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Simultaneous equidistribution of toric periods and fractional moments of L-functions

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Abstract. The embedding of a torus into an inner form of PGL_2 defines an adelic toric period. A general version of Duke's theorem states that this period equidistributes as the discriminant of the splitting field tends to infinity. In this paper we consider a torus embedded diagonally into two distinct inner forms of PGL_2. Assuming the Generalized Riemann Hypothesis (and some additional technical assumptions), we show simultaneous equidistribution as the discriminant tends to infinity, with an effective logarithmic rate. Our proof is based on a probabilistic approach to estimating fractional moments of L-functions twisted by extended class group characters.

Keywords. Toric periods, equidistribution, Rankin–Selberg L-functions, class group, Heegner points, sums of three squares, elliptic curves

1. Introduction

A well-known theorem of Legendre (proved by Gauß) states that a positive integer d is a sum of three integer squares precisely when d is not of the form $d = 4^a(8b + 7)$ for $a, b \in \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}^3_{\text{prim}}$ denote the subset of $\mathbb{Z}^3$ whose coordinates are relatively prime. Then the Legendre–Gauß theorem can be reformulated to state that the set

$$\mathscr{R}_d = \{x \in \mathbb{Z}^3_{\text{prim}} \mid x_1^2 + x_2^2 + x_3^2 = d\} \tag{1.1}$$

is non-empty precisely when d lies in $\mathbb{D} = \{d \in \mathbb{N} \mid d \not\equiv 0, 4, 7 \pmod 8\}$, the set of locally admissible values. Gauß proved in fact much more; quite remarkably, he established an exact formula for the cardinality $|\mathscr{R}_d|$ in terms of class numbers of quadratic orders in the number field $E = \mathbb{Q}(\sqrt{-d})$ [21, Section 291]. For example, if $\mathbb{D}_{\text{fund}} = \mathbb{D} \cap \mathbb{F}$, where $\mathbb{F}$ denotes the set of squarefree integers, then for $d > 3$ in $\mathbb{D}_{\text{fund}}$ he showed that

$$|\mathscr{R}_d| = \begin{cases} 24|\mathrm{Cl}_E|, & d \equiv 3 \pmod 8, \\ 12|\mathrm{Cl}_E|, & d \equiv 1, 2 \pmod 4, \end{cases} \tag{1.2}$$

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where Cl_E is the class group of the ring of integers of E. In particular, it follows from the work of Dirichlet and Siegel that $|\mathscr{R}_d| = d^{1/2+o(1)}$ for $d \in \mathbb{D}_{\mathrm{fund}}$ (in fact for all $d \in \mathbb{D}$).

With the issues of cardinality settled, one can then investigate the distribution of the points in $\mathscr{R}_d$. Using his ergodic method, Linnik [33] in the late 1950s proved that the projection $d^{-1/2}\mathscr{R}_d$ equidistributes on S^2 – the Euclidean sphere in $\mathbb{R}^3$ – with respect to the rotationally invariant Lebesgue probability measure m_{S^2}, for $d \to \infty$ along a sequence in $\mathbb{D}$ such that

$$-d \text{ is a non-zero square modulo } p, \tag{1.3}$$

where p is a fixed odd prime. This *Linnik condition*, as it is called, is equivalent to the splitting at p of a stabilizer subgroup of an acting orthogonal group, which in turn allows for the use of measure classification results arising from homogeneous dynamics. The rate of convergence in Linnik's proof can be made sufficiently uniform in p so as to allow for the following reformulation: the set $d^{-1/2}\mathscr{R}_d$ equidistributes on S^2 provided that $d \to \infty$ along a sequence in $\mathbb{D}$ for which there exists a prime p which splits in $E = \mathbb{Q}(\sqrt{-d})$ such that $\log p \ll \log d/\log\log d$. The existence of such small split primes is in fact known to hold under the Generalized Riemann Hypothesis (GRH) for L-functions of quadratic Dirichlet characters.

Gauß' formula (1.2) hints at an interesting structural relation between $\mathscr{R}_d$ and the class group Cl_E. In fact, it was Venkov [43] who first explicated Gauß' formula in terms of quaternion algebras, and we shall adopt this perspective. Let $\mathbf{B} = \mathbf{B}^{(2,\infty)}$ denote the rational quaternion algebra ramified at 2 and ∞. A solution $x = (x_1, x_2, x_3) \in \mathscr{R}_d$ can then be identified with a trace-zero integral quaternion $x_1 i + x_2 j + x_3 k$ of reduced norm d. When $d \in \mathbb{D}_{\mathrm{fund}}$, the choice of a base point $x_0 \in \mathscr{R}_d$ yields an optimal embedding $\iota : E \to \mathbf{B}(\mathbb{Q})$, $a + b\sqrt{-d} \mapsto a + bx_0$, relative to the maximal order $\mathscr{O}$ of Hurwitz quaternions. Since $\mathscr{O}$ is principal, if $\mathfrak{a}$ is a fractional ideal in E then the $\mathbb{Z}$-module $\iota(\mathfrak{a})\mathscr{O}$ is a principal ideal (q) in $\mathscr{O}$. Letting Γ denote the order 12 group of projective units $\mathscr{O}^\times/\{\pm 1\}$, which acts on the coordinate lines in $\mathbb{R}^3$ via even permutations, we may define an action of Cl_E on the quotient $\mathscr{R}_d^* = \Gamma\backslash\mathscr{R}_d$ by $[\mathfrak{a}] \cdot x = q^{-1}xq$. This action is free and has one or two orbits, according to the two types of congruence classes in (1.2).

The problem of equidistribution of integer points on the sphere admits many variants, such as the distribution of Heegner points on the modular curve [32], packets of closed geodesics [39] on the modular curve, as well as the supersingular reduction of CM elliptic curves [34]. We will review these examples later in Section 3. In each case, the underlying set (or "packet") of arithmetic objects admits an action by the class group of a quadratic order. From a modern perspective, what they have in common is the equidistribution of the adelic quotient $[\mathbf{T}] = \mathbf{T}(\mathbb{Q})\backslash\mathbf{T}(\mathbb{A})$ of an algebraic torus $\mathbf{T}$ of large discriminant inside the automorphic space $[\mathbf{G}] = \mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})$ of an inner form $\mathbf{G}$ of PGL_2.

Several decades after Linnik's fundamental contributions, Iwaniec [27] developed an innovative technique to bound the Fourier coefficients of half-integral weight modular forms which paved the way for removing Linnik's condition (1.3), without recourse to GRH. By extending Iwaniec's methods, Duke, in his famous paper [10] (and also Fomenko–Golubeva [22]), proved the equidistribution of $d^{-1/2}\mathscr{R}_d$ on S^2 for all

$d \in \mathbb{D}_{\text{fund}}$, with a power savings rate of convergence. This result, and the others he treated in [10], now go collectively under the name of *Duke's Theorems* and cover in particular the case of Heegner points, closed geodesics and integer points on general ellipsoids [12]. The method is very different from Linnik's ergodic approach and based on harmonic analysis and automorphic forms. The most uniform treatment uses Waldspurger's theorem on toric periods [45] and subconvex bounds on quadratic character twists of degree two L-functions, due to Duke, Friedlander, and Iwaniec [11] when the base field is $\mathbb{Q}$, and to Michel–Venkatesh [36] in general. This family of results belongs to the landmark achievements in analytic number theory in the past 30 years.

1.1. Simultaneous equidistribution

A natural generalization of the above setting, first put forward by Michel and Venkatesh in their 2006 ICM proceedings article [35], is to consider the diagonal embedding of a quadratic number field E into two distinct quaternion algebras $\mathbf{B}_1$ and $\mathbf{B}_2$. A dynamical analogue of Goursat's lemma[1] [23] would suggest that, since the projective unit groups $\mathbf{G}_1 = \mathbf{PB}_1^\times$ and $\mathbf{G}_2 = \mathbf{PB}_2^\times$ are non-isomorphic, and the adelic quotient $[\mathbf{T}]$ of the projective torus $\mathbf{T}$ defined by E equidistributes on each $[\mathbf{G}_i]$ by Duke's theorem, then the diagonal embedding of $[\mathbf{T}]$ should equidistribute on the product $[\mathbf{G}_1] \times [\mathbf{G}_2]$ as the discriminant of E gets large.

This problem was approached through ergodic theory by Einsiedler and Lindenstrauss [13], by bootstrapping Duke's theorem in each $\mathbf{G}_i$ by means of their joinings theorem. However, their method, like that of Linnik, requires an auxiliary congruence condition on the allowed set of discriminants, similar to (1.3) but with *two* auxiliary primes. This *double Linnik condition* guarantees the existence of an action by a higher rank torus, which can be shown to enjoy decisive measure rigidity properties. Arithmetic applications of the joinings theorem have been explicated in [1–3, 28], with many still to come.

There is an important quantitative distinction between Linnik's result and that of Einsiedler and Lindenstrauss. While Linnik's condition (1.3) can be removed under GRH, the joint equidistribution statements of Einsiedler and Lindenstrauss are ineffective (without a rate of convergence), and it is presently unknown whether their methods can be strengthened to allow for a replacement of the double Linnik condition by the assumption of GRH.

The main result of this paper is a proof of this conjecture conditionally on the Generalized Riemann Hypothesis (and some minor simplifying assumptions). Like Duke, we approach the problem through automorphic forms and L-functions, using Waldspurger's formula as a crucial input. Unlike his setting, subconvexity is not enough, and one must estimate fractional moments of a certain family of Rankin–Selberg L-functions. The assumption of GRH allows us to use methods in probabilistic number theory pioneered by

[1] Recall that this classical-group theoretic result states, as a special case, that if two groups G_1 and G_2 have no non-trivial isomorphic factors, then any subgroup of their product which projects surjectively onto both factors is necessarily equal to $G_1 \times G_2$.

Soundararajan [40]. The analytic heart of the paper is Theorem 3 in Section 4. A special case of this result can be rephrased as a non-trivial bound for Bessel periods of Yoshida lifts on GSp(4) (cf. Corollary 4).

We delay the full statement of our main theorems until Section 1.3, since they require substantial notational preliminaries. Instead, we provide an illustrative special case, building upon Linnik's sphere problem. More examples will be given in Section 3.

Example. We let $\mathbf{B}$ be one of the five quaternion algebras over $\mathbb{Q}$ having class number 1 (cf. [44, Theorem 25.4.1]), with discriminants 2, 3, 5, 7, or 13. Let $Q = \mathrm{Nm}|_{\mathbf{B}^0}$ be the reduced norm form Nm restricted to the trace-zero elements $\mathbf{B}^0$ of $\mathbf{B}$. Choose a maximal order $\mathscr{O}$ in $\mathbf{B}(\mathbb{Q})$ and write $\mathscr{O}^0 = \mathscr{O} \cap \mathbf{B}^0(\mathbb{Q})$. If $\mathscr{O}^0_{\mathrm{prim}}$ denotes the subset of primitive elements we put, for any $d \in \mathbb{N}$,

$$\mathscr{R}_d(Q) = \{x \in \mathscr{O}^0_{\mathrm{prim}} \mid Q(x) = d\}.$$

Let $\mathbb{D}(Q) = \{d \in \mathbb{N} \mid \mathscr{R}_d(Q) \neq \emptyset\}$ and $\mathbb{D}_{\mathrm{fund}}(Q) = \mathbb{D}(Q) \cap \mathbb{F}$. Shemanske [38] extended the method of Venkov to show that for $d \in \mathbb{D}_{\mathrm{fund}}(Q)$, with $d > 3$, the class group Cl_E of $E = \mathbb{Q}(\sqrt{-d})$ acts freely with one, two, four, or eight orbits on $\mathscr{R}^*_d(Q) = \Gamma\backslash\mathscr{R}_d(Q)$, where Γ is the group of projective units $\mathscr{O}^\times/\{\pm 1\}$.

Define the ellipse $V_Q = \{x \in \mathbf{B}_0(\mathbb{R}) \mid Q(x) = 1\}$, endowed with the probability measure m induced by assigning to any $\Omega \subset V_Q$ the Lebesgue volume of $\bigcup_{x\in\Omega}[0,1]x$. Duke and Schulze-Pillot [12] proved that $\mathscr{R}^*_d(Q)$ equidistributes on $\Gamma\backslash V_Q$ relative to the measure m as $d \to \infty$ in $\mathbb{D}_{\mathrm{fund}}(Q)$.

Now take two *distinct* quaternion algebras $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbb{Q}$ of class number 1. Fixing base points $x_i \in \mathscr{R}^*_d(Q_i)$ we can consider the joint orbit

$$\Delta\mathscr{R}^*_d(Q_1, Q_2) = \{(tx_1, tx_2) \mid t \in \mathrm{Cl}_E\} \subseteq \mathscr{R}^*_d(Q_1) \times \mathscr{R}^*_d(Q_2). \tag{1.4}$$

A special case of Theorem 1 below is the following result. *Assume the Generalized Riemann Hypothesis. Then* $d^{-1/2}\Delta\mathscr{R}^*_d(Q_1, Q_2)$ *equidistributes in* $\Gamma_1\backslash V_{Q_1} \times \Gamma_2\backslash V_{Q_2}$ *with respect to the product measure* $m_1 \times m_2$ *as* $d \to \infty$ *in* $\mathbb{D}_{\mathrm{fund}}(Q_1) \cap \mathbb{D}_{\mathrm{fund}}(Q_2)$.

1.2. The conjecture of Michel–Venkatesh

We now pass to the adelic language. To set up the stage, we wish to review the most general form of Duke's theorems, as they have been refined and generalized in recent years, most notably by Einsiedler, Lindenstrauss, Michel, and Venkatesh.

Let $\mathbf{B}$ be a quaternion algebra defined over a number field F and write $\mathbf{PB}^\times$ for its group of projective units. A *homogeneous toral subset* inside the automorphic quotient space $[\mathbf{PB}^\times] = \mathbf{PB}^\times(F)\backslash\mathbf{PB}^\times(\mathbb{A}_F)$ is a set of the form

$$[\mathbf{T}_\iota.\mathrm{g}] = \mathbf{T}_\iota(F)\backslash\mathbf{T}_\iota(\mathbb{A}_F).\mathrm{g},$$

where $\mathbf{T}_\iota \subset \mathbf{PB}^\times$ is the image under a rational embedding $\iota : \mathbf{T} \hookrightarrow \mathbf{PB}^\times$ of an anisotropic algebraic torus over F and $\mathrm{g} \in \mathbf{PB}^\times(\mathbb{A}_F)$. One can associate with $[\mathbf{T}_\iota.\mathrm{g}]$ two important objects:

(1) a positive number D called the *discriminant*, defined in [15, §4.2], which encodes the arithmetic complexity of $[\mathbf{T}_\iota.\mathrm{g}]$;

(2) a $\mathrm{g}^{-1}\mathbf{T}_\iota(\mathbb{A}_F)\mathrm{g}$-invariant probability measure μ on $[\mathbf{T}_\iota.\mathrm{g}]$, given by the pushforward under $t \mapsto \iota(t)\mathrm{g}$ of the normalized Haar measure on $\mathbf{T}_\iota(F)\backslash\mathbf{T}_\iota(\mathbb{A}_F)$.

Now let $[\mathbf{T}_{\iota_n}.\mathrm{g}_n]$ be a sequence of homogeneous toral subsets, with associated probability measures μ_n, such that $D_n \to \infty$. The following theorem, which was stated in [15, Theorem 4.6], gives a minimal set of conditions under which $[\mathbf{T}_{\iota_n}.\mathrm{g}_n]$ equidistributes with respect to a convex combination of homogeneous probability measures on $[\mathbf{PB}^\times]$.

Theorem. *Any weak-* limit of μ_n is a homogeneous probability measure on $[\mathbf{PB}^\times]$ invariant under $\mathbf{PB}^\times(\mathbb{A}_F)^+$. Here, $\mathbf{PB}^\times(\mathbb{A}_F)^+$ is the image of $\mathbf{B}^{(1)}(\mathbb{A}_F) \to \mathbf{PB}^\times(\mathbb{A}_F)$, where $\mathbf{B}^{(1)}$ is the simply connected cover of $\mathbf{PB}^\times$. Moreover, the rate of convergence can be quantified, with an error term of the form $O(D_n^{-\delta})$ for some $\delta > 0$.*

In their 2006 ICM address, Michel and Venkatesh [35, §2.3, §6.4.1] considered the following simultaneous equidistribution problem. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two non-isomorphic quaternion algebras over F. For $j = 1, 2$ let $\mathbf{G}_j = \mathbf{PB}_j^\times$, and let $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$. Let $\mathbf{T}$ be an anisotropic algebraic torus over F equipped with rational embeddings $\iota_j : \mathbf{T} \hookrightarrow \mathbf{G}_j$ for $j = 1, 2$. We may define a *diagonal homogeneous toral subset* to be the subset of the product space $[\mathbf{G}] = [\mathbf{G}_1] \times [\mathbf{G}_2]$ given by

$$[\Delta\mathbf{T}_\iota.\mathrm{g}] = \Delta\mathbf{T}_\iota(F)\backslash\Delta\mathbf{T}_\iota(\mathbb{A}_F).\mathrm{g},$$

where $\Delta\mathbf{T}_\iota \subset \mathbf{G}$ is the image under the diagonal embedding $\iota = (\iota_1, \iota_2) : \mathbf{T} \hookrightarrow \mathbf{G}$ and $\mathrm{g} \in \mathbf{G}(\mathbb{A}_F)$. As before, one may associate with $[\Delta\mathbf{T}_\iota.\mathrm{g}]$ its discriminant $D = \min(D_1, D_2)$ and a natural probability measure $\Delta\mu$ on $[\mathbf{G}]$. Let $\mathbf{G}(\mathbb{A}_F)^+ = \mathbf{G}_1(\mathbb{A}_F)^+ \times \mathbf{G}_2(\mathbb{A}_F)^+$.

Conjecture (Michel–Venkatesh). *Let $[\Delta\mathbf{T}_{\iota_n}.\mathrm{g}_n]$ be a sequence of diagonal homogeneous toral subsets satisfying $D_n \to \infty$. Then any weak-* limit of $\Delta\mu_n$ is a homogeneous probability measure on $[\mathbf{G}]$ invariant under $\mathbf{G}(\mathbb{A})^+$.*

By choosing a finite level structure $K_f \subset \mathbf{G}(\mathbb{A}_f)$ and a compact subgroup $K_\infty \subset \mathbf{G}(\mathbb{R})$, we can reinterpret the above conjecture more classically in the double quotient space

$$[\mathbf{G}]_K = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)/K \quad (K = K_f K_\infty)$$

by considering the distribution of the images

$$[\Delta\mathbf{T}_\iota.\mathrm{g}]_K = \mathbf{G}(F)\Delta\mathbf{T}_\iota(\mathbb{A}_F)\mathrm{g}K$$

of $[\Delta\mathbf{T}_\iota.\mathrm{g}]$ under the natural projection $[\mathbf{G}] \to [\mathbf{G}]_K$. We shall call these projections *diagonal packets*, extending the terminology [15] to this setting.

1.3. Main results

We are now ready to state our principal result. In it, we establish the conjecture of Michel–Venkatesh for diagonal packets, under the assumption (most notably) of the Generalized

Riemann Hypothesis. To simplify the presentation, we have made further restrictions, including maximal level structure, optimal embeddings, and the number field $\mathbb{Q}$. Moreover, we test convergence only against functions in the discrete spectrum, an assumption only relevant if one of $\mathbf{B}_j$ is the matrix algebra (for more on this assumption, see Remark 4). More precisely, we give ourselves the following data.

Let $\mathbf{B}_1, \mathbf{B}_2$ be non-isomorphic quaternion algebras over $\mathbb{Q}$ and write $\mathbf{G}_j = \mathbf{PB}_j^\times$ for $j = 1, 2$. Let $\mathscr{O}_j$ be a maximal order in $\mathbf{B}_j(\mathbb{Q})$. For a prime p let $\mathbf{G}_j(\mathbb{Z}_p)$ denote the projective unit group of the local maximal order $\mathscr{O}_{j,p} = \mathscr{O}_j \otimes \mathbb{Z}_p$. Let $\mathbf{G}_j(\widehat{\mathbb{Z}}) = \prod_p \mathbf{G}_j(\mathbb{Z}_p)$. Let $\mathbf{G}_j(\mathbb{A}_\mathbb{Q})$ denote the adelic points relative to the subgroups $\mathbf{G}_j(\mathbb{Z}_p)$. Let $K_{\infty,j} \subset \mathbf{G}_j(\mathbb{R})$ be either

- $\mathbf{G}_j(\mathbb{R})$ itself if $\mathbf{B}_j \otimes_\mathbb{Q} \mathbb{R}$ is non-split, or
- a maximal compact torus if $\mathbf{B}_j \otimes_\mathbb{Q} \mathbb{R}$ is split or non-split.

When $\mathbf{B}_j$ is split at infinity, we fix an isomorphism of $\mathbf{B}_j(\mathbb{R})$ with $M_2(\mathbb{R})$ which induces an isomorphism of $\mathbf{G}_j^\times(\mathbb{R})$ with $\mathrm{PGL}_2(\mathbb{R})$ sending $K_{\infty,j}$ to PSO(2).

Let E be a quadratic field extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_E$. Let $\iota_j : E \hookrightarrow \mathbf{B}_j$ be an optimal embedding of $\mathcal{O}_E$ into some maximal order $\mathbb{O}_j$ of $\mathbf{B}_j(\mathbb{Q})$, depending on ι_j. This induces optimal local embeddings at each prime p, in the following sense. Let v be a finite place of E lying over p. Let E_v be the v-adic completion of E (a quadratic étale algebra) and write $\mathcal{O}_{E,v}$ for its maximal order. Let $\mathbb{O}_{j,p} = \mathbb{O}_j \otimes \mathbb{Z}_p$. Then $\iota_j(\mathcal{O}_{E_v}) = \iota_j(E_v) \cap \mathbb{O}_{j,p}$. Since $\mathscr{O}_j$ is everywhere locally isomorphic to $\mathbb{O}_j$, there are $g_{j,p} \in \mathbf{G}_j(\mathbb{Q}_p)$ such that $\mathscr{O}_{j,p} = g_{j,p}\mathbb{O}_{j,p}g_{j,p}^{-1}$. In this way we obtain

$$\iota_j(\mathcal{O}_{E_v}) = \iota_j(E_v) \cap g_{j,p}^{-1}\mathscr{O}_{j,p}g_{j,p}. \tag{1.5}$$

Let $g_{j,f} = (g_{j,p})_p \in \mathbf{G}_j(\mathbb{A}_f)$. Let $\mathbf{T} = (\mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_\mathrm{m})/\mathbb{G}_\mathrm{m}$. Then ι_j induces an embedding $\iota_j : \mathbf{T} \hookrightarrow \mathbf{G}_j$ which, in view of (1.5), satisfies

$$\iota_j(\mathbf{T}(\widehat{\mathbb{Z}})) = \iota_j(\mathbf{T}(\mathbb{A}_f)) \cap g_{j,f}^{-1}\mathbf{G}_j(\widehat{\mathbb{Z}})g_{j,f}. \tag{1.6}$$

We write $\mathbf{T}_{\iota_j} = \iota_j(\mathbf{T}) \subset \mathbf{G}_j$. Assume furthermore that $g_{j,\infty} \in \mathbf{G}_j(\mathbb{R})$ is such that

$$\begin{cases} g_{j,\infty}\mathbf{T}_{\iota_j}(\mathbb{R})g_{j,\infty}^{-1} \subset K_{\infty,j} \text{ when } \mathbf{T}(\mathbb{R}) \text{ is anisotropic;} \\ g_{j,\infty}\mathbf{T}_{\iota_j}(\mathbb{R})g_{j,\infty}^{-1} \text{ is, under the fixed identification of } \mathbf{G}_j(\mathbb{R}) \text{ with } \mathrm{PGL}_2(\mathbb{R}) \text{ above,} \\ \qquad \text{the group of projective diagonal matrices when } \mathbf{T}(\mathbb{R}) \text{ is isotropic.} \end{cases} \tag{1.7}$$

Put $g_j = (g_{j,f}, g_{j,\infty}) \in \mathbf{G}_j(\mathbb{A}_\mathbb{Q})$.

Put $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$ and let dg be the right $\mathbf{G}(\mathbb{A}_\mathbb{Q})$-invariant probability measure. Let $\mathbf{G}(\widehat{\mathbb{Z}}) = \mathbf{G}_1(\widehat{\mathbb{Z}}) \times \mathbf{G}_2(\widehat{\mathbb{Z}})$ and $K_\infty = K_{\infty,1} \times K_{\infty,2}$. Put $K = \mathbf{G}(\widehat{\mathbb{Z}})K_\infty$. Let $g = (g_1, g_2) \in \mathbf{G}(\mathbb{A}_\mathbb{Q})$. Let $[\Delta\mathbf{T}_\iota.g]$ be a diagonal homogeneous toral subset in $[\mathbf{G}]$, where $\iota = (\iota_1, \iota_2) : \mathbf{T} \hookrightarrow \mathbf{G}$. Then $[\Delta\mathbf{T}_\iota.g]$ is endowed with its invariant probability measure $\Delta\mu$, which we shall write simply as dt. Denote by D the discriminant of the packet $[\Delta\mathbf{T}_\iota.g]_K$. Conditions (1.5) and (1.7), together with suitable choices of archimedean metric normalizations, imply that D is the absolute value of the discriminant of E; see Section 2.2 for details.

Let $C_c^\infty([\mathbf{G}]_K)$ denote the space of right K-invariant compactly supported functions $f : [\mathbf{G}] \to \mathbb{C}$ such that $g \mapsto f(xg)$ is in $C_c^\infty(\mathbf{G}(\mathbb{R}))$ for every $x \in [\mathbf{G}]$. Let

$$C_{c,\mathrm{disc}}^\infty([\mathbf{G}]_K) = C_c^\infty([\mathbf{G}]_K) \cap L^2_{\mathrm{disc}}([\mathbf{G}]_K).$$

We have $C_{c,\mathrm{disc}}^\infty([\mathbf{G}]_K) = C_c^\infty([\mathbf{G}]_K)$ unless one of $\mathbf{G}_j$ is PGL_2. For $f \in C_c^\infty([\mathbf{G}]_K)$ let $\mathcal{S}_{\infty,d}(f) = \sum_{\mathrm{ord}(\mathcal{D}) \leq d} \|\mathcal{D} f\|_{L^\infty([\mathbf{G}]_K)}$, where $\mathcal{D}$ runs over monomials of degree at most d in a fixed basis of $\mathrm{Lie}(\mathbf{G}(\mathbb{R}))$.

Theorem 1. *Let the notations be as above. Then,* under the Generalized Riemann Hypothesis, *the diagonal packets $[\Delta\mathbf{T}_\iota.\mathrm{g}]_K$ equidistribute on $[\mathbf{G}]_K$ relative to $C_{c,\mathrm{disc}}^\infty([\mathbf{G}]_K)$, with an effective rate of convergence of the form $O_\varepsilon((\log D)^{-1/4+\varepsilon})$ as $D \to \infty$. More precisely, there is $d \in \mathbb{N}$ such that for every $f \in C_{c,\mathrm{disc}}^\infty([\mathbf{G}]_K)$, and every $\varepsilon > 0$, we have*

$$\int_{[\Delta\mathbf{T}_\iota]} f(t\mathrm{g})\, dt = \int_{[\mathbf{G}]} f(g)\, dg + O_\varepsilon\big(\mathcal{S}_{\infty,d}(f)(\log D)^{-1/4+\varepsilon}\big).$$

A variety of situations in which our theorem applies will be given in Section 3. Our effective error term also allows applications to equidistribution on (very slowly) shrinking subsets of $[\mathbf{G}]_K$.

We may also prove an equidistribution statement in the case $\mathbf{B}_1 = \mathbf{B}_2$ (or otherwise) if we twist the diagonal embedding by allowing each component embedding to travel through $\mathbf{T}$ with different speeds:

$$\iota_{\alpha,\beta} : \mathbf{T} \to \mathbf{PB}_1^\times \times \mathbf{PB}_2^\times, \quad t \mapsto (\iota_1(t)^\alpha, \iota_2(t)^\beta),$$

where $\alpha, \beta \in \mathbb{N}$ are distinct integers. Let $\Delta_{\alpha,\beta}\mathbf{T}_\iota$ denote the image of $\mathbf{T}$ under $\iota_{\alpha,\beta}$ and write $[\Delta_{\alpha,\beta}\mathbf{T}_\iota.\mathrm{g}]_K$ for the image of $\Delta_{\alpha,\beta}\mathbf{T}_\iota(\mathbb{Q})\backslash\Delta_{\alpha,\beta}\mathbf{T}_\iota(\mathbb{A}_\mathbb{Q}).\mathrm{g}$ in $[\mathbf{G}]_K$.

Theorem 2. *Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be quaternion algebras over $\mathbb{Q}$ (not necessarily distinct), and otherwise keep the assumptions and notations of Theorem 1. Let $\alpha, \beta \in \mathbb{N}$ be two distinct integers. Suppose that the class group Cl_E of the field E associated with the torus $\mathbf{T}$ has no p-torsion, for all $p \mid 2\alpha\beta$. Then,* under the Generalized Riemann Hypothesis, *there is $d \in \mathbb{N}$ such that for every $f \in C_{c,\mathrm{disc}}^\infty([\mathbf{G}])$, and every $\varepsilon > 0$, we have*

$$\int_{[\Delta_{\alpha,\beta}\mathbf{T}_\iota]} f(t\mathrm{g})\, dt = \int_{[\mathbf{G}]} f(g)\, dg + O_{\varepsilon,\alpha,\beta}\big(\mathcal{S}_{\infty,d}(f)(\log D)^{-1/4+\varepsilon}\big).$$

Several assumptions in Theorems 1 and 2 can be relaxed, using the same methods, but at the cost of a greater technical effort. For instance, instead of maximal orders one can take Eichler orders (and even more general orders, such as in [3, Example 10.5]), the difficulty being in treating oldforms. The assumption in Theorem 2 on the 2β torsion can also be relaxed, for instance it would suffice for its cardinality to be bounded independently of D; cf. Section 4.2 and Remark 6. Note that if E is real, a conjecture of Gauß (quantified by Hooley [26]) says that for "many" discriminants the group Cl_E is trivial, in particular torsion-free.

1.4. Beyond sparse equidistribution

As discussed in the opening paragraphs, the modern approach to proving Duke's theorem passes through Waldspurger's formula, which relates the square of the toric period of an automorphic form to an associated L-function. A separate problem is then to prove subconvex bounds on these L-functions, a time-honored subject in analytic number theory. In fact, one can consider a natural refinement of Duke's theorem, where one seeks to prove the equidistribution of the orbit of a subgroup of the class group Cl_E of large enough index. This type of problem has been referred to in the literature as *sparse equidistribution* [42], and it is solved by again appealing to Waldspurger's formula and subconvex bounds, this time on L-functions twisted by class group characters [34].

A fundamental property underlying the proof of Waldspurger's formula is that the subgroup pair $(\mathbf{T}, \mathbf{PB}^\times)$ is a strong Gelfand pair. This is not the case with $(\Delta\mathbf{T}, \mathbf{PB}_1^\times \times \mathbf{PB}_2^\times)$, as the diagonal torus $\Delta\mathbf{T}$ is too small relative to the product group $\mathbf{PB}_1^\times \times \mathbf{PB}_2^\times$, and one no longer expects the corresponding diagonal period to be directly related to a single L-function. Following Bernstein and Reznikov (see [37] for an overview) one can nevertheless form the following *Gelfand formation*:

$$\begin{array}{c} \mathbf{PB}_1^\times \times \mathbf{PB}_2^\times \\ | \\ \mathbf{T} \times \mathbf{T} \\ | \\ \Delta\mathbf{T} \end{array}$$

in which the intermediate subgroup pairs $(\Delta\mathbf{T}, \mathbf{T} \times \mathbf{T})$ and $(\mathbf{T} \times \mathbf{T}, \mathbf{PB}_1^\times \times \mathbf{PB}_2^\times)$ are strong Gelfand pairs. In such a situation the diagonal period should be related to a *family of twisted L-functions*. We establish this link more precisely in Section 4.

The equidistribution problems of Theorems 1 and 2 therefore go beyond even the sparse equidistribution refinements of Duke's theorem. From this perspective it is perhaps less surprising that one should need the deeper statistical information provided by GRH, which we take as a working assumption. The analytic tools we develop for families of L-functions, as expressed in our main analytic-number-theoretic result, Theorem 3, should (we believe) provide a new paradigm for treating equidistribution problems in the absence of direct period formulae.

1.5. The plan of the paper

In Section 2 we take some time to translate the content of Sections 1.2 and 1.3 into classical language and in particular discuss general versions of Duke's theorem. This prepares the ground to give classical applications of Theorems 1 and 2 in Section 3:

- on simultaneous equidistribution on pairs of quadrics,
- on simultaneous equidistribution by genus classes,
- on simultaneous supersingular reduction of CM elliptic curves.

Section 4 uses Waldspurger's theorem and Parseval to reduce the proof of Theorems 1 and 2 to the proof of Theorem 3, a mean value estimate for fractional moments of twisted L-functions. It offers an independent consequence, stated as Corollary 4, on Bessel periods of Yoshida lifts. Before we start with the proof of Theorem 3, we give a heuristic argument in Section 5. Section 6 compiles general results on L-functions. The combinatorial input of the proof of Theorem 3 is provided in Section 7, while the analytic input is the content of Sections 8 and 9.

2. Converting from adelic to classical language

We begin by converting from the adelic language in which we have expressed Theorems 1 and 2 to more classical language. Throughout this section we shall give ourselves only *one* quaternion algebra; concrete examples of Theorems 1 and 2 with two (distinct) quaternion algebras will be given in Section 3.

The following notation will be in place: let $\mathbf{B}$ be a quaternion algebra over $\mathbb{Q}$. Let $\mathbf{PB}^\times$ be its group of projective units. Let $\mathscr{O} \subset \mathbf{B}(\mathbb{Q})$ be a maximal order and write $\mathbf{PB}^\times(\mathbb{Z})$, resp. $\mathbf{PB}^\times(\widehat{\mathbb{Z}})$ for the projective unit group of $\mathscr{O}$, resp. $\widehat{\mathscr{O}} = \mathscr{O} \otimes \widehat{\mathbb{Z}}$. Write $K = \mathbf{PB}^\times(\widehat{\mathbb{Z}})K_\infty$, where K_∞ is a maximal compact torus of $\mathbf{PB}^\times(\mathbb{R})$. (The case of $\mathbf{B}$ definite and $K_\infty = \mathbf{PB}^\times(\mathbb{R})$, which is allowed in the setting of the main theorem in Section 1.3, will be excluded in this expository section. We will, however, discuss important examples in Section 3, where this case admits a lovely arithmetic interpretation.)

2.1. *Viewing* $[\mathbf{T}]_K$ *classically*

Let E be quadratic field extension of $\mathbb{Q}$, with ring of integers $\mathcal{O}_E$, which is not split wherever $\mathbf{B}$ is ramified. This condition assures that E embeds into $\mathbf{B}(\mathbb{Q})$ [44, Prop. 14.6.7]. Let $\mathbf{T} = (\mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_\mathrm{m})/\mathbb{G}_\mathrm{m}$ and let $\iota : \mathbf{T} \hookrightarrow \mathbf{PB}^\times$ be an optimal embedding of $\mathcal{O}_E$ into $\mathbb{O}_\iota$. Denote by $K_{\mathbf{T}_\iota,\infty}$ the maximal compact subgroup of $\mathbf{T}_\iota(\mathbb{R})$.

The map
$$\mathbf{T}_\iota(\mathbb{A}_\mathbb{Q}) \to [\mathbf{T}_\iota.\mathrm{g}]_K, \quad t \mapsto \mathbf{PB}^\times(\mathbb{Q})tgK,$$
induces a bijection $\mathbf{T}_\iota(\mathbb{Q})\backslash\mathbf{T}_\iota(\mathbb{A}_\mathbb{Q})/(\mathbf{T}_\iota(\mathbb{A}_\mathbb{Q}) \cap \mathrm{g}K\mathrm{g}^{-1}) \xrightarrow{\sim} [\mathbf{T}_\iota.\mathrm{g}]_K$. By (1.6) and the hypothesis (which can be deduced from (1.7)) that $\mathbf{T}_\iota(\mathbb{R}) \cap g_\infty^{-1}K_\infty g_\infty = K_{\mathbf{T}_\iota,\infty}$, the preceding adelic double quotient may be rewritten as $\mathbf{T}_\iota(\mathbb{Q})\backslash\mathbf{T}_\iota(\mathbb{A}_\mathbb{Q})/\iota(\mathbf{T}(\widehat{\mathbb{Z}}))K_{\mathbf{T}_\iota,\infty}$.

We now observe that the latter group can naturally be identified with the *Arakelov class group* $\widetilde{\mathrm{Cl}}_E$, in the sense of [18], of the ring of integers $\mathcal{O}_E$ of E. Indeed, we put $\widetilde{\mathbf{T}} = \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_\mathrm{m}$ and let $K_{\widetilde{\mathbf{T}},\infty}$ denote the maximal compact subgroup of $\widetilde{\mathbf{T}}(\mathbb{R})$ and recall that
$$\widetilde{\mathrm{Cl}}_E = E^\times\backslash\mathbb{A}_E^\times/\widehat{\mathcal{O}}_E^\times K_{\widetilde{\mathbf{T}},\infty}$$
is the usual class group Cl_E of E if E is imaginary, and the extension of Cl_E by the circle $\mathbb{R}^\times\backslash E_\infty^\times/\mathcal{O}_E^\times$ if E is real. We see that

$$\widetilde{\mathrm{Cl}}_E = \widetilde{\mathbf{T}}(\mathbb{Q})\backslash\widetilde{\mathbf{T}}(\mathbb{A}_\mathbb{Q})/\widetilde{\mathbf{T}}(\widehat{\mathbb{Z}})K_{\widetilde{\mathbf{T}},\infty} \xrightarrow{\sim} \mathbf{T}_\iota(\mathbb{Q})\backslash\mathbf{T}_\iota(\mathbb{A}_\mathbb{Q})/\iota(\mathbf{T}(\widehat{\mathbb{Z}}))K_{\mathbf{T}_\iota,\infty} \cong [\mathbf{T}_\iota.g]_K, \quad (2.1)$$

the middle arrow being an isomorphism, since the kernel is $\mathbb{Q}^\times\backslash\mathbb{A}_f^\times/\widehat{\mathbb{Z}}^\times$ and $\mathbb{Q}$ has class number 1.

We now put a finite measure on $\widetilde{\mathrm{Cl}}_E$ whose volume behaves regularly in the discriminant of E. If E is real, we have

$$1 \to \mu_2 \to \mathbf{T}(\mathbb{R}) = \mathbb{R}^\times\backslash E_\infty^\times \xrightarrow{\log|x_1/x_2|} \mathbb{R} \to 1$$

and $\mathcal{O}_E^\times \simeq \mu_2 \times \langle \log\epsilon\rangle$, where $\epsilon > 1$ is a totally positive fundamental unit. We deduce an isomorphism between $\mathbf{T}(\mathbb{R})/\mathcal{O}_E^\times$ and $\mathbb{R}/\langle\log\epsilon\rangle$, with which we transport the Lebesgue measure on the circle of arclength $\log\epsilon$ to $\mathbf{T}(\mathbb{R})/\mathcal{O}_E^\times$. Using the counting measure on Cl_E we obtain a measure on $\widetilde{\mathrm{Cl}}_E$ with total volume

$$\mathrm{vol}(\widetilde{\mathrm{Cl}}_E) = \begin{cases} |\mathrm{Cl}_E|, & E \text{ imaginary quadratic},\\ |\mathrm{Cl}_E|\log\epsilon, & E \text{ real quadratic}.\end{cases}$$

Let η_E be the quadratic character of conductor D associated to $E/\mathbb{Q}$ by class field theory. By the Dirichlet class number formula we have

$$\mathrm{vol}(\widetilde{\mathrm{Cl}}_E) = cL(1,\eta_E)D^{1/2}, \tag{2.2}$$

where $c > 0$ is an absolutely bounded constant depending only on the signature of E at infinity and the number of roots of unity of $\mathcal{O}_E$.

Since $\widetilde{\mathrm{Cl}}_E$ is compact, its dual $\widetilde{\mathrm{Cl}}_E^\vee$ is discrete: finite and equal to $\mathrm{Cl}_E^\vee$ if E is imaginary, and infinite and isomorphic to $\mathrm{Cl}_E^\vee \times \mathbb{Z}$ if E is real.

2.2. *Viewing the discriminant D classically*

Next we show that the discriminant D of the homogeneous torus subset $[\mathbf{T}_\iota.\mathrm{g}]$ is the absolute value of the discriminant of E. We shall use an equivalent description of $D = \prod_v D_v$ from [15, §4.2] as described in [28, §2.4.4].

The discriminant at finite places. In this case D_v is the discriminant of the maximal quadratic order $\iota(\mathcal{O}_{E,v}) = \iota(E_v) \cap \mathrm{g}_v^{-1}\mathscr{O}_v\mathrm{g}_v$ inside $\iota(E_v)$, where we have used the optimality assumption (1.5). The latter discriminant is equal to the discriminant of $\mathcal{O}_{E,v}$ inside E_v.

The discriminant at the archimedean place.

When $\mathbf{B}(\mathbb{R})$ *is indefinite.* When $\mathbf{B}(\mathbb{R})$ is indefinite, we follow [15, §6.1], which explicates the case of $\mathrm{PGL}_n(\mathbb{R})$. We use the fixed isomorphism of $\mathbf{PB}^\times(\mathbb{R})$ with $\mathrm{PGL}_2(\mathbb{R})$ from Section 1.3 to identify the Lie algebra of $\mathbf{PB}^\times(\mathbb{R})$ with the quotient $\mathfrak{g} = M_2(\mathbb{R})/\mathbb{R}$. Let $\|\cdot\|_\infty^2$ denote the norm on $\mathfrak{g}$ which descends from the norm $\mathrm{tr}(X^2)/2$ on $M_2(\mathbb{R})$.

For any quadratic étale subalgebra $\tilde{\mathfrak{h}}$ of $M_2(\mathbb{R})$ let $\{1,\tilde{f}\}$ be an $\mathbb{R}$-basis for $\tilde{\mathfrak{h}}$ which is orthonormal with respect to $\|\cdot\|_\infty$. Let $\mathfrak{h}$ be the image of $\tilde{\mathfrak{h}}$ in $\mathfrak{g}$ and f the image of $\tilde{f}$ in $\mathfrak{g}$. Then we put $D_\infty(\mathfrak{h}) = \|f\|_\infty^{-2}$. Note that when $\mathfrak{h} = \mathfrak{a}$ is the diagonal subalgebra of $\mathfrak{g}$ or when $\mathfrak{h} = \mathfrak{k}$ is the Lie algebra of PSO(2), then $D_\infty(\mathfrak{h}) = 1$.

Following [15, p. 841] we put $D_\infty = D_\infty(\mathrm{g}_\infty \iota(E_\infty)\mathrm{g}_\infty^{-1})$. From (1.7), and the above remark, we see that $D_\infty = 1$.

When $\mathbf{B}(\mathbb{R})$ *is definite.* When $\mathbf{B}(\mathbb{R})$ is definite we follow the discussion in [28, p. 166]. We let $\|\cdot\|_\infty^2$ be the reduced norm Nm on $\mathbf{B}(\mathbb{R})$ and write

$$\mathscr{O}_\infty = \{g \in \mathbf{B}(\mathbb{R}) \mid \|g\|_\infty \leq 1\}.$$

We fix the volume form vol_∞ on $\mathrm{g}_\infty \iota(E_\infty)\mathrm{g}_\infty^{-1}$ induced by the metric $|\cdot|_\infty$ for which $\mathrm{g}_\infty \mathbf{T}(\mathbb{R})\mathrm{g}_\infty^{-1}$ acts by isometries, and normalized so that the unit disc is of volume 1. Let

$$\Lambda_\infty = \mathrm{g}_\infty \iota(E_\infty)\mathrm{g}_\infty^{-1} \cap \mathscr{O}_\infty.$$

Then we set $D_\infty = \mathrm{vol}_\infty(\Lambda_\infty)^2$.

We now calculate D_∞, given the above choice of data. The essential point is that K_∞ preserves $\|\cdot\|_\infty$, since it sits inside the projective image in $\mathbf{PB}^\times(\mathbb{R})$ of $\mathscr{O}_\infty^\times = \{g \in \mathbf{B}^\times(\mathbb{R}) \mid \|g\|_\infty = 1\}$, which acts by conjugation on $\mathbf{B}(\mathbb{R})$ as the full group of orientation preserving isometries. From this it follows, using (1.7), that $\mathrm{g}_\infty \mathbf{T}(\mathbb{R})\mathrm{g}_\infty^{-1}$ preserves $\|\cdot\|_\infty$, so that the restriction of $\|\cdot\|_\infty$ to $\mathrm{g}_\infty \iota(E_\infty)\mathrm{g}_\infty^{-1}$ is $|\cdot|_\infty$. Thus Λ_∞ is the unit disc for $|\cdot|_\infty$, proving $D_\infty = 1$.

2.3. *Viewing Duke's theorems classically*

We now explicate Duke's theorems, converting from the adelic language of Section 1.2 to the classical arithmetic setting of integral points on (unions of) quadrics, as in the papers of Linnik [32] and Skubenko [39]. This will be helpful for generating examples of Theorems 1 and 2 in the next section.

Let $Q = \mathrm{Nm}|_{\mathbf{B}^0}$ be the restriction of the reduced norm Nm to the trace zero quaternions $\mathbf{B}^0$. For a non-zero integer d let $\mathbf{X}_{Q,d}$ denote the level set $\{x \in \mathbf{B}^0 : Q(x) = d\}$. This is an affine $\mathbf{PB}^\times$-variety, under the action of conjugation.

Let $\mathbb{D}(Q)$ denote the set of non-zero integers which are everywhere locally integrally represented by Q, and let $\mathbb{D}_{\mathrm{fund}}(Q) = \mathbb{D}(Q) \cap \mathbb{F}$. For $d \in \mathbb{D}_{\mathrm{fund}}(Q)$ we let $E = \mathbb{Q}(\sqrt{-d})$ and choose an optimal embedding ι of $\mathcal{O}_E$ into a maximal order $\mathbb{O}$ of $\mathbf{B}$, as in Section 1.3. Since ι preserves the trace and the norm, the point

$$x_0 = \iota(\sqrt{-d})$$

lies in $\mathbf{X}_{Q,d}(\mathbb{Q})$. Note that the stabilizer of x_0 in $\mathbf{PB}^\times$ consists of (projective) invertible elements of the form $a + bx_0$; this stabilizer is then seen to be $\mathbf{T}_\iota = \iota(\mathbf{T})$, where $\mathbf{T} = (\mathrm{Res}_{E/\mathbb{Q}}\, \mathbb{G}_{\mathrm{m}})/\mathbb{G}_{\mathrm{m}}$. By Witt's theorem we have $\mathbf{X}_{Q,d}(\mathbb{Q}) = \mathbf{PB}^\times(\mathbb{Q}).x_0$, so that $\mathbf{X}_{Q,d}(\mathbb{Q})$ is identified with the quotient $\mathbf{PB}^\times(\mathbb{Q})/\mathbf{T}_\iota(\mathbb{Q})$ through the orbit map on x_0.

We shall be interested in the distribution of the *integral points* of $\mathbf{X}_{Q,d}$. To this end, let $\mathbf{PB}^\times(\mathbb{Q})\backslash\mathbf{PB}^\times(\mathbb{A}_f)/\mathbf{PB}^\times(\widehat{\mathbb{Z}})$ be the *class set* of $\mathbf{PB}^\times$, for which we fix representatives $\{b_1 = [e], \ldots, b_h\}$. Then $\{\mathbb{O}_i = b_i \mathbb{O} b_i^{-1} \cap \mathbf{B}(\mathbb{Q})\}$ forms a complete set of representatives for the $\mathbf{PB}^\times(\mathbb{Q})$-conjugacy classes of maximal orders of $\mathbf{B}(\mathbb{Q})$.

Define the lattice $\Lambda_i = \mathbb{O}_i \cap \mathbf{B}^0(\mathbb{Q})$ and let $d_i = \operatorname{disc}(\Lambda_i)$ be its discriminant. The restriction of $d_i.Q$ to Λ_i yields integral ternary quadratic forms q_i, forming a full $\mathbf{PB}^\times$-genus class [44, Ch. 22], which we denote by $\operatorname{Gen}(Q)$. Let $\mathscr{R}_d(q_i) = \mathbf{PB}^\times(\mathbb{Q}).x_0 \cap \Lambda_i$ be the Λ_i-integral points of $\mathbf{X}_{Q,d}$, the set of all Λ_i-integral representations of d by q_i. Considering them all together yields

$$\mathscr{R}_d(\mathrm{gen}_Q) = \coprod_{q_i \in \operatorname{Gen}(Q)} \mathscr{R}_d(q_i).$$

The *equidistribution problems of Linnik's type* are the study of the distribution of the projection $|d|^{-1/2}\mathscr{R}_d(\mathrm{gen}_Q)$ on the union of quadrics

$$\coprod_{q_i \in \operatorname{Gen}(Q)} \mathbf{X}_{q_i,\operatorname{sgn}(d)}(\mathbb{R}) \tag{2.3}$$

as $d \to \pm\infty$ in $\mathbb{D}_{\mathrm{fund}}(Q)$, where $\operatorname{sgn}(d)$ is $+1$ or -1 according to the sign of d. To formulate this precisely one must prescribe the relevant measures. The quadric $\mathbf{X}_{Q,\operatorname{sgn}(d)}(\mathbb{R})$ has a unique (up to non-zero scaling) $\mathbf{PB}^\times(\mathbb{R})$-invariant measure coming from its structure as a homogeneous space for $\mathbf{PB}^\times(\mathbb{R})$. When Q is indefinite, $\mathbf{X}_{Q,\operatorname{sgn}(d)}(\mathbb{R})$ is non-compact and the measure will then be infinite. In this case, we let $\mu_{Q,\operatorname{sgn}(d)}$ be any such choice, and equidistribution relative to this measure is taken in the sense of [35, (1.1)]. In the definite case we let $\mu_{Q,\operatorname{sgn}(d)}$ be the probability measure which assigns to the ith copy of $\mathbf{X}_{Q,\operatorname{sgn}(d)}(\mathbb{R})$ the $\mathbf{PB}^\times(\mathbb{R})$-invariant measure with volume $|\operatorname{Aut}(q_i)|^{-1}$.

Then Duke's theorem [10, 12] states that $|d|^{-1/2}\mathscr{R}_d(\mathrm{gen}_Q)$ equidistributes on (2.3) relative to $\mu_{Q,\operatorname{sgn}(d)}$ as $d \to \pm\infty$ in $\mathbb{D}_{\mathrm{fund}}(Q)$. In particular, every large enough $d \in \mathbb{D}_{\mathrm{fund}}(Q)$ is integrally represented by every genus member, solving (up to issues of effectivity) the last remaining case of Hilbert's 11th problem over $\mathbb{Q}$ (see [5] for the number field case).

2.4. *Finiteness of equivalence classes*

In this subsection we explicate the action of the class group of $\mathbf{T}_\iota$ on certain equivalence classes of integral representations $\mathscr{R}_d(\mathrm{gen}_Q)$.

We write $\Gamma = \mathbf{PB}^\times(\mathbb{Z})$. Then $\operatorname{Gen}(\Gamma) = \{\Gamma_1 = \Gamma, \ldots, \Gamma_h\}$, where Γ_i denotes the projective units of the maximal order $\mathscr{O}_i$, is the *genus class* of Γ. Since Λ_i is stable under the action of Γ_i by conjugation, we may form the quotient $\mathscr{R}_d^*(q_i) = \Gamma_i\backslash\mathscr{R}_d(q_i)$ and put

$$\mathscr{R}_d^*(\mathrm{gen}_Q) = \coprod_{q_i \in \operatorname{Gen}(Q)} \mathscr{R}_d^*(q_i).$$

We now give an adelic parametrization of $\mathscr{R}_d^*(\mathrm{gen}_Q)$ for $d \in \mathbb{D}_{\mathrm{fund}}(Q)$. This will show, in particular, that $\mathscr{R}_d^*(\mathrm{gen}_Q)$ is finite, a fact which is not *a priori* evident when $\mathbf{B}$ is indefinite.

To this end, we introduce purely local analogues of the global problem of parametrizing Λ-integral points on $\mathbf{X}_d$, where $\Lambda = \Lambda_1$. Namely, with $q = q_1$ and $\widehat{\Lambda} = \Lambda \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, we put

$$\mathbf{PB}^\times(\mathbb{A}_f)^{\mathrm{int}} = \{g \in \mathbf{PB}^\times(\mathbb{A}_f) \mid g.x_0 \in \widehat{\Lambda}\}.$$

In this way, the local analogues $\mathbf{PB}^\times(\mathbb{A}_f).x_0 \cap \widehat{\Lambda}$ and $\mathbf{PB}^\times(\widehat{\mathbb{Z}})\backslash(\mathbf{PB}^\times(\mathbb{A}_f).x_0 \cap \widehat{\Lambda})$ are identified, under the orbit map through x_0, with the group quotients

$$\mathbf{PB}^\times(\mathbb{A}_f)^{\mathrm{int}}/\mathbf{T}_\iota(\mathbb{Q}) \quad \text{and} \quad \mathbf{PB}^\times(\widehat{\mathbb{Z}})\backslash\mathbf{PB}^\times(\mathbb{A}_f)^{\mathrm{int}}/\mathbf{T}_\iota(\mathbb{Q}),$$

respectively. To compute the latter, one must determine the orbit structure of $\mathbf{PB}^\times(\mathbb{Z}_p) = \mathscr{O}_p^\times/\{\pm 1\}$ acting by conjugation on the $\mathbb{Z}_p$-solutions $\mathbf{X}_d(\mathbb{Z}_p) = \{x_p \in \mathbf{B}^0(\mathbb{Z}_p) \mid \mathrm{Nm}(x_p) = d\}$ for every p. This has been done in [17, Prop. 10.1]. In particular, for almost all p the action is transitive.

With this notation in place, it follows from [19, §3.2.2] that $\mathscr{R}_d^*(\mathrm{gen}_Q)$ is naturally identified with

$$\bigcup_{[\mathrm{h}_f]\in\mathbf{PB}^\times(\widehat{\mathbb{Z}})\backslash\mathbf{PB}^\times(\mathbb{A}_f)^{\mathrm{int}}/\mathbf{T}_\iota(\mathbb{Q})} \mathbf{T}_\iota(\mathbb{Q})\backslash\mathbf{T}_\iota(\mathbb{A}_f)/\big(\mathbf{T}_\iota(\mathbb{A}_f) \cap \mathrm{h}_f^{-1}\mathbf{PB}^\times(\widehat{\mathbb{Z}})\mathrm{h}_f\big).$$

Each of the above quotients is a finite group, being the class group of a quadratic order in E.

Remark 1. The lack of non-transitivity (for certain congruence classes mod d) of the class group action, as described in the introduction in the context of Linnik's sphere problem, is "explained" by the local parametrizing set $\mathscr{R}_d^*(x^2+y^2+z^2)_{\mathrm{loc}}$. In many other sources, such as [17], the integral structure for $\mathbf{PB}^\times$ is induced by the orthogonal group for the maximal order, rather than its projective units, via the $\mathbb{Q}$-isomorphism of $\mathbf{PB}^\times$ with SO_Q. The resulting quotient then receives a transitive action of Cl_E for *all* congruence classes, but possibly with non-trivial stabilizers of finite order.

Note that (1.5) implies $x_0 \in \mathbf{X}_{Q,d}(\mathbb{Q}) \cap \mathrm{g}_f^{-1}\widehat{\Lambda}\mathrm{g}_f$. Thus $\mathrm{g}_f.x_0 \in \mathrm{g}_f\mathbf{X}_{Q,d}(\mathbb{Q})\mathrm{g}_f^{-1} \cap \widehat{\Lambda}$, so $\mathrm{g}_f \in \mathbf{PB}^\times(\mathbb{A}_f)^{\mathrm{int}}$. Similarly to (2.1), we may identity Cl_E with $\mathbf{T}_\iota(\mathbb{Q})\backslash\mathbf{T}_\iota(\mathbb{A}_f)/\mathbf{T}_\iota(\widehat{\mathbb{Z}})$, where $\mathbf{T}_\iota(\widehat{\mathbb{Z}})$ satisfies (1.6). Taking $[\mathrm{h}_f] = [\mathrm{g}_f]$, we get an embedding

$$\mathrm{Cl}_E \to \mathscr{R}_d^*(\mathrm{gen}_Q), \tag{2.4}$$

given by the orbit map through x_0.

2.5. The dual picture and the modular formulation of Duke's theorem

When $\mathbf{B}$ is indefinite, it is more convenient to formulate (and prove!) Duke's theorem using a dual formulation [35] involving packets of Heegner points or closed geodesics on a Shimura or modular curve.

In this section we shall take $\mathbf{B}$ indefinite; for simplicity, we shall furthermore assume that the genus class of Q is a singleton. In particular, one can assume that $\mathbb{O} = \mathscr{O}$ and the element g_f from Section 1.3 can be taken to be the identity. According to whether d is positive or negative, we let $H < \mathbf{PB}^\times(\mathbb{R})$ be either K_∞ or the pullback of the projective diagonal matrices via the fixed isomorphism of $\mathbf{B}(\mathbb{R})$ with $M_2(\mathbb{R})$ from Section 1.3. We fix $x_\infty \in \mathbf{X}_{Q,\mathrm{sgn}(d)}(\mathbb{R})$ such that $H = \mathrm{Stab}_{\mathbf{PB}^\times(\mathbb{R})}(x_\infty)$. We then identify $\mathbf{X}_{Q,\mathrm{sgn}(d)}(\mathbb{R})$

with $\mathbf{PB}^\times(\mathbb{R})/H$ via the orbit map through x_∞. The image of (2.4) is given by a finite union of $\mathbf{PB}^\times(\mathbb{Z})$-orbits

$$\coprod_{\tau\in \mathrm{Cl}_E} \mathbf{PB}^\times(\mathbb{Z})x_\tau.$$

We project these onto the quadric $\mathbf{X}_{Q,\mathrm{sgn}(d)}(\mathbb{R})$ by rescaling by $|d|^{-1/2}$. In this way we produce elements $g_\tau \in \mathbf{PB}^\times(\mathbb{R})$ satisfying $g_\tau.x_\infty = |d|^{-1/2}x_\tau$.

The following equidistribution statements are equivalent [14, Prop. 2.1]:

(1) (*arithmetic equidistribution statement*) the finite union of left $\mathbf{PB}^\times(\mathbb{Z})$-orbits

$$\coprod_{\tau\in \mathrm{Cl}_E} \mathbf{PB}^\times(\mathbb{Z})g_\tau H/H$$

equidistribute on the quadric $\mathbf{X}_{Q,\mathrm{sgn}(d)}(\mathbb{R})$;

(2) (*modular equidistribution statement*) the finite set of right H-orbits

$$\{y_\tau H : \tau\in \mathrm{Cl}_E\}, \quad \text{where} \quad y_\tau = \mathbf{PB}^\times(\mathbb{Z})\backslash \mathbf{PB}^\times(\mathbb{Z})g_\tau,$$

equidistribute on $\mathbf{PB}^\times(\mathbb{Z})\backslash\mathbf{PB}^\times(\mathbb{R})$.

We remark that by [16, §2.4.1] the right H-orbits in (2) are periodic.

Let us now examine the modular equidistribution statement (2). Let $\mathbb{H}^\pm = \mathbb{H}\cup -\mathbb{H}$ be the union of the upper and lower half-planes. We identify $\mathbf{PB}^\times(\mathbb{R})$ with $\mathrm{PGL}_2(\mathbb{R})$ as in Section 2.2 so that

$$\mathbf{PB}^\times(\mathbb{R})/K_\infty \simeq \mathrm{PGL}_2(\mathbb{R})/\mathrm{PSO}(2)\simeq \mathbb{H}^\pm.$$

Then, with $K = \mathbf{PB}^\times(\widehat{\mathbb{Z}})K_\infty$, we have

$$[\mathbf{PB}^\times]_K = \mathbf{PB}^\times(\mathbb{Q})\backslash\mathbf{PB}^\times(\mathbb{A}_\mathbb{Q})/K = \mathbf{PB}^\times(\mathbb{Z})\backslash\mathbf{PB}^\times(\mathbb{R})/K_\infty \simeq X_\Gamma,$$

where $X_\Gamma = \mathbf{PB}^\times(\mathbb{Z})\backslash\mathbb{H}^\pm$ is a Shimura curve when $\mathbf{B}$ is a division algebra and the modular curve $\mathrm{PGL}_2(\mathbb{Z})\backslash\mathbb{H}^\pm = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ when $\mathbf{B}$ is the matrix algebra. In the modular statement, we may identify the quotient $\mathbf{PB}^\times(\mathbb{Z})\backslash\mathbf{PB}^\times(\mathbb{R})$ with the unit tangent bundle $T^1(X_\Gamma)$ of X_Γ, equipped with the Liouville measure. However, in view of the archimedean restrictions in Theorems 1 and 2 we wish rather to examine (2) on X_Γ itself.[2] In this case, X_Γ is equipped with the Poincaré measure, normalized to have volume 1.

Now when $H = K_\infty$ is compact (so that $E = \mathbb{Q}(\sqrt{-d})$, with $d>0$, is imaginary quadratic), (2) is equivalent to the equidistribution of the *Heegner points*

$$\mathscr{H}_d(X_\Gamma) = \{z_\tau = y_\tau K_\infty/K_\infty \mid \tau\in \mathrm{Cl}_E\}$$

on X_Γ. For example, when $\mathbf{B} = M_2$ is the matrix algebra so that $Q(a,b,c) = b^2-4ac$ is the discriminant form, we obtain the Heegner points

$$\mathscr{H}_d(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}) = \mathrm{PSL}_2(\mathbb{Z})\backslash\left\{\frac{-b+\sqrt{-d}}{2a}\;\middle|\; b^2\equiv -d \pmod{4a}\right\}$$

[2] The stronger version of Duke's original theorem, which upgrades X_Γ to $T^1(X_\Gamma)$, was proved by Chelluri [7].

on the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. When H is non-compact (so that $E = \mathbb{Q}(\sqrt{-d})$, with $d < 0$, is real quadratic), (2) states that the packet of *closed geodesics*

$$\mathscr{G}_d(X_\Gamma) = \{\gamma_\tau = y_\tau H/K_\infty \mid \tau \in \mathrm{Cl}_E\}$$

equidistributes on X_Γ. Taking the modular surface again as our example, each γ_τ is the projection to $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ of the unique geodesic in $\mathbb{H}$ with endpoints $\frac{-b+\sqrt{-d}}{2a}$, where $b^2 \equiv -d \pmod{4a}$ (cf. [16, §2.3]).

3. Instances of Theorems 1 and 2

We now give explicit examples of pairings which fit into the simultaneous equidistribution statement of our Theorems 1 and 2.

Example 1 (Simultaneous equidistribution on pairs of quadrics). We begin by giving an example with class number 1 algebras; we shall emphasize equidistribution across genus classes shortly.

Let $\mathbf{B}_1$ be the split algebra M_2 and $\mathbf{B}_2 = \mathbf{B}^{(2,\infty)}$. Any imaginary quadratic extension E of $\mathbb{Q}$ of discriminant $-D$ with $D \not\equiv 7 \pmod 8$ embeds diagonally into $\mathbf{B}_1 \times \mathbf{B}_2$. Theorem 1 states that, under GRH, the corresponding packets of pairings, as in (1.4), between Heegner points $\mathscr{H}_D$ on the modular curve and the integral points $D^{-1/2}\mathscr{R}_D$ on the sphere (cf. (1.1)) equidistribute, relative to the discrete spectrum, to the product of the uniform probability measures on $\mathrm{SO}_3^+(\mathbb{Z})\backslash S^2 \times \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ as $D \to \infty$.

If we replace $\mathbf{B}_1 = M_2$ with an indefinite division algebra, we obtain the same result with Heegner points on a Shimura curve. Since the automorphic spectrum of the latter is discrete, the restriction to test functions in the discrete spectrum in Theorem 1 holds automatically in this case.

Example 2 (A variation). The previous example has an interesting geometric variation, treated in [1]. To each point in $x \in \mathscr{R}_D$ we can associate the rank 2 lattice $x^\perp \cap \mathbb{Z}^3$; after rotating to a fixed reference plane and rescaling by the volume, it can be viewed as a Heegner point of discriminant D via the isomorphism of $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ with isometry classes of unimodular lattices in $\mathbb{R}^2$. As explained in [17, §5.2], this corresponds to the twisted diagonal embedding $\iota_{\alpha,\beta}$ with $\alpha = 1$, $\beta = 2$; indeed, the projection of the image onto the second factor meets only one coset of squares in the class group (cf. [17, footnote 11 on p. 151]). We may deduce from Theorem 2 that, under GRH, the set of pairs

$$(x/\|x\|, x^\perp \cap \mathbb{Z}^3)$$

equidistributes (relative to the discrete spectrum) to the product of the uniform measures on $\mathrm{SO}_3^+(\mathbb{Z})\backslash S^2 \times \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ as $D \to \infty$ through prime discriminants (so that by Gauß' genus theory the 2-torsion is trivial).

Remark 2. This orthogonal complement construction works in greater generality. For instance, as explained in Section 2.5, if $\mathbf{B} = M_2$, a (necessarily primitive) integral element x on the discriminant variety $\mathbf{X}_{Q,d}$, associated with the determinant form $Q(a,b,c) = b^2 - 4ac$, corresponds to a Heegner point or a closed geodesic on the modular curve, according to whether $d < 0$ or $d > 0$, respectively. The restriction of Q to $\langle x \rangle^\perp$ then has signature $(2,0)$ if $d < 0$ and signature $(1,1)$ if $d > 0$, yielding a positive, respectively indefinite, binary lattice of discriminant d.

Example 3 (Simultaneous representation by genus classes). Next we assume $\mathbf{B}$ is definite and let $K_\infty = \mathbf{PB}^\times(\mathbb{R})$. In this case the adelic double quotient $[\mathbf{PB}^\times]_K$ is simply a finite union of singletons, indexed by the class set of $\mathbf{PB}^\times$. We give two very concrete examples of Theorem 1 in this setting in this and the following subsection.

Let $\mathbf{B}_1 = \mathbf{B}^{(11,\infty)}$ and $\mathbf{B}_2 = \mathbf{B}^{(19,\infty)}$, both genus 2 definite quaternion algebras. For $i = 1,2$, let $Q_i = \mathrm{Nm}|_{\mathbf{B}_i^0}$ be the trace zero norm form and choose two non-conjugate maximal orders $\mathbb{O}_i^{(1)}, \mathbb{O}_i^{(2)}$ in $\mathbf{B}_i(\mathbb{Q})$. Restricting Q_i to the trace zero dual lattices $\Lambda_i^{(1)}, \Lambda_i^{(2)}$ yields, as in Section 2.3, a pair of genus equivalent, integral, positive definite ternary quadratic forms $Q_i^{(1)}, Q_i^{(2)}$. Explicitly, we have

$$\mathrm{gen}_{Q_1} = \{Q_1^{(1)}, Q_1^{(2)}\},$$
$$\begin{cases} Q_1^{(1)} = 3x^2 + 11y^2 + xz + z^2, & |\mathrm{Aut}(Q_1^{(1)})| = 4, \\ Q_1^{(2)} = 3x^2 + 4y^2 + 4z^2 + 2xy + 2xz - 3yz, & |\mathrm{Aut}(Q_1^{(2)})| = 6, \end{cases}$$

and

$$\mathrm{gen}_{Q_2} = \{Q_2^{(1)}, Q_2^{(2)}\},$$
$$\begin{cases} Q_2^{(1)} = x^2 + 5y^2 + 19z^2 + xy, & |\mathrm{Aut}(Q_2^{(1)})| = 8, \\ Q_2^{(2)} = 4x^2 + 5y^2 + 6z^2 + 5yz + xz + 2xy, & |\mathrm{Aut}(Q_2^{(2)})| = 4. \end{cases}$$

We define probability measures m_i on gen_{Q_i} by weighting each form by the reciprocal of the order of its automorphism group. Thus

$$m_1 = \tfrac{3}{5}\delta_{Q_1^{(1)}} + \tfrac{2}{5}\delta_{Q_1^{(2)}} \quad \text{and} \quad m_2 = \tfrac{1}{3}\delta_{Q_2^{(1)}} + \tfrac{2}{3}\delta_{Q_2^{(2)}}.$$

It follows from the theorem of Duke and Schulze-Pillot [12] that any large enough $d \in \mathbb{D}_{\mathrm{fund}}(Q_i)$ is integrally represented both by $Q_i^{(1)}$ and $Q_i^{(2)}$, and that the relative share of such representations as $d \to \infty$ is governed by m_i. Theorem 1 states that, under GRH, any large enough $d \in \mathbb{D}_{\mathrm{fund}}(Q_1) \cap \mathbb{D}_{\mathrm{fund}}(Q_2)$ is represented *simultaneously* by every pair $(Q_1^{(i)}, Q_2^{(j)})$, and that the relative share of such representations as $d \to \infty$ is governed by the product measure $m_1 \times m_2$.

Example 4 (Simultaneous supersingular reduction of CM elliptic curves). While the above example featured two explicit definite quaternion algebras ramified at a single prime, this was only for computational simplicity. By contrast, the next example exploits the special arithmetic significance of the quaternion algebra $\mathbf{B}^{(p,\infty)}$.

In this case, the class set of $\mathbf{B}^{(p,\infty)}$ can be identified with the set $\mathscr{E}_p^{\mathrm{ss}}$ of isomorphism classes of supersingular elliptic curves defined over $\overline{\mathbb{F}}_p$. Under this identification, $\mathscr{E}_p^{\mathrm{ss}}$ is endowed with a natural probability measure m_p which, as in the previous example assigns to each $e \in \mathscr{E}_p^{\mathrm{ss}}$ a weight proportional to $|\mathrm{End}(e)^\times|^{-1}$.

Let E be an imaginary quadratic number field with ring of integers $\mathcal{O}_E$ such that p is inert in E; then E embeds into $\mathbf{B}^{(p,\infty)}$. Let $\mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}}$ be the set of elliptic curves defined over $\mathbb{C}$ with complex multiplication by $\mathcal{O}_E$. Any $e \in \mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}}$ is defined over the Hilbert class field H_E. Fix a prime $\mathfrak{p} \mid p$ of H_E. Then the reduction of $e \in \mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}}$ modulo $\mathfrak{p}$ is a supersingular elliptic curve defined over $\overline{\mathbb{F}}_p$ [8, p. 41]. In this way we obtain a reduction map

$$\mathrm{red}_p : \mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}} \to \mathscr{E}_p^{\mathrm{ss}}, \quad e \mapsto e \bmod \mathfrak{p}.$$

It was observed by Michel [34] that this example fits into the paradigm of Duke's theorems of the previous examples. (We refer to the recent preprint [3] for a detailed description of the relation with toric packets.) Namely, for fixed p, the map red_p is surjective if the discriminant of E is large enough, and in fact one has the equidistribution statement

$$\frac{|\{e \in \mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}} : \mathrm{red}_p(e) = e_0\}|}{|\mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}}|} \to m_p(e_0) \quad (e_0 \in \mathscr{E}_p^{\mathrm{ss}}),$$

as E varies along the imaginary quadratic fields for which p is inert.

Having reprised Duke's theorem in this setting, we now take an imaginary quadratic extension E of $\mathbb{Q}$, in which two distinct odd primes p, q are inert. Then E embeds diagonally into the product $\mathbf{B}^{(p,\infty)} \times \mathbf{B}^{(q,\infty)}$. We may then consider the simultaneous supersingular reduction map

$$\mathrm{red}_{p,q} : \mathrm{Ell}_{\mathcal{O}_E}^{\mathrm{CM}} \to \mathscr{E}_p^{\mathrm{ss}} \times \mathscr{E}_q^{\mathrm{ss}}, \quad e \mapsto (e \bmod \mathfrak{p}, e \bmod \mathfrak{q}).$$

Our main result, Theorem 1, shows that, under GRH, $\mathrm{red}_{p,q}$ is surjective for E of large enough discriminant D, and that the pushforward of the counting measure on $\mathrm{Ell}_{\mathcal{O}_K}^{\mathrm{CM}}$ tends to the product measure $m_p \times m_q$ on $\mathscr{E}_p^{\mathrm{ss}} \times \mathscr{E}_q^{\mathrm{ss}}$ as $D \to \infty$. Note that, in contrast to the spectral approach we have adopted, dynamical methods can handle $n \geq 2$ distinct copies of $\mathbf{B}_i$.

4. Reduction to half-integral mixed moment

In this subsection we reduce the proofs of Theorem 1 and 2 to statements about a half-integral mixed moment of L-functions. Throughout, we retain the notation and hypotheses of Theorems 1 and 2. For $f \in L^2([\mathbf{G}]_K)$ let

$$\mathcal{P}_{\Delta\mathbf{T}}(f) = \int_{[\Delta\mathbf{T}_\iota]} f(t)\,dt \quad \text{and} \quad \mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(f) = \int_{[\Delta_{\alpha,\beta}\mathbf{T}_\iota]} f(t)\,dt \tag{4.1}$$

be the global toric period integral relative to the subgroups $\Delta\mathbf{T}_\iota \subset \mathbf{G}$ and $\Delta_{\alpha,\beta}\mathbf{T}_\iota \subset \mathbf{G}$. When $\alpha = \beta = 1$ these two subgroups and their corresponding periods coincide. We recall that the measures on $[\Delta\mathbf{T}_\iota]$ and $[\Delta_{\alpha,\beta}\mathbf{T}_\iota]$ are normalized to have volume 1.

Let $L^2_0([\mathbf{G}]_K)$ denote the orthocomplement of the character spectrum in $L^2_{\mathrm{disc}}([\mathbf{G}]_K)$. Since K_j is the projective unit group of the maximal order $\mathscr{O}_j$, the character spectrum of K_j, and thus of $K = K_1 \times K_2$, is trivial (see [3, §9.2–9.3], where this is shown to hold for Eichler orders).

4.1. Setting up Weyl's criterion

It suffices, for proving Theorems 1 and 2, to study the periods (4.1) when f is an element of a given orthonormal basis of $L^2_0([\mathbf{G}]_K)$. We now describe a particularly nice such orthonormal basis, coming from the theory of new vectors for inner forms of GL_2.

We begin with the group $\mathbf{G}_j$ for $j = 1, 2$. Let $L^2_0([\mathbf{G}_j]_{K_j})$ denote the orthocomplement of the trivial character $1_{\mathbf{G}_j}$ in $L^2_{\mathrm{disc}}([\mathbf{G}_j]_{K_j})$. We have the Hilbert space direct sum decomposition

$$L^2_0([\mathbf{G}_j]_{K_j}) = \bigoplus_{\sigma_j \subset L^2_0([\mathbf{G}_j])} \sigma_j^{K_j} \tag{4.2}$$

into irreducible discrete automorphic subrepresentations having non-trivial invariants under $K_j = \mathbf{G}_j(\widehat{\mathbb{Z}})K_{\infty,j}$, where we have used the Multiplicity One Theorem. Since K_j is a maximal compact subgroup at each finite place, and either a maximal compact or maximal compact proper subgroup (if $\mathbf{B}$ is definite) at infinity, we have $\dim \sigma_j^{K_j} \leq 1$. For each σ_j appearing in the decomposition (4.2), there is therefore a unique up to unitary scaling choice of an L^2-normalized vector ϕ_{σ_j} in the line of K_j-invariants, and we let $\mathscr{B}_{0,j} = \{\phi_{\sigma_j}\}$ be the resulting orthonormal basis of $L^2_0([\mathbf{G}_j]_{K_j})$.

Note that, in the case of $\mathbf{B}_j$ definite and $K_{\infty,j}$ a maximal compact torus, each ϕ_{σ_j} is an eigenfunction for the sphere Laplacian with an eigenvalue of the form $k(k+1)$. When $\mathbf{B}_j$ is indefinite, each ϕ_{σ_j} is an eigenfunction for the hyperbolic Laplacian of the form $1/4 + t^2$. In either case, we denote the Laplacian eigenvalue by $\lambda^2_{\sigma_j}$ and call λ_{σ_j} the *spectral parameter*. Note that $\lambda_{\sigma_j} > 1/3$ is bounded away from 0 by [29] and comparable in size to the archimedean conductor of σ_j. In the remaining case where $\mathbf{B}_j$ is definite and $K_{\infty,j} = \mathbf{G}_j(\mathbb{R})$, we simply put $\lambda_{\sigma_j} = 1$.

We now return to the product group $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$. Let $L^2_{00}([\mathbf{G}]_K) = L^2_0([\mathbf{G}_1]_{K_1}) \otimes L^2_0([\mathbf{G}_2]_{K_2})$. Then

$$L^2_0([\mathbf{G}]_K) = (\mathbb{C}1_{\mathbf{G}_1} \otimes L^2_0([\mathbf{G}_2]_{K_2})) \oplus (L^2_0([\mathbf{G}_1]_{K_1} \otimes \mathbb{C}1_{\mathbf{G}_2}) \oplus L^2_{00}([\mathbf{G}]_K).$$

We deduce that $L^2_0([\mathbf{G}]_K)$ admits an orthonormal basis of the form $\mathscr{B}_0 = \mathscr{B}_{01} \cup \mathscr{B}_{02} \cup \mathscr{B}_{00}$, where

$$\mathscr{B}_{01} = \mathscr{B}_{0,1} \otimes 1_{\mathbf{G}_2}, \quad \mathscr{B}_{02} = 1_{\mathbf{G}_1} \otimes \mathscr{B}_{0,2}, \quad \mathscr{B}_{00} = \mathscr{B}_{0,1} \otimes \mathscr{B}_{0,2}.$$

For each $\phi \in \mathscr{B}_0$ we let λ_ϕ denote λ_{σ_j} or $\lambda_{\sigma_1}\lambda_{\sigma_2}$ according to whether $\phi \in \mathscr{B}_{0j}$ or $\phi \in \mathscr{B}_{00}$.

Now let $f \in C_c^\infty([\mathbf{G}]_K) \cap L^2_0([\mathbf{G}]_K)$ with L^2-spectral expansion

$$f = \sum_{\phi \in \mathscr{B}_0} \langle f, \phi \rangle \phi.$$

This expansion is finite whenever both $\mathbf{B}_j$ are definite and $K_{\infty,j} = \mathbf{G}_j(\mathbb{R})$. Otherwise the expansion is absolutely convergent, and by self-adjointness of the Laplace operator, the L^2-inner products satisfy $\langle f, \phi\rangle \ll_d \mathcal{S}_{\infty,2d}(f)\lambda_\phi^{-d}$ for all $d \in \mathbb{N}$. In all cases, we deduce that Theorems 1 and 2 follow from

$$\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(\mathrm{g}.\phi) \ll (\log D)^{-1/4+\varepsilon} \tag{4.3}$$

for all $\phi \in \mathcal{B}_0$, where the implied constant may depend on $\alpha, \beta, \varepsilon$ and *polynomially* on λ_ϕ. Note that

$$\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(\mathrm{g}.(\phi_{\sigma_1} \otimes 1_{\mathbf{G}_2})) = \int_{[\mathbf{T}]} \phi_{\sigma_1}(t^\alpha \mathrm{g}_1)\, dt,$$
$$\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(\mathrm{g}.(1_{\mathbf{G}_2} \otimes \phi_{\sigma_2})) = \int_{[\mathbf{T}]} \phi_{\sigma_2}(t^\beta \mathrm{g}_1)\, dt.$$

Thus the bound (4.3) for test functions in $\mathcal{B}_{0j}$ is covered by Duke's theorem as stated in Section 1.2 (note that multiplication by α or β in the class group is an isomorphism), even with a power saving rate, so that it suffices to prove (4.3) on the basis elements in $\mathcal{B}_{00}$.

4.2. Parseval and absolute values

For $\phi_j \in L^2([\mathbf{G}_j]_{K_j})$ and $\chi \in \widetilde{\mathrm{Cl}}_E^\vee$ we define

$$\mathcal{P}_{\mathbf{T}}^\chi(\phi_j) = \int_{[\mathbf{T}_{\iota_j}]} \phi_j(t)\chi(t)\, dt \tag{4.4}$$

to be the global $(\mathbf{T}_{\iota_j}, \chi)$-period integral. We now convert the problem of estimating the diagonal period $\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}$ to one of bounding an average of these twisted periods.

In the context of Theorem 1 this is a straightforward task. We recall from Section 2.1 that $[\mathbf{T}_{\iota_j}]_{K_j}$ for $j = 1, 2$ can naturally be identified with the Arakelov class group $\widetilde{\mathrm{Cl}}_E$. Using the identification (2.1), we may view the integral over $[\Delta\mathbf{T}_\iota]_K$ in (4.1) as an inner product over the Arakelov class group $\widetilde{\mathrm{Cl}}_E$. Plancherel's identity then gives

$$\mathcal{P}_{\Delta\mathbf{T}}(\phi_\sigma^\circ) = \int_{[\mathbf{T}]} \phi_{\sigma_1}^\circ(\iota_1(t))\overline{\phi_{\sigma_2}^\circ(\iota_2(t))}\, dt = \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \mathcal{P}_{\mathbf{T}}^\chi(\phi_{\sigma_1}^\circ)\overline{\mathcal{P}_{\mathbf{T}}^\chi(\phi_{\sigma_2}^\circ)}, \tag{4.5}$$

where $\phi_{\sigma_j}^\circ = \mathrm{g}_j.\phi_{\sigma_j}$. More generally, working in the context of Theorem 2, we apply Fourier inversion to obtain

$$\begin{aligned}\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(\phi_\sigma^\circ) &= \int_{[\mathbf{T}]} \Big(\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \mathcal{P}_{\mathbf{T}}^\chi(\phi_{\sigma_1}^\circ)\overline{\chi^\alpha(t)} \Big) \overline{\phi_{\sigma_2}^\circ(\iota_2(t)^\beta)}\, dt \\ &= \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \mathcal{P}_{\mathbf{T}}^\chi(\phi_{\sigma_1}^\circ) \int_{[\mathbf{T}]} \overline{\phi_{\sigma_2}^\circ(\iota_2(t)^\beta)\chi^\alpha(t)}\, dt.\end{aligned}$$

Since by assumption Cl_E has no β-torsion, the integral vanishes unless $\chi^\alpha = \psi^\beta$ for some $\psi \in \widetilde{\mathrm{Cl}}_E^\vee$ (which is unique). Writing $\alpha = \alpha'\delta$, $\beta = \beta'\delta$ with $(\alpha', \beta') = 1$, we see that $\chi = \psi^{\beta'}$ for some $\psi \in \widetilde{\mathrm{Cl}}_E^\vee$, so $\chi^\alpha(t) = \psi^{\alpha'}(t^\beta)$. Changing variables $t \leftarrow t^\beta$ in the t-integral, we obtain

$$\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(\phi_\sigma^\circ) = \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \mathcal{P}_{\mathbf{T}}^{\chi^{\beta'}}(\phi_{\sigma_1}^\circ)\overline{\mathcal{P}_{\mathbf{T}}^{\chi^{\alpha'}}(\phi_{\sigma_2}^\circ)},$$

which specializes to (4.5) when $\alpha = \beta = 1$.

Having no way of accessing the sign of the periods $\mathcal{P}_{\mathbf{T}}^{\chi}(\phi_\sigma^\circ)$ as χ varies, we now apply the triangle inequality to the χ-sum, to get

$$|\mathcal{P}_{\Delta_{\alpha,\beta}\mathbf{T}}(\phi_\sigma^\circ)| \leq \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} |\mathcal{P}_{\mathbf{T}}^{\chi^{\beta'}}(\phi_{\sigma_1}^\circ)\mathcal{P}_{\mathbf{T}}^{\chi^{\alpha'}}(\phi_{\sigma_2}^\circ)|. \tag{4.6}$$

Remark 3. The bound (4.6) sacrifices all cancellation in the χ-sum. Something similar was done in the pioneering work of Holowinsky [25], and later Lester and Radziwiłł [30], with respect to unipotent periods, in which $\mathbf{T}$ is replaced by the unipotent subgroup $\mathbf{N}$ of upper triangular matrices in PGL_2. At first sight this looks like a hopeless gambit, in view of the loss of information incurred, but these breakthrough papers demonstrated that it is reasonable nonetheless to hope for some small savings.

4.3. Fractional moments of L-functions

We now convert the right-hand side of (4.6) to a fractional moment of L-functions using an explicit form of Waldspurger's theorem [45], which relates the twisted period (4.4) to the central Rankin–Selberg L-value $L(1/2, \pi_j \times \chi)$, where π_j on $\mathrm{PGL}_2(\mathbb{A})$ is associated with σ_j by the Jacquet–Langlands correspondence and χ is viewed as the automorphic induction to GL_2 over $\mathbb{Q}$. This will lead us to the statement of our main analytic result, Theorem 3, which will be seen to imply both Theorems 1 and 2.

The dual group $\widetilde{\mathrm{Cl}}_E^\vee$ consists of everywhere unramified Hecke characters, trivial at infinity if E is imaginary and totally even at infinity if E is real. Let $\psi \in \widetilde{\mathrm{Cl}}_E^\vee$ denote either of the characters $\chi^{\alpha'}$ or $\chi^{\beta'}$ appearing in (4.6). We may assume that $\mathrm{Hom}_{\mathbf{T}_{\iota_j}(\mathbb{A}_\mathbb{Q})}(\sigma_j, \psi) \neq 0$ since otherwise the twisted period $\mathcal{P}_{\mathbf{T}}^{\psi}(\phi_{\sigma_j}^\circ)$ in (4.6) vanishes. Moreover, since ψ is unramified, and $\phi_{\sigma_j}^\circ$ is invariant under $\mathrm{g}_{j,f}^{-1}\mathbf{G}_j(\widehat{\mathbb{Z}})\mathrm{g}_{j,f}$ in which $\mathbf{T}$ is optimally embedded, the vector $\phi_{\sigma_j}^\circ$ is the global Gross–Prasad [24] vector. This observation allows us to use the explicit Waldspurger formula from [20, Theorem 1.1], which states that

$$|\mathcal{P}_{\mathbf{T}}^{\psi}(\phi_{\sigma_j}^\circ)|^2 = C_{\mathbf{G}_j} C_{\mathrm{Ram}}(\pi_j, \psi) \frac{1}{\sqrt{D}} \frac{1}{L(1, \eta_E)^2} \frac{L(1/2, \pi_j \times \psi)}{L(1, \mathrm{Ad}\, \pi_j)} F(\pi_{j,\infty}, \psi_\infty) \tag{4.7}$$

for a constant $C_{\mathbf{G}_j} > 0$ depending only on $\mathbf{G}_j$, a constant $C_{\mathrm{Ram}}(\pi_j, \psi) > 0$ depending only on ψ and on the local ramified local components of π_j, and a function F of the spectral parameters of $\pi_{j,\infty}$ and ψ_∞.

In Appendix A we prove bounds on $C_{\mathrm{Ram}}(\pi_j, \psi)$ and $F(\pi_{j,\infty}, \psi_\infty)$. To state them here, let N_{π_j} denote the (arithmetic) conductor of π_j and write λ_{π_j} for the spectral parameter λ_{σ_j} from Section 4.1. If ψ_∞ is the archimedean component of ψ, we let $\lambda_\psi \in \mathbb{R}$ be its frequency as a character on $\mathbf{T}(\mathbb{R})/K_{\mathbf{T},\infty}$. In particular, when E is imaginary quadratic, we have $\psi_\infty = 1$ and thus $\lambda_\psi = 0$. If E is real, then $\psi_\infty(x_1, x_2) = |x_1|^{i\lambda_\psi}|x_2|^{-i\lambda_\psi}$ with $\lambda_\psi \in \frac{\pi}{\log \epsilon}\mathbb{Z}$. We show in Appendix A that

$$C_{\mathrm{Ram}}(\pi_j, \psi) \ll_\varepsilon N_{\pi_j}^{\varepsilon} \quad \text{and} \quad F(\pi_{j,\infty}, \psi_\infty) \ll \exp(-c_0|\lambda_\psi|/\lambda_{\pi_j}) \tag{4.8}$$

for some absolute constant $c_0 > 0$.

We now take *positive* square-roots in (4.7) and insert this into the right-hand side of (4.6), which we are to bound. Using (2.2), as well as the bound $1/L(1, \eta_E) \ll \log\log D$, which is known to hold under GRH (see (6.9) below), we obtain

$$\mathcal{P}_{\Delta \mathbf{T}_{\alpha,\beta}}(\phi_\sigma^\circ) \ll_\varepsilon \frac{N_\pi^\varepsilon \log\log D}{\mathrm{vol}(\widetilde{\mathrm{Cl}}_E)} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \exp\left(-\frac{c_0|\lambda_\chi|}{\lambda_\pi}\right)\left(\frac{L(1/2, \pi_1 \times \chi^{\beta'})L(1/2, \pi_2 \times \chi^{\alpha'})}{L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_2)}\right)^{1/2}$$

for some $c_0 > 0$ (which may depend on α, β), where we have put $N_\pi = N_{\pi_1}N_{\pi_2}$, and $\lambda_\pi = \lambda_{\pi_1}\lambda_{\pi_2}$. For notational simplicity we drop the prime in the above estimate and relabel $\alpha' \to \alpha$, $\beta' \to \beta$.

Now that we have converted the original problem to one on L-functions, the key observation is that on average over χ, the central values $L(1/2, \pi_1 \times \chi^\alpha)$ and $L(1/2, \pi_2 \times \chi^\beta)$ are a little less than 1, and that these small values occur *independently of each other*. This will just suffice to obtain the equidistribution results in Theorems 1 and 2. The following result makes this precise.

Theorem 3. *Let E be a quadratic field extension of $\mathbb{Q}$, of discriminant D. Let π_1, π_2 be irreducible cuspidal automorphic representations on $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$, of squarefree level N_{π_j}. Let λ_{π_j} be the spectral parameter of π_j and put $\lambda_\pi = \lambda_{\pi_1}\lambda_{\pi_2}$, $N_\pi = N_{\pi_1}N_{\pi_2}$. Let $c_0 > 0$. Assume the Generalized Riemann Hypothesis.*

(a) *If $\pi_1 \neq \pi_2$ then*

$$\frac{1}{\mathrm{vol}(\widetilde{\mathrm{Cl}}_E)} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \exp\left(-\frac{c_0|\lambda_\chi|}{\lambda_\pi}\right)\left(\frac{L(1/2, \pi_1 \times \chi)L(1/2, \pi_2 \times \chi)}{L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_2)}\right)^{1/2} \ll_{\varepsilon, c_0} (\log D)^{-1/4+\varepsilon}$$

for any $\varepsilon > 0$.

(b) *Let $\alpha, \beta \in \mathbb{N}$ be distinct positive integers. Assume that $\widetilde{\mathrm{Cl}}_E$ has no 2β torsion. Then*

$$\frac{1}{\mathrm{vol}(\widetilde{\mathrm{Cl}}_E)} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \exp\left(-\frac{c_0|\lambda_\chi|}{\lambda_\pi}\right)\left(\frac{L(1/2, \pi_1 \times \chi^\alpha)L(1/2, \pi_2 \times \chi^\beta)}{L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_2)}\right)^{1/2} \ll_{\alpha,\beta,\varepsilon, c_0} (\log D)^{-1/4+\varepsilon}$$

for any $\varepsilon > 0$.

All implied constants depend polynomially on $\lambda_\pi N_\pi$.

This theorem is the technical heart of the paper. The bound is best possible, up to the power of ε, and in the following section we explain the probabilistic model behind it.

As promised in the introduction, we give an immediate application to Fourier coefficients and Bessel periods of Yoshida lifts. Let f, g be distinct holomorphic cusp forms of weight 2 and $2k-2$, respectively, and level N where N is squarefree and has an odd number of prime factors. To f and g one can associate a non-zero holomorphic Siegel cusp form F such that $L(s,F) = L(s,f)L(s,g)$ and for all $p \mid N$ the local representation π_p associated with F is of type VIb (cf. [9]). For a fundamental discriminant $D < 0$ such that all $p \mid N$ remain inert in $\mathbb{Q}(\sqrt{D})$, let $\mathrm{Sym}^2_D(\mathbb{Z})$ be the set of positive definite symmetric semi-integral matrices with integral diagonal in M_2 of determinant $D/4$. We may identify the set of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of elements in $\mathrm{Sym}^2_D(\mathbb{Z})$ with the class group Cl_E of $E = \mathbb{Q}(\sqrt{D})$. Let $a(F,S)$ denote the Fourier coefficient of F at the matrix $S \in \mathrm{Sym}^2_D(\mathbb{Z})$; as it depends only on the $\mathrm{SL}_2(\mathbb{Z})$-equivalence class of S, we may write $a(F,S_t)$ for $t \in \mathrm{Cl}_E$. For a fundamental discriminant $D < 0$ and a class group character $\chi \in \mathrm{Cl}_E^\vee$ we define the Bessel period

$$R(F,D,\chi) = \sum_{t \in \mathrm{Cl}_E} a(F,S_t)\bar{\chi}(t).$$

From [9, Prop. 3.14], and Fourier inversion over Cl_E, we obtain the following corollary of Theorem 3.

Corollary 4. *Assume GRH. Let F be a Yoshida lift and $D < 0$ a fundamental discriminant satisfying the above assumptions. Then*

$$\max_{t \in \mathrm{Cl}_E} |a(F,S_t)| \leq \frac{1}{|\mathrm{Cl}_E|} \sum_{\chi \in \mathrm{Cl}_E^\vee} |R(F,D,\chi)| \ll_{\varepsilon,F} \frac{|D|^{(k-1)/2}}{(\log|D|)^{1/4-\varepsilon}}$$

for any $\varepsilon > 0$.

On GRH, the trivial bound on the left-hand side and the sum in the middle is $|D|^{(k-1)/2+\varepsilon}$. The bound on the right-hand side is sharp, up to the value of ε, since the same is true of Theorem 3. One expects a bound of size $a(F,S_t) \ll_\varepsilon |D|^{k/2-3/4+\varepsilon}$, based on both GRH and square-root cancellation in the χ-sum in (4.5).

5. Heuristics

In this section we give a heuristic argument for Theorem 3 (a). For simplicity we assume that E is imaginary quadratic (so that $\widetilde{\mathrm{Cl}}_E$ is just the ideal class group Cl_E), and we also drop the adjoint L-values. We aim to explain what one should expect for the average

$$\frac{1}{|\mathrm{Cl}_E|} \sum_{\chi \in \mathrm{Cl}_E^\vee} \big(L(1/2, \pi_1 \times \chi) L(1/2, \pi_2 \times \chi)\big)^{1/2}.$$

For a function $f : \mathrm{Cl}_E^\vee \to \mathbb{C}$ let

$$\mathbb{E}(f) = \frac{1}{|\mathrm{Cl}_E|} \sum_{\chi \in \mathrm{Cl}_E^\vee} f(\chi).$$

The basic idea is that on GRH we can express $\log L(1/2, \pi_j \times \chi)$ by a short sum over primes, namely

$$\log L(1/2, \pi_j \times \chi) \approx S_{\mathrm{sp}}(\pi_j, \chi) + S_{\mathrm{in}}(\pi_j, \chi) + O(1), \tag{5.1}$$

where

$$S_{\mathrm{sp}}(\pi_j, \chi) = \sum_{\substack{(p)=\mathfrak{p}\bar{\mathfrak{p}} \\ p \le x}} \frac{(\chi(\mathfrak{p}) + \bar{\chi}(\mathfrak{p}))\lambda_j(p)}{p^{1/2}} + \frac{1}{2} \sum_{\substack{(p)=\mathfrak{p}\bar{\mathfrak{p}} \\ p^2 \le x}} \frac{(\chi(\mathfrak{p})^2 + \bar{\chi}(\mathfrak{p})^2)(\lambda_j(p^2) - 1)}{p},$$

$$S_{\mathrm{in}}(\pi_j, \chi) = \frac{1}{2} \sum_{\substack{\eta_E(p)=-1 \\ p^2 \le x}} \frac{2(\lambda_j(p^2) - 1)}{p},$$

$\lambda_j(n)$ denotes the Hecke eigenvalues of π_j, and we think of $x \approx \exp(\log D/\log\log D)$. For the sake of exposition we ignore ramified primes.

Asymptotically we have $\mathbb{E}(S_{\mathrm{sp}}(\pi_j, \chi)) \sim 0$ by orthogonality of characters, and the variance of $S_{\mathrm{sp}}(\pi_j, \chi)$ equals

$$\begin{aligned}
\mathbb{E} \sum_{\substack{(p)=\mathfrak{p}\bar{\mathfrak{p}} \\ p \le x}} \frac{(\chi(\mathfrak{p}) + \bar{\chi}(\mathfrak{p}))^2 \lambda_j(p)^2}{p} + O(1) \\
&= \mathbb{E} \sum_{\substack{(p)=\mathfrak{p}\bar{\mathfrak{p}} \\ p \le x}} \frac{(\chi(\mathfrak{p})^2 + \bar{\chi}(\mathfrak{p})^2 + 2)(\lambda_j(p^2) + 1)}{p} + O(1) \\
&= \sum_{p \le x} \frac{(\lambda_j(p^2) + 1)(1 + \eta_E(p))}{p} + O(1).
\end{aligned} \tag{5.2}$$

Recall that η_E is the quadratic character associated to the extension $E/\mathbb{Q}$ by class field theory, and put $\theta_E = 1 \boxplus \eta_E$. Then (5.2) is $\mathrm{var}_{j,D}(x) + O(1)$, where

$$\mathrm{var}_{j,D}(x) = \log\log x + \log L(1, \eta_E) + \log(1, \mathrm{Ad}\,\pi_j \times \theta_E).$$

The inert sum $S_{\mathrm{in}}(\pi_j, \chi)$ is independent of χ and can be written as

$$S_{\mathrm{in}}(\pi_j, \chi) = \sum_{p^2 \le x} \frac{(\lambda_j(p^2) - 1)\frac{1}{2}(1 - \eta_E(p))}{p} = \mu_{j,D}(x) + O(1),$$

where

$$\mu_{j,D}(x) = \tfrac{1}{2} \log L(1, \eta_E) + \tfrac{1}{2} \log L(1, \mathrm{Ad}\,\pi_j) - \tfrac{1}{2} \log L(1, \mathrm{Ad}\,\pi_j \times \eta_E) - \tfrac{1}{2} \log\log x. \tag{5.3}$$

Assuming that the sums over p behave like independent Gaußian random variables, we conclude

$$\begin{aligned}\mathbb{E}(L(1/2,\pi_j\times\chi)) &\asymp \mathbb{E}\big(\exp(S_{\mathrm{sp}}(\pi_j,\chi))\big)\mathbb{E}\big(\exp(S_{\mathrm{in}}(\pi_j,\chi))\big)\\ &\asymp \exp\big(\tfrac{1}{2}\mathrm{var}_{j,D}(x)\big)\exp(\mu_{j,D}(x)) = L(1,\eta_E)L(1,\mathrm{Ad}\,\pi_j).\end{aligned}$$

Assuming now that the random variables $\log L(1/2,\pi_1\times\chi)$ and $\log L(1/2,\pi_2\times\chi)$ are independent, we obtain by the same argument

$$\begin{aligned}&\mathbb{E}\big((L(1/2,\pi_1\times\chi)L(1/2,\pi_2\times\chi))^{1/2}\big)\\ &\asymp \mathbb{E}\big(\exp(\tfrac{1}{2}S_{\mathrm{sp}}(\pi_1,\chi))\big)\mathbb{E}\big(\exp(\tfrac{1}{2}S_{\mathrm{sp}}(\pi_2,\chi))\big)\mathbb{E}\big(\exp(\tfrac{1}{2}S_{\mathrm{in}}(\pi_1,\chi))\big)\mathbb{E}\big(\exp(\tfrac{1}{2}S_{\mathrm{in}}(\pi_2,\chi))\big)\\ &\asymp \exp\left(\frac{\mathrm{var}_{1,D}(x)+\mathrm{var}_{2,D}(x)}{8}\right)\exp\left(\frac{\mu_{1,D}(x)+\mu_{2,D}(x)}{2}\right)\asymp\frac{(\mathcal{L}_1\mathcal{L}_2)^{1/8}}{(\log x)^{1/4}},\end{aligned}\tag{5.4}$$

where

$$\mathcal{L}_j=\frac{L(1,\eta_E)^3L(1,\mathrm{Ad}\,\pi_j)^3}{L(1,\mathrm{Ad}\,\pi_j\times\eta_E)}.$$

This probabilistic model suggests a final saving of $(\log x)^{-1/4}=(\log D)^{1/4+o(1)}$, in agreement with Theorem 3 (a).

Remark 4. The above computations are sensitive to π_j being cuspidal. In particular, the important term $-\frac{1}{2}\log\log x$ in (5.3) is a reflection of the fact that $\lambda_j(p^2)$ oscillates unlike, for instance, $\tau(p^2)$.

The probabilistic model just described shows what one can reasonably expect, and as such it is a useful tool in stress-testing a proof strategy. But of course these heuristics are far from a proof. In an important paper on moments of the Riemann zeta function [40], Soundararajan made the key observation that (5.1) can in fact be made precise as an *upper bound* for relatively small x. Then one proceeds by computing very high moments

$$\frac{1}{|\mathrm{Cl}_E|}\sum_{\chi\in\mathrm{Cl}_E^\vee}\big(\log L(1/2,\pi_1\times\chi)+\log L(1/2,\pi_2\times\chi)\big)^k,$$

for k as large as about $\log\log D$. From the moments we get sufficient information on the distribution function of $\log L(1/2,\pi_1\times\chi)+\log L(1/2,\pi_2\times\chi)$ (which supports the Gaußian heuristics), and hence also the desired $k=1/2$-moment. We will formalize this argument in Section 9.

Remark 5. We will see later that the probabilistic model of this section is not entirely correct: the random variables $\log L(1/2,\pi_1\times\chi)$ and $\log L(1/2,\pi_2\times\chi)$ are *not* independent, and their correlation is measured by the L-value $L(1,\pi_1\times\pi_2\times\theta_E)$ of degree 8. That this value is well-defined is a consequence of our assumption in Theorem 1 that π_1 and π_2 are distinct (and at least one is cuspidal). The presence of this correlation L-value has no influence on the power of $\log x$. We see, however, that it is important to analyze the L-values at 1 very carefully, which is even on GRH a subtle matter. We refer the reader to the discussion in Sections 6.2 and 9.7.

6. L-functions

In this section, we summarize (and in some cases prove) the various analytic properties of L-functions which will be necessary in the proof of Theorem 3. Throughout, we consider two cuspidal automorphic representations π_1, π_2 for PGL_2 of squarefree levels N_1, N_2 and analytic conductors $Q_{\pi_1} \asymp N_1\lambda_{\pi_2}^2$, $Q_{\pi_2} \asymp N_2\lambda_{\pi_1}^2$, where the spectral parameters λ_{π_i} are defined in §4.1.

6.1. Generalities

We denote by $\lambda_j(n)$, $j = 1, 2$, the Hecke eigenvalues of π_j and record the Hecke relation

$$\lambda_j(p^2) = \lambda_j(p)^2 - \psi_j(p) \tag{6.1}$$

for a prime p where ψ_j is the trivial character modulo N_j. This relation will be used repeatedly throughout our argument.

Recall the Arakelov class group $\widetilde{\mathrm{Cl}}_E$ discussed in Section 2.1. Let $\chi \in \widetilde{\mathrm{Cl}}_E^\vee$ be an everywhere unramified idele class character. By automorphic induction, we may view χ, when convenient, as a theta series of weight 1 if E is imaginary and of weight 0 if E is real. We denote its Dirichlet coefficients by $a_\chi(n)$. Note that χ is in particular trivial on ideals $(n) \subseteq \mathcal{O}_E$ with $n \in \mathbb{N}$, i.e. ideals induced from $\mathbb{Q}$. Since N_1, N_2 are squarefree, the Dirichlet series expansions are given by

$$\begin{aligned} L(s, \pi_j \times \chi) &= L(2s, \eta_E) \sum_n \frac{\lambda_j(n) a_\chi(n)}{n^s}, \\ L(s, \mathrm{Ad}\,\pi_j) &= \zeta^{(N_j)}(2s) \sum_n \frac{\lambda_j(n^2)}{n^s}, \\ L(s, \mathrm{Ad}\,\pi_j \times \eta_E) &= \zeta^{(DN_j)}(2s) \sum_n \frac{\lambda_j(n^2)\eta_E(n)}{n^s} \end{aligned} \tag{6.2}$$

(see [31, §2] and [4, §2.3.3]). We have the Euler product

$$L(s, \pi_j \times \chi) = \prod_p \prod_{i,k=1,2} \left(1 - \frac{\alpha_j(p,i)\xi_\chi(p,k)}{p^s}\right)^{-1} \tag{6.3}$$

where

$$\{\xi_\chi(p,1), \xi_\chi(p,2)\} = \begin{cases} \{\chi(\mathfrak{p}), \bar{\chi}(\mathfrak{p})\}, & (p) = \mathfrak{p}\bar{\mathfrak{p}}, \mathfrak{p} \neq \bar{\mathfrak{p}}, \\ \{\chi(\mathfrak{p}), 0\}, & (p) = \mathfrak{p}^2, \\ \{-1, 1\}, & \eta_E(p) = -1, \end{cases} \tag{6.4}$$

are the Satake parameters of χ and $\alpha_j(p,i)$ are the Satake parameters of π_j. If $(p) = \mathfrak{p}^2$, then $\chi(\mathfrak{p})^2 = 1$, and we conclude

$$\xi_\chi(p,1)^2 + \xi_\chi(p,2)^2 = \begin{cases} \xi_{\chi^2}(p,1) + \xi_{\chi^2}(p,2), & p \text{ split}, \\ 1, & p \text{ ramified}, \\ 2, & p \text{ inert}. \end{cases} \tag{6.5}$$

Let $\theta_E = 1 \boxplus \eta_E$. We use the explicit computations in [31] again to conclude

$$\begin{aligned} L(s, \pi_1 \times \pi_2 \times \theta_E) &\\ &= L(s, \pi_1 \times \pi_2) L(s, \pi_1 \times \pi_2 \times \eta_E) \\ &= \zeta^{(N_1N_2)}(2s)\zeta^{(N_1N_2D)}(2s) \sum_n \frac{\lambda_1(n)\lambda_2(n)}{n^s} \sum_n \frac{\lambda_1(n)\lambda_2(n)\eta_E(n)}{n^s} \\ &\quad \times \prod_{\substack{p \mid (N_1,N_2) \\ p \nmid D}} \Big(1 - \frac{\lambda_1(p)\lambda_2(p)}{p^{s-1}}\Big)\Big(1 - \frac{\lambda_1(p)\lambda_2(p)\eta_E(p)}{p^{s-1}}\Big). \end{aligned}$$

Note that $\lambda_p := \lambda_1(p)\lambda_2(p)p \in \{\pm 1\}$ for $p \mid (N_1, N_2)$, so that

$$\begin{aligned} L(s, \pi_1 \times \pi_2 \times \theta_E) = \sum_n \frac{\lambda_1(n)\lambda_2(n)}{n^s} \sum_n \frac{\lambda_1(n)\lambda_2(n)\eta_E(n)}{n^s} \\ \times \zeta^{(N_1N_2)}(2s)\zeta^{(N_1N_2D)}(2s) \prod_{\substack{p \mid (N_1,N_2) \\ p \nmid D}} \Big(1 - \frac{\lambda_p(1+\eta_E(p))}{p^s} + \frac{\eta_E(p)}{p^{2s}}\Big). \end{aligned} \tag{6.6}$$

6.2. L-functions at $s = 1$ on GRH

We must be very careful with bounds for L-functions on the 1-line. On GRH *and* the Ramanujan conjecture all of them are $(\log\log D)^{O(1)}$ from above and below. Without the Ramanujan conjecture, existing bounds are exponential in $\log D$, which is problematic. Luckily, we do have good bounds in some situations and this suffices for our application. The following lemma is well-known (see e.g. [30, Lemma 5.3] for a special case) and goes essentially back to Littlewood. For convenience we provide a complete proof.

Lemma 5. *Let $L(s, \pi)$ be a holomorphic L-function of fixed degree d and analytic conductor Q_π in the extended[3] Selberg class, not necessarily primitive, with Dirichlet coefficients $\lambda_\pi(n)$. Assume GRH for $L(s, \pi)$ and assume that the Satake parameters $\alpha_\pi(p, j)$, $1 \le j \le d$, satisfy $|\alpha_\pi(p, j)| \le p^{1/2-\rho}$ for some $\rho > 0$. Then*

$$\sum_{p \le x} \frac{\lambda_\pi(p)}{p} = \log L(1, \pi) + O_\varepsilon(1), \quad x \ge (\log Q_\pi)^{2+\varepsilon}. \tag{6.7}$$

Moreover, for $\alpha > 0$ we have

$$\begin{aligned} L(1, \pi) &\gg_\alpha (\log\log Q_\pi)^{-\alpha} \quad \text{if } \lambda_\pi(p) \ge -\alpha \text{ for all } p, \\ L(1, \pi) &\ll_\alpha (\log\log Q_\pi)^{\alpha} \quad \text{if } \lambda_\pi(p) \le \alpha \text{ for all } p. \end{aligned} \tag{6.8}$$

Proof. Let $T > 2$ and $s = \sigma + it$. We start with Perron's formula

$$\frac{1}{2\pi i} \int_{1-iT}^{1+iT} \log L(s+1, \pi) \frac{x^s}{s}\, ds = \sum_{p \le x} \frac{\lambda_\pi(p)}{p} + O\Big(1 + \frac{x \log x}{T}\Big).$$

[3] That is, without assuming the Ramanujan bounds.

By the Borel-Carathéodory inequality and the convexity bound we have

$$\log L(s+1,\pi) \ll (\sigma+1/2)^{-1}\log(Q_\pi(1+|t|))$$

for $\sigma > -1/2$. We shift the contour to $\operatorname{Re} s = -1/2+\delta$ for some small $\delta > 0$. At $s = 0$ we collect a simple pole with residue $\log L(1,\pi)$. The horizontal contours contribute

$$\ll \int_{-1/2+\delta}^{1} |\log L(\sigma+1+iT,\pi)| \frac{x^\sigma}{T}\, d\sigma \ll \frac{x\log(Q_\pi T)}{\delta T}.$$

The vertical contour contributes

$$\ll \int_{-T}^{T} |\log L(1/2+\delta+it,\pi)| \frac{x^{-1/2+\delta}}{1+|t|}\, dt \ll \frac{\log(Q_\pi T)\log T}{\delta x^{1/2-\delta}}.$$

With $T = x^2\log Q_\pi$ and $\delta = 1/\log x$ we conclude

$$\begin{aligned}\log L(1,\pi) &= \sum_{p\le x}\frac{\lambda_\pi(p)}{p} + O\left(1+\frac{x\log x}{T}+\frac{x\log(Q_\pi T)}{\delta T}+\frac{\log(Q_\pi T)\log T}{\delta x^{1/2-\delta}}\right)\\ &= \sum_{p\le x}\frac{\lambda_\pi(p)}{p} + O\left(1+\frac{\log x\,(\log Q_\pi+\log x)(\log x+\log\log Q_\pi)}{x^{1/2}}\right).\end{aligned}$$

The error term is $O_\varepsilon(1)$ if $x \ge (\log Q_\pi)^{2+\varepsilon}$. If $\lambda_\pi(p) \ge -\alpha$ we choose $x = (\log Q_\pi)^3$, getting

$$L(1,\pi) \gg_\alpha \exp(-\alpha\log\log x) \gg_\alpha (\log\log Q_\pi)^{-\alpha}.$$

An analogous argument works for $\lambda_\pi(p) \le \alpha$. ■

In particular, we have

$$\frac{1}{\log\log D} \ll L(1,\eta_E) \ll \log\log D \tag{6.9}$$

(which was proved by Littlewood). Moreover, for π_j as at the beginning of this section, a key observation is that the Hecke relations (6.1) imply that $\lambda_j(p^2) \ge -1$, and then obviously also $\lambda_j(p^2)(1+\eta_E(p)) \ge -2$. Thus

$$\frac{1}{\log\log Q_{\pi_j}} \ll L(1,\operatorname{Ad}\pi_j), \qquad \frac{1}{(\log\log Q_{\pi_j}D)^2} \ll L(1,\operatorname{Ad}\pi_j\times\theta_E). \tag{6.10}$$

We complement this with the additional bounds (cf. [30, Lemma 5.5])

$$\sum_{p\le x}\frac{\lambda_j(p)^2}{p}, \quad \sum_{p\le x}\frac{\lambda_j(p^2)}{p} \ll \log\log x + (\log Q_{\pi_j})^{1/3}. \tag{6.11}$$

6.3. L-functions at $s = 1/2$ on GRH

We now prove a precise version (on GRH) of the heuristic formula (5.1). A crucial ingredient will be the following special case of [6, Theorem 2.1].

Lemma 6. *Assume GRH for $L(1/2, \pi_j \times \chi)$. Then for any $x > 1$ we have*

$$\log L(1/2, \pi_j \times \chi) \leq \sum_{p^n \leq x} \sum_{i=1}^{4} \frac{\alpha_{\pi_j \times \chi}(p,i)^n}{n p^{n(1/2+1/\log x)}} \frac{\log(x/p^n)}{\log x} + 10 \frac{\log Q_{\pi_j \times \chi}}{\log x},$$

where $\{\alpha_{\pi_j \times \chi}(p,i)\}$ are the four Satake parameters of $\pi_j \times \chi$ which can be read off from (6.3)–(6.4).

Upon choosing $x = \log Q_{\pi_j \times \chi}$ and using the prime number theorem, we conclude

$$\log L(1/2, \pi_j \times \chi) \ll \frac{\log Q_{\pi_j \times \chi}}{\log \log Q_{\pi_j \times \chi}}. \tag{6.12}$$

Corollary 7. *Assume GRH for $L(1/2, \pi_j \times \chi)$. Let $\varepsilon > 0$ and suppose that*

$$(\log Q_{\pi_j \times \chi})^{4+\varepsilon} \leq x.$$

Then

$$\begin{aligned} \log L(1/2, \pi_j \times \chi) \leq & \sum_{p \leq x} \frac{a_\chi(p)\lambda_j(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x} \\ & + \frac{1}{2} \sum_{\substack{\eta_E(p)=1 \\ p^2 \leq x}} \frac{a_{\chi^2}(p)(\lambda_j(p^2) - \psi_j(p))}{p^{1+2/\log x}} \frac{\log(x/p^2)}{\log x} \\ & + \mu_{j,D}(x) + 10 \frac{\log Q_{\pi_j \times \chi}}{\log x} + O_\varepsilon(\log \log \log Q_{\pi_j}), \end{aligned} \tag{6.13}$$

where

$$\begin{aligned} \mu_{j,D}(x) = & \tfrac{1}{2} \log L(1, \eta_E) + \tfrac{1}{2} \log L(1, \mathrm{Ad}\, \pi_j) \\ & - \tfrac{1}{2} \log L(1, \mathrm{Ad}\, \pi_j \times \eta_E) - \tfrac{1}{2} \log \log x \end{aligned} \tag{6.14}$$

satisfies

$$\mu_{j,D}(x) \ll_\varepsilon \log \log x + (\log Q_{\pi_j})^{1/3}. \tag{6.15}$$

We also have

$$\sum_{\substack{\eta_E(p)=1 \\ p^2 \leq x}} \frac{a_{\chi^2}(p)(\lambda_j(p^2) - \psi_j(p))}{p^{1+2/\log x}} \frac{\log(x/p^2)}{\log x} \ll_\varepsilon \log \log x + (\log Q_{\pi_j})^{1/3}. \tag{6.16}$$

Proof. We can spell out the main term in Lemma 6 explicitly. Indeed, the contribution from $n = 1$ is the first term on the right-hand side of (6.13). The terms corresponding to $n \geq 3$ contribute $O(1)$ in view of the bound $\alpha_j(p,i) \ll p^\delta$ with $\delta < 1/6$ [29]. We use the Hecke relations (6.1) for the terms corresponding to $n = 2$. Using (6.5), the split primes contribute the second term on the right-hand side of (6.13), while the ramified and inert primes contribute

$$\frac{1}{2}\sum_{\substack{p|D\\p^2\le x}}\frac{(\lambda_j(p^2)-\psi_j(p))}{p^{1+2/\log x}}\frac{\log(x/p^2)}{\log x}+\frac{1}{2}\sum_{\substack{\eta_E(p)=-1\\p^2\le x}}\frac{2(\lambda_j(p^2)-\psi_j(p))}{p^{1+2/\log x}}\frac{\log(x/p^2)}{\log x}$$
$$=\sum_{p^2\le x}\frac{(\lambda_j(p^2)-\psi_j(p))\frac{1}{2}(1-\eta_E(p))+2-2}{p^{1+2/\log x}}\frac{\log(x/p^2)}{\log x}.$$

Since $(\lambda_j(p^2)-\psi_j(p))\frac{1}{2}(1-\eta_E(p))+2\ge 0$, the previous display is at most

$$\sum_{p\le\sqrt{x}}\frac{(\lambda_j(p^2)-\psi_j(p))\frac{1}{2}(1-\eta_E(p))}{p}+\sum_{p\le\sqrt{x}}\frac{2}{p}-\sum_{p\le\sqrt{x}}\frac{2\log(x/p^2)}{p^{1+2/\log x}\log x}. \tag{6.17}$$

From (6.7) and $\log\log\sqrt{x}=\log\log x+O(1)$, the first term in (6.17) is

$$\frac{1}{2}\log\frac{L(1,\eta_E)L(1,\operatorname{Ad}\pi_j)}{L(1,\operatorname{Ad}\pi_j\times\eta_E)\log x}+O_\varepsilon\Big(1+\sum_{p|N_j}\frac{1}{p}\Big)$$

provided that $x\ge(\log Q_{\pi_j\times\chi})^{4+\varepsilon}$, while the second and third terms in (6.17) are

$$2\log\log\sqrt{x}-2\sum_{p\le\sqrt{x}}\frac{1}{p}\Big(1-O\Big(\frac{\log p}{\log x}\Big)\Big)+O(1)=O(1).$$

This establishes (6.13), observing that $\sum_{p|N_j}1/p\ll\log\log\log N_j\le\log\log\log Q_{\pi_j}$. Reversing the analysis, we deduce from (6.7) and (6.11) that

$$|\mu_{j,D}(x)|=\Big|\sum_{p\le\sqrt{x}}\frac{(\lambda_j(p^2)-\psi_j(p))\frac{1}{2}(1-\eta_E(p))}{p}+O(1)\Big|$$
$$\le\sum_{p\le\sqrt{x}}\frac{\lambda_j(p)^2+2}{p}+O_\varepsilon(1)\ll\log\log x+(\log Q_{\pi_j})^{1/3}$$

which establishes (6.15), and by the same argument we conclude (6.16). ■

7. Orthogonality

As outlined in Section 5, our methods are ultimately based on computing high moments of $\log L(1/2,\pi_1\times\chi)L(1/2,\pi_2\times\chi)$, or more generally $\log L(1/2,\pi_1\times\chi^\alpha)L(1/2,\pi_2\times\chi^\beta)$, and by the results of the previous section these values can be upper bounded by short sums over primes. Starting from basic orthogonality relations, the lemmas in this section estimate averages of increasing complexity over the Arakelov class group $\widetilde{\mathrm{Cl}}_E^\vee$ of these short sums over primes. The principal results here are Lemmas 10 and 12, which serve for Theorems 1 and 2, respectively. We emphasize that this section does not invoke GRH, nor in fact do the cusp forms π_j play any role.

For the rest of this paper we fix an even, non-negative Schwartz class function F on $\mathbb{R}$ whose Fourier transform has support in $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$. This will be of use in smoothly truncating the non-compact part of $\widetilde{\mathrm{Cl}}_E^\vee$ for real quadratic fields E. We recall the notation λ_χ as defined in Section 4.3, which in the case of real quadratic fields satisfies $\chi_\infty(x_1, x_2) = |x_1|^{i\lambda_\chi}|x_2|^{-i\lambda_\chi}$.

Lemma 8. *There exists a constant $c > 1$ with the following property. If $\{0\} \neq \mathfrak{a} \subseteq \mathcal{O}_E$ is an ideal with $N\mathfrak{a} < D/4$ and $D > c$, then*

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)\chi(\mathfrak{a}) = 0$$

unless $\mathfrak{a} = (a)$, with $a \in \mathbb{N}$, is an ideal induced from $\mathbb{Q}$.

Proof. We put $\sigma = \eta_E(-1)$, so that the discriminant of E is σD. Recall that $\mathcal{O}_K = \mathbb{Z} + \frac{1}{2}(\sqrt{\sigma D} + \kappa)\mathbb{Z}$ where $\kappa = 0$ if $D \equiv 0 \pmod 4$ and $\kappa = 1$ if $\sigma D \equiv 1 \pmod 4$. The lemma is easy to see if E is imaginary. The sum can only be non-zero if $\mathfrak{a} = (\alpha)$ is a principal ideal. Now

$$\frac{D}{4} > N\alpha = N\left(a + b\frac{\sqrt{\sigma D} + \kappa}{2}\right) = a^2 + \kappa ab + b^2\frac{D+\kappa}{4} \geq b^2\frac{D}{4}$$

implies $b = 0$.

Let us now assume that E is real. Again the sum can only be non-zero if $\mathfrak{a} = (\alpha)$ is principal. Thus our sum becomes

$$\sum_{n \in \mathbb{Z}} F\left(\frac{\pi n}{\log \epsilon}\right)\left|\frac{\alpha}{\alpha'}\right|^{in\pi/\log\epsilon}. \tag{7.1}$$

Changing $\alpha = a + b(\sqrt{D} + \kappa)/2$ by a sign and replacing α with α' if necessary (without loss of generality since F is even), we may assume that $b \geq 0$ and $a + \kappa b/2 \geq 0$. Now changing α by a power of ϵ, we can also assume $\epsilon^{-1} \leq |\alpha/\alpha'| < \epsilon$. By Poisson summation the previous display equals

$$\frac{\log\epsilon}{\pi}\sum_{m \in \mathbb{Z}} \widehat{F}\left(\frac{m}{\pi}\log\epsilon - \frac{1}{2\pi}\log\left|\frac{\alpha}{\alpha'}\right|\right).$$

Since $\log\epsilon \geq \log D + O(1)$ and $D > c$, the support of $\widehat{F}$ implies that the sum consists only of one term and $\epsilon^{-1} \leq |\alpha/\alpha'| < \epsilon$ implies that this term corresponds to $m = 0$. We have

$$\frac{\alpha}{\alpha'} = \frac{a + b(\sqrt{D} + \kappa)/2}{a - b(\sqrt{D} - \kappa)/2} = 1 + \frac{b\sqrt{D}\,(a + b(\sqrt{D} + \kappa)/2)}{N\alpha} \geq 1 + \frac{b^2 D}{N\alpha} \geq 1 + 4b^2,$$

so the term $m = 0$ is outside the support of $\widehat{F}$ for $b \neq 0$ (since $\log 5 > 1$). Hence (7.1) vanishes unless $b = 0$, in which case $\mathfrak{a} = (\alpha)$ is induced from $\mathbb{Q}$. ∎

7.1. High moments of short Dirichlet polynomials

Let $\tilde{a}_\chi$ be the completely multiplicative function with $\tilde{a}_\chi(p) = a_\chi(p)$ where as in the beginning of Section 6.1 we write $a_\chi(n)$ for the Dirichlet coefficients of the theta function induced by χ. Let R be the multiplicative function with

$$R(p^\alpha) = \begin{cases} \binom{\alpha}{\alpha/2}, & \alpha \text{ even, } p \text{ split,} \\ 1, & \alpha \text{ even, } p \text{ ramified,} \\ 0, & \text{otherwise.} \end{cases}$$

In the following we assume that D is sufficiently large in order to apply the previous lemma.

Lemma 9. *Let $\nu \in \mathbb{N}$. Assume that $n < (D/4)^{1/\nu}$ and if ν is even suppose that n is only composed of split primes. Then*

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)\tilde{a}_{\chi^\nu}(n) = R(n) \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi).$$

Proof. If n is divisible by an inert prime, then $\tilde{a}_\chi(n) = 0$ for any $\chi \in \widetilde{\mathrm{Cl}}_E^\vee$, so both sides vanish. We may therefore assume that n is supported on non-inert primes and write $n = \prod p_j^{\alpha_j} \prod q_k^{\beta_k}$ with pairwise distinct split primes $p_j = \mathfrak{p}_j \bar{\mathfrak{p}}_j$ and pairwise distinct ramified primes $q_k = \mathfrak{q}_k^2$. We obtain

$$\begin{aligned} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)\tilde{a}_{\chi^\nu}(n) &= \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \prod (\chi^\nu(\mathfrak{p}_j) + \bar{\chi}^\nu(\mathfrak{p}_j))^{\alpha_j} \prod \chi^\nu(\mathfrak{q}_k)^{\beta_k} \\ &= \sum_{\gamma_1 \le \alpha_1, \gamma_2 \le \alpha_2, \dots} \binom{\alpha_1}{\gamma_1}\binom{\alpha_2}{\gamma_2} \cdots \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)\chi^\nu(\mathfrak{p}_1^{\gamma_1}\bar{\mathfrak{p}}_1^{\alpha_1-\gamma_1} \cdots \mathfrak{q}_1^{\beta_1} \cdots). \end{aligned}$$

By Lemma 8, the inner sum vanishes unless $(\mathfrak{p}_1^{\gamma_1}\bar{\mathfrak{p}}_1^{\alpha_1-\gamma_1} \cdots \mathfrak{q}_1^{\beta_1} \cdots)^\nu$ is a rational integer. This is the case precisely when $2\gamma_j = \alpha_j$ for all j, and the β_k are even for all k (here we use the assumption $\beta_k = 0$ if ν is even). These conditions give exactly the definition of $R(n)$. ■

The following is inspired by [30, Lemma 4.3]. It is a central ingredient in the proof of Theorem 1.

Lemma 10. *Let $\nu \in \mathbb{N}$. Let $x \ge 2$, $k \in \mathbb{N}$ with $x^{2k} < (D/4)^{1/\nu}$. For any sequence of complex numbers $b(p)$ indexed by primes (split primes if ν is even) we have*

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)\Big(\sum_{p \le x} \frac{a_{\chi^\nu}(p)b(p)}{\sqrt{p}}\Big)^{2k} \le \frac{(2k)!}{k!}\Big(\frac{1}{2}\sum_{p \le x} \frac{(1+\eta_E(p))b(p)^2}{p}\Big)^k \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi).$$

Proof. We extend $b(p)$ to all integers as a completely multiplicative function. Let $p_j(n)$ be the characteristic function on numbers with j prime factors (counted with multiplicity) and ν the multiplicative function with $\nu(p^\alpha) = \alpha!$. With these notational conventions we have

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \Big(\sum_{p \le x} \frac{a_{\chi^\nu}(p) b(p)}{\sqrt{p}} \Big)^{2k} = \sum_{\substack{n \ge 1 \\ p \mid n \Rightarrow p \le x}} \frac{(2k)!}{\nu(n)} \frac{b(n) p_{2k}(n)}{\sqrt{n}} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \tilde{a}_{\chi^\nu}(n).$$

We conclude from Lemma 9 that the inner sum vanishes unless n is a square, so that the previous display equals

$$(2k)! \sum_{\substack{n \ge 1 \\ p \mid n \Rightarrow p \le x}} \frac{b(n)^2 p_k(n)}{n} \frac{R(n^2)}{\nu(n^2)} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi).$$

It follows from the definitions that

$$\frac{R(p^{2\alpha})}{\nu(p^{2\alpha})} = \left. \begin{cases} (\alpha!)^{-2} \le (\alpha!)^{-1} = \nu(p^\alpha)^{-1}, & p \text{ split} \\ (2\alpha)!^{-1} \le (2^\alpha \alpha!)^{-1} = (2^\alpha \nu(p^\alpha))^{-1}, & p \text{ ramified} \end{cases} \right\} \le \frac{1 + \eta_E(p)}{2\nu(p^\alpha)}.$$

Denoting by $r(n)$ the completely multiplicative function extending $r(p) = 1 + \eta_E(p)$, we obtain

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \Big(\sum_{p \le x} \frac{a_{\chi^\nu}(p) b(p)}{\sqrt{p}} \Big)^{2k} \le \frac{(2k)!}{k!} \Bigg(\frac{1}{2^k} \sum_{\substack{n \ge 1 \\ p \mid n \Rightarrow p \le x}} \frac{k!}{\nu(n)} \frac{r(n) b(n)^2 p_k(n)}{n} \Bigg) \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi).$$

The claim follows. ∎

7.2. *High moments of short mixed Dirichlet polynomials*

For the proof of Theorem 2 we need slightly more advanced combinatorics. We fix two *distinct* positive integers $\alpha, \beta \in \mathbb{N}$. For $n \in \mathbb{N}$, $0 \le m \le n$ we define

$$B_{\alpha,\beta}(n, m) = \sum_{\substack{r=0 \\ 2\alpha r + (\beta - \alpha) m - \beta n + 2\beta s = 0}}^{m} \sum_{s=0}^{n-m} \binom{n}{m} \binom{m}{r} \binom{n-m}{s}.$$

One checks directly that

$$B_{\alpha,\beta}(n, 0) = \delta_{2 \mid n} \binom{n}{n/2}, \quad B_{\alpha,\beta}(n, m) = B_{\beta,\alpha}(n, n-m). \tag{7.2}$$

We need to compute a few more values by hand: for distinct positive integers α, β we have

$$B_{\alpha,\beta}(4, 2) = 6 \sum_{\substack{r=0 \\ \alpha r + \beta s = \alpha + \beta}}^{2} \sum_{s=0}^{2} \binom{2}{r} \binom{2}{s} = 6 \binom{2}{1} \binom{2}{1} = 24 \tag{7.3}$$

and

$$B_{\alpha,\beta}(6,2) = 15 \sum_{\substack{r=0 \\ \alpha r+\beta s=\alpha+3\beta}}^{2} \sum_{s=0}^{4} \binom{2}{r}\binom{4}{s}$$
$$\le 15\left(\binom{2}{0}\binom{4}{3} + \binom{2}{1}\binom{4}{3} + \binom{2}{2}\binom{4}{2}\right)$$
$$\le 270. \tag{7.4}$$

Estimating even more coarsely (the middle binomial coefficient is the largest), we have

$$B_{\alpha,\beta}(8,2) \le \binom{8}{2}2^2\binom{6}{3} = 2240, \quad B_{\alpha,\beta}(8,4) \le \binom{8}{4}2^4\binom{4}{2} = 6720. \tag{7.5}$$

Finally, we record the trivial bound

$$B_{\alpha,\beta}(n,m) \le \binom{n}{m}2^n. \tag{7.6}$$

From (7.2)–(7.5) for $r \le 4$ and (7.6) for $r > 4$ we conclude

$$\frac{r!}{(2r)!}B_{\alpha,\beta}(2r,2m) \le \binom{r}{m}. \tag{7.7}$$

Let v_2 the denote the usual 2-adic valuation. Then

$$B_{\alpha,\beta}(n,m) = 0 \quad \text{unless} \quad \begin{cases} 2 \mid n, & v_2(\alpha) = v_2(\beta), \\ 2 \mid m, & v_2(\alpha) < v_2(\beta), \\ 2 \mid n-m, & v_2(\alpha) > v_2(\beta). \end{cases} \tag{7.8}$$

The following lemma should be compared with Lemma 9.

Lemma 11. *Let $b(p)$, $c(p)$ be any sequences indexed by split primes. Let f_χ be the completely multiplicative function whose values at primes are given by $f_\chi(p) = a_{\chi^\alpha}(p)b(p) + a_{\chi^\beta}(p)c(p)$. For $n < (D/4)^{1/\max(\alpha,\beta)}$ we have*

$$\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)f_\chi(n) = H_{\alpha,\beta}(n)\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \tag{7.9}$$

where $H_{\alpha,\beta}$ is multiplicative and given by

$$H_{\alpha,\beta}(p^\nu) = \sum_{m=0}^{\nu} B_{\alpha,\beta}(\nu,m)b(p)^m c(p)^{\nu-m}.$$

Remark 6. For n consisting only of split primes, (7.2) ensures that (7.9) is supported only on squarefull n, but this property fails if n has ramified prime factors. This would make later estimates in Section 9.8 more cumbersome. For simplicity we exclude ramified primes factors which is reflected in the assumption on trivial 2-torsion in Theorem 2.

Proof of Lemma 11. For a split prime $p = \mathfrak{p}\bar{\mathfrak{p}}$ and $\nu \in \mathbb{N}$ we have

$$f_\chi(p^\nu) = \sum_{m=0}^{\nu} \binom{\nu}{m} \big(a_{\chi^\alpha}(p)b(p)\big)^m \big(a_{\chi^\beta}(p)c(p)\big)^{\nu-m}$$
$$= \sum_{m=0}^{\nu} \binom{\nu}{m} b(p)^m c(p)^{\nu-m} \sum_{r=0}^{m} \binom{m}{r} \sum_{s=0}^{\nu-m} \binom{\nu-m}{s} \chi(\mathfrak{p}^{\alpha r+\beta s}\bar{\mathfrak{p}}_j^{\alpha(m-r)+\beta(\nu-m-s)}).$$

We write $n = \prod_j p_j^{\nu_j}$ with pairwise distinct split primes $p_j = \mathfrak{p}_j \bar{\mathfrak{p}}_j$, so that $f_\chi(n)$ equals

$$\prod_j \sum_{m_j=0}^{\nu_j} \sum_{r_j=0}^{m_j} \sum_{s_j=0}^{\nu_j-m_j} \binom{\nu_j}{m_j}\binom{m_j}{r_j}\binom{\nu_j-m_j}{s_j} b(p_j)^{m_j} c(p_j)^{\nu_j-m_j} \times \chi(\mathfrak{p}_j^{\alpha r_j+\beta s_j}\bar{\mathfrak{p}}_j^{\alpha(m_j-r_j)+\beta(\nu_j-m_j-s_j)}).$$

Summing $F(\lambda_\chi) f_\chi(n)$ over χ, we see by Lemma 8 that only those terms with

$$2\alpha r_j + (\beta-\alpha)m_j - \beta\nu_j + 2\beta s_j = 0$$

survive, so that the left-hand side of (7.9) equals

$$\prod_j \sum_{m_j=0}^{\nu_j} B_{\alpha,\beta}(\nu_j, m_j) b(p_j)^{m_j} c(p_j)^{\nu_j-m_j} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi).$$

This features precisely the function $H_{\alpha,\beta}$ specified in the lemma. ■

The following result should of course be compared to Lemma 10.

Lemma 12. *Let $\alpha, \beta \in \mathbb{N}$ be distinct positive integers. Let $b(p)$, $c(p)$ be any sequences of real numbers indexed by split primes. Let $x \geq 2$, $k \in \mathbb{N}$ with $x^{2k} < (D/4)^{1/\max(\alpha,\beta)}$. Then*

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \Big(\sum_{p \leq x} \frac{a_{\chi^\alpha}(p)b(p) + a_{\chi^\beta}(p)c(p)}{\sqrt{p}} \Big)^{2k}$$

is bounded by

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi) \begin{cases} \displaystyle\sum_{2\ell_1+3\ell_2=2k} \frac{(2k)!}{(\ell_1)!(\ell_2)!} \Big(\sum_{p\leq x} \frac{b(p)^2+c(p)^2}{p} \Big)^{\ell_1} \Big(c_0 \sum_{p \leq x} \frac{b(p)^2|c(p)|}{p^{3/2}} \Big)^{\ell_2}, & v_2(\alpha) \neq v_2(\beta), \\ \displaystyle\sum_{2\ell_1+4\ell_2=2k} \frac{(2k)!}{(\ell_1)!(\ell_2)!} \Big(\sum_{p\leq x} \frac{b(p)^2+c(p)^2}{p} \Big)^{\ell_1} \Big(\sum_{p \leq x} \frac{(|b(p)|+|c(p)|)^4}{p^2} \Big)^{\ell_2}, & v_2(\alpha) = v_2(\beta). \end{cases}$$

for some absolute constant c_0 (one can take $c_0 = 4$).

Proof. We use the notation from Lemmas 10–11 and their proofs. We have

$$\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi)\Big(\sum_{p\le x}\frac{a_{\chi^\alpha}(p)b(p)+a_{\chi^\beta}(p)c(p)}{\sqrt{p}}\Big)^{2k} = (2k)!\sum_{p|n\Rightarrow p\le x}\frac{p_{2k}(n)}{\nu(n)\sqrt{n}}H_{\alpha,\beta}(n)\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee}F(\lambda_\chi).$$

Let us now consider the right-hand side of the claimed inequality according to the two different cases.

Case 1. Suppose that $v_2(\alpha)\neq v_2(\beta)$, without loss of generality $v_2(\alpha)<v_2(\beta)$. Let Φ,Ψ be the completely multiplicative function extending $\Phi(p)=b(p)^2+c(p)^2$ and $\Psi(p)=b(p)^2|c(p)|$. Note that Φ,Ψ are non-negative. Now the right-hand side of the bound in Lemma 12 equals

$$(2k)!\sum_{2\ell_1+3\ell_2=2k}\sum_{p|n_1\Rightarrow p\le x}\frac{p_{\ell_1}(n_1)\Phi(n_1)}{\nu(n_1)n_1}\sum_{p|n_2\Rightarrow p\le x}\frac{p_{\ell_2}(n_2)\Psi(n_2)}{\nu(n_2)n_2^{3/2}}.$$

Note that $H_{\alpha,\beta}$ is supported only on squarefull numbers (cf. (7.2)). Decompose uniquely $n=n_1^2n_2^3$ with $\mu(n_2)^2=1$. It then suffices to show

$$\frac{H_{\alpha,\beta}(n)}{\nu(n)}\le\frac{\Phi(n_1)\Psi(n_2)}{\nu(n_1)\nu(n_2)}$$

and again by multiplicativity

$$\frac{H_{\alpha,\beta}(p^{2r})}{(2r)!}\le\frac{(b(p)^2+c(p)^2)^r}{r!},\qquad \frac{H_{\alpha,\beta}(p^{2r+3})}{(2r+3)!}\le\frac{(b(p)^2+c(p)^2)^r\cdot b^2(p)|c(p)|}{r!}.$$

Both inequalities of the last display follow from (7.8) and (7.7):

$$\frac{r!}{(2r)!}H_{\alpha,\beta}(p^{2r})\le\sum_{m=0}^{r}\binom{r}{m}b(p)^{2m}c(p)^{2r-2m}=(b(p)^2+c(p)^2)^r$$

and

$$\begin{aligned}\frac{r!}{(2r+3)!}H_{\alpha,\beta}(p^{2r+3})&\le\frac{r!}{(2r+3)!}\sum_{m=0}^{r}\binom{2r+3}{2m+2}2^{2r+3}b(p)^{2m+2}c(p)^{2r+1-2m}\\&\le c_0\sum_{m=0}^{r}\binom{r}{m}b(p)^{2m}c(p)^{2r-2m}b^2(p)|c(p)|\\&=c_0(b(p)^2+c(p)^2)^rb^2(p)|c(p)|,\end{aligned}$$

for an absolute constant $c_0>0$.

Case 2. Next suppose that $v_2(\alpha)=v_2(\beta)$. We argue similarly. Let $\tilde\Psi$ be the completely multiplicative function extending $\tilde\Psi(p)=(|b(p)|+|c(p)|)^4$. The right-hand side of the bound in Lemma 12 equals

$$(2k)! \sum_{2\ell_1+4\ell_2=2k} \sum_{p|n_1 \Rightarrow p\le x} \frac{p_{\ell_1}(n_1)\Phi(n_1)}{\nu(n_1)n_1} \sum_{p|n_2\Rightarrow p\le x} \frac{p_{\ell_2}(n_2)\tilde{\Psi}(n_2)}{\nu(n_2)n_2^2}.$$

Note that $H_{\alpha,\beta}$ is supported only on squares (cf. (7.8)). It then suffices to show

$$\frac{H_{\alpha,\beta}(p^2)}{2!} \le b(p)^2 + c(p)^2, \quad \frac{H_{\alpha,\beta}(p^{2r})}{(2r)!} \le \frac{(b(p)^2+c(p)^2)^{r-2}\cdot(|b(p)|+|c(p)|)^4}{(r-2)!}$$

for $r \ge 2$. We have $H_{\alpha,\beta}(p^2) = 2(b(p)^2 + c(p)^2)$ [this uses $\alpha \ne \beta$ and is actually the only point where this assumption is used] and

$$H_{\alpha,\beta}(p^{2r}) \le \sum_{m=0}^{2r} \binom{2r}{m} 2^{2r} |b(p)|^m |c(p)|^{2r-m} = 2^{2r}(|b(p)|+|c(p)|)^{2r}.$$

As $b(p)^2 + c(p)^2 \ge \frac{1}{2}(|b(p)|+|c(p)|)^2$, the last inequality follows from

$$\frac{2^{2r}}{(2r)!} \le \frac{1}{2^{r-2}(r-2)!}$$

for $r \ge 2$. This completes the proof. ∎

8. A bound on the second moment

In preparation for the proof of Theorem 3, we start with a bound for the second moment of $L(1/2, \pi_j \times \chi)$. This will only serve a technical purpose to exclude very large values of L-functions. We continue to denote by F an even non-negative Schwartz-class function whose Fourier transform has support in $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$. Note that for $Q > 1$ the function $x \mapsto F(x/Q)$ is still an even non-negative Schwartz-class function whose Fourier transform has support in $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$. For later purposes we record for $k \in \mathbb{N}$ the elementary estimate

$$\frac{(2k)!}{k!} \le \sqrt{2}\left(\frac{4}{e}k\right)^k. \tag{8.1}$$

(The constant $\sqrt{2}$ plays no role in the following.) We observe that, when E is imaginary, the conductor of $\pi \times \chi$ is constant within the family of $\chi \in \widetilde{\mathrm{Cl}}_E^\vee$ and depend only on D. If E is real, the conductor of $\pi \times \chi$ does depend on χ, via its archimedean component χ_∞. In either case, it is a consequence of the class number formula, and the fact that λ_χ runs through a one-dimensional lattice of volume $\pi/\log\epsilon$ when E is real, that

$$\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi/Q) \ll QH_E, \quad H_E := L(1,\eta_E)\sqrt{D} \tag{8.2}$$

for $Q \ge 1$. Note that $H_E \asymp \mathrm{vol}(\widetilde{\mathrm{Cl}}_E)$ by (2.2).

Since we allow polynomial dependence on the representations π_1, π_2 in the bound of Theorem 3, we will assume throughout this section that

$$Q_\pi \leq D^{1/10},$$

so that

$$\log Q_{\pi_j \times \chi} \ll \log D$$

provided that $\lambda_\chi \leq D$. Moreover, we assume for the rest of the paper that D is sufficiently large (in terms of ε). We also fix $j \in \{1, 2\}$ and drop it from the notation. With this in mind we define

$$\mathscr{C} = \log\log D + (\log Q_\pi)^{1/3}. \tag{8.3}$$

We keep the general assumptions and notations from Section 6.1.

Lemma 13. *Assume that* $Q_\pi \leq D^{1/10}$*, and let* $1 \leq Q \leq D^{1/5}$*. For* $\varepsilon > 0$ *we have*

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q}\right) L(1/2, \pi \times \chi)^2 \ll \exp(O_\varepsilon(\mathscr{C}^{1+\varepsilon}))QH_E.$$

Proof. For $\mathcal{V} \in \mathbb{R}$ we define

$$\mathcal{S}(\mathcal{V}) := \sum_{L(1/2, \pi \times \chi) > e^{\mathcal{V}}} F^*\left(\frac{\lambda_\chi}{Q}\right),$$

where $F^*(x) = F(x)\delta_{|x| \leq D/Q}$. Using the convexity bound for $L(1/2, \pi \times \chi)$ and the rapid decay of F and then partial summation, we have

$$\begin{aligned}\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F(\lambda_\chi/Q) L(1/2, \pi \times \chi)^2 &= \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F^*(\lambda_\chi/Q) L(1/2, \pi \times \chi)^2 + O(D^{-10}) \\ &= \int_{\mathbb{R}} e^{\mathcal{V}} \mathcal{S}(\mathcal{V}/2)\, d\mathcal{V} + O(D^{-10}).\end{aligned}$$

We may truncate the integral at $\mathcal{V} \ll \log D / \log\log D$, in view of (6.12), since otherwise $\mathcal{S}(\mathcal{V}/2) = 0$. Moreover, using the trivial bound $\mathcal{S}(\mathcal{V}/2) \leq \sum F(\lambda_\chi/Q)$ and (8.2), we have

$$\int_{-\infty}^{A\mathscr{C}^{1+\varepsilon}} e^{\mathcal{V}} \mathcal{S}(\mathcal{V}/2)\, d\mathcal{V} \leq \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q}\right) \int_{-\infty}^{A\mathscr{C}^{1+\varepsilon}} e^{\mathcal{V}}\, d\mathcal{V} \ll QH_E e^{A\mathscr{C}^{1+\varepsilon}}$$

for $A > 0$. Put $V = \mathcal{V}/2$. From the above considerations, we may now assume that V satisfies

$$A\mathscr{C}^{1+\varepsilon} \leq V \leq B \frac{\log D}{\log\log D} \tag{8.4}$$

for some sufficiently large constants A, B. We shall show that

$$\mathcal{S}(V) \leq \exp(-c(\varepsilon) V \log V) QH_E \tag{8.5}$$

for V satisfying (8.4) and some $c(\varepsilon) > 0$. In this way,

$$\int_{2A\mathscr{C}^{1+\varepsilon}}^{\infty} e^{\mathcal{V}} \mathcal{S}(\mathcal{V}/2)\, d\mathcal{V} \ll_{\varepsilon} QH_E,$$

which suffices for the proof of the lemma.

The rest of the proof is devoted to (8.5). Choose

$$x = D^{5B/V} = \exp\left(\frac{5B}{V} \log D\right), \tag{8.6}$$

which by (8.4) implies $x \geq \exp(5 \log\log D) = (\log D)^5 \geq (\log Q_{\pi\times\chi})^{4+\epsilon}$. For χ counted by $\mathcal{S}(V)$ we apply Corollary 7, and conclude from (6.13), (6.15) and (6.16) that

$$\begin{aligned} V &\leq \log L(1/2, \pi \times \chi) \\ &\leq \sum_{p \leq x} \frac{a_\chi(p)\lambda_\pi(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x} + O\left(\log\log x + (\log Q_\pi)^{1/3} + \frac{\log D}{\log x}\right) \end{aligned} \tag{8.7}$$

Recalling (8.3) and taking B sufficiently large in (8.6) we find

$$V \leq \sum_{p \leq x} \frac{a_\chi(p)\lambda_\pi(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x} + \frac{1}{4} V + O(\mathscr{C}).$$

Hence if A in (8.4) is sufficiently large, we have

$$\frac{1}{2} V < \sum_{p \leq x} \frac{a_\chi(p)\lambda_\pi(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x}.$$

For any $k \geq 0$, this implies (by positivity)

$$\mathcal{S}(V) \leq \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q}\right) \frac{2^{2k}}{V^{2k}} \left(\sum_{p \leq x} \frac{a_\chi(p)\lambda_\pi(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x}\right)^{2k}.$$

Lemma 10 (with $\nu = 1$) then shows that, as long as k satisfies $x^{2k} < D/4$, we have

$$\mathcal{S}(V) \leq \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q}\right) \frac{(2k)!}{k!} \frac{2^{2k}}{V^{2k}} \left(\sum_{p \leq x} \frac{\lambda_\pi(p)^2}{p}\right)^{k},$$

where we have used the simple inequality $|1 + \eta_E(p)| \leq 2$. Now by (6.11) and (8.6) we have

$$\sum_{p \leq x} \frac{\lambda_\pi(p)^2}{p} \ll \mathscr{C}.$$

Recalling (8.1) and (8.2) we deduce

$$\mathcal{S}(V) \ll QH_E \frac{(2k)!}{k!} \frac{2^{2k}}{V^{2k}} (O(\mathscr{C}))^k \leq QH_E \left(\frac{16k}{eV^2} O(\mathscr{C})\right)^{k}.$$

The condition $x^{2k} < D/4$ allows us to take

$$k < \frac{V}{11B} \asymp V.$$

For such k we have $k\mathscr{C}/V^2 \ll V^{-\varepsilon/(1+\varepsilon)}$ by (8.4). Inserting this proves (8.5). ∎

9. Proof of Theorem 3

In this section we finally prove Theorem 3. We first prove part (a), and then in Section 9.8 pass to the proof of part (b). In both cases, the proof scheme is similar to the proof of Lemma 13, but the estimates are more delicate.

9.1. Setting up the proof of Theorem 3 (a)

For $\chi \in \widetilde{\mathrm{Cl}}_E^\vee$ let

$$\mathcal{L}(\chi) = L(1/2, \pi_1 \times \chi)L(1/2, \pi_2 \times \chi). \tag{9.1}$$

We continue to assume that D is sufficiently large and $Q_{\pi_1}, Q_{\pi_2} \leq D^{1/10}$. For $x > 0$ we define

$$\begin{aligned}\mu_D(x) = \tfrac{1}{2}\log L(1,\eta_E) + \tfrac{1}{4}\log L(1,\mathrm{Ad}\,\pi_1) + \tfrac{1}{4}\log L(1,\mathrm{Ad}\,\pi_2)\\ - \tfrac{1}{4}\log L(1,\mathrm{Ad}\,\pi_1\times\eta_E) - \tfrac{1}{4}\log L(1,\mathrm{Ad}\,\pi_2\times\eta_E) - \tfrac{1}{2}\log\log x\end{aligned} \tag{9.2}$$

and note that by (6.14) we have $\mu_D(x) = \frac{1}{2}(\mu_{1,D}(x) + \mu_{2,D}(x))$. We also define

$$\begin{aligned}\mathrm{var}_D(x) = \tfrac{1}{2}\log\log x + \tfrac{1}{2}\log L(1,\eta_E) + \tfrac{1}{4}\log L(1,\mathrm{Ad}\pi_1\times\theta_E)\\ + \tfrac{1}{4}\log L(1,\mathrm{Ad}\pi_2\times\theta_E) + \tfrac{1}{2}\log L(1,\pi_1\times\pi_2\times\theta_E).\end{aligned} \tag{9.3}$$

We write $\mu_D = \mu_D(D)$ and $\mathrm{var}_D = \mathrm{var}_D(D)$.

Recalling the notation H_E from (8.2), our primary goal is to prove the bound

$$\frac{1}{H_E}\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right)\mathcal{L}(\chi)^{1/2} \ll_\varepsilon \exp\bigl(\mu_D + (1/2+\varepsilon)\,\mathrm{var}_D\bigr) + \exp\bigl(-\tfrac{1}{3}\sqrt{\log D}\bigr) \tag{9.4}$$

provided that $Q_{\pi_1}, Q_{\pi_2} \leq D^{1/10}$. Here and henceforth all implied constants are allowed to depend polynomially on Q_{π_1} and Q_{π_2}. The proof of (9.4) extends over the next few subsections and will be completed in Section 9.6. We then show in Section 9.7 how to deduce Theorem 3 (a) from (9.4).

9.2. Some useful bounds

We now relate μ_D and var_D to short Dirichlet polynomials. Let

$$\begin{aligned}\mathscr{L}(s) := \tfrac{1}{2}\log L(s,\eta_E) + \tfrac{1}{4}\log L(s,\mathrm{Ad}\,\pi_1\times\theta_E) + \tfrac{1}{4}\log L(s,\mathrm{Ad}\,\pi_2\times\theta_E)\\ + \tfrac{1}{2}\log L(s,\pi_1\times\pi_2\times\theta_E).\end{aligned}$$

By (6.2), (6.6) and (6.1) the Dirichlet series coefficients of $\mathscr{L}(s)$ at primes $p \nmid N_1N_2$ are

$$\frac{2\eta_E(p) + (1+\eta_E(p))(\lambda_1(p^2) + 2\lambda_1(p)\lambda_2(p) + \lambda_2(p^2))}{4} = \frac{(1+\eta_E(p))(\lambda_1(p)+\lambda_2(p))^2 - 2}{4}.$$

Similar formulae hold if p divides exactly one of N_1, N_2 and if it divides both N_1 and N_2. We write this collectively as

$$\frac{(1+\eta_E(p))(\lambda_1(p)+\lambda_2(p))^2 - \kappa_1(p)}{4}, \quad \kappa_1(p) \in \{2, 1-\eta_E(p), 4\eta_E - 2\}.$$

We deduce from this that

$$\tfrac{1}{2}\log\log D + O(\log\log\log D) \leq \mathrm{var}_D \ll (\log D)^{1/3}, \tag{9.5}$$

where we use (6.8) for the lower bound and (6.7), (6.9) and (6.11) for the upper bound. Moreover, using (6.7), we deduce that

$$\begin{aligned}\sum_{p\leq x} \frac{(1+\eta_E(p))(\lambda_1(p)+\lambda_2(p))^2}{2p} &= 2\mathscr{L}(1) + O_\varepsilon(1) + \log\log x + O\Big(\sum_{p\mid N_1N_2}\frac{1}{p}\Big)\\ &= 2\mathrm{var}_D(x) + O_\varepsilon(\log\log\log N_1N_2) \qquad (9.6)\end{aligned}$$

provided that $x \geq (\log Q_{\pi_1}Q_{\pi_2}D)^{2+\varepsilon}$.

By a similar computation, the Dirichlet coefficients of

$$-\log L(s, \mathrm{Ad}\,\pi_1 \times \theta_E) - \log L(s, \mathrm{Ad}\,\pi_2 \times \theta_E) + 2\log L(s, \pi_1 \times \pi_2 \times \theta_E)$$

at primes p are

$$-(1+\eta_E)\big((\lambda_1(p) - \lambda_2(p))^2 - \kappa(p)\big), \quad \kappa_2(p) \in \{2, 1, 0\}.$$

We conclude by (6.8) that

$$\frac{L(1, \pi_1 \times \pi_2 \times \theta_E)}{L(1, \mathrm{Ad}\,\pi_1 \times \theta_E)L(1, \mathrm{Ad}\,\pi_2 \times \theta_E)} \ll (\log\log Q_{\pi_1}Q_{\pi_2}D)^4. \tag{9.7}$$

Analogously the Dirichlet coefficients of

$$-\log L(s, \mathrm{Ad}\,\pi_1) - \log L(s, \mathrm{Ad}\,\pi_2) + \log L(s, \pi_1 \times \pi_2 \times \theta_E)$$

at primes p are

$$\kappa_3(p) + (1+\eta_E(p))\lambda_1(p)\lambda_2(p) - \lambda_1(p)^2 - \lambda_2(p)^2$$

with $\kappa_3(p) \leq 2$, so that

$$\frac{L(1, \pi_1 \times \pi_2 \times \theta_E)}{L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_2)} \ll (\log\log Q_{\pi_1}Q_{\pi_2}D)^2. \tag{9.8}$$

9.3. Preliminary reductions

As in the proof of Lemma 13, we can restrict to characters with $|\lambda_\chi| \leq D$ by the rapid decay of F. Again we write $F^*(x) = F(x)\delta_{|x|\leq D/(Q_{\pi_1}Q_{\pi_2})}$. Recalling the notation (9.1), we define, similarly to the previous section,

$$\mathcal{T}(V) := \sum_{\mathcal{L}(\chi)>e^V} F^*\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big).$$

We first deal with large values of V. For $U \geq 1$ we have

$$\sum_{\mathcal{L}(\chi)\geq U} F\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big)\mathcal{L}(\chi)^{1/2} \leq \frac{1}{U^{1/2}} \sum_{\mathcal{L}(\chi)\geq U} F\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big)\mathcal{L}(\chi)$$
$$\leq \frac{1}{U^{1/2}} \sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big)\big(L(1/2,\pi_1\times\chi)^2 + L(1/2,\pi_2\times\chi)^2\big).$$

We now invoke Lemma 13, and recall the definition of $\mathscr{C}$ from (8.3), to obtain

$$\sum_{\mathcal{L}(\chi)\geq U} F\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big)\mathcal{L}(\chi)^{1/2} \ll U^{-1/2}\exp((\log D)^{2/5})H_E.$$

We may therefore treat all χ with $\mathcal{L}(\chi) > \exp(\sqrt{\log D})$ trivially and estimate their contribution by

$$\sum_{\mathcal{L}(\chi)>\exp(\sqrt{\log D})} F\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big)\mathcal{L}(\chi)^{1/2} \ll H_E\exp\big(-\tfrac{1}{3}\sqrt{\log D}\big).$$

This is admissible for (9.4).

By partial summation it now remains to estimate

$$\int_{-\infty}^{(\log D)^{1/2}} e^{V/2}\mathcal{T}(V)\,dV = \exp(\mu_D)\int_{-\infty}^{(\log D)^{1/2}-2\mu_D} e^{V/2}\mathcal{T}(V+2\mu_D)\,dV.$$

The contribution of $V \leq \varepsilon\log\log D$ can be estimated trivially by

$$\exp(\mu_D)\int_{-\infty}^{\varepsilon\log\log D} e^{V/2}\mathcal{T}(V+2\mu_D)\,dV$$
$$\leq \sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F\Big(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Big)\exp(\mu_D)\int_{-\infty}^{\varepsilon\log\log D} e^{V/2}\,dV$$
$$\ll H_E\exp\Big(\mu_D + \frac{\varepsilon}{2}\log\log D\Big) \ll H_E\exp(\mu_D + O(\varepsilon)\,\mathrm{var}_D),$$

where we have used (9.5) in the last step. This is again admissible for (9.4), perhaps after redefining ε.

The hardest part is to estimate

$$\exp(\mu_D) \int_{\varepsilon \log\log D}^{(\log D)^{1/2} - 2\mu_D} e^{V/2} \mathcal{T}(V + 2\mu_D)\, dV. \tag{9.9}$$

Recalling the definition of μ_D in §9.1, and applying (6.15), we have

$$(\log D)^{1/2} - 2\mu_D \asymp (\log D)^{1/2}. \tag{9.10}$$

Henceforth we restrict V to the interval

$$\varepsilon \log\log D \le V \le (\log D)^{1/2} - 2\mu_D \ll (\log D)^{1/2}. \tag{9.11}$$

9.4. Application of Corollary 7

For V as in (9.11) we choose

$$x = D^{A/\varepsilon V} = \exp\left(\frac{A}{\varepsilon V} \log D\right) \tag{9.12}$$

for a sufficiently large constant A, so that in particular $\log x \gg (\log D)^{1/2}$ and so $x \ge (\log D)^5 \gg (\log Q_{\pi_j \times \chi})^{4+\varepsilon}$ for all χ in the support of $F^*(\lambda_\chi/(Q_{\pi_1} Q_{\pi_2}))$. We may now apply Corollary 7 for $j = 1$ and $j = 2$ in a similar way to (8.7) to conclude that

$$\log \mathcal{L}(\chi) - 2\mu_D \le \sum_{p \le x} \frac{a_\chi(p)(\lambda_1(p) + \lambda_2(p))}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x}$$
$$+ \frac{1}{2} \sum_{\substack{\eta_E(p)=1 \\ p^2 \le x}} \frac{a_{\chi^2}(p)(\lambda_1(p^2) + \lambda_2(p^2) - \psi_1(p) - \psi_2(p))}{p^{1+2/\log x}} \frac{\log(x/p^2)}{\log x} + O(\varepsilon V). \tag{9.13}$$

Here we have used $\mu_D(x) - \mu_D = -\frac{1}{2}(\log\log x + \log\log D) = \log V + O_\varepsilon(1) \le \frac{1}{10}\varepsilon V$, and also (9.11) and (9.12) to bound the remaining terms in Corollary 7 by $O(\varepsilon V)$. Hence if χ is counted by $\mathcal{T}(V + 2\mu_D)$ we have

$$(1 - \varepsilon)V \le \sum_{p \le x} \frac{a_\chi(p)(\lambda_1(p) + \lambda_2(p))}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x}$$
$$+ \frac{1}{2} \sum_{\substack{\eta_E(p)=1 \\ p^2 \le x}} \frac{a_{\chi^2}(p)(\lambda_1(p^2) + \lambda_2(p^2) - \psi_1(p) - \psi_2(p))}{p^{1+2/\log x}} \frac{\log(x/p^2)}{\log x}.$$

We write the right-hand side as

$$\sum_{p \le z} + \sum_{z < p \le x} + \sum_{p^2 \le x} = S_1(\chi) + S_2(\chi) + S_3(\chi),$$

say, for some $z \geq (\log D)^{2+\varepsilon}$. We choose

$$\Delta = (\log\log D)^{1/3}, \quad z = x^{1/\Delta} = \exp\left(\frac{A}{\varepsilon V}\frac{\log D}{\Delta}\right), \tag{9.14}$$

which is clearly $\geq (\log D)^{2+\varepsilon}$ in view of (9.12) and (9.11). We now estimate, using Lemma 10 as a crucial input, the quantities

$$M_1 = \sum_{S_1(\chi)\geq(1-3\varepsilon)V} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right), \quad M_2 = \sum_{S_2(\chi)\geq\varepsilon V} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right),$$
$$M_3 = \sum_{S_3(\chi)\geq\varepsilon V} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right)$$

so that $\mathcal{T}(V + 2\mu_D) \leq M_1 + M_2 + M_3$ by (9.13), where we emphasize that each M_j depends in particular on V.

9.5. *Bounding M_1, M_2, M_3*

We decompose the interval (9.11) as $I_1 \cup I_2$, where

$$I_1 = \left[\varepsilon\log\log D, \frac{\varepsilon}{A}\Delta\cdot\mathrm{var}_D\right], \quad I_2 = \left[\frac{\varepsilon}{A}\Delta\cdot\mathrm{var}_D, (\log D)^{1/2} - 2\mu_D\right], \tag{9.15}$$

and we recall (9.10) and (8.2).

Lemma 14. *We have*

$$M_1 \ll \begin{cases} H_E\exp\left(-\frac{((1-5\varepsilon)V)^2}{8\,\mathrm{var}_D}\right), & V \in I_1,\\ H_E\exp\left(-c(\varepsilon)V\log\frac{V}{\mathrm{var}_D}\right), & V \in I_2.\end{cases}$$

Proof. By Lemma 10 (with $\nu = 1$) we have

$$M_1 \leq \frac{2k!}{k!}\frac{1}{((1-4\varepsilon)V)^{2k}}\left(\frac{1}{2}\sum_{p\leq z}\frac{(\lambda_1(p)+\lambda_2(p))^2(1+\eta_E(p))}{p}\right)^k\sum_{\chi\in\widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right),$$

provided that $z^{2k} < D/4$. Using (8.1), (9.6) and (9.5) along with the obvious fact $\mathrm{var}_D(z) \leq \mathrm{var}_D$, we conclude

$$M_1 \ll H_E\left(\frac{8k(\mathrm{var}_D(z) + O_\varepsilon(\log\log\log D))}{e((1-4\varepsilon)V)^2}\right)^k \ll H_E\left(\frac{8k\ \mathrm{var}_D}{e((1-5\varepsilon)V)^2}\right)^k. \tag{9.16}$$

Our choice of z in (9.14) allows us to take

$$k \leq \frac{\varepsilon V\Delta}{3A}. \tag{9.17}$$

We now choose

$$k = \begin{cases}\left[\frac{((1-5\varepsilon)V)^2}{8\,\mathrm{var}_D}\right], & V\in I_1,\\ [V], & V \in I_2,\end{cases} \tag{9.18}$$

in agreement with (9.17), which completes the proof of the lemma. ∎

The following bounds are similar, but easier.

Lemma 15. *There is* $c(\varepsilon) > 0$ *such that*

$$M_2, M_3 \ll H_E \exp(-c(\varepsilon)V \log V) \tag{9.19}$$

for $V \in I_1 \cup I_2$.

Proof. By Lemma 10 (with $\nu = 2$ noting that the sum contains only split primes) we obtain

$$M_3 \leq \frac{2k!}{k!}\frac{1}{(\varepsilon V)^{2k}}\Bigg(\sum_{p \leq \sqrt{x}} \frac{(\lambda_1(p^2)+\lambda_2(p^2)-\psi_1(p)-\psi_2(p))^2}{4p^2}\Bigg)^k \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\Bigg(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\Bigg)$$

whenever $x^k < (D/4)^{1/2}$. Thus

$$M_3 \ll H_E \bigg(\frac{ck}{(\varepsilon V)^2}\bigg)^k$$

for some constant $c > 0$. Our choice (9.12) implies that we can choose $k = [\frac{\varepsilon}{3A}V]$, yielding the stated bound for M_3 in (9.19).

Next, by Lemma 10 (with $\nu = 1$), the Hecke relations (6.1), and (6.7), we have (using $1 + \eta_E(p) \leq 2$ and $(r+s)^2 \leq 2(r^2+s^2)$ for $r, s \in \mathbb{R}$)

$$\begin{aligned} M_2 &\ll H_E \frac{(2k)!}{k!}\frac{1}{(\varepsilon V)^{2k}}\Bigg(\sum_{z<p\leq x} \frac{(\lambda_1(p)+\lambda_2(p))^2}{p}\Bigg)^k \\ &\leq H_E \frac{(2k)!}{k!(\varepsilon V)^{2k}}\Bigg(\sum_{z<p\leq x} \frac{2(2+\lambda_1(p^2)+\lambda_2(p^2))}{p}\Bigg)^k \\ &= H_E \frac{(2k)!}{k!(\varepsilon V)^{2k}}\bigg(4\log\frac{\log x}{\log z} + O_\varepsilon(1)\bigg)^k \end{aligned} \tag{9.20}$$

provided that $x^{2k} \leq D/4$ and $z \geq (\log D)^{2+\varepsilon}$. We choose $k = [\frac{\varepsilon}{3A}V]$ and recall (8.1) and (9.14), getting

$$M_2 \ll H_E \bigg(\frac{16k}{e\varepsilon^2 V^2}(\log\log\log D + O_\varepsilon(1))\bigg)^k.$$

By (9.11) this is at most $H_E \exp(-c(\varepsilon)V \log V)$ for some $c(\varepsilon) > 0$, again confirming (9.19). ∎

9.6. *Completion of the proof of* (9.4)

We substitute the bounds of the previous two lemmas back into (9.9). We start with the contribution of M_1. The interval I_2 contributes

$$\begin{aligned} &H_E \exp(\mu_D) \int_{I_2} \exp\bigg(\frac{1}{2}V - c(\varepsilon)V \log \frac{V}{\mathrm{var}_D}\bigg)\, dV \\ &\quad \ll H_E \exp(\mu_D) \int_{I_2} \exp\bigg(-\frac{1}{4}c(\varepsilon)V \log\log\log D\bigg)\, dV \ll H_E \exp(\mu_D), \end{aligned} \tag{9.21}$$

which is admissible for (9.4). The contribution of I_1 is

$$H_E \exp(\mu_D) \int_{I_1} \exp\left(\frac{1}{2}V - \frac{((1-5\varepsilon)V)^2}{8 \operatorname{var}_D}\right) dV.$$

We extend the range of integration to all of $\mathbb{R}$ and use the formula

$$\int_{\mathbb{R}} e^{-\alpha x^2 + \beta x}\, dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right), \quad \alpha, \beta > 0,$$

getting the upper bound

$$H_E \exp(\mu_D) \frac{(8\pi \operatorname{var}_D)^{1/2}}{1-5\varepsilon} \exp\left(\frac{\operatorname{var}_D}{2(1-5\varepsilon)^2}\right) \ll H_E \exp\big(\mu_D + (1/2 + O(\varepsilon)) \operatorname{var}_D\big)$$

in agreement with (9.4), potentially after redefining ε.

For M_2, M_3 we obtain the same contribution as in (9.21), which completes the proof of (9.4).

9.7. *The endgame*

We have now prepared the scene to complete the proof of Theorem 3 (a). Recall that we need to establish the bound

$$\frac{1}{H_E} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} \exp\left(-\frac{c_0|\lambda_\chi|}{Q_{\pi_1} Q_{\pi_2}}\right) \left(\frac{L(1/2, \pi_1 \times \chi) L(1/2, \pi_2 \times \chi)}{L(1, \mathrm{Ad}\, \pi_1) L(1, \mathrm{Ad}\, \pi_2)}\right)^{1/2} \ll_{\varepsilon, c_0} (\log D)^{-1/4+\varepsilon},$$

with polynomial dependence on Q_{π_1}, Q_{π_2}.

By (6.10), (8.2) and the convexity bound, the left-hand side is trivially (and crudely) bounded by $(DQ_{\pi_1} Q_{\pi_2})^{10} \ll (Q_{\pi_1} Q_{\pi_2})^{10^3} (\log D)^{-1/4}$ if $\max(Q_{\pi_1}, Q_{\pi_2}) \geq D^{1/10}$. So from now on we can assume

$$Q_{\pi_1}, Q_{\pi_2} \leq D^{1/10}. \tag{9.22}$$

Next we majorize $x \mapsto \exp(-c_0|x|)$ by a non-negative, even Schwartz-class function F whose Fourier transform has support in $[-\frac{1}{2\pi}, \frac{1}{2\pi}]$ (see [41] for an explicit construction of such a function). Recalling (9.2) and (9.3) and using (6.10) for the error term, the bound (9.4) yields

$$\frac{1}{H_E} \sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q_{\pi_1} Q_{\pi_2}}\right) \left(\frac{L(1/2, \pi_1 \times \chi) L(1/2, \pi_2 \times \chi)}{L(1, \mathrm{Ad}\, \pi_1) L(1, \mathrm{Ad}\, \pi_1)}\right)^{1/2}$$

$$\ll_\varepsilon \frac{L(1, \eta_E)^{3/4+\varepsilon/2} L(1, \pi_1 \times \pi_2 \times \theta_E)^{1/4+\varepsilon/2}}{L(1, \mathrm{Ad}\, \pi_1 \times \theta_E)^{1/8-\varepsilon/4} L(1, \mathrm{Ad}\, \pi_1 \times \theta_E)^{1/8-\varepsilon/4} (\log D)^{1/4-\varepsilon/2}} + \exp\big(-\tfrac{1}{4}\sqrt{\log D}\big)$$

whenever $Q_{\pi_1}, Q_{\pi_2} \leq D^{1/10}$. By (9.7) and (6.9), the right-hand side is

$$\ll_\varepsilon \frac{L(1, \pi_1 \times \pi_2 \times \theta_E)^\varepsilon}{(\log D)^{1/4-\varepsilon}} + \exp\big(-\tfrac{1}{4}\sqrt{\log D}\big). \tag{9.23}$$

Let us temporarily make the additional assumption

$$Q_{\pi_1}, Q_{\pi_2} \leq (\log D)^{10}. \tag{9.24}$$

In this case, (6.7) and (6.11) with $x = (\log\log D)^3$, say, imply the existence of a constant C such that

$$L(1, \mathrm{Ad}\,\pi_j) \ll \exp(C(\log\log D)^{1/3}) \ll (\log D)^\varepsilon.$$

Together with (9.8) we obtain

$$L(1, \pi_1 \times \pi_2 \times \theta_E) = L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_2)\frac{L(1, \pi_1 \times \pi_2 \times \theta_E)}{L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_2)} \ll (\log D)^\varepsilon,$$

which when inserted in (9.23) is admissible for Theorem 3 (a).

Let us now assume that (9.24) fails, but (9.22) holds, so that

$$(\log D)^{10} < \max(Q_{\pi_1}, Q_{\pi_2}) \leq D^{1/10}.$$

Then we use the Cauchy–Schwarz inequality and Lemma 13 together with (8.2), (8.3) and (6.10) to obtain

$$\frac{1}{H_E}\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right)\left(\frac{L(1/2, \pi_1 \times \chi)L(1/2, \pi_2 \times \chi)}{L(1, \mathrm{Ad}\,\pi_1)L(1, \mathrm{Ad}\,\pi_1)}\right)^{1/2}$$
$$\ll Q_{\pi_1}Q_{\pi_2}(\log\log D)\exp\Big(\max_j(\log Q_{\pi_j})^{2/5}\Big) \ll (Q_{\pi_1}Q_{\pi_2})^2(\log D)^{-1/4}$$

as desired. This completes the proof of Theorem 3 (a) in all cases.

9.8. *Proof of Theorem 3 (b)*

This is similar and we highlight only the relevant changes. We keep the definition (9.2) of $\mu_D(x)$, but we redefine (9.3) as follows:

$$\begin{aligned}\mathrm{var}^*_D(x) = \tfrac{1}{2}\log\log x + \tfrac{1}{2}\log L(1, \eta_E) \\ + \tfrac{1}{4}\log L(1, \mathrm{Ad}\pi_1 \times \theta_E) + \tfrac{1}{4}\log L(1, \mathrm{Ad}\pi_2 \times \theta_E).\end{aligned}$$

Remark 7. This differs from $\mathrm{var}_D(x)$ by the term $\frac{1}{2}\log L(1, \pi_1 \times \pi_2 \times \theta_E)$. The reason for this can be traced back to a comparison of Lemma 10 and Lemma 12. For $\alpha \neq \beta$ we have $(B_{\alpha,\beta}(2,0), B_{\alpha,\beta}(2,1), B_{\alpha,\beta}(2,2)) = (2,0,2)$ whereas for $\alpha = \beta$ this is $(2,4,2)$. Consequently, Lemma 12 features a "main term" $b(p)^2 + c(p)^2$, whereas the analogous situation in Lemma 10 gives $(b(p) + c(p))^2 = b(p)^2 + 2b(p)c(p) + c(p)^2$. It is the extra mixed term $2b(p)c(p)$ that is responsible for the term $\frac{1}{2}\log L(1, \pi_1 \times \pi_2 \times \theta_E)$, which of course only makes sense in the situation $\pi_1 \neq \pi_2$ of Theorem 1, but not in the potentially allowed situation $\pi_1 = \pi_2$ of Theorem 2.

We write $\mu_D = \mu_D(D)$ and $\mathrm{var}^*_D = \mathrm{var}^*_D(D)$. The analogues of (9.5) and (9.6) are

$$\tfrac{1}{2}\log\log D + O(\log\log\log D) \le \mathrm{var}^*_D \ll (\log D)^{1/3}$$

and

$$\sum_{p\le x} \frac{(1+\eta_E(p))(\lambda_1(p)^2+\lambda_2(p)^2)}{2p} = 2\mathrm{var}^*_D(x) + O_\varepsilon(\log\log\log N_1N_2)$$

provided that $x \ge (\log Q_{\pi_1}Q_{\pi_2}D)^{2+\varepsilon}$. We generalize (9.1) to

$$\mathcal{L}^*(\chi) = L(1/2, \pi_1 \times \chi^\alpha)L(1/2, \pi_2 \times \chi^\beta).$$

We follow the argument up to (9.13), which now reads

$$\log \mathcal{L}^*(\chi) - 2\mu_D \le S_1^*(\chi) + S_2^*(\chi) + S_3^*(\chi) + O(\varepsilon V),$$

where

$$S_1^*(\chi) = \sum_{p\le z} \frac{a_{\chi^\alpha}(p)\lambda_1(p)+a_{\chi^\beta}(p)\lambda_2(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x},$$

$$S_2^*(\chi) = \sum_{z<p\le x} \frac{a_{\chi^\alpha}(p)\lambda_1(p)+a_{\chi^\beta}(p)\lambda_2(p)}{p^{1/2+1/\log x}} \frac{\log(x/p)}{\log x},$$

$$S_3^*(\chi) = \frac{1}{2}\sum_{\substack{\eta_E(p)=1\\ p^2\le x}} \frac{a_{\chi^{2\alpha}}(p)(\lambda_1(p^2)-\psi_1(p))+a_{\chi^{2\beta}}(p)(\lambda_2(p^2)-\psi_2(p))}{p^{1+2/\log x}} \frac{\log(x/p^2)}{\log x}.$$

Correspondingly we define M_j^* for $j = 1, 2, 3$ (which as before depend on V). In order to bound M_j^* we apply Lemma 12 instead of Lemma 10. For notational simplicity we study the case $v_2(\alpha) = v_2(\beta)$, the other case being almost identical. In the following let $c > 0$ denote a sufficiently large constant, *not necessarily the same at every occurrence.*

We have

$$M_3^* \ll H_E \sum_{2\ell_1+4\ell_2=2k} \frac{(2k)!}{(\ell_1)!(\ell_2)!}\frac{1}{(\varepsilon V)^{2k}} c^k$$

if $x^k < (D/4)^{1/\max(2\alpha,2\beta)}$. Since $(\ell_1)!(\ell_2)! \ge [k/3]!$, by Stirling's formula we obtain

$$M_3^* \ll H_E \left(\frac{ck^{5/3}}{(\varepsilon V)^2}\right)^k.$$

Choosing $k \asymp V$, we obtain the analogue of (9.19) for M_3^*. In the same way, we obtain

$$M_2^* \ll H_E \sum_{2\ell_1+4\ell_2=2k} \frac{(2k)!}{(\ell_1)!(\ell_2)!}\frac{1}{(\varepsilon V)^{2k}}\left(4\log\frac{\log x}{\log z} + O_\varepsilon(1)\right)^{\ell_1} c^{\ell_2}$$

as an analogue of (9.20), and it is easy to confirm (9.19) also for M_2^*.

The estimation of M_1^* is only slightly more difficult. As in (9.16) we obtain

$$M_1^* \ll H_E \sum_{2\ell_1+4\ell_2=2k} \frac{(2k)!}{(\ell_1)!(\ell_2)!} \frac{(2\mathrm{var}_D^*)^{\ell_1} c^{\ell_2}}{((1-5\varepsilon)V)^{2k}}.$$

We write $\ell_1 = ak$, $\ell_2 = bk$ with $a + 2b = 1$, so that by Stirling's formula we have (with the convention $0^0 = 1$)

$$\frac{(2k)!}{(\ell_1)!(\ell_2)!} \ll \left(\frac{4k^{2-a-b}}{e^{2-a-b}a^a b^b}\right)^k,$$

uniformly in a, b (one can take 2 as an implied constant). We conclude

$$M_1^* \ll H_E \sum_{2\ell_1+4\ell_2=2k} c^{\ell_2}\left(\frac{4k^{2-a-b}(2\mathrm{var}_D^*)^a}{e^{2-a-b}a^a b^b((1-5\varepsilon)V)^2}\right)^k.$$

Note that $1/2 \le \xi^\xi \le 1$ for $0 \le \xi \le 1$. We make the same choice for k as in (9.18) and we make the same choices of I_1, I_2 as in (9.15) but in all cases with var_D^* in place of var_D.

If $V \in I_2$, then with $k = [V]$ we obtain

$$\begin{aligned} M_1^* &\ll H_E \sum_{2\ell_1+4\ell_2=2k} \frac{(\mathrm{var}_D^*)^{\ell_1}}{V^{\ell_1+\ell_2}} c^k \le H_E \sum_{2\ell_1+4\ell_2=2k} \frac{(\mathrm{var}_D^*)^{\ell_1+\ell_2}}{V^{\ell_1+\ell_2}} c^k \\ &\ll H_E \exp\left(-c(\varepsilon)V \log \frac{V}{\mathrm{var}_D^*}\right) \end{aligned}$$

since $k/2 \le \ell_1 + \ell_2 \le k$.

If $V \in I_1$ then with $k = [((1-4\varepsilon)V)^2/(8\mathrm{var}_D^*)]$ and $b = (1-a)/2$ we obtain

$$\begin{aligned} M_1^* &\ll H_E \sum_{2\ell_1+4\ell_2=2k} \left(\frac{1}{ea^a b^b}\left(\frac{cV}{(\mathrm{var}_D^*)^{3/2}}\right)^{1-a}\right)^k \\ &\ll H_E \sum_{2\ell_1+4\ell_2=2k} (ea^a b^b)^{-k} (\log\log D)^{-\frac{1}{7}\cdot 2\ell_2} \end{aligned}$$

as $cV(\mathrm{var}_D^*)^{-3/2} \ll \Delta(\mathrm{var}_D^*)^{-1/2} \ll (\log\log D)^{-1/6}$ for $V \in I_1$. Since all of the terms in the previous display are less than $(e-\varepsilon)^{-k}$ for D sufficiently large (in terms of ε), we conclude altogether

$$M_1^* \ll H_E \exp\left(-\frac{((1-5\varepsilon)V)^2}{8\mathrm{var}_D^*}\right)$$

for $V \in I_1$. Having recovered the bounds from Lemmas 14 and 15, the analogue of the basic bound (9.4) is now

$$\sum_{\chi \in \widetilde{\mathrm{Cl}}_E^\vee} F\left(\frac{\lambda_\chi}{Q_{\pi_1}Q_{\pi_2}}\right)\mathcal{L}^*(\chi)^{1/2} \ll H_E\Big(\exp\big(\mu_D + (1/2+\varepsilon)\,\mathrm{var}_D^*\big) + \exp\big(-\tfrac{1}{3}\sqrt{\log D}\big)\Big),$$

so that Theorem 3 (b) now follows by adopting the argument in Section 9.7.

Appendix A. Some explicit computations related to Waldspurger's formula

Our aim in this appendix is to justify the bound (4.8). We begin by explicating the shape of Waldspurger's formula, in the explicit form given by [20], which leads to the expression (4.7).

Let $\mathbf{B}$, $\mathscr{O}$ and K be as in Section 2. Let N denote the discriminant of $\mathscr{O}$; then N is squarefree. Fix an optimal embedding ι of the quadratic field E into $\mathbf{B}(\mathbb{Q})$ satisfying (1.5) and (1.7). As usual we write $\mathbf{T}_\iota$ for the associated torus in $\mathbf{PB}^\times$.

Let $\sigma \subset L^2_{\mathrm{disc}}([\mathbf{PB}^\times])$ be irreducible and have non-zero invariants by K. Let ϕ_σ be a non-zero vector in the line σ^K, normalized to have L^2-norm 1. Fix $\chi \in \widetilde{\mathrm{Cl}}_E^\vee$ and assume that $\mathrm{Hom}_{\mathbf{T}_\iota}(\sigma, \chi) \neq 0$. Recall from the discussion in Section 4.3 that $\phi_\sigma^\circ = g.\phi_\sigma$ is the global Gross–Prasad vector, with respect to the pair $(\mathbf{T}_\iota, \chi)$.

We recall the twisted adelic torus period $\mathscr{P}^\chi_{\mathbf{T}}(\phi_\sigma^\circ)$ from (4.4), where the measure is normalized to have volume 1. Note that the toric period in [20] is taken with respect to the Tamagawa measure $\mu^{\mathbf{T}}_{\mathrm{Tam}}$ on $[\mathbf{T}]$. Moreover, the L^2-normalization of the test vectors in [20] is itself taken with respect to the Tamagawa measure $\mu^{\mathbf{G}}_{\mathrm{Tam}}$ on $[\mathbf{G}]$.

Let π be the irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$ of level N corresponding to σ via the Jacquet–Langlands correspondence. We apply [20, Theorem 1.1] with

$$S(\Omega) = S_2(\pi) = \emptyset, \quad S(\pi) = S_1(\pi) = \{p \mid N\}, \quad S_0(\pi) = \{p \mid (D, N)\}.$$

From [20, Theorem 1.1] we obtain

$$\begin{aligned}|\mathscr{P}^\chi_{\mathbf{T}}(\phi_\sigma^\circ)|^2 &= \frac{\mu^{\mathbf{G}}_{\mathrm{Tam}}([\mathbf{G}])^2}{\mu^{\mathbf{T}}_{\mathrm{Tam}}([\mathbf{T}])^2}\,\frac{1}{2}\,\frac{1}{\sqrt{D}} L_{S(\pi)}(1, \eta_E)\zeta^{S(\pi)}(2) \\ &\quad \times \prod_{p \mid N} e(E_p/\mathbb{Q}_p) C_\infty(E, \pi, \chi) \frac{L^{S_0(\pi)}(1/2, \pi \times \chi)}{L^{S_0(\pi)}(1, \mathrm{Ad}\,\pi)},\end{aligned}$$

where $C_\infty(E, \pi, \chi)$ is defined in [20, §7B] and recalled below. Note that if S is a finite (possibly empty) set of primes, the superscript notation L^S includes the local factor at infinity. Using $\mu^{\mathbf{T}}_{\mathrm{Tam}}([\mathbf{T}]) = \mathrm{Res}_{s=1}\,\zeta_E(s) = L(1, \eta_E)$ and reorganizing we obtain

$$|\mathscr{P}^\chi_{\mathbf{T}}(\phi_\sigma^\circ)|^2 = C_{\mathbf{G}} C_{\mathrm{Ram}}(\pi, \chi) \frac{1}{L(1, \eta_E)^2}\,\frac{1}{\sqrt{D}}\,\frac{L(1/2, \pi \times \chi)}{L(1, \mathrm{Ad}\,\pi)} F(\pi_\infty, \chi_\infty),$$

where $C_{\mathbf{G}} = \mu^{\mathbf{G}}_{\mathrm{Tam}}([\mathbf{G}])^2 \frac{1}{2}\xi(2)$,

$$C_{\mathrm{Ram}}(\pi, \chi) = \frac{L_{S_0(\pi)}(1, \mathrm{Ad}\,\pi)}{L_{S_0(\pi)}(1/2, \pi \times \chi)} \prod_{p \mid N} e(E_p/\mathbb{Q}_p) \frac{1 - p^{-2}}{1 - \eta_E(p) p^{-1}},$$

and

$$F(\pi_\infty, \chi_\infty) = C_\infty(E, \pi, \chi) \frac{L_\infty(1/2, \pi \times \chi)}{L_\infty(1, \mathrm{Ad}\,\pi)}.$$

Recall the notation λ_π for the spectral parameter of π in Section 4.1 and $\lambda_\chi \in \mathbb{R}$ for the frequency of χ from Section 4.3.

Lemma 16. *We have*

$$C_{\mathrm{Ram}}(\pi,\chi) \ll_\varepsilon N^\varepsilon \quad \textit{and} \quad F(\pi_\infty,\chi_\infty) \ll \exp(-c_0|\lambda_\chi|/\lambda_\pi).$$

Proof. At the finite places the local factors are given by (6.2)–(6.3). Since $|\alpha(p,i)| \le p^{-1/2}$ for $p \mid N$ we obtain $|L_p(1/2,\pi\times\chi)|^{-1} \le (1-p^{-1})^{-4}$ and $|L_p(1,\mathrm{Ad}\,\pi)| \le 1$, and so $|C_{\mathrm{Ram}}(\pi,\chi)| \le \prod_{p\mid N} 2(1-1/p)^{-5} \ll N^\varepsilon$.

For the second estimate, recall that π_∞ can be either discrete series of weight k or principal series with spectral parameter t. In the latter case, it suffices to assume that $t \in \mathbb{R}$, so that π_∞ is tempered. In this notation, the archimedean L-factors take the form

$$L_\infty(s,\pi\times\chi) = \begin{cases} 4(2\pi)^{-2s}\prod_\pm \Gamma(s\pm it), & \pi_\infty \text{ principal series, } E \text{ imaginary,} \\ \pi^{-2s}\prod_{\pm,\pm}\Gamma\big(\tfrac12(s\pm it\pm i\lambda_\chi)\big), & \pi_\infty \text{ principal series, } E \text{ real,} \\ 4(2\pi)^{-2s-(k-1)}\prod_\pm \Gamma\big(s+\tfrac12(k-1)\pm i\lambda_\chi\big), & \pi_\infty \text{ discrete series,} \end{cases}$$

and

$$L_\infty(s,\mathrm{Ad}\,\pi) = \begin{cases} \pi^{-3s/2}\Gamma(s/2)\prod_\pm \Gamma(s/2\pm it), & \pi_\infty \text{ principal series,} \\ 2^{2-k-s}\pi^{(1-2k-3s)/2}\Gamma(s+k-1)\Gamma((s+1)/2), & \pi_\infty \text{ discrete series.} \end{cases}$$

Moreover, by definition, we have

$$C_\infty(E,\pi,\chi) = \begin{cases} 1, & \pi_\infty \text{ principal series,} \\ \frac{\Gamma(k)}{\pi\Gamma(k/2)^2}, & \pi_\infty \text{ discrete series, } E \text{ imaginary,} \\ 2^k, & \pi_\infty \text{ discrete series, } E \text{ real.} \end{cases}$$

By Stirling's formula we have

$$C_\infty(E,\pi,\chi)\frac{L_\infty(1/2,\pi\times\chi)}{L_\infty(1,\mathrm{Ad}\,\pi)} \ll \begin{cases} 1, & E \text{ imaginary;} \\ e^{-\pi\max(0,|\lambda_\chi|-|t|)}, & \pi_\infty \text{ principal series, } E \text{ real;} \\ e^{-c\min(|\lambda_\chi|,|\lambda_\chi|^2/k)}, & \pi_\infty \text{ discrete series, } E \text{ real,} \end{cases}$$

for some absolute constant $c>0$. Using $\lambda_\pi^2 = k(k+1)$ or $\lambda_\pi^2 = 1/4+t^2$ according to whether π_∞ is discrete or principal series, we estimate this very crudely by $\exp(-c_0|\lambda_\chi|/\lambda_\pi)$. ■

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