-1}\,\mathrm{cl}(\mathsf{c}_0\Lambda_i - \mathsf{c}_i\Lambda_0),$$

where the central element c is equal to $\sum_{i\in I} \mathsf{c}_i h_i$. For any $i \in I_0$, we denote by $V(\varpi_i)$ the *i-th fundamental representation*. Note that the dominant extremal weight of $V(\varpi_i)$ is ϖ_i.

2.4. R-matrices

In this subsection we review the notion of R-matrices on $U_q'(\mathfrak{g})$-modules and their coefficients (see [9], [1, Appendices A and B] and [25, Section 8] for details).

For a module $M \in \mathscr{C}_\mathfrak{g}$, we denote by M^{aff} the *affinization* of M and by $z_M \colon M^{\mathrm{aff}} \to M^{\mathrm{aff}}$ the $U_q'(\mathfrak{g})$-module automorphism of weight δ. Note that $M^{\mathrm{aff}} \simeq \mathbf{k}[z^{\pm 1}] \otimes_\mathbf{k} M$ with the action

$$e_i(a \otimes v) = z^{\delta_{i,0}} a \otimes e_i v \quad \text{for } a \in \mathbf{k}[z^{\pm 1}] \text{ and } v \in M.$$

We sometimes write M_z instead of M^{aff} to emphasize the endomorphism z. For $x \in \mathbf{k}^\times$, we define

$$M_x := M^{\mathrm{aff}}/(z_M - x)M^{\mathrm{aff}}.$$

We call x a *spectral parameter* (see [25, Section 4.2] for details).

Take a basis $\{P_\nu\}_\nu$ of $U_q^+(\mathfrak{g})$ and a basis $\{Q_\nu\}_\nu$ of $U_q^-(\mathfrak{g})$ dual to each other with respect to a suitable coupling between $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$. For $U_q'(\mathfrak{g})$-modules M and N, we define

$$R^{\mathrm{univ}}_{M,N}(u \otimes v) := q^{(\mathrm{wt}(u),\mathrm{wt}(v))} \sum_\nu P_\nu v \otimes Q_\nu u \quad \text{for } u \in M \text{ and } v \in N,$$

so that $R^{\mathrm{univ}}_{M,N}$ gives a $U_q'(\mathfrak{g})$-linear homomorphism $M \otimes N \to N \otimes M$, called the *universal R-matrix*, provided that the infinite sum has a meaning. As $R^{\mathrm{univ}}_{M,N_z}$ converges in the z-adic topology for $M, N \in \mathscr{C}_\mathfrak{g}$, we have a morphism of $\mathbf{k}((z)) \otimes U_q'(\mathfrak{g})$-modules

$$R^{\mathrm{univ}}_{M,N_z} \colon \mathbf{k}((z)) \underset{\mathbf{k}[z^{\pm 1}]}{\otimes} (M \otimes N_z) \to \mathbf{k}((z)) \underset{\mathbf{k}[z^{\pm 1}]}{\otimes} (N_z \otimes M).$$

Note that $R^{\mathrm{univ}}_{M,N_z}$ is an isomorphism. For non-zero $M, N \in \mathscr{C}_\mathfrak{g}$, we say that the universal R-matrix $R^{\mathrm{univ}}_{M,N_z}$ is *rationally renormalizable* if there exists $f(z) \in \mathbf{k}((z))^\times$ such that

$$f(z) R^{\mathrm{univ}}_{M,N_z}(M \otimes N_z) \subset N_z \otimes M.$$

In this case, we can choose $c_{M,N}(z) \in \mathbf{k}((z))^\times$ such that, for any $x \in \mathbf{k}^\times$, the specialization of $R^{\mathrm{ren}}_{M,N_z} := c_{M,N}(z) R^{\mathrm{univ}}_{M,N_z} \colon M \otimes N_z \to N_z \otimes M$ at $z = x$,

$$R^{\mathrm{ren}}_{M,N_z}|_{z=x} \colon M \otimes N_x \to N_x \otimes M,$$

does not vanish. Note that R^{ren}_{M,N_z} and $c_{M,N}(z)$ are unique up to a multiple of $\mathbf{k}[z^{\pm 1}]^\times = \bigsqcup_{n\in\mathbb{Z}} \mathbf{k}^\times z^n$. We call R^{ren}_{M,N_z} *the renormalized R-matrix* and $c_{M,N}(z)$ the *renormalizing coefficient*. We denote by $\mathbf{r}_{M,N}$ the specialization at $z = 1$,

$$\mathbf{r}_{M,N} := R^{\mathrm{ren}}_{M,N_z}|_{z=1} \colon M \otimes N \to N \otimes M, \tag{2.5}$$

and call it the *R-matrix*. The R-matrix $\mathbf{r}_{M,N}$ is well-defined up to a constant multiple whenever $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable. By the definition, $\mathbf{r}_{M,N}$ never vanishes.

Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$ and let u and v be dominant extremal weight vectors of M and N, respectively. Then there exists $a_{M,N}(z) \in \mathbf{k}[[z]]^{\times}$ such that

$$R^{\mathrm{univ}}_{M,N_z}(u \otimes v_z) = a_{M,N}(z)(v_z \otimes u).$$

Thus we have a unique $\mathbf{k}(z) \otimes U'_q(\mathfrak{g})$-module isomorphism

$$R^{\mathrm{norm}}_{M,N_z} := a_{M,N}(z)^{-1} R^{\mathrm{univ}}_{M,N_z}\big|_{\mathbf{k}(z)\otimes_{\mathbf{k}[z^{\pm 1}]}(M\otimes N_z)}$$

from $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (M \otimes N_z)$ to $\mathbf{k}(z) \otimes_{\mathbf{k}[z^{\pm 1}]} (N_z \otimes M)$ which satisfies

$$R^{\mathrm{norm}}_{M,N_z}(u \otimes v_z) = v_z \otimes u.$$

We call $a_{M,N}(z)$ the *universal coefficient* of M and N, and $R^{\mathrm{norm}}_{M,N_z}$ the *normalized R-matrix*.

Let $d_{M,N}(z) \in \mathbf{k}[z]$ be a monic polynomial of the smallest degree such that the image of $d_{M,N}(z)R^{\mathrm{norm}}_{M,N_z}(M \otimes N_z)$ is contained in $N_z \otimes M$; the polynomial is called the *denominator of* $R^{\mathrm{norm}}_{M,N_z}$. Then we have

$$R^{\mathrm{ren}}_{M,N_z} = d_{M,N}(z)R^{\mathrm{norm}}_{M,N_z} \colon M \otimes N_z \to N_z \otimes M \quad \text{up to a multiple of } \mathbf{k}[z^{\pm 1}]^{\times}.$$

Thus

$$R^{\mathrm{ren}}_{M,N_z} = a_{M,N}(z)^{-1} d_{M,N}(z) R^{\mathrm{univ}}_{M,N_z} \quad \text{and} \quad c_{M,N}(z) = \frac{d_{M,N}(z)}{a_{M,N}(z)}$$

up to a multiple of $\mathbf{k}[z^{\pm 1}]^{\times}$. In particular, $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable whenever M and N are simple.

The following proposition was one of the main results of [22].

Proposition 2.6 ([22, Theorem 3.12]). *Let M and N be simple modules, and assume that one of them is real. Then $\mathrm{Im}(\mathbf{r}_{M,N})$ is a simple module and it coincides with the head of $M \otimes N$ and with the socle of $N \otimes M$.*

Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. Suppose that one of them is real. Thanks to Proposition 2.6, the diagram

$$\begin{tikzcd}
M \otimes N \arrow[r, two heads] \arrow[rrr, bend left, "\mathbf{r}_{M,N}"] & M \nabla N \arrow[r, no head, "\sim"] & N \Delta M \arrow[r, tail] & N \otimes M
\end{tikzcd} \tag{2.6}$$

commutes. Here $\twoheadrightarrow$ denotes the natural projection and $\rightarrowtail$ denotes the embedding.

Lemma 2.7 ([22, Corollary 3.13]). *Let L be a real simple module. Then for any simple module X, we have*

$$(L \nabla X) \nabla \mathscr{D}L \simeq X, \quad \mathscr{D}^{-1}L \nabla (X \nabla L) \simeq X,$$
$$L \nabla (X \nabla \mathscr{D}L) \simeq X, \quad (\mathscr{D}^{-1}L \nabla X) \nabla L \simeq X.$$

Lemma 2.8 ([22, Corollary 3.14]). *Let X, Y and L be simple modules in $\mathscr{C}_{\mathfrak{g}}$. Suppose that L is real.*

(i) $X \simeq L \nabla Y$ *if and only if* $X \nabla \mathscr{D}L \simeq Y$,

(ii) $X \simeq Y \nabla L$ *if and only if* $(\mathscr{D}^{-1}L) \nabla X \simeq Y$.

In the following theorem, we refer to [25] for the notion of good modules. We only note that the fundamental module $V(\varpi_i)$ is a good module.

Theorem 2.9 ([1,4,22,25]).

(i) *For good modules M and N, the zeroes of $d_{M,N}(z)$ belong to $\mathbb{C}[[q^{1/m}]]q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.*

(ii) *For simple modules M and N such that one of them is real, M_x and N_y strongly commute if and only if $d_{M,N}(z)d_{N,M}(1/z)$ does not vanish at $z = y/x$.*

(iii) *Let M_k be a good module with a dominant extremal vector u_k of weight λ_k, and $a_k \in \mathbf{k}^\times$ for $k = 1, \dots, t$. Assume that a_j/a_i is not a zero of $d_{M_i,M_j}(z)$ for any $1 \le i < j \le t$. Then the following statements hold:*

  (a) *$(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is generated by $u_1 \otimes \cdots \otimes u_t$.*

  (b) *The head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ is simple.*

  (c) *Any non-zero submodule of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ contains the vector $u_t \otimes \cdots \otimes u_1$.*

  (d) *The socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ is simple.*

  (e) *Let $\mathbf{r}\colon (M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t} \to (M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$ be $\mathbf{r}_{(M_1)_{a_1},\dots,(M_t)_{a_t}} := \prod_{1 \le j < k \le t} \mathbf{r}_{(M_j)_{a_j},(M_k)_{a_k}}$. Then the image of $\mathbf{r}$ is simple and it coincides with the head of $(M_1)_{a_1} \otimes \cdots \otimes (M_t)_{a_t}$ and with the socle of $(M_t)_{a_t} \otimes \cdots \otimes (M_1)_{a_1}$.*

(iv) *For any simple module $M \in \mathscr{C}_{\mathfrak{g}}$, there exists a finite sequence $\{(i_k, a_k)\}_{k=1}^t$ in $\sigma(\mathfrak{g})$ (see (2.7) below) such that M has $\sum_{k=1}^t \varpi_{i_k}$ as a dominant extremal weight and it is isomorphic to a simple subquotient of $V(\varpi_{i_1})_{a_1} \otimes \cdots \otimes V(\varpi_{i_t})_{a_t}$. Moreover, the sequence $\{(i_k, a_k)\}_{k=1}^t$ is unique up to a permutation.*

We call $\sum_{k=1}^t (i_k, a_k) \in \widehat{\mathsf{P}} := \mathbb{Z}^{\oplus \sigma(\mathfrak{g})}$ the affine highest weight *of M.*

2.5. *Hernandez–Leclerc categories*

For $i \in I_0$, let m_i be a positive integer such that

$$\mathsf{W}\pi_i \cap (\pi_i + \mathbb{Z}\delta) = \pi_i + \mathbb{Z}m_i\delta,$$

where π_i is an element of P such that $\mathrm{cl}(\pi_i) = \varpi_i$. Note that $m_i = (\alpha_i, \alpha_i)/2$ when $\mathfrak{g}$ is the dual of an untwisted affine algebra, and $m_i = 1$ otherwise. Then $V(\varpi_i)_x \simeq V(\varpi_i)_y$ if and only if $x^{m_i} = y^{m_i}$ for $x, y \in \mathbf{k}^\times$ [1, Section 1.3]. We define

$$\sigma(\mathfrak{g}) := I_0 \times \mathbf{k}^\times/\sim, \tag{2.7}$$

where the equivalence relation $\sim$ is given by

$$(i,x) \sim (j,y) \iff V(\varpi_i)_x \simeq V(\varpi_j)_y \iff i = j \text{ and } x^{m_i} = y^{m_j}.$$

We denote by $[(i,a)]$ the equivalence class of (i,a) in $\sigma(\mathfrak{g})$. When no confusion can arise, we simply write (i,a) for the equivalence class $[(i,a)]$. For $(i,x),(j,y) \in \sigma(\mathfrak{g})$, we draw d arrows from (i,x) to (j,y), where d is the order of the zero of $d_{V(\varpi_i),V(\varpi_j)}(z_{V(\varpi_j)}/z_{V(\varpi_i)})$ at $z_{V(\varpi_j)}/z_{V(\varpi_i)} = y/x$. Thus, $\sigma(\mathfrak{g})$ has a quiver structure.

We choose a connected component $\sigma_0(\mathfrak{g})$ of $\sigma(\mathfrak{g})$. Since a connected component of $\sigma(\mathfrak{g})$ is unique up to a spectral parameter shift, $\sigma_0(\mathfrak{g})$ is uniquely determined up to a quiver isomorphism. We define $\mathscr{C}^0_{\mathfrak{g}}$ to be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}$ that

(a) contains $V(\varpi_i)_x$ for all $(i,x) \in \sigma_0(\mathfrak{g})$,

(b) is stable by taking subquotients, extensions and tensor products.

For symmetric affine types, this category was introduced in [15]. Note that every simple module in $\mathscr{C}_{\mathfrak{g}}$ is isomorphic to a tensor product of certain spectral parameter shifts of some simple modules in $\mathscr{C}^0_{\mathfrak{g}}$ [15, Section 3.7].

2.6. *Invariants related to R-matrices*

Let us recall the new invariants introduced in [27]. We set

$$\varphi(z) := \prod_{s=0}^{\infty}(1-\tilde{p}^s z) = \sum_{n=0}^{\infty} \frac{(-1)^n\, \tilde{p}^{n(n-1)/2}}{\prod_{k=1}^{n}(1-\tilde{p}^k)} z^n \in \mathbf{k}[[z]],$$

where $p^* := (-1)^{\langle \rho^\vee,\delta\rangle} q^{\langle c,\rho\rangle}$ and $\tilde{p} := (p^*)^2 = q^{2\langle c,\rho\rangle}$. We consider the subgroup $\mathscr{G}$ of $\mathbf{k}((z))^\times$ given by

$$\mathscr{G} := \left\{ cz^m \prod_{a\in\mathbf{k}^\times} \varphi(az)^{\eta_a} \;\middle|\; \begin{array}{c} c \in \mathbf{k}^\times,\ m\in\mathbb{Z}, \\ \eta_a \in \mathbb{Z} \text{ vanishes for all but finitely many } a\text{'s} \end{array} \right\}.$$

For a subset S of $\mathbb{Z}$, let $\tilde{p}^S := \{\tilde{p}^k \mid k \in S\}$. We define the group homomorphisms

$$\mathrm{Deg}\colon \mathscr{G} \to \mathbb{Z} \quad \text{and} \quad \mathrm{Deg}^\infty\colon \mathscr{G} \to \mathbb{Z}$$

by

$$\mathrm{Deg}(f(z)) = \sum_{a\in\tilde{p}^{\mathbb{Z}_{\le 0}}} \eta_a - \sum_{a\in\tilde{p}^{\mathbb{Z}_{>0}}} \eta_a \quad \text{and} \quad \mathrm{Deg}^\infty(f(z)) = \sum_{a\in\tilde{p}^{\mathbb{Z}}} \eta_a$$

for $f(z) = cz^m \prod_{a\in\mathbf{k}^\times} \varphi(az)^{\eta_a} \in \mathscr{G}$.

Note that

$$\mathrm{Deg}(f(z)) = 2\,\mathrm{zero}_{z=1} f(z) \quad \text{for } f(z) \in \mathbf{k}(z)^\times \subset \mathscr{G} \tag{2.8}$$

(see [27, Lemma 3.4]).

Definition 2.10. For non-zero $U_q'(\mathfrak{g})$-modules M and N such that $R^{\mathrm{univ}}_{M,N_z}$ is rationally renormalizable, we define

$$\begin{aligned}\Lambda(M,N) &:= \mathrm{Deg}(c_{M,N}(z)),\\ \Lambda^\infty(M,N) &:= \mathrm{Deg}^\infty(c_{M,N}(z)),\\ \mathfrak{d}(M,N) &:= \tfrac{1}{2}(\Lambda(M,N)+\Lambda(N,M)).\end{aligned}$$

Note that $\Lambda(M,N) \equiv \Lambda^\infty(M,N) \bmod 2$.

Proposition 2.11 ([27, Lemma 3.7, 3.8 and Corollary 3.23]). *Let M, N be simple modules in $\mathscr{C}_\mathfrak{g}$.*

(i) $\Lambda^\infty(M,N) = -\mathrm{Deg}^\infty(a_{M,N}(z))$.

(ii) $\Lambda^\infty(M,N) = \Lambda^\infty(N,M)$.

(iii) $\Lambda^\infty(M,N) = -\Lambda^\infty(\mathscr{D}M,N) = -\Lambda^\infty(M,\mathscr{D}N)$.

(iv) *In particular,* $\Lambda^\infty(M,N) = \Lambda^\infty(\mathscr{D}M,\mathscr{D}N)$.

Proposition 2.12 ([27, Lemma 3.7 and Proposition 3.18]). *Let M, N be simple modules in $\mathscr{C}_\mathfrak{g}$.*

(i) $\Lambda(M,N) = \Lambda(N,\mathscr{D}M) = \Lambda(\mathscr{D}^{-1}N,M)$.

(ii) *In particular,*

$$\Lambda(M,N) = \Lambda(\mathscr{D}M,\mathscr{D}N) = \Lambda(\mathscr{D}^{-1}M,\mathscr{D}^{-1}N).$$

Proposition 2.13 ([27, Proposition 3.11]). *Let M, N and L be non-zero modules in $\mathscr{C}_\mathfrak{g}$, and let S be a non-zero subquotient of $M \otimes N$.*

(i) *Assume that $R^{\mathrm{univ}}_{M,L_z}$ and $R^{\mathrm{univ}}_{N,L_z}$ are rationally renormalizable. Then $R^{\mathrm{univ}}_{S,L_z}$ is rationally renormalizable and*

$$\Lambda(S,L) \le \Lambda(M,L)+\Lambda(N,L) \quad \textit{and} \quad \Lambda^\infty(S,L) = \Lambda^\infty(M,L)+\Lambda^\infty(N,L).$$

(ii) *Assume that $R^{\mathrm{univ}}_{L,M_z}$ and $R^{\mathrm{univ}}_{L,N_z}$ are rationally renormalizable. Then $R^{\mathrm{univ}}_{L,S_z}$ is rationally renormalizable and*

$$\Lambda(L,S) \le \Lambda(L,M)+\Lambda(L,N) \quad \textit{and} \quad \Lambda^\infty(L,S) = \Lambda^\infty(L,M)+\Lambda^\infty(L,N).$$

Proposition 2.14 ([27, Proposition 3.16]). *Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$.*

(i) $\mathfrak{d}(M,N) = \mathrm{zero}_{z=1}(d_{M,N}(z)d_{N,M}(z^{-1}))$,

(ii) $\mathfrak{d}(M,N) = \mathfrak{d}(N,M)$.

In particular, $\mathfrak{d}(M,N) \in \mathbb{Z}_{\ge 0}$.

Corollary 2.15 ([27, Corollaries 3.17 and 3.20]). *Let M and N be simple modules in $\mathscr{C}_\mathfrak{g}$.*

(i) *Suppose that M or N is real. Then M and N strongly commute if and only if* $\mathfrak{d}(M,N) = 0$.

(ii) *In particular, if M is real, then* $\Lambda(M,M) = 0$.

Proposition 2.16 (i, ii) below was proved in [27, Proposition 3.22], and Proposition 2.16 (iii) is new. We give the whole proof of Proposition 2.16 for the reader's convenience.

Proposition 2.16. *For simple modules M and N in $\mathscr{C}_{\mathfrak{g}}$, we have the following:*

(i) $\Lambda(M,N) = \sum_{k\in\mathbb{Z}}(-1)^{k+\delta(k<0)}\,\mathfrak{d}(M,\mathscr{D}^kN) = \sum_{k\in\mathbb{Z}}(-1)^{k+\delta(k>0)}\,\mathfrak{d}(\mathscr{D}^kM,N)$,

(ii) $\Lambda^\infty(M,N) = \sum_{k\in\mathbb{Z}}(-1)^k\,\mathfrak{d}(M,\mathscr{D}^kN)$,

(iii) $\mathrm{zero}_{z=1}\,c_{M,N}(z) = \sum_{k=0}^{\infty}(-1)^k\,\mathfrak{d}(M,\mathscr{D}^kN)$.

Proof. We write $c_{M,N}(z) \equiv \prod_{a\in\mathbf{k}^\times}\varphi(az)^{\eta_a} \mod \mathbf{k}[z^{\pm1}]^\times$. Then

$$\frac{c_{M,N}(z)}{c_{M,N}(\tilde{p}z)} \equiv \prod_{a\in\mathbf{k}^\times}(1-az)^{\eta_a},$$

which yields

$$\begin{aligned}
\eta_{\tilde{p}^{-k}} &= \mathrm{zero}_{z=\tilde{p}^k}\left(\frac{c_{M,N}(z)}{c_{M,N}(\tilde{p}z)}\right) = \mathrm{zero}_{z=1}\left(\frac{c_{M,N}(\tilde{p}^kz)}{c_{M,N}(\tilde{p}^{k+1}z)}\right)\\
&= \mathrm{zero}_{z=1}\left(\frac{c_{M,N_{\tilde{p}^k}}(z)}{c_{M,N_{\tilde{p}^k}}(\tilde{p}z)}\right) \underset{(*)}{=} \mathrm{zero}_{z=1}\left(\frac{d_{M,N_{\tilde{p}^k}}(z)d_{N_{\tilde{p}^k},M}(z^{-1})}{d_{\mathscr{D}^{-1}M,N_{\tilde{p}^k}}(z)d_{N_{\tilde{p}^k},\mathscr{D}^{-1}M}(z^{-1})}\right)\\
&\underset{(**)}{=} \mathfrak{d}(M,N_{\tilde{p}^k}) - \mathfrak{d}(\mathscr{D}^{-1}M,N_{\tilde{p}^k}) = \mathfrak{d}(M,\mathscr{D}^{2k}N) - \mathfrak{d}(M,\mathscr{D}^{2k+1}N),
\end{aligned}$$

where $(*)$ follows from [27, Lemma 3.15] and $(**)$ from Proposition 2.14. Therefore,

$$\begin{aligned}
\Lambda(M,N) &= \sum_{k\in\mathbb{Z}}(-1)^{\delta(k>0)}\eta_{\tilde{p}^k} = \sum_{k\in\mathbb{Z}}(-1)^{\delta(k<0)}\eta_{\tilde{p}^{-k}}\\
&= \sum_{k\in\mathbb{Z}}(-1)^{\delta(k<0)}\big(\mathfrak{d}(M,\mathscr{D}^{2k}N) - \mathfrak{d}(M,\mathscr{D}^{2k+1}N)\big)\\
&= \sum_{k\in\mathbb{Z}}(-1)^{k+\delta(k<0)}\,\mathfrak{d}(M,\mathscr{D}^kN),
\end{aligned}$$

which implies the first assertion (i). Similarly, we obtain (ii):

$$\begin{aligned}
\Lambda^\infty(M,N) &= \sum_{k\in\mathbb{Z}}\eta_{\tilde{p}^k} = \sum_{k\in\mathbb{Z}}\big(\mathfrak{d}(M,\mathscr{D}^{-2k}N) - \mathfrak{d}(M,\mathscr{D}^{-2k+1}N)\big)\\
&= \sum_{k\in\mathbb{Z}}(-1)^k\,\mathfrak{d}(M,\mathscr{D}^kN).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{zero}_{z=1}\,c_{M,N}(z) &= \sum_{k=0}^{\infty}\eta_{\tilde{p}^{-k}} = \sum_{k=0}^{\infty}\big(\mathfrak{d}(M,\mathscr{D}^{2k}N) - \mathfrak{d}(M,\mathscr{D}^{2k+1}N)\big)\\
&= \sum_{k=0}^{\infty}(-1)^k\,\mathfrak{d}(M,\mathscr{D}^kN),
\end{aligned}$$

which gives (iii). ■

Proposition 2.17 ([27, Corollary 4.12]). *Let L be a real simple module, and M a simple module. Assume that $\mathfrak{d}(L, M) > 0$. Then*

$$\mathfrak{d}(L, S) < \mathfrak{d}(L, M)$$

for any simple subquotient S of $L \otimes M$ and also for any simple subquotient S of $M \otimes L$.

The assumption in the following definition is slightly weaker than the one in [27, Definition 4.14]. Under this weak assumption, the statements as in [27, Lemma 4.15–4.18] can be proved in the same manner.

Definition 2.18. Let $L_1, \dots, L_r$ be simple modules that are all real except possibly one. The sequence $(L_1, \dots, L_r)$ is called a *normal sequence* if the composition of the R-matrices

$$\begin{aligned}\mathbf{r}_{L_1,\dots,L_r} &:= \prod_{1 \le i < j \le r} \mathbf{r}_{L_i,L_j} \\ &= (\mathbf{r}_{L_{r-1},L_r}) \circ \cdots \circ (\mathbf{r}_{L_2,L_r} \circ \cdots \circ \mathbf{r}_{L_2,L_3}) \circ (\mathbf{r}_{L_1,L_r} \circ \cdots \circ \mathbf{r}_{L_1,L_2}) \colon \\ &\qquad\qquad L_1 \otimes L_2 \otimes \cdots \otimes L_r \to L_r \otimes \cdots \otimes L_2 \otimes L_1\end{aligned}$$

does not vanish.

Lemma 2.19 ([27, Lemma 4.15]). *Let $(L_1, \dots, L_r)$ be a normal sequence of simple modules that are all real except possibly one. Then $\operatorname{Im}(\mathbf{r}_{L_1,\dots,L_r})$ is simple and it coincides with the head of $L_1 \otimes \cdots \otimes L_r$ and with the socle of $L_r \otimes \cdots \otimes L_1$.*

Lemma 2.20 ([27, Lemma 4.16]). *Let $L_1, \dots, L_r$ be simple modules that are all real except possibly one.*

- (i) *If $(L_1, \dots, L_r)$ is normal, then*
  - (a) *$(L_2, \dots, L_r)$ and $(L_1, \dots, L_{r-1})$ are normal,*
  - (b) $\Lambda(L_1, \mathrm{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{j=2}^{r} \Lambda(L_1, L_j)$,
    $\Lambda(\mathrm{hd}(L_1 \otimes \cdots \otimes L_{r-1}), L_r) = \sum_{j=1}^{r-1} \Lambda(L_j, L_r)$.
- (ii) *If L_1 is real, $(L_2, \dots, L_r)$ is normal and*

$$\Lambda(L_1, \mathrm{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{j=2}^{r} \Lambda(L_1, L_j),$$

  then $(L_1, \dots, L_r)$ is normal.
- (iii) *If L_r is real, $(L_1, \dots, L_{r-1})$ is normal and*

$$\Lambda(\mathrm{hd}(L_1 \otimes \cdots \otimes L_{r-1}), L_r) = \sum_{j=1}^{r-1} \Lambda(L_j, L_r),$$

  then $(L_1, \dots, L_r)$ is normal.

Lemma 2.21 ([27, Lemmas 4.3 and 4.17]). *Let L, M, N be simple modules that are all real except possibly one. If one of the following conditions holds:*

(a) $\mathfrak{d}(L, M) = 0$ *and L is real,*

(b) $\mathfrak{d}(M, N) = 0$ *and N is real,*

(c) $\mathfrak{d}(L, \mathscr{D}^{-1}N) = \mathfrak{d}(\mathscr{D}L, N) = 0$ *and L or N is real,*

then (L, M, N) is a normal sequence, i.e.,

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N), \quad \Lambda(L \nabla M, N) = \Lambda(L, N) + \Lambda(M, N).$$

Lemma 2.22 ([27, Corollary 4.18]). *Let L, M, N be simple modules. Assume that L is real and one of M and N is real. Then (L, M, N) is a normal sequence if and only if $(M, N, \mathscr{D}L)$ is a normal sequence.*

In [27, Corollary 4.18], a stronger condition is assumed, but the same proof still works without change.

Lemma 2.23. *Let $L_1, \dots, L_r$ be simple modules that are all real except possibly one. Suppose that the sequence $(L_1, \dots, L_r)$ is normal. For any $m \in \mathbb{Z}$, we have*

$$\mathscr{D}^m(\mathrm{hd}(L_1 \otimes \cdots \otimes L_r)) \simeq \mathrm{hd}(\mathscr{D}^m L_1 \otimes \cdots \otimes \mathscr{D}^m L_r).$$

Proof. It suffices to handle the cases $m = \pm 1$. We assume that $m = 1$. Since the sequence $(\mathscr{D}L_1, \dots, \mathscr{D}L_r)$ is normal, by Lemma 2.19 we have

$$\mathscr{D}(\mathrm{hd}(L_1 \otimes \cdots \otimes L_r)) \simeq \mathrm{soc}(\mathscr{D}L_r \otimes \cdots \otimes \mathscr{D}L_1) \simeq \mathrm{hd}(\mathscr{D}L_1 \otimes \cdots \otimes \mathscr{D}L_r).$$

The case $m = -1$ can be proved in the same manner. ∎

Lemma 2.24. *Let L, M, N be simple modules. Assume that L is real and one of M and N is real. Then $\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N)$ if and only if (L, M, N) and (M, N, L) are normal sequences.*

Proof. By the assumption, we have

$$\begin{aligned} 2\big(\mathfrak{d}(L, M) + \mathfrak{d}(L, N) - \mathfrak{d}(L, M \nabla N)\big) = \big(&(\Lambda(L, M) + \Lambda(L, N) - \Lambda(L, M \nabla N)\big) \\ &+ \big(\Lambda(M, L) + \Lambda(N, L) - \Lambda(M \nabla N, L)\big). \end{aligned}$$

Since $\Lambda(L, M) + \Lambda(L, N) - \Lambda(L, M \nabla N)$ and $\Lambda(M, L) + \Lambda(N, L) - \Lambda(M \nabla N, L)$ are non-negative by Proposition 2.13, we have

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N) \quad \text{and} \quad \Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L)$$

if and only if $\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N)$. Then the assertion follows from Lemma 2.20. ∎

Corollary 2.25. *Let L and M be real simple modules and X a simple module.*

(i) *If* $\mathfrak{d}(L, M) = \mathfrak{d}(\mathscr{D}^{-1}L, M) = 0$, *then* $\mathfrak{d}(L, M \nabla X) = \mathfrak{d}(L, X)$.

(ii) *If* $\mathfrak{d}(L, M) = \mathfrak{d}(\mathscr{D}L, M) = 0$, *then* $\mathfrak{d}(L, X \nabla M) = \mathfrak{d}(L, X)$.

Proof. (i) The triples (L, M, X) and (M, X, L) are normal by Lemma 2.21, and hence $\mathfrak{d}(L, M \nabla X) = \mathfrak{d}(L, M) + \mathfrak{d}(L, X) = \mathfrak{d}(L, X)$ by the preceding lemma.

(ii) can be proved similarly. ∎

The following lemma can be proved similarly to [24, Proposition 3.2.17], and we do not repeat the proof here.

Lemma 2.26. *Let M and N be simple modules, and assume that one of them is real. If $\mathfrak{d}(M, N) = 1$, then $M \otimes N$ has length 2 and we have an exact sequence*

$$0 \to N \nabla M \to M \otimes N \to M \nabla N \to 0.$$

The following lemma gives a criterion for a simple module to be real.

Lemma 2.27. *Let X be a simple module such that $\mathfrak{d}(X, X) = 0$ and $X \otimes X$ has a simple head. Then X is real.*

Proof. Since $\mathfrak{d}(X, X) = 0$, we have $R^{\mathrm{ren}}_{X,X_z} \circ R^{\mathrm{ren}}_{X_z,X} = f(z)\,\mathrm{id}$ for some $f(z) \in \mathbf{k}(z)$ which is invertible at $z = 1$. Thus $\mathbf{r}^2_{X,X} \in \mathbf{k}^\times \mathrm{id}$. By normalizing, we may assume that $\mathbf{r}^2_{X,X} = \mathrm{id}$. Then

$$X \otimes X = \mathrm{Ker}(\mathbf{r}_{X,X} - \mathrm{id}) \oplus \mathrm{Ker}(\mathbf{r}_{X,X} + \mathrm{id}).$$

Since $X \otimes X$ has a simple head, we conclude that $\mathbf{r}_{X,X}$ should be $\pm\,\mathrm{id}$, which implies the assertion by [22, Corollary 3.3 and Theorem 3.12]. ∎

Lemma 2.28. *Let M, N be real simple modules such that $\mathfrak{d}(M, N) = 1$. Then $M \nabla N$ is real.*

Proof. It follows from Proposition 2.17 that

$$\mathfrak{d}(M, M \nabla N) < \mathfrak{d}(M, N) = 1, \quad \mathfrak{d}(N, M \nabla N) < \mathfrak{d}(M, N) = 1,$$

which implies that $\mathfrak{d}(M, M \nabla N) = \mathfrak{d}(N, M \nabla N) = 0$. We set $X := M \nabla N$. Since $\mathfrak{d}(M, X) = \mathfrak{d}(N, X) = 0$, we have $0 \le \mathfrak{d}(X, X) \le \mathfrak{d}(M, X) + \mathfrak{d}(N, X) = 0$, i.e.,

$$\mathfrak{d}(X, X) = 0.$$

Since N is real and $X \otimes M$ is simple, $(X \otimes M) \otimes N$ has a simple head. Thus the surjection

$$(X \otimes M) \otimes N \twoheadrightarrow X \otimes X$$

tells us that $X \otimes X$ has a simple head. Then the assertion follows from Lemma 2.27. ∎

Lemma 2.29. *Let M and N be real simple modules such that $\mathfrak{d}(M, N) = 1$. Then*

(i) *$M \nabla N$ commutes with M and N,*

(ii) *for any $m, n \in \mathbb{Z}_{\ge 0}$, we have*

$$M^{\otimes m} \nabla N^{\otimes n} \simeq \begin{cases} (M \nabla N)^{\otimes m} \otimes N^{\otimes (n-m)} & \text{if } m \le n, \\ (M \nabla N)^{\otimes n} \otimes M^{\otimes (m-n)} & \text{if } m \ge n. \end{cases}$$

Proof. (i) follows from $\mathfrak{d}(M, M \nabla N) \le \mathfrak{d}(M, N) - 1 = 0$ and $\mathfrak{d}(N, M \nabla N) \le \mathfrak{d}(N, M) - 1 = 0$.

(ii) We shall prove only the first isomorphism. We argue by induction on $m \le n$. If $m = 0$ the assertion is obvious. Assume that $m > 0$. Then

$$\begin{aligned} M^{\otimes m} \otimes N^{\otimes n} &\twoheadrightarrow M^{\otimes(m-1)} \otimes (M \nabla N) \otimes N^{\otimes(n-1)} \\ &\simeq (M \nabla N) \otimes M^{\otimes(m-1)} \otimes N^{\otimes(n-1)} \\ &\twoheadrightarrow (M \nabla N) \otimes ((M \nabla N)^{\otimes(m-1)} \otimes N^{\otimes(n-m)}) \\ &\simeq (M \nabla N)^{\otimes m} \otimes N^{\otimes(n-m)}. \end{aligned}$$

Now the assertion follows from the fact that $(M \nabla N)^{\otimes m} \otimes N^{\otimes(n-m)}$ is a simple quotient of $M^{\otimes m} \otimes N^{\otimes n}$ which has a simple head. ∎

Lemma 2.30. *Let M and N be real simple modules such that $\mathfrak{d}(M, N) = 1$. Then for any simple module X,*

(i) *the simple module $M \nabla (N \nabla X)$ is isomorphic to either $(M \nabla N) \nabla X$ or $(N \nabla M) \nabla X$,*

(ii) *the simple module $(X \nabla M) \nabla N$ is isomorphic to either $X \nabla (M \nabla N)$ or $X \nabla (N \nabla M)$.*

Proof. Since the proof is similar, we prove only (i). Let us consider a commutative diagram with an exact row:

$$\begin{tikzcd} 0 \arrow[r] & (N \nabla M) \otimes X \arrow[r] \arrow[dr, "f"'] & M \otimes N \otimes X \arrow[r] \arrow[d, two heads] & (M \nabla N) \otimes X \arrow[r] & 0 \\ & & M \nabla (N \nabla X) & & \end{tikzcd}$$

The exactness follows from Lemma 2.26. By Lemma 2.28, $M \nabla N$ and $N \nabla M$ are real simple modules. If f does not vanish, then we have $(N \nabla M) \nabla X \simeq M \nabla (N \nabla X)$.

If f vanishes then there exists an epimorphism $(M \nabla N) \otimes X \twoheadrightarrow M \nabla (N \nabla X)$ and hence $(M \nabla N) \nabla X \simeq M \nabla (N \nabla X)$. ∎

3. Root modules

In this section, we investigate properties of *root modules*.

Definition 3.1. A module $L \in \mathscr{C}_{\mathfrak{g}}$ is called a *root module* if L is a real simple module such that

$$\mathfrak{d}(L, \mathscr{D}^k L) = \delta(k = \pm 1) \quad \text{for any } k \in \mathbb{Z}. \tag{3.1}$$

Note that for a root module L, we have

$$\Lambda^\infty(L, L) = -2$$

by Proposition 2.16.

The name “root module” comes from Lemma 4.15 below.

Example 3.2. Using the denominators for fundamental modules (see [31, Appendix A] for example), one can easily prove that any fundamental module $V(\varpi_i)_a$ ($i \in I_0, a \in \mathbf{k}^\times$) is a root module.

3.1. Properties of root modules

Lemma 3.3. *Let L be a root module and let X be a simple module.*

(i) *For $k \in \mathbb{Z}$, we have*

$$\mathfrak{d}(\mathscr{D}^k L, X) - \delta(k = 0, 2) \le \mathfrak{d}(\mathscr{D}^k L, L \nabla X) \le \mathfrak{d}(\mathscr{D}^k L, X) + \delta(k = \pm 1).$$

In particular, $\mathfrak{d}(\mathscr{D}^k L, L \nabla X) = \mathfrak{d}(\mathscr{D}^k L, X)$ for $k \ne -1, 0, 1, 2$, and

$$\mathfrak{d}(\mathscr{D}^k L, L \nabla X) - \mathfrak{d}(\mathscr{D}^k L, X) \in \begin{cases} \{0, 1\} & \text{for } k = \pm 1, \\ \{0, -1\} & \text{for } k = 0, 2. \end{cases}$$

Moreover,

$$\begin{aligned}&\big(\mathfrak{d}(\mathscr{D}^{-1} L, L \nabla X) - \mathfrak{d}(\mathscr{D}^{-1} L, X)\big) + \big(\mathfrak{d}(L, X) - \mathfrak{d}(L, L \nabla X)\big) \\ &\qquad + \big(\mathfrak{d}(\mathscr{D} L, L \nabla X) - \mathfrak{d}(\mathscr{D} L, X)\big) + \big(\mathfrak{d}(\mathscr{D}^2 L, X) - \mathfrak{d}(\mathscr{D}^2 L, L \nabla X)\big) = 2.\end{aligned}$$

(ii) *For $k \in \mathbb{Z}$, we have*

$$\mathfrak{d}(\mathscr{D}^k L, X) - \delta(k = -2, 0) \le \mathfrak{d}(\mathscr{D}^k L, X \nabla L) \le \mathfrak{d}(\mathscr{D}^k L, X) + \delta(k = \pm 1).$$

In particular, $\mathfrak{d}(\mathscr{D}^k L, X \nabla L) = \mathfrak{d}(\mathscr{D}^k L, X)$ for $k \ne -2, -1, 0, 1$, and

$$\mathfrak{d}(\mathscr{D}^k L, X \nabla L) - \mathfrak{d}(\mathscr{D}^k L, X) \in \begin{cases} \{0, 1\} & \text{for } k = \pm 1, \\ \{0, -1\} & \text{for } k = 0, -2. \end{cases}$$

Moreover,

$$\begin{aligned}&\big(\mathfrak{d}(\mathscr{D}^{-2} L, X) - \mathfrak{d}(\mathscr{D}^{-2} L, X \nabla L)\big) + \big(\mathfrak{d}(\mathscr{D}^{-1} L, X \nabla L) - \mathfrak{d}(\mathscr{D}^{-1} L, X)\big) \\ &\qquad + \big(\mathfrak{d}(L, X) - \mathfrak{d}(L, X \nabla L)\big) + \big(\mathfrak{d}(\mathscr{D} L, X \nabla L) - \mathfrak{d}(\mathscr{D} L, X)\big) = 2.\end{aligned}$$

Proof. (i) By [27, Proposition 4.2], we have

$$\mathfrak{d}(\mathscr{D}^k L, L \nabla X) \le \mathfrak{d}(\mathscr{D}^k L, X) + \mathfrak{d}(\mathscr{D}^k L, L) = \mathfrak{d}(\mathscr{D}^k L, X) + \delta(k = \pm 1).$$

For the same reason, it follows from $X \simeq (L \nabla X) \nabla \mathscr{D} L$ (see Lemma 2.7) that

$$\begin{aligned}\mathfrak{d}(\mathscr{D}^k L, X) &= \mathfrak{d}(\mathscr{D}^k L, (L \nabla X) \nabla \mathscr{D} L) \\ &\le \mathfrak{d}(\mathscr{D}^k L, L \nabla X) + \mathfrak{d}(\mathscr{D}^k L, \mathscr{D} L) \\ &= \mathfrak{d}(\mathscr{D}^k L, L \nabla X) + \delta(k = 0, 2).\end{aligned}$$

Hence we obtain the first assertion. Since

$$\Lambda^\infty(L, L \nabla X) = \Lambda^\infty(L, L) + \Lambda^\infty(L, X) = -2 + \Lambda^\infty(L, X), \tag{3.2}$$

it follows from (3.2) and Proposition 2.16 that

$$\begin{aligned} 2 &= \Lambda^\infty(L, X) - \Lambda^\infty(L, L \nabla X) \\ &= \sum_{k\in\mathbb{Z}} (-1)^k (\mathfrak{d}(\mathscr{D}^k L, X) - \mathfrak{d}(\mathscr{D}^k L, L \nabla X)) \\ &= \big(\mathfrak{d}(\mathscr{D}^{-1} L, L \nabla X) - \mathfrak{d}(\mathscr{D}^{-1} L, X)\big) + \big(\mathfrak{d}(L, X) - \mathfrak{d}(L, L \nabla X)\big) \\ &\quad + \big(\mathfrak{d}(\mathscr{D} L, L \nabla X) - \mathfrak{d}(\mathscr{D} L, X)\big) + \big(\mathfrak{d}(\mathscr{D}^2 L, X) - \mathfrak{d}(\mathscr{D}^2 L, L \nabla X)\big), \end{aligned}$$

which yields the last assertion.

(ii) can be proved in the same manner using the fact that $X \simeq (\mathscr{D}^{-1} L) \nabla (X \nabla L)$. ■

Lemma 3.4. *Let L be a root module and let X be a simple module. Suppose that*

$$\mathfrak{d}(L, X) > 0.$$

Then

(i) $\mathfrak{d}(L, L \nabla X) = \mathfrak{d}(L, X) - 1$ *and* $\mathfrak{d}(\mathscr{D}^{-1} L, L \nabla X) = \mathfrak{d}(\mathscr{D}^{-1} L, X)$,

(ii) $\mathfrak{d}(L, X \nabla L) = \mathfrak{d}(L, X) - 1$ *and* $\mathfrak{d}(\mathscr{D} L, X \nabla L) = \mathfrak{d}(\mathscr{D} L, X)$.

Proof. We shall prove only (i) since the proof of (ii) is similar.

Since $\mathfrak{d}(L, X) > 0$, L does not commute with X. By Proposition 2.17, we have

$$\mathfrak{d}(L, L \nabla X) < \mathfrak{d}(L, X).$$

On the other hand, Lemma 3.3 implies $\mathfrak{d}(L, X) \le \mathfrak{d}(L, L \nabla X) + 1$, which implies the first assertion.

Let us show the second equation in (i). By Lemma 3.3, we have $\mathfrak{d}(\mathscr{D}^{-1} L, L \nabla X) = \mathfrak{d}(\mathscr{D}^{-1} L, X)$ or $\mathfrak{d}(\mathscr{D}^{-1} L, X) + 1$. If

$$\mathfrak{d}(\mathscr{D}^{-1} L, L \nabla X) = \mathfrak{d}(\mathscr{D}^{-1} L, X) + 1,$$

then Lemma 2.24 says that $(\mathscr{D}^{-1} L, L, X)$ is a normal sequence, and hence (L, X, L) is also a normal sequence by Lemma 2.22, which implies that

$$L \nabla X \simeq X \nabla L.$$

But this contradicts $\mathfrak{d}(L, X) > 0$, and therefore $\mathfrak{d}(\mathscr{D}^{-1} L, L \nabla X) = \mathfrak{d}(\mathscr{D}^{-1} L, X)$. ■

Lemma 3.5. *Let L be a root module and X a simple module.*

(i) *Assume one of the following conditions:*

  (a) $\mathfrak{d}(\mathscr{D} L, L \nabla X) > 0$,

  (b) $\mathfrak{d}(\mathscr{D} L, X) > 0$,

  (c) $\mathfrak{d}(\mathscr{D}^2 L, X) = 0$.

  Then $\mathfrak{d}(\mathscr{D} L, L \nabla X) = \mathfrak{d}(\mathscr{D} L, X) + 1$.

(ii) *Assume one of the following conditions:*

(a) $\mathfrak{d}(\mathscr{D}^{-1}L, X \nabla L) > 0$,

(b) $\mathfrak{d}(\mathscr{D}^{-1}L, X) > 0$,

(c) $\mathfrak{d}(\mathscr{D}^{-2}L, X) = 0$.

Then $\mathfrak{d}(\mathscr{D}^{-1}L, X \nabla L) = \mathfrak{d}(\mathscr{D}^{-1}L, X) + 1$.

Proof. We shall only prove (i) since the proof of (ii) is similar.

(a) Assume that $\mathfrak{d}(\mathscr{D}L, L \nabla X) > 0$. Setting $Y = L \nabla X$, we have $X \simeq Y \nabla \mathscr{D}L$. Hence Lemma 3.4 (ii) implies $\mathfrak{d}(\mathscr{D}L, Y \nabla \mathscr{D}L) = \mathfrak{d}(\mathscr{D}L, Y) - 1$.

(b) If $\mathfrak{d}(\mathscr{D}L, X) > 0$, then $\mathfrak{d}(\mathscr{D}L, L \nabla X) \geq \mathfrak{d}(\mathscr{D}L, X) > 0$ by Lemma 3.3.

(c) Finally, assume that $\mathfrak{d}(\mathscr{D}^2L, X) = 0$. If $\mathfrak{d}(L, X) = 0$, then

$$\mathfrak{d}(\mathscr{D}L, L \nabla X) = \mathfrak{d}(\mathscr{D}L, L) + \mathfrak{d}(\mathscr{D}L, X) = \mathfrak{d}(\mathscr{D}L, X) + 1.$$

Suppose that $\mathfrak{d}(L, X) > 0$. Since $\mathfrak{d}(\mathscr{D}^2L, X) = 0$ and $\mathfrak{d}(\mathscr{D}^2L, L) = 0$, we have

$$\mathfrak{d}(\mathscr{D}^2L, L \nabla X) = 0. \tag{3.3}$$

Moreover, Lemma 3.4 tells us that

$$\mathfrak{d}(L, L \nabla X) = \mathfrak{d}(L, X) - 1 \quad \text{and} \quad \mathfrak{d}(\mathscr{D}^{-1}L, L \nabla X) = \mathfrak{d}(\mathscr{D}^{-1}L, X). \tag{3.4}$$

By Lemma 3.3, (3.3) and (3.4), we have

$$\begin{aligned} 2 &= \big(\mathfrak{d}(\mathscr{D}^{-1}L, L \nabla X) - \mathfrak{d}(\mathscr{D}^{-1}L, X)\big) + \big(\mathfrak{d}(L, X) - \mathfrak{d}(L, L \nabla X)\big) \\ &\quad + \big(\mathfrak{d}(\mathscr{D}L, L \nabla X) - \mathfrak{d}(\mathscr{D}L, X)\big) + \big(\mathfrak{d}(\mathscr{D}^2L, X) - \mathfrak{d}(\mathscr{D}^2L, L \nabla X)\big) \\ &= 1 + \big(\mathfrak{d}(\mathscr{D}L, L \nabla X) - \mathfrak{d}(\mathscr{D}L, X)\big), \end{aligned}$$

which yields the desired result. ■

Lemma 3.6. *Let L be a root module, X a simple module, and $k \in \mathbb{Z}_{\geq 0}$.*

(i) *Suppose that one of the following conditions is true:*

(a) $\mathfrak{d}(\mathscr{D}L,\ L^{\otimes k} \nabla X) \geq k$,

(b) $\mathfrak{d}(\mathscr{D}^2L, X) = 0$.

Then $\mathfrak{d}(\mathscr{D}L,\ L^{\otimes k} \nabla X) = \mathfrak{d}(\mathscr{D}L, X) + k$.

(ii) *Suppose that one of the following is true:*

(a) $\mathfrak{d}(\mathscr{D}^{-1}L, X \nabla L^{\otimes k}) \geq k$,

(b) $\mathfrak{d}(\mathscr{D}^{-2}L, X) = 0$.

Then $\mathfrak{d}(\mathscr{D}^{-1}L,\ X \nabla L^{\otimes k}) = \mathfrak{d}(\mathscr{D}^{-1}L, X) + k$.

Proof. This follows from the preceding lemma by induction on k. Note that we have $\mathfrak{d}(\mathscr{D}^2L, L^{\otimes k} \nabla X) = 0$ as long as $\mathfrak{d}(\mathscr{D}^2L, X) = 0$. ■

Proposition 3.7. *Let L be a root module and X a simple module. If $\mathfrak{d}(L,X) \neq 1$, then $(L \nabla X) \nabla L$ is isomorphic to $L \nabla (X \nabla L)$.*

Proof. If $\mathfrak{d}(L,X) = 0$, then this is obvious. Hence we may assume that $\mathfrak{d}(L,X) \geq 2$.

Set $Y := (L \nabla X) \nabla L$. Then

$$L \nabla X \simeq \mathscr{D}^{-1}L \nabla Y \quad \text{and} \quad X \simeq (\mathscr{D}^{-1}L \nabla Y) \nabla \mathscr{D}L.$$

Lemma 3.4 says that $\mathfrak{d}(L, L \nabla X) = \mathfrak{d}(L,X) - 1 > 0$ and $\mathfrak{d}(L,Y) = \mathfrak{d}(L, L \nabla X) - 1$. Hence

$$\mathfrak{d}(L, \mathscr{D}^{-1}L \nabla Y) = \mathfrak{d}(L,Y) + 1 = \mathfrak{d}(L, \mathscr{D}^{-1}L) + \mathfrak{d}(L,Y).$$

Then Lemma 2.24 implies that $(L, \mathscr{D}^{-1}L, Y)$ is a normal sequence, and $(\mathscr{D}^{-1}L, Y, \mathscr{D}L)$ is also a normal sequence by Lemma 2.22. Hence

$$(\mathscr{D}^{-1}L \nabla Y) \nabla \mathscr{D}L \simeq \mathscr{D}^{-1}L \nabla (Y \nabla \mathscr{D}L).$$

Since $X \simeq \mathscr{D}^{-1}L \nabla (Y \nabla \mathscr{D}L)$, we obtain $Y \simeq L \nabla (X \nabla L)$. ■

3.2. *Properties of pairs of root modules*

Let L and L' be root modules. Throughout this subsection, we assume that

$$\mathfrak{d}(\mathscr{D}^k L, L') = \delta(k = 0) \quad \text{for } k \in \mathbb{Z}. \tag{3.5}$$

Note that, by Proposition 2.16,

$$\Lambda(L, L') = \Lambda^\infty(L, L') = 1.$$

Lemma 3.8. *The simple module $L \nabla L'$ is a root module.*

Proof. Set $L'' := L \nabla L'$. By Lemma 2.28, L'' is real.

It is obvious that $\mathfrak{d}(\mathscr{D}^k L, L'') = \mathfrak{d}(\mathscr{D}^k L', L'') = 0$ for $k \neq 0, \pm 1$. On the other hand, Lemma 3.4 implies that $\mathfrak{d}(L, L'') = \mathfrak{d}(L', L'') = 0$. Hence $\mathfrak{d}(\mathscr{D}^k L'', L'') = 0$ for $k \neq \pm 1$. Now, we have

$$\Lambda^\infty(L'', L'') = \Lambda^\infty(L, L) + 2\Lambda^\infty(L, L') + \Lambda^\infty(L', L') = (-2) + 2 + (-2) = -2.$$

Then Proposition 2.16 implies that

$$-2 = \sum_{k \in \mathbb{Z}} (-1)^k \, \mathfrak{d}(\mathscr{D}^k L'', L'') = -\mathfrak{d}(\mathscr{D}L'', L'') - \mathfrak{d}(\mathscr{D}^{-1}L'', L'').$$

Since $\mathfrak{d}(\mathscr{D}L'', L'') = \mathfrak{d}(\mathscr{D}^{-1}L'', L'')$, we obtain $\mathfrak{d}(\mathscr{D}^{\pm 1}L'', L'') = 1$. ■

Lemma 3.9. *We have*

$$\mathfrak{d}(\mathscr{D}^k L, L \nabla L') = \delta(k = 1) \quad \textit{and} \quad \mathfrak{d}(\mathscr{D}^k L, L' \nabla L) = \delta(k = -1).$$

Proof. Since $\mathfrak{d}(\mathscr{D}^k L, L \nabla L') \le \mathfrak{d}(\mathscr{D}^k L, L) + \mathfrak{d}(\mathscr{D}^k L, L')$, we have $\mathfrak{d}(\mathscr{D}^k L, L \nabla L') = 0$ for $k \neq -1, 0, 1$. It follows from Lemma 3.4 that

$$\begin{aligned} \mathfrak{d}(L, L \nabla L') &= \mathfrak{d}(L, L') - 1 = 0, \\ \mathfrak{d}(\mathscr{D}^{-1} L, L \nabla L') &= \mathfrak{d}(\mathscr{D}^{-1} L, L') = 0. \end{aligned}$$

On the other hand, since

$$\mathfrak{d}(\mathscr{D} L, L \nabla L') \le \mathfrak{d}(\mathscr{D} L, L) + \mathfrak{d}(\mathscr{D} L, L') = 1,$$

we have $\mathfrak{d}(\mathscr{D} L, L \nabla L') \in \{0, 1\}$.

If $\mathfrak{d}(\mathscr{D} L, L \nabla L') = 0$, then

$$L' \simeq (L \nabla L') \nabla \mathscr{D} L \simeq (L \nabla L') \otimes \mathscr{D} L,$$

which implies that

$$0 = \mathfrak{d}(\mathscr{D}^2 L, L') = \mathfrak{d}(\mathscr{D}^2 L, (L \nabla L') \otimes \mathscr{D} L) = \mathfrak{d}(\mathscr{D}^2 L, L \nabla L') + \mathfrak{d}(\mathscr{D}^2 L, \mathscr{D} L) \ge 1.$$

This is a contradiction, so $\mathfrak{d}(\mathscr{D} L, L \nabla L') = 1$. Thus we obtained the first equality.

The second equality can be proved similarly. ■

Lemma 3.10. *Let X be a simple module.*

(i) *If $k \neq 0, 1$, then*

$$\mathfrak{d}(\mathscr{D}^k L, L' \nabla X) = \mathfrak{d}(\mathscr{D}^k L, X).$$

As for $k = 0$ and 1, one and only one of the following two statements is true:

(a) $\mathfrak{d}(L, L' \nabla X) = \mathfrak{d}(L, X)$ *and* $\mathfrak{d}(\mathscr{D} L, L' \nabla X) = \mathfrak{d}(\mathscr{D} L, X) - 1$,

(b) $\mathfrak{d}(L, L' \nabla X) = \mathfrak{d}(L, X) + 1$ *and* $\mathfrak{d}(\mathscr{D} L, L' \nabla X) = \mathfrak{d}(\mathscr{D} L, X)$.

(ii) *If $k \neq -1, 0$, then*

$$\mathfrak{d}(\mathscr{D}^k L, X \nabla L') = \mathfrak{d}(\mathscr{D}^k L, X).$$

As for $k = -1$ and 0, one and only one of the following two statements is true:

(a) $\mathfrak{d}(L, X \nabla L') = \mathfrak{d}(L, X)$ *and* $\mathfrak{d}(\mathscr{D}^{-1} L, X \nabla L') = \mathfrak{d}(\mathscr{D}^{-1} L, X) - 1$,

(b) $\mathfrak{d}(L, X \nabla L') = \mathfrak{d}(L, X) + 1$ *and* $\mathfrak{d}(\mathscr{D}^{-1} L, X \nabla L') = \mathfrak{d}(\mathscr{D}^{-1} L, X)$.

Proof. (i) By [27, Proposition 4.2], we have

$$\begin{aligned} \mathfrak{d}(\mathscr{D}^k L, L' \nabla X) &\le \mathfrak{d}(\mathscr{D}^k L, X) + \delta(k = 0), \\ \mathfrak{d}(\mathscr{D}^k L, X) &\le \mathfrak{d}(\mathscr{D}^k L, L' \nabla X) + \delta(k = 1), \end{aligned}$$

where the second inequality follows from $X \simeq (L' \nabla X) \nabla \mathscr{D} L'$. The above inequalities give the first assertion and

$$\begin{aligned} \mathfrak{d}(L, L' \nabla X) &= \mathfrak{d}(L, X) \text{ or } \mathfrak{d}(L, X) + 1, \\ \mathfrak{d}(\mathscr{D} L, L' \nabla X) &= \mathfrak{d}(\mathscr{D} L, X) \text{ or } \mathfrak{d}(\mathscr{D} L, X) - 1. \end{aligned} \tag{3.6}$$

By the assumption (3.5), we have $\Lambda^\infty(L, L') = 1$, which implies

$$1 = \Lambda^\infty(L, L' \nabla X) - \Lambda^\infty(L, X) = \sum_{k\in\mathbb{Z}} (-1)^k \big(\mathfrak{d}(\mathscr{D}^k L, L' \nabla X) - \mathfrak{d}(\mathscr{D}^k L, X)\big)$$
$$= \big(\mathfrak{d}(L, L' \nabla X) - \mathfrak{d}(L, X)\big) + \big(\mathfrak{d}(\mathscr{D} L, X) - \mathfrak{d}(\mathscr{D} L, L' \nabla X)\big).$$

Then (3.6) implies the second assertion.

(ii) can be proved similarly by using $X \simeq \mathscr{D}^{-1} L' \nabla (X \nabla L')$. ■

Proposition 3.11. *Let X be a simple module.*

(i) *If $\mathfrak{d}(\mathscr{D} L, X) = 0$, then*

$$\mathfrak{d}(L, L' \nabla X) = \mathfrak{d}(L, X) + 1, \quad \mathfrak{d}(\mathscr{D} L, L' \nabla X) = 0.$$

(ii) *If $\mathfrak{d}(\mathscr{D}^{-1} L, X) = 0$, then*

$$\mathfrak{d}(L, X \nabla L') = \mathfrak{d}(L, X) + 1, \quad \mathfrak{d}(\mathscr{D}^{-1} L, X \nabla L') = 0.$$

Proof. (i) Since $\mathfrak{d}(\mathscr{D} L, L' \nabla X) \le \mathfrak{d}(\mathscr{D} L, L') + \mathfrak{d}(\mathscr{D} L, X) = 0$, Lemma 3.10 tells us that $\mathfrak{d}(L, L' \nabla X) = \mathfrak{d}(L, X) + 1$.

(ii) can be proved in the same manner as above. ■

Corollary 3.12. *Let $n \in \mathbb{Z}_{\ge 0}$ and let X be a simple module.*

(i) *If $\mathfrak{d}(\mathscr{D} L, X) = 0$, then*

$$\mathfrak{d}(L, L'^{\otimes n} \nabla X) = \mathfrak{d}(L, X) + n \quad \textit{and} \quad \mathfrak{d}(\mathscr{D} L, L'^{\otimes n} \nabla X) = 0.$$

(ii) *If $\mathfrak{d}(\mathscr{D}^{-1} L, X) = 0$, then*

$$\mathfrak{d}(L, X \nabla L'^{\otimes n}) = \mathfrak{d}(L, X) + n \quad \textit{and} \quad \mathfrak{d}(\mathscr{D}^{-1} L, X \nabla L'^{\otimes n}) = 0.$$

Proof. These follow easily from Proposition 3.11 by induction on n. ■

Proposition 3.13. *Let $m \in \mathbb{Z}_{\ge 0}$ and let Y be a simple module. Set*

$$X := L^{\otimes m} \nabla Y.$$

Suppose that

$$\mathfrak{d}(L, X) = 0, \quad \mathfrak{d}(\mathscr{D} L', Y) = 0, \quad \mathfrak{d}(\mathscr{D}^t L, Y) = 0 \quad \textit{for } t = 1, 2.$$

Then

(i) *$\mathfrak{d}(\mathscr{D} L, X) = m$,*

(ii) *for any integer k such that $0 \le k \le m$, we have*

$$\mathfrak{d}(L, L'^{\otimes k} \nabla X) = 0, \quad \mathfrak{d}(\mathscr{D} L, L'^{\otimes k} \nabla X) = m - k,$$

(iii) *for any integer $k \geq m$, we have*

$$\mathfrak{d}(L, L'^{\otimes k} \nabla X) = k - m, \quad \mathfrak{d}(\mathscr{D}L, L'^{\otimes k} \nabla X) = 0.$$

Proof. (i) As $\mathfrak{d}(\mathscr{D}^t L, Y) = 0$ for $t = 1, 2$, we have

$$\mathfrak{d}(\mathscr{D}L, X) = \mathfrak{d}(\mathscr{D}L, L^{\otimes m} \nabla Y) = \mathfrak{d}(\mathscr{D}L, Y) + m = m$$

by Lemma 3.6 (i).

(ii) We see that

- $L' \nabla L$ commutes with L, L' and $\mathscr{D}L$ by Lemma 3.9,
- $\mathfrak{d}(\mathscr{D}L', X) = 0$ because $\mathfrak{d}(\mathscr{D}L', Y) = 0$ and $\mathfrak{d}(\mathscr{D}L', L) = 0$,
- the triple $(L'^{\otimes a}, A, X)$ is normal because $\mathfrak{d}(\mathscr{D}L', X) = 0$,
- the triple $(L'^{\otimes a}, B, L)$ is normal because $\mathfrak{d}(\mathscr{D}L', L) = 0$,
- the triples $(L'^{\otimes a}, A, Y)$, $(L^{\otimes a}, A, Y)$ and $((L' \nabla L)^{\otimes a}, A, Y)$ are normal because $\mathfrak{d}(\mathscr{D}L', Y) = 0$ and $\mathfrak{d}(\mathscr{D}L, Y) = 0$,

Here A is a real simple module, B is a simple module, and a is an arbitrary non-negative integer. We will use these facts freely in the subsequent arguments.

For any k such that $0 \leq k \leq m$, we have

$$\begin{aligned} L'^{\otimes k} \nabla X &\simeq L'^{\otimes k} \nabla (L^{\otimes m} \nabla Y) \simeq (L'^{\otimes k} \nabla L^{\otimes m}) \nabla Y \\ &\simeq ((L' \nabla L)^{\otimes k} \otimes L^{\otimes (m-k)}) \nabla Y \simeq (L^{\otimes (m-k)} \otimes (L' \nabla L)^{\otimes k}) \nabla Y \\ &\simeq L^{\otimes (m-k)} \nabla ((L' \nabla L)^{\otimes k} \nabla Y), \end{aligned}$$

where the third isomorphism follows from Lemma 2.29. Thus, for $1 \leq k \leq m$,

$$\begin{aligned} L \nabla (L'^{\otimes k} \nabla X) &\simeq L \nabla \big((L^{\otimes (m-k)} \otimes (L' \nabla L)^{\otimes k}) \nabla Y\big) \\ &\simeq \big(L^{\otimes (m-k+1)} \otimes (L' \nabla L)^{\otimes k}\big) \nabla Y \\ &\simeq \big((L' \nabla L) \otimes L^{\otimes (m-k+1)} \otimes (L' \nabla L)^{\otimes (k-1)}\big) \nabla Y \\ &\simeq (L' \nabla L) \nabla ((L^{\otimes (m-(k-1))} \otimes (L' \nabla L)^{\otimes (k-1)}) \nabla Y) \\ &\simeq (L' \nabla L) \nabla (L'^{\otimes (k-1)} \nabla X), \end{aligned}$$

and

$$\begin{aligned} (L'^{\otimes k} \nabla X) \nabla L &\simeq L'^{\otimes k} \nabla (X \otimes L) \simeq L'^{\otimes k} \nabla (L \otimes X) \\ &\simeq (L'^{\otimes k} \nabla L) \nabla X \simeq \big((L' \nabla L) \otimes L'^{\otimes (k-1)}\big) \nabla X \\ &\simeq (L' \nabla L) \nabla (L'^{\otimes (k-1)} \nabla X). \end{aligned}$$

This tells us that

$$L \nabla (L'^{\otimes k} \nabla X) \simeq (L'^{\otimes k} \nabla X) \nabla L,$$

which implies that $\mathfrak{d}(L, L'^{\otimes k} \nabla X) = 0$.

On the other hand, since $\mathfrak{d}(\mathscr{D}^2 L, Y) = 0$ and $\mathfrak{d}(\mathscr{D}^2 L, L' \nabla L) = 0$, we have

$$\mathfrak{d}\big(\mathscr{D}^2 L, (L' \nabla L)^{\otimes k} \nabla Y\big) = 0.$$

Then, Lemma 3.6 implies that

$$\begin{aligned}\mathfrak{d}(\mathscr{D} L, L'^{\otimes k} \nabla X) &= \mathfrak{d}\big(\mathscr{D} L,\ L^{\otimes (m-k)} \nabla ((L' \nabla L)^{\otimes k} \nabla Y)\big)\\ &= \mathfrak{d}(\mathscr{D} L, (L' \nabla L)^{\otimes k} \nabla Y) + m - k = m - k,\end{aligned}$$

where the last equality follows from $\mathfrak{d}(\mathscr{D} L, L' \nabla L) = 0$ and $\mathfrak{d}(\mathscr{D} L, Y) = 0$.

(iii) By (ii), we have

$$\mathfrak{d}(L, L'^{\otimes m} \nabla X) = 0, \quad \mathfrak{d}(\mathscr{D} L, L'^{\otimes m} \nabla X) = 0.$$

Since

$$L'^{\otimes k} \nabla X \simeq L'^{\otimes (k-m)} \nabla (L'^{\otimes m} \nabla X),$$

we have the assertion by Corollary 3.12 (i). ■

4. Quantum affine Schur–Weyl duality

Let $\mathcal{D} := \{\mathsf{L}_i\}_{i \in J} \subset \mathscr{C}_\mathfrak{g}$ be a family of simple modules of $\mathscr{C}_\mathfrak{g}$. The family $\mathcal{D}$ is called a *duality datum* associated with a generalized Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$ of symmetric type if it satisfies the following:

(a) for each $i \in J$, L_i is a real simple module,

(b) for any $i, j \in J$ such that $i \neq j$, $\mathfrak{d}(\mathsf{L}_i, \mathsf{L}_j) = -c_{i,j}$.

Then one can construct a monoidal functor

$$\mathcal{F}_\mathcal{D} \colon R_\mathsf{C}\text{-gmod} \to \mathscr{C}_\mathfrak{g}$$

using the duality datum $\mathcal{D}$ [21, 35].

The functor $\mathcal{F}_\mathcal{D}$ is called a *quantum affine Schur–Weyl duality functor* or briefly a *duality functor*.

In Section 4.2 below, we slightly modify the definition of quantum affine Schur–Weyl duality functor in order that it commutes with affinization.

4.1. Affinizations

4.1.1. Pro-objects. Let $\mathbf{k}$ be a base field and let $\mathcal{C}$ be an essentially small $\mathbf{k}$-abelian category. Let Pro($\mathcal{C}$) be the category of pro-objects of $\mathcal{C}$ (see [36] for details). One can show that

$$\operatorname{Pro}(\mathcal{C}) \simeq \big\{\text{left exact } \mathbf{k}\text{-linear functors from } \mathcal{C} \text{ to } \mathbf{k}\text{-Mod}\big\}^{\text{opp}}$$

by means of the functor

$$\text{“}\varprojlim_i\text{” } M_i \to \big(\mathcal{C} \ni X \mapsto \varinjlim \operatorname{Hom}_\mathcal{C}(M_i, X)\big).$$

Here, $\mathbf{k}$-Mod is the category of vector spaces over $\mathbf{k}$, and "$\varprojlim$" denotes the *pro-lim* (see [36, Section 2.6 and Proposition 6.1.7] for notations and details). Then, $\mathrm{Pro}(\mathscr{C})$ is a $\mathbf{k}$-abelian category which admits small projective limits. If no confusion can arise, we regard $\mathscr{C}$ as a full subcategory of $\mathrm{Pro}(\mathscr{C})$, which is stable by extensions and subquotients. Any functor $F\colon\mathscr{C}\to\mathscr{C}'$ extends to $\mathrm{P}F\colon\mathrm{Pro}(\mathscr{C})\to\mathrm{Pro}(\mathscr{C}')$ which commutes with small filtrant projective limits:

$$\mathrm{P}F\Big(\text{“}\varprojlim_{i}\text{”}\, M_i\Big)\simeq \text{“}\varprojlim_{i}\text{”}\, F(M_i).$$

4.1.2. Affinization in quiver Hecke algebra case. Let R be a symmetric quiver Hecke algebra. Note that

$$R(\beta)\text{-gMod}\to\mathrm{Pro}(R(\beta)\text{-gmod}).$$

Recall that $R(\beta)$-gMod is the category of graded $R(\beta)$-modules. Let

$$\mathrm{Pro}(R):=\bigoplus_{\beta\in\mathsf{Q}^+}\mathrm{Pro}(R(\beta)\text{-gmod}),$$

which is a monoidal category. Let z be an indeterminate of homogeneous degree 2, and we set

$$R(\beta)^{\mathrm{aff}}:=\mathbf{k}[z]\mathop{\otimes}_{\mathbf{k}}\mathbf{1}_z R(\beta),$$

which has the graded $R(\beta)$-bimodule structure. Here $\mathbf{1}_z R(\beta)$ is a free right $R(\beta)$-module of rank 1 and the left module structure is given by

$$e(\nu)\mathbf{1}_z=\mathbf{1}_z e(\nu),\quad x_k\mathbf{1}_z=\mathbf{1}_z x_k+z\mathbf{1}_z,\quad \tau_k\mathbf{1}_z=\mathbf{1}_z\tau_k.$$

Hence we have

$$\mathbf{1}_z x_k=(x_k-z)\mathbf{1}_z. \tag{4.1}$$

For $X\in R(\beta)$-gmod, the affinization X^{aff} of X is isomorphic to $R(\beta)^{\mathrm{aff}}\otimes_{R(\beta)}X$. Since X^{aff} is not in $R(\beta)$-gmod, we set

$$X^{\mathrm{Aff}}:=\text{“}\varprojlim_{m}\text{”}\, X^{\mathrm{aff}}/z^m X^{\mathrm{aff}}\in\mathrm{Pro}(R(\beta)\text{-gmod}).$$

Note that

$$X^{\mathrm{Aff}}\simeq\mathbf{k}[[z]]\mathop{\otimes}_{\mathbf{k}}X$$

as an object of $\mathrm{Pro}(\mathbf{k}\text{-mod})$ forgetting the action of $R(\beta)$. Here we regard $\mathbf{k}[[z]]$ as the object of $\mathrm{Pro}(\mathbf{k}\text{-mod})$:

$$\text{“}\varprojlim_{m}\text{”}\,\mathbf{k}[z]\;/\;\mathbf{k}[z]z^m.$$

Similarly we set

$$R(\beta)^{\mathrm{Aff}}:=\text{“}\varprojlim_{m}\text{”}\, R(\beta)^{\mathrm{aff}}/R(\beta)^{\mathrm{aff}}(z,x_1,\ldots,x_{\mathrm{ht}(\beta)})^m,$$

which is an object of $\mathrm{Pro}(R(\beta)\text{-gmod})$ with a right $R(\beta)$-action. Here, $(z, x_1, \dots, x_m)$ is the ideal of $\mathbf{k}[z, x_1, \dots, x_m]$ generated by $z, x_1, \dots, x_m$. Then

$$M^{\mathrm{Aff}} \simeq R(\beta)^{\mathrm{Aff}} \otimes_{R(\beta)} M \quad \text{for any } M \in R(\beta)\text{-gmod}.$$

For $M, N \in R$-gmod, we have

$$M^{\mathrm{Aff}} \underset{z}{\circ} N^{\mathrm{Aff}} \simeq (M \circ N)^{\mathrm{Aff}},$$

where

$$M^{\mathrm{Aff}} \underset{z}{\circ} N^{\mathrm{Aff}} := \mathrm{Coker}\big(M^{\mathrm{Aff}} \circ N^{\mathrm{Aff}} \xrightarrow{z_M - z_N} M^{\mathrm{Aff}} \circ N^{\mathrm{Aff}}\big).$$

We remark that, in this paper, we use the language of pro-objects instead of the completion in [21, Section 3.1] and [12].

4.1.3. Affinization in quantum affine algebra case. Let $U_q'(\mathfrak{g})$ be a quantum affine algebra and let $\mathscr{C}_\mathfrak{g}$ be the category of finite-dimensional integrable $U_q'(\mathfrak{g})$-modules. We embed $\mathscr{C}_\mathfrak{g}$ into $\mathrm{Pro}(\mathscr{C}_\mathfrak{g})$. Note that $\mathrm{Pro}(\mathscr{C}_\mathfrak{g})$ is a $\mathbf{k}$-abelian monoidal category. For $M \in \mathscr{C}_\mathfrak{g}$, let M^{aff} be the affinization of M. Recall that

$$M^{\mathrm{aff}} \simeq \mathbf{k}[\mathbf{z}_M^{\pm 1}] \underset{\mathbf{k}}{\otimes} M$$

with the action

$$e_i(a \otimes v) = \mathbf{z}_M^{\delta_{i,0}} a \otimes e_i v \quad \text{for } a \in \mathbf{k}[\mathbf{z}_M^{\pm 1}] \text{ and } v \in M.$$

Here we use $\mathbf{z}$ to distinguish from z in the quiver Hecke algebra setting. We set

$$M^{\mathrm{Aff}} := \text{“}\varprojlim_m\text{”}\, M^{\mathrm{aff}}/(\mathbf{z}_M - 1)^m M^{\mathrm{aff}} \in \mathrm{Pro}(\mathscr{C}_\mathfrak{g}).$$

Note that there is a canonical algebra homomorphism

$$\mathbf{k}[[\mathbf{z}_M - 1]] \to \mathrm{End}_{\mathrm{Pro}(\mathscr{C}_\mathfrak{g})}(M^{\mathrm{Aff}}).$$

For $M, N \in \mathscr{C}_\mathfrak{g}$, we have

$$M^{\mathrm{Aff}} \underset{\mathbf{z}}{\otimes} N^{\mathrm{Aff}} \simeq (M \otimes N)^{\mathrm{Aff}},$$

where

$$M^{\mathrm{Aff}} \underset{\mathbf{z}}{\otimes} N^{\mathrm{Aff}} := \mathrm{Coker}\big(M^{\mathrm{Aff}} \otimes N^{\mathrm{Aff}} \xrightarrow{\mathbf{z}_M - \mathbf{z}_N} M^{\mathrm{Aff}} \otimes N^{\mathrm{Aff}}\big).$$

For simple modules M, N in $\mathscr{C}_\mathfrak{g}$, we can define the renormalized R-matrix

$$R^{\mathrm{ren}}_{M,N}(\mathbf{z}_N/\mathbf{z}_M)\colon M^{\mathrm{Aff}} \otimes N^{\mathrm{Aff}} \to N^{\mathrm{Aff}} \otimes M^{\mathrm{Aff}}.$$

4.2. Quantum affine Schur–Weyl duality functor

We now consider a duality datum $\mathcal{D} = \{\mathsf{L}_i\}_{i \in J}$ associated with a symmetric generalized Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$. For $i, j \in J$, we choose $\mathrm{c}_{i,j}(x) \in \mathbf{k}[[x]]$ such that

$$\mathrm{c}_{i,j}(x)\mathrm{c}_{j,i}(-x) = 1 \quad \text{and} \quad \mathrm{c}_{i,i}(0) = 1.$$

We set

$$P_{ij}(u,v) := \mathsf{c}_{ij}(u-v)\cdot(u-v)^{d_{i,j}},$$

where $d_{i,j} := \mathrm{zero}_{z=1}\, d_{\mathsf{L}_i,\mathsf{L}_j}(z)$.

Let $\mathsf{Q}^+_{\mathsf{C}}$ be the positive root lattice associated with C. For $\beta \in \mathsf{Q}^+_{\mathsf{C}}$ with $\ell = \mathrm{ht}(\beta)$ and $\nu = (\nu_1,\ldots,\nu_\ell) \in J^\beta$, we set

$$\widehat{\mathsf{L}}_\nu := \mathsf{L}_{\nu_1}^{\mathrm{Aff}} \otimes \cdots \otimes \mathsf{L}_{\nu_\ell}^{\mathrm{Aff}}, \quad \widehat{\mathsf{L}}(\beta) := \bigoplus_{\nu\in J^\beta} \widehat{\mathsf{L}}_\nu \in \mathrm{Pro}(\mathscr{C}_\mathfrak{g}).$$

The algebra $R(\beta)$ acts $\widehat{\mathsf{L}}(\beta)$ from the right as follows:

(a) $e(\nu)$ is the projection to $\widehat{\mathsf{L}}_\nu$,

(b) $x_k \in R(\beta)$ acts by $\log \mathbf{z}_{L_{\nu_k}}$, where $\log \mathbf{z}_{L_{\nu_k}} \in \mathbf{k}[[\mathbf{z}_{L_{\nu_k}} - 1]] \subset \mathrm{End}(\widehat{\mathsf{L}}_\nu)$,

(c) $e(\nu)\tau_k$ $(1 \le k < \ell)$ acts on $\widehat{\mathsf{L}}_\nu$ by

$$\begin{cases} R^{\mathrm{norm}}_{\mathsf{L}_{\nu_k},\mathsf{L}_{\nu_{k+1}}} \circ P_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) & \text{if } \nu_k \neq \nu_{k+1}, \\ (x_k - x_{k+1})^{-1}\big(R^{\mathrm{norm}}_{\mathsf{L}_{\nu_k},\mathsf{L}_{\nu_{k+1}}} \circ P_{\nu_k,\nu_k}(x_k,x_{k+1}) - \mathrm{id}_{\widehat{\mathsf{L}}_\nu}\big) & \text{if } \nu_k = \nu_{k+1}. \end{cases}$$

Note that we used $\mathbf{z}-1$ instead of $\log \mathbf{z}$ in [21]. We have also relaxed the condition on $\mathsf{c}_{i,i}(u)$. We have changed the definition so as to have Theorem 4.2 below.

Then $\widehat{\mathsf{L}}(\beta)$ gives the monoidal functor

$$\widehat{\mathscr{F}}_\mathfrak{D}\colon R\text{-gmod} \to \mathrm{Pro}(\mathscr{C}_\mathfrak{g})$$

defined by

$$\widehat{\mathscr{F}}_\mathfrak{D}(M) = \widehat{\mathsf{L}}(\beta) \underset{R(\beta)}{\otimes} M \quad \text{for } M \in R(\beta)\text{-gmod}.$$

It extends to

$$\widehat{\mathscr{F}}_\mathfrak{D}\colon \mathrm{Pro}(R) \to \mathrm{Pro}(\mathscr{C}_\mathfrak{g})$$

such that $\widehat{\mathscr{F}}_\mathfrak{D}$ commutes with filtrant projective limits.

The following proposition can be proved in a similar manner to [21].

Proposition 4.1. *$\widehat{\mathscr{F}}_\mathfrak{D}$ is a monoidal functor and it induces a monoidal functor $\mathscr{F}_\mathfrak{D}$: R-*gmod *$\to \mathscr{C}_\mathfrak{g}$.*

Then the following theorem tells us that the functor $\widehat{\mathscr{F}}_\mathfrak{D}$ preserves affinizations.

Theorem 4.2. *Functorially in $M \in R$-*gmod*, we have an isomorphism*

$$\widehat{\mathscr{F}}_\mathfrak{D}(M^{\mathrm{Aff}}) \simeq (\mathscr{F}_\mathfrak{D}(M))^{\mathrm{Aff}}.$$

Moreover,

(i) *the action of z_M on the left term coincides with $\log \mathbf{z}_{\mathscr{F}_\mathfrak{D}(M)}$ on the right term,*

(ii) *for $M, N \in R$-*gmod*, the following diagram commutes:*

$$\begin{tikzcd}
\widehat{\mathscr{F}}_\mathfrak{D}(M^{\mathrm{Aff}} \circ N^{\mathrm{Aff}}) \arrow[r, "\sim"] \arrow[d] & \widehat{\mathscr{F}}_\mathfrak{D}(M^{\mathrm{Aff}}) \otimes \widehat{\mathscr{F}}_\mathfrak{D}(N^{\mathrm{Aff}}) \arrow[r, "\sim"] & (\mathscr{F}_\mathfrak{D}(M))^{\mathrm{Aff}} \otimes (\mathscr{F}_\mathfrak{D}(N))^{\mathrm{Aff}} \arrow[d] \\
\widehat{\mathscr{F}}((M \circ N)^{\mathrm{Aff}}) \arrow[r, "\sim"] & (\mathscr{F}_\mathfrak{D}(M \circ N))^{\mathrm{Aff}} \arrow[r, "\sim"] & (\mathscr{F}_\mathfrak{D}(M) \otimes \mathscr{F}_\mathfrak{D}(N))^{\mathrm{Aff}}
\end{tikzcd}$$

Proof. Let us show (i). Since $\widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)^{\mathrm{Aff}}) \otimes_{R(\beta)} M \simeq \widehat{\mathscr{F}}_{\mathfrak{D}}(M^{\mathrm{Aff}})$ for any $M \in R$-gmod, it is enough to show that

$$\widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)^{\mathrm{Aff}}) \simeq (\widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)))^{\mathrm{Aff}}$$

compatible with the right actions of $R(\beta)$.

Set $\ell := \mathrm{ht}(\beta)$ and $x_k := \log \mathbf{z}_{L_{\nu_k}} \in \mathrm{End}(\widehat{\mathsf{L}}(\beta))$ for $k = 1, \ldots, \ell$. Then we have $\widehat{\mathsf{L}}_\nu = \mathbf{k}[[x_1, \ldots, x_\ell]] \otimes \mathsf{L}_\nu$. Here we set

$$\mathsf{L}_\nu = \mathsf{L}_{\nu_1} \otimes \cdots \otimes \mathsf{L}_{\nu_\ell}, \quad \mathsf{L}(\beta) = \bigoplus_{\nu \in J^\beta} \mathsf{L}_\nu.$$

Then e_i acts on $\widehat{\mathsf{L}}_\nu$ by

$$\sum_{k=1}^{\ell} \mathrm{e}^{\delta_{i,0} x_k} (e_i)_k. \tag{4.2}$$

Here e^x is the exponential function and $(e_i)_k$ denotes the action on L_ν given by

$$\underbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{k-1 \text{ times}} \otimes e_i \otimes \underbrace{K_i^{-1} \otimes \cdots \otimes K_i^{-1}}_{\ell - k \text{ times}}.$$

Then we have:

(i) $\widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)^{\mathrm{Aff}}) \simeq \mathbf{k}[[z, x_1, \ldots, x_\ell]] \otimes \mathsf{L}(\beta)$. Here e_i acts by (4.2). The right action of $x_k \in R(\beta)$ is given by $x_k - z$ by (4.1).

(ii) $(\widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)))^{\mathrm{Aff}} \simeq \mathbf{k}[[z, x_1, \ldots, x_\ell]] \otimes \mathsf{L}(\beta)$. Here e_i acts by

$$\mathrm{e}^{\delta_{i,0} z} \sum_{k=1}^{\ell} \mathrm{e}^{\delta_{i,0} x_k} (e_i)_k = \sum_{k=1}^{\ell} \mathrm{e}^{\delta_{i,0}(x_k + z)} (e_i)_k. \tag{4.3}$$

The right action of $x_k \in R(\beta)$ is given by x_k.

Hence, the morphism

$$f : \widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)^{\mathrm{Aff}}) \to (\widehat{\mathscr{F}}_{\mathfrak{D}}(R(\beta)))^{\mathrm{Aff}}$$

given by $a(z, x) \otimes v \mapsto a(z, x_1 + z, \ldots, x_\ell + z) \otimes v$ (with $a(z, x) \in \mathbf{k}[[z, x_1, \ldots, x_\ell]]$ and $v \in \mathsf{L}(\beta)$) gives an isomorphism in $\mathrm{Pro}(\mathscr{C}_\mathfrak{g})$ and the right action of $x_k \in R(\beta)$ commutes with it. The compatibility of the right action of $\tau_k \in R(\beta)$ easily follows from the fact that $P_{i,j}(u, v)$ is a function of $u - v$.

The second assertion (ii) is immediate. ■

4.3. *Quantum affine Schur–Weyl duality with simply-laced Cartan matrix*

Hereafter, we assume that $\mathsf{C} = (c_{i,j})_{i,j \in J}$ *is a simply-laced Cartan matrix of finite type.*

Let R_C be the symmetric quiver Hecke algebra associated with C. If no confusion can arise, we simply write R for R_C.

Let $\mathscr{D} = \{\mathsf{L}_i\}_{i \in J}$ be a duality datum associated with the Cartan matrix C.

Proposition 4.3 ([21]). (i) *$\widehat{\mathscr{F}}_{\mathscr{D}}$ is an exact functor and it commutes with projective limits.*

(ii) *$\mathscr{F}_{\mathscr{D}}$ sends a simple module to a simple module or zero.*

Lemma 4.4. *Let $M \in R$-gmod be a real simple module, and assume that $\mathscr{F}_{\mathscr{D}}(M)$ is simple. Then $\mathscr{F}_{\mathscr{D}}(M)$ is also a real simple module.*

Proof. Since $\mathscr{F}_{\mathscr{D}}(M) \otimes \mathscr{F}_{\mathscr{D}}(M) \simeq \mathscr{F}_{\mathscr{D}}(M \circ M)$ and $M \circ M$ is simple, $\mathscr{F}_{\mathscr{D}}(M) \otimes \mathscr{F}_{\mathscr{D}}(M)$ is simple, i.e., $\mathscr{F}_{\mathscr{D}}(M)$ is real. ∎

Lemma 4.5. *Let $M, N \in R$-gmod be simple modules such that $\mathscr{F}_{\mathscr{D}}(M)$ and $\mathscr{F}_{\mathscr{D}}(N)$ are simple modules. Assume that M or N is real.*

(i) $\mathfrak{d}(\mathscr{F}_{\mathscr{D}}(M), \mathscr{F}_{\mathscr{D}}(N)) \le \mathfrak{d}(M, N)$.

(ii) *The following conditions are equivalent:*

  (a) $\mathfrak{d}(\mathscr{F}_{\mathscr{D}}(M), \mathscr{F}_{\mathscr{D}}(N)) = \mathfrak{d}(M, N)$.

  (b) *$\mathscr{F}_{\mathscr{D}}(M \nabla N)$ and $\mathscr{F}_{\mathscr{D}}(N \nabla M)$ are simple.*

  If these conditions hold, then

  (1) *$\mathscr{F}_{\mathscr{D}}(\mathbf{r}_{M,N}) \neq 0$ and $\mathscr{F}_{\mathscr{D}}(\mathbf{r}_{N,M}) \neq 0$,*

  (2) *$\mathscr{F}_{\mathscr{D}}(M) \nabla \mathscr{F}_{\mathscr{D}}(N) \simeq \mathscr{F}_{\mathscr{D}}(M \nabla N)$ and $\mathscr{F}_{\mathscr{D}}(N) \nabla \mathscr{F}_{\mathscr{D}}(M) \simeq \mathscr{F}_{\mathscr{D}}(N \nabla M)$.*

Proof. Set $z = \log \mathbf{z}$ and $d := \mathfrak{d}(M, N)$. By the definition of $\mathfrak{d}(M, N)$, we have the following commutative diagram (up to a constant multiple):

$$\begin{tikzcd}
M_z \circ N \arrow[r, "R^{\mathrm{ren}}_{M_z, N}"] \arrow[rr, bend left, "z^d \, \mathrm{id}"] & N \circ M_z \arrow[r, "R^{\mathrm{ren}}_{N, M_z}"] & M_z \circ N.
\end{tikzcd}$$

Applying $\widehat{\mathscr{F}}$ to the above diagram, by Proposition 4.1 and Theorem 4.2 we obtain

$$\begin{tikzcd}
\mathscr{F}_{\mathscr{D}}(M)_{\mathbf{z}} \otimes \mathscr{F}_{\mathscr{D}}(N) \arrow[r, "\widehat{\mathscr{F}}(R^{\mathrm{ren}}_{M_z, N})"] \arrow[rr, bend left, "z^d \, \mathrm{id}"] & \mathscr{F}_{\mathscr{D}}(N) \otimes \mathscr{F}_{\mathscr{D}}(M)_{\mathbf{z}} \arrow[r, "\widehat{\mathscr{F}}(R^{\mathrm{ren}}_{N, M_z})"] & \mathscr{F}_{\mathscr{D}}(M)_{\mathbf{z}} \otimes \mathscr{F}_{\mathscr{D}}(N).
\end{tikzcd} \tag{4.4}$$

Since z^d id is non-zero, $\widehat{\mathscr{F}}(R^{\mathrm{ren}}_{M_z,N})$ and $\widehat{\mathscr{F}}(R^{\mathrm{ren}}_{N,M_z})$ are non-zero. Note that

$$\mathrm{Hom}_{\mathbf{k}[\mathbf{z}^{\pm 1}] \otimes U_q'(\mathfrak{g})}(U \otimes V_{\mathbf{z}}, V_{\mathbf{z}} \otimes U) = \mathbf{k}[\mathbf{z}^{\pm 1}] R^{\mathrm{ren}}_{U, V_{\mathbf{z}}},$$
$$\mathrm{Hom}_{\mathbf{k}[\mathbf{z}^{\pm 1}] \otimes U_q'(\mathfrak{g})}(U_{\mathbf{z}} \otimes V, V \otimes U_{\mathbf{z}}) = \mathbf{k}[\mathbf{z}^{\pm 1}] R^{\mathrm{ren}}_{U_{\mathbf{z}}, V},$$

for any simple modules $U, V \in \mathscr{C}_\mathfrak{g}$ by [25, Proposition 9.5]. Hence

$$\widehat{\mathcal{F}}_\mathfrak{D}(R^{\mathrm{ren}}_{M_z,N}) = z^a f(z) R^{\mathrm{ren}}_{\mathcal{F}_\mathfrak{D}(M)_z,N}, \quad \widehat{\mathcal{F}}_\mathfrak{D}(R^{\mathrm{ren}}_{N,M_z}) = z^b g(z) R^{\mathrm{ren}}_{N,\mathcal{F}_\mathfrak{D}(M)_z}$$

for some $a, b \geq 0$ and $f(z), g(z) \in \mathbf{k}[[z]]^\times$. Hence it follows from (4.4) that

$$d = a + b + \mathfrak{d}(\mathcal{F}_\mathfrak{D}(M), \mathcal{F}_\mathfrak{D}(N)).$$

Thus $\mathfrak{d}(\mathcal{F}_\mathfrak{D}(M), \mathcal{F}_\mathfrak{D}(N)) \leq d$.

Moreover, $d = \mathfrak{d}(\mathcal{F}_\mathfrak{D}(M), \mathcal{F}_\mathfrak{D}(N))$ if and only if $a = b = 0$. Since $a = b = 0$ is equivalent to $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{M,N}) = \widehat{\mathcal{F}}_\mathfrak{D}(R^{\mathrm{ren}}_{M_z,N})|_{z=0} \neq 0$ and $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{N,M}) = \widehat{\mathcal{F}}_\mathfrak{D}(R^{\mathrm{ren}}_{N,M_z})|_{z=0} \neq 0$. The last two conditions conditions are equivalent to $\mathrm{Im}(\mathcal{F}_\mathfrak{D}(\mathbf{r}_{M,N})) \simeq \mathcal{F}_\mathfrak{D}(M \nabla N) \not\simeq 0$ and $\mathrm{Im}(\mathcal{F}_\mathfrak{D}(\mathbf{r}_{N,M})) \simeq \mathcal{F}_\mathfrak{D}(N \nabla M) \not\simeq 0$. ■

Lemma 4.6. *Let $\mathfrak{D} = \{\mathsf{L}_i\}_{i \in J}$ be a duality datum associated with a simply-laced finite Cartan matrix C. Let L, M, N be simple R_C-modules and S a simple subquotient of $M \circ N$. Assume that $\mathcal{F}_\mathfrak{D}(M)$, $\mathcal{F}_\mathfrak{D}(N)$ and $\mathcal{F}_\mathfrak{D}(S)$ are simple.*

(i) *Assume that $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{M,L})$ and $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{N,L})$ are non-zero. Then*

$$\begin{aligned}\Lambda(\mathcal{F}_\mathfrak{D}(M), \mathcal{F}_\mathfrak{D}(L)) + \Lambda(\mathcal{F}_\mathfrak{D}(N), \mathcal{F}_\mathfrak{D}(L)) - \Lambda(\mathcal{F}_\mathfrak{D}(S), \mathcal{F}_\mathfrak{D}(L)) \\ \geq \Lambda(M, L) + \Lambda(N, L) - \Lambda(S, L).\end{aligned}$$

Equality holds if and only if $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{S,L})$ does not vanish.

(ii) *Assume that $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{L,M})$ and $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{L,N})$ are non-zero. Then*

$$\begin{aligned}\Lambda(\mathcal{F}_\mathfrak{D}(L), \mathcal{F}_\mathfrak{D}(M)) + \Lambda(\mathcal{F}_\mathfrak{D}(L), \mathcal{F}_\mathfrak{D}(N)) - \Lambda(\mathcal{F}_\mathfrak{D}(L), \mathcal{F}_\mathfrak{D}(S)) \\ \geq \Lambda(L, M) + \Lambda(L, N) - \Lambda(L, S).\end{aligned}$$

Equality holds if and only if $\mathcal{F}_\mathfrak{D}(\mathbf{r}_{L,S})$ does not vanish.

Proof. Since the proof of (ii) is similar, we shall prove only (i).

As S is a simple subquotient of $M \circ N$, there exists a submodule K of $M \circ N$ such that S is a quotient of K. We consider the following commutative diagram in R-gmod:

$$\begin{tikzcd}
(M \circ N) \circ L_z \arrow[r, "R^{\mathrm{ren}}_{M \circ N, L_z}"] & L_z \circ (M \circ N) \\
K \circ L_z \arrow[u, hook] \arrow[r] \arrow[d, two heads] & L_z \circ K \arrow[u, hook] \arrow[d, two heads] \\
S \circ L_z \arrow[r, "z^c R^{\mathrm{ren}}_{S, L_z}"] & L_z \circ S
\end{tikzcd}$$

for some $c \in \mathbb{Z}_{\geq 0}$. Comparing the homogeneous degrees of morphisms in the above diagram, we have

$$2c = \Lambda(M, L) + \Lambda(N, L) - \Lambda(S, L). \tag{4.5}$$

We set $z = \log \mathbf{z}$. Applying the duality functor $\widehat{\mathscr{F}}_{\mathfrak{D}}$ to the above diagram, we obtain

$$\begin{tikzcd}
(\widetilde{M} \otimes \widetilde{N}) \otimes \widetilde{L}_{\mathbf{z}} \arrow[rr, "\widehat{\mathscr{F}}_{\mathfrak{D}}(R^{\mathrm{ren}}_{M\circ N,L_z})"] & & \widetilde{L}_{\mathbf{z}} \otimes (\widetilde{M} \otimes \widetilde{N}) \\
\widetilde{K} \otimes \widetilde{L}_{\mathbf{z}} \arrow[u, tail] \arrow[rr] \arrow[d, two heads] & & \widetilde{L}_{\mathbf{z}} \otimes \widetilde{K} \arrow[u, tail] \arrow[d, two heads] \\
\widetilde{S} \otimes \widetilde{L}_{\mathbf{z}} \arrow[rr, "z^c \widehat{\mathscr{F}}_{\mathfrak{D}}(R^{\mathrm{ren}}_{S,L_z})"] & & \widetilde{L}_{\mathbf{z}} \otimes \widetilde{S}
\end{tikzcd}$$

where $\widetilde{X}$ denotes $\mathscr{F}_{\mathfrak{D}}(X)$ for a simple R_{C}-module X. There exist $a \in \mathbb{Z}_{\ge 0}$ and $f(z) \in \mathbf{k}[[z]]^{\times}$ such that

$$\widehat{\mathscr{F}}_{\mathfrak{D}}(R^{\mathrm{ren}}_{S,L_z}) = z^a f(z) R^{\mathrm{ren}}_{\widetilde{S},\widetilde{L}}.$$

Since $\mathscr{F}_{\mathfrak{D}}(\mathbf{r}_{M,L})$ and $\mathscr{F}_{\mathfrak{D}}(\mathbf{r}_{N,L})$ do not vanish, we have

$$\widehat{\mathscr{F}}_{\mathfrak{D}}(R^{\mathrm{ren}}_{M,L_z}) \equiv R^{\mathrm{ren}}_{\widetilde{M},\widetilde{L}_{\mathbf{z}}}, \quad \widehat{\mathscr{F}}_{\mathfrak{D}}(R^{\mathrm{ren}}_{N,L_z}) \equiv R^{\mathrm{ren}}_{\widetilde{N},\widetilde{L}_{\mathbf{z}}},$$

up to a multiple of $\mathbf{k}[[z]]^{\times}$. The above diagram tells us that

$$\frac{c_{\widetilde{M},\widetilde{L}}(\mathbf{z})c_{\widetilde{N},\widetilde{L}}(\mathbf{z})}{(\mathbf{z}-1)^{c+a}\, c_{\widetilde{S},\widetilde{L}}(\mathbf{z})}$$

is a rational function in $\mathbf{z}$ which is regular and invertible at $\mathbf{z} = 1$. Hence, by [27, Lemma 3.4], we have

$$\operatorname{Deg}\left(\frac{c_{\widetilde{M},\widetilde{L}}(\mathbf{z})c_{\widetilde{N},\widetilde{L}}(\mathbf{z})}{c_{\widetilde{S},\widetilde{L}}(\mathbf{z})}\right) = 2 \cdot \operatorname{zero}_{\mathbf{z}=1}\left(\frac{c_{\widetilde{M},\widetilde{L}}(\mathbf{z})c_{\widetilde{N},\widetilde{L}}(\mathbf{z})}{c_{\widetilde{S},\widetilde{L}}(\mathbf{z})}\right) = 2(c+a).$$

Therefore, by (4.5), we conclude that

$$\begin{aligned}
\Lambda(M,L) + \Lambda(N,L) - \Lambda(S,L) = 2c &= \operatorname{Deg}\left(\frac{c_{\widetilde{M},\widetilde{L}}(\mathbf{z})c_{\widetilde{N},\widetilde{L}}(\mathbf{z})}{c_{\widetilde{S},\widetilde{L}}(\mathbf{z})}\right) - 2a \\
&= \Lambda(\widetilde{M},\widetilde{L}) + \Lambda(\widetilde{N},\widetilde{L}) - \Lambda(\widetilde{S},\widetilde{L}) - 2a.
\end{aligned}$$

Hence

$$\Lambda(M,L) + \Lambda(N,L) - \Lambda(S,L) \le \Lambda(\widetilde{M},\widetilde{L}) + \Lambda(\widetilde{N},\widetilde{L}) - \Lambda(\widetilde{S},\widetilde{L}).$$

Equality holds if and only if $a = 0$, which is equivalent to $\mathscr{F}_{\mathfrak{D}}(\mathbf{r}_{S,L}) \neq 0$. ■

4.4. Strong duality datum

Definition 4.7. A *strong* duality datum $\mathfrak{D} = \{\mathsf{L}_i\}_{i \in J}$ is a duality datum associated with a simply-laced finite Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$ such that all L_i's are root modules and

$$\mathfrak{d}(\mathsf{L}_i, \mathscr{D}^k(\mathsf{L}_j)) = -\delta(k=0)c_{i,j}$$

for any $k \in \mathbb{Z}$ and $i, j \in J$ such that $i \neq j$.

In particular, we have

$$\begin{aligned} \Lambda(\mathsf{L}_i, \mathsf{L}_j) &= -c_{i,j} \quad \text{for } i \neq j, \\ \Lambda^\infty(\mathsf{L}_i, \mathsf{L}_j) &= -c_{i,j} \quad \text{for all } i, j \in J. \end{aligned}$$

Let $\mathscr{D} = \{\mathsf{L}_i\}_{i \in J}$ be a strong duality datum associated with a Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$ of finite ADE type. Let R_{C} be the symmetric quiver Hecke algebra associated with C. If no confusion can arise, we simply write R for R_{C}. We denote by

$$\mathscr{F}_{\mathscr{D}} \colon R_{\mathsf{C}}\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}$$

the duality functor arising from $\mathscr{D}$. Recall that $\mathscr{F}_{\mathscr{D}}$ sends simples to simples or zero. However, if $\mathscr{D}$ is strong, we can say more as we see below.

Throughout this subsection, we assume that $\mathscr{D}$ is a strong duality datum.

Lemma 4.8. *For $w \in \mathsf{W}$, $\Lambda \in \mathsf{P}^+$ and $i \in J$, we have*

$$\varepsilon_i(\mathsf{D}(w\Lambda, \Lambda)) = \begin{cases} -(\alpha_i, w\Lambda) & \text{if } s_i w < w, \\ 0 & \text{if } s_i w > w, \end{cases}$$

$$\varepsilon_i^*(\mathsf{D}(w\Lambda, \Lambda)) = \begin{cases} (\alpha_i, \Lambda) & \text{if } w \geq s_i, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathfrak{d}(L(i), \mathsf{D}(w\Lambda, \Lambda)) = \begin{cases} 0 & \text{if } s_i w < w, \\ (\alpha_i, w\Lambda) & \text{if } s_i w > w \text{ and } w \geq s_i, \\ (\alpha_i, w\Lambda - \Lambda) & \text{otherwise}, \end{cases}$$

where $\mathsf{D}(w\Lambda, \Lambda)$ is the determinantial module appearing in Section 2.2.

Proof. The equality for ε_i is proved in [24, Proposition 10.2.4]. Let us show the equality for ε_i^*. If $w \geq s_i$, then $w\Lambda \preceq s_i\Lambda$. Hence $\varepsilon_i^*(\mathsf{D}(w\Lambda, s_i\Lambda)) = 0$ by the same proposition.

Since

$$\mathsf{D}(w\Lambda, \Lambda) \simeq \mathsf{D}(w\Lambda, s_i\Lambda) \nabla \mathsf{D}(s_i\Lambda, \Lambda) \simeq \mathsf{D}(w\Lambda, s_i\Lambda) \nabla L(i)^{\circ(\alpha_i, \Lambda)}$$

by [24, Theorem 10.3.1], we have $\varepsilon_i^*(\mathsf{D}(w\Lambda, \Lambda)) = (\alpha_i, \Lambda)$.

Assume that $w \not\geq s_i$. Then $\Lambda - w\Lambda$ does not contain α_i, and hence $\varepsilon_i^*(\mathsf{D}(w\Lambda, \Lambda)) = 0$.

The equality for $\mathfrak{d}$ follows immediately from

$$\mathfrak{d}(L(i), M) = \varepsilon_i(M) + \varepsilon_i^*(M) + (\alpha_i, \mathrm{wt}(M)) \tag{4.6}$$

(see [26, Corollary 3.8]). ■

Theorem 4.9. *Let $w \in \mathsf{W}$, $\Lambda \in \mathsf{P}^+$, and set*

$$V_w(\Lambda) := \mathscr{F}_{\mathscr{D}}(\mathsf{D}(w\Lambda, \Lambda)).$$

Then $V_w(\Lambda)$ is simple and

$$\begin{aligned}\mathfrak{d}(\mathsf{L}_i, V_w(\Lambda)) &= \mathfrak{d}(L(i), \mathsf{D}(w\Lambda, \Lambda)),\\ \mathfrak{d}(\mathscr{D}\mathsf{L}_i, V_w(\Lambda)) &= \varepsilon_i(\mathsf{D}(w\Lambda, \Lambda)),\\ \mathfrak{d}(\mathscr{D}^2\mathsf{L}_i, V_w(\Lambda)) &= 0.\end{aligned}$$

Proof. First note that once we prove that $V_w(\Lambda)$ is a simple module, we know that $\mathfrak{d}(\mathscr{D}^2\mathsf{L}_i, V_w(\Lambda)) = 0$, since $\mathscr{D}^2\mathsf{L}_i$ commutes with all L_j's.

Since $\mathsf{D}(w\Lambda, \Lambda) \circ \mathsf{D}(w\Lambda', \Lambda') \simeq \mathsf{D}(w(\Lambda + \Lambda'), \Lambda + \Lambda')$ up to a grading shift for any $\Lambda, \Lambda \in \mathsf{P}^+$ and $w \in \mathsf{W}$ [26, Proposition 4.2], we may assume that $\Lambda = \Lambda_t$ for some $t \in J$. We may assume further that Λ is w-regular, that is, $\ell(w) \le \ell(w')$ for any $w' \in \mathsf{W}$ such that $w'\Lambda = w\Lambda$. Then, by the preceding lemma,

$$\mathfrak{d}(L(i), \mathsf{D}(w\Lambda, \Lambda)) = \begin{cases} 0 & \text{if } s_i w < w,\\ (\alpha_i, w\Lambda) & \text{if } s_i w > w,\end{cases}$$
$$\varepsilon_i(\mathsf{D}(w\Lambda, \Lambda)) = \begin{cases} -(\alpha_i, w\Lambda) & \text{if } s_i w < w,\\ 0 & \text{if } s_i w > w,\end{cases}$$

if $w\Lambda \neq \Lambda$.

We shall argue by induction on $\ell(w)$. If $\ell(w) = 0$, then there is nothing to prove. If $\ell(w) = 1$, then $V_w(\Lambda) = \mathsf{L}_t$, and it is straightforward that the assertion is true.

We now assume that $\ell(w) \ge 2$.

Case 1: Assume that $s_i w < w$. We set

$$w' = s_i w, \quad n := (\alpha_i, w'\Lambda) \in \mathbb{Z}_{\ge 0}.$$

Then Λ is w'-regular and $w'\Lambda \neq \Lambda$. Hence, by the induction hypothesis,

$$\mathfrak{d}(\mathscr{D}\mathsf{L}_i, V_{w'}(\Lambda)) = 0, \quad \mathfrak{d}(\mathsf{L}_i, V_{w'}(\Lambda)) = \mathfrak{d}(L(i), \mathsf{D}(w'\Lambda, \Lambda)) = n.$$

Since $\mathsf{D}(w\Lambda, \Lambda) \simeq L(i)^{\circ n} \nabla \mathsf{D}(w'\Lambda, \Lambda)$, we have

$$V_w(\Lambda) \simeq \mathsf{L}_i^{\otimes n} \nabla V_{w'}(\Lambda) \tag{4.7}$$

by Lemma 4.5. In particular, $V_w(\Lambda)$ is simple.

It follows from Lemma 3.4 that $\mathfrak{d}(\mathsf{L}_i, V_w(\Lambda)) = 0$. Moreover, $\mathfrak{d}(\mathscr{D}^2\mathsf{L}_i, V_{w'}(\Lambda)) = 0$. Applying Lemma 3.6 (i) to $L = \mathsf{L}_i$ and $X = V_{w'}(\Lambda)$, we obtain

$$\mathfrak{d}(\mathscr{D}\mathsf{L}_i, V_w(\Lambda)) = \mathfrak{d}(\mathscr{D}\mathsf{L}_i, V_{w'}(\Lambda)) + n = n,$$

which gives the assertion.

Case 2: Assume that $s_i w > w$. Since $\ell(w) \ge 2$, there exists $j \in J$ such that $s_j w < w$. We set

$$w' := s_j w, \quad n := (\alpha_j, w'\Lambda) \in \mathbb{Z}_{\ge 0}.$$

Note that Λ is w'-regular and $w'\Lambda \neq \Lambda$. By (4.7), we have

$$V_w(\Lambda) \simeq \mathsf{L}_j^{\otimes n} \nabla V_{w'}(\Lambda).$$

We set $Z := V_w(\Lambda)$ and $Z' := V_{w'}(\Lambda)$. Hence

$$Z \simeq \mathsf{L}_j^{\otimes n} \nabla Z'. \tag{4.8}$$

(1) Suppose that $c_{i,j} = 0$. Then $s_i w' > w'$ since $s_i s_j = s_j s_i$. By the induction hypothesis, we have

$$\mathfrak{d}(\mathsf{L}_i, V_{w'}(\Lambda)) = (\alpha_i, w'\Lambda) = (s_j(\alpha_i), w\Lambda) = (\alpha_i, w\Lambda), \quad \mathfrak{d}(\mathscr{D}\mathsf{L}_i, V_{w'}(\Lambda)) = 0.$$

Since $\mathfrak{d}(\mathscr{D}^k \mathsf{L}_i, \mathsf{L}_j) = 0$ for any $k \in \mathbb{Z}$, it follows from (4.8) and Corollary 2.25 that

$$\mathfrak{d}(\mathscr{D}^k \mathsf{L}_i, Z) = \mathfrak{d}(\mathscr{D}^k \mathsf{L}_i, Z') \quad \text{for any } k \in \mathbb{Z}.$$

In particular,

$$\mathfrak{d}(\mathsf{L}_i, Z) = \mathfrak{d}(\mathsf{L}_i, Z') = (\alpha_i, w\Lambda), \quad \mathfrak{d}(\mathscr{D}\mathsf{L}_i, Z) = \mathfrak{d}(\mathscr{D}\mathsf{L}_i, Z') = 0.$$

(2) We now assume that $c_{i,j} = -1$. Then we have two cases: $s_i w' > w'$ or $s_i w' < w'$.

Assume that $s_i w' > w'$. Then $\mathfrak{d}(\mathscr{D}\mathsf{L}_i, Z') = 0$ by the induction hypothesis. Hence, by (4.8) and Corollary 3.12 (i), we have $\mathfrak{d}(\mathscr{D}\mathsf{L}_i, Z) = 0$ and

$$\mathfrak{d}(\mathsf{L}_i, Z) = \mathfrak{d}(\mathsf{L}_i, Z') + n = (\alpha_i, w'\Lambda) + n = (\alpha_i, w'\Lambda - n\alpha_j) = (\alpha_i, w\Lambda),$$

where the second identity follows from the induction hypothesis.

Assume now $s_i w' < w'$. Letting $w'' := s_i w'$, we have $w = s_j s_i w''$ and $\ell(w) = 2 + \ell(w'')$. If $s_j w'' < w''$, then $\ell(w) = 3 + \ell(s_j w'')$ and $w = s_j s_i s_j (s_j w'') = s_i s_j s_i (s_j w'')$. This implies that $s_i w < w$, which contradicts the assumption of Case 2. Hence $s_j w'' > w''$, which tells us that

$$(\alpha_j, s_i w'\Lambda) = (\alpha_j, w''\Lambda) \geq 0.$$

Set $m := (\alpha_i, s_i w'\Lambda) \in \mathbb{Z}_{\geq 0}$. Then

$$(\alpha_j, s_i w'\Lambda) = (\alpha_j, w'\Lambda + m\alpha_i) = n - m,$$

which says that $n - m \geq 0$.

Set $Z'' := V_{w''}(\Lambda)$. By the induction hypothesis,

$$\mathfrak{d}(\mathsf{L}_i, Z') = 0, \quad \mathfrak{d}(\mathscr{D}\mathsf{L}_j, Z'') = 0, \quad \mathfrak{d}(\mathscr{D}^2 \mathsf{L}_j, Z'') = 0.$$

Applying Proposition 3.13 (iii) to $L := \mathsf{L}_i$, $L' := \mathsf{L}_j$ and $X := Z'$, $Y := Z''$ and $k := n$, we have

$$\mathfrak{d}(\mathsf{L}_i, Z) = \mathfrak{d}(\mathsf{L}_i, \mathsf{L}_j^{\otimes n} \nabla Z') = n - m,$$
$$\mathfrak{d}(\mathscr{D}\mathsf{L}_i, Z) = 0.$$

Since $(\alpha_i, w\Lambda) = (\alpha_i, w'\Lambda - n\alpha_j) = -(\alpha_i, s_i w'\Lambda) + n = n - m$, we conclude that

$$\mathfrak{d}(\mathsf{L}_i, Z) = n - m = (\alpha_i, w\Lambda),$$

which completes the proof. ∎

Theorem 4.10. *Let $\mathfrak{D} = \{\mathsf{L}_i\}_{i\in J}$ be a* strong *duality datum associated with a simply-laced finite Cartan matrix* C. *Then the duality functor $\mathcal{F}_{\mathfrak{D}}$ sends simple modules to simple modules.*

Proof. Since the duality functor $\mathcal{F}_{\mathfrak{D}}$ sends a simple module to a simple module or zero, it suffices to show that $\mathcal{F}_{\mathfrak{D}}(X)$ is non-zero for any simple module $X \in R$-gmod.

Let w_0 be the longest element of the Weyl group W of C. Note that the category $\mathscr{C}_{w_0}$ is equal to R-gmod. For $i \in J$, we set $\mathsf{C}_i := \mathsf{D}(w_0\Lambda_i, \Lambda_i)$ and denote by $(\mathsf{C}_i, R_{\mathsf{C}_i})$ the *non-degenerate braider* induced from R-matrices [30, Proposition 4.1]. It is proved in [30, Section 5] that there is a localization $\widetilde{R} := R\text{-gmod}[\mathsf{C}_i^{\circ -1} \mid i \in J]$ of R-gmod by the braiders C_i. Moreover, $\widetilde{R}$ is left rigid [30, Corollary 5.11]. Thus, for any simple module $X \in R$-gmod, there exists a module $Y \in R$-gmod and $\Lambda \in \mathsf{P}^+$ such that there exists a surjective homomorphism

$$Y \circ X \twoheadrightarrow \mathsf{D}(w_0\Lambda, \Lambda).$$

Applying the duality functor $\mathcal{F}_{\mathfrak{D}}$ to the above surjection, we have

$$\mathcal{F}_{\mathfrak{D}}(Y) \otimes \mathcal{F}_{\mathfrak{D}}(X) \twoheadrightarrow \mathcal{F}_{\mathfrak{D}}(\mathsf{D}(w_0\Lambda, \Lambda)).$$

Since $\mathcal{F}_{\mathfrak{D}}(\mathsf{D}(w_0\Lambda, \Lambda))$ is simple by Theorem 4.9, $\mathcal{F}_{\mathfrak{D}}(X)$ does not vanish. ∎

Corollary 4.11. *Let $\mathfrak{D}$ be a strong duality datum associated with a simply-laced finite Cartan matrix* C. *Then $\mathcal{F}_{\mathfrak{D}}$ is faithful, i.e., for any non-zero morphism f in* R-gmod, *$\mathcal{F}_{\mathfrak{D}}(f)$ is non-zero.*

Theorem 4.12. *Let $\mathfrak{D} = \{\mathsf{L}_i\}_{i\in J}$ be a strong duality datum associated with a simply-laced finite Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j\in J}$. Then, for any simple modules M, N in* R_{C}-gmod,

(i) $\Lambda(M, N) = \Lambda(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N))$,

(ii) $\mathfrak{d}(M, N) = \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N))$,

(iii) $(\mathrm{wt}\, M, \mathrm{wt}\, N) = -\Lambda^{\infty}(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N))$,

(iv) $\mathfrak{d}(\mathscr{D}^k \mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) = 0$ *for any* $k \neq 0, \pm 1$,

(v) $\widetilde{\Lambda}(M, N) = \mathfrak{d}(\mathscr{D}\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) = \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}^{-1}\mathcal{F}_{\mathfrak{D}}(N))$.

Proof. Set $\beta := -\mathrm{wt}(M)$ and $\gamma := -\mathrm{wt}(N)$ and write $m := \mathrm{ht}(\beta)$ and $n := \mathrm{ht}(\gamma)$.

(i) We shall use induction on $m + n$. If $m = 0$ or $n = 0$, then the assertion is obvious. Hence we assume that $m, n \geq 1$.

If $m + n = 2$, then $M = L(i)$ and $N = L(j)$ for some $i, j \in J$. Since the assertion is obvious for $i = j$, we assume that $i \neq j$. Since $\mathcal{F}_{\mathfrak{D}}(M) \simeq \mathsf{L}_i$ and $\mathcal{F}_{\mathfrak{D}}(N) \simeq \mathsf{L}_j$, we have

$$\Lambda(\mathsf{L}_i, \mathsf{L}_j) = \mathfrak{d}(\mathsf{L}_i, \mathsf{L}_j) = -c_{i,j} = \Lambda(L(i), L(j)).$$

Suppose that $m+n \geq 3$. If $m \geq 2$, then there exist simple modules M_1 and M_2 such that

(a) $\mathrm{wt}(M_1) \neq 0$ and $\mathrm{wt}(M_2) \neq 0$,

(b) M_1 or M_2 is real,

(c) $M \simeq M_1 \nabla M_2$.

Hence, by Lemma 4.6 together with Corollary 4.11, we obtain

$$\begin{aligned}&\Lambda(M_1, N) + \Lambda(M_2, N) - \Lambda(M, N)\\ &\quad = \Lambda(\mathcal{F}_{\mathfrak{D}}(M_1), \mathcal{F}_{\mathfrak{D}}(N)) + \Lambda(\mathcal{F}_{\mathfrak{D}}(M_2), \mathcal{F}_{\mathfrak{D}}(N)) - \Lambda(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)).\end{aligned}$$

Since $\Lambda(M_k, N) = \Lambda(\mathcal{F}_{\mathfrak{D}}(M_k), \mathcal{F}_{\mathfrak{D}}(N))$ for $k = 1, 2$ by the induction hypothesis, we have

$$\Lambda(M, N) = \Lambda(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)).$$

The case where $n \geq 2$ can be handled similarly.

(ii) follows immediately from (i).

(iii) There exist sequences $(i_1, \dots, i_m)$ and $(j_1, \dots, j_n)$ in J such that M and N appear as quotients of $L(i_1) \circ \cdots \circ L(i_m)$ and $L(j_1) \circ \cdots \circ L(j_n)$, respectively. Note that $\beta = -\sum_{p=1}^{m} \alpha_{i_p}$ and $\gamma = -\sum_{q=1}^{n} \alpha_{j_q}$. Since $\mathcal{F}_{\mathfrak{D}}$ is exact and $\mathcal{F}_{\mathfrak{D}}(M)$ and $\mathcal{F}_{\mathfrak{D}}(N)$ are simple, $\mathcal{F}_{\mathfrak{D}}(M)$ and $\mathcal{F}_{\mathfrak{D}}(N)$ appear as quotients in $\mathsf{L}_{i_1} \otimes \cdots \otimes \mathsf{L}_{i_m}$ and $\mathsf{L}_{j_1} \otimes \cdots \otimes \mathsf{L}_{j_n}$, respectively. Therefore, by [27, Proposition 3.11], we have

$$-\Lambda^{\infty}(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) = -\sum_{p,q} \Lambda^{\infty}(\mathsf{L}_{i_p}, \mathsf{L}_{j_q}) = \sum_{p,q} c_{i_p, j_q} = (\beta, \gamma).$$

(iv) follows from $\mathfrak{d}(\mathscr{D}^k(\mathsf{L}_i), \mathsf{L}_j) = 0$ for any i, j and $|k| \geq 2$.

(v) By (i), (iii) and (iv), we have

$$\begin{aligned}\Lambda(M, N) &= \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) - \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}\mathcal{F}_{\mathfrak{D}}(N)) + \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}^{-1}\mathcal{F}_{\mathfrak{D}}(N)),\\ (\beta, \gamma) &= -\mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)) + \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}\mathcal{F}_{\mathfrak{D}}(N)) + \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}^{-1}\mathcal{F}_{\mathfrak{D}}(N)).\end{aligned}$$

Thus

$$\begin{aligned}\widetilde{\Lambda}(M, N) &= \tfrac{1}{2}(\Lambda(M, N) + (\beta, \gamma))\\ &= \mathfrak{d}(\mathcal{F}_{\mathfrak{D}}(M), \mathscr{D}^{-1}\mathcal{F}_{\mathfrak{D}}(N)) = \mathfrak{d}(\mathscr{D}\mathcal{F}_{\mathfrak{D}}(M), \mathcal{F}_{\mathfrak{D}}(N)).\end{aligned}$$

■

Corollary 4.13. *Let $\mathfrak{D} = \{\mathsf{L}_i\}_{i \in J}$ be a* strong *duality datum. For any $i \in J$ and any simple module $M \in R_{\mathsf{C}}$-gmod, we have*

(i) $\varepsilon_i(M) = \mathfrak{d}(\mathscr{D}\,\mathsf{L}_i, \mathcal{F}_{\mathfrak{D}}(M))$,

(ii) $\varepsilon_i^*(M) = \mathfrak{d}(\mathscr{D}^{-1}\mathsf{L}_i, \mathcal{F}_{\mathfrak{D}}(M))$.

Proof. This follows from [26, Corollary 3.8] and Theorem 4.12 (v). ■

Corollary 4.14. *Let* $\mathfrak{D} = \{\mathsf{L}_i\}_{i\in J}$ *be a* strong *duality datum associated with a simply-laced finite Cartan matrix* C*. Then the duality functor* $\mathcal{F}_{\mathfrak{D}}$ *induces an injective ring homomorphism*

$$K_{q=1}(R_{\mathsf{C}}\text{-gmod}) \rightarrowtail K(\mathscr{C}_{\mathfrak{g}}),$$

where $K_{q=1}(R_{\mathsf{C}}\text{-gmod})$ *is the specialization of* $K(R_{\mathsf{C}}\text{-gmod})$ *at* $q = 1$.

Proof. Thanks to Theorem 4.10, it is enough to show that $\mathcal{F}_{\mathfrak{D}}(M) \not\simeq \mathcal{F}_{\mathfrak{D}}(N)$ for any non-isomorphic simple R-modules M and N. Let M and N be simple R-modules such that

$$\mathcal{F}_{\mathfrak{D}}(M) \simeq \mathcal{F}_{\mathfrak{D}}(N).$$

We set $\beta := -\mathrm{wt}(M)$ and $\gamma := -\mathrm{wt}(N)$. We shall show $M \simeq N$.

We first assume that $\mathcal{F}_{\mathfrak{D}}(M) = \mathcal{F}_{\mathfrak{D}}(N) = \mathbf{1}$. Then $(\beta,\beta) = -\Lambda^\infty(M,M) = 0$, which implies $\beta = 0$. Hence $M \simeq \mathbf{1}$. Similarly, $N \simeq \mathbf{1}$.

We now assume that $\mathcal{F}_{\mathfrak{D}}(M) \simeq \mathcal{F}_{\mathfrak{D}}(N) \not\simeq \mathbf{1}$. Since $M \not\simeq \mathbf{1}$, there exists $i \in J$ such that $\varepsilon_i(M) > 0$. By Corollary 4.13,

$$\varepsilon_i(M) = \Lambda(\mathscr{D}\mathsf{L}_i, \mathcal{F}_{\mathfrak{D}}(M)) = \Lambda(\mathscr{D}\mathsf{L}_i, \mathcal{F}_{\mathfrak{D}}(N)) = \varepsilon_i(N),$$

which tells us that $\tilde{e}_i(M) \neq 0$ and $\tilde{e}_i(N) \neq 0$. Setting $M' := \tilde{e}_i(M)$ and $N' := \tilde{e}_i(N)$, we have

$$\mathsf{L}_i \mathbin{\nabla} \mathcal{F}_{\mathfrak{D}}(M') \simeq \mathcal{F}_{\mathfrak{D}}(M) \simeq \mathcal{F}_{\mathfrak{D}}(N) \simeq \mathsf{L}_i \mathbin{\nabla} \mathcal{F}_{\mathfrak{D}}(N'),$$

which implies that $\mathcal{F}_{\mathfrak{D}}(M') \simeq \mathcal{F}_{\mathfrak{D}}(N')$ by Lemma 2.8. Thus, by the standard induction argument, we conclude that

$$\tilde{e}_i(M) = M' \simeq N' = \tilde{e}_i(N),$$

which yields $M \simeq N$. ∎

Lemma 4.15. *Let* M *be a real simple module in* R-gmod*. Then* $\mathcal{F}_{\mathfrak{D}}(M)$ *is a root module if and only if* $\mathrm{wt}(M)$ *is a root of* $\mathfrak{g}_{\mathrm{fin}}$.

Proof. Set $V = \mathcal{F}_{\mathfrak{D}}(M)$. Then $\mathfrak{d}(\mathscr{D}^k V, V) = 0$ for $k \neq \pm 1$. Hence

$$(\mathrm{wt}(M), \mathrm{wt}(M)) = -\Lambda^\infty(V,V) = 2\mathfrak{d}(\mathscr{D}V, V).$$

Therefore

$$V \text{ is a root module} \Leftrightarrow \mathfrak{d}(\mathscr{D}V, V) = 1 \Leftrightarrow (\mathrm{wt}(M), \mathrm{wt}(M)) = 2 \Leftrightarrow \mathrm{wt}(M) \text{ is a root.}$$ ∎

5. Strong duality datum and affine cuspidal modules

5.1. *Unmixed pairs*

The notion of an unmixed pair of modules over quiver Hecke algebras has an analogue for modules over quantum affine algebras.

Definition 5.1. Let (M, N) be an ordered pair of simple modules in $\mathscr{C}_{\mathfrak{g}}$. We call it *unmixed* if

$$\mathfrak{d}(\mathscr{D}M, N) = 0,$$

and *strongly unmixed* if

$$\mathfrak{d}(\mathscr{D}^k M, N) = 0 \quad \text{for any } k \in \mathbb{Z}_{\geq 1}.$$

Lemma 5.2. *Let M and N be simple modules in $\mathscr{C}_{\mathfrak{g}}$. If (M, N) is strongly unmixed, then*

$$\Lambda^{\infty}(M, N) = \Lambda(M, N).$$

Proof. It follows from Definition 5.1 and Proposition 2.16 that

$$\Lambda(M, N) = \sum_{k \in \mathbb{Z}} (-1)^{k+\delta(k>0)} \mathfrak{d}(\mathscr{D}^k M, N) = \sum_{k \in \mathbb{Z}} (-1)^k \mathfrak{d}(\mathscr{D}^k M, N) = \Lambda^{\infty}(M, N). \ \blacksquare$$

Lemma 5.3. *Let $L_1, \dots, L_r$ be real simple modules in $\mathscr{C}_{\mathfrak{g}}$ for $r \in \mathbb{Z}_{>1}$. If (L_a, L_b) is unmixed for any $a < b$, then $(L_1, \dots, L_r)$ is normal.*

Proof. We argue by induction on r. Since the assertion is obvious when $r = 2$, we assume that $r > 2$. By the induction hypothesis, $(L_1, \dots, L_{r-1})$ is normal. Set $X = \mathrm{hd}(L_2 \otimes \cdots \otimes L_{r-1})$. Then Lemma 2.20 implies that

$$\Lambda(L_1, X) = \sum_{k=2}^{r-1} \Lambda(L_1, L_k).$$

Since (L_1, L_r) is unmixed, Lemma 2.21 implies that (L_1, X, L_r) is normal. Hence

$$\Lambda(L_1, \mathrm{hd}(L_2 \otimes \cdots \otimes L_r)) = \Lambda(L_1, X \nabla L_r) = \Lambda(L_1, X) + \Lambda(L_1, L_r),$$

which implies that

$$\Lambda(L_1, \mathrm{hd}(L_2 \otimes \cdots \otimes L_r)) = \sum_{k=2}^{r} \Lambda(L_1, L_k).$$

Since $(L_2, \dots, L_r)$ is normal, Lemma 2.20 implies that $(L_1, \dots, L_r)$ is normal. ■

5.2. *Affine cuspidal modules*

Let $\mathcal{D} = \{\mathsf{L}_i\}_{i \in J}$ be a strong duality datum in $\mathscr{C}_{\mathfrak{g}}^0$ associated with a simply-laced finite Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j \in J}$. Let R_{C} be the symmetric quiver Hecke algebra associated with C.

We define $\mathscr{C}_{\mathcal{D}}$ to be the smallest full subcategory of $\mathscr{C}_{\mathfrak{g}}^0$ which

(a) contains $\mathcal{F}_{\mathcal{D}}(L)$ for any simple R_{C}-module L,

(b) is stable by taking subquotients, extensions, and tensor products.

Since $\mathfrak{d}(\mathscr{D}^k \mathsf{L}_i, \mathsf{L}_j) = 0$ for any $i, j \in J$ and $k \geq 2$, it follows from Theorem 4.12 that

$$\mathfrak{d}(\mathscr{D}^k M, N) = 0 \quad \text{for any simple modules } M, N \in \mathscr{C}_{\mathfrak{D}} \text{ and } k \geq 2. \tag{5.1}$$

For $k \in \mathbb{Z}$, let $\mathscr{D}^k(\mathscr{C}_{\mathfrak{D}})$ be the full subcategory of $\mathscr{C}^0_{\mathfrak{g}}$ whose objects are $\mathscr{D}^k M$ for all $M \in \mathscr{C}_{\mathfrak{D}}$.

Proposition 5.4. *Let $k \in \mathbb{Z}$ with $k \neq 0$. If a simple module M is in $\mathscr{C}_{\mathfrak{D}} \cap \mathscr{D}^k(\mathscr{C}_{\mathfrak{D}})$, then $M \simeq \mathbf{1}$.*

Proof. We may assume $k > 0$ without loss of generality. Let M be a simple module in $\mathscr{C}_{\mathfrak{D}} \cap \mathscr{D}^k(\mathscr{C}_{\mathfrak{D}})$. By Theorem 4.10, there exists a simple module $V \in R_{\mathsf{C}}$-gmod such that $\mathscr{F}_{\mathfrak{D}}(V) \simeq M$. By Corollary 4.13 and Theorem 4.12 (iv), for any $i \in J$ we have

$$\varepsilon_i^*(V) = \mathfrak{d}(\mathscr{D}^{-1} \mathsf{L}_i, M) = \mathfrak{d}(\mathsf{L}_i, \mathscr{D} M) = 0.$$

Thus V should be in $R_{\mathsf{C}}(0)$-gmod, which says that $V \simeq \mathbf{1}$. ■

Lemma 5.5. *Let M, N be simple modules in R_{C}-gmod. If (M, N) is unmixed, then $(\mathscr{F}_{\mathfrak{D}}(M), \mathscr{F}_{\mathfrak{D}}(N))$ is strongly unmixed.*

Proof. By (5.1), we know that $\mathfrak{d}(\mathscr{D}^k \mathscr{F}_{\mathfrak{D}}(M), \mathscr{F}_{\mathfrak{D}}(N)) = 0$ for $k \geq 2$. It follows from [26, Proposition 2.12] that $\Lambda(M, N) = -(\mathrm{wt}(M), \mathrm{wt}(N))$, i.e., $\widetilde{\Lambda}(M, N) = 0$. Thus, by Theorem 4.12 (v), we obtain

$$\mathfrak{d}(\mathscr{D} \mathscr{F}_{\mathfrak{D}}(M), \mathscr{F}_{\mathfrak{D}}(N)) = \widetilde{\Lambda}(M, N) = 0,$$

which completes the proof. ■

Let $\mathfrak{g}_{\mathsf{C}}$ be the simple Lie algebra associated with C. Let Φ^+_{C} be the set of positive roots of $\mathfrak{g}_{\mathsf{C}}$ and let W_{C} be the Weyl group associated with $\mathfrak{g}_{\mathsf{C}}$. Let w_0 be the longest element of W_{C}, and let ℓ denote its length. We choose an arbitrary reduced expression $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ of w_0. We extend $\{i_k\}_{k=1}^{\ell}$ to $\{i_k\}_{k \in \mathbb{Z}}$ by

$$i_{k+\ell} = (i_k)^* \quad \text{for any } k \in \mathbb{Z}. \tag{5.2}$$

(Recall that, for $i \in J$, i^* is a unique element of J such that $\alpha_{i^*} = -w_0 \alpha_i$.)

We can easily see that $s_{i_{a+1}} \cdots s_{i_{a+\ell}}$ is also a reduced expression of w_0 for any $a \in \mathbb{Z}$. Let

$$\{\mathsf{V}_k\}_{k=1}^{\ell} \subset R_{\mathsf{C}}\text{-gmod}$$

be the cuspidal modules associated with the reduced expression $\underline{w_0}$. Under the categorification, the cuspidal module V_k corresponds to the dual PBW vector $E^*(\beta_k)$ corresponding to $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in \Phi^+_{\mathsf{C}}$ for $k = 1, \ldots, \ell$ (see Section 2.2).

We now introduce the notion of affine cuspidal modules for quantum affine algebras.

Definition 5.6. We define a sequence $\{\mathsf{S}_k\}_{k \in \mathbb{Z}}$ of simple $U_q'(\mathfrak{g})$-modules in $\mathscr{C}_{\mathfrak{g}}$ as follows:

(a) $\mathsf{S}_k = \mathcal{F}_{\mathfrak{D}}(\mathsf{V}_k)$ for any $k = 1, \dots, \ell$, and we extend the definition to all $k \in \mathbb{Z}$ by

(b) $\mathsf{S}_{k+\ell} = \mathscr{D}(\mathsf{S}_k)$ for any $k \in \mathbb{Z}$.

The modules S_k $(k \in \mathbb{Z})$ are called the *affine cuspidal modules* corresponding to $\mathfrak{D}$ and $\underline{w_0}$.

Proposition 5.7. *The affine cuspidal modules satisfy the following properties:*

(i) S_a *is a root module for any* $a \in \mathbb{Z}$.

(ii) *For any* $a, b \in \mathbb{Z}$ *with* $a > b$, *the pair* $(\mathsf{S}_a, \mathsf{S}_b)$ *is strongly unmixed.*

(iii) *Let* $k_1 > \cdots > k_t$ *be integers and* $(a_1, \dots, a_t) \in \mathbb{Z}^t_{\geq 0}$. *Then*

  (a) *the sequence* $(\mathsf{S}_{k_1}^{\otimes a_1}, \dots, \mathsf{S}_{k_t}^{\otimes a_t})$ *is normal,*

  (b) *the head of the tensor product* $\mathsf{S}_{k_1}^{\otimes a_1} \otimes \cdots \otimes \mathsf{S}_{k_t}^{\otimes a_t}$ *is simple.*

Proof. (i) follows immediately from Lemma 4.15.

(ii) Without loss of generality, we may assume that $1 \leq b \leq \ell$. We write $a = \ell \cdot t + r$ for some $t \in \mathbb{Z}_{\geq 0}$ and $1 \leq r \leq \ell$. By the definition, we have $\mathsf{S}_a = \mathscr{D}^t \mathsf{S}_r$. If $t \geq 1$, then

$$\mathfrak{d}(\mathscr{D}^k \mathsf{S}_a, \mathsf{S}_b) = \mathfrak{d}(\mathscr{D}^{k+t} \mathsf{S}_r, \mathsf{S}_b) = 0 \quad \text{for any } k \geq 1,$$

by (5.1).

Suppose that $t = 0$. As $\ell \geq a > b \geq 1$, the pair $(\mathsf{V}_a, \mathsf{V}_b)$ is unmixed. Thus Lemma 5.5 says that $(\mathsf{S}_a, \mathsf{S}_b)$ is strongly unmixed.

(iii) follows from Lemmas 5.3 and 2.19. ■

Example 5.8. Let $U_q'(\mathfrak{g})$ be the quantum affine algebra of affine type $A_2^{(1)}$, and let $\mathscr{C}_{\mathfrak{g}}^0$ be the Hernandez–Leclerc category corresponding to

$$\sigma_0(\mathfrak{g}) = \{(1, (-q)^{2k}), (2, (-q)^{2k+1}) \mid k \in \mathbb{Z}\}.$$

For $i \in I_0$ and $m \in \mathbb{Z}_{>0}$, we denote the *Kirillov–Reshetikhin module* by

$$V(i^m) := \operatorname{hd}\big(V(\varpi_i)_{(-q)^{m-1}} \otimes V(\varpi_i)_{(-q)^{m-3}} \otimes \cdots \otimes V(\varpi_i)_{(-q)^{-m+1}}\big).$$

We simply write $V(i)$ instead of $V(i^1)$, which is the i-th fundamental module $V(\varpi_i)$.

Let $\mathsf{L}_1 := V(1)$ and $\mathsf{L}_2 := V(1)_{(-q)^2}$, and define $\mathfrak{D} := \{\mathsf{L}_1, \mathsf{L}_2\} \subset \mathscr{C}_{\mathfrak{g}}^0$. Then $\mathfrak{D}$ is a strong duality datum (see [21, Section 4.1]). Let C be the Cartan matrix of finite type A_2. Then we have the duality functor $\mathcal{F}_{\mathfrak{D}}\colon R_{\mathsf{C}}\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}^0$.

(i) We choose a reduced expression $\underline{w_0} = s_1 s_2 s_1$. Then

$$\beta_1 := \alpha_1, \quad \beta_2 := s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad \beta_3 = s_1 s_2(\alpha_1) = \alpha_2,$$

and the affine cuspidal modules corresponding to $\mathfrak{D}$ and $\underline{w_0}$ are

$$\mathsf{S}_1 = \mathcal{F}_{\mathfrak{D}}(L(1)) = \mathsf{L}_1 = V(1),$$
$$\mathsf{S}_2 = \mathcal{F}_{\mathfrak{D}}(L(1) \nabla L(2)) = \mathsf{L}_1 \nabla \mathsf{L}_2 = V(1) \nabla V(1)_{(-q)^2} = V(2)_{-q},$$
$$\mathsf{S}_3 = \mathcal{F}_{\mathfrak{D}}(L(2)) = \mathsf{L}_2 = V(1)_{(-q)^2},$$

and $\mathsf{S}_{k+3} = \mathscr{D}(\mathsf{S}_k)$ for $k \in \mathbb{Z}$. Here $L(i)$ is the self-dual 1-dimensional simple $R(\alpha_i)$-module. It is easy to see that the set $\{\mathsf{S}_k \mid k \in \mathbb{Z}\}$ of all affine cuspidal modules is equal to the set of all fundamental modules in $\mathscr{C}_{\mathfrak{g}}^0$.

(ii) We choose another reduced expression $\underline{w_0}' = s_2 s_1 s_2$. Then

$$\beta_1' := \alpha_2, \quad \beta_2' := s_2(\alpha_1) = \alpha_1 + \alpha_2, \quad \beta_3' = s_2 s_1(\alpha_2) = \alpha_1,$$

and the affine cuspidal modules corresponding to $\mathfrak{D}$ and $\underline{w_0}'$ are

$$\begin{aligned}
\mathsf{S}_1' &= \mathscr{F}_{\mathfrak{D}}(L(2)) = \mathsf{L}_2 = V(1)_{(-q)^2}, \\
\mathsf{S}_2' &= \mathscr{F}_{\mathfrak{D}}(L(2) \nabla L(1)) = \mathsf{L}_2 \nabla \mathsf{L}_1 = V(1)_{(-q)^2} \nabla V(1) = V(1^2)_{-q}, \\
\mathsf{S}_3' &= \mathscr{F}_{\mathfrak{D}}(L(1)) = \mathsf{L}_1 = V(1),
\end{aligned}$$

and $\mathsf{S}_{k+3}' = \mathscr{D}(\mathsf{S}_k')$ for $k \in \mathbb{Z}$. Note that the affine cuspidal modules S_{2+3t}' $(t \in \mathbb{Z})$ are not fundamental modules.

5.3. *Reflections*

For any $k \in J$, we set

$$\mathscr{S}_k(\mathfrak{D}) := \{\mathscr{S}_k(\mathsf{L}_i)\}_{i \in J} \quad \text{and} \quad \mathscr{S}_k^{-1}(\mathfrak{D}) := \{\mathscr{S}_k^{-1}(\mathsf{L}_i)\}_{i \in J}, \tag{5.3}$$

where

$$\mathscr{S}_k(\mathsf{L}_i) := \begin{cases} \mathscr{D}\mathsf{L}_i & \text{if } i = k, \\ \mathsf{L}_k \nabla \mathsf{L}_i & \text{if } c_{i,k} = -1, \\ \mathsf{L}_i & \text{if } c_{i,k} = 0, \end{cases} \qquad \mathscr{S}_k^{-1}(\mathsf{L}_i) := \begin{cases} \mathscr{D}^{-1}\mathsf{L}_i & \text{if } i = k, \\ \mathsf{L}_i \nabla \mathsf{L}_k & \text{if } c_{i,k} = -1, \\ \mathsf{L}_i & \text{if } c_{i,k} = 0. \end{cases}$$

It is easy to see that $\mathscr{S}_k \circ \mathscr{S}_k^{-1}(\mathfrak{D}) = \mathfrak{D}$ and $\mathscr{S}_k^{-1} \circ \mathscr{S}_k(\mathfrak{D}) = \mathfrak{D}$ for any $k \in J$.

Proposition 5.9. *Let $k \in J$.*

(i) *For any $i \in J$, $\mathscr{S}_k(\mathsf{L}_i)$ and $\mathscr{S}_k^{-1}(\mathsf{L}_i)$ are root modules.*

(ii) *$\mathscr{S}_k(\mathfrak{D})$ and $\mathscr{S}_k^{-1}(\mathfrak{D})$ are strong duality data associated with the Cartan matrix* C.

Proof. We focus on the case of $\mathscr{S}_k$ since the case of $\mathscr{S}_k^{-1}$ can be proved in a similar manner.

Set $\mathsf{L}_i' := \mathscr{S}_k(\mathsf{L}_i)$ for $i \in J$. For $i, j \in J$, we write $i \sim j$ if $c_{i,j} = -1$ and $i \not\sim j$ if $c_{i,j} = 0$. Note that, for real simple modules L, M and N, Lemma 2.21 says that if one of the following conditions holds:

- $\mathfrak{d}(L, M) = 0$,
- $\mathfrak{d}(M, N) = 0$,
- $\mathfrak{d}(L, \mathscr{D}^{-1}N) = \mathfrak{d}(\mathscr{D}L, N) = 0$,

then

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N), \quad \Lambda(L \nabla M, N) = \Lambda(L, N) + \Lambda(M, N),$$

which will be used several times in the proof.

(i) follows from Lemma 3.8.

(ii) Thanks to (i), it suffices to prove that

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = -\delta(t=0)c_{i,j} \quad \text{for } t \in \mathbb{Z} \text{ and } i \neq j.$$

Let $i, j \in J$ with $i \neq j$. We shall prove it case by case.

Case 1: If $i \nsim k$ and $j \nsim k$, then

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_j) = -\delta(t=0)c_{i,j} \quad \text{for } t \in \mathbb{Z}.$$

Case 2: If $i \nsim k$ and $j = k$, then $c_{i,j} = 0$ and

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathscr{D}\mathsf{L}_j) = \mathfrak{d}(\mathscr{D}^{t-1} \mathsf{L}_i, \mathsf{L}_j) = 0 = -\delta(t=0)c_{i,j}.$$

Case 3: Suppose that $i \nsim k$ and $j \sim k$. Then

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k \nabla \mathsf{L}_j).$$

We have

$$\begin{aligned}\Lambda(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k \nabla \mathsf{L}_j) &= \Lambda(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k) + \Lambda(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_j),\\ \Lambda(\mathsf{L}_k \nabla \mathsf{L}_j, \mathscr{D}^t \mathsf{L}_i) &= \Lambda(\mathsf{L}_k, \mathscr{D}^t \mathsf{L}_i) + \Lambda(\mathsf{L}_j, \mathscr{D}^t \mathsf{L}_i),\end{aligned}$$

where the first equality follows from $\mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k) = 0$ and the second from $\mathfrak{d}(\mathscr{D}\mathsf{L}_k, \mathscr{D}^t \mathsf{L}_i) = 0$. Hence we obtain

$$\begin{aligned}\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) &= \mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k) + \mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_j) = \mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_j)\\ &= -\delta(t=0)c_{i,j}.\end{aligned}$$

Case 4: Suppose that $i \sim k$ and $j \sim k$. Then, by Lemma 2.23,

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathscr{D}^t(\mathsf{L}_k \nabla \mathsf{L}_i), \mathsf{L}_k \nabla \mathsf{L}_j) = \mathfrak{d}(\mathscr{D}^t \mathsf{L}_k \nabla \mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k \nabla \mathsf{L}_j).$$

Since C is of finite type, we have $c_{i,j} = 0$, i.e., $\mathfrak{d}(\mathscr{D}^t \mathsf{L}_i, \mathsf{L}_j) = 0$ for any $t \in \mathbb{Z}$.

If $t \neq 0, \pm 1$, then

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathscr{D}^t \mathsf{L}_k \nabla \mathscr{D}^t \mathsf{L}_i, \mathsf{L}_k \nabla \mathsf{L}_j) = 0$$

since $\mathfrak{d}(\mathscr{D}^t(\mathsf{L}_a), \mathsf{L}_b) = 0$ for $a, b = i, j, k$ by Theorem 4.12 (iv).

Suppose that $t = 0$. Then

$$\begin{aligned}\Lambda(\mathsf{L}_k \nabla \mathsf{L}_i, \mathsf{L}_k \nabla \mathsf{L}_j) &= \Lambda(\mathsf{L}_k \nabla \mathsf{L}_i, \mathsf{L}_k) + \Lambda(\mathsf{L}_k \nabla \mathsf{L}_i, \mathsf{L}_j)\\ &= -\Lambda(\mathsf{L}_k, \mathsf{L}_k \nabla \mathsf{L}_i) + \Lambda(\mathsf{L}_k \nabla \mathsf{L}_i, \mathsf{L}_j)\\ &= -\Lambda(\mathsf{L}_k, \mathsf{L}_i) + \Lambda(\mathsf{L}_k, \mathsf{L}_j) + \Lambda(\mathsf{L}_i, \mathsf{L}_j).\end{aligned}$$

Here the first and second identities follow from $\mathfrak{d}(\mathsf{L}_k, \mathsf{L}_k \nabla \mathsf{L}_i) = 0$ by Lemma 3.9, and the third follows from $\mathfrak{d}(\mathsf{L}_i, \mathsf{L}_j) = 0$. Exchanging i and j, we have

$$\Lambda(\mathsf{L}_k \nabla \mathsf{L}_j, \mathsf{L}_k \nabla \mathsf{L}_i) = -\Lambda(\mathsf{L}_k, \mathsf{L}_j) + \Lambda(\mathsf{L}_k, \mathsf{L}_i) + \Lambda(\mathsf{L}_j, \mathsf{L}_i),$$

which tells us that

$$\mathfrak{d}(\mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathsf{L}_k \nabla \mathsf{L}_j, \mathsf{L}_k \nabla \mathsf{L}_i) = \mathfrak{d}(\mathsf{L}_i, \mathsf{L}_j) = -c_{i,j}.$$

Suppose that $t = \pm 1$. We have

$$\begin{aligned}&\Lambda^\infty(\mathsf{L}'_i, \mathsf{L}'_j) = \Lambda^\infty(\mathsf{L}_k \nabla \mathsf{L}_i, \mathsf{L}_k \nabla \mathsf{L}_j)\\ &= \Lambda^\infty(\mathsf{L}_k, \mathsf{L}_k) + \Lambda^\infty(\mathsf{L}_k, \mathsf{L}_j) + \Lambda^\infty(\mathsf{L}_i, \mathsf{L}_k) + \Lambda^\infty(\mathsf{L}_i, \mathsf{L}_j) = (-2) + 1 + 1 + 0 = 0.\end{aligned}$$

Hence

$$0 = \sum_{t \in \mathbb{Z}} (-1)^t \, \mathfrak{d}(\mathscr{D}^t \mathsf{L}'_i, \mathsf{L}'_j) = -\mathfrak{d}(\mathscr{D} \mathsf{L}'_i, \mathsf{L}'_j) - \mathfrak{d}(\mathscr{D}^{-1} \mathsf{L}'_i, \mathsf{L}'_j).$$

Therefore $\mathfrak{d}(\mathscr{D} \mathsf{L}'_i, \mathsf{L}'_j) = \mathfrak{d}(\mathscr{D}^{-1} \mathsf{L}'_i, \mathsf{L}'_j) = 0$.

Case 5: Suppose that $i \sim k$ and $j = k$. Then

$$\mathfrak{d}(\mathscr{D}^t \mathsf{L}'_j, \mathsf{L}'_i) = \mathfrak{d}(\mathscr{D}^{t+1} \mathsf{L}_k, \mathsf{L}_k \nabla \mathsf{L}_i) \quad \text{for } t \in \mathbb{Z},$$

which is equal to $\delta(t = 0)$ by Lemma 3.9. ■

Proposition 5.10. *Let* $\{\mathsf{S}_k\}_{k \in \mathbb{Z}}$ *be the sequence of the affine cuspidal modules corresponding to* $\mathfrak{D}$ *and a reduced expression* $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ *of* w_0. *Set* $\mathsf{S}'_k = \mathsf{S}_{k+1}$ *for* $k \in \mathbb{Z}$. *Then* $\{\mathsf{S}'_k\}_{k \in \mathbb{Z}}$ *is the sequence of the affine cuspidal modules corresponding to* $\mathscr{S}_{i_1}\mathfrak{D}$ *and the reduced expression* $\underline{w_0}' = s_{i_2} \cdots s_{i_{\ell+1}}$ *(see* (5.2)*).*

Proof. Set $i = i_1$. We denote by $\prec$, $\{\beta_k\}_{k=1}^{\ell}$ and $\{\mathsf{V}_k\}_{k=1}^{\ell}$ the convex order, the ordered set of positive roots and the cuspidal modules in R_{C}-gmod corresponding to $\underline{w_0}$ as in Section 2.2. Similarly, we write $\prec'$, $\{\beta'_k\}_{k=1}^{\ell}$ and $\{\mathsf{V}'_k\}_{k=1}^{\ell}$ for the ones corresponding to $\underline{w_0}'$. It is enough to show that

$$\mathcal{F}_{\mathscr{S}_i(\mathfrak{D})}(\mathsf{V}'_k) \simeq \mathsf{S}_{k+1} \quad \text{for } 1 \le k \le \ell.$$

It is easy to see that

- $\beta_{k+1} = s_i \beta'_k$ for $k = 1, \ldots, \ell - 1$, (5.4)
- $\mathsf{V}_{k+1} \simeq \mathcal{T}_i(\mathsf{V}'_k)$ for $k = 1, \ldots, \ell - 1$, (5.5)
- α_i is smallest (resp. largest) with respect to $\prec$ (resp. $\prec'$). (5.6)

It follows from (5.6) that $\mathsf{V}_1 \simeq L(i) \simeq \mathsf{V}'_\ell$. Thus

$$\mathcal{F}_{\mathscr{S}_i(\mathfrak{D})}(\mathsf{V}'_\ell) \simeq \mathscr{D}\mathsf{L}_i \simeq \mathscr{D}(\mathcal{F}_{\mathfrak{D}}(\mathsf{V}_1)) = \mathsf{S}_{\ell+1}. \tag{5.7}$$

It remains to prove that

$$\mathcal{F}_{\mathscr{S}_i(\mathfrak{D})}(\mathsf{V}'_k) \simeq \mathcal{F}_{\mathfrak{D}}(\mathsf{V}_{k+1}) \quad \text{for } k = 1, \ldots, \ell - 1. \tag{5.8}$$

We shall use induction on $\mathrm{ht}(\beta'_k)$.

If $\mathrm{ht}(\beta'_k) = 1$, then $\beta'_k = \alpha_j$ for some $j \in J$. Note that $j \neq i$ because $k < \ell$. Thus, $\beta_{k+1} = s_i(\beta'_k) = s_i(\alpha_j)$ and

$$\mathsf{V}_{k+1} = \begin{cases} L(i) \nabla L(j) & \text{if } c_{i,j} = -1, \\ L(j) & \text{otherwise.} \end{cases}$$

By the definition of $\mathscr{S}_i$, we have

$$\mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_k) \simeq \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(L(j)) \simeq \mathscr{S}_i(\mathsf{L}_j) \simeq \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{k+1}).$$

Suppose that $\mathrm{ht}(\beta'_k) > 1$. We take a minimal pair (β'_a, β'_b) of β'_k with respect to $\prec'$. It follows from (2.2) that

$$\mathsf{V}'_a \nabla \mathsf{V}'_b \simeq \mathsf{V}'_k. \tag{5.9}$$

Case 1: Suppose that $b \neq \ell$. Applying $\mathscr{T}_i$ to (5.9), it follows from (5.5) that

$$\mathsf{V}_{a+1} \nabla \mathsf{V}_{b+1} \simeq \mathsf{V}_{k+1}. \tag{5.10}$$

Applying $\mathscr{F}_{\mathscr{D}}$ to (5.10) and using the induction hypothesis, we have

$$\begin{aligned} \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{k+1}) &\simeq \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{a+1}) \nabla \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{b+1}) \simeq \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_a) \nabla \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_b) \\ &\simeq \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_k). \end{aligned}$$

Case 2: Suppose that $b = \ell$. Since $\mathsf{V}'_b = L(i)$, by applying $\mathscr{F}_{\mathscr{S}_i(\mathscr{D})}$ to (5.9) we have

$$\mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_a) \nabla \mathscr{D}\mathsf{L}_i \simeq \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_k). \tag{5.11}$$

On the other hand, it follows from $\tilde{f}_i^*(\mathsf{V}'_a) = \mathsf{V}'_a \nabla L(i) \simeq \mathsf{V}'_k$ that

$$\varepsilon_i^*(\mathsf{V}'_a) + 1 = \varepsilon_i^*(\mathsf{V}'_k), \quad \varphi_i^*(\mathsf{V}'_a) = \varphi_i^*(\mathsf{V}'_k) + 1.$$

Thus, by (2.3) and (5.5), we have

$$\begin{aligned} \mathsf{V}_{a+1} = \mathscr{T}_i(\mathsf{V}'_a) &\simeq \tilde{f}_i^{\varphi_i^*(\mathsf{V}'_k)+1} \, \tilde{e}_i^{*\varepsilon_i^*(\mathsf{V}'_a)+1} \, \tilde{f}_i^* \mathsf{V}'_a \\ &\simeq L(i) \nabla (\tilde{f}_i^{\varphi_i^*(\mathsf{V}'_k)} \, \tilde{e}_i^{*\varepsilon_i^*(\mathsf{V}'_k)} \mathsf{V}'_k) \simeq L(i) \nabla \mathscr{T}_i(\mathsf{V}'_k) = L(i) \nabla \mathsf{V}_{k+1}, \end{aligned}$$

which implies that

$$\mathscr{F}_{\mathscr{D}}(\mathsf{V}_{a+1}) \simeq \mathsf{L}_i \nabla \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{k+1}). \tag{5.12}$$

Then

$$\begin{aligned} \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{k+1}) &\simeq \mathscr{F}_{\mathscr{D}}(\mathsf{V}_{a+1}) \nabla \mathscr{D}\mathsf{L}_i && \text{by (5.12) and Lemma 2.7} \\ &\simeq \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_a) \nabla \mathscr{D}\mathsf{L}_i && \text{by the induction hypothesis} \\ &\simeq \mathscr{F}_{\mathscr{S}_i(\mathscr{D})}(\mathsf{V}'_k) && \text{by (5.11),} \end{aligned}$$

which completes the proof of (5.8). ■

Proposition 5.11. *Let $i \in J$, and let S be a simple module in $\mathscr{C}_{\mathfrak{g}}$.*

(i) *The following conditions are equivalent:*

  (a) $S \in \mathcal{F}_{\mathfrak{D}}(\mathcal{C}_{*,s_i})$,

  (b) $S \in \mathscr{C}_{\mathfrak{D}}$ *and* $\mathfrak{d}(\mathscr{D}^{-1}\mathsf{L}_i, S) = 0$,

  (c) $S \in \mathscr{C}_{\mathscr{S}_i(\mathfrak{D})} \cap \mathscr{C}_{\mathfrak{D}}$.

(ii) *The following conditions are equivalent:*

  (a) $S \in \mathcal{F}_{\mathfrak{D}}(\mathcal{C}_{s_i w_0})$,

  (b) $S \in \mathscr{C}_{\mathfrak{D}}$ *and* $\mathfrak{d}(\mathscr{D}\mathsf{L}_i, S) = 0$,

  (c) $S \in \mathscr{C}_{\mathscr{S}_i^{-1}(\mathfrak{D})} \cap \mathscr{C}_{\mathfrak{D}}$.

Here, $\mathcal{C}_{*,s_i}$ *and* $\mathcal{C}_{s_i w_0}$ *are the subcategories of* R_{C}-gmod *that appeared in* Section 2.2.

Proof. We focus on proving (i) since (ii) can be proved in a similar manner.

Let us take a reduced expression $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ of w_0 such that $i_1 = i$. Let $\{\mathsf{V}_k\}_{k=1}^{\ell}$ be the cuspidal modules in R_{C}-gmod corresponding to $\underline{w_0}$. Let $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ be the affine cuspidal modules corresponding to $\mathfrak{D}$ and $\underline{w_0}$. Set $\mathsf{S}'_k = \mathsf{S}_{k+1}$ or $k \in \mathbb{Z}$. Then $\{\mathsf{S}'_k\}_{k\in\mathbb{Z}}$ are the cuspidal modules corresponding to $\mathscr{S}_{i_1}\mathfrak{D}$ and $\underline{w_0}' = s_{i_2} \cdots s_{i_{\ell+1}}$ by Proposition 5.10.

Now we prove (i). It is known that

(1) any simple module in $\mathcal{C}_{*,s_i}$ is isomorphic to the head of a convolution product of copies of $\mathsf{V}_2, \ldots, \mathsf{V}_\ell$,

(2) for any simple module $M \in R_{\mathsf{C}}$-gmod, $M \in \mathcal{C}_{*,s_i}$ if and only if $\varepsilon_i^*(M) = 0$

(see [26, Proposition 2.18 and Theorem 2.20]). Hence (a)⇔(b) follows from (2) and Corollary 4.13.

Note that

$$\mathcal{F}_{\mathfrak{D}}(\mathsf{V}_k) = \mathsf{S}_k = \mathsf{S}'_{k-1} \in \mathscr{C}_{\mathscr{S}_i(\mathfrak{D})} \quad \text{for } 2 \le k \le \ell.$$

Hence, by (1), we have

$$\mathcal{F}_{\mathfrak{D}}(\mathcal{C}_{*,s_i}) \subset \mathscr{C}_{\mathscr{S}_i(\mathfrak{D})} \cap \mathscr{C}_{\mathfrak{D}},$$

that is, (a)⇒(c).

Let S be a simple module in $\mathscr{C}_{\mathscr{S}_i(\mathfrak{D})} \cap \mathscr{C}_{\mathfrak{D}}$. Since $\mathscr{D}^{-1}\mathsf{L}_i \in \mathscr{D}^{-2}\mathscr{C}_{\mathscr{S}_i(\mathfrak{D})}$, we have

$$\mathfrak{d}(\mathscr{D}^{-1}\mathsf{L}_i, S) = 0.$$

Thus we obtain (c)⇒(b). ■

Example 5.12. We use the same notations as in Example 5.8.

(i) We shall apply $\mathscr{S}_1$ to the duality datum $\mathfrak{D} = \{\mathsf{L}_1, \mathsf{L}_2\}$. Let

$$\widetilde{\mathsf{L}}_1 := \mathscr{S}_1(\mathsf{L}_1) = \mathscr{D}\mathsf{L}_1 = V(2)_{(-q)^3},$$
$$\widetilde{\mathsf{L}}_2 := \mathscr{S}_1(\mathsf{L}_2) = \mathsf{L}_1 \nabla \mathsf{L}_2 = V(2)_{-q}.$$

Then $\mathscr{S}_1(\mathcal{D}) = \{\widetilde{\mathsf{L}}_1, \widetilde{\mathsf{L}}_2\}$. The affine cuspidal modules $\widetilde{\mathsf{S}}_k$ corresponding to $\mathscr{S}_1(\mathcal{D})$ and the reduced expression $s_2s_1s_2$ are

$$\begin{aligned}
\widetilde{\mathsf{S}}_1 &= \mathcal{F}_{\mathscr{S}_1\mathcal{D}}(L(2)) = \widetilde{\mathsf{L}}_2 = V(2)_{-q},\\
\widetilde{\mathsf{S}}_2 &= \mathcal{F}_{\mathscr{S}_1\mathcal{D}}(L(2) \nabla L(1)) = \widetilde{\mathsf{L}}_2 \nabla \widetilde{\mathsf{L}}_1 = V(2)_{-q} \nabla V(2)_{(-q)^3} = V(1)_{(-q)^2},\\
\widetilde{\mathsf{S}}_3 &= \mathcal{F}_{\mathscr{S}_1\mathcal{D}}(L(1)) = \widetilde{\mathsf{L}}_1 = V(2)_{(-q)^3},
\end{aligned}$$

and $\widetilde{\mathsf{S}}_{k+3} = \mathscr{D}(\widetilde{\mathsf{S}}_k)$ for $k \in \mathbb{Z}$. Note that $\widetilde{\mathsf{S}}_k = \mathsf{S}_{k+1}$ for any $k \in \mathbb{Z}$ (see Proposition 5.10).

(ii) We shall apply $\mathscr{S}_2$ to the duality datum $\mathcal{D} = \{\mathsf{L}_1, \mathsf{L}_2\}$. Let

$$\begin{aligned}
\widehat{\mathsf{L}}_1 &:= \mathscr{S}_2(\mathsf{L}_1) = \mathsf{L}_2 \nabla \mathsf{L}_1 = V(1)_{(-q)^2} \nabla V(1) = V(1^2)_{-q},\\
\widehat{\mathsf{L}}_2 &:= \mathscr{S}_2(\mathsf{L}_2) = \mathscr{D}\mathsf{L}_2 = V(2)_{(-q)^5}.
\end{aligned}$$

Then $\mathscr{S}_2(\mathcal{D}) = \{\widehat{\mathsf{L}}_1, \widehat{\mathsf{L}}_2\}$. As can be seen, the duality datum $\mathscr{S}_2(\mathcal{D})$ has a root module which is not fundamental.

The affine cuspidal modules $\widehat{\mathsf{S}}_k$ corresponding to $\mathscr{S}_2(\mathcal{D})$ and the reduced expression $s_1s_2s_1$ are

$$\begin{aligned}
\widehat{\mathsf{S}}_1 &= \mathcal{F}_{\mathscr{S}_2\mathcal{D}}(L(1)) = \widehat{\mathsf{L}}_1 = V(1^2)_{-q},\\
\widehat{\mathsf{S}}_2 &= \mathcal{F}_{\mathscr{S}_2\mathcal{D}}(L(1) \nabla L(2)) = \widehat{\mathsf{L}}_1 \nabla \widehat{\mathsf{L}}_2 = V(1^2)_{-q} \nabla V(2)_{(-q)^5}\\
&= (V(1)_{(-q)^2} \nabla V(1)) \nabla V(2)_{(-q)^5} = V(1) \qquad \text{by Lemma 2.7},\\
\widehat{\mathsf{S}}_3 &= \mathcal{F}_{\mathscr{S}_2\mathcal{D}}(L(2)) = \widehat{\mathsf{L}}_2 = V(2)_{(-q)^5},
\end{aligned}$$

and $\widehat{\mathsf{S}}_{k+3} = \mathscr{D}(\widehat{\mathsf{S}}_k)$ for $k \in \mathbb{Z}$. Note that $\widehat{\mathsf{S}}_k = \mathsf{S}'_{k+1}$ for any $k \in \mathbb{Z}$ (see Proposition 5.10).

6. PBW-theoretic approach

6.1. Complete duality datum

Definition 6.1. A duality datum $\mathcal{D}$ is called *complete* if it is strong and, for any simple module $M \in \mathscr{C}^0_{\mathfrak{g}}$, there exist simple modules $M_k \in \mathscr{C}_{\mathcal{D}}$ ($k \in \mathbb{Z}$) such that

(a) $M_k \simeq \mathbf{1}$ for all but finitely many k,

(b) $M \simeq \mathrm{hd}(\cdots \otimes \mathscr{D}^2 M_2 \otimes \mathscr{D} M_1 \otimes M_0 \otimes \mathscr{D}^{-1} M_{-1} \otimes \cdots)$.

In [31], we associate to the category $\mathscr{C}^0_{\mathfrak{g}}$ a simply-laced finite type root system in a canonical way. For a simple module $M \in \mathscr{C}_{\mathfrak{g}}$, define $\mathsf{E}(M) \in \mathrm{Hom}(\sigma(\mathfrak{g}), \mathbb{Z})$ by

$$\mathsf{E}(M)(i,a) := \Lambda^{\infty}(M, V(\varpi_i)_a) \quad \text{for } (i,a) \in \sigma(\mathfrak{g}).$$

Let

$$\mathcal{W}_0 := \{\mathsf{E}(M) \mid M \text{ is simple in } \mathscr{C}^0_{\mathfrak{g}}\} \quad \text{and} \quad \Delta_0 := \{\mathsf{s}_{i,a} \mid (i,a) \in \sigma_0(\mathfrak{g})\} \subset \mathcal{W}_0,$$

where we set $\mathsf{s}_{i,a} := \mathsf{E}(V(\varpi_i)_a)$. Then $\Upsilon_{\mathfrak{g}} := (\mathcal{W}_0, \Delta_0)$ forms a root system, and the type of $\Upsilon_{\mathfrak{g}}$ is as follows (see [31, Theorem 4.6]):

Type of $\mathfrak{g}$	$A_n^{(1)}$ $(n \geq 1)$	$B_n^{(1)}$ $(n \geq 2)$	$C_n^{(1)}$ $(n \geq 3)$	$D_n^{(1)}$ $(n \geq 4)$	$A_{2n}^{(2)}$ $(n \geq 1)$	$A_{2n-1}^{(2)}$ $(n \geq 2)$	$D_{n+1}^{(2)}$ $(n \geq 3)$
Type of $\Upsilon_{\mathfrak{g}}$	A_n	A_{2n-1}	D_{n+1}	D_n	A_{2n}	A_{2n-1}	D_{n+1}

Type of $\mathfrak{g}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$	$E_6^{(2)}$	$D_4^{(3)}$
Type of $\Upsilon_{\mathfrak{g}}$	E_6	E_7	E_8	E_6	D_4	E_6	D_4

(6.1)

We denote by $X_{\mathfrak{g}}$ the type of $\Upsilon_{\mathfrak{g}}$.

We define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathcal{W}_0$ by $(\mathsf{E}(M),\mathsf{E}(N)) = -\Lambda^{\infty}(M,N)$ for simple modules M and N. Then $(\cdot,\cdot)$ is a Weyl group invariant positive definite bilinear form and $\Delta_0 = \{\alpha \in \mathcal{W}_0 \mid (\alpha,\alpha) = 2\}$.

Proposition 6.2. *Let $\mathfrak{D} := \{\mathsf{L}_i\}_{i\in J} \subset \mathscr{C}_{\mathfrak{g}}^0$ be a complete duality datum associated with a simply-laced finite Cartan matrix C. Then C is of type $X_{\mathfrak{g}}$.*

Proof. We denote by Q_{C} and Φ_{C} the root lattice and the set of roots associated with C.

It follows from Proposition 2.11, Proposition 2.13, Theorem 4.10 and Definition 6.1 that the abelian group $\mathcal{W}_0$ is generated by $\mathsf{E}(M)$ for $M \in \mathscr{C}_{\mathfrak{D}}$. Moreover, $\mathsf{E}(\mathcal{F}_{\mathfrak{D}}(M))$ depends only on $\mathrm{wt}(M)$ by Theorem 4.12 (iii). Hence the functor $\mathcal{F}_{\mathfrak{D}}$ induces the surjective additive map

$$[\mathcal{F}_{\mathfrak{D}}] \colon \mathsf{Q}_{\mathsf{C}} \twoheadrightarrow \mathcal{W}_0$$

given by $[\mathcal{F}_{\mathfrak{D}}](\alpha_i) = \mathsf{E}(\mathsf{L}_i)$ for $i \in J$. Moreover, $[\mathcal{F}_{\mathfrak{D}}]$ preserves the positive definite pairing $(\cdot,\cdot)$. Hence $[\mathcal{F}_{\mathfrak{D}}]$ is bijective. Since both Φ_{C} and Δ_0 are characterized by the condition $(X,X) = 2$ ([31, Corollary 4.8] and [19, Proposition 5.10]), the set $\{\mathsf{E}(\mathsf{L}_i)\}_{i\in J}$ becomes a basis of the root system $\Upsilon_{\mathfrak{g}}$. Since $c_{i,j} = (\alpha_i,\alpha_j) = (\mathsf{E}(\mathsf{L}_i),\mathsf{E}(\mathsf{L}_j))$ for any $i,j \in J$ by Theorem 4.12 (iii), we conclude that the Cartan matrix $\mathsf{C} = (c_{i,j})_{i,j\in J}$ is of type $X_{\mathfrak{g}}$. ∎

Theorem 6.3. *Let $\mathfrak{D} := \{\mathsf{L}_i\}_{i\in J}$ be a complete duality datum. For any $i \in J$, $\mathscr{S}_i(\mathfrak{D})$ and $\mathscr{S}_i^{-1}(\mathfrak{D})$ are complete.*

Proof. We focus on the case of $\mathscr{S}_i$ since the other case is similar. Since $\mathscr{S}_i(\mathfrak{D})$ is strong by Proposition 5.9, it suffices to show that $\mathscr{S}_i(\mathfrak{D})$ satisfies the conditions of Definition 6.1.

Let $i \in J$ and choose a reduced expression $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ of the longest element w_0 of W_{C} with $i_1 = i$. Define $\{i_k\}_{k\in\mathbb{Z}}$ and the cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ corresponding to $\mathfrak{D}$ and $\underline{w_0}$ as in Section 5.2. Let M be a simple module in $\mathscr{C}_{\mathfrak{g}}^0$. As $\mathfrak{D}$ is complete, there exist simple modules $M_k \in \mathscr{C}_{\mathfrak{D}}$ $(k \in \mathbb{Z})$ such that $M_k \simeq \mathbf{1}$ for all but finitely many k and

$$M \simeq \mathrm{hd}(\cdots \otimes \mathscr{D}^2 M_2 \otimes \mathscr{D} M_1 \otimes \ M_0 \otimes \mathscr{D}^{-1} M_{-1} \otimes \cdots). \tag{6.2}$$

For each $k \in \mathbb{Z}$, there exist $a_{k,1}, \dots, a_{k,\ell} \in \mathbb{Z}_{\geq 0}$ such that

$$M_k \simeq \operatorname{hd}(\mathsf{S}_\ell^{\otimes a_{k,\ell}} \otimes \cdots \otimes \mathsf{S}_1^{\otimes a_{k,1}}).$$

Set $c_{s+k\ell} = a_{k,s}$ for $1 \leq s \leq \ell$ and $k \in \mathbb{Z}$. Then, by Lemma 2.23, we have

$$\mathscr{D}^k M_k \simeq \operatorname{hd}(\mathsf{S}_{k\ell+\ell}^{\otimes c_{k\ell+\ell}} \otimes \cdots \otimes \mathsf{S}_{k\ell+1}^{\otimes c_{k\ell+1}}).$$

Hence we have

$$M \simeq \operatorname{hd}(\cdots \otimes \mathsf{S}_1^{\otimes c_1} \otimes \mathsf{S}_0^{\otimes c_0} \otimes \mathsf{S}_{-1}^{\otimes c_{-1}} \otimes \cdots).$$

Set

$$N_k = \operatorname{hd}(\mathsf{S}_{\ell+1}^{\otimes c_{k\ell+\ell+1}} \otimes \cdots \otimes \mathsf{S}_2^{\otimes c_{k\ell+2}}).$$

Then $N_k \in \mathscr{C}_{\mathscr{S}_i \mathfrak{D}}$ by Proposition 5.10, and we have

$$\mathscr{D}^k N_k \simeq \operatorname{hd}(\mathsf{S}_{k\ell+\ell+1}^{\otimes c_{k\ell+\ell+1}} \otimes \cdots \otimes \mathsf{S}_{k\ell+2}^{\otimes c_{k\ell+2}}).$$

Hence we obtain

$$M \simeq \operatorname{hd}(\cdots \otimes \mathscr{D}^1 N_1 \otimes \mathscr{D}^0 N_0 \otimes \mathscr{D}^{-1} N_{-1} \otimes \cdots). \qquad \blacksquare$$

6.2. *Duality datum arising from Q-datum*

The subcategory $\mathscr{C}_{\mathscr{Q}}$ of $\mathscr{C}_{\mathfrak{g}}^0$ was introduced in [16] for simply-laced affine type ADE, in [23] for twisted affine types $A^{(2)}$ and $D^{(2)}$, in [34, 51] for untwisted affine types $B^{(1)}$ and $C^{(1)}$, and in [47] for exceptional affine type. Let $\mathfrak{g}_{\mathrm{fin}}$ be the simple Lie algebra of type $X_{\mathfrak{g}}$ defined in (6.1) and I_{fin} the index set of $\mathfrak{g}_{\mathrm{fin}}$. The category $\mathscr{C}_{\mathscr{Q}}$ categorifies the coordinate ring $\mathbb{C}[N]$ of the maximal unipotent group N associated with $\mathfrak{g}_{\mathrm{fin}}$. This category is defined by a Q-datum. A *Q-datum* is a triple $\mathscr{Q} := (\triangle, \sigma, \xi)$ consisting of the Dynkin diagram $\triangle$ of $\mathfrak{g}_{\mathrm{fin}}$, an automorphism σ on $\triangle$ and a height function ξ, which satisfy certain conditions (see [13] for details, and also [33, Section 6]). When $\mathfrak{g}$ is of untwisted affine type ADE, σ is the identity and $\mathscr{Q}$ is equal to a Dynkin quiver with a height function. To a Q-datum $\mathscr{Q}$, we can associate a subset $\sigma_{\mathscr{Q}}(\mathfrak{g})$ of $\sigma_0(\mathfrak{g})$. This set $\sigma_{\mathscr{Q}}(\mathfrak{g})$ is in 1-1 correspondence with the set Φ_{fin}^+ of positive roots of $\mathfrak{g}_{\mathrm{fin}}$, which is denoted by

$$\phi_{\mathscr{Q}} \colon \Phi_{\mathrm{fin}}^+ \xrightarrow{\sim} \sigma_{\mathscr{Q}}(\mathfrak{g}). \tag{6.3}$$

Set

$$\mathfrak{D}_{\mathscr{Q}} := \{\mathsf{L}_i\}_{i \in I_{\mathrm{fin}}},$$

where L_i is the fundamental module corresponding to $\phi_{\mathscr{Q}}(\alpha_i)$ for $i \in I_{\mathrm{fin}}$. Then $\mathfrak{D}_{\mathscr{Q}}$ becomes a strong duality datum [11, 13, 20, 23, 33, 34, 47], which gives the duality functor $\mathscr{F}_{\mathfrak{D}_{\mathscr{Q}}}$. By the definition, we have $\mathscr{C}_{\mathscr{Q}} = \mathscr{C}_{\mathfrak{D}_{\mathscr{Q}}}$. We simply write $\mathscr{F}_{\mathscr{Q}}$ for $\mathscr{F}_{\mathfrak{D}_{\mathscr{Q}}}$:

$$\mathscr{F}_{\mathscr{Q}} \colon R^{\mathfrak{g}_{\mathrm{fin}}}\text{-gmod} \to \mathscr{C}_{\mathfrak{g}}.$$

We refer the reader to [13, 33, 51] for the notion of (twisted) $\mathscr{Q}$-adapted reduced expressions of the longest element w_0 of the Weyl group of $\mathfrak{g}_{\rm fin}$.

Let $\mathsf{W}_{\rm fin}$ be the Weyl group of $\mathfrak{g}_{\rm fin}$. For a $\mathscr{Q}$-adapted reduced expression $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ of the longest element w_0 of $\mathsf{W}_{\rm fin}$, we define $\beta_k \in \Phi^+_{\rm fin}$ $(1 \le k \le \ell)$ by (2.1). Then there exist a sequence $\{(i_k, a_k)\}_{k\in\mathbb{Z}} \subset I_{\rm fin} \times \mathbf{k}^\times$ and $\pi\colon I_{\rm fin} \to I_0$ such that $(\pi(i_k), a_k) = \phi_{\mathscr{Q}}(\beta_k) \in \sigma_{\mathscr{Q}}(\mathfrak{g})$ for $k = 1, \ldots, \ell$ and

$$(\pi(i_{s+m\ell}), a_{s+m\ell}) = \delta^m((\pi(i_s), a_s)) \quad \text{for } 1 \le s \le \ell \text{ and } m \in \mathbb{Z}.$$

Here we set

$$\delta^m((i,a)) := \begin{cases} (i, (p^*)^m a) & \text{if } m \text{ is even,} \\ (i^*, (p^*)^m a) & \text{if } m \text{ is odd.} \end{cases}$$

(See [33, Section 6].)

We define the affine cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ as in Definition 5.6.

Collecting results in [13, 16, 20, 23, 34, 47, 51], we obtain Proposition 6.4 below. In the proposition, the symmetric cases follow from [16, 20], the untwisted $B^{(1)}$ and $C^{(1)}$ cases follow from [34, 51], the twisted $A^{(2)}$ and $D^{(2)}$ cases follow from [23], and the exceptional cases follow from [47]. The uniform approach is given in [13]. See also [33, Section 6].

Proposition 6.4 ([13, 16, 20, 23, 34, 47, 51]). *Let $\mathscr{Q}$ be a Q-datum.*

(i) $\sigma_0(\mathfrak{g}) = \bigsqcup_{m\in\mathbb{Z}} \delta^m\, \sigma_{\mathscr{Q}}(\mathfrak{g})$ (see e.g. [13, Proposition 4.21]).

(ii) *There exists a $\mathscr{Q}$-adapted reduced expression of w_0* (see e.g. [51, Section 3]).

(iii) *For a $\mathscr{Q}$-adapted reduced expression $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ of w_0, let $\{(i_k, a_k)\}_{k\in\mathbb{Z}}$ be the sequence as above, and let $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$ be the affine cuspidal modules corresponding to $\mathfrak{D}_{\mathscr{Q}}$ and $\underline{w_0}$. Then*

  (a) $\mathsf{S}_k \simeq V(\varpi_{\pi(i_k)})_{a_k}$,

  (b) $d_{V(\varpi_{\pi(i_s)}), V(\varpi_{\pi(i_t)})}(a_t/a_s) \neq 0$ *for $t, s \in \mathbb{Z}$ such that $s > t$. Here, d is the denominator of the R-matrix.*

  (See [20, Theorem 4.3.4], [23, Theorem 5.1 and Lemma 5.2] , [34, Theorems 6.3, 6.4] and [47, Section 6]).

Proposition 6.5. *The duality datum $\mathfrak{D}_{\mathscr{Q}}$ is a complete duality datum.*

Proof. Recall that $\sigma_0(\mathfrak{g}) = \{(\pi(i_k), a_k) \mid k \in \mathbb{Z}\}$. For a simple module M in $\mathscr{C}^0_{\mathfrak{g}}$, let $\lambda = \sum_{s=1}^r (\pi(i_{k_s}), a_{k_s})$ be the affine highest weight of M (see Theorem 2.9 (iv)). We may assume that $\{k_s\}_{s=1}^r$ is a decreasing sequence. Then, by Proposition 6.4 and Theorem 2.9, we have $M \simeq \operatorname{hd}(\mathsf{S}_{k_1} \otimes \cdots \otimes \mathsf{S}_{k_r})$. ■

Thanks to Theorem 6.3, we have the following.

Corollary 6.6. *The duality datum obtained from $\mathfrak{D}_{\mathscr{Q}}$ by applying a finite sequence of $\mathscr{S}_i$ and $\mathscr{S}_i^{-1}$ $(i \in I_{\rm fin})$ is a complete duality datum.*

Example 6.7. We use the same notations as in Examples 5.8 and 5.12. Let $\triangle$ be the Dynkin diagram of finite type A_2.

(i) Let ξ be the height function on $\triangle$ defined by $\xi(1) = 0$ and $\xi(2) = 1$, and let $\mathscr{Q}$ be the Q-datum consisting of $\triangle$ and ξ. Then $\mathfrak{D}$ is equal to the duality datum arising from the Q-datum $\mathscr{Q}$, which says that $\mathfrak{D}$ is complete. The reduced expression $s_1 s_2 s_1$ is $\mathscr{Q}$-adapted, but $s_2 s_1 s_2$ is not.

(ii) By Corollary 6.6, $\mathscr{S}_1(\mathfrak{D})$ and $\mathscr{S}_2(\mathfrak{D})$ are complete duality data. The duality datum $\mathscr{S}_1(\mathfrak{D})$ arises from the Q-datum consisting of $\triangle$ and the height function ξ' defined by $\xi'(1) = 2$ and $\xi'(2) = 1$, but $\mathscr{S}_2(\mathfrak{D})$ does not come from any Q-datum.

6.3. *PBW for quantum affine algebras*

In this subsection, we develop the PBW theory for $\mathscr{C}_{\mathfrak{g}}^0$ using a complete duality datum. This generalizes the ordinary standard modules and related results [14,25,45,46,53]. Note that the ordinary standard modules are cyclic tensor products of fundamental modules.

Let $\mathsf{C} = (c_{i,j})_{i.j\in J}$ be a simply-laced finite Cartan matrix. Throughout this subsection, we assume that

$$\mathfrak{D} = \{\mathsf{L}_i\}_{i\in J} \text{ is a } \textit{complete} \text{ duality datum associated with } \mathsf{C}.$$

Proposition 6.2 says that C is of type $X_{\mathfrak{g}}$ and $J = I_{\mathrm{fin}}$. Let W_{C} be the Weyl group associated with C. We fix a reduced expression $\underline{w_0} = s_{i_1} \cdots s_{i_\ell}$ of the longest element w_0 of W_{C}, and let S_k $(k \in \mathbb{Z})$ be the affine cuspidal modules corresponding to $\mathfrak{D}$ and $\underline{w_0}$. We define

$$\mathbf{Z} := \mathbb{Z}_{\geq 0}^{\oplus \mathbb{Z}} = \{(a_k)_{k\in\mathbb{Z}} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}} \mid a_k = 0 \text{ for all but finitely many } k\text{'s}\}. \tag{6.4}$$

We denote by $\prec$ the bi-lexicographic order on $\mathbf{Z}$, i.e., for any $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$ and $\mathbf{a}' = (a'_k)_{k\in\mathbb{Z}}$ in $\mathbf{Z}$, $\mathbf{a} \prec \mathbf{a}'$ if and only if the following conditions hold:

$$\begin{cases} \text{(a) there exists } r \in \mathbb{Z} \text{ such that } a_k = a'_k \text{ for any } k < r \text{ and } a_r < a'_r, \\ \text{(b) there exists } s \in \mathbb{Z} \text{ such that } a_k = a'_k \text{ for any } k > s \text{ and } a_s < a'_s. \end{cases} \tag{6.5}$$

Similarly, we let $\prec_r$ (resp. $\prec_l$) be the right (resp. left) lexicographic order on $\mathbf{Z}$, i.e., for any $\mathbf{a}, \mathbf{a}' \in \mathbf{Z}$, $\mathbf{a} \prec_r \mathbf{a}'$ (resp. $\mathbf{a} \prec_l \mathbf{a}'$) if and only if condition (a) (resp. (b)) in (6.5) holds. Hence

$$\mathbf{a} \prec \mathbf{a}' \iff \mathbf{a} \prec_l \mathbf{a}' \text{ and } \mathbf{a} \prec_r \mathbf{a}'. \tag{6.6}$$

For $\mathbf{a} = (a_k)_{k\in\mathbb{Z}} \in \mathbf{Z}$, we define

$$\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}) := \bigotimes_{k=+\infty}^{-\infty} \mathsf{S}_k^{\otimes a_k} = \cdots \otimes \mathsf{S}_2^{\otimes a_2} \otimes \mathsf{S}_1^{\otimes a_1} \otimes \mathsf{S}_0^{\otimes a_0} \otimes \mathsf{S}_{-1}^{\otimes a_{-1}} \otimes \mathsf{S}_{-2}^{\otimes a_{-2}} \otimes \cdots .$$

Here $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(0)$ should be understood as the trivial module $\mathbf{1}$. We call the modules $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ *standard modules* with respect to the cuspidal modules $\{\mathsf{S}_k\}_{k\in\mathbb{Z}}$.

Lemma 6.8. *Let $k \in \mathbb{Z}$ and $a \in \mathbb{Z}_{>0}$, and let M be a simple module in $\mathscr{C}_{\mathfrak{g}}^0$.*

(i) *If $\mathfrak{d}(\mathscr{D}^t \mathsf{S}_k, M) = 0$ for $t = 1, 2$, then $a = \mathfrak{d}(\mathscr{D}\mathsf{S}_k, \mathsf{S}_k^{\otimes a} \nabla M)$.*

(ii) *If $\mathfrak{d}(\mathscr{D}^t \mathsf{S}_k, M) = 0$ for $t = -1, -2$, then $a = \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, M \nabla \mathsf{S}_k^{\otimes a})$.*

Proof. (i) Note that S_k is a root module by Proposition 5.7. Applying Lemma 3.6 (i) to the setting $L := \mathsf{S}_k$ and $X := M$, we have

$$\mathfrak{d}(\mathscr{D}\mathsf{S}_k, \mathsf{S}_k^{\otimes a} \nabla M) = a + \mathfrak{d}(\mathscr{D}\mathsf{S}_k, M) = a.$$

(ii) can be proved in the same manner. ■

Lemma 6.9. *Let $m, l \in \mathbb{Z}$ with $m \geq l$ and $a_m, a_{m-1}, \ldots, a_l \in \mathbb{Z}_{\geq 0}$. Set*

$$M := \mathrm{hd}(\mathsf{S}_m^{\otimes a_m} \otimes \mathsf{S}_{m-1}^{\otimes a_{m-1}} \otimes \cdots \otimes \mathsf{S}_l^{\otimes a_l}).$$

(i) *$\mathfrak{d}(\mathscr{D}\mathsf{S}_k, M) = 0$ for any $k > m$.*

(ii) *Set $M_m := M$ and define inductively*

$$d_k := \mathfrak{d}(\mathscr{D}\mathsf{S}_k, M_k) \quad \textit{and} \quad M_{k-1} := M_k \nabla \mathscr{D}(\mathsf{S}_k^{\otimes d_k})$$

for $k = m, \ldots, l$. Then

$$d_k = a_k \quad \textit{and} \quad M_k \simeq \mathrm{hd}(\mathsf{S}_k^{\otimes a_k} \otimes \mathsf{S}_{k-1}^{\otimes a_{k-1}} \otimes \cdots \otimes \mathsf{S}_l^{\otimes a_l}) \quad \textit{for } k = m, \ldots, l.$$

(iii) *$\mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, M) = 0$ for any $k < l$.*

(iv) *Set $N_l := M$ and define inductively*

$$e_k := \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, N_k) \quad \textit{and} \quad N_{k+1} := \mathscr{D}^{-1}(\mathsf{S}_k^{\otimes e_k}) \nabla N_k$$

for $k = l, \ldots, m$. Then

$$e_k = a_k \quad \textit{and} \quad N_k \simeq \mathrm{hd}(\mathsf{S}_m^{\otimes a_m} \otimes \cdots \otimes \mathsf{S}_{k+1}^{\otimes a_{k+1}} \otimes \mathsf{S}_k^{\otimes a_k}) \quad \textit{for } k = m, \ldots, l.$$

Proof. (i) By Proposition 5.7 (ii), $(\mathsf{S}_k, \mathsf{S}_t)$ is strongly unmixed for any $k > m$ and $t = m, \ldots, l$. Thus $\mathfrak{d}(\mathscr{D}\mathsf{S}_k, \mathsf{S}_t) = 0$ for $t = m, \ldots, l$, which implies that $\mathfrak{d}(\mathscr{D}\mathsf{S}_k, M) = 0$.

(ii) By induction on k, we may assume that $k = m$. Set $N := \mathrm{hd}(\mathsf{S}_{m-1}^{\otimes a_{m-1}} \otimes \cdots \otimes \mathsf{S}_l^{\otimes a_l})$. By (i), we have $\mathfrak{d}(\mathscr{D}^t \mathsf{S}_m, N) = 0$ for $t = 1, 2$. Proposition 5.7 (iii) tells us that $M \simeq \mathsf{S}_m^{\otimes a_m} \nabla N$. Thus, by Lemmas 2.7 and 6.8, we have

$$\begin{aligned} d_m &= \mathfrak{d}(\mathscr{D}\mathsf{S}_m, M) = \mathfrak{d}(\mathscr{D}\mathsf{S}_m, \mathsf{S}_m^{\otimes a_m} \nabla N) = a_m, \\ M \nabla \mathscr{D}(\mathsf{S}_k^{\otimes a_m}) &\simeq (\mathsf{S}_m^{\otimes a_m} \nabla N) \nabla \mathscr{D}(\mathsf{S}_k^{\otimes a_m}) \simeq N. \end{aligned}$$

Assertions (iii) and (iv) can be proved in the same manner. ■

Theorem 6.10. (i) *For any $\mathbf{a} \in \mathbf{Z}$, the head of $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ is simple; denote it by*

$$\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}) := \mathrm{hd}(\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})).$$

(ii) *For any simple module $M \in \mathscr{C}_{\mathfrak{g}}^0$, there exists a unique $\mathbf{a} \in \mathbf{Z}$ such that*

$$M \simeq \mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}).$$

Therefore, the set $\{\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Z}\}$ is a complete and irredundant set of simple modules of $\mathscr{C}_{\mathfrak{g}}^0$ up to isomorphism.

Proof. (i) follows from Proposition 5.7.

(ii) Let M be a simple module in $\mathscr{C}_{\mathfrak{g}}^0$. Since $\mathfrak{D}$ is complete, there exist simple modules $M_k \in \mathscr{C}_{\mathfrak{D}}$ $(k \in \mathbb{Z})$ such that $M_k \simeq \mathbf{1}$ for all but finitely many k and

$$M \simeq \operatorname{hd}(\cdots \otimes \mathscr{D}^2 M_2 \otimes \mathscr{D} M_1 \otimes M_0 \otimes \mathscr{D}^{-1} M_{-1} \otimes \cdots).$$

Since $M_k \in \mathscr{C}_{\mathfrak{D}}$, there exist $b_1^k, \dots, b_\ell^k \in \mathbb{Z}_{\geq 0}$ such that $M_k \simeq \operatorname{hd}(\mathsf{S}_\ell^{b_\ell^k} \otimes \cdots \otimes \mathsf{S}_1^{b_1^k})$, which yields

$$\mathscr{D}^k M_k \simeq \operatorname{hd}(\mathsf{S}_{k\ell+\ell}^{b_\ell^k} \otimes \cdots \otimes \mathsf{S}_{k\ell+1}^{b_1^k})$$

by Lemma 2.23. For $t \in \mathbb{Z}$, we define $a_t := b_r^k$, where $t = k\ell + r$ for some $k \in \mathbb{Z}$ and $r = 1, \dots, \ell$, and set $\mathbf{a} := (a_t)_{t \in \mathbb{Z}}$. By Proposition 5.7, we have

$$M \simeq \mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}).$$

The uniqueness for $\mathbf{a}$ follows from Lemma 6.9. This completes the proof. ∎

The element $\mathbf{a} \in \mathbf{Z}$ associated with a simple module M in Theorem 6.10 (ii) is called the *cuspidal decomposition* of M with respect to the cuspidal modules $\{\mathsf{S}_k\}_{k \in \mathbb{Z}}$, and it is denoted by

$$\mathbf{a}_{\mathfrak{D},\underline{w_0}}(M) := \mathbf{a}. \tag{6.7}$$

Lemma 6.11. *Let L, M, N be simple modules in $\mathscr{C}_{\mathfrak{g}}$ and assume that L is real.*

(i) *If (L, M) and (L, N) are strongly unmixed and $L \nabla N$ appears in $L \otimes M$ as a subquotient, then $M \simeq N$.*

(ii) *If (M, L) and (N, L) are strongly unmixed and $N \nabla L$ appears in $M \otimes L$ as a subquotient, then $M \simeq N$.*

Proof. (i) Since (L, M) and (L, N) are strongly unmixed,

$$\Lambda(L, M) = \Lambda^\infty(L, M) \quad \text{and} \quad \Lambda(L, N) = \Lambda^\infty(L, N)$$

by Lemma 5.2. Since $L \nabla N$ appears in $L \otimes M$, Proposition 2.13 tells us that

$$\Lambda(L, M) = \Lambda^\infty(L, M) = \Lambda^\infty(L, N) = \Lambda(L, N) = \Lambda(L, L \nabla N).$$

Thus it follows from [27, Theorem 4.11] that $L \nabla M \simeq L \nabla N$, which implies that $M \simeq N$ by Lemma 2.7.

(ii) can be proved in the same manner. ∎

For $\mathbf{c} = (c_k)_{k\in\mathbb{Z}} \in \mathbf{Z}$, we set $l(\mathbf{c})$ (resp. $r(\mathbf{c})$) to be the integer t such that

$$c_t \neq 0, \quad c_k = 0 \quad \text{for any } k > t \text{ (resp. } k < t\text{)}. \tag{6.8}$$

Theorem 6.12. *Let* $\mathbf{a}$ *be an element of* $\mathbf{Z}$.

(i) *The simple module* $\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ *appears only once in* $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$.

(ii) *If* V *is a simple subquotient of* $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ *which is not isomorphic to* $\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$*, then* $\mathbf{a}_{\mathfrak{D},\underline{w_0}}(V) \prec \mathbf{a}$.

(iii) *In the Grothendieck ring, we have*

$$[\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})] = [\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})] + \sum_{\mathbf{a}'\prec\mathbf{a}} c(\mathbf{a}')[\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}')] \quad \textit{for some } c(\mathbf{a}') \in \mathbb{Z}_{\geq 0}.$$

Proof. We focus on proving (ii) because (i) and (iii) follow from (ii).

Let $\mathbf{a} = (a_k)_{k\in\mathbb{Z}}$ and set

$$l := l(\mathbf{a}) \quad \text{and} \quad r := r(\mathbf{a}).$$

Let V be a simple subquotient of $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ which is not isomorphic to $\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$. We set

$$\mathbf{b} = (b_k)_{k\in\mathbb{Z}} := \mathbf{a}_{\mathfrak{D},\underline{w_0}}(V).$$

For $k > l$ and $r > t$, since $(\mathsf{S}_k, \mathsf{S}_l)$ and $(\mathsf{S}_r, \mathsf{S}_t)$ are strongly unmixed by Proposition 5.7, we have

$$\mathfrak{d}(\mathscr{D}\mathsf{S}_k, \mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})) = 0, \quad \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_t, \mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})) = 0,$$

which implies that $\mathfrak{d}(\mathscr{D}\mathsf{S}_k, V) = 0$ and $\mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_t, V) = 0$ by [27, Proposition 4.2]. Thus, Lemma 6.9 tells us that

$$l \geq l(\mathbf{b}) \quad \text{and} \quad r(\mathbf{b}) \geq r.$$

We shall now prove $\mathbf{b} \prec_l \mathbf{a}$, where $\prec_l$ is the left lexicographical order on $\mathbf{Z}$. Note that, by Lemma 6.9, Proposition 5.7 and [27, Proposition 4.2], we have

$$b_l = \mathfrak{d}(\mathscr{D}\mathsf{S}_l, V) \leq \mathfrak{d}(\mathscr{D}\mathsf{S}_l, \mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})) = \mathfrak{d}(\mathscr{D}\mathsf{S}_l, \mathsf{S}_l^{\otimes a_l}) = a_l.$$

When either $l > l(\mathbf{b})$ or $l = l(\mathbf{b})$ and $b_l < a_l$, it is obvious that $\mathbf{b} \prec_l \mathbf{a}$ by the definition. We assume that $l = l(\mathbf{b})$ and $b_l = a_l$. Set

$$c := b_l = a_l, \quad \mathbf{a}^- = (a_k^-)_{k\in\mathbb{Z}}, \quad \text{where} \quad a_k^- := \begin{cases} 0 & \text{if } k = l, \\ a_k & \text{otherwise,} \end{cases}$$

and

$$P^- := \mathsf{S}_{l-1}^{\otimes a_{l-1}} \otimes \cdots \otimes \mathsf{S}_r^{\otimes a_r}, \quad V^- := \mathrm{hd}(\mathsf{S}_{l-1}^{\otimes b_{l-1}} \otimes \cdots \otimes \mathsf{S}_{r(\mathbf{b})}^{\otimes b_{r(\mathbf{b})}}).$$

Note that

$$P^- = \mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}^-), \quad \mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}) = \mathsf{S}_l^{\otimes c} \otimes P^-, \quad V \simeq (\mathsf{S}_l^{\otimes c}) \nabla V^-, \tag{6.9}$$

where the third relation follows from Proposition 5.7 (iii). As V appears in $\mathsf{S}_l^{\otimes c} \otimes P^-$ as a simple subquotient, there exists a simple subquotient L of P^- such that

$$V \text{ appears in } \mathsf{S}_l^{\otimes c} \otimes L \text{ as a simple subquotient.}$$

By Proposition 5.7 (ii), we know that (S_l, V^-) and (S_l, L) are strongly unmixed. Hence, by Lemma 6.11, we conclude that

$$V^- \simeq L.$$

If V^- is isomorphic to $\mathrm{hd}(P^-)$, then $V \simeq \mathrm{hd}(\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}))$ by (6.9), which contradicts the assumption. Hence V^- is not isomorphic to $\mathrm{hd}(P^-)$. Applying the standard induction argument to the setting V^- and P^-, we obtain

$$\mathbf{a}_{\mathfrak{D},\underline{w_0}}(V^-) \prec_l \mathbf{a}^-,$$

which implies that $\mathbf{b} \prec_l \mathbf{a}$.

In the same manner, one can prove that $\mathbf{b} \prec_r \mathbf{a}$. Therefore it follows from (6.6) that $\mathbf{b} \prec \mathbf{a}$. ■

Remark 6.13. Let V be a simple subquotient of $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$. Theorem 6.12 says that $\mathbf{a}_{\mathfrak{D},\underline{w_0}}(V) \prec \mathbf{a}$. There is another condition which V should satisfy. By Proposition 2.13,

$$\mathsf{E}(V) = \mathsf{E}(\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})), \tag{6.10}$$

where E is given in Section 6.1. Thus they are in the same block of $\mathscr{C}_{\mathfrak{g}}$.

Remark 6.14. There is a well-known partial ordering, called the *Nakajima partial ordering*, in the q-character theory. For simplicity, we assume that $U_q'(\mathfrak{g})$ is of untwisted affine ADE type. Let $Y_{i,a}$ be an indeterminate for $i \in I_0$ and $a \in \mathbf{k}^\times$. For $i \in I_0$ and $a \in \mathbf{k}^\times$, set $A_{i,a} := Y_{i,aq^{-1}} Y_{i,aq} \prod_{(\alpha_i,\alpha_j)=-1} Y_{j,a}^{-1}$. Then one can define a partial ordering $\leq$ on the set of monomials in $\mathbb{Z}[Y_{i,a}^{\pm} \mid i \in I_0,\ a \in \mathbf{k}^\times]$ as follows: for monomials m and m', $m \leq m'$ if and only if $m^{-1}m'$ is a product of elements of $\{A_{i,a} \mid i \in I_0,\ a \in \mathbf{k}^\times\}$ [10,46]. The simple modules and ordinary standard modules in $\mathscr{C}_{\mathfrak{g}}$ are parameterized by dominant monomials, which are denoted by $L(m)$ and $M(m)$ respectively for a dominant monomial m. Note that the fundamental module $V(\varpi_i)_a$ corresponds to $Y_{i,a}$. From the viewpoint of (q,t)-characters, it was shown in [45,46] that

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m,m'}[L(m')] \tag{6.11}$$

in the Grothendieck ring $K(\mathscr{C}_{\mathfrak{g}})$ and the multiplicity $P_{m,m'}$ can be understood as the specialization at $t = 1$ of an analogue $P_{m,m'}(t)$ of the Kazhdan–Lusztig polynomial.

Let $\mathcal{Q}$ be a Q-datum and let $\underline{w_0}$ be a $\mathcal{Q}$-adapted reduced expression. In this case, the affine cuspidal modules S_k are all fundamental modules in $\mathscr{C}_{\mathfrak{g}}^0$ and $\mathsf{P}_{\mathfrak{D}_{\mathcal{Q}},\underline{w_0}}(\mathbf{a})$ are ordinary standard modules (see Example 5.8 (i) for instance). Let m and m' be dominant monomials and set $\mathbf{a} := \mathbf{a}_{\mathfrak{D}_{\mathcal{Q}},\underline{w_0}}(L(m))$ and $\mathbf{a}' := \mathbf{a}_{\mathfrak{D}_{\mathcal{Q}},\underline{w_0}}(L(m'))$. Considering the definition of $A_{i,a}$ and [33, Proposition 6.11], one can show that if $m \le m'$ in the partial ordering, then $\mathbf{a} \preceq \mathbf{a}'$ in the ordering (6.5). From this observation about two orders $\le$ and $\preceq$, Theorem 6.12 is compatible with (6.11). Since affine cuspidal modules need not be fundamental in general (see Example 5.8 (ii) for instance), Theorem 6.12 can be viewed as a generalization of (6.11).

Remark 6.13 says that condition (6.10) holds when V is a simple subquotient of $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$. Thus it is interesting to ask under what conditions the ordering (6.5) is equal to the ordering $\le$.

For $a, b \in \mathbb{Z} \sqcup \{\pm\infty\}$, an *interval* $[a, b]$ is the set of integers between a and b:

$$[a, b] := \{s \in \mathbb{Z} \mid a \le s \le b\}.$$

If $a > b$, we understand $[a, b] = \emptyset$.

For an interval $[a, b]$, we define $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$ to be the full subcategory of $\mathscr{C}_{\mathfrak{g}}$ whose objects have all their composition factors V satisfying the following condition:

$$b \ge l(\mathbf{a}_{\mathfrak{D},\underline{w_0}}(V)) \quad \text{and} \quad r(\mathbf{a}_{\mathfrak{D},\underline{w_0}}(V)) \ge a. \tag{6.12}$$

Thanks to Theorem 6.12, we have the following proposition.

Proposition 6.15. *The category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$ is stable by taking subquotients, extensions, and tensor products.*

It is easy to show that the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$ is equal to the smallest full subcategory of $\mathscr{C}_0$ satisfying the following conditions:

(i) it is stable under taking subquotients, extensions, tensor products,

(ii) it contains S_s for all $a \le s \le b$ and the trivial module $\mathbf{1}$.

If no confusion can arise, we simply write $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ instead of $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$.

For an interval $[a, b]$, we set

$$\mathbf{Z}^{[a,b]} := \{\mathbf{a} = (a_k)_{k\in\mathbb{Z}} \in \mathbf{Z} \mid a_k = 0 \text{ if either } k > b \text{ or } a > k\}.$$

The theorem below follows directly from Lemma 6.9 and Theorems 6.10 and 6.12.

Theorem 6.16. *Let $[a, b]$ be an interval.*

(i) *The set $\{\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}) \mid \mathbf{a} \in \mathbf{Z}^{[a,b]}\}$ is a complete and irredundant set of simple modules of $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ up to isomorphism.*

(ii) *Let M be a simple module in $\mathscr{C}_{\mathfrak{g}}^{0}$. Then M belongs to $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ if and only if*

$$\mathfrak{d}(\mathscr{D}\mathsf{S}_k, M) = 0 \; \textit{for } k > b \quad \textit{and} \quad \mathfrak{d}(\mathscr{D}^{-1}\mathsf{S}_k, M) = 0 \; \textit{for } k < a.$$

(iii) *For $\mathbf{a} \in \mathbf{Z}^{[a,b]}$, the standard module $\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})$ is contained in $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ and, in the Grothendieck ring, we have*

$$[\mathsf{P}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})] = [\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a})] + \sum_{\mathbf{a}' \prec \mathbf{a}} c(\mathbf{a}')[\mathsf{V}_{\mathfrak{D},\underline{w_0}}(\mathbf{a}')] \quad \textit{for some } c(\mathbf{a}') \in \mathbb{Z}_{\ge 0}.$$

Example 6.17. We use the same notations as in Example 5.8.

(i) We consider the affine cuspidal modules S_k given in Example 5.8 (i). Let $l \in \mathbb{Z}_{\geq 0}$. The category $\mathscr{C}_{\mathfrak{g}}^{[1,2(l+1)]}$ is determined by S_k for $k \in [1, 2(l+1)]$. It follows from

$$\{\mathsf{S}_k \mid k \in [1, 2(l+1)]\} = \{V(1)_{(-q)^{2t}}, V(2)_{(-q)^{2t+1}} \mid t \in [0, l]\}$$

that the category $\mathscr{C}_{\mathfrak{g}}^{[1,2(l+1)]}$ is equal to the Hernandez–Leclerc category $\mathscr{C}_l$ defined in [15, Section 3.8].

(ii) Let us take the affine cuspidal modules S'_k given in Example 5.8 (ii). In this case, the category $\mathscr{C}_{\mathfrak{g}}^{[a,b]}$ is not equal to $\mathscr{C}_l$ in general. From this viewpoint, the category $\mathscr{C}_{\mathfrak{g}}^{[a,b],\mathfrak{D},\underline{w_0}}$ is a generalization of the category $\mathscr{C}_l$.

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