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The Coleman correspondence at the free fermion point

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Abstract. We prove that the truncated correlation functions of the charge and gradient fields associated with the massless sine-Gordon model on \mathbb{R}^2 with $\beta = 4\pi$ exist for all coupling constants and are equal to those of the chiral densities and vector current of free massive Dirac fermions. This is an instance of Coleman's prediction that the massless sine-Gordon model and the massive Thirring model are equivalent (in the above sense of correlation functions). Our main novelty is that we prove this correspondence starting from the Euclidean path integral in the nonperturbative regime of the infinite volume models. We use this correspondence to show that the correlation functions of the massless sine-Gordon model with $\beta = 4\pi$ decay exponentially and that the corresponding probabilistic field is localized.

Keywords. Coleman, bosonization, sine-Gordon model, Thirring model, free fermions, Coulomb gas

1. Introduction and main results

Statistical and quantum field theory in two (Euclidean) dimensions is rich and special in various ways. This manifests itself, for example, through the existence of the powerful theory of conformal field theory (CFT), the possibility of quasiparticles which are neither bosons nor fermions but instead have anyonic particle statistics, or the perhaps surprising possibility of equivalence of fermionic and bosonic field theories – known as bosonization. The two-dimensional setting also provides one of the main test cases for the understanding of strongly interacting field theories. The massless sine-Gordon model is a principal example of a *nonconformal* perturbation of a CFT in two dimensions. Despite the absence of conformal symmetry, there is a detailed but almost entirely conjectural description of many of its physical features, not accessible by perturbation theory, including the prediction of a mass gap for all coupling constants. These features pertain to the infinite volume theory. In this paper, we study the arguably most fundamental (and

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simplest) instance of this – the Coleman correspondence at the free fermion point, which we prove starting from the path integral formulation in the *nonperturbative* regime of the infinite volume models and for all coupling constants.

1.1. Coleman correspondence

The Coleman correspondence is a prototype for bosonization [16]. It states that the massless sine-Gordon model with parameters (β, z) and the massive Thirring model with parameters (g, μ) are equivalent in the sense of correlation functions when (β, z) and (g, μ) are appropriately related. This is an instance of bosonization because the sine-Gordon model is a bosonic quantum field theory while the massive Thirring model is a fermionic quantum field theory. The equivalence is especially striking when $\beta = 4\pi$, which corresponds to parameters of the massive Thirring model for which it becomes noninteracting (free massive Dirac fermions), while the sine-Gordon model is interacting (non-Gaussian).

Previous mathematical results have established variants of the Coleman correspondence in the *perturbative* regime, i.e., for small coupling constants and with finite volume interaction term (or with external mass term); see [7, 21, 33, 55] and Section 1.3. In this article, we prove that, for $\beta = 4\pi$, the Coleman correspondence holds in the *nonperturbative* regime of the infinite volume models (without external mass term). Unlike the previous results, our proof has thus strong implications for the massless sine-Gordon model with $\beta = 4\pi$: we show exponential decay of correlations for all $z \neq 0$ and that the field is probabilistically localized – results that are nonperturbative in the coupling constant (and false for $z = 0$).

Before stating our results, we first introduce the sine-Gordon model and free Dirac fermions (both in their Euclidean versions). The *massless sine-Gordon model* with coupling constants $\beta \in (0, 8\pi)$ and $z \in \mathbb{R}$ is defined in terms of $\varepsilon \rightarrow 0, m \rightarrow 0, L \rightarrow \infty$ the limit of the probability measures

$$\nu^{\text{SG}(\beta, z|\varepsilon, m, L)}(d\varphi) \propto \exp\left[2z \int_{\Lambda_L} \varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta} \varphi(x)) dx\right] \nu^{\text{GFF}(\varepsilon, m)}(d\varphi), \quad (1.1)$$

where $\Lambda_L = \{x \in \mathbb{R}^2 : |x| \leq L\}$ is the Euclidean disk of radius L , and $\nu^{\text{GFF}(\varepsilon, m)}$ is the Gaussian free field (GFF) on \mathbb{R}^2 with mass $m > 0$ regularized at scale $\varepsilon > 0$. Here the precise choice of the regularization of the GFF is not important, but to be concrete, we choose $\nu^{\text{GFF}(\varepsilon, m)}$ as the Gaussian measure supported on $C^\infty(\mathbb{R}^2)$ with covariance kernel

$$\int_{\varepsilon^2}^{\infty} ds e^{-s(-\Delta+m^2)}(x, y). \quad (1.2)$$

We denote the expectation with respect to the measure $\nu^{\text{SG}(\beta, z|\varepsilon, m, L)}$ by $\langle \cdot \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)}$. The gradient correlation functions are the moments of $\partial\varphi$ and $\bar{\partial}\varphi$ in the limit $\varepsilon \rightarrow 0, m \rightarrow 0, L \rightarrow \infty$. The charge correlation functions are the limits (when they exist) of linear combinations of expectations of products of

$$:e^{\pm i\sqrt{\beta}\varphi(x)}:_\varepsilon := \varepsilon^{-\beta/4\pi} e^{\pm i\sqrt{\beta}\varphi(x)} \quad (1.3)$$

or their smeared versions, defined for $f \in L^\infty(\mathbb{R}^2)$ with compact support by

$$:e^{\pm i\sqrt{\beta}\varphi}:\varepsilon(f) := \varepsilon^{-\beta/4\pi} \int_{\mathbb{R}^2} dx f(x) e^{\pm i\sqrt{\beta}\varphi(x)}. \quad (1.4)$$

The relevant linear combinations are the truncated correlation functions (or joint cumulants). For example, for $\beta \geq 4\pi$, the charge one-point function $\langle :e^{i\sqrt{\beta}\varphi}:\varepsilon(f) \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)}$ diverges when $z \neq 0$ and $\int_{\mathbb{R}^2} f \neq 0$ (see Proposition 1.4), but we will see that the truncated charge two-point function, defined by

$$\begin{aligned} \langle :e^{i\sqrt{\beta}\varphi}:\varepsilon(f_1) :e^{-i\sqrt{\beta}\varphi}:\varepsilon(f_2) \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)}^T &= \langle :e^{i\sqrt{\beta}\varphi}:\varepsilon(f_1) :e^{-i\sqrt{\beta}\varphi}:\varepsilon(f_2) \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)} \\ &\quad - \langle :e^{i\sqrt{\beta}\varphi}:\varepsilon(f_1) \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)} \langle :e^{-i\sqrt{\beta}\varphi}:\varepsilon(f_2) \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)}, \end{aligned} \quad (1.5)$$

has a nontrivial limit for test functions f_1 and f_2 with disjoint supports. In general, the truncated correlation function of observables O_1, \dots, O_n can be defined inductively by

$$\langle O_1 \cdots O_n \rangle^T = \langle O_1 \cdots O_n \rangle - \sum_{P \in \mathfrak{P}_n} \prod_j \left\langle \prod_{i \in P_j} O_i \right\rangle^T, \quad \langle O_i \rangle^T = \langle O_i \rangle, \quad (1.6)$$

where the sum is over proper partitions $P = (P_j) \in \mathfrak{P}_n$ of $[n] = \{1, \dots, n\}$. Note that $\langle O_1 \cdots O_n \rangle^T$ does not only depend on the product of the O_i , and a more precise notation would be $\langle O_1; \dots; O_n \rangle^T$. However, the formal product notation without semicolons is standard and more convenient for our purposes. Equivalent to (1.6), the truncated correlations are Taylor coefficients of the logarithm of the joint moment generating function of O_1, \dots, O_n if it exists; see Appendix A.

Free fermions are defined in terms of their correlation kernel. The correlation kernel of *free Dirac fermions* of mass $\mu \in \mathbb{R}$ is the fundamental solution of the massive Dirac operator on \mathbb{R}^2 for which we use the representation

$$\not{d} + \mu = \begin{pmatrix} \mu & 2\bar{\partial} \\ 2\partial & \mu \end{pmatrix}, \quad (1.7)$$

where $\partial = \frac{1}{2}(-i\partial_0 + \partial_1)$ and $\bar{\partial} = \frac{1}{2}(i\partial_0 + \partial_1)$ and we identify $x = (x_0, x_1) \in \mathbb{R}^2$ with $ix_0 + x_1 \in \mathbb{C}$. In terms of the modified Bessel function K_0 , this fundamental solution is explicitly given by

$$\begin{aligned} S(x, y) &= -\frac{1}{2\pi} \begin{pmatrix} -\mu K_0(|\mu||x-y|) & 2\bar{\partial}_x K_0(|\mu||x-y|) \\ 2\partial_x K_0(|\mu||x-y|) & -\mu K_0(|\mu||x-y|) \end{pmatrix} \\ &\sim \frac{1}{2\pi} \begin{pmatrix} 0 & 1/(\bar{x} - \bar{y}) \\ 1/(x - y) & 0 \end{pmatrix}, \end{aligned} \quad (1.8)$$

where \sim holds as $\mu \rightarrow 0$; see Section 6. For distinct points $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}^2$, and any $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \{1, 2\}$, we then denote the correlation functions of free Dirac fermions by

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i}(x_i) \psi_{\beta_i}(y_i) \right\rangle_{\text{FF}(\mu)} = \det(S_{\alpha_i \beta_j}(x_i, y_j))_{i,j=1}^n. \quad (1.9)$$

The right-hand side is regarded as the definition of the left-hand side. In Appendix A, some standard operational tools for free fermions that we will use later are collected. Because S is singular, the correlations of $\bar{\psi}_{\alpha_i}(x_i)\psi_{\beta_i}(x_i)$ are not defined, in general, but for distinct points x_1, \dots, x_n with $n > 1$, the truncated correlations of $\bar{\psi}_{\alpha_i}\psi_{\beta_i}(x_i)$ formally make sense and are given by

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i}\psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(\mu)}^T = (-1)^{n+1} \sum_{\pi} \prod_{i=1}^n S_{\alpha_{\pi^i(1)}\beta_{\pi^{i+1}(1)}}(x_{\pi^i(1)}, x_{\pi^{i+1}(1)}) \quad (1.10)$$

where the sum is over cyclic permutations π of $[n] = \{1, \dots, n\}$. For our purposes, we again regard the right-hand side of (1.10) as the definition of the left-hand side of (1.10). (As explained in Appendix A, if S was not singular, then (1.10) would be an identity that follows from (1.6) and (1.9) without restriction to distinct points. Alternatively, one could thus define the truncated correlations as limits of regularized correlations and arrive at the same result as in our definition.) Finally, for any $f_1, \dots, f_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the following integrand is (absolutely) integrable, we will write

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i}\psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu)}^T = \int dx_1 \cdots dx_n f_1(x_1) \cdots f_n(x_n) \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i}\psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(\mu)}^T. \quad (1.11)$$

Since $S(x, y)$ has singularity $O(1/|x - y|)$, for $n \geq 3$ the above integrand is integrable for all bounded f_i with compact support. For $n = 2$, this is true for f_1 and f_2 with disjoint compact supports.

For $\beta = 4\pi$, the Coleman correspondence is the following theorem, our first main result.

Theorem 1.1. *Let $\beta = 4\pi$ and $z \in \mathbb{R}$. Then the $\varepsilon \rightarrow 0$, $m \rightarrow 0$, $L \rightarrow \infty$ limits of the truncated correlation functions of $\partial\varphi$, $\bar{\partial}\varphi$, $:e^{+i\sqrt{\beta}\varphi}:$, $:e^{-i\sqrt{\beta}\varphi}:$ of the sine-Gordon model exist (under the restrictions below), and they are equal to the correlation functions of free massive Dirac fermions with mass $\mu = Az$ (the constant A is defined below): for $n + n' + q + q' \geq 2$ and all test functions $f_1^+, \dots, f_n^+, f_1^-, \dots, f_{n'}^- \in L^\infty(\mathbb{R}^2)$ and $g_1^+, \dots, g_q^+, g_1^-, \dots, g_{q'}^- \in C_c^\infty(\mathbb{R}^2)$, all with disjoint compact supports,*

$$\begin{aligned} & \lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{+i\sqrt{4\pi}\varphi}:\varepsilon(f_k^+) \prod_{k'=1}^{n'} :e^{-i\sqrt{4\pi}\varphi}:\varepsilon(f_{k'}^-) \right. \\ & \quad \left. \times \prod_{j=1}^q (-i\partial\varphi(g_j^+)) \prod_{j'=1}^{q'} (-i\bar{\partial}\varphi(g_{j'}^-)) \right\rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)}^T \\ & = A^{n+n'} B^{q+q'} \left\langle \prod_{k=1}^n \bar{\psi}_1\psi_1(f_k^+) \prod_{k'=1}^{n'} \bar{\psi}_2\psi_2(f_{k'}^-) \prod_{j=1}^q \bar{\psi}_2\psi_1(g_j^+) \prod_{j'=1}^{q'} \bar{\psi}_1\psi_2(g_{j'}^-) \right\rangle_{\text{FF}(\mu)}^T, \end{aligned} \quad (1.12)$$

where $A = 4\pi e^{-\gamma/2}$ and $B = \sqrt{\pi}$ (and where γ is the Euler–Mascheroni constant).

Moreover, for $n + n' + q + q' \geq 3$ and $n + n' = q + q' = 1$, the statement is true without the disjoint support assumption.

We emphasize that the right-hand side is the explicit polynomial in $S_{\alpha\beta}(x, y)$ given by (1.10) which is integrated over the test functions as in (1.11). To lighten notation, we will write the limit on the left-hand side of (1.12) as

$$\left\langle \prod_{k=1}^n :e^{+i\sqrt{4\pi}\varphi}:(f_k^+) \prod_{k'=1}^{n'} :e^{-i\sqrt{4\pi}\varphi}:(f_{k'}^-) \prod_{j=1}^q (-i\partial\varphi(g_j^+)) \prod_{j'=1}^{q'} (-i\bar{\partial}\varphi(g_{j'}^-)) \right\rangle_{\text{SG}(4\pi, z)}^T. \quad (1.13)$$

By choosing $n + n' = 0$ and $q + q' = 2$, respectively $n + n' = 2$ and $q + q' = 0$, the gradient and charge two-point functions of the sine-Gordon model are in particular given, for test functions f_1 and f_2 with disjoint support, by

$$\begin{aligned} & \langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{SG}(4\pi, z)} \\ &= -\frac{B^2}{\pi^2} \int dx_1 dx_2 f_1(x_1)f_2(x_2)(\partial_{x_1} K_0(A|z||x_1 - x_2|))^2, \end{aligned} \quad (1.14)$$

$$\begin{aligned} & \langle \partial\varphi(f_1)\bar{\partial}\varphi(f_2) \rangle_{\text{SG}(4\pi, z)} \\ &= -\frac{B^2 A^2 z^2}{4\pi^2} \int dx_1 dx_2 f_1(x_1)f_2(x_2)(K_0(A|z||x_1 - x_2|))^2, \end{aligned} \quad (1.15)$$

and

$$\begin{aligned} & \langle :e^{i\sqrt{4\pi}\varphi}:(f_1) :e^{-i\sqrt{4\pi}\varphi}:(f_2) \rangle_{\text{SG}(4\pi, z)}^T \\ &= \frac{A^2}{\pi^2} \int dx_1 dx_2 f_1(x_1)f_2(x_2)|\partial_{x_1} K_0(A|z||x_1 - x_2|)|^2, \end{aligned} \quad (1.16)$$

$$\begin{aligned} & \langle :e^{i\sqrt{4\pi}\varphi}:(f_1) :e^{i\sqrt{4\pi}\varphi}:(f_2) \rangle_{\text{SG}(4\pi, z)}^T \\ &= -\frac{A^4 z^2}{4\pi^2} \int dx_1 dx_2 f_1(x_1)f_2(x_2)(K_0(A|z||x_1 - x_2|))^2. \end{aligned} \quad (1.17)$$

Indeed, for example, by (1.10) and (1.8),

$$\langle \bar{\psi}_2\psi_1(x)\bar{\psi}_2\psi_1(y) \rangle_{\text{FF}(\mu)}^T = -S_{21}(x, y)S_{21}(y, x) = \frac{1}{(2\pi)^2} (2\partial_x K_0(|\mu||x - y|))^2, \quad (1.18)$$

so that (1.12) gives (1.14), noting that for the gradient two-point function we can drop the truncation of the expectation since $\langle \partial\varphi(f_i) \rangle_{\text{SG}(\beta, z)} = 0$ by symmetry. The equalities (1.15)–(1.17) are analogous.

Note that the right-hand side of (1.16) is not integrable for overlapping test functions, explaining the restriction in the $(n + n', q + q') = (2, 0)$ case in Theorem 1.1. For the gradient two-point functions, i.e., the case $(n + n', q + q') = (0, 2)$, the statement can be extended to test functions with overlapping support, but the singular integrals on the right-hand side of (1.14) and (1.15) then require a more careful interpretation, as in the following theorem. Similarly as before, we write

$$\langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{SG}(4\pi, z)} := \lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)} \quad (1.19)$$

when the limits exist, and similarly for its complex conjugate and $\langle \partial\varphi(f_1)\bar{\partial}\varphi(f_2) \rangle_{\text{SG}(4\pi, z)}$.

Theorem 1.2. *Let $\beta = 4\pi$ and $z \in \mathbb{R}$. Then for $f_1, f_2 \in C_c^\infty(\mathbb{R}^2)$,*

$$\langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{SG}(4\pi,z)} = -\frac{B^2}{\pi^2} \text{p.v.} \int dx_1 dx_2 f_1(x_1) f_2(x_2) (\partial_{x_1} K_0(A|z| |x_1 - x_2|))^2, \quad (1.20)$$

$$\begin{aligned} \langle \partial\varphi(f_1)\bar{\partial}\varphi(f_2) \rangle_{\text{SG}(4\pi,z)} &= -\frac{B^2 A^2 z^2}{4\pi^2} \int dx_1 dx_2 f_1(x_1) f_2(x_2) (K_0(A|z| |x_1 - x_2|))^2 \\ &\quad + \frac{1}{4} \int dx f_1(x) f_2(x), \end{aligned} \quad (1.21)$$

where p.v. \int denotes the Cauchy principal value integral, $\lim_{\delta \rightarrow 0} \int_{|x_1 - x_2| \geq \delta}$.

In particular, the limits defining the left-hand sides exist.

In particular, since the modified Bessel function K_0 and its derivative decay exponentially, the massless sine-Gordon correlation functions decay exponentially whenever $z \neq 0$, when $\beta = 4\pi$. It is conjectured (but in general open) that the massless sine-Gordon model has exponential decay of correlations for all $\beta \in (0, 8\pi)$ and $z \in \mathbb{R} \setminus \{0\}$, with an explicit conjectured relation between the rate of exponential decay (mass) and the parameters of the sine-Gordon model [61]. For further discussion of this problem, see also the last paragraph of [7, p.717] and Section 1.3 below.

The exponential decay is, of course, in contrast to the well-known situation for the GFF (i.e., the case $z = 0$). It is an elementary computation that GFF correlations decay polynomially:

$$\langle \partial\varphi(x)\partial\varphi(y) \rangle_{\text{GFF}} = \frac{-1}{4\pi(x-y)^2}, \quad (1.22)$$

$$\langle \partial\varphi(x)\bar{\partial}\varphi(y) \rangle_{\text{GFF}} = 0, \quad (1.23)$$

$$\langle :e^{i\sqrt{4\pi}\varphi(x)} : :e^{-i\sqrt{4\pi}\varphi(y)} : \rangle_{\text{GFF}} = \frac{4e^{-\gamma}}{|x-y|^2}, \quad (1.24)$$

$$\langle :e^{i\sqrt{4\pi}\varphi(x)} : :e^{i\sqrt{4\pi}\varphi(y)} : \rangle_{\text{GFF}} = 0, \quad (1.25)$$

and that the one-point functions exist and vanish; see, for example, the computations in Section 2.2. The free field correlations $\langle \cdot \rangle_{\text{GFF}}$ are defined as in (1.13) with $z = 0$.

While the above results are for the charge and gradient correlation functions, as a consequence we can also construct the (probabilistic) massless sine-Gordon field itself when $\beta = 4\pi$ and $z \neq 0$. Note that the assumption $z \neq 0$ is essential as the massless GFF on \mathbb{R}^2 only exists up to an additive constant – not in the sense of the following theorem.

Theorem 1.3. *Let $\beta = 4\pi$ and $z \in \mathbb{R}$, $z \neq 0$. Then there exists a probability measure on $\mathcal{S}'(\mathbb{R}^2)$ (not restricted to test functions with mean 0) whose expectation we denote by $\langle \cdot \rangle_{\text{SG}(4\pi,z)}$ with the following properties. For any $f, g \in C_c^\infty(\mathbb{R}^2)$ with $\int f = 0 = \int g$,*

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(4\pi,z)} = \lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle e^{i\varphi(f)} \rangle_{\text{SG}(4\pi,z|\varepsilon,m,L)}, \quad (1.26)$$

$$\langle \varphi(f)\varphi(g) \rangle_{\text{SG}(4\pi,z)} = \lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle \varphi(f)\varphi(g) \rangle_{\text{SG}(4\pi,z|\varepsilon,m,L)}. \quad (1.27)$$

For $f, g \in C_c^\infty(\mathbb{R}^2)$, one has $\langle \varphi(f) \rangle_{\text{SG}(4\pi, z)} = 0$ and the two-point function is given by

$$\langle \varphi(f)\varphi(g) \rangle_{\text{SG}(4\pi, z)} = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \hat{f}(p)\hat{g}(-p)\hat{C}_{Az}(p), \quad (1.28)$$

where $\hat{f}(p) = \int_{\mathbb{R}^2} dx f(x)e^{-ip \cdot x}$ is the Fourier transform of f ,

$$\hat{C}_\mu(p) = \mu^{-2}F(|p|/\mu), \quad \text{where} \quad F(x) = \frac{1}{x^2} - 4\frac{\text{arsinh}(x/2)}{x^3\sqrt{4+x^2}}, \quad (1.29)$$

and $\text{arsinh}(x) = \log(x + \sqrt{x^2 + 1})$ is the inverse hyperbolic sine.

In particular, the above massless sine-Gordon field on \mathbb{R}^2 is localized and has exponential decay of correlation: for any $f \in C_c^\infty(\mathbb{R}^2)$,

$$\sup_{x, y \in \mathbb{R}^2} \langle (\varphi(f_x) - \varphi(f_y))^2 \rangle_{\text{SG}(4\pi, z)} < \infty, \quad (1.30)$$

and

$$\langle \varphi(f_x)\varphi(f_y) \rangle_{\text{SG}(4\pi, z)} \text{ decays exponentially as } |x - y| \rightarrow \infty, \quad (1.31)$$

where $f_x(y) = f(y - x)$ denotes the translation of f to $x \in \mathbb{R}^2$.

Finally, we comment on the exclusion of the one-point functions in Theorem 1.1. While the charge one-point functions vanish in the massless free field case, the following proposition shows that they typically diverge when $z \neq 0$.

Proposition 1.4. *Let $\beta = 4\pi$ and $z \in \mathbb{R}$. For $f \in L^\infty(\mathbb{R}^2)$ with compact support, the charge one-point functions satisfy*

$$\lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\log \varepsilon^{-1}} \langle :e^{\pm i\sqrt{\beta}\varphi \cdot_\varepsilon}(f) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)} \right] = 2\pi z e^{-\gamma} \int_{\mathbb{R}^2} dx f(x), \quad (1.32)$$

while the gradient one-point functions vanish (by symmetry):

$$\langle \partial\varphi(f) \rangle_{\text{SG}(4\pi, z)} = \langle \bar{\partial}\varphi(f) \rangle_{\text{SG}(4\pi, z)} = 0. \quad (1.33)$$

The above divergence of the charge one-point functions is shown in Theorem 3.1 (iv), in fact more generally for all $\beta \in [4\pi, 6\pi)$. As a consequence of this and of the existence of the truncated charge correlation functions, none of the untruncated charge correlation functions involving a test function with $\int_{\mathbb{R}^2} f \neq 0$ converge as $\varepsilon \rightarrow 0$. On the other hand, since the gradient one-point functions exist, the existence of the truncated gradient correlation functions also implies that of the untruncated gradient correlation functions

$$\left\langle \prod_{j=1}^q \partial\varphi(g_j^+) \prod_{j'=1}^{q'} \bar{\partial}\varphi(g_{j'}^-) \right\rangle_{\text{SG}(4\pi, z)}, \quad (1.34)$$

with explicit expressions given by inverting (1.6).

Before discussing consequences of Theorems 1.1–1.2 and our more general analysis in their proofs, we remark on the physical interpretation of the fermionic side of the Coleman correspondence.

Remark 1.5. The Coleman correspondence can be written in terms of Dirac matrices γ^μ satisfying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta_{\mu\nu} \mathbf{1}$. In the representation we have chosen, these are

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.35)$$

Thus $\gamma^0 = -\sigma_2$, $\gamma^1 = \sigma_1$, $\gamma^5 = \sigma_3$, where the σ_i are the Pauli matrices. In terms of these, the Dirac operator can be written as

$$\not{D} = \gamma^0 \partial_0 + \gamma^1 \partial_1 = \begin{pmatrix} 0 & i\partial_0 + \partial_1 \\ -i\partial_0 + \partial_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\bar{\partial} \\ 2\partial & 0 \end{pmatrix}. \quad (1.36)$$

The Coleman correspondence can then be regarded as the following equivalence of the fields:

$$:e^{i\sqrt{4\pi}\varphi}: \leftrightarrow A\bar{\psi}_1\psi_1 = \frac{A}{2}\bar{\psi}(\mathbf{1} + \gamma_5)\psi, \quad (1.37)$$

$$:e^{-i\sqrt{4\pi}\varphi}: \leftrightarrow A\bar{\psi}_2\psi_2 = \frac{A}{2}\bar{\psi}(\mathbf{1} - \gamma_5)\psi, \quad (1.38)$$

$$-i\partial\varphi \leftrightarrow B\bar{\psi}_2\psi_1 = \frac{B}{2}\bar{\psi}(i\gamma^0 + \gamma^1)\psi, \quad (1.39)$$

$$-i\bar{\partial}\varphi \leftrightarrow B\bar{\psi}_1\psi_2 = \frac{B}{2}\bar{\psi}(-i\gamma^0 + \gamma^1)\psi. \quad (1.40)$$

The right-hand sides of (1.37)–(1.38) have the interpretation of being the chiral densities associated with the spinor field ψ , and the right-hand sides of (1.39)–(1.40) that of the vector current $\bar{\psi}\gamma^\mu\psi$ (written in complex coordinates); see, for example, [30, Section 3].

1.2. Further results

Our estimates for the sine-Gordon model together with the correlation inequalities from [32] also imply the following results for the infinite volume limit for $\beta \in (0, 6\pi)$.

The first theorem is for the infinite volume limit of the massless sine-Gordon field modulo constants (the “gradient field”). Let $\mathcal{S}'(\mathbb{R}^2)/\text{constants}$ denote the topological dual of the (closed) subspace of integral-0 functions of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

Theorem 1.6. *Let $\beta \in (0, 6\pi)$ and $z \in \mathbb{R}$. Then for any $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f = 0$, the limit*

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z)} := \lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)} \quad (1.41)$$

exists, and extends to the characteristic functional of a probability measure on the space $\mathcal{S}'(\mathbb{R}^2)/\text{constants}$ whose expectation we denote by $\langle \cdot \rangle_{\text{SG}(\beta, z)}$. This measure is invariant under Euclidean transformations and satisfies

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z)} \geq e^{-\frac{1}{2}(f, (-\Delta)^{-1}f)}, \quad \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z)} \leq (f, (-\Delta)^{-1}f). \quad (1.42)$$

For $m > 0$ fixed and $z > 0$, we similarly obtain the existence of the infinite volume of limit of the massive sine-Gordon field.

Theorem 1.7. *Let $\beta \in (0, 6\pi)$ and $m, z > 0$. For any $f \in C_c^\infty(\mathbb{R}^2)$ (not assuming $\int f = 0$), the limit*

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|m)} := \lim_{L \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} \langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)} \quad (1.43)$$

exists, and extends to the characteristic functional of a probability measure on $\mathcal{S}'(\mathbb{R}^2)$ whose expectation we denote by $\langle \cdot \rangle_{\text{SG}(\beta, z|m)}$. This measure is invariant under Euclidean transformations and satisfies

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|m)} \geq e^{-\frac{1}{2}\langle f, (-\Delta + m^2)^{-1} f \rangle}, \quad \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z|m)} \leq \langle f, (-\Delta + m^2)^{-1} f \rangle, \quad (1.44)$$

and

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|m)} \geq \langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z)}, \quad \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z|m)} \leq \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z)}, \quad (1.45)$$

where the right-hand sides of the last two bounds are as in Theorem 1.6 and hold if $\int f = 0$.

For $\beta = 4\pi$, we can then deduce using the localization bound (1.30) that the $m \rightarrow 0$ limit can be taken after the infinite volume limit, which means that the formal $\varphi \mapsto \varphi + \frac{2\pi}{\sqrt{\beta}} \mathbb{Z}$ -symmetry of the massless sine-Gordon model is spontaneously broken in the infinite volume limit.

Corollary 1.8. *Let $\beta = 4\pi$ and $z > 0$. Then for any $f \in C_c^\infty(\mathbb{R}^2)$ (not assuming $\int f = 0$),*

$$\langle \varphi(f)^2 \rangle_{\text{SG}(4\pi, z|0^+)} := \lim_{m \rightarrow 0} \lim_{L \rightarrow \infty} \langle \varphi(f)^2 \rangle_{\text{SG}(4\pi, z|m, L)} \leq \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} |\hat{f}(p)|^2 \hat{C}_{Az}(p) < \infty, \quad (1.46)$$

where $\hat{C}_\mu(p)$ is as in Theorem 1.2.

Moreover, the limit $\langle \cdot \rangle_{\text{SG}(4\pi, z|0^+)} := \lim_{m \rightarrow 0} \lim_{L \rightarrow \infty} \langle \cdot \rangle_{\text{SG}(4\pi, z|m, L)}$ exists in the sense of characteristic functionals and defines a probability measure on $\mathcal{S}'(\mathbb{R}^2)$ (not dividing out constants).

We expect that $\langle \cdot \rangle_{\text{SG}(4\pi, z|0^+)}$ is the same as $\langle \cdot \rangle_{\text{SG}(4\pi, z)}$ but our arguments do not imply this.

1.3. Heuristics and previous results

The formal equivalence of the massless sine-Gordon model and the massive Thirring model was observed by Coleman [16]. The massive Thirring model with parameters (g, μ) is formally given by a fermionic path integral with “density”

$$\exp \left[- \int_{\mathbb{R}^2} dx (\psi \not{\partial} \bar{\psi} + \mu(\psi_1 \bar{\psi}_1 + \psi_2 \bar{\psi}_2) + 2g \psi_1 \bar{\psi}_2 \psi_2 \bar{\psi}_1) \right]. \quad (1.47)$$

Coleman observed that, order by order in a formal expansion, the massless sine-Gordon model with parameters (β, z) is related to the massive Thirring model if the parameters

of the two models are related by

$$g = B^2 \left(1 - \frac{4\pi}{\beta}\right), \quad \mu = Az. \quad (1.48)$$

Heuristically this prediction is not difficult to understand from the type of massless Gaussian free field and massless free fermion computations we derive in Section 2; these are versions of the identifications (1.37)–(1.40) in the elementary situation of the massless Gaussian free field and massless free fermions. Indeed, after rescaling φ by $\sqrt{4\pi/\beta}$, the measure of the sine-Gordon model with parameters (β, z) has formal density

$$\begin{aligned} & \exp \left[- \int_{\mathbb{R}^2} dx \left(\frac{8\pi}{\beta} (\partial\varphi)(\bar{\partial}\varphi) - z (:e^{i\sqrt{4\pi}\varphi}: + :e^{-i\sqrt{4\pi}\varphi}:) \right) \right] \\ &= \exp \left[- \int_{\mathbb{R}^2} dx 2(\partial\varphi)(\bar{\partial}\varphi) \right. \\ & \quad \left. - \int_{\mathbb{R}^2} dx \left(2 \left(1 - \frac{4\pi}{\beta}\right) (-i\partial\varphi)(-i\bar{\partial}\varphi) - z (:e^{i\sqrt{4\pi}\varphi}: + :e^{-i\sqrt{4\pi}\varphi}:) \right) \right]. \quad (1.49) \end{aligned}$$

Thus relative to the massless free fermion “measure” respectively the massless Gaussian free field measure, formally, the massive Thirring model and the sine-Gordon model are weighted by

$$\exp \left[\int_{\mathbb{R}^2} dx (-2g\bar{\psi}_1\psi_2\bar{\psi}_2\psi_1 + \mu(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2)) \right], \quad (1.50)$$

$$\exp \left[\int_{\mathbb{R}^2} dx \left(-2 \left(1 - \frac{4\pi}{\beta}\right) (-i\partial\varphi)(-i\bar{\partial}\varphi) + z (:e^{i\sqrt{4\pi}\varphi}: + :e^{-i\sqrt{4\pi}\varphi}:) \right) \right], \quad (1.51)$$

and one can see the order by order correspondence of the models with parameters (1.48), using the equivalence of the correlations of (1.37)–(1.40) with respect to the noninteracting measures. To directly apply these identities, note that we changed the order of the Grassmann variables in (1.50) compared to (1.47) explaining the change of the sign of the quadratic term.

Mathematically, this formal argument is however far from a proof. To start with, the probabilistic or analytic existence of the massless sine-Gordon model and the massive Thirring model is a nontrivial problem. Both the ultraviolet (short-distance) and infrared (long-distance) behavior of both models cause significant difficulties, while both regimes need to be handled to establish the Coleman correspondence for the infinite volume models. We summarize the most relevant previous results on these problems now.

Concerning the ultraviolet stability of the sine-Gordon model, we note that various constructions of the finite volume sine-Gordon model exist under different assumptions (see in particular [4, 8, 13, 22, 23, 27, 33, 44, 47, 52]), but none of these covers all $\beta \in (0, 8\pi)$ and all $z \in \mathbb{R}$. For the ultraviolet construction of the Thirring model, for $|g|$ small, we refer in particular to [6] which considers the massive case using previous results on the massless case including [9, 10]; see also the further references therein. In preparation for later discussion, we stress that it is a technically important ingredient of these analyses that the finite volume regularizations of these models are defined on a torus with spatially constant mass term.

Concerning the infrared behavior of the massless sine-Gordon and the massive Thirring models, the following previous results are particularly relevant. For the sine-Gordon model with $\beta > 0$ small, exponential decay of the charge correlation functions was proved for the model constructed with Dirichlet boundary conditions [60]; see also the discussion on Debye screening further below. For the massive Thirring model, stretched exponential correlation decay was proved for $|g|$ small (corresponding to $|\beta - 4\pi|$ small on the sine-Gordon side), with antiperiodic boundary conditions [6]. For a potential application of these results to a proof of the Coleman correspondence in the regimes they apply to (and thus to transfer the results from one side to the other as we do for $\beta = 4\pi$ in this article), we emphasize the boundary conditions the former results are proved for. Indeed, the generalization of the argument we use for $\beta = 4\pi$ (which is in line with Coleman’s original proposal) would require “free” boundary versions of the former results, by which we mean that the sine-Gordon model is defined in terms of the infinite volume free field but with finite volume interaction, and that the massive Thirring model is defined with infinite volume quartic interaction term but with finite volume mass term, all with uniform dependence on the volume. We expect such estimates are true, but due to lack of translation invariance, they are significantly more difficult to obtain and pose interesting problems for future work.

Concerning the Coleman correspondence, i.e., the equivalence of both models, we mention that in view of the restrictions in the domains of construction of the two models, previous results are restricted to models with finite volume sine-Gordon interaction or with fixed external “bare” mass $m > 0$. In particular, to avoid any doubt, we stress that the Coleman correspondence for the *massless* sine-Gordon model in *infinite volume* remains open for $\beta \neq 4\pi$. In the presence of a bare mass or finite volume interaction, the relevant previous results are as follows. For $\beta < 4\pi$, a variant of the Coleman correspondence between the *massive* sine-Gordon model (i.e., $m > 0$ fixed) and the massive Thirring–Schwinger model (QED₂) was proved in [33] for z/m^2 sufficiently small; see also [29] for a review. Also for $\beta < 4\pi$, but now with finite volume interaction instead of with an external mass term, a version of the Coleman correspondence was shown in [55]. In the same regime, $\beta < 4\pi$ and finite volume sine-Gordon interaction, a construction of Haag–Kastler nets of the sine-Gordon model with finite volume interaction in Lorentzian signature was carried out in [2], and a version of the Coleman correspondence was verified in this setting of algebraic QFT. For $\beta = 4\pi$ but with finite volume interaction, a version of the Coleman correspondence applying to the sine-Gordon model with small coupling constant z (depending on the volume) was proved in [21]. Finally, for β in a neighborhood of 4π , but again in finite volume and with all coupling constants small depending on the volume, the Coleman correspondence was proved in [7].

The integrability of two-dimensional conformal field theories is celebrated and well known. That nonconformal perturbations of conformal field theories are in some cases expected to remain integrable is perhaps more surprising. The sine-Gordon and massive Thirring models are such examples, and our result confirms the most fundamental (and arguably simplest) instance of this integrability. In the physics literature, many other exact results have been predicted by employing various techniques. For example, at the free

fermion point $\beta = 4\pi$, exact expressions for the fractional charge two-point functions, i.e., $\langle :e^{i\sqrt{\alpha_1}\varphi(x_1)} : :e^{i\sqrt{\alpha_2}\varphi(x_2)} : \rangle_{\text{SG}(4\pi, z)}$ with $\alpha_1, \alpha_2 \in (0, 4\pi)$, were derived in [11], the mass was determined for general β by using a mapping to the continuum limit of an inhomogeneous six-vertex model and the Bethe ansatz in [20, 61], and exact expressions for the fractional charge one-point function $\langle :e^{i\sqrt{\alpha}\varphi(x)} : \rangle_{\text{SG}(\beta, z)}$ for $\alpha \in (0, 4\pi)$ and general β were derived in [48] by extrapolation of exact results for $\beta = 4\pi$ and in the asymptotics $\beta \rightarrow 0$. Further references are [19, 25, 49], and for a review, see also [59]. All of these integrability results in infinite volume remain conjectural (except for our results at $\beta = 4\pi$). In finite volume, we mention the rigorous connection to the XOR Ising model at $\beta = 2\pi$ proved in [43].

It is also well known that the sine-Gordon model is exactly related to the classical two-dimensional (two-component) Coulomb gas. For this, we refer in particular to [28] and also [32] where, using this relation, many fundamental properties of the Coulomb gas have been derived when $\beta < 4\pi$ including existence of the pressure and correlation functions, the exact equation of state for the pressure, and the exact scaling behaviour in z of the rate of exponential decay of correlations assuming its existence in a suitable sense. The latter exponential decay of correlations is in general open. For the related three-dimensional Coulomb gas, exponential decay (Debye screening) was proved for $\beta > 0$ and z both small in [12]. The methods have also been partially extended to the two-dimensional Coulomb gas in [60]. This last result is incomplete in the sense that it requires small coupling constants and more significantly that it relies on Dirichlet boundary conditions. On the other hand, the relation between the sine-Gordon model and the Thirring model only holds for “free” boundary conditions in finite volume in the previously discussed sense. Thus the proof of Debye screening of the two-dimensional Coulomb gas with free boundary condition (and its equivalence with Dirichlet boundary conditions) remains an interesting problem. For related results in the three-dimensional setting, see also [26]. Correlation inequalities for the Coulomb gas and the sine-Gordon model as well as their applications are discussed in [31, 32, 53]; we make some use of these in Section 3. Assuming the validity of the Coleman correspondence at the free fermion point (which we prove here), its implications for the Coulomb gas at $\beta = 4\pi$ are discussed in [17]; see also [34].

Next, we mention a few related bosonization results. The concept of bosonization goes at least back to [50]; see also [58] for a review. In the free field case, the boson-fermion correspondence has been extended by disorder operators [30]. Second, while the bosonization relations in this paper rely essentially on the precise asymptotics of the correlations in the continuum limit, in the massless free field case, exact discrete versions have been found as well; see in particular [24]. Some bosonization results are also expected to extend to Riemann surfaces [30]. For applications of bosonization of free fermions, see, for example, [42, Chapter 10.5].

Finally, let us emphasize that the massless sine-Gordon model is an essential example of a two-dimensional *nonconformal* perturbation of a CFT. For *conformal* field theories, a lot of recent progress has been made, in particular for the Ising model (see [14, 15, 41] and references therein) and for the Liouville CFT (see [39, 45] and references therein).

Moreover, we mention that models related to the *massless* Thirring model have also been studied in detail, in particular recently in the form of interacting dimers [36, 37].

1.4. Outline of the paper

The paper is structured as follows.

In Section 2, we derive the Coleman correspondence in the (noninteracting) massless case $z = \mu = 0$. This analysis is elementary and the result is well known, but lacking a reference providing exactly what we need later we include the short and instructive proofs. This is also an opportunity for us to introduce notation as well as to collect various estimates for Gaussian free fields and massless fermions for later use.

In Section 3, we state our estimates for the sine-Gordon model and free fermions, and then prove our main theorems assuming these estimates. As discussed already briefly in Section 1.3, it is important that these estimates apply to “free” boundary versions of both models. The remainder of the paper is mainly devoted to proving these estimates.

In Sections 4 and 5, we consider the sine-Gordon side. In particular, we construct the renormalized potential for the regularized sine-Gordon model in Section 4, and then use it, in Section 5, to prove the analyticity of the partition function of the sine-Gordon model and the convergence of the correlations functions, for any finite volume interaction. The analysis in Section 4 extends the continuous renormalization group approach of [13] by allowing space dependent coupling constants and extraction of the precise estimates needed subsequently; similar results could presumably be obtained using the related methods of [5, 7]. The analyticity and convergence results of Section 5 rely on the combination of the expansions for the renormalized potential up a finite scale at which they converge with qualitative bounds and concentration estimates for Gaussian measures, which provide sufficient control in the regime where the expansions fail to converge.

In Section 6, we prove the corresponding results on the free fermion side. Our main work here goes into the analysis of the Green’s function of the Dirac operator with finite volume mass term. Due to lack of the maximum principle or a random walk representation for the Dirac operator as well as lack of translation invariance, we rely on a series construction by expansion in a carefully chosen basis. This basis is related to eigenfunctions of the Laplacian on the disk and the spherical geometry is convenient here, but we expect that analogous results hold in more general geometry.

In Appendix A, we collect a few (well-known) operational tools for cumulants and free fermions that we use in various places throughout the paper.

1.5. Notation

We will write $f \in L_c^\infty(\mathbb{R}^2)$ if f is compactly supported and essentially bounded. We write similarly $f \in L_c^\infty(\mathbb{R}^2 \times \{\pm 1\})$ if $f(x, \pm 1)$ is compactly supported and essentially bounded. We often write $\xi = (x, \sigma) \in \mathbb{R}^2 \times \{\pm 1\}$ and

$$\int d\xi f(\xi) \equiv \int_{\mathbb{R}^2 \times \{-1, 1\}} d\xi f(\xi) \equiv \sum_{\sigma \in \{-1, 1\}} \int_{\mathbb{R}^2} dx f(x, \sigma). \quad (1.52)$$

Throughout the paper, $|\cdot|$ denotes the Euclidean norm, and we will often make use of the identification of \mathbb{R}^2 and \mathbb{C} . More precisely, we will denote the components of a point $x \in \mathbb{R}^2$ by $x = (x_0, x_1)$ and its identification with an element in \mathbb{C} by $x_1 + ix_0$. We will also repeatedly write $[n] := \{1, \dots, n\}$. We write $A \subset B$ to indicate that A is any subset of B (no need to be proper).

2. Free field estimates and bosonization of massless fermions

A well-understood (but essential) step in the proof of Theorem 1.1 is to verify (1.12) when $z = \mu = 0$. Results of this flavor exist in the literature – see [30, Section 3] or [21, Section 2.2], for example, but since neither of these references provides the exact statements that we need, we will give a derivation in our set-up in this section. Along the way we will also collect estimates for the correlations of the free field that we require for the proof of the Coleman correspondence with $z \neq 0$.

2.1. Fermionic side: massless free fermion correlations

We start with computation of the correlation functions of free massless Dirac fermions whose correlation kernel S is given by (1.8) with $\mu = 0$, i.e.,

$$S(x, y) = \frac{1}{2\pi} \begin{pmatrix} 0 & 1/(\bar{x} - \bar{y}) \\ 1/(x - y) & 0 \end{pmatrix}. \quad (2.1)$$

In this section, the fermionic correlation functions are then defined by

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i}(x_i) \psi_{\beta_i}(y_i) \right\rangle_{\text{FF}(0)} = \det(S_{\alpha_i \beta_j}(x_i, y_j))_{i,j=1}^n \quad (2.2)$$

whenever the determinant on the right-hand side is well defined, i.e., for all $i, j \in [n]$, either $x_i \neq y_j$ or $\alpha_i = \beta_j$. The Coleman correspondence is in terms of truncated correlation functions, and importantly, we shall require the setting where $x_i = y_i$. These truncated correlation functions are defined by

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(0)}^T = (-1)^{n+1} \sum_{\pi} \prod_{i=1}^n S_{\alpha_{\pi^i(1)} \beta_{\pi^{i+1}(1)}}(x_{\pi^i(1)}, x_{\pi^{i+1}(1)}), \quad (2.3)$$

where the sum is over cyclic permutations π , whenever the right-hand side is well defined, i.e., for all $i, j \in [n]$, either $x_i \neq x_j$ or $\alpha_i = \beta_j$ and $\alpha_j = \beta_i$. These definitions are consistent with (1.9) and (1.10) but slightly more general. (This generality is required for the proof of Theorem 1.1.)

We will need various identities for these determinants defining our correlation functions. These identities are conveniently seen in the representation of these determinants in terms of Grassmann integrals. We discuss the details of this representation and prove the required (well-known) properties in Appendix A. The connection between our discussion

there and that here is that to study the correlation functions $\langle \prod_{1 \leq i \leq n} \bar{\psi}_{\alpha_i}(x_i) \psi_{\beta_i}(y_i) \rangle_{\text{FF}(0)}$ the matrix (K_{ij}) in Lemma A.2 can be defined to be $S_{\alpha_i \beta_j}(x_i, y_j)$ off the diagonal and on the diagonal to be a constant real number chosen so large that K is invertible – the exact value of this constant is irrelevant (see also Remark A.3). This definition and Lemma A.2 then allow us to deduce that the properties of Lemma A.4 hold also for the correlation functions we are considering here. Based on this representation, we can also use Lemma A.5 to see that (2.3) is also consistent with (1.6), i.e.,

$$\langle \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \rangle_{\text{FF}(0)}^T = \langle \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \rangle_{\text{FF}(0)} = 0 \quad (\text{assuming } \alpha_i = \beta_i), \quad (2.4)$$

and, for $n \geq 2$,

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(0)}^T = \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(0)} - \sum_{P \in \mathfrak{P}_n} \prod_j \left\langle \prod_{i \in P_j} \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(0)}^T \quad (2.5)$$

when the right-hand sides exist. Thus when the untruncated correlation functions exist, they determine the truncated ones by (2.5). In view of this fact, the next lemma determines the truncated correlation functions

$$\left\langle \prod_{k=1}^n \bar{\psi}_1 \psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2 \psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T \quad (2.6)$$

when $x_k \neq y_{k'}$ for all k and k' .

Lemma 2.1. *For any $x_1, \dots, x_n, y_1, \dots, y_{n'}$ in \mathbb{R}^2 with $x_k \neq y_{k'}$ for all $k \in [n]$ and $k' \in [n']$,*

$$\left\langle \prod_{k=1}^n \bar{\psi}_1 \psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2 \psi_2(y_{k'}) \right\rangle_{\text{FF}(0)} = \mathbf{1}_{n=n'} \frac{1}{(2\pi)^{2n}} \left| \det \left(\frac{1}{x_k - y_{k'}} \right)_{k,k'=1}^n \right|^2. \quad (2.7)$$

Proof. First consider $n \neq n'$. Then every term in the expansion of the determinant (2.2) that defines the left-hand side must contain a factor S_{11} or S_{22} and hence vanish. Let us assume now that $n = n'$. Then, by anticommutativity (see (A.22)),

$$\begin{aligned} & \left\langle \prod_{k=1}^n \bar{\psi}_1(x_k) \psi_1(x_k) \prod_{k=1}^n \bar{\psi}_2(y_k) \psi_2(y_k) \right\rangle_{\text{FF}(0)} \\ &= (-1)^n \left\langle \prod_{k=1}^n \bar{\psi}_1(x_k) \psi_2(y_k) \prod_{k=1}^n \bar{\psi}_2(y_k) \psi_1(x_k) \right\rangle_{\text{FF}(0)}. \end{aligned} \quad (2.8)$$

Since $S_{11} = S_{22} = 0$ the right-hand side factorizes (see (A.23)), and by (2.2) it is hence equal to

$$\begin{aligned} & (-1)^n \left\langle \prod_{k=1}^n \bar{\psi}_1(x_k) \psi_2(y_k) \right\rangle_{\text{FF}(0)} \left\langle \prod_{k=1}^n \bar{\psi}_2(y_k) \psi_1(x_k) \right\rangle_{\text{FF}(0)} \\ &= \frac{(-1)^n}{(2\pi)^{2n}} \det \left(\frac{1}{\bar{x}_k - \bar{y}_{k'}} \right)_{k,k'=1}^n \det \left(\frac{1}{y_k - x_{k'}} \right)_{k,k'=1}^n, \end{aligned} \quad (2.9)$$

which gives the right-hand side of the claim. \blacksquare

The next two lemmas then allow computing all truncated correlation functions involving also the factors $\bar{\psi}_2\psi_1$ and $\bar{\psi}_1\psi_2$.

Lemma 2.2. *For $n + n' + q + q' \geq 2$ and any distinct $x_1, \dots, x_n, y_1, \dots, y_{n'}, z_1, \dots, z_q, w_1, \dots, w_{q'}, z, w$ in \mathbb{R}^2 , the following identities hold:*

$$\begin{aligned} & \left\langle \bar{\psi}_2\psi_1(z) \prod_{j=1}^q \bar{\psi}_2\psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1\psi_2(w_{j'}) \prod_{k=1}^n \bar{\psi}_1\psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2\psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T \\ &= \frac{\mathbf{1}_{n=n'}}{2\pi} \sum_{i=1}^n \left(\frac{1}{x_i - z} - \frac{1}{y_i - z} \right) \\ & \quad \times \left\langle \prod_{j=1}^q \bar{\psi}_2\psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1\psi_2(w_{j'}) \prod_{k=1}^n \bar{\psi}_1\psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2\psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} & \left\langle \bar{\psi}_1\psi_2(w) \prod_{j=1}^q \bar{\psi}_2\psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1\psi_2(w_{j'}) \prod_{k=1}^n \bar{\psi}_1\psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2\psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T \\ &= \frac{\mathbf{1}_{n=n'}}{2\pi} \sum_{i=1}^n \left(\frac{1}{\bar{x}_i - \bar{w}} - \frac{1}{\bar{y}_i - \bar{w}} \right) \\ & \quad \times \left\langle \prod_{j=1}^q \bar{\psi}_2\psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1\psi_2(w_{j'}) \prod_{k=1}^n \bar{\psi}_1\psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2\psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T. \end{aligned} \quad (2.11)$$

The right-hand sides are interpreted as 0 when $n = n' = 0$.

Proof. Since the proofs of (2.10) and (2.11) are analogous, we only consider (2.10). By (2.3), when $n + n' + q + q' \geq 2$, we have

$$\begin{aligned} & \left\langle \prod_{j=1}^q \bar{\psi}_2\psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1\psi_2(w_{j'}) \prod_{k=1}^n \bar{\psi}_1\psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2\psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T \\ &= (-1)^{n+n'+q+q'+1} \sum_{\pi \in C_{n+n'+q+q'}} \prod_{i=1}^{n+n'+q+q'} S_{\alpha_{\pi^i(1)} \beta_{\pi^{i+1}(1)}} (u_{\pi^i(1)}, u_{\pi^{i+1}(1)}) \end{aligned} \quad (2.12)$$

where C_p is the set of cyclic permutations of $\{1, \dots, p\}$ and we have defined

$$(\alpha_i, \beta_i, u_i) = \begin{cases} (1, 1, x_i) & (1 \leq i \leq n), \\ (2, 2, y_{i-n}) & (n < i \leq n + n'), \\ (2, 1, z_{i-n-n'}) & (n + n' < i \leq n + n' + q), \\ (1, 2, w_{i-n-n'-q}) & (n + n' + q < i \leq n + n' + q + q'). \end{cases} \quad (2.13)$$

By (2.1), all terms that contain a factor S_{11} or S_{22} vanish. Therefore it is necessary that the number of factors of $\bar{\psi}_1$ equals that of ψ_2 , which implies that $n = n'$ if (2.12) is nonzero, which we thus assume from now on. The truncated correlation function

$$\left\langle \bar{\psi}_2 \psi_1(z) \prod_{j=1}^q \bar{\psi}_2 \psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1 \psi_2(w_{j'}) \prod_{k=1}^n \bar{\psi}_1 \psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2 \psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}^T \quad (2.14)$$

is given by replacing one of the factors $S_{\alpha\beta'}(u, u')$ in (2.12) by

$$-S_{\alpha 1}(u, z) S_{2\beta'}(z, u') \quad (2.15)$$

and then summing over the choice of which factor gets replaced. Using again $S_{11} = S_{22} = 0$, the last term vanishes unless $(\alpha, \beta') = (2, 1)$, and in this case,

$$\begin{aligned} -S_{21}(u, z) S_{21}(z, u') &= \frac{1}{(2\pi)^2 (u-z)(u'-z)} = \frac{1}{(2\pi)^2} \left(\frac{1}{u-z} - \frac{1}{u'-z} \right) \frac{1}{u'-u} \\ &= \frac{1}{2\pi} \left(\frac{1}{u'-z} - \frac{1}{u-z} \right) S_{21}(u, u'). \end{aligned} \quad (2.16)$$

Thus the replacement of the factor $S_{21}(u, u')$ is equivalent to multiplying it by

$$\frac{1}{2\pi} \left(\frac{1}{u'-z} - \frac{1}{u-z} \right). \quad (2.17)$$

The possibilities for (u, u') that are compatible with the constraint $(\alpha, \beta') = (2, 1)$ are

$$(u, u') = (y_i, x_j), \quad (u, u') = (y_i, z_k), \quad (u, u') = (z_k, x_j), \quad (u, u') = (z_k, z_l), \quad (2.18)$$

for some $i, j \in [n]$ and $k, l \in [q]$ with $k \neq l$. In these cases we obtain factors of, respectively,

$$\begin{aligned} &\frac{1}{2\pi} \left(\frac{1}{x_j - z} - \frac{1}{y_i - z} \right), \quad \frac{1}{2\pi} \left(\frac{1}{z_l - z} - \frac{1}{z_k - z} \right), \\ &\frac{1}{2\pi} \left(\frac{1}{z_k - z} - \frac{1}{y_i - z} \right), \quad \frac{1}{2\pi} \left(\frac{1}{x_j - z} - \frac{1}{z_k - z} \right). \end{aligned} \quad (2.19)$$

In the sum over cycles in (2.12), we may restrict to cycles which give a nonvanishing contribution, and we will do this in the following. Then by symmetry, given any pair $(i, j) \in [n]^2$, the proportion r of such cycles giving the factor $S_{21}(y_i, x_j)$ is independent of (i, j) ; given any pair $(i, k) \in [n] \times [q]$, the proportion s of such cycles giving the factor $S_{21}(y_i, z_k)$ is independent of (i, k) and the same as the proportion of cycles giving the factor $S_{21}(z_k, x_i)$; and given any pair $(k, l) \in [q]^2$ with $k \neq l$ the proportion t of cycles giving the factor $S_{21}(z_k, z_l)$ is independent of (k, l) . Therefore (2.14) is obtained from (2.12) by multiplication by $1/2\pi$ and

$$\begin{aligned} &r \sum_{i,j} \left(\frac{1}{x_j - z} - \frac{1}{y_i - z} \right) + s \sum_{i,k} \left(\frac{1}{z_k - z} - \frac{1}{y_i - z} \right) + s \sum_{i,k} \left(\frac{1}{x_i - z} - \frac{1}{z_k - z} \right) \\ &+ t \sum_{k,l} \left(\frac{1}{z_l - z} - \frac{1}{z_k - z} \right) = (rn + sq) \sum_i \left(\frac{1}{x_i - z} - \frac{1}{y_i - z} \right). \end{aligned} \quad (2.20)$$

Since for any cycle π with nonvanishing contribution, each of the points y_i must appear once as the first argument of $S_{\alpha\beta}$ (and then necessarily $\alpha = 2$) and each x_i once as the second argument of $S_{\alpha\beta}$ (and then necessarily $\beta = 1$) in the product in (2.12), we also see that $rn + sq = 1$. Thus we have recovered (2.10) in the case $n + n' \neq 0$.

For the case $n = n' = 0$, the same argument shows that the only possibility for u, u' is now $(u, u') = (z_k, z_l)$, and as before, this gives a zero contribution since the sum $\sum_{k,l} ((z_k - z)^{-1} - (z_l - z)^{-1})$ vanishes. This concludes the proof. ■

Lemma 2.3. *For $q + q' \geq 2$ and any distinct $z_1, \dots, z_q, w_1, \dots, w_{q'}$ in \mathbb{R}^2 ,*

$$\left\langle \prod_{j=1}^q \bar{\psi}_2 \psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1 \psi_2(w_{j'}) \right\rangle_{\text{FF}(0)}^T = \begin{cases} \frac{1}{(2\pi)^2 (z_1 - z_2)^2} & (q = 2, q' = 0), \\ \frac{1}{(2\pi)^2 (\bar{w}_1 - \bar{w}_2)^2} & (q = 0, q' = 2), \\ 0 & \text{else.} \end{cases} \quad (2.21)$$

Proof. Lemma 2.2 implies that the left-hand side is 0 when $q + q' > 2$. In the case $(q, q') = (1, 1)$, any of the products in (2.3) must contain factors S_{11} or S_{22} , and thus vanish as well. In the case $(q, q') = (2, 0)$, by (2.3), we get

$$-S_{21}(z_1, z_2) S_{21}(z_2, z_1) = -\frac{1}{(2\pi)^2} \frac{1}{z_1 - z_2} \frac{1}{z_2 - z_1} = \frac{1}{(2\pi)^2} \frac{1}{(z_1 - z_2)^2}. \quad (2.22)$$

The case $(q, q') = (0, 2)$ is analogous. ■

2.2. Bosonic side: free field correlations

For the computation of the free field correlations, we first recall that, for $\varepsilon > 0$ and $m > 0$, our regularized GFF is the centered Gaussian field with covariance

$$\int_{\varepsilon^2}^{\infty} ds e^{-s(-\Delta + m^2)}(x, y) = \int_{\varepsilon^2}^{\infty} ds \frac{e^{-\frac{|x-y|^2}{4s}}}{4\pi s} e^{-m^2 s}. \quad (2.23)$$

We write $\nu^{\text{GFF}(\varepsilon, m)}$ for the (centered) Gaussian measure with this covariance. It is a basic fact that this measure is supported on smooth functions and that the covariance of the derivatives of the field is given by the derivatives of the covariance; see e.g. [46, Appendix B]. We also recall the definition

$$:e^{\pm i\sqrt{\beta}\varphi(x)}:_\varepsilon := \varepsilon^{-\beta/4\pi} e^{\pm i\sqrt{\beta}\varphi(x)}. \quad (2.24)$$

Our goal is to compute the truncated correlation functions

$$\begin{aligned} & \left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)}:_\varepsilon \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})}_\varepsilon \right\rangle_{\text{GFF}}^T \\ & := \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)}:_\varepsilon \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})}_\varepsilon \right\rangle_{\text{GFF}(\varepsilon, m)}^T \end{aligned} \quad (2.25)$$

as well as smeared versions of them.

The following estimates for the covariance of φ and its derivatives will be useful. (As before, γ is the Euler–Mascheroni constant.)

Lemma 2.4. *Uniformly on compact subsets of $x \neq y \in \mathbb{R}^2$, as $\varepsilon \rightarrow 0$,*

$$\langle \varphi(x)^2 \rangle_{\text{GFF}(\varepsilon, m)} + \frac{1}{2\pi} \log \varepsilon \rightarrow -\frac{1}{2\pi} \log m - \frac{\gamma}{4\pi}, \quad (2.26)$$

$$\langle \varphi(x)\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow -\frac{1}{2\pi} \log m - \frac{1}{2\pi} \log \frac{|x-y|}{2} - \frac{\gamma}{2\pi} + O(m|x-y|). \quad (2.27)$$

Moreover, uniformly on compact sets of $x \neq y$, as $\varepsilon \rightarrow 0$ and then $m \rightarrow 0$,

$$-\langle \partial\varphi(x)\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)} = \langle \varphi(x)\partial\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow \frac{1}{4\pi} \frac{1}{x-y}, \quad (2.28)$$

and

$$\langle \partial\varphi(x)\partial\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow -\frac{1}{4\pi} \frac{1}{(x-y)^2}, \quad \langle \partial\varphi(x)\bar{\partial}\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow 0. \quad (2.29)$$

Moreover, for any $g \in L_c^\infty(\mathbb{R}^2)$, uniformly on compact subsets of $u \in \mathbb{R}^2$,

$$\langle \varphi(u)\partial\varphi(g) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow -\int_{\mathbb{R}^2} dx g(x) \frac{1}{4\pi} \frac{1}{x-u}, \quad (2.30)$$

and for all $f, g \in L_c^\infty(\mathbb{R}^2)$ with disjoint supports,

$$\langle \partial\varphi(f)\partial\varphi(g) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow -\frac{1}{4\pi} \int_{\mathbb{R}^2} dx dy f(x) g(y) \frac{1}{(x-y)^2}. \quad (2.31)$$

Finally, for any $f \in L_c^\infty(\mathbb{R}^2)$ with $\int f = 0$, uniformly on compact subsets of $x \in \mathbb{R}^2$,

$$\langle \varphi(x)\varphi(f) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow -\int_{\mathbb{R}^2} dy \frac{1}{2\pi} \log|x-y| f(y). \quad (2.32)$$

The limits above also exist when $\varepsilon \rightarrow 0$ with $m > 0$ fixed and have the same local uniformity.

Proof. The estimates here are largely routine, so we sketch the main ideas and leave the full details to the reader. Let us consider separately pointwise estimates and smeared estimates.

Pointwise estimates: For (2.26), we note that by definition,

$$\begin{aligned} \langle \varphi(x)^2 \rangle_{\text{GFF}(\varepsilon, m)} &= \int_{\varepsilon^2}^{\infty} dt \frac{e^{-m^2 t}}{4\pi t} = \int_{m^2 \varepsilon^2}^{\infty} dt \frac{e^{-t}}{4\pi t} \\ &= \frac{1}{4\pi} \Gamma(0, m^2 \varepsilon^2) \\ &= -\frac{1}{2\pi} \log(m\varepsilon) - \frac{\gamma}{4\pi} + O(\varepsilon^2 m^2), \end{aligned} \quad (2.33)$$

where Γ is the incomplete Gamma function and we have used its well-known asymptotics. Similarly, for (2.27), we note that as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \langle \varphi(x)\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)} &\rightarrow \int_0^\infty dt \frac{e^{-|x-y|^2/4t}}{4\pi t} e^{-m^2 t} \\ &= \int_0^\infty dt \frac{e^{-(m|x-y|/2)(t+1/t)}}{4\pi t} = \frac{1}{2\pi} K_0(m|x-y|) \\ &= -\frac{1}{2\pi} \log \frac{m|x-y|}{2} - \frac{\gamma}{2\pi} + O(m|x-y|), \end{aligned} \quad (2.34)$$

where K_0 is the modified Bessel function of the second kind and we have used its well-known asymptotics. The proofs of (2.28) and (2.29) are similar, and make use of standard asymptotics of Bessel functions – we omit further details.

Smearred estimates: Consider next (2.30). For $g \in L_c^\infty(\mathbb{R}^2)$ and $u \in \mathbb{R}^2$, we have

$$\begin{aligned} \langle \varphi(u)\nabla\varphi(g) \rangle_{\text{GFF}(\varepsilon, m)} &= \int_{\mathbb{R}^2} dx g(x) \int_{\varepsilon^2}^\infty ds \left(-\frac{x-u}{2s} \right) \frac{e^{-|x-u|^2/4s}}{4\pi s} e^{-m^2 s} \\ &= - \int_{\mathbb{R}^2} dy y e^{-|y|^2/4} \int_{\varepsilon^2}^\infty ds \frac{e^{-m^2 s}}{8\pi\sqrt{s}} g(u + \sqrt{s} y). \end{aligned} \quad (2.35)$$

Thus if $u \in \mathbb{R}^2$ is in some fixed compact set, say a disk of radius r_1 , and we choose $r_2 > 0$ such that $\text{supp}(g) \subset B(0, r_2)$ (where both r_1, r_2 are independent of ε, m), then one readily checks via the triangle inequality that we have $|g(u + \sqrt{s} y)| \leq \|g\|_{L^\infty(\mathbb{R}^2)} \mathbf{1}\{s \leq (r_1 + r_2)^2/|y|^2\}$. Applying this type of bound in the above integral representation, it follows that as $\varepsilon, m \rightarrow 0$, $\langle \varphi(u)\nabla\varphi(g) \rangle_{\text{GFF}(\varepsilon, m)}$ converges uniformly in u in a fixed compact set. On the other hand, this type of estimate can be readily used to justify the use of the dominated convergence theorem, so using (2.28), we see that in fact as $\varepsilon, m \rightarrow 0$,

$$\langle \varphi(u)\nabla\varphi(g) \rangle_{\text{GFF}(\varepsilon, m)} \rightarrow - \int_{\mathbb{R}^2} dx g(x) \frac{1}{2\pi} \frac{x-u}{|x-u|^2}, \quad (2.36)$$

and that this is a locally bounded function of u .

The bound (2.31) follows directly from (2.29), while (2.32) follows from (2.27) through similar estimates as above (and making use of our assumption that $\int f = 0$). This concludes our proof. \blacksquare

Next, we record a basic estimate for the charge correlation functions.

Lemma 2.5. *For any $\beta > 0$ and any distinct $x_1, \dots, x_n, y_1, \dots, y_{n'}$ in \mathbb{R}^2 , where $n + n' \geq 1$, the limits*

$$\begin{aligned} \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} : \right\rangle_{\text{GFF}(m)} \\ = \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} :_\varepsilon \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} :_\varepsilon \right\rangle_{\text{GFF}(\varepsilon, m)}, \end{aligned} \quad (2.37)$$

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} : \right\rangle_{\text{GFF}} \\ &= \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} : \right\rangle_{\text{GFF}(\varepsilon, m)} \end{aligned} \quad (2.38)$$

exist, and

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} : \right\rangle_{\text{GFF}} \\ &= \mathbf{1}_{n=n'} (4e^{-\gamma})^{\beta n/4\pi} \frac{\prod_{i < j} |x_i - x_j|^{\beta/2\pi} |y_i - y_j|^{\beta/2\pi}}{\prod_{i, j} |x_i - y_j|^{\beta/2\pi}} \end{aligned} \quad (2.39)$$

where the empty product $\prod_{i < j}$ is interpreted as 1 if $n = n' = 1$.

Proof. Since φ is Gaussian under $\nu^{\text{GFF}(\varepsilon, m)}$ with covariance $c(x, y) = \langle \varphi(x)\varphi(y) \rangle_{\text{GFF}(\varepsilon, m)}$,

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} : \right\rangle_{\text{GFF}(\varepsilon, m)} \\ &= \varepsilon^{-(n+n')(\beta/4\pi)} e^{-\frac{\beta}{2}[\sum_{i, j=1}^n c(x_i, x_j) + \sum_{i, j=1}^{n'} c(y_i, y_j) - 2\sum_{i=1}^n \sum_{j=1}^{n'} c(x_i, y_j)]} \\ &= (\varepsilon^{-1/2\pi} e^{-c(0,0)})^{\beta(n+n')/2} e^{-\beta[\sum_{i < j} c(x_i, x_j) + \sum_{i < j} c(y_i, y_j) - \sum_{i, j} c(x_i, y_j)]}. \end{aligned} \quad (2.40)$$

By Lemma 2.4, the $\varepsilon \rightarrow 0$ and $m \rightarrow 0$ limits both exist, and the $\varepsilon, m \rightarrow 0$ limit is given by

$$\begin{aligned} & \lim_{m \downarrow 0} m^{(\beta/4\pi)(n-n')^2} e^{(\beta\gamma/4\pi)(n+n')/2} (2\beta/2\pi e^{-\gamma\beta/2\pi})^n \frac{\prod_{i < j} |x_i - x_j|^{\beta/2\pi} |y_i - y_j|^{\beta/2\pi}}{\prod_{i, j} |x_i - y_j|^{\beta/2\pi}} \\ &= \mathbf{1}_{n=n'} (4e^{-\gamma})^{\beta n/4\pi} \frac{\prod_{i < j} |x_i - x_j|^{\beta/2\pi} |y_i - y_j|^{\beta/2\pi}}{\prod_{i, j} |x_i - y_j|^{\beta/2\pi}} \end{aligned} \quad (2.41)$$

as claimed. \blacksquare

By definition, the truncated correlation functions of $:e^{\pm i\sqrt{\beta}\varphi}:$ are determined by (2.39) and (1.6). The next two lemmas give the general truncated correlations also involving factors $\partial\varphi$ or $\bar{\partial}\varphi$.

Lemma 2.6. *Let $\beta > 0$. For $n \geq 1$, $q, q' \geq 0$, $x_1, \dots, x_n, z_1, \dots, z_q, w_1, \dots, w_{q'} \in \mathbb{R}^2$ distinct, and $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$, the limits*

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(m)}^T \\ &= \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T, \end{aligned} \quad (2.42)$$

$$\begin{aligned}
& \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}}^T \\
&= \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} :_{\varepsilon} \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \quad (2.43)
\end{aligned}$$

exist uniformly on compact subsets of $u_i \neq u_j$ for $i \neq j$ (where the u_i are an enumeration of the points $x_k, z_j, w_{j'}$), and we have

$$\begin{aligned}
& \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}}^T \\
&= \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \right\rangle_{\text{GFF}}^T \prod_{j=1}^q \left(i \frac{\sqrt{\beta}}{4\pi} \sum_{k=1}^n \frac{\sigma_k}{x_k - z_j} \right) \prod_{j'=1}^{q'} \left(i \frac{\sqrt{\beta}}{4\pi} \sum_{k=1}^n \frac{\sigma_k}{\bar{x}_k - \bar{w}_{j'}} \right). \quad (2.44)
\end{aligned}$$

Proof. By Lemma A.1, when $\varepsilon, m > 0$, the truncated correlation functions are given by

$$\begin{aligned}
& \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} :_{\varepsilon} \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\
&= \prod_{k=1}^n \frac{\partial}{\partial\mu_k} \Big|_{\mu_k=0} \prod_{j=1}^q \frac{\partial}{\partial v_j} \Big|_{v_j=0} \prod_{j'=1}^{q'} \frac{\partial}{\partial \eta_{j'}} \Big|_{\eta_{j'}=0} \\
& \log \left\langle \exp \left[\sum_{k=1}^n \mu_k :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} :_{\varepsilon} + \sum_{j=1}^q v_j \partial\varphi(z_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial}\varphi(w_{j'}) \right] \right\rangle_{\text{GFF}(\varepsilon, m)}^T. \quad (2.45)
\end{aligned}$$

We would like to use the Girsanov–Cameron–Martin theorem to get rid of the $\partial\varphi$ and $\bar{\partial}\varphi$ terms at the expense of replacing μ_k by something which depends on $v_j, \eta_{j'}, z_j$, and $w_{j'}$ as well. We need to be slightly careful here as $\partial\varphi$ and $\bar{\partial}\varphi$ are complex-valued, and Girsanov’s theorem holds a priori only for real-valued Gaussian random variables.

To justify the use of Girsanov’s theorem in our setting, assume we have some real-valued Gaussian random variables X_1, \dots, X_N and (possibly complex) constants $\gamma_1, \dots, \gamma_N$. Then by a routine combination of the dominated convergence theorem (to justify continuity), Fubini’s theorem, and Morera’s theorem, one finds that

$$(\lambda_1, \dots, \lambda_N) \mapsto \left\langle e^{\sum_{j=1}^N \gamma_j e^{iX_j}} e^{\sum_{j=1}^N \lambda_j X_j} \right\rangle \quad (2.46)$$

is an entire function. Then by an elementary version of Girsanov’s theorem for finite-dimensional Gaussian vectors (which is just completion of the square and change of variables), we find for real λ_i that

$$\left\langle e^{\sum_{j=1}^N \gamma_j e^{iX_j}} e^{\sum_{j=1}^N \lambda_j X_j} \right\rangle = \left\langle e^{\sum_{j=1}^N \gamma_j e^{iX_j} + i \sum_{k=1}^N \lambda_k \langle X_j X_k \rangle} \right\rangle e^{\frac{1}{2} \langle (\sum_{j=1}^N \lambda_j X_j)^2 \rangle}. \quad (2.47)$$

Using a similar argument as before, one checks that this also defines an entire function of the λ_i , so since as these entire functions agree on real values, they must be the same:

$$\langle e^{\sum_{j=1}^N \gamma_j e^{iX_j}} e^{\sum_{j=1}^N \lambda_j X_j} \rangle = \langle e^{\sum_{j=1}^N \gamma_j e^{iX_j + i \sum_{k=1}^N \lambda_k (X_j X_k)} \rangle e^{\frac{1}{2} (\sum_{j=1}^N \lambda_j X_j)^2}, \quad (2.48)$$

also for complex λ_i .

Applying this to our setting (taking X_k to consist of $\varphi(x_k)$ and the real and imaginary parts of $\partial\varphi(z_j)$ and $\bar{\partial}\varphi(w_{j'})$ – the values of γ_i and λ_i are chosen accordingly), we see that the expectation on the right-hand side of (2.45) equals (when the expectation is nonzero – this is true at least for small enough parameter values, and in the end, we evaluate derivatives at zero)

$$\left\langle \exp \left[\sum_{k=1}^n \mu_k : e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} \cdot_{\varepsilon} \times e^{i\sqrt{\beta} \sigma_k [\sum_{j=1}^q v_j \langle \varphi(x_k) \partial\varphi(z_j) \rangle_{\text{GFF}(\varepsilon, m)} + \sum_{j'=1}^{q'} \eta_{j'} \langle \varphi(x_k) \bar{\partial}\varphi(w_{j'}) \rangle_{\text{GFF}(\varepsilon, m)}]} \right] \right\rangle_{\text{GFF}(\varepsilon, m)} \\ \times \exp \left[\frac{1}{2} \left\langle \left(\sum_{j=1}^q v_j \partial\varphi(z_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial}\varphi(w_{j'}) \right)^2 \right\rangle_{\text{GFF}(\varepsilon, m)} \right]. \quad (2.49)$$

The last term does not contribute when we take derivatives with respect to μ_k so we can ignore it. Therefore, using the last identity and rewriting the result in terms of the truncated charge correlations given by (2.45) with $q = q' = 0$ yields

$$\prod_{k=1}^n \frac{\partial}{\partial \mu_k} \Big|_{\mu_k=0} \\ \log \left\langle \exp \left[\sum_{k=1}^n \mu_k : e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} \cdot_{\varepsilon} + \sum_{j=1}^q v_j \partial\varphi(z_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial}\varphi(w_{j'}) \right] \right\rangle_{\text{GFF}(\varepsilon, m)} \\ = \left\langle \prod_{k=1}^n : e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} \cdot_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ \times \prod_{k=1}^n e^{i\sqrt{\beta} \sigma_k [\sum_{j=1}^q v_j \langle \varphi(x_k) \partial\varphi(z_j) \rangle_{\text{GFF}(\varepsilon, m)} + \sum_{j'=1}^{q'} \eta_{j'} \langle \varphi(x_k) \bar{\partial}\varphi(w_{j'}) \rangle_{\text{GFF}(\varepsilon, m)}]}. \quad (2.50)$$

Thus, carrying out the v_j and $\eta_{j'}$ differentiations, we obtain

$$\left\langle \prod_{k=1}^n : e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} \cdot_{\varepsilon} \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ = \left\langle \prod_{k=1}^n : e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} \cdot_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}^T \times \prod_{j=1}^q \left(i\sqrt{\beta} \sum_{k=1}^n \sigma_k \langle \varphi(x_k) \partial\varphi(z_j) \rangle_{\text{GFF}(\varepsilon, m)} \right) \\ \times \prod_{j'=1}^{q'} \left(i\sqrt{\beta} \sum_{k=1}^n \sigma_k \langle \varphi(x_k) \bar{\partial}\varphi(w_{j'}) \rangle_{\text{GFF}(\varepsilon, m)} \right). \quad (2.51)$$

Using the covariance estimate (2.28) (and its complex conjugate version), we obtain (2.44) by taking $\varepsilon \rightarrow 0$ and $m \rightarrow 0$. \blacksquare

Lemma 2.7. For $q + q' \geq 1$ and $z_1, \dots, z_q, w_1, \dots, w_{q'} \in \mathbb{R}^2$ distinct, the limits

$$\left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(m)}^T = \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T, \quad (2.52)$$

$$\left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}}^T = \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \quad (2.53)$$

exist, and

$$\left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}}^T = \begin{cases} -\frac{1}{4\pi} \frac{1}{(z_1 - z_2)^2} & (q = 2, q' = 0), \\ -\frac{1}{4\pi} \frac{1}{(\bar{w}_1 - \bar{w}_2)^2} & (q = 0, q' = 2), \\ 0 & \text{else.} \end{cases} \quad (2.54)$$

Proof. Since $\partial\varphi(z_j)$ and $\bar{\partial}\varphi(w_{j'})$ are Gaussian variables, only the second order cumulants (truncated correlation functions) are nonzero and given by the covariance

$$\left\langle \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T = \begin{cases} \langle \partial\varphi(z_1) \partial\varphi(z_2) \rangle_{\text{GFF}(\varepsilon, m)} & (q = 2, q' = 0), \\ \langle \bar{\partial}\varphi(w_1) \bar{\partial}\varphi(w_2) \rangle_{\text{GFF}(\varepsilon, m)} & (q = 0, q' = 2), \\ \langle \partial\varphi(z_1) \bar{\partial}\varphi(w_1) \rangle_{\text{GFF}(\varepsilon, m)} & (q = 1, q' = 1), \\ 0 & \text{else.} \end{cases} \quad (2.55)$$

Their limits as $\varepsilon \rightarrow 0$ and $m \rightarrow 0$ are given by (2.29) (and its complex conjugate version). \blacksquare

We are ultimately interested in smeared correlation functions, and there is some care to be taken on the diagonal of the pointwise correlation functions. The following result describes what happens with the truncated charge correlation functions.

Lemma 2.8. For $\beta \in (0, 6\pi)$ and $n \neq 2$, the truncated charge correlations are locally integrable: for any $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$ and any compact set $K \subset (\mathbb{R}^2)^n$,

$$\int_K dx_1 \cdots dx_n \left| \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)}: \right\rangle_{\text{GFF}}^T \right| < \infty. \quad (2.56)$$

Moreover, if $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$, then for $n \neq 2$,

$$\begin{aligned} & \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)}:_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ &= \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)}: \right\rangle_{\text{GFF}}^T. \end{aligned} \quad (2.57)$$

If K and the set $\{x_k = x_{k'} \text{ for some } k \neq k'\}$ are disjoint and if the f_k have disjoint supports, the statements also hold for $n = 2$.

The proof of this lemma is not completely straightforward from the direct definition of the truncated charge correlation functions. For example, in [21, Lemma 3], the analogous statement is only shown for $\beta < 4\pi$ (and the need for the statement at $\beta = 4\pi$ is circumvented there by defining the sine-Gordon model with $\beta = 4\pi$ in terms of the limit as $\beta \uparrow 4\pi$). For us, Lemma 2.8 follows immediately as a by-product of our later analysis, and we thus postpone its proof to Section 5.4.

For the gradient fields, we have the following smeared analogue of Lemma 2.7.

Lemma 2.9. *For $q, q' \geq 0$ with $q + q' \geq 1$ and $g_1, \dots, g_q, h_1, \dots, h_{q'} \in C_c^\infty(\mathbb{R}^2)$, the limits*

$$\left\langle \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{GFF}(m)}^T := \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T, \quad (2.58)$$

$$\left\langle \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{GFF}}^T := \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \quad (2.59)$$

exist, and

$$\begin{aligned} & \left\langle \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{GFF}}^T \\ &= \begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \partial g_1(x) \partial g_2(y) \log|x-y|^{-1} & (q=2, q'=0), \\ \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy \bar{\partial} h_1(x) \bar{\partial} h_2(y) \log|x-y|^{-1} & (q=0, q'=2), \\ \frac{1}{4} \int_{\mathbb{R}^2} dx g_1(x) h_1(x) & (q=q'=1), \\ 0 & \text{else.} \end{cases} \quad (2.60) \end{aligned}$$

For $(q, q') = (2, 0), (0, 2)$, the right-hand sides are also equal to the Cauchy principal value integrals

$$\frac{-1}{4\pi} \text{p.v.} \int dx dy \frac{g_1(x)g_2(y)}{(x-y)^2}, \quad \frac{-1}{4\pi} \text{p.v.} \int dx dy \frac{h_1(x)h_2(y)}{(\bar{x}-\bar{y})^2}. \quad (2.61)$$

Proof. The fact that the truncated correlation function vanishes if $q + q' = 1$ or $q + q' \geq 3$ follows from the fact that we are dealing with centered Gaussian random variables.

We thus need to only focus on the $q + q' = 2$ case. The $q = 2, q' = 0$ and $q' = 2, q = 0$ cases follow readily from (2.32) (note that $\int \partial g_i = \int \bar{\partial} h_j = 0$). For the $q = q' = 1$ case, we find again from (2.32) and integrating by parts that

$$\begin{aligned} \langle \partial\varphi(g_1) \bar{\partial}\varphi(h_1) \rangle_{\text{GFF}(\varepsilon, m)} &\rightarrow \frac{1}{2\pi} \int_{\mathbb{R}^2} dx dy \partial g_1(x) \bar{\partial} h_1(y) \log|x-y|^{-1} \\ &= \frac{1}{4\pi} \int_{\mathbb{R}^2} dx dy \partial g_1(x) h_1(y) \frac{1}{\bar{y}-\bar{x}}, \end{aligned} \quad (2.62)$$

from which the claim follows after noting that $\partial_x \frac{1}{\pi(\bar{x}-\bar{y})} = \delta(x-y)$. For smooth test functions, it is well known that (2.61) follows by integration by parts. \blacksquare

With Lemmas 2.8 and 2.9 in hand, we can describe the smeared free field correlation functions in the generality we need.

Lemma 2.10. *Let $\beta \in (0, 6\pi)$, $n = 1$ or $n \geq 3$, and $q, q' \geq 0$. Suppose $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ and $g_1, \dots, g_q, h_1, \dots, h_{q'} \in C_c^\infty(\mathbb{R}^2)$. Then*

$$\begin{aligned} & (\xi_1, \dots, \xi_n, z_1, \dots, z_q, w_1, \dots, w_{q'}) \\ & \mapsto f_1(\xi_1) \cdots f_n(\xi_n) g_1(z_1) \cdots g_q(z_q) h_1(w_1) \cdots h_{q'}(w_{q'}) \\ & \quad \times \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}}^T \\ & \in L^1((\mathbb{R}^2 \times \{-1, 1\})^n \times (\mathbb{R}^2)^{q+q'}), \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} & \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi} : (f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ & = \int \prod_{i=1}^n d\xi_i f_i(\xi_i) \prod_{j=1}^q dz_j g_j(z_j) \prod_{j'=1}^{q'} dw_{j'} h_{j'}(w_{j'}) \\ & \quad \times \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi(x_k)} : \prod_{j=1}^q \partial\varphi(z_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(w_{j'}) \right\rangle_{\text{GFF}}^T. \end{aligned} \quad (2.64)$$

Moreover, if the f_j have disjoint supports or if $q + q' \geq 1$, then the claims hold also for $n = 2$.

Proof. If $n \neq 2$ or the f_k have disjoint supports, the claim follows immediately from Lemmas 2.6 and 2.8. Thus the only slightly delicate case is the claim that the supports of f_1 and f_2 need not be disjoint for $n = 2$ if $q + q' \geq 1$. For this, note first that if $\sigma_1 = \sigma_2$, then the charge correlation function vanishes and there is nothing to prove. For $\sigma_1 \neq \sigma_2$, let us only prove that the limiting quantity is integrable – justifying convergence can be readily deduced with a similar argument. By (2.39), the truncated charge two-point function is proportional to

$$\frac{1}{|x_1 - x_2|^{\beta/2\pi}}, \quad (2.65)$$

and, by Lemma 2.6, the correlation function in the claim is thus proportional to

$$\begin{aligned} & \frac{1}{|x_1 - x_2|^{\beta/2\pi}} \prod_{j=1}^q \left(i \frac{\sqrt{\beta}}{4\pi} \left(\frac{1}{x_1 - z_j} - \frac{1}{x_2 - z_j} \right) \right) \\ & \quad \times \prod_{j'=1}^{q'} \left(i \frac{\sqrt{\beta}}{4\pi} \left(\frac{1}{\bar{x}_1 - \bar{w}_{j'}} - \frac{1}{\bar{x}_2 - \bar{w}_{j'}} \right) \right). \end{aligned} \quad (2.66)$$

It thus suffices to show that

$$\begin{aligned} & \frac{1}{|x_1 - x_2|^{\beta/2\pi}} \prod_{j=1}^q \left| \frac{1}{x_1 - z_j} - \frac{1}{x_2 - z_j} \right| \prod_{j'=1}^{q'} \left| \frac{1}{x_1 - w_{j'}} - \frac{1}{x_2 - w_{j'}} \right| \\ & \leq |x_1 - x_2|^{-\beta/2\pi + q + q'} \prod_{j=1}^q \frac{1}{|z_j - x_1| |z_j - x_2|} \prod_{j'=1}^{q'} \frac{1}{|w_{j'} - x_1| |w_{j'} - x_2|} \end{aligned} \quad (2.67)$$

is locally integrable. One readily checks that since we are integrating over given compact sets, each z_j -integral gives a bound of the form $1 + |\log |x_1 - x_2||$ and analogously for the $w_{j'}$ -integrals. Thus it suffices to check the local integrability of

$$(1 + |\log |x_1 - x_2||)^{q+q'} |x_1 - x_2|^{-\beta/2\pi + q + q'}. \quad (2.68)$$

As we are in two dimensions, this certainly holds for $\beta < 6\pi$ when $q + q' \geq 1$. \blacksquare

2.3. Bosonization in the massless case

That the Coleman correspondence (1.12) holds in the noninteracting case $z = \mu = 0$ follows by matching the above computations of the correlation functions of massless free fermions and of the massless Gaussian free field, together with the following well-known identity for Cauchy–Vandermonde matrices:

$$\det \left(\frac{1}{x_i - y_j} \right)_{i,j=1}^n = \frac{\prod_{1 \leq i < i' \leq n} (x_i - x_{i'}) \prod_{1 \leq j < j' \leq n} (y_j - y_{j'})}{\prod_{1 \leq i \leq n} \prod_{1 \leq j \leq n} (x_i - y_j)}. \quad (2.69)$$

This allows us to prove the Coleman correspondence in the case $\mu = z = 0$.

Corollary 2.11. *Let $\beta = 4\pi$, $z = \mu = 0$. For $n, n', q, q' \geq 0$ with $n + n' + q + q' = 1$ or $n + n' + q + q' \geq 3$, $f_1^+, \dots, f_n^+, f_1^-, \dots, f_{n'}^- \in L_c^\infty(\mathbb{R}^2)$, and $g_1^+, \dots, g_q^+, g_1^-, \dots, g_{q'}^- \in C_c^\infty(\mathbb{R}^2)$, the identity (1.12) holds:*

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{+i\sqrt{4\pi}\varphi}:(f_k^+) \prod_{k'=1}^{n'} :e^{-i\sqrt{4\pi}\varphi}:(f_{k'}^-) \prod_{j=1}^q (-i\partial\varphi(g_j^+)) \prod_{j'=1}^{q'} (-i\bar{\partial}\varphi(g_{j'}^-)) \right\rangle_{\text{GFF}}^T \\ & = A^{n+n'} B^{q+q'} \left\langle \prod_{k=1}^n \bar{\psi}_1 \psi_1(f_k^+) \prod_{k'=1}^{n'} \bar{\psi}_2 \psi_2(f_{k'}^-) \prod_{j=1}^q \bar{\psi}_2 \psi_1(g_j^+) \prod_{j'=1}^{q'} \bar{\psi}_1 \psi_2(g_{j'}^-) \right\rangle_{\text{FF}(0)}^T, \end{aligned} \quad (2.70)$$

where A and B are as in Theorem 1.1.

Moreover, if $n + n' + q + q' = 2$, we have the following statements: (i) for $n + n' = 2$, $q + q' = 0$, the claim holds if f_i^\pm have disjoint supports, (ii) if $n + n' = 1$ and $q + q' = 1$, the claim holds in the same generality as for $n + n' + q + q' \geq 3$ (both sides vanish), and (iii) if $n + n' = 0$ and $q + q' = 2$, the claim holds either if g_j^\pm have disjoint supports, or if the right hand side is understood as that given by Lemma 2.9.

Proof. Let $q = q' = 0$. Then applying (2.39) with $\beta = 4\pi$, the determinant identity (2.69), and finally (2.7), we find that for any distinct points,

$$\begin{aligned}
& \left\langle \prod_{k=1}^n :e^{+i\sqrt{\beta}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{\beta}\varphi(y_{k'})} : \right\rangle_{\text{GFF}} \\
&= \mathbf{1}_{n=n'} (4e^{-\gamma})^{\beta n/4\pi} \frac{\prod_{i<j} |x_i - x_j|^{\beta/2\pi} |y_i - y_j|^{\beta/2\pi}}{\prod_{i,j} |x_i - y_j|^{\beta/2\pi}} \\
&= \mathbf{1}_{n=n'} (4e^{-\gamma})^n \frac{\prod_{i<j} |x_i - x_j|^2 |y_i - y_j|^2}{\prod_{i,j} |x_i - y_j|^2} \\
&= (4\pi e^{-\gamma/2})^{n+n'} \left\langle \prod_{k=1}^n \bar{\psi}_1 \psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2 \psi_2(y_{k'}) \right\rangle_{\text{FF}(0)}. \quad (2.71)
\end{aligned}$$

Using this, if $q + q' > 0$ then (2.10)–(2.11) and (2.21) for the fermionic side respectively (2.44) for the bosonic side imply that, for any distinct points,

$$\begin{aligned}
& \left\langle \prod_{k=1}^n :e^{+i\sqrt{4\pi}\varphi(x_k)} : \prod_{k'=1}^{n'} :e^{-i\sqrt{4\pi}\varphi(y_{k'})} : \prod_{j=1}^q (-i\partial\varphi(z_j)) \prod_{j'=1}^{q'} (-i\bar{\partial}\varphi(w_{j'})) \right\rangle_{\text{GFF}}^T \\
&= (4\pi e^{-\gamma/2})^{n+n'} \sqrt{\pi}^{q+q'} \\
&\quad \times \left\langle \prod_{k=1}^n \bar{\psi}_1 \psi_1(x_k) \prod_{k'=1}^{n'} \bar{\psi}_2 \psi_2(y_{k'}) \prod_{j=1}^q \bar{\psi}_2 \psi_1(z_j) \prod_{j'=1}^{q'} \bar{\psi}_1 \psi_2(w_{j'}) \right\rangle_{\text{FF}(0)}^T. \quad (2.72)
\end{aligned}$$

The claim (along with the relevant restrictions for the $n + n' + q + q' = 2$ case) now follows from Lemma 2.9 (possibly using integration by parts) and Lemma 2.10. \blacksquare

3. Estimates for the sine-Gordon model and free fermions; proof of main theorems

In this section, we record our main estimates for sine-Gordon correlation functions as well as those for free fermions with a finite volume mass term. The proofs of these estimates are presented in the remainder of the paper. Assuming these estimates, we then give our proofs of the theorems of Section 1. The intuition for Theorems 1.1–1.2 is as outlined in Section 1.3. Namely, in view of the Coleman correspondence when $z = \mu = 0$, i.e., Corollary 2.11, the sine-Gordon measure which is formally obtained from the GFF measure by weighting it by

$$e^{2\mu \int dx : \cos \sqrt{4\pi}\varphi(x) :} \mathbf{1}_{\Lambda_L}(x) \quad (3.1)$$

should correspond to the massless free fermion “Grassmann measure” weighted by

$$e^{Az \int dx (\bar{\psi}_1 \psi_1(x) + \bar{\psi}_2 \psi_2(x))} \mathbf{1}_{\Lambda_L}(x). \quad (3.2)$$

Our estimates stated in this section provide the required analyticity and convergence to make this correspondence rigorously. Our main innovation here is that our estimates hold for all z in a complex neighborhood of the entire real axis (not just a neighborhood of the origin) and for all $L > 0$, and that we control the infinite volume limit as $L \rightarrow \infty$.

The main analyticity results for the sine-Gordon model stated in this section do not cause additional difficulties for general $\beta \in (0, 6\pi)$, so we state them in this generality. Together with well-known correlation inequalities they then imply the remaining results stated in Section 1.

3.1. The sine-Gordon model and estimates for its correlation functions

To state our estimates for the sine-Gordon model, we begin with the precise definition of our regularization of the continuum, finite volume, massless sine-Gordon model.

For $\varepsilon, m > 0$, we define the probability measure $\nu^{\text{GFF}(\varepsilon, m)}$ of the regularized GFF as in Section 2.2 and recall that $\nu^{\text{GFF}(\varepsilon, m)}$ is supported on $C^\infty(\mathbb{R}^2)$. We then take as a regularization of the sine-Gordon model the probability measure

$$\nu^{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}(d\varphi) = \frac{1}{Z(\beta, z|\varepsilon, m, \Lambda)} \exp\left[2z \int_{\Lambda} dx \varepsilon^{-\beta/4\pi} \cos(\sqrt{\beta} \varphi)\right] \nu^{\text{GFF}(\varepsilon, m)}(d\varphi), \quad (3.3)$$

where $\Lambda \subset \mathbb{R}^2$ is a compact set, $\beta \in (0, 6\pi)$, $z \in \mathbb{R}$, and Z is the partition function – a normalization constant. We will also write $\langle \cdot \rangle_{\text{GFF}(\varepsilon, m)}$ for integration with respect to $\nu^{\text{GFF}(\varepsilon, m)}$ and $\langle \cdot \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}$ for integration with respect to $\nu^{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}$. For $\Lambda = \Lambda_L = \{x \in \mathbb{R}^2 : |x| \leq L\}$ we of course recover our definition of $\langle \cdot \rangle_{\text{SG}(\beta, z|\varepsilon, m, L)}$ in (1.1), but we allow more general Λ here because this allows us to obtain the Euclidean invariance of the infinite volume limits in Theorems 1.6 and 1.7.

Let us comment briefly on some of the restrictions we have imposed here. As mentioned earlier, the continuum sine-Gordon model is interesting for $\beta \in (0, 8\pi)$. While we are mainly interested in proving the Coleman correspondence for $\beta = 4\pi$, the sine-Gordon estimates we prove hold for all $\beta \in (0, 6\pi)$, so we present the results in this generality. The regime $\beta \in [6\pi, 8\pi)$ is also interesting, but would require finer estimates. For $\beta \in (0, 4\pi)$, the sine-Gordon measure is absolutely continuous with respect to the GFF when Λ is compact. The free fermion point, $\beta = 4\pi$, is precisely where this fails.

We now state our main result about the sine-Gordon correlation functions that are important for the Coleman correspondence.

Theorem 3.1. *Let $\beta \in (0, 6\pi)$, $z \in \mathbb{R}$, $n, q, q' \geq 0$, and let $\Lambda \subset \mathbb{R}^2$ be compact. Suppose $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2)$, $g_1, \dots, g_q, h_1, \dots, h_{q'} \in C_c^\infty(\mathbb{R}^2)$ and $\sigma_1, \dots, \sigma_n \in \{-1, 1\}$.*

- (i) *If either $(n, q + q') \neq (1, 0)$ and $(n, q + q') \neq (2, 0)$ or if f_1, f_2 have disjoint supports, the limit*

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi}:(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z|\Lambda)}^T \\ & := \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi}:\varepsilon(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}^T \end{aligned} \quad (3.4)$$

exists and is finite.

(ii) Under the assumptions of item (i), the function

$$z \mapsto \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi}:(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z|\Lambda)}^T \quad (3.5)$$

has an analytic continuation into a Λ -dependent neighborhood of the real axis. Moreover, it is even in z when $n = 0$.

(iii) Under the assumptions of item (i), for any $l \geq 0$,

$$\begin{aligned} & \frac{d^l}{dz^l} \Big|_{z=0} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi}:(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z|\Lambda)}^T \\ &= \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi}:(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) (:e^{i\sqrt{\beta}\varphi}(\mathbf{1}_\Lambda): + :e^{-i\sqrt{\beta}\varphi}(\mathbf{1}_\Lambda):)^l \right\rangle_{\text{GFF}}^T. \end{aligned} \quad (3.6)$$

(iv) For any $f \in L_c^\infty(\mathbb{R}^2)$ with support in Λ we have, for $\beta \in (4\pi, 6\pi)$,

$$\begin{aligned} & \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} [\varepsilon^{\frac{\beta}{2\pi}-2} \langle :e^{\pm i\sqrt{\beta}\varphi} \cdot_\varepsilon(f) \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}] \\ &= 2\pi z e^{-\frac{\gamma\beta}{4\pi}} \int_{\Lambda} dx f(x) \int_0^\infty dr r^{-\frac{\beta}{2\pi}+1} e^{-\frac{\beta}{4\pi}\Gamma(0, r^2)}, \end{aligned} \quad (3.7)$$

where Γ is the incomplete gamma function, and for $\beta = 4\pi$,

$$\lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \left[\frac{1}{\log \varepsilon^{-1}} \langle :e^{\pm i\sqrt{\beta}\varphi} \cdot_\varepsilon(f) \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} \right] = 2\pi z e^{-\gamma} \int_{\mathbb{R}^2} dx f(x). \quad (3.8)$$

By essentially the same proof, we also obtain the existence of the φ field.

Theorem 3.2. Let $\beta \in (0, 6\pi)$, $z \in \mathbb{R}$, $m \in (0, \infty)$, and let $\Lambda \subset \mathbb{R}^2$ be compact. Then for any $f \in C_c^\infty(\mathbb{R}^2)$ and $w \in \mathbb{C}$, the limit

$$\langle e^{w\varphi(f)} \rangle_{\text{SG}(\beta, z|m, \Lambda)} = \lim_{\varepsilon \rightarrow 0} \langle e^{w\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} \quad (3.9)$$

exists and is entire in w . If also $\int f = 0$, then the limit

$$\langle e^{w\varphi(f)} \rangle_{\text{SG}(\beta, z|\Lambda)} = \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle e^{w\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} \quad (3.10)$$

also exists and is an even function of z and an entire function of w .

Before turning to fermions, we comment here on a few facts the reader might want to keep in mind concerning these theorems.

First of all, we recall from (1.6) and the discussion following it that the product notation in the truncated correlation functions $\langle \prod_{i=1}^n X_i \rangle^T$ means $\langle X_1; \dots; X_n \rangle^T$, and that correspondingly, in item (iii), terms involving powers should be interpreted as

$$\left\langle \left(\prod_{i=1}^n X_i \right) Y^l \right\rangle^T = \langle X_1; \dots; X_n; Y; \dots; Y \rangle^T, \quad (3.11)$$

with l copies of Y .

Next we mention that by Lemma 2.10 and our assumptions on n, q, q' the derivatives in item (iii) are indeed finite as they should be.

Finally, we mention that in the literature, there certainly exist some results that are similar to parts of this theorem – see in particular [7, 21, 23, 47]. What we believe is truly new, and critical to our proof of the Coleman correspondence, is that we are able to treat all values of $z \in \mathbb{R}$ and prove analyticity in a neighborhood of the real axis – not just in a neighborhood of the origin.

We now turn to describing what we need to know about free massive fermions with a finite volume mass term.

3.2. Free fermion estimates

As discussed at the beginning of Section 3, we will establish the equivalence of the sine-Gordon measure with finite volume interaction with that of Dirac fermions with a finite volume mass term. We will here choose $\Lambda = \Lambda_L$ to be a disk of radius $L > 0$ centered at the origin. We again take the pragmatic approach of defining the free fermion model with a finite volume mass term, formally represented by the fermionic path integral with weight (3.2), directly through its correlation functions. Namely, given a corresponding propagator $S_{\mu\mathbf{1}_{\Lambda_L}}$ constructed in Theorem 3.3 below, the correlation functions are defined by formulas like (1.9) and (1.10) but now with $S_{\mu\mathbf{1}_{\Lambda_L}}$ instead of S . In particular, given $n \geq 3$, $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2)$ and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n \in \{1, 2\}$, the smeared truncated correlation functions are defined by

$$\begin{aligned} & \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T \\ & := (-1)^{n+1} \sum_{\pi} \int_{(\mathbb{R}^2)^n} \prod_{i=1}^n f_i(x_i) \prod_{i=1}^n S_{\mu\mathbf{1}_{\Lambda_L}; \alpha_{\pi^i(1)} \beta_{\pi^{i+1}(1)}}(x_{\pi^i(1)}, x_{\pi^{i+1}(1)}), \end{aligned} \quad (3.12)$$

where we sum over cyclic permutations π – as we see in our proof of Theorem 3.3, this is finite for $n \geq 3$. For $n = 2$, the same definition applies to f_1 and f_2 with disjoint compact supports. For $n = 2$ and f_1 and f_2 with overlapping supports the above integral is no longer necessarily finite and we will instead consider the two-point function with the singularity subtracted, i.e.,

$$\begin{aligned} & \int dx_1 dx_2 f_1(x_1) f_2(x_2) \\ & \times \left(-S_{\mu\mathbf{1}_{\Lambda_L}; \alpha_1 \beta_2}(x_1, x_2) S_{\mu\mathbf{1}_{\Lambda_L}; \alpha_2 \beta_1}(x_2, x_1) + S_{0; \alpha_1 \beta_2}(x_1, x_2) S_{0; \alpha_2 \beta_1}(x_2, x_1) \right), \end{aligned} \quad (3.13)$$

with S_0 given by the right-hand side of (2.1). This is formally equal to

$$\langle \bar{\psi}_{\alpha_1} \psi_{\beta_1}(f_1) \bar{\psi}_{\alpha_2} \psi_{\beta_2}(f_2) \rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda})}^T - \langle \bar{\psi}_{\alpha_1} \psi_{\beta_1}(f_1) \bar{\psi}_{\alpha_2} \psi_{\beta_2}(f_2) \rangle_{\text{FF}(0)}^T. \quad (3.14)$$

Therefore the existence of the propagator $S_{\mu\mathbf{1}_{\Lambda_L}}$ and some of its basic properties are our main result concerning such models – this is summarized in the following theorem. Here recall our definition of the Dirac operator \not{D} from (1.7).

Theorem 3.3. *For each $\mu \in \mathbb{R}$ and $L > 0$, the Dirac operator with finite volume mass term, $i\not{D} + \mu\mathbf{1}_{\Lambda_L}$, where $\Lambda_L = \{x \in \mathbb{R}^2 : |x| \leq L\}$, has a fundamental solution $S_{\mu\mathbf{1}_{\Lambda_L}}(x, y)$, $x \neq y$, with values in $\mathbb{C}^{2 \times 2}$, that is,*

$$(i\not{D}_x + \mu\mathbf{1}_{\Lambda_L}(x))S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) = \delta(x - y) \quad \text{and} \quad \lim_{x \rightarrow \infty} S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) = 0, \quad (3.15)$$

such that given $n \geq 3$, $f_1, \dots, f_n \in L_c^\infty(\Lambda_L)$ and $\alpha_1, \beta_1, \dots, \alpha_n, \beta_n \in \{1, 2\}$, the smeared truncated correlation functions (3.12) satisfy the following properties:

(i) *The function*

$$\mu \mapsto \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T \quad (3.16)$$

has an analytic continuation into an L -dependent neighborhood of the real axis. In particular, the smeared truncated correlation function is finite. (For $\mu = 0$, $S_{\mu\mathbf{1}_{\Lambda}} = S_0$.)

(ii) *For $l \geq 1$,*

$$\begin{aligned} \frac{d^l}{d\mu^l} \Big|_{\mu=0} \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T \\ = \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) (\bar{\psi}_1 \psi_1(\mathbf{1}_{\Lambda_L}) + \bar{\psi}_2 \psi_2(\mathbf{1}_{\Lambda_L}))^l \right\rangle_{\text{FF}(0)}^T. \end{aligned} \quad (3.17)$$

(iii) *For any $\mu \in \mathbb{R}$, as $L \rightarrow \infty$,*

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T \rightarrow \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu)}^T. \quad (3.18)$$

On the right-hand side, the correlation functions with index $\text{FF}(\mu)$ are defined by the propagator (1.8) of Dirac fermions with infinite volume mass term μ .

For $n = 2$, the same statements remain true if f_1 and f_2 have disjoint compact supports, or if the truncated two-point function is replaced by (3.13) in (i) and on the left-hand sides of (ii) and (iii), and analogously on the right-hand side of (iii).

We again comment on some issues regarding this theorem.

First of all, one could readily formulate a nonsmeared version of this result as well, but for the proof of Theorem 1.1, the smeared versions of the correlation functions are the relevant ones.

Secondly, in item (ii), the correct way to understand the term on the right-hand side is that one expands the power, uses multilinearity, and (1.9). Moreover, the fact that the right-hand side is finite follows from the last statement in Corollary 2.11.

Finally, we mention that given that this theorem is essentially about controlling a finite volume approximation to massive free fermions, we expect that at least parts of this result are well known to some experts. Unfortunately, we have been unable to find a suitable

reference for the results we need. Our proof makes use of the convenient domain of a disk, but we expect that the result also holds for much more general domains.

3.3. Proofs of Theorems 1.1 and 1.2

Assuming Theorems 3.1 and 3.3, we are now in a position to prove the Coleman correspondence at $\beta = 4\pi$.

Proof of Theorem 1.1. It suffices to show that the correlation functions of the massless sine-Gordon model with an interaction term supported in Λ_L at $\beta = 4\pi$ and those of free Dirac fermions with a mass term supported in Λ_L agree for all $z \in \mathbb{R}$ (and corresponding $\mu = Az$) and all $L < \infty$. Indeed, by Theorem 3.3 (iii), the smeared truncated correlation functions of free Dirac fermions with a Λ_L mass term converge as $L \rightarrow \infty$ to their infinite volume versions, and hence the identification in finite volume implies that the sine-Gordon correlation functions converge to the same limit.

For the equivalence in finite volume, let us write O_k^B for one of the quantities $:e^{\pm i\sqrt{4\pi}\varphi}:$, $\partial\varphi$, or $\bar{\partial}\varphi$ on the sine-Gordon side, and write O_k^F for the corresponding one on the fermionic side – the correspondence being the one given by the statement of Theorem 1.1. Thus $:e^{i\sqrt{4\pi}\varphi}:$ corresponds to $A\bar{\psi}_1\psi_1$, $:e^{-i\sqrt{4\pi}\varphi}:$ corresponds to $A\bar{\psi}_2\psi_2$, $\partial\varphi$ corresponds to $iB\bar{\psi}_2\psi_1$, and $\bar{\partial}\varphi$ corresponds to $iB\bar{\psi}_1\psi_2$. We also let f_j be compactly supported and either essentially bounded or smooth (depending on whether it is a charge or gradient observable that is acting on it) that O_k^B and O_k^F act on.

Let us first focus on the case where $n + n' + q + q' \geq 3$ and let us not assume that the supports of the test functions are disjoint. To see that the truncated correlation functions agree for all $z \in \mathbb{R}$ when $L < \infty$, we use that both are analytic in z respectively μ in a complex neighborhood of the real axis, by Theorems 3.1 (ii) and 3.3 (i). By unique analytic continuation, it therefore suffices to verify that they agree in a complex neighborhood of $z = \mu = 0$. This in turn holds if the truncated correlation functions agree at $z = 0$ and all z -derivatives at $z = 0$ agree. That the correlation functions agree at $z = 0$ is Corollary 2.11. On the fermionic side, the μ -derivatives at $\mu = 0$ are given by Theorem 3.3 (ii) as

$$\begin{aligned} \frac{d^l}{dz^l} \left\langle \prod_{k=1}^{n+n'+q+q'} O_k^F(f_k) \right\rangle_{\text{FF}(Az\mathbf{1}_{\Lambda_L})}^T \Big|_{z=0} \\ = A^l \left\langle \prod_{k=1}^{n+n'+q+q'} O_k^F(f_k) (\bar{\psi}_1\psi_1(\mathbf{1}_{\Lambda_L}) + \bar{\psi}_2\psi_2(\mathbf{1}_{\Lambda_L})) \right\rangle_{\text{FF}(0)}^l. \end{aligned} \quad (3.19)$$

On the sine-Gordon side, the z -derivatives at $z = 0$ are given by Theorem 3.1 (iii) as

$$\begin{aligned} \frac{d^l}{dz^l} \left\langle \prod_{k=1}^{n+n'+q+q'} O_k^B(f_k) \right\rangle_{\text{SG}(\beta, z|\Lambda_L)}^T \Big|_{z=0} \\ = \left\langle \prod_{k=1}^{n+n'+q+q'} O_k^B(f_k) (:e^{i\sqrt{\beta}\varphi}(\mathbf{1}_{\Lambda_L}): + :e^{-i\sqrt{\beta}\varphi}(\mathbf{1}_{\Lambda_L}):) \right\rangle_{\text{GFF}}^l. \end{aligned} \quad (3.20)$$

That these are equal when $\beta = 4\pi$ again follows from Corollary 2.11.

The same argument is valid for $n + n' = q + q' = 1$. Moreover, if we further assume that the f_k have disjoint supports, then the same argument works also for general $n + n' + q + q' = 2$. ■

The remaining $(n + n', q + q') = (0, 2)$ case with overlapping test functions, i.e., Theorem 1.2, works similarly, as follows.

Proof of Theorem 1.2. By Theorem 3.1, the relevant finite volume $\varepsilon, m \rightarrow 0$ limits exist on the sine-Gordon side. Moreover, by Lemma 2.9, it suffices to show that

$$\begin{aligned} & \langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{SG}(4\pi, z)} - \langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{GFF}} \\ &= -\frac{B^2}{\pi^2} \int dx_1 dx_2 f_1(x_1)f_2(x_2) \left((\partial_{x_1} K_0(A|z||x_1 - x_2|))^2 - \frac{1}{4(x_1 - x_2)^2} \right), \end{aligned} \quad (3.21)$$

$$\begin{aligned} & \langle \partial\varphi(f_1)\bar{\partial}\varphi(f_2) \rangle_{\text{SG}(4\pi, z)} - \langle \partial\varphi(f_1)\bar{\partial}\varphi(f_2) \rangle_{\text{GFF}} \\ &= -\frac{B^2 A^2 z^2}{4\pi^2} \int dx_1 dx_2 f_1(x_1)f_2(x_2) (K_0(A|z||x_1 - x_2|))^2. \end{aligned} \quad (3.22)$$

The claim then follows from the result about the GFF two-point function from Lemma 2.9.

The proof of (3.21) and (3.22) is analogous to that of Theorem 1.1, as follows. To be concrete, we focus on the proof of (3.21); the other one is analogous. By Theorem 3.3 (iii), it suffices to show that

$$\begin{aligned} & \langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{SG}(4\pi, z|\Lambda_L)} - \langle \partial\varphi(f_1)\partial\varphi(f_2) \rangle_{\text{GFF}} = B^2 \int dx_1 dx_2 f_1(x_1)f_2(x_2) \\ & \quad \times (-S_{Az\mathbf{1}_\Lambda; 2\mathbf{1}}(x_1, x_2)S_{Az\mathbf{1}_\Lambda; 2\mathbf{1}}(x_2, x_1) + S_{0; 2\mathbf{1}}(x_1, x_2)S_{0; 2\mathbf{1}}(x_1, x_2)). \end{aligned} \quad (3.23)$$

For $z = 0$, this claim is trivial as both sides vanish then. Theorem 3.1 and the special $n = 2$ case of Theorem 3.3 now again imply that both sides are analytic in z and that their derivatives are identical, using Corollary 2.11. ■

3.4. Proofs of Theorems 1.6 and 1.7

For the proofs of the results stated in Section 1.2, we need the following correlation inequalities from [32].

First note that the $\varphi \mapsto -\varphi$ symmetry of the measure implies $\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} = \langle \cos(\varphi(f)) \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}$. For $z > 0$, it then follows from [32, Corollary 3.2] that, as a function of $m > 0$ and $z > 0$ and the set Λ ,

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} \text{ is increasing and } \langle \varphi(g)^2 \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} \text{ is decreasing.} \quad (3.24)$$

Indeed, by rescaling φ by $\sqrt{\beta}$, in the notation of [32, Section 3], one has

$$\langle F(\varphi/\sqrt{\beta}) \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} = \langle F(\varphi) \rangle_{C, \rho}, \quad (3.25)$$

where

$$C = \beta \int_{\varepsilon^2}^{\infty} dt e^{t\Delta - tm^2}, \quad \rho(dx) = 2z\varepsilon^{-\beta/4\pi} \mathbf{1}_\Lambda(x) dx, \quad (3.26)$$

and [32, Corollary 3.2] states that if $\rho_1 \leq \rho_2$ and $C_2 \leq C_1$ then

$$\langle \cos(\varphi(g)) \rangle_{C_2, \rho_2} \geq \langle \cos(\varphi(g)) \rangle_{C_1, \rho_1}, \quad \langle \varphi(g)^2 \rangle_{C_2, \rho_2} \leq \langle \varphi(g)^2 \rangle_{C_1, \rho_1}. \quad (3.27)$$

The monotonicity (3.24) is immediate from this.

As a particular case of (3.24) we get the following infrared bound: for any $f \in C_c^\infty(\mathbb{R}^2)$, $m, z > 0$ and Λ , we have

$$\langle \varphi(f)^2 \rangle_{SG(\beta, z | \varepsilon, m, L)} \leq \langle \varphi(f)^2 \rangle_{GFF(\varepsilon, m)}. \quad (3.28)$$

Proof of Theorems 1.6 and 1.7. Since the proofs of Theorems 1.6 and 1.7 are essentially identical, we focus on the first theorem and leave the modifications for the second theorem to the reader.

By Theorem 3.2, for any $f \in C_c^\infty(\mathbb{R}^2)$ with $\int f = 0$, the limit

$$\langle e^{i\varphi(f)} \rangle_{SG(\beta, z | \Lambda)} := \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle e^{i\varphi(f)} \rangle_{SG(\beta, z | \varepsilon, m, \Lambda)} \quad (3.29)$$

exists and is invariant under $z \mapsto -z$. Thus without loss of generality we can and will assume $z > 0$. By (3.24), it follows that $\langle e^{i\varphi(f)} \rangle_{SG(\beta, z | \Lambda)}$ is monotone in Λ , and thus converges as $\Lambda \uparrow \mathbb{R}^2$ to a limit which we denote by $\langle e^{i\varphi(f)} \rangle_{SG(\beta, z)}$.

The limit is trivially bounded above by 1 and the map $f \mapsto \langle e^{i\varphi(f)} \rangle_{SG(\beta, z)}$ satisfies the following continuity estimate: for any $g \in C_c^\infty(\mathbb{R}^2)$ with $\int g = 0$,

$$|\langle e^{i\varphi(f+g)} \rangle_{SG(\beta, z)} - \langle e^{i\varphi(f)} \rangle_{SG(\beta, z)}| \leq \frac{1}{2} \langle \varphi(g)^2 \rangle_{SG(\beta, z)} \leq \frac{1}{2} \langle g, (-\Delta)^{-1} g \rangle. \quad (3.30)$$

Indeed, for $\varepsilon, m > 0$ and Λ finite, the analogue of the first inequality is immediate, and the second inequality follows from (3.24). The claimed inequality then follows by taking the limits in ε, m, Λ .

In particular, if functions $g_k \in C_c^\infty(\mathbb{R}^2)$ with $\int g_k = 0$ converge to 0 in the topology of $\mathcal{S}(\mathbb{R}^2)$, the right-hand side of (3.30) converges to 0. Since $C_c^\infty(\mathbb{R}^2)$ is dense in $\mathcal{S}(\mathbb{R}^2)$ (and likewise for the subspaces of functions which integrate to 0), it follows that $\langle e^{i\varphi(f)} \rangle_{SG(\beta, z)}$ extends to a continuous functional on $\mathcal{S}'(\mathbb{R}^2)/\text{constants}$ (the topological dual space of the closed subspace of integral-0 functions in $\mathcal{S}(\mathbb{R}^2)$). Minlos's theorem then implies that $\langle e^{i\varphi(f)} \rangle_{SG(\beta, z)}$ is the characteristic functional of a probability measure on $\mathcal{S}'(\mathbb{R}^2)/\text{constants}$.

That the limit is Euclidean invariant is a standard argument that follows from the Euclidean invariance of the GFF and the monotonicity of $\langle e^{i\varphi(f)} \rangle_{SG(\beta, z | \Lambda)}$ in Λ for any increasing family of sets; see, e.g., [57, Section VIII.6].

Finally, the bounds (1.42)–(1.45) are immediate from the monotonicity of (3.24). ■

3.5. Proof of Theorem 1.3 and Corollary 1.8

Proof of Theorem 1.3. The main step of the proof will be to show that (1.28) holds for functions with integral 0, i.e., for all $f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$ with $\int f_i = 0$,

$$\langle \varphi(f_1)\varphi(f_2) \rangle_{SG(4\pi, z)} = \lim_{L \rightarrow \infty} \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \langle \varphi(f_1)\varphi(f_2) \rangle_{SG(4\pi, z | \varepsilon, m, L)}. \quad (3.31)$$

To this end, let us first further assume that there are $g_i, h_i \in C_c^\infty(\mathbb{R}^2)$ such that

$$f_i = \partial g_i + \bar{\partial} h_i. \quad (3.32)$$

In this case, we note that, by integrating by parts,

$$\begin{aligned} \langle \varphi(f_1)\varphi(f_2) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)} &= \langle \partial\varphi(g_1)\partial\varphi(g_2) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)} \\ &\quad + \langle \bar{\partial}\varphi(h_1)\bar{\partial}\varphi(h_2) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)} \\ &\quad + \langle \partial\varphi(g_1)\bar{\partial}\varphi(h_2) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)} \\ &\quad + \langle \bar{\partial}\varphi(h_1)\partial\varphi(g_2) \rangle_{\text{SG}(4\pi, z|\varepsilon, m, L)}, \end{aligned} \quad (3.33)$$

and we find that, by Theorem 1.2, the $\varepsilon, m \rightarrow 0, L \rightarrow \infty$ limits exist and

$$\begin{aligned} \langle \varphi(f_1)\varphi(f_2) \rangle_{\text{SG}(4\pi, z)} &= \langle \partial\varphi(g_1)\partial\varphi(g_2) \rangle_{\text{SG}(4\pi, z)} \\ &\quad + \langle \bar{\partial}\varphi(h_1)\bar{\partial}\varphi(h_2) \rangle_{\text{SG}(4\pi, z)} \\ &\quad + \langle \partial\varphi(g_1)\bar{\partial}\varphi(h_2) \rangle_{\text{SG}(4\pi, z)} \\ &\quad + \langle \bar{\partial}\varphi(h_1)\partial\varphi(g_2) \rangle_{\text{SG}(4\pi, z)}. \end{aligned} \quad (3.34)$$

To express the right-hand side as in (1.28)–(1.29), let us first look at the g_1, g_2 term – the remaining terms are similar. Recalling that $\frac{1}{2\pi}K_0(A|z||x-y|)$ is the covariance of the massive free field, we have the following Fourier space representation of K_0 :

$$K_0(A|z||x-y|) = \int_{\mathbb{R}^2} \frac{dp}{2\pi} \frac{e^{-ip \cdot (x-y)}}{|p|^2 + A^2|z|^2}, \quad (3.35)$$

where the integral is understood either in principal value sense or in the sense of distributions. Thus with the convention $\hat{f}(p) = \int_{\mathbb{R}^2} f(x)e^{-ip \cdot x} dp$ for the Fourier transform and $2\hat{\partial} = i\bar{p}$, Theorem 1.2 and a routine calculation show that (with integrals understood in principal value sense)

$$\begin{aligned} &\langle \partial\varphi(g_1)\partial\varphi(g_2) \rangle_{\text{SG}(4\pi, z)} \\ &= \frac{1}{16\pi^3} \int_{\mathbb{R}^2 \times \mathbb{R}^2} dp_1 dp_2 \hat{g}_1(p_1 + p_2)\hat{g}_2(-p_1 - p_2) \frac{\bar{p}_1 \bar{p}_2}{(|p_1|^2 + A^2|z|^2)(|p_2|^2 + A^2|z|^2)} \\ &= \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \hat{g}_1(p)\hat{g}_2(-p) \int_{\mathbb{R}^2} \frac{dq}{4\pi} \frac{\bar{q}(\bar{p} - \bar{q})}{(|q|^2 + A^2|z|^2)(|p - q|^2 + A^2|z|^2)} \\ &= \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \widehat{\partial g_1}(p)\widehat{\partial g_2}(-p) \frac{1}{\pi \bar{p}^2} \int_{\mathbb{R}^2} dq \frac{\bar{q}(\bar{p} - \bar{q})}{(|q|^2 + A^2|z|^2)(|p - q|^2 + A^2|z|^2)} \\ &=: \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \widehat{\partial g_1}(p)\widehat{\partial g_2}(-p)\hat{C}_{A|z|}(p). \end{aligned} \quad (3.36)$$

The \hat{C} on the right-hand side can be computed as follows. Going into polar coordinates, scaling the radial variable, and translating the angular variable shows that

$$\hat{C}_{A|z|}(p) = \frac{2}{|p|^2} \int_0^\infty dr \frac{r}{r^2 + \mu_p^2} \int_0^{2\pi} \frac{dt}{2\pi} \frac{e^{-it} r(1 - re^{-it})}{1 + r^2 - 2r \cos t + \mu_p^2}. \quad (3.37)$$

where $\mu_p = A|z|/|p|$. To evaluate the t -integral through the residue theorem, we note that out of the two poles for $\eta = e^{-it}$,

$$\eta = \frac{1 + \mu_p^2 + r^2 \pm \sqrt{(1 + \mu_p^2 + r^2)^2 - 4r^2}}{2r}, \quad (3.38)$$

only the minus-one is inside the unit disk, and we thus find

$$\begin{aligned} \hat{C}_{A|z|}(p) &= \frac{2}{|p|^2} \int_0^\infty dr \frac{r}{r^2 + \mu_p^2} \oint_{|\eta|=1} \frac{d\eta}{2\pi i \eta} \frac{\eta r(1-r\eta)}{1 + r^2 - r(\eta + \eta^{-1}) + \mu_p^2} \\ &= \frac{2}{|p|^2} \int_0^\infty dr \frac{r}{r^2 + \mu_p^2} \frac{-\mu_p^2 + r^2 - (\mu_p^2 + r^2)^2 + (\mu_p^2 + r^2)\sqrt{(1 + \mu_p^2 + r^2)^2 - 4r^2}}{2\sqrt{(1 + \mu_p^2 + r^2)^2 - 4r^2}}. \end{aligned} \quad (3.39)$$

A straightforward (but slightly tedious) calculation shows that the last integrand can be written as

$$\begin{aligned} &\frac{1}{4} \partial_r \left(r^2 - \sqrt{(1 + \mu_p^2)^2 + 2(\mu_p^2 - 1)r^2 + r^4} - \frac{2\mu_p^2 \log(r^2 + \mu_p^2)}{\sqrt{1 + 4\mu_p^2}} \right. \\ &\quad \left. + \frac{2\mu_p^2 \log(1 + 3\mu_p^2 - r^2 + \sqrt{1 + 4\mu_p^2} \sqrt{(1 + \mu_p^2)^2 + 2(\mu_p^2 - 1)r^2 + r^4})}{\sqrt{1 + 4\mu_p^2}} \right) \\ &= \frac{r}{r^2 + \mu_p^2} \frac{-\mu_p^2 + r^2 - (\mu_p^2 + r^2)^2 + (\mu_p^2 + r^2)\sqrt{(1 + \mu_p^2 + r^2)^2 - 4r^2}}{2\sqrt{(1 + \mu_p^2 + r^2)^2 - 4r^2}}, \end{aligned} \quad (3.40)$$

from which we see after another slightly tedious calculation that $\hat{C}_{A|z|}(p)$ equals

$$\frac{1}{|p|^2} \left(1 + \frac{\mu_p^2}{\sqrt{1 + 4\mu_p^2}} [\log \mu_p^2 + \log(\sqrt{1 + 4\mu_p^2} - 1) - \log(1 + 3\mu_p^2 + (1 + \mu_p^2)\sqrt{1 + 4\mu_p^2})] \right). \quad (3.41)$$

Finally, an elementary calculation shows that

$$\frac{x^3 + 3x + (x^2 + 1)\sqrt{x^2 + 4}}{\sqrt{x^2 + 4} - x} = \left(\frac{x}{2} + \sqrt{\frac{x^2}{4} + 1} \right)^4, \quad (3.42)$$

from which we can deduce (1.29) with another routine calculation.

We see in particular from this that $\hat{C}_{A|z|}$ is bounded for $|z| > 0$. A similar calculation shows that

$$\langle \partial\varphi(g_1) \bar{\partial}\varphi(h_2) \rangle_{\text{SG}(4\pi, z)} = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \widehat{\partial g_1}(p) \widehat{\bar{\partial} g_2}(-p) \hat{C}_{A|z|}(p), \quad (3.43)$$

with the same $\hat{C}_{A|z|}$. Thus taking complex conjugates of these identities, we find that for our f_i given by $f_i = \partial g_i + \bar{\partial} h_i$,

$$\langle \varphi(f_1)\varphi(f_2) \rangle_{SG(4\pi, z)} = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \hat{f}_1(p) \hat{f}_2(-p) \hat{C}_{A|z|}(p), \quad (3.44)$$

which is precisely the claim for the f_i which can be represented this way.

Finally, to extend the statement to arbitrary $f_i \in C_c^\infty(\mathbb{R}^2)$ or $f_i \in \mathcal{S}(\mathbb{R}^2)$ with $\int f_i = 0$, we note that such f_i can be written as in (3.32) but with g_i and h_i in $\mathcal{S}(\mathbb{R}^2)$, by Taylor expanding \hat{f}_i . Thus it remains to extend our argument to Schwartz functions. For this, given $f_i \in \mathcal{S}(\mathbb{R}^2)$ satisfying $\int f_i = 0$, let $g_i, h_i \in \mathcal{S}(\mathbb{R}^2)$ be such that we have the representation (3.32). Let us take $\chi \in C_c^\infty(\mathbb{R}^2)$ nonnegative, bounded by 1, supported in $\Lambda_{2R} = \{x \in \mathbb{R}^2 : |x| < 2R\}$, equal to 1 in Λ_R , and with gradient bounded as a function of R . Then write $g_i = \chi g_i + (1 - \chi)g_i$ and similarly for h_i . We then have

$$\langle \varphi(f_1)\varphi(f_2) \rangle_{SG(4\pi, z|\varepsilon, m, L)} = \Sigma_1(R|\varepsilon, m, L) + \Sigma_2(R|\varepsilon, m, L), \quad (3.45)$$

where in Σ_1 , we have kept only the $\chi g_i, \chi h_i$ terms, while in Σ_2 we have at least one $(1 - \chi)g_i$ or $(1 - \chi)h_i$ term.

Using the initial part of this proof and a routine dominated convergence argument, we see that when we let $\varepsilon \rightarrow 0, m \rightarrow 0, L \rightarrow \infty$, and finally $R \rightarrow \infty$, Σ_1 converges to our target – namely (3.44) (which is perfectly well defined for $f_i \in \mathcal{S}(\mathbb{R}^2)$). Thus we need to show that Σ_2 tends to zero in the same limit. For this, using (3.28) and routine Cauchy–Schwarz arguments shows that (in the $\varepsilon \rightarrow 0, m \rightarrow 0, L \rightarrow \infty$ limit) we end up estimating e.g. quantities of the form

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy |\nabla((1 - \chi(x))g_1(x))| |\nabla((1 - \chi(y))g_1(y))| |\log|x - y||. \quad (3.46)$$

By dominated convergence, this tends to zero as $R \rightarrow \infty$, and one finds that Σ_2 tends to zero in our limit. This shows that (3.44) is true also for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^2)$ satisfying $\int f_i = 0$.

The localization bound (1.30) now follows easily by observing that $\int (f_x - f_y) = 0$ so by (1.28) for integral-0 test functions (for which we have now established (1.28)), the left-hand side of (1.30) is given by

$$\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} |\hat{f}(p)|^2 (2 - 2\cos(p \cdot x)) \hat{C}_{Az}(p). \quad (3.47)$$

This is uniformly bounded since $\hat{C}_\mu(p)$ is bounded for $\mu \neq 0$.

Finally, we construct the required probability measure $\langle \cdot \rangle_{SG(4\pi, z)}$ on $\mathcal{S}'(\mathbb{R}^2)$. In Theorem 1.6, we have already constructed such a measure on $\mathcal{S}'(\mathbb{R}^2)/\text{constants}$, i.e., for test functions $f \in \mathcal{S}(\mathbb{R}^2)$ with $\int f = 0$. Using the uniform bound on \hat{C}_μ for $\mu \neq 0$ we can extend this measure to all test functions in $\mathcal{S}(\mathbb{R}^2)$ as follows. Let $\gamma_N(x) = (2\pi N)^{-1} e^{-|x|^2/2N}$ be the density of the two-dimensional Gaussian probability measure of variance N and Fourier transform $\hat{\gamma}_N(p) = e^{-\frac{1}{2}N|p|^2}$. For any $f \in \mathcal{S}(\mathbb{R}^2)$, the function $f - \hat{f}(0)\gamma_N \in \mathcal{S}(\mathbb{R}^2)$ then has integral 0, and

$$\langle e^{i\varphi(f - \hat{f}(0)\gamma_N)} \rangle_{SG(4\pi, z)} \quad (3.48)$$

is well defined by Theorem 1.6. For $\mu \neq 0$, we will show that it is a Cauchy sequence in N , as a consequence of the boundedness of \hat{C}_μ . Indeed,

$$\begin{aligned} & |\langle e^{i\varphi(f-\hat{f}(0)\gamma_N)} \rangle_{\text{SG}(4\pi,z)} - \langle e^{i\varphi(f-\hat{f}(0)\gamma_M)} \rangle_{\text{SG}(4\pi,z)}| \\ & \leq \frac{|\hat{f}(0)|^2}{2} \langle \varphi(\gamma_N - \gamma_M)^2 \rangle_{\text{SG}(4\pi,z)} \\ & = \frac{|\hat{f}(0)|^2}{2} \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} |e^{-\frac{1}{2}N|p|^2} - e^{-\frac{1}{2}M|p|^2}|^2 \hat{C}_{A|z|}(p) \rightarrow 0 \end{aligned} \quad (3.49)$$

as $N, M \rightarrow \infty$. For $f \in \mathcal{S}(\mathbb{R}^2)$, we may thus define

$$\langle e^{i\varphi(f)} \rangle_{\text{SG}(4\pi,z)} = \lim_{N \rightarrow \infty} \langle e^{i\varphi(f-\hat{f}(0)\gamma_N)} \rangle_{\text{SG}(4\pi,z)}. \quad (3.50)$$

That this is indeed the characteristic functional of a probability measure on $\mathcal{S}'(\mathbb{R}^2)$ again follows from Minlos's theorem and the continuity of $f \mapsto \langle e^{i\varphi(f)} \rangle_{\text{SG}(4\pi,z)}$, which follows from the boundedness of \hat{C}_μ by an argument analogous to the above Cauchy sequence argument. This argument also shows that the covariance is given by

$$\langle \varphi(f_1)\varphi(f_2) \rangle_{\text{SG}(4\pi,z)} = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \hat{f}_1(p)\hat{f}_2(-p)\hat{C}_{A|z|}(p). \quad (3.51)$$

The exponential decay (1.31) now follows (see e.g. [54, Theorem IX.14]) from the fact that $\hat{C}_\mu(p)$ is uniformly bounded and that, as one readily checks from (1.29), it has an analytic continuation into a strip $|\text{Im}(p_0)|, |\text{Im}(p_1)| < \eta$ for some $\eta > 0$ (proportional to $|\mu|$). ■

For the proof of Corollary 1.8, we need the following observation from [32] adapted to our setting.

Lemma 3.4. *Let $\beta \in (0, 6\pi)$ and $m, z > 0$. Then for any $f \in C_c^\infty(\mathbb{R}^2)$, with $f_x(y) = f(y-x)$,*

$$\langle \varphi(f)\varphi(f_x) \rangle_{\text{SG}(\beta,z|m)} \rightarrow 0 \quad (|x| \rightarrow \infty). \quad (3.52)$$

Proof. The argument is as in the proof of [32, Theorem 4.4]. Indeed, by Theorem 1.7, the measure $\langle \cdot \rangle_{\text{SG}(\beta,z|m)}$ is translation invariant and satisfies, for any $f \in C_c^\infty(\mathbb{R}^2)$,

$$\langle \varphi(f)^2 \rangle_{\text{SG}(\beta,z|m)} \leq \langle f, (-\Delta + m^2)^{-1} f \rangle = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} \frac{|\hat{f}(p)|^2}{|p|^2 + m^2}. \quad (3.53)$$

Therefore $C_f(x) = \langle \varphi(f)\varphi(f_x) \rangle_{\text{SG}(\beta,z|m)}$ satisfies

$$0 \leq \hat{C}_f(p) \leq \frac{|\hat{f}(p)|^2}{|p|^2 + m^2} \in L^1(\mathbb{R}^2) \quad (3.54)$$

in the distributional sense. Indeed, this follows from

$$\langle (\varphi * f)(g)^2 \rangle_{\text{SG}(\beta,z)} = \int_{\mathbb{R}^2 \times \mathbb{R}^2} dx dy g(x)C_f(x-y)g(y) = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} |\hat{g}(p)|^2 \hat{C}_f(p). \quad (3.55)$$

Thus the Riemann–Lebesgue lemma implies that

$$\langle \varphi(f)\varphi(f_x) \rangle = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} e^{ip \cdot x} \hat{C}_f(p) \rightarrow 0 \quad (|x| \rightarrow \infty) \quad (3.56)$$

as claimed. \blacksquare

Proof of Corollary 1.8. By (3.52), for $m > 0$,

$$\langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z|m)} = \frac{1}{2} \lim_{|x| \rightarrow \infty} \langle (\varphi(f) - \varphi(f_x))^2 \rangle_{\text{SG}(\beta, z|m)}. \quad (3.57)$$

By monotonicity in m and L due to (3.24), the $m \rightarrow 0$ and $L \rightarrow \infty$ limits exist in both orders, and if $\int f = 0$,

$$\sup_{m>0} \lim_{L \rightarrow \infty} \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z|m, \Lambda_L)} \leq \lim_{L \rightarrow \infty} \sup_{m>0} \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z|m, \Lambda_L)} = \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z)}. \quad (3.58)$$

In conclusion, we get

$$\begin{aligned} \sup_{m>0} \langle \varphi(f)^2 \rangle_{\text{SG}(\beta, z|m)} &= \frac{1}{2} \sup_{m>0} \lim_{|x| \rightarrow \infty} \langle (\varphi(f) - \varphi(f_x))^2 \rangle_{\text{SG}(\beta, z|m)} \\ &\leq \frac{1}{2} \limsup_{|x| \rightarrow \infty} \sup_{m>0} \langle (\varphi(f) - \varphi(f_x))^2 \rangle_{\text{SG}(\beta, z|m)} \\ &\leq \frac{1}{2} \limsup_{|x| \rightarrow \infty} \langle (\varphi(f) - \varphi(f_x))^2 \rangle_{\text{SG}(\beta, z)}. \end{aligned} \quad (3.59)$$

For $\beta = 4\pi$, the right-hand side is finite by Theorem 1.3. In fact, since $|\hat{f}|^2 \hat{C}_\mu$ is integrable for $\mu \neq 0$, by the Riemann–Lebesgue lemma it is equal to

$$\limsup_{x \rightarrow \infty} \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} |\hat{f}(p)|^2 (1 - \cos(p \cdot x)) \hat{C}_{Az}(p) = \int_{\mathbb{R}^2} \frac{dp}{(2\pi)^2} |\hat{f}(p)|^2 \hat{C}_{Az}(p) \quad (3.60)$$

as claimed.

The proof of the existence of the infinite volume measure as $m \rightarrow 0$ is now exactly as in the proof of Theorems 1.6 and 1.7, only using the now proved bound (1.46) instead of the last bound in (3.30) for the continuity of the characteristic functional. \blacksquare

4. The sine-Gordon model: the renormalized potential

One of our main tools in the proof of Theorem 3.1 is estimates for a renormalized version of the sine-Gordon potential, and we turn to studying it now. For $\varphi \in C_b(\mathbb{R}^2)$ and $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}, \mathbb{C})$, we define

$$v_0(\zeta, \varphi|\varepsilon) = \varepsilon^{-\frac{\beta}{4\pi}} \int_{\mathbb{R}^2 \times \{-1, 1\}} d\xi \zeta(\xi) e^{i\sqrt{\beta} \sigma \varphi(x)}, \quad (4.1)$$

which we refer to as the *microscopic* (sine-Gordon) *potential*. In terms of this microscopic potential, we introduce the following generalized partition function that can be seen as a

generating function for charge correlation functions:

$$Z(\zeta|\varepsilon, m) = \langle e^{-v_0(\zeta, \varphi|\varepsilon)} \rangle_{\text{GFF}(\varepsilon, m)}, \quad (4.2)$$

where the GFF expectation is over φ . For $\zeta = -z\mathbf{1}_\Lambda$ this would just be the partition function of the (regularized) sine-Gordon model.

Our analysis of the generating function $Z(\zeta|\varepsilon, m)$ relies on a convenient decomposition of the regularized free field GFF(ε, m). More precisely, we define, for any $t, m > 0$ and $x, y \in \mathbb{R}^2$ with $x \neq y$,

$$c_t^{m^2}(x-y) := \int_0^t ds c_s^{m^2}(x-y) := \int_0^t ds e^{-m^2 s} \frac{e^{-\frac{|x-y|^2}{4s}}}{4\pi s}. \quad (4.3)$$

For any $t > \varepsilon^2$, note that $c_t^{m^2} - c_{\varepsilon^2}^{m^2}$ and $c_\infty^{m^2} - c_t^{m^2}$ are covariances, so the fact that the sum of two independent Gaussian processes is a Gaussian process whose covariance is the sum of the covariances of the two processes implies that we can in fact write (4.2) as

$$Z(\zeta|\varepsilon, m) = \langle e^{-v_t(\zeta, \varphi|\varepsilon, m)} \rangle_{\text{GFF}(\sqrt{t}, m)} \quad (4.4)$$

where we have defined the renormalized potential v_t by

$$e^{-v_t(\zeta, \varphi|\varepsilon, m)} = \mathbf{E}_{c_t^{m^2} - c_{\varepsilon^2}^{m^2}}(e^{-v_0(\zeta, \varphi + \eta|\varepsilon)}), \quad (4.5)$$

and have written $\mathbf{E}_{c_t^{m^2} - c_{\varepsilon^2}^{m^2}}$ for the expectation with respect to the law of the Gaussian process with covariance $c_t^{m^2} - c_{\varepsilon^2}^{m^2}$ and the last integral is over η .

The analysis of the $\varepsilon, m \rightarrow 0$ behavior of the generating function $Z(\zeta|\varepsilon, m)$ can thus be rephrased in terms of $\varepsilon, m \rightarrow 0$ asymptotics of the renormalized potential $v_t(\zeta, \cdot|\varepsilon, m)$. Note that as ζ is complex, only $e^{-v_t(\zeta, \varphi|\varepsilon, m)}$ is a priori well defined, but we will see in this section that for any given $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ and t small enough, its logarithm $v_t(\zeta, \varphi|\varepsilon, m)$ is also well defined. Moreover, the goal of this section is to prove bounds for $v_t(\zeta, \varphi|\varepsilon, m)$ that are uniform in $\varepsilon > 0$ and $m > 0$. Our analysis follows the approach of [13] as presented in [3, Section 3], but it permits space-dependent coupling constants and we also work directly in the continuum. As discussed in Section 1.4, we expect that similar results could be obtained by using the methods of [5, 7]. The $\varepsilon \rightarrow 0$ and $m \rightarrow 0$ limits will be studied in Section 5.

To control v_t we will show in this section that the following expansion is convergent and agrees with $v_t(\varphi, \zeta|\varepsilon, m)$ for $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ and suitable t :

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m) e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j)} \quad (4.6)$$

where the coefficients \tilde{v}_t^n are determined recursively as follows. For $t > \varepsilon^2$ and for $\xi \in \mathbb{R}^2 \times \{-1, 1\}$, we set

$$\tilde{v}_t^1(\xi|\varepsilon, m) = e^{-\frac{\beta}{2} \int_{\varepsilon^2}^t ds c_s^{m^2}(0) + \frac{1}{4\pi} \log \varepsilon^2}, \quad (4.7)$$

and for $n \geq 2$ and $\xi_j = (x_j, \sigma_j) \in \mathbb{R}^2 \times \{-1, 1\}$,

$$\begin{aligned} & \tilde{v}_t^n(\xi_1, \dots, \xi_n | \varepsilon, m) \\ &= \frac{1}{2} \int_{\varepsilon^2}^t ds \sum_{I_1 \dot{\cup} I_2 = [n]} \sum_{i \in I_1, j \in I_2} \dot{u}_s^{m^2}(\xi_i, \xi_j) \tilde{v}_s^{|I_1|}(\xi_{I_1} | \varepsilon, m) \tilde{v}_s^{|I_2|}(\xi_{I_2} | \varepsilon, m) \\ & \quad \times e^{-(w_t^{m^2}(\xi_1, \dots, \xi_n) - w_s^{m^2}(\xi_1, \dots, \xi_n))}, \end{aligned} \quad (4.8)$$

where

$$\dot{u}_s^{m^2}(\xi_1, \xi_2) = \beta \sigma_1 \sigma_2 \dot{c}_s^{m^2}(x_1 - x_2), \quad (4.9)$$

$$w_t^{m^2}(\xi_1, \dots, \xi_n) - w_s^{m^2}(\xi_1, \dots, \xi_n) = \frac{1}{2} \sum_{i,j=1}^n \int_s^t dr \dot{u}_r^{m^2}(\xi_i, \xi_j). \quad (4.10)$$

We have also written $[n] = \{1, \dots, n\}$ and $I_1 \dot{\cup} I_2 = [n]$ to indicate that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = [n]$. To control the expansion (4.6), we introduce the following norms for $f : (\mathbb{R}^2 \times \{-1, 1\})^n \rightarrow \mathbb{C}$:

$$\|f\|_n = \begin{cases} \sup_{\xi \in \mathbb{R}^2 \times \{-1, 1\}} |f(\xi)| & \text{if } n = 1, \\ \sup_{\xi_1 \in \mathbb{R}^2 \times \{-1, 1\}} \int_{(\mathbb{R}^2 \times \{-1, 1\})^{n-1}} d\xi_2 \cdots d\xi_n |f(\xi_1, \dots, \xi_n)| & \text{if } n \geq 2. \end{cases} \quad (4.11)$$

The goal of the rest of this section is to prove the following proposition. In its statement, the condition $\beta < 6\pi$ necessitates the exclusion of the $n = 2$ term as the analogous estimate fails when $\beta \geq 4\pi$; see also Remark 4.2 below. The $n = 2$ term will be considered explicitly later.

Proposition 4.1. *For $\beta \in (0, 6\pi)$, $t > 0$, and $n \neq 2$, there exist functions $h_t^n : (\mathbb{R}^2 \times \{-1, 1\})^n \rightarrow [0, \infty]$ which are independent of ε, m and for $0 < \varepsilon^2 < t < m^{-2}$, one has*

$$|\tilde{v}_t^n(\xi_1, \dots, \xi_n | \varepsilon, m)| \leq h_t^n(\xi_1, \dots, \xi_n) \quad (4.12)$$

for all $\xi_1, \dots, \xi_n \in (\mathbb{R}^2 \times \{-1, 1\})^n$ and

$$\|h_t^n\|_n \leq n^{n-2} t^{-1} (C_\beta t^{1-\frac{\beta}{8\pi}})^n \quad (4.13)$$

for some constant C_β depending only on β .

Remark 4.2. It remains a conjecture [5, p. 672] that similar estimates remain valid for all $\beta < 8\pi$ when not only the $n = 2$ term is excluded but when the first n_0 terms are excluded where n_0 is the largest integer such that $2(n_0 - 1) - \beta n_0 / 4\pi \leq 0$. (The results of [23, 52] which do construct the (massive) sine-Gordon model for all $\beta < 8\pi$ do not proceed by this expansion and instead rely on probabilistic estimates on large gradients, thus leaving this stronger conjecture open.)

Proposition 4.1 allows us to identify the expansion (4.6) with the renormalized potential as follows.

Corollary 4.3. For all $0 < \varepsilon^2 < t < m^{-2} < \infty$, $\varphi \in C^\infty(\mathbb{R}^2)$, and $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}, \mathbb{C})$ satisfying

$$\sup_{\xi \in \mathbb{R}^2 \times \{-1, 1\}} |\zeta(\xi)| < \frac{1}{e C_\beta t^{1-\beta/8\pi}} \quad (4.14)$$

where C_β is the constant from Proposition 4.1, the sums and integrals in the expansion (4.6) converge absolutely and equal $v_t(\zeta, \varphi|\varepsilon, m)$ defined in (4.5):

$$v_t(\zeta, \varphi|\varepsilon, m) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \\ \times \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m) e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j)}. \quad (4.15)$$

Proof. Throughout the proof, we fix $\varepsilon, m > 0$ and $\zeta \in L^\infty(\mathbb{R}^2 \times \{-1, 1\})$ with support in a compact set $\Lambda \times \{-1, 1\} \subset \mathbb{R}^2 \times \{-1, 1\}$, and we will always assume that $t \in (\varepsilon^2, t_0)$ where t_0 is the supremum over $t > \varepsilon^2$ such that (4.14) holds. Then, for $n \geq 3$,

$$\frac{1}{n!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n |\zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m) e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j)}| \\ \leq \frac{1}{n!} 2|\Lambda| \|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{\pm 1\})}^n \|h_t^n\|_n \leq 2|\Lambda| n^{-2} t^{-1} (t/t_0)^{(1-\beta/8\pi)n} \quad (4.16)$$

where we have used $n^n/n! \leq e^n$. The $n = 1, 2$ terms are trivially bounded with ε, m -dependent constants when $t > \varepsilon^2$, uniformly in $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$, by the definitions (4.7)–(4.8). For $t < t_0$, it follows that the sum over n in (4.6) converges geometrically, again uniformly in $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$. We denote this sum by $a_t(\varphi)$ and note that $a_t(\varphi)$ only depends on $\varphi|_\Lambda$, so that we can consider $\varphi \mapsto a_t(\varphi)$ as a function $a_t : C(\Lambda) \rightarrow \mathbb{R}$. We will denote the supremum norm on $C(\Lambda)$ by $\|\cdot\|$ below. From the geometric convergence,

$$\left| e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j f(x_j)} - 1 - i\sqrt{\beta} \sum_{j=1}^n \sigma_j f(x_j) \right| \leq \frac{1}{2} \beta n^2 \|f\|^2, \quad (4.17)$$

and similar estimates for higher derivatives, we then see that $a_t : C(\Lambda) \rightarrow \mathbb{R}$ is actually smooth, i.e., Fréchet differentiable to any order, for $t \in (\varepsilon^2, t_0)$. Its first two derivatives are given by

$$Da_t(\varphi; f_1) = i\sqrt{\beta} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \\ \times \sum_{k=1}^n \sigma_k f_1(x_k) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m) e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j)} \\ =: \int dx_1 f_1(x_1) \nabla a_t(\varphi, x_1), \quad (4.18)$$

$$D^2 a_t(\varphi; f_1, f_2) = -\beta \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \\ \times \sum_{k=1}^n \sigma_k f_1(x_k) \sum_{l=1}^n \sigma_l f_2(x_l) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m) e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j)} \\ =: \int dx_1 dx_2 f_1(x_1) f_2(x_2) \text{Hess } a_t(\varphi, x_1, x_2), \quad (4.19)$$

where $f_1, f_2 \in C(\Lambda)$. As in (4.16), $\|\nabla a_t(\varphi, \cdot)\|_{L^\infty(\mathbb{R}^2)}$ and $\|\text{Hess } a_t(\varphi, \cdot, \cdot)\|_{L^1 L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)}$ are bounded independently of φ , and since ζ has support in Λ , it is also clear that $\nabla a_t(\varphi, \cdot)$ has support in Λ and that $\text{Hess } a_t(\varphi, \cdot, \cdot)$ has support in Λ^2 . Defining

$$\Delta_{\dot{c}_t} a_t(\varphi) = \int_{\Lambda^2} dx_1 dx_2 \dot{c}_t^{m^2}(x_1 - x_2) \text{Hess } a_t(\varphi, x_1, x_2), \quad (4.20)$$

$$(\nabla a_t(\varphi), \dot{c}_t^{m^2} \nabla a_t(\varphi)) = \int_{\Lambda^2} dx_1 dx_2 \dot{c}_t^{m^2}(x_1 - x_2) \nabla a_t(\varphi, x_1) \nabla a_t(\varphi, x_2), \quad (4.21)$$

it then follows from (4.8) that, for $t \in (\varepsilon^2, t_0)$,

$$\frac{\partial}{\partial t} a_t(\varphi) = \frac{1}{2} \Delta_{\dot{c}_t} a_t(\varphi) - \frac{1}{2} (\nabla a_t(\varphi), \dot{c}_t^{m^2} \nabla a_t(\varphi)). \quad (4.22)$$

Let $h_t(\varphi) = e^{-a_t(\varphi)}$. Then by the chain rule, h_t is also twice Fréchet differentiable with (using similar notation)

$$\nabla h_t(\varphi, x_1) = -\nabla a_t(\varphi, x_1) e^{-a_t(\varphi)}, \quad (4.23)$$

$$\text{Hess } h_t(\varphi, x_1, x_2) = [-\text{Hess } a_t(\varphi, x_1, x_2) + \nabla a_t(\varphi, x_1) \nabla a_t(\varphi, x_2)] e^{-a_t(\varphi)}, \quad (4.24)$$

and hence

$$\begin{aligned} & \frac{1}{2} \Delta_{\dot{c}_t} h_t(\varphi) - \frac{\partial}{\partial t} h_t(\varphi) \\ &= \left[-\frac{1}{2} \Delta_{\dot{c}_t} a_t(\varphi) + \frac{1}{2} (\nabla a_t(\varphi), \dot{c}_t^{m^2} \nabla a_t(\varphi)) + \frac{\partial}{\partial t} a_t(\varphi) \right] e^{-a_t(\varphi)} = 0. \end{aligned} \quad (4.25)$$

We will show that e^{-v_t} satisfies the same heat equation (with the same initial data at $t = \varepsilon^2$) and argue that the solution must be unique, so $v_t = a_t$, which will then yield the proof.

The Laplacian $\Delta_{\dot{c}_t} h_t$ can alternatively be expressed as follows. Since Λ is bounded, assume that $\Lambda \subset [-L, L]^2$. Let χ be a smooth function with $\chi(t) = 1$ for $t \leq 4L$ and $\chi(t) = 0$ for $t \geq 8L$. We then choose a torus Λ' of period $16L$ and define $\dot{c}'_t(x) = \sum_{n \in \mathbb{Z}^2} \dot{c}_t^{m^2}(x + 16Ln) \chi(|x + 16Ln|)$. Thus \dot{c}'_t is a smooth (periodic) function on Λ' , and we note that $\dot{c}'_t(x) = \dot{c}_t^{m^2}(x)$ if $|x| \leq 4L$. Thus if we regard Λ as a subset of Λ' (by embedding it into a fundamental domain centered at 0 in the obvious way), we have $\dot{c}'_t(x - y) = \dot{c}_t^{m^2}(x - y)$ for $x, y \in \Lambda$. In particular, there are $\dot{\lambda}_{t,k} \geq 0$ decaying rapidly in k for each $t > 0$ such that

$$\dot{c}_t^{m^2}(x - y) = \dot{c}'_t(x - y) = \sum_k \dot{\lambda}_{k,t} f_k(x) f_k(y) \quad \text{for } x, y \in \Lambda, \quad (4.26)$$

where (f_k) is the real orthonormal Fourier basis of $L^2(\Lambda')$ consisting of sin and cos functions, so in particular satisfying $\|f_k\| \leq C$. For a general function $g \in C_b^2(C(\Lambda))$ and $t > 0$ we can now define

$$\Delta_{\dot{c}_t} g(\varphi) = \sum_k \dot{\lambda}_{k,t} D^2 g(\varphi; f_k, f_k). \quad (4.27)$$

By Fubini (whose application is justified by rapid convergence of all sums and integrals), this definition is consistent with (4.20).

Let $\Pi_N \varphi$ be the $L^2(\Lambda')$ projection of $\varphi|_{\Lambda'}$ to Fourier modes $k \leq N$. For any N , the above implies that $h_t^N(\varphi) = h_t(\Pi_N \varphi)$ satisfies the finite-dimensional heat equation

$$\partial_t h_t^N(\varphi) = \frac{1}{2} \sum_{k \leq N} \dot{\lambda}_{k,t} D^2 h_t^N(\varphi; f_k, f_k), \quad h_{\varepsilon^2}^N(\varphi) = h_{\varepsilon^2}(\Pi_N \varphi). \quad (4.28)$$

Next we will verify that $g_t(\varphi) = e^{-v_t(\varphi)}$ defined in (4.5) also satisfies the heat equation $\partial_t g_t = \frac{1}{2} \Delta_{\dot{c}_t} g_t$ with the same initial condition $g_{\varepsilon^2} = h_{\varepsilon^2}$. To see this, first observe that the definition of $g_t(\varphi)$ in (4.5) only depends on $\eta|_{\Lambda}$. The Gaussian field $\eta|_{\Lambda}$ has covariance $(c_t^{m^2} - c_{\varepsilon^2}^{m^2})|_{\Lambda \times \Lambda}$ and can be realized in terms of independent standard Gaussian random variables $(X_k)_{k \in \mathbb{N}}$ as

$$\eta|_{\Lambda} = \sum_k \sqrt{\lambda_{k,t}} X_k f_k|_{\Lambda} \quad (4.29)$$

where $\dot{\lambda}_{k,t}$ above is the t -derivative of these $\lambda_{k,t}$. (This follows from the fact that $\lambda_{k,t} = (f_k, c'_t f_k)$ and the differentiability of c'_t in t .) From this representation we again see that $g_t^N(\varphi) = g_t(\Pi_N \varphi)$ satisfies

$$\partial_t g_t^N(\varphi) = \frac{1}{2} \sum_{k \leq N} \dot{\lambda}_{k,t} D^2 g_t^N(\varphi; f_k, f_k), \quad g_{\varepsilon^2}^N(\varphi) = g_{\varepsilon^2}(\Pi_N \varphi) = h_{\varepsilon^2}(\Pi_N \varphi). \quad (4.30)$$

By the standard uniqueness of bounded solutions to such equations (finite-dimensional heat equations), we conclude that $h_t^N(\varphi) = g_t^N(\varphi)$ for all $t \in (\varepsilon^2, t_0)$ and $N \in \mathbb{N}$. It remains to deduce that this implies that $g_t(\varphi) = h_t(\varphi)$ for all smooth φ . Indeed, we have $\|\Pi_N \varphi - \varphi\| \rightarrow 0$ for any smooth φ and since both g_t and h_t are continuous in $\varphi \in C(\Lambda)$, it follows that $h_t^N(\varphi) = h_t(\Pi_N \varphi) \rightarrow h_t(\varphi)$ as $N \rightarrow \infty$ and analogously $g_t^N(\varphi) \rightarrow g_t(\varphi)$. ■

4.1. Covariance and (massive) heat kernel estimates

For the proof of Proposition 4.1, we require some basic estimates for the covariance $c_t^{m^2}$ and the (massive) heat kernel $\dot{c}_t^{m^2}$. We turn to recording these now. The most basic estimate we shall have use for is just for $c_t^{m^2}(x)$.

Lemma 4.4. *There exists a universal constant $C > 0$ such that for $0 < t < m^{-2}$ and $x \in \mathbb{R}^2$, we have the estimate*

$$\left| c_t^{m^2}(x) + \frac{1}{2\pi} \log\left(\frac{|x|}{\sqrt{t}} \wedge 1\right) \right| \leq C. \quad (4.31)$$

Proof. Let us write

$$c_t^{m^2}(x) = \int_0^t ds \frac{e^{-m^2 s}}{4\pi s} e^{-\frac{|x|^2}{4s}} = \int_0^{\frac{t}{|x|^2}} ds \frac{e^{-m^2 s |x|^2}}{4\pi s} e^{-\frac{1}{4s}}. \quad (4.32)$$

For $|x|/\sqrt{t} \geq 1$, we see that

$$0 \leq c_t^{m^2}(x) \leq \int_0^1 ds \frac{e^{-\frac{1}{4s}}}{4\pi s} < \infty, \quad (4.33)$$

so it is sufficient to focus on the regime $|x| < \sqrt{t}$. Here (using $|e^{-x} - 1| \leq x$ for $x > 0$)

$$\begin{aligned} \left| c_t^{m^2}(x) + \frac{1}{2\pi} \log \frac{|x|}{\sqrt{t}} \right| &\leq \int_{\frac{t}{|x|^2}}^{\frac{t}{|x|^2}} ds \frac{|e^{-m^2 s |x|^2 - \frac{1}{4s}} - 1|}{4\pi s} + \int_0^1 ds \frac{e^{-\frac{1}{4s}}}{4\pi s} \\ &\leq \frac{m^2 t}{4\pi} + \frac{1}{16\pi} \int_1^\infty \frac{ds}{s^2} + \int_0^1 ds \frac{e^{-\frac{1}{4s}}}{4\pi s}. \end{aligned} \quad (4.34)$$

Recalling that we are assuming that $m^2 t \leq 1$, this concludes the proof. \blacksquare

The next estimate is slightly more involved.

Lemma 4.5. For $\beta \in (0, 6\pi)$, $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^2 \times \{-1, 1\}$, and $0 < \varepsilon^2 \leq t \leq m^{-2}$ we have

$$|\dot{u}_t^{m^2}(\xi_1, \xi_3) + \dot{u}_t^{m^2}(\xi_2, \xi_3)| \left| 1 - e^{-\sigma_1 \sigma_2 \beta \int_{\varepsilon^2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)} \right| \leq F_t(\xi_1, \xi_2, \xi_3) \quad (4.35)$$

for some function $F_t : (\mathbb{R}^2 \times \{-1, 1\})^3 \rightarrow [0, \infty]$ which is invariant under permutations of the coordinates, independent of ε, m , and in the notation (4.11), satisfies

$$\|F_t\|_3 \leq C_\beta t \quad (4.36)$$

for some constant C_β depending only on β .

Proof. The proof is slightly lengthy and we split it into two parts.

Case 1: $\sigma_1 \neq \sigma_2$. In this case, we bound the quantity $|\dot{u}_t^{m^2}(\xi_1, \xi_3) + \dot{u}_t^{m^2}(\xi_2, \xi_3)| = \beta |\dot{c}_t^{m^2}(x_1 - x_3) - \dot{c}_t^{m^2}(x_2 - x_3)|$. Let us write $[x_1, x_2]$ for the line segment from x_1 to x_2 . Then by the mean value theorem (recalling that $t \leq m^{-2}$),

$$\begin{aligned} |\dot{c}_t^{m^2}(x_1 - x_3) - \dot{c}_t^{m^2}(x_2 - x_3)| &= \frac{e^{-m^2 t}}{4\pi t} \left| e^{-\frac{|x_1 - x_3|^2}{4t}} - e^{-\frac{|x_2 - x_3|^2}{4t}} \right| \\ &\leq \frac{1}{4\pi t} |x_1 - x_2| \sup_{u \in [x_1, x_2]} \left| \nabla_u e^{-\frac{|u - x_3|^2}{4t}} \right|. \end{aligned} \quad (4.37)$$

To bound the gradient, we use that for any $\alpha > 0$, there exists $A(\alpha)$ (depending only on α) such that

$$\left| \nabla_u e^{-\frac{|u - x_3|^2}{4t}} \right| \leq \frac{|u - x_3|}{t} e^{-\frac{|u - x_3|^2}{4t}} \leq A(\alpha) t^{-1/2} e^{-\alpha \frac{|u - x_3|}{\sqrt{t}}}. \quad (4.38)$$

We used here the estimate that there exists $A(\alpha)$ such that $x e^{-x^2/4} \leq A(\alpha) e^{-\alpha x}$ for all $x > 0$.

From the triangle inequality we find that, for $u \in [x_1, x_2]$,

$$-|u - x_3| \leq |x_1 - u| - |x_1 - x_3| \leq |x_1 - x_2| - |x_1 - x_3|. \quad (4.39)$$

This leads to the following bound: for any $\alpha > 0$,

$$|\dot{c}_t^{m^2}(x_1 - x_3) - \dot{c}_t^{m^2}(x_2 - x_3)| \leq A(\alpha) \frac{1}{4\pi t^{3/2}} |x_1 - x_2| e^{\alpha \frac{|x_1 - x_2|}{\sqrt{t}}} e^{-\alpha \frac{|x_1 - x_3|}{\sqrt{t}}}. \quad (4.40)$$

The second term in our statement we bound with the following estimate which is a consequence of Lemma 4.4 (recall that $\int_{\varepsilon_2}^t \dot{c}_r^{m^2}(x_1 - x_2) dr \leq c_t^{m^2}(x_1 - x_2)$):

$$\begin{aligned} |1 - e^{\beta \int_{\varepsilon_2}^t dr \dot{c}_r^{m^2}(x_1 - x_2)}| &\leq e^{\beta c_t^{m^2}(x_1 - x_2)} - 1 \\ &= \int_0^t dr \beta \dot{c}_r^{m^2}(x_1 - x_2) e^{\beta c_r^{m^2}(x_1 - x_2)} \\ &\leq C \int_0^t \frac{dr}{r} e^{-\frac{|x_1 - x_2|^2}{4r}} e^{-\frac{\beta}{2\pi} \log[\frac{|x_1 - x_2|}{\sqrt{r}} \wedge 1]}, \end{aligned} \quad (4.41)$$

where the constant is universal.

Combining our estimates, we see that for each $\alpha > 0$, there exist $A(\alpha)$, $\tilde{A}(\alpha)$ (depending only on α and possibly different from our previous $A(\alpha)$) such that

$$\begin{aligned} &|\dot{u}_t^{m^2}(\xi_1, \xi_3) + \dot{u}_t^{m^2}(\xi_2, \xi_3)| |1 - e^{-\sigma_1 \sigma_2 \beta \int_{\varepsilon_2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)}| \\ &\leq A(\alpha) t^{-3/2} \int_0^t \frac{dr}{r} |x_1 - x_2| e^{\alpha \frac{|x_1 - x_2|}{\sqrt{t}}} e^{-\alpha \frac{|x_1 - x_3|}{\sqrt{t}}} e^{-\frac{|x_1 - x_2|^2}{4r}} e^{-\frac{\beta}{2\pi} \log[\frac{|x_1 - x_2|}{\sqrt{r}} \wedge 1]} \\ &\leq \tilde{A}(\alpha) t^{-3/2} \int_0^t \frac{dr}{r} |x_1 - x_2| e^{\alpha \frac{|x_1 - x_2|}{\sqrt{t}}} e^{-\alpha \frac{|x_1 - x_3|}{\sqrt{t}}} e^{-2\alpha \frac{|x_1 - x_2|}{\sqrt{r}}} e^{-\frac{\beta}{2\pi} \log[\frac{|x_1 - x_2|}{\sqrt{r}} \wedge 1]} \\ &\leq \tilde{A}(\alpha) t^{-3/2} \int_0^t \frac{dr}{r} |x_1 - x_2| e^{-\alpha \frac{|x_1 - x_3|}{\sqrt{t}}} e^{-\alpha \frac{|x_1 - x_2|}{\sqrt{r}}} e^{-\frac{\beta}{2\pi} \log[\frac{|x_1 - x_2|}{\sqrt{r}} \wedge 1]}, \end{aligned} \quad (4.42)$$

where we made use of the facts that for some $A(\alpha)$, $e^{-x^2} \leq A(\alpha) e^{-4\alpha x}$ for $x > 0$, and that for $r \leq t$, $e^{\alpha \frac{|x_1 - x_2|}{\sqrt{t}} - \alpha \frac{|x_1 - x_2|}{\sqrt{r}}} \leq 1$. To summarize, choosing $\alpha = 1$ we have the bound

$$\begin{aligned} &|\dot{u}_t^{m^2}(\xi_1, \xi_3) + \dot{u}_t^{m^2}(\xi_2, \xi_3)| |1 - e^{-\sigma_1 \sigma_2 \beta \int_{\varepsilon_2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)}| \\ &\leq C t^{-3/2} \int_0^t \frac{dr}{r} |x_1 - x_2| e^{-\frac{|x_1 - x_3|}{\sqrt{t}}} e^{-\frac{|x_1 - x_2|}{\sqrt{r}}} e^{-\frac{\beta}{2\pi} \log[\frac{|x_1 - x_2|}{\sqrt{r}} \wedge 1]} \end{aligned} \quad (4.43)$$

for some universal constant C and we can then define F_t (at least in the case $\sigma_1 \neq \sigma_2$) to be the function obtained by symmetrizing the above function with respect to the variables x_i . Note in particular that this is independent of ε, m .

To control $\|F_t\|_3$, let us in all of our terms (coming from symmetrization) shift x_2 and x_3 by x_1 so we are left with the estimate

$$\|F_t \mathbf{1}_{\sigma_1 \neq \sigma_2}\|_3 \leq C t^{-3/2} \int_0^t \frac{dr}{\sqrt{r}} \int_{\mathbb{R}^2} dx e^{-\frac{|x|}{\sqrt{t}}} \int_{\mathbb{R}^2} dy \frac{|y|}{\sqrt{r}} e^{-\frac{|y|}{\sqrt{r}}} \left(\frac{|y|}{\sqrt{r}} \wedge 1 \right)^{-\frac{\beta}{2\pi}} \quad (4.44)$$

for some universal constant C . By a change of integration variables, the x -integral is some universal constant times t , while the y -integral is some constant depending on β times r

(note that the singularity at the origin is integrable precisely for $\beta < 6\pi$). Thus for some constant C_β (depending only on β),

$$\|F_t \mathbf{1}_{\sigma_1 \neq \sigma_2}\|_3 \leq C_\beta t^{-1/2} \int_0^t dr r^{1/2} \leq C_\beta t, \quad (4.45)$$

which was the claim.

Case 2: $\sigma_1 = \sigma_2$. In this case, we simply write

$$|\dot{u}_t^{m^2}(\xi_1, \xi_3) + \dot{u}_t^{m^2}(\xi_2, \xi_3)| \leq \frac{\beta}{4\pi t} \left(e^{-\frac{|x_1-x_3|^2}{4t}} + e^{-\frac{|x_2-x_3|^2}{4t}} \right), \quad (4.46)$$

while for the exponential, we have by Lemma 4.4 (for some universal constant C)

$$\begin{aligned} |1 - e^{-\beta \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_1-x_2)}| &\leq 1 - e^{-\beta c_t^{m^2}(x_1-x_2)} \\ &= \int_0^t dr \beta \dot{c}_r^{m^2}(x_1-x_2) e^{-\beta c_r^{m^2}(x_1-x_2)} \\ &\leq C \int_0^t dr \frac{e^{-\frac{|x_1-x_2|^2}{4r}}}{r} e^{\frac{\beta}{2\pi} \log[\frac{|x_1-x_2|}{\sqrt{r}} \wedge 1]}. \end{aligned} \quad (4.47)$$

Combining the estimates, we have (for some possibly different universal constant)

$$\begin{aligned} |\dot{u}_t^{m^2}(\xi_1, \xi_3) + \dot{u}_t^{m^2}(\xi_2, \xi_3)| |1 - e^{-\beta \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_1-x_2)}| \\ \leq C t^{-1} \left(e^{-\frac{|x_1-x_3|^2}{4t}} + e^{-\frac{|x_2-x_3|^2}{4t}} \right) \int_0^t dr \frac{e^{-\frac{|x_1-x_2|^2}{4r}}}{r} e^{\frac{\beta}{2\pi} \log[\frac{|x_1-x_2|}{\sqrt{r}} \wedge 1]}. \end{aligned} \quad (4.48)$$

The relevant function F_t is again obtained by symmetrizing with respect to x_1, x_2, x_3 .

To estimate the norm, we can again get rid of x_1 by a shift of the integration variables. One is left with the estimate

$$\|F_t \mathbf{1}_{\sigma_1=\sigma_2}\|_3 \leq C \int_{\mathbb{R}^2} dx \frac{e^{-\frac{|x|^2}{4t}}}{t} \int_0^t dr \int_{\mathbb{R}^2} \frac{dy}{r} e^{-\frac{|y|^2}{4r}} \left(\frac{|y|}{\sqrt{r}} \wedge 1 \right)^{\frac{\beta}{2\pi}} \leq \tilde{C} t \quad (4.49)$$

now for universal constants C, \tilde{C} . This concludes the proof. \blacksquare

The final estimate we shall need involves four points.

Lemma 4.6. For $\beta \in (0, 6\pi)$, $0 < \varepsilon^2 \leq t < m^{-2}$, and $\xi_1, \dots, \xi_4 \in \mathbb{R}^2 \times \{-1, 1\}$, we have

$$\left| \sum_{i \in \{1,2\}, j \in \{3,4\}} \dot{u}_t^{m^2}(\xi_i, \xi_j) \right| |1 - e^{-\beta \sigma_1 \sigma_2 \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_1-x_2)}| |1 - e^{-\beta \sigma_3 \sigma_4 \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_3-x_4)}| \leq G_t(\xi_1, \xi_2, \xi_3, \xi_4) \quad (4.50)$$

for some function G_t which is independent of ε, m and is symmetric in the arguments. Moreover, there exists a constant C_β depending only on β such that, in the notation (4.11),

$$\|G_t\|_4 \leq C_\beta t^2. \quad (4.51)$$

Proof. The proof is very similar to that of Lemma 4.5. We again split it into two cases.

Case 1: $\sigma_1 \neq \sigma_2$ and $\sigma_3 \neq \sigma_4$. Arguing as in (4.37), but noting that now we are dealing with a kind of second order difference, we find bounds in terms of the second order derivative of the heat kernel. Using again an elementary estimate bounding $x^2 e^{-x^2} + e^{-x^2}$ in terms of $e^{-\alpha x}$ times a constant depending only on $\alpha > 0$, we find that for each $\alpha > 0$ there exists a constant $A(\alpha)$ (depending only on α) such that

$$\left| \sum_{\substack{i \in \{1,2\} \\ j \in \{3,4\}}} \dot{u}_t^{m^2}(\xi_i, \xi_j) \right| = \left| \dot{c}_t^{m^2}(x_1 - x_3) - \dot{c}_t^{m^2}(x_2 - x_3) - \dot{c}_t^{m^2}(x_1 - x_4) + \dot{c}_t^{m^2}(x_2 - x_4) \right| \leq A(\alpha) \frac{|x_1 - x_2| |x_3 - x_4|}{t^2} \sup_{\substack{u \in [x_1, x_2] \\ v \in [x_3, x_4]}} e^{-\alpha \frac{|u-v|}{\sqrt{t}}}. \quad (4.52)$$

Instead of the bound (4.39), we now use the fact (again following from the triangle inequality) that

$$-|u - v| \leq |x_1 - u| + |x_3 - v| - |x_1 - x_3| \leq |x_1 - x_2| + |x_3 - x_4| - |x_1 - x_3|, \quad (4.53)$$

which implies for our choice of σ 's that for each $\alpha > 0$, there exists $A(\alpha)$ such that

$$\left| \sum_{i \in \{1,2\}, j \in \{3,4\}} \dot{u}_t^{m^2}(\xi_i, \xi_j) \right| \leq t^{-2} A(\alpha) |x_1 - x_2| |x_3 - x_4| e^{\alpha \frac{|x_1 - x_2|}{\sqrt{t}} + \alpha \frac{|x_3 - x_4|}{\sqrt{t}} - \alpha \frac{|x_1 - x_3|}{\sqrt{t}}}. \quad (4.54)$$

We estimate the exponentials as in (4.41) and arguing as in the proof of Lemma 4.5 (choosing α' and α in a similar way etc.), we arrive at the bound

$$\begin{aligned} & \left| \sum_{i \in \{1,2\}, j \in \{3,4\}} \dot{u}_t^{m^2}(\xi_i, \xi_j) \right| \left| 1 - e^{-\beta \sigma_1 \sigma_2 \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_1 - x_2)} \right| \left| 1 - e^{-\beta \sigma_3 \sigma_4 \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_3 - x_4)} \right| \\ & \leq C t^{-2} e^{-\frac{|x_1 - x_3|}{\sqrt{t}}} \int_0^t \frac{dr}{r^{1/2}} \frac{|x_1 - x_2|}{\sqrt{r}} e^{-\frac{|x_1 - x_2|}{\sqrt{r}}} \left(\frac{|x_1 - x_2|}{\sqrt{r}} \wedge 1 \right)^{-\frac{\beta}{2\pi}} \\ & \quad \times \int_0^t \frac{ds}{\sqrt{s}} \frac{|x_3 - x_4|}{\sqrt{s}} e^{-\frac{|x_3 - x_4|}{\sqrt{s}}} \left(\frac{|x_3 - x_4|}{\sqrt{s}} \wedge 1 \right)^{-\frac{\beta}{2\pi}}. \end{aligned} \quad (4.55)$$

Symmetrizing with respect to the x_i yields our function G_t . Its norm can be estimated with similar scaling arguments as in the proof of Lemma 4.5 and we find

$$\|G_t \mathbf{1}_{\sigma_1 \neq \sigma_2} \mathbf{1}_{\sigma_3 \neq \sigma_4}\|_4 \leq C \beta t^{-1} \left(\int_0^t dr r^{1/2} \right)^2 \leq C \beta t^2, \quad (4.56)$$

which was the claim.

Case 2: $\sigma_1 = \sigma_2$ or $\sigma_3 = \sigma_4$ or both. Let us assume (by symmetry) that $\sigma_3 = \sigma_4$. We can then use Lemma 4.5 (and the triangle inequality) to write

$$\begin{aligned} & \left| \sum_{i \in \{1,2\}, j \in \{3,4\}} \dot{u}_t^{m^2}(\xi_i, \xi_j) \right| \left| 1 - e^{-\beta \sigma_1 \sigma_2 \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_1 - x_2)} \right| \left| 1 - e^{-\beta \sigma_3 \sigma_4 \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(x_3 - x_4)} \right| \\ & \leq (F_t(\xi_1, \xi_2, \xi_3) + F_t(\xi_1, \xi_2, \xi_4)) (1 - e^{-\beta \int_0^t dr \dot{c}_r^{m^2}(x_3 - x_4)}) \\ & \leq \beta (F_t(\xi_1, \xi_2, \xi_3) + F_t(\xi_1, \xi_2, \xi_4)) \int_0^t \frac{dr}{4\pi r} e^{-\frac{|x_3 - x_4|^2}{4r}}. \end{aligned} \quad (4.57)$$

The claim follows now by symmetrizing and Lemma 4.5, which implies

$$\|G_t(1 - \mathbf{1}_{\sigma_1 \neq \sigma_2} \mathbf{1}_{\sigma_3 \neq \sigma_4})\|_4 \leq C_\beta t \int_{\mathbb{R}^2} dx \int_0^t \frac{dr}{4\pi r} e^{-\frac{|x|^2}{4r}} \leq C_\beta t^2 \quad (4.58)$$

as needed. \blacksquare

4.2. Proof of Proposition 4.1

We begin our proof of Proposition 4.1 with the remark that (4.7) implies (using $x > 0$, $1 - e^{-x} \leq x$ and $m^2 t \leq 1$)

$$0 \leq \tilde{v}_t^1(\xi|\varepsilon, m) = e^{-\frac{\beta}{2}(\int_{\varepsilon^2}^t ds \frac{e^{-m^2 s} - 1}{4\pi s} + \frac{1}{4\pi} \log t)} \leq t^{-\frac{\beta}{8\pi}} e^{\frac{\beta}{8\pi} m^2 t} \leq C_\beta t^{-\frac{\beta}{8\pi}} =: h_t^1(\xi) \quad (4.59)$$

for a constant C_β depending only on β . This verifies the bound in Proposition 4.1 for $n = 1$. We will verify the claimed bound explicitly also for $n = 3$ and $n = 4$, but prove the rest of it by induction. We will make use of the following explicit form of the $n = 2$ term:

$$\begin{aligned} \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m) &= \beta\sigma_1\sigma_2 \int_{\varepsilon^2}^t ds (\dot{c}_s^{m^2}(x_1 - x_2) e^{-\beta(\int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(0) + \frac{1}{4\pi} \log \varepsilon^2)} \\ &\quad \cdot e^{-\beta \int_s^t dr \dot{c}_r^{m^2}(0) - \beta\sigma_1\sigma_2 \int_s^t dr \dot{c}_r^{m^2}(x_1 - x_2)}) \\ &= e^{-\beta(\int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(0) + \frac{1}{4\pi} \log \varepsilon^2)} (1 - e^{-\beta\sigma_1\sigma_2 \int_{\varepsilon^2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)}). \end{aligned} \quad (4.60)$$

Indeed, this equality follows from a straightforward calculation (using (4.7) and (4.8)). This allows us to prove Proposition 4.1 in the special case $n = 3$.

Lemma 4.7. *For $\beta \in (0, 6\pi)$ and $t > 0$, there exists a function h_t^3 which is independent of $\varepsilon, m > 0$ and for $0 < \varepsilon^2 < t < m^{-2}$,*

$$|\tilde{v}_t^3(\xi_1, \xi_2, \xi_3|\varepsilon, m)| \leq h_t^3(\xi_1, \xi_2, \xi_3) \quad (4.61)$$

and, in the notation (4.11), $\|h_t^3\|_3 \leq C_\beta t^{-1} (t^{1-\beta/8\pi})^3$ for a constant C_β depending only on β .

Proof. From the definitions of \tilde{v}_t^n in (4.7) and (4.8) and the expression for \tilde{v}_t^2 from (4.60), a straightforward calculation shows that, for any ξ_1, ξ_2, ξ_3 ,

$$\begin{aligned} \tilde{v}_t^3(\xi_1, \xi_2, \xi_3|\varepsilon, m) &= \beta(\tilde{v}_t^1(\xi|\varepsilon, m))^3 \int_{\varepsilon^2}^t ds [(\dot{u}_s^{m^2}(\xi_1, \xi_2) + \dot{u}_s^{m^2}(\xi_1, \xi_3))(1 - e^{-\sigma_2\sigma_3\beta \int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(x_2 - x_3)}) \\ &\quad + (\dot{u}_s^{m^2}(\xi_2, \xi_1) + \dot{u}_s^{m^2}(\xi_2, \xi_3))(1 - e^{-\sigma_1\sigma_3\beta \int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(x_1 - x_3)}) \\ &\quad + (\dot{u}_s^{m^2}(\xi_3, \xi_1) + \dot{u}_s^{m^2}(\xi_3, \xi_2))(1 - e^{-\sigma_1\sigma_2\beta \int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(x_1 - x_2)})] \\ &\quad \times e^{-\beta \sum_{1 \leq i < j \leq 3} \sigma_i \sigma_j \int_s^t dr \dot{c}_r^{m^2}(x_i - x_j)}. \end{aligned} \quad (4.62)$$

We claim that we have the following estimate: for $t > s$, for any $\sigma_1, \sigma_2, \sigma_3$ and x_1, x_2, x_3 and for some universal constant $C > 0$,

$$\begin{aligned} \sigma_1 \sigma_2 \int_s^t dr \dot{c}_r^{m^2}(x_1 - x_2) + \sigma_1 \sigma_3 \int_s^t dr \dot{c}_r^{m^2}(x_1 - x_3) + \sigma_2 \sigma_3 \int_s^t dr \dot{c}_r^{m^2}(x_2 - x_3) \\ \geq -\frac{1}{4\pi} \log \frac{t}{s} - C. \end{aligned} \quad (4.63)$$

Indeed, the worst case scenario is when $\sigma_1 = \sigma_2 \neq \sigma_3$ and $|x_1 - x_2| \geq |x_1 - x_3|, |x_2 - x_3|$ (or the same with a permutation of indices). By the triangle inequality, at least one of $|x_1 - x_3|$ and $|x_2 - x_3|$ is greater than $\frac{1}{2}|x_1 - x_2|$. Thus from Lemma 4.4 we have

$$\begin{aligned} \sigma_1 \sigma_2 \int_s^t dr \dot{c}_r^{m^2}(x_1 - x_2) + \sigma_1 \sigma_3 \int_s^t dr \dot{c}_r^{m^2}(x_1 - x_3) + \sigma_2 \sigma_3 \int_s^t dr \dot{c}_r^{m^2}(x_2 - x_3) \\ \geq \inf_{x \in \mathbb{R}^2} \left(\int_s^t dr (\dot{c}_r^{m^2}(x) - \dot{c}_r^{m^2}(x/2)) \right) - \frac{1}{4\pi} \int_s^t \frac{dr}{r} \\ = \inf_{x \in \mathbb{R}^2} \left(-\frac{1}{2\pi} \log \left(\frac{|x|}{\sqrt{t}} \wedge 1 \right) + \frac{1}{2\pi} \log \left(\frac{|x|}{2\sqrt{t}} \wedge 1 \right) \right) - \frac{1}{4\pi} \log \frac{t}{s} + O(1), \end{aligned} \quad (4.64)$$

where the implied constant is universal. Going through the various cases ($|x| < \sqrt{t}$, $\sqrt{t} \leq |x| < 2\sqrt{t}$, and $|x| \geq 2\sqrt{t}$), one readily checks that the infimum is $-\frac{1}{2\pi} \log 2$ and we have the bound (4.63).

Now making use of Lemma 4.5, (4.63), and (4.59), we find that for a constant \tilde{C}_β (depending only on β)

$$|\tilde{v}_t^3(\xi_1, \xi_2, \xi_3 | \varepsilon, m)| \leq \tilde{C}_\beta t^{-\frac{3\beta}{8\pi}} \int_0^t ds \left(\frac{t}{s} \right)^{\frac{\beta}{4\pi}} F_s(\xi_1, \xi_2, \xi_3) =: h_t^3(\xi_1, \xi_2, \xi_3), \quad (4.65)$$

which is independent of ε, m as required. Finally, by Lemma 4.5, there is another constant C_β depending only on β such that

$$\|h_t^3\|_3 \leq C_\beta t^{-\frac{3\beta}{8\pi}} \int_0^t ds \left(\frac{t}{s} \right)^{\frac{\beta}{4\pi}} s \leq C_\beta t^{2-\frac{3\beta}{8\pi}}, \quad (4.66)$$

which was precisely the claim. \blacksquare

We now turn to \tilde{v}_t^4 .

Lemma 4.8. *For $\beta \in (0, 6\pi)$ and $t > 0$ there exists a function h_t^4 , which is independent of m and ε , such that for $0 < \varepsilon^2 < t < m^{-2}$,*

$$|\tilde{v}_t^4(\xi_1, \xi_2, \xi_3, \xi_4 | \varepsilon, m)| \leq h_t^4(\xi_1, \xi_2, \xi_3, \xi_4) \quad (4.67)$$

and $\|h_t^4\|_4 \leq C_\beta t^{-1} (t^{1-\beta/8\pi})^4$ for a constant C_β depending only on β .

Proof. We begin with the recursion (4.8). We see that there are two types of contributions: either $|I_1| = |I_2| = 2$ or $|I_1|, |I_2| \in \{1, 3\}$ (with $|I_1| + |I_3| = 4$). Let us consider

the latter case first. Here we can use (4.59) and Lemma 4.7 along with the remark that $w_t^{m^2} - w_s^{m^2} \geq 0$ (since $\dot{c}_r^{m^2}$ is a covariance), to get the simple upper bound

$$\begin{aligned} & \frac{1}{2} \left| \sum_{\substack{I_1 \dot{\cup} I_2 = [4] \\ |I_1|, |I_2| \neq 2}} \int_{\varepsilon^2}^t ds \sum_{i \in I_1, j \in I_2} \dot{u}_s^{m^2}(\xi_i, \xi_j) \tilde{v}_s^{|I_1|}(\xi_{I_1} | \varepsilon, m) \tilde{v}_s^{|I_2|}(\xi_{I_2} | \varepsilon, m) \right. \\ & \qquad \qquad \qquad \left. \times e^{-(w_t^{m^2}(\xi_1, \dots, \xi_4) - w_s^{m^2}(\xi_1, \dots, \xi_4))} \right| \\ & \leq C_\beta \sum_{k=1}^4 \int_0^t \frac{ds}{s} \sum_{l \neq k} e^{-\frac{|x_k - x_l|^2}{4s}} h_s^1(\xi_k) h_s^3(\xi_{[4] \setminus \{k\}}), \end{aligned} \quad (4.68)$$

which is the contribution to h_t^4 from the $|I_1|, |I_2| \neq 2$ case. Note that using (4.59) and Lemma 4.7, one can check readily that the $\|\cdot\|_4$ -norm of this quantity is bounded by (for some constants C_β, \tilde{C}_β depending only on β)

$$C_\beta \int_0^t ds s^{-\frac{\beta}{8\pi}} s^{-1} s^{3(1-\frac{\beta}{8\pi})} \leq \tilde{C}_\beta t^{-1} t^{4(1-\frac{\beta}{8\pi})}, \quad (4.69)$$

which is precisely of the required form (note that the integral here is convergent since $\beta < 6\pi$).

It remains to control the $|I_1| = |I_2| = 2$ case. A typical term that one encounters in the sum is of the form

$$\begin{aligned} & \int_{\varepsilon^2}^t ds (\dot{u}_s^{m^2}(\xi_1, \xi_3) + \dot{u}_s^{m^2}(\xi_2, \xi_3) + \dot{u}_s^{m^2}(\xi_1, \xi_4) + \dot{u}_s^{m^2}(\xi_2, \xi_4)) e^{-2\beta \int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(0) + \frac{1}{4\pi} \log \varepsilon^2} \\ & \quad \times (1 - e^{-\beta \sigma_1 \sigma_2 \int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(x_1 - x_2)}) (1 - e^{-\beta \sigma_3 \sigma_4 \int_{\varepsilon^2}^s dr \dot{c}_r^{m^2}(x_3 - x_4)}) \\ & \quad \times e^{-\frac{1}{2}\beta \sum_{i,j \in [4]} \sigma_i \sigma_j \int_s^t dr \dot{c}_r^{m^2}(x_i - x_j)}. \end{aligned} \quad (4.70)$$

The last exponential term can again be dropped by positive definiteness of $\dot{c}_r^{m^2}$, so using Lemma 4.6 and (4.59), we see that for some C_β depending only on β , such terms can be bounded by

$$C_\beta \int_0^t ds s^{-\frac{\beta}{2\pi}} G_s(\xi_1, \xi_2, \xi_3, \xi_4), \quad (4.71)$$

where G_s is as in Lemma 4.6. Summing over the other contributions shows that all of the $|I_1| = |I_2|$ terms can be bounded by such quantities. Combining this with the $|I_1|, |I_2| \neq 2$ case gives the definition of h_t^4 . Moreover, we note from Lemma 4.6 that

$$\int_0^t ds s^{-\frac{\beta}{2\pi}} \|G_s\|_4 \leq C_\beta \int_0^t ds s^{2-\frac{\beta}{2\pi}} \leq \tilde{C}_\beta t^{-1} t^{4(1-\frac{\beta}{8\pi})} \quad (4.72)$$

for some constants C_β, \tilde{C}_β depending only on β . Again, $\beta < 6\pi$ played an important role here. Together with the estimate from the previous case, we see that $\|h_t^4\|_4 \leq C_\beta t^{-1} t^{4(1-\beta/8\pi)}$ as required. This concludes the proof. \blacksquare

We turn now to the proof of the general case.

Proof of Proposition 4.1. As mentioned already, the proof is by induction. To propagate the induction, we find it convenient to prove the claim in a slightly different form. More precisely, we will prove the existence of functions h_t^n (independent of ε, m) for which $|\tilde{v}_t^n(\cdot|\varepsilon, m)| \leq h_t^n$ and for some C_β depending only on β and some universal constant $C > 0$,

$$\|h_t^n\|_n \leq n^{n-2} t^{-1} C_\beta^{n-1} (C t^{1-\frac{\beta}{8\pi}})^n, \quad (4.73)$$

which of course implies the claim (with a possibly different C_β). For $n = 1$, (4.73) is (4.59) and for $n = 3$ and $n = 4$, (4.73) is proved in Lemmas 4.7 and 4.8. Let us now assume as our induction hypothesis that for some $n \geq 5$, the estimate (4.73) holds for all $k \leq n - 1$ with $k \neq 2$. As mentioned, this has been verified for $n = 5$. To advance the induction, we plug the hypothesis into (4.8), and need to be slightly careful about the contributions from $|I_1| = 2$ or $|I_2| = 2$.

Let us consider the terms in (4.8) with $|I_1| \neq 2$ and $|I_2| \neq 2$ first. In (4.8), it will be sufficient to just drop the $w_t^{m^2} - w_s^{m^2}$ term (which, as before, is allowed due to the positive definiteness of $\hat{c}_r^{m^2}$). Then one readily checks (from (4.8) and our induction hypothesis) that the $|I_1|, |I_2| \neq 2$ contribution can be bounded by

$$h_t^{n,1}(\xi_1, \dots, \xi_n) := \frac{\beta}{8\pi} \sum_{\substack{I_1 \dot{\cup} I_2 = [n] \\ |I_1|, |I_2| \neq 2}} \sum_{i \in I_1, j \in I_2} \int_0^t \frac{ds}{s} e^{-\frac{|x_i - x_j|^2}{4s}} h_s^{|I_1|}(\xi_{I_1}) h_s^{|I_2|}(\xi_{I_2}). \quad (4.74)$$

Note that this is indeed independent of ε, m as required. Using the fact that $\int_{\mathbb{R}^2} dx \frac{e^{-|x|^2/4t}}{4\pi t} = 1$ and our induction hypothesis, we find for the norm of this the bound

$$\begin{aligned} \|h_t^{n,1}\|_n &\leq \frac{\beta}{2} \sum_{\substack{I_1 \dot{\cup} I_2 = [n] \\ |I_1|, |I_2| \neq 2}} |I_1| |I_2| \int_0^t ds \|h_s^{|I_1|}\|_{|I_1|} \|h_s^{|I_2|}\|_{|I_2|} \\ &\leq \frac{\beta}{2} C_\beta^{n-2} C^n \sum_{\substack{I_1 \dot{\cup} I_2 = [n] \\ |I_1|, |I_2| \neq 2}} |I_1|^{|I_1|-1} |I_2|^{|I_2|-1} \int_0^t ds s^{-2+n(1-\frac{\beta}{8\pi})} \\ &= \frac{\beta}{2} \frac{1}{-1+n(1-\frac{\beta}{8\pi})} C_\beta^{n-2} C^n t^{-1} t^{n(1-\frac{\beta}{8\pi})} \sum_{k=1}^{n-1} \binom{n}{k} k^{k-1} (n-k)^{n-k-1} \\ &= \frac{\beta}{2} \frac{2(n-1)}{-1+n(1-\frac{\beta}{8\pi})} n^{n-2} C_\beta^{n-2} t^{-1} (C t^{1-\frac{\beta}{8\pi}})^n \end{aligned} \quad (4.75)$$

where in the last equality we made use of the identity $\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1} (n-k)^{n-k-1} = 2(n-1)n^{n-2}$. (This identity has the following combinatorial interpretation. The number of trees on $[n]$ is n^{n-2} . Thus $2(n-1)n^{n-2}$ represents the number of trees on $[n]$ together with a choice of a directed edge. Such trees rooted by a directed edge can also be obtained by connecting two disjoint vertex rooted trees with k and $n-k$ vertices by an edge between their roots.) Now for $n \geq 5$ and $\beta \in (0, 6\pi)$, $0 \leq \frac{\beta}{2} \frac{2(n-1)}{n(1-\beta/8\pi)-1}$ is bounded

by a universal constant, so possibly increasing C_β verifies that the bound (4.73) holds for the contribution coming from $|I_1|, |I_2| \neq 2$.

Let us now turn to the case where $|I_1| = 2$ or $|I_2| = 2$. We again drop the $w_t^{m^2} - w_s^{m^2}$ term from the exponential by positive definiteness. In terms of the notation of Lemma 4.5, we find (using the lemma and (4.60)) that the contribution from the $|I_1| = 2$ or $|I_2| = 2$ case can be bounded by

$$\begin{aligned} \tilde{C}_\beta \sum_{1 \leq a < b \leq n} \sum_{j \in [n] \setminus \{a, b\}} \int_0^t ds |\dot{u}_s^{m^2}(\xi_a, \xi_j) + \dot{u}_s^{m^2}(\xi_b, \xi_j)| h_s^{n-2}(\xi_{[n] \setminus \{a, b\}}) \\ \times s^{-\frac{\beta}{4\pi}} |1 - e^{-\beta \sigma_a \sigma_b c_s^{m^2}(x_a - x_b)}| \\ \leq \sum_{1 \leq a < b \leq n} \sum_{j \in [n] \setminus \{a, b\}} \int_0^t ds s^{-\frac{\beta}{4\pi}} F_s(\xi_a, \xi_b, \xi_j) h_s^{n-2}(\xi_{[n] \setminus \{a, b\}}) \\ =: h_t^{n,2}(\xi_1, \dots, \xi_n) \end{aligned} \quad (4.76)$$

for some constant \tilde{C}_β depending only on β . For the norm of this, we readily find from Lemma 4.5 and our induction hypothesis that (for some possibly different \tilde{C}_β , still depending only on β)

$$\begin{aligned} \|h_t^{n,2}\|_n &\leq \tilde{C}_\beta C_\beta^{n-3} C^{n-2} n^2 (n-2)(n-2)^{n-4} \int_0^t ds s^{-\frac{\beta}{4\pi}} s^{(n-2)(1-\frac{\beta}{8\pi})} \\ &= \tilde{C}_\beta C_\beta^{n-3} C^{n-2} n^{n-2} \frac{(n-2)}{1 - \frac{\beta}{4\pi} + (n-2)(1 - \frac{\beta}{8\pi})} t^{(n-2)(1-\frac{\beta}{8\pi})+1-\frac{\beta}{4\pi}}. \end{aligned}$$

The ratio here is again bounded by a universal constant, so possibly increasing C_β (to account for this universal constant and \tilde{C}_β) then yields the bound we are after.

In particular, choosing $h_t^n = h_t^{n,1} + h_t^{n,2}$ gives the required function and concludes the proof. \blacksquare

5. The sine-Gordon model: the partition and correlation functions

The goal of this section is to prove Theorem 3.1, which is our main statement about the correlation functions of the sine-Gordon model. As already suggested in the previous section, a central tool in our proof of Theorem 3.1 is a suitable generating function for the correlation functions. To reiterate, the generating function we consider is (as in (4.2)) for $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ given by

$$Z(\zeta|\varepsilon, m) = Z(\beta, \zeta|\varepsilon, m) = \left\langle \exp \left[- \int d\xi \varepsilon^{-\frac{\beta}{4\pi}} \zeta(\xi) e^{i\sqrt{\beta} \sigma \varphi(x)} \right] \right\rangle_{\text{GFF}(\varepsilon, m)} \quad (5.1)$$

with $\xi = (x, \sigma)$ and $\int d\xi = \sum_{\sigma \in \{\pm 1\}} \int_{\mathbb{R}^2} dx$ as before. Of course, $\zeta(\xi) = -z \mathbf{1}_\Lambda(x)$ is admissible and $Z(\zeta|\varepsilon, m)$ then reduces to the normalization constant in (3.3). In general, note that we allow complex-valued functions ζ , and that $Z(\zeta|\varepsilon, m)$ is then not necessarily a normalizing constant for a positive measure. The purpose of introducing $Z(\zeta|\varepsilon, m)$ is

that by choosing ζ to depend on suitable external parameters, we can obtain (smeared) sine-Gordon correlation functions from logarithmic derivatives of $Z(\zeta|\varepsilon, m)$ with respect to these parameters. Thus if we can control $Z(\zeta|\varepsilon, m)$ in the $\varepsilon, m \rightarrow 0$ limit, we can also control the correlation functions.

A significant part of our analysis will rely on properties of the free field correlation functions studied in Section 2.2. Particularly important for us will be charge correlation functions. Their importance can be seen, for example, from the fact that since $:e^{i\sqrt{\beta}\varphi(x)}:_\varepsilon$ is a bounded random variable for any $\varepsilon > 0$ one finds (for more details, see Lemma 5.5)

$$Z(\zeta|\varepsilon, m) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int d\xi_1 \cdots d\xi_k \zeta(\xi_1) \cdots \zeta(\xi_k) \left\langle \prod_{j=1}^k :e^{i\sqrt{\beta}\sigma_j\varphi(x_j)}:_\varepsilon \right\rangle_{\text{GFF}(\varepsilon, m)}. \quad (5.2)$$

It turns out that for $\beta \geq 4\pi$ (so in particular for $\beta = 4\pi$), $Z(\zeta|\varepsilon, m)$ does not converge as $\varepsilon \rightarrow 0$. Heuristic evidence for this can be seen from Lemma 2.5 combined with the expansion (5.2): one expects to have divergence already at order $k = 2$ in the expansion since

$$\langle :e^{i\sqrt{\beta}\varphi(x)} : :e^{-i\sqrt{\beta}\varphi(y)} : \rangle_{\text{GFF}} \propto |x - y|^{-\frac{\beta}{2\pi}} \quad (5.3)$$

is not integrable for $\beta \geq 4\pi$. It turns out that for $\beta \in [4\pi, 6\pi)$, this is in a sense the only type of divergence that occurs and a nontrivial limit can be obtained once Z is multiplied by an explicit counterterm. This counterterm and the limit theorem for the partition function are most conveniently expressed in terms of truncated free field correlation functions, which we again recall from Section 2.2.

The counterterm is then defined as follows: for $\xi_1, \xi_2 \in \mathbb{R}^2 \times \{-1, 1\}$ let

$$A(\xi_1, \xi_2|\varepsilon, m) = \langle :e^{i\sqrt{\beta}\sigma_1\varphi(x_1)}:_\varepsilon :e^{i\sqrt{\beta}\sigma_2\varphi(x_2)}:_\varepsilon \rangle_{\text{GFF}(\varepsilon, m)}^T. \quad (5.4)$$

We then define our renormalized partition function as

$$\mathcal{Z}(\zeta|\varepsilon, m) := Z(\zeta|\varepsilon, m) \exp \left[-\frac{1}{2} \int d\xi_1 d\xi_2 \zeta(\xi_1)\zeta(\xi_2) A(\xi_1, \xi_2|\varepsilon, m) \right]. \quad (5.5)$$

It follows from Lemma 2.5 that

$$\lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} A(\xi_1, \xi_2|\varepsilon, m) = \delta_{\sigma_1 + \sigma_2, 0} e^{-\frac{\gamma\beta}{4\pi}} \left(\frac{2}{|x_1 - x_2|} \right)^{\frac{\beta}{2\pi}}, \quad (5.6)$$

and since this is nonintegrable for $\beta \geq 4\pi$, our counterterm at least has a chance to cure the divergence of the partition function. This is indeed true, in that \mathcal{Z} turns out to have a finite limit for $\beta < 6\pi$, and thus in particular for $\beta = 4\pi$, which is the case we are interested in. For $\beta \geq 6\pi$ further counterterms, which turn out to involve higher order truncated correlation functions, would be required; see [8, 23, 52].

Before stating our result about the convergence of $\mathcal{Z}(\zeta|\varepsilon, m)$, recall from Lemma 2.8 that while the truncated charge two-point function is not integrable, all higher order charge correlation functions are integrable. With this notation and fact in hand, we are in a position to state our main result about $\mathcal{Z}(\zeta|\varepsilon, m)$. For $\beta < 4\pi$ the conclusions also follow

from [28], but our extension to $\beta < 6\pi$ (crucially including the free fermion point $\beta = 4\pi$) relies on new ideas. We prove this in Section 5.4 and then deduce Theorem 3.1 in Section 5.6.

Theorem 5.1. *For $\beta \in (0, 6\pi)$, $m \in (0, \infty)$, and $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}, \mathbb{C})$ the following claims hold:*

(i) *The limits*

$$\mathcal{Z}(\zeta|m) = \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(\zeta|\varepsilon, m), \quad \mathcal{Z}(\zeta) = \lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathcal{Z}(\zeta|\varepsilon, m), \quad (5.7)$$

exist and are finite.

(ii) *The functions $z \mapsto \mathcal{Z}(z\zeta|m)$ and $z \mapsto \mathcal{Z}(z\zeta)$ are entire functions of $z \in \mathbb{C}$ and $\mathcal{Z}(z\zeta) = \mathcal{Z}(-z\zeta)$.*

(iii) *If $\zeta(x, 1) = \overline{\zeta(x, -1)}$ for almost all $x \in \mathbb{R}^2$, then $\mathcal{Z}(\zeta|m) > 0$ and $\mathcal{Z}(\zeta) > 0$.*

(iv) *Finally, if $\zeta_\alpha \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ depends on some complex parameters $\alpha \in \mathbb{C}^N$ and $\zeta_\alpha(\cdot|\varepsilon, m) \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ depends also on $\varepsilon, m > 0$ and these complex parameters α in such a way that for some $K \subset \mathbb{C}^N$ compact,*

$$\lim_{m \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} \sup_{\alpha \in K} \|\zeta_\alpha(\cdot|\varepsilon, m) - \zeta_\alpha\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} = 0, \quad (5.8)$$

then

$$\lim_{m \rightarrow 0} \limsup_{\varepsilon \rightarrow 0} |\mathcal{Z}(\zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m) - \mathcal{Z}(\zeta_\alpha)| = 0 \quad (5.9)$$

and the convergence is uniform in $\alpha \in K$. An analogous statement holds for $m \in (0, \infty)$ fixed.

As a preliminary remark, we note that by rescaling space it suffices to prove the statements for fixed $m > 0$ in this theorem only for $m \in (0, 1)$; we will henceforth assume this.

Before we turn to the actual proofs, we need to recall some basic facts about regularity and extrema of Gaussian processes.

5.1. Preliminaries: regularity and extrema of Gaussian processes

In this section, we record some basic facts we need to know about the regularization of the GFF to a scale \sqrt{t} which is of order 1, namely we look at the Gaussian process with law $\nu^{\text{GFF}(\sqrt{t}, m)}$ – in particular in the $m \rightarrow 0$ limit. Given (4.4), this will be useful to control the renormalized partition function. The main fact we will prove in this section is the following.

Lemma 5.2. *For $0 < t < m^{-2}$, $\Lambda \subset \mathbb{R}^2$ compact, and $p > 0$, we have*

$$\langle e^{p\|\nabla\varphi\|_{L^\infty(\Lambda)}} \rangle_{\text{GFF}(\sqrt{t}, m)} \leq C_{p,t,\Lambda} \quad (5.10)$$

for some constant $C_{p,t,\Lambda} < \infty$ which is independent of m .

We will apply this estimate with $t > 0$ fixed as in Corollary 4.3. Clearly, the constant $C_{p,t,\Lambda}$ must diverge as $t \rightarrow 0$ or $|\Lambda| \rightarrow \infty$ (as the limiting Gaussian free field is not differentiable as $t \rightarrow 0$ or bounded as $m \rightarrow 0$); these divergences are not important for our application.

First of all, using arguments based on Kolmogorov–Chentsov-type results (see e.g. [46, Appendix B]), one can check that the smoothness of $c_\infty^{m^2} - c_t^{m^2}$ (recall the notation (4.3)) implies that we can regard φ as a smooth function, and $\nabla\varphi$ is a centered Gaussian process with covariance

$$\langle \partial_i \varphi(x) \partial_j \varphi(y) \rangle_{\text{GFF}(\sqrt{t}, m)} = \int_t^\infty ds \frac{e^{-m^2 s}}{4\pi s} \left(\delta_{i,j} \frac{1}{2s} - \frac{(x_i - y_i)(x_j - y_j)}{4s^2} \right) e^{-\frac{|x-y|^2}{4s}}. \quad (5.11)$$

To estimate the exponential moments in Lemma 5.2, we rely on two classical theorems about Gaussian processes. The first one is Dudley’s theorem (see e.g. [1, Theorem 1.3.3]) which states that if for a centered real-valued Gaussian process X on say a compact metric space T we define a new (pseudo) metric by setting $d_X(t, s) = \sqrt{\mathbf{E}[(X(t) - X(s))^2]}$, then

$$\mathbf{E} \left(\sup_{t \in T} X(t) \right) \leq C \int_0^\infty d\varepsilon \sqrt{\log N_X(\varepsilon)}, \quad (5.12)$$

where C is a universal constant, and $N_X(\varepsilon)$ is the minimal number of (closed) d_X -radius ε balls required to cover T .

The second result we need is the Borell-TIS inequality (see e.g. [1, Theorem 2.1.1]), which states that in the same setting as in Dudley’s theorem, if X is further assumed to be almost surely bounded on T , and if $\sigma_T^2 := \sup_{t \in T} \mathbf{E} X(t)^2$, then for all $u > 0$,

$$\mathbf{P} \left(\sup_{t \in T} X(t) - \mathbf{E} \left[\sup_{t \in T} X(t) \right] > u \right) \leq e^{-\frac{u^2}{2\sigma_T^2}}. \quad (5.13)$$

With these tools, we can prove our claim about $\nabla\varphi$.

Proof of Lemma 5.2. First of all, we note that by a simple Cauchy–Schwarz argument, it is enough for us to prove the claim for $\|\partial_0\varphi\|_{L^\infty(\Lambda)}$ (or $\|\partial_1\varphi\|_{L^\infty(\Lambda)}$ as they both have the same distribution) instead of $\|\nabla\varphi\|_{L^\infty(\Lambda)}$. Then noting that

$$\begin{aligned} \langle e^{p\|\partial_0\varphi\|_{L^\infty(\Lambda)}} \rangle_{\text{GFF}(\sqrt{t}, m)} &\leq \langle e^{p \sup_{x \in \Lambda} \partial_0\varphi(x)} \rangle_{\text{GFF}(\sqrt{t}, m)} + \langle e^{p \sup_{x \in \Lambda} (-\partial_0\varphi(x))} \rangle_{\text{GFF}(\sqrt{t}, m)} \\ &= 2 \langle e^{p \sup_{x \in \Lambda} \partial_0\varphi(x)} \rangle_{\text{GFF}(\sqrt{t}, m)}, \end{aligned} \quad (5.14)$$

we see that it is enough to consider only $\sup_{x \in \Lambda} \partial_0\varphi(x)$ instead of $\sup_{x \in \Lambda} |\partial_0\varphi(x)|$. In this set-up we can use Dudley’s theorem and Borell-TIS.

To apply Dudley’s theorem, we note that

$$\begin{aligned} d_{\partial_0\varphi}(x, y)^2 &= 2 \int_t^\infty ds \frac{e^{-m^2 s}}{8\pi s^2} \left(1 - e^{-\frac{|x-y|^2}{4s}} \right) + 2(x_0 - y_0)^2 \int_t^\infty ds \frac{e^{-m^2 s}}{16\pi s^3} e^{-\frac{|x-y|^2}{4s}} \\ &\leq C_t^2 |x - y|^2 \end{aligned} \quad (5.15)$$

for a constant C_t independent of m . Thus we have $\{y \in \Lambda : d_{\varphi_1}(x, y) \leq \varepsilon\} \supset \{y \in \Lambda : C_t|x - y| \leq \varepsilon\}$. So the number of radius- ε $d_{\partial_0\varphi}$ -balls it takes to cover Λ is less than the number of radius- ε/C_t Euclidean balls it takes to cover Λ . It thus follows from Dudley's theorem, (5.12), that $\langle \sup_{x \in \Lambda} \partial_0\varphi(x) \rangle_{\text{GFF}(\sqrt{t}, m)} \leq \tilde{C}_{t, \Lambda}$ for some constant $\tilde{C}_{t, \Lambda}$ which is independent of m .

Now $\sigma_\Lambda^2 = \int_t^\infty ds \frac{e^{-m^2s}}{8\pi s^2} \leq \int_t^\infty \frac{ds}{8\pi s^2} =: \hat{C}_t$. In particular, we have, for say $u > 2\tilde{C}_{t, \Lambda}$,

$$\frac{(u - \langle \sup_{x \in \Lambda} \partial_0\varphi(x) \rangle_{\text{GFF}(\sqrt{t}, m)})^2}{\sigma_\Lambda^2} \geq \frac{(u - \tilde{C}_{t, \Lambda})^2}{\hat{C}_t} \geq \frac{u^2}{4\hat{C}_t}. \quad (5.16)$$

Thus we find from the Borell-TIS inequality (5.13) that, for $u > 2\tilde{C}_{t, \Lambda}$,

$$\nu^{\text{GFF}(\sqrt{t}, m)}\left(\sup_{x \in \Lambda} \partial_0\varphi(x) > u\right) \leq e^{-\frac{u^2}{4\hat{C}_t}} \quad (5.17)$$

and

$$\begin{aligned} \langle e^{p \sup_{x \in \Lambda} \partial_0\varphi(x)} \rangle_{\text{GFF}(\sqrt{t}, m)} &= \int_{\mathbb{R}} du pe^{pu} \nu^{\text{GFF}(\sqrt{t}, m)}\left(\sup_{x \in \Lambda} \partial_0\varphi(x) > u\right) \\ &\leq \int_{-\infty}^{2\tilde{C}_{t, \Lambda}} du pe^{pu} + \int_{2\tilde{C}_{t, \Lambda}}^\infty du pe^{pu} e^{-\frac{u^2}{4\hat{C}_t}}, \end{aligned} \quad (5.18)$$

which yields the desired claim for the exponential moments of $\sup_{x \in \Lambda} \partial_0\varphi(x)$ and by our preliminary considerations, also for $\|\nabla\varphi\|_{L^\infty(\Lambda)}$. This concludes the proof. \blacksquare

5.2. Uniform bounds for the renormalized partition function

We are now ready to turn to analysis of the partition function. We begin with the bounds for \mathcal{Z} stated in the following proposition. The estimate of item (ii) applies to uniformly small coupling constants ζ and is thus a standard consequence of the expansion of the renormalized potential. The estimates of items (i) and (iii) on the other hand apply to arbitrarily large ζ and make in addition use of the Gaussian concentration estimate of Lemma 5.2.

Proposition 5.3. *Fix $\Lambda \subset \mathbb{R}^2$ compact and $\beta \in (0, 6\pi)$.*

(i) *For any fixed $M > 0$,*

$$\sup_{\varepsilon, m \in (0, 1)} \sup_{\substack{\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}) \\ \text{supp}(\zeta(\cdot, \pm 1)) \subset \Lambda \\ \|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq M}} |\mathcal{Z}(\zeta|\varepsilon, m)| < \infty. \quad (5.19)$$

(ii) *There exists a $\delta = \delta_{\Lambda, \beta} > 0$ independent of ε, m such that*

$$\inf_{\varepsilon, m \in (0, 1)} \inf_{\substack{\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}) \\ \text{supp}(\zeta(\cdot, \pm 1)) \subset \Lambda \\ \|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq \delta}} |\mathcal{Z}(\zeta|\varepsilon, m)| > 0. \quad (5.20)$$

(iii) For any fixed $M > 0$,

$$\inf_{\varepsilon, m \in (0, 1)} \inf_{\substack{\xi \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}) \\ \text{supp}(\xi(\cdot, \pm 1)) \subset \Lambda \\ \|\xi\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq M \\ \xi(\cdot, 1) = \xi(\cdot, -1)}} \mathcal{Z}(\xi|\varepsilon, m) > 0. \quad (5.21)$$

Before we turn to the proof, we record a simple estimate that we will have use for in the proof and also later on.

Lemma 5.4. For each $t > 0$ and $\beta \in (0, 6\pi)$, there exists a function $g_t \in L_{\text{loc}}^1(\mathbb{R}^2 \times \mathbb{R}^2)$ which is independent of $\varepsilon, m \in (0, 1)$ such that for all $\xi_1, \xi_2 \in \mathbb{R}^2 \times \{-1, 1\}$, with $A(\xi_1, \xi_2|\varepsilon, m)$ from (5.4) and $\tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m)$ from (4.60),

$$|A(\xi_1, \xi_2|\varepsilon, m) + \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m)| \leq g_t(x_1, x_2). \quad (5.22)$$

Proof. By the definitions in (5.4) and (4.60), we have

$$\begin{aligned} & A(\xi_1, \xi_2|\varepsilon, m) + \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m) \\ &= e^{-\beta \int_{\varepsilon^2}^1 ds \frac{e^{-m^2 s} - 1}{4\pi s} - \beta \int_1^\infty ds \frac{e^{-m^2 s}}{4\pi s}} (e^{-\beta \sigma_1 \sigma_2 \int_{\varepsilon^2}^\infty ds \dot{c}_s^{m^2}(x_1 - x_2)} - 1) \\ &\quad - e^{-\beta \int_{\varepsilon^2}^1 ds \frac{e^{-m^2 s} - 1}{4\pi s} - \beta \int_1^t ds \frac{e^{-m^2 s}}{4\pi s}} (e^{-\beta \sigma_1 \sigma_2 \int_{\varepsilon^2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)} - 1). \end{aligned} \quad (5.23)$$

Note first of all that the $\int_{\varepsilon^2}^1$ -integral is common to both terms and bounded for $m < 1$, so that we can ignore it. Moreover, the contribution of the 1's is also uniformly bounded in m (for any fixed t), so that we can also ignore them. Finally, if $\sigma_1 = \sigma_2$, then also the $e^{-\beta \sigma_1 \sigma_2 \int(\dots)}$ terms are uniformly bounded, so there is nothing to prove in this case. The remaining question is to prove the required estimate for

$$\begin{aligned} & \left| e^{-\beta \int_1^\infty ds \frac{e^{-m^2 s}}{4\pi s}} e^{\beta \int_{\varepsilon^2}^\infty ds \dot{c}_s^{m^2}(x_1 - x_2)} - e^{-\beta \int_1^t ds \frac{e^{-m^2 s}}{4\pi s}} e^{\beta \int_{\varepsilon^2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)} \right| \\ &= e^{\beta \int_{\varepsilon^2}^1 ds \dot{c}_s^{m^2}(x_1 - x_2)} e^{\beta \int_1^t ds \frac{e^{-m^2 s}}{4\pi s}} \left(e^{-\frac{|x_1 - x_2|^2}{4s}} - 1 \right) \left| 1 - e^{\beta \int_t^\infty ds \frac{e^{-m^2 s}}{4\pi s}} \left(e^{-\frac{|x_1 - x_2|^2}{4s}} - 1 \right) \right|. \end{aligned} \quad (5.24)$$

Using repeatedly the estimate $|1 - e^{-x}| \leq x$ for $x > 0$ and Lemma 4.4, along with some elementary considerations, we find that, for some universal constant C ,

$$\begin{aligned} & \left| e^{-\beta \int_1^\infty ds \frac{e^{-m^2 s}}{4\pi s}} e^{\beta \int_{\varepsilon^2}^\infty ds \dot{c}_s^{m^2}(x_1 - x_2)} - e^{-\beta \int_1^t ds \frac{e^{-m^2 s}}{4\pi s}} e^{\beta \int_{\varepsilon^2}^t ds \dot{c}_s^{m^2}(x_1 - x_2)} \right| \\ & \leq C |x_1 - x_2|^2 (|x_1 - x_2| \wedge 1)^{-\frac{\beta}{2\pi}} e^{\frac{\beta}{4\pi} \int_{\min(1, t)}^{\max(1, t)} \frac{ds}{s}} \int_t^\infty \frac{ds}{s^2}. \end{aligned} \quad (5.25)$$

Note that this function is locally integrable for $\beta < 6\pi$ (in fact for $\beta < 8\pi$), so we are done. \blacksquare

With this in hand, we can turn to the proof of Proposition 5.3.

Proof of Proposition 5.3 (i). For $M > 0$ and $\beta \in (0, 6\pi)$, let us choose $t = t_{M,\beta} \in (0, m^{-2})$ independent of ε (so that Lemma 5.2 is applicable) but small enough that Corollary 4.3 is applicable: for C_β as in Proposition 4.1, we assume that

$$eC_\beta M t^{1-\frac{\beta}{8\pi}} \leq 1/2. \quad (5.26)$$

With this choice of t , we use (4.4) and Corollary 4.3 to write for $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ satisfying $\text{supp}(\zeta(\cdot, \pm 1)) \subset \Lambda$, $\|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq M$,

$$\begin{aligned} \mathcal{Z}(\zeta|\varepsilon, m) &= \left\langle e^{-\sum_{n=1}^\infty \frac{1}{n!} \int d\xi \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} [e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}] \right\rangle_{\text{GFF}(\sqrt{t}, m)} \\ &\quad \times e^{-\frac{1}{2} \int_{(\mathbb{R}^2 \times \{-1, 1\})^2} d\xi_1 d\xi_2 \zeta(\xi_1) \zeta(\xi_2) (A(\xi_1, \xi_2|\varepsilon, m) + \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m))}, \end{aligned} \quad (5.27)$$

where $\int d\xi$ stands for $\int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n$. The second factor in (5.27) is bounded (uniformly in ε, m, ζ) by Lemma 5.4, our assumption that $\zeta(\cdot, \pm 1)$ has support in Λ , and our assumption $\|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq M$, so we only need to consider the first factor. For this, Proposition 4.1 yields

$$\begin{aligned} &\left\langle e^{-\sum_{n=1}^\infty \frac{1}{n!} \int d\xi \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} [e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}] \right\rangle_{\text{GFF}(\sqrt{t}, m)} \\ &\leq e^{\sum_{n \neq 2} 2|\Lambda| \frac{M^n}{n!} n^{n-2} (C_\beta t^{1-\frac{\beta}{8\pi}})^n t^{-1}} \\ &\quad \times \left\langle e^{\frac{M^2}{2} \int_{(\Lambda \times \{-1, 1\})^2} d\xi_1 d\xi_2 |\tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m)| |e^{i\sqrt{\beta} \sum_{j=1}^2 \sigma_j \varphi(x_j) - 1}|} \right\rangle_{\text{GFF}(\sqrt{t}, m)}. \end{aligned} \quad (5.28)$$

Since $n^n/n! \leq e^n$, by our choice of t in (5.26), the $n \neq 2$ sum in the first term on the right-hand side above depends only on β, Λ, M (in particular, it does not depend on ε, m, ζ), so it only remains to control the $n = 2$ contribution, i.e., the second term on the right-hand side. For this, we note from (4.60) that the $\sigma_1 = \sigma_2$ contribution is uniformly bounded by a quantity independent of ε, m, ζ , so again using (4.60) and the elementary estimate $|e^{i\sqrt{\beta}(\varphi(x_1) - \varphi(x_2))} - 1| \leq \sqrt{\beta} \|\nabla\varphi\|_{L^\infty(\Lambda)} |x_1 - x_2|$ for $x_1, x_2 \in \Lambda$, we see that, for some constant $C = C_{\beta, \Lambda, M}$ (in particular, independent of ε, m, ζ),

$$\begin{aligned} &\left\langle e^{-\sum_{n=1}^\infty \frac{1}{n!} \int d\xi \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} [e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}] \right\rangle_{\text{GFF}(\sqrt{t}, m)} \leq \\ &C \left\langle e^{\sqrt{\beta} M^2 \|\nabla\varphi\|_{L^\infty(\Lambda)} e^{-\beta \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(0) + \frac{1}{4\pi} \log \varepsilon^2} \int_{\Lambda^2} dx_1 dx_2 |e^{\beta c_t^{m^2}(x_1 - x_2)} - 1| |x_1 - x_2|} \right\rangle_{\text{GFF}(\sqrt{t}, m)}. \end{aligned} \quad (5.29)$$

From Lemma 4.4, we see that $\int_{\Lambda^2} dx_1 dx_2 |e^{\beta c_t^{m^2}(x_1 - x_2)} - 1| |x_1 - x_2|$ can be bounded by a quantity depending only on Λ, t, β , while on the other hand, recalling (4.59) (and that we chose in addition to (5.26) that $t \leq m^{-2}$)

$$e^{-\beta \int_{\varepsilon^2}^t dr \dot{c}_r^{m^2}(0) + \frac{1}{4\pi} \log \varepsilon^2} \leq \tilde{C}_\beta t^{-(1-\frac{\beta}{8\pi})} \quad (5.30)$$

for a constant \tilde{C}_β depending only on β . In summary,

$$\begin{aligned} &\left\langle e^{-\sum_{n=1}^\infty \frac{1}{n!} \int d\xi \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} [e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}] \right\rangle_{\text{GFF}(\sqrt{t}, m)} \\ &\leq C \left\langle e^{\beta \|\nabla\varphi\|_{L^\infty(\Lambda)}} \right\rangle_{\text{GFF}(\sqrt{t}, m)}, \end{aligned} \quad (5.31)$$

for some constants $C = C_{\beta, \Lambda, M}$ and $p = p_{\beta, \Lambda, M}$ independent of ε, m, ζ . Thus Lemma 5.2 shows that the expectation part of (5.27) is bounded by a quantity independent of m, ε, ζ . ■

Proof of Proposition 5.3 (ii). Consider first $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ with $\|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} = 1$ (say) and let us look at the function $z \mapsto \mathcal{Z}(z\zeta|\varepsilon, m)$. Since for $z = 0$, $\mathcal{Z}(z\zeta|\varepsilon, m) = 1$, it will turn out to be sufficient to bound the derivative of $\mathcal{Z}(z\zeta|\varepsilon, m)$ uniformly in some neighborhood of the origin – then taking a small enough neighborhood (independent of ε, m), we can bound the distance to zero in this neighborhood. Translating this into a statement about ζ with small enough L^∞ -norm will follow from a scaling argument.

For the derivative, note that for $\varepsilon, m > 0$, the relevant random variables are deterministically bounded, so $\mathcal{Z}(z\zeta|\varepsilon, m)$ is an entire function. Thus for any compact $K' \subset B(0, R)$ and $z \in K'$, by Cauchy's integral formula we have

$$\frac{d}{dz} \mathcal{Z}(z\zeta|\varepsilon, m) = \frac{2}{2\pi i} \oint_{|w|=R} dw \frac{\mathcal{Z}(w\zeta|\varepsilon, m)}{(w-z)^2}. \quad (5.32)$$

By Proposition 5.3 (i), we can bound the numerator uniformly in ε, m, ζ (recall that we normalized $\|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} = 1$) and we can assume $|w-z|$ to be uniformly bounded from below, so we see that also the derivative is uniformly bounded in compact subsets. Thus there exists some $\delta > 0$ (independent of ε, m, ζ) for which we have

$$\inf_{\varepsilon, m \in (0, 1)} \inf_{|z| < \delta} \inf_{\substack{\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}) \\ \text{supp}(\zeta(\cdot, \pm 1)) \subset \Lambda \\ \|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} = 1}} |\mathcal{Z}(z\zeta|\varepsilon, m)| > 0. \quad (5.33)$$

By scaling, we note that this can be translated into precisely the claim of the proposition. ■

Proof of Proposition 5.3 (iii). Note from (4.6) that under the condition $\zeta(x, 1) = \overline{\zeta(x, -1)}$, the renormalized potential $v_t(\zeta, \cdot|\varepsilon, m)$ is real. Thus making the same choices as in the proof of Proposition 5.3 (i), we can write

$$\begin{aligned} & \mathcal{Z}(\zeta|\varepsilon, m) \\ & \geq \left\langle e^{-\sum_{n=1}^{\infty} \frac{1}{n!} \int d\xi_1 \cdots d\xi_n |\zeta(\xi_1) \cdots \zeta(\xi_n)| \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} [e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j)} - \delta_{n,2}] \right\rangle_{\text{GFF}(\sqrt{t}, m)} \\ & \quad \times e^{-\int d\xi_1 d\xi_2 |\zeta(\xi_1)\zeta(\xi_2)| (A(\xi_1, \xi_2|\varepsilon, m) + \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m))}. \end{aligned} \quad (5.34)$$

We can now argue exactly as in the proof of Proposition 5.3 (i): the sum of the $n \neq 2$ terms is deterministically bounded uniformly in ε, m, ζ . Similarly, Lemma 5.4 lets us deduce that that the $A + \tilde{v}_t^2(\cdot|\varepsilon, m)$ term can be bounded from below by a uniform constant. It remains to argue that

$$\left\langle e^{-\frac{1}{2} \int d\xi_1 d\xi_2 |\zeta(\xi_1)\zeta(\xi_2)| \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m)} [e^{i\sqrt{\beta} \sum_{j=1}^2 \sigma_j \varphi(x_j)} - 1] \right\rangle_{\text{GFF}(\sqrt{t}, m)} > C > 0 \quad (5.35)$$

for some constant C independent of ε, m, ζ . As in the proof of Proposition 5.3, this follows from Lemma 5.2, though now combined with Jensen's inequality (used in the form $\mathbb{E} \frac{1}{X} \geq \frac{1}{\mathbb{E} X}$ for a positive random variable X). This concludes the proof. ■

Before turning to the proof of Theorem 5.1, we introduce some notation and make some preliminary remarks about the renormalized partition function as an analytic function.

5.3. Expansion of the renormalized partition function

Our proof of convergence of the renormalized partition function and entirety of the limit will go through analyzing the series expansion of $z \mapsto \mathcal{Z}(z\zeta|\varepsilon, m)$ and making use of the estimates from Propositions 4.1 and 5.3. For this purpose, we introduce some notation for the series expansion of $z \mapsto \mathcal{Z}(z\zeta|\varepsilon, m)$. We formulate this as the following lemma.

Lemma 5.5. *For fixed $\varepsilon, m > 0$ and $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$, the function $z \mapsto \mathcal{Z}(z\zeta|\varepsilon, m)$ is entire and we have*

$$\mathcal{Z}(z\zeta|\varepsilon, m) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{M}_n(\zeta|\varepsilon, m), \tag{5.36}$$

where

$$\mathcal{M}_n(\zeta|\varepsilon, m) = \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m) \tag{5.37}$$

with (recall the definition of A from (5.4))

$$\begin{aligned} \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m) &= \frac{1}{n!} \sum_{\tau \in S_n} \left[\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} (-1)^{n-j} 2^{-j} \right. \\ &\quad \left. \times \left(\prod_{l=1}^j A(\xi_{\tau_{2l-1}}, \xi_{\tau_{2l}}|\varepsilon, m) \right) \left\langle \prod_{l'=2j+1}^n :e^{i\sqrt{\beta}\sigma_{\tau_{l'}}\varphi(x_{\tau_{l'}})} :_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)} \right], \end{aligned} \tag{5.38}$$

where S_n denotes the group of permutations of the set $[n]$. Moreover, for each $\delta, M > 0$ and $\Lambda \subset \mathbb{R}^2$ compact, there exists a constant $C(\delta, M, \Lambda)$ independent of ε, m, ζ, n such that

$$\sup_{\varepsilon, m \in (0, 1)} \sup_{\substack{\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}) \\ \text{supp}(\zeta(\cdot, \pm 1)) \subset \Lambda \\ \|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq M}} |\mathcal{M}_n(\zeta|\varepsilon, m)| \leq C(\delta, M, \Lambda) \delta^n n!. \tag{5.39}$$

Proof. Let us recall from (5.1) and (5.5) that

$$\begin{aligned} \mathcal{Z}(z\zeta|\varepsilon, m) &= \left\langle e^{-z \int_{\mathbb{R}^2 \times \{-1, 1\}} d\xi \zeta(\xi) :e^{i\sqrt{\beta}\sigma\varphi(x)} :_{\varepsilon}} \right\rangle_{\text{GFF}(\varepsilon, m)} \\ &\quad \times e^{-\frac{z^2}{2} \int_{(\mathbb{R}^2 \times \{-1, 1\})^2} d\xi_1 d\xi_2 \zeta(\xi_1)\zeta(\xi_2) A(\xi_1, \xi_2|\varepsilon, m)}. \end{aligned} \tag{5.40}$$

As mentioned in the proof of Proposition 5.3, for each $\varepsilon, m > 0$, the expectation is an entire function of z since

$$\int_{\mathbb{R}^2 \times \{-1, 1\}} d\xi \zeta(\xi) :e^{i\sqrt{\beta}\sigma\varphi(x)} :_{\varepsilon} \tag{5.41}$$

is a bounded random variable. The second factor in (5.40) is trivially an entire function of z (for any fixed $\varepsilon, m > 0$). Thus we see that indeed $\mathcal{Z}(z\zeta|\varepsilon, m)$ is entire.

For the expansion coefficients, by series expanding both terms (and interchanging the order of summation/integration and expectation which is justified by the boundedness of the relevant random variables and a routine Fubini argument), we find

$$\begin{aligned} \mathcal{Z}(z\zeta|\varepsilon, m) &= \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{2^j j!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^{2j}} d\xi_1 \cdots d\xi_{2j} \zeta(\xi_1) \cdots \zeta(\xi_{2j}) \prod_{l=1}^j A(\xi_{2l-1}, \xi_{2l}|\varepsilon, m) \\ &\quad \times \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} \int_{(\mathbb{R}^2 \times \{-1, 1\})^k} d\xi'_1 \cdots d\xi'_k \zeta(\xi'_1) \cdots \zeta(\xi'_k) \left\langle \prod_{l'=1}^k :e^{i\sqrt{\beta} \sigma_{l'} \varphi(x_{l'})} :_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}. \end{aligned} \quad (5.42)$$

The claim about the representation of $\mathcal{M}_n(\zeta|\varepsilon, m)$ now follows by relabeling our integration variables (write $n = k + 2j$ and $\xi'_{l'} = \xi_{2j+l'}$) and then symmetrizing in the arguments.

Finally, for the proof of the bound (5.39), note that as $\mathcal{Z}(z\zeta|\varepsilon, m)$ is entire, Cauchy's integral formula implies that, for any $R > 0$,

$$\mathcal{M}_n(\zeta|\varepsilon, m) = \frac{n!}{2\pi i} \oint_{|w|=R} dw \frac{\mathcal{Z}(w\zeta|\varepsilon, m)}{w^{n+1}}. \quad (5.43)$$

The claim now follows by choosing $R = \delta^{-1}$ and (by Proposition 5.3)

$$\begin{aligned} C(\delta, M, \Lambda) &\geq \sup_{\varepsilon, m \in (0, 1)} \sup_{\substack{\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\}) \\ \text{supp}(\zeta(\cdot, \pm 1)) \subset \Lambda \\ \|\zeta\|_{L^\infty(\mathbb{R}^2 \times \{-1, 1\})} \leq \delta^{-1} M}} |\mathcal{Z}(\zeta|\varepsilon, m)|. \end{aligned} \quad (5.44) \quad \blacksquare$$

In the course of our proof of convergence of $\mathcal{Z}(\zeta|\varepsilon, m)$, we will have use for an alternative representation for $\mathcal{M}_n(\zeta|\varepsilon, m)$ in terms of the renormalized potential. To control the kernel $\tilde{\mathcal{M}}$ in terms of this alternative representation, we record the following simple fact.

Lemma 5.6. *For $\varepsilon, m > 0$, $\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m)$ is the unique continuous function of ξ_1, \dots, ξ_n for which*

$$\begin{aligned} &\frac{1}{n!} \frac{\partial}{\partial \delta_1} \cdots \frac{\partial}{\partial \delta_n} \mathcal{M}_n(\delta_1 f_1 + \cdots + \delta_n f_n|\varepsilon, m) \\ &= \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m) \end{aligned} \quad (5.45)$$

for all $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$.

Proof. Uniqueness can be seen, for example, by choosing f_i to be of the form

$$e^{2\pi i k_1 x_1/2L} e^{2\pi i k_2 x_2/2L} \mathbf{1}_{\{|x_1|, |x_2| \leq L\}} \quad (5.46)$$

in the x -variables and to be a Kronecker δ in the σ -variable. This shows that any two functions $\tilde{\mathcal{M}}$ satisfying this relation have the same Fourier series in an arbitrary square, so by continuity they must be the same in this square. Since the square was arbitrary, they must be the same in all of \mathbb{R}^2 .

To see that $\tilde{\mathcal{M}}$ actually satisfies this relation, write

$$\mathcal{M}_n\left(\sum_{l=1}^n \delta_l f_l \mid \varepsilon, m\right) = \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n \prod_{j=1}^n \left(\sum_{l=1}^n \delta_l f_l(\xi_j)\right) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n \mid \varepsilon, m). \quad (5.47)$$

Say the δ_n -derivative can hit any of the n terms in the j -product and produce a factor of $f_n(\xi_j)$ for some j . The δ_{n-1} -derivative can hit any of the $n-1$ remaining terms in the j -product (and produce a factor of $f_{n-1}(\xi_{j'})$ for some $j' \neq j$) etc. We see that

$$\begin{aligned} & \frac{\partial}{\partial \delta_1} \cdots \frac{\partial}{\partial \delta_n} \mathcal{M}_n(\delta_1 f_1 + \cdots + \delta_n f_n \mid \varepsilon, m) \\ &= \sum_{\tau \in S_n} \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_{\tau_1}) \cdots f_n(\xi_{\tau_n}) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n \mid \varepsilon, m) \\ &= n! \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n \mid \varepsilon, m), \end{aligned} \quad (5.48)$$

where in the last step we simply relabeled our integration variables. \blacksquare

As a final ingredient before we turn to proving convergence of the renormalized partition function, we will construct an integrable upper bound on $|\tilde{\mathcal{M}}|$ by representing $\mathcal{M}_n(\zeta \mid \varepsilon, m)$ in terms of the renormalized partition function. Before stating the bound, we state the representation of $\mathcal{M}_n(\zeta \mid \varepsilon, m)$ in terms of the renormalized potential $v_t(\zeta, \cdot \mid \varepsilon, m)$.

Lemma 5.7. *Let $\beta \in (0, 6\pi)$, let $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$, and choose $t \in (\varepsilon^2, m^{-2})$ as in Corollary 4.3. Then $\mathcal{M}_n(\zeta \mid \varepsilon, m)$ defined in Lemma 5.5 can be expressed as*

$$\begin{aligned} \mathcal{M}_n(\zeta \mid \varepsilon, m) &= \int d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \\ &\times \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!} 2^{-j} (-1)^j \prod_{l=1}^j (A(\xi_{2l-1}, \xi_{2l} \mid \varepsilon, m) + \tilde{v}_t^2(\xi_{2l-1}, \xi_{2l} \mid \varepsilon, m)) \\ &\times \sum_{k=1}^{n-2j} \frac{(-1)^k}{k!} \sum_{\substack{1 \leq n_1, \dots, n_k \leq n-2j \\ n_1 + \dots + n_k = n-2j}} \frac{1}{n_1! \cdots n_k!} \\ &\times \prod_{l=1}^k \tilde{v}_t^{n_l}(\xi_{2j+n_1+\dots+n_{l-1}+1}, \dots, \xi_{2j+n_1+\dots+n_l} \mid \varepsilon, m) \\ &\times \left\langle \prod_{l=1}^k (e^{i\sqrt{\beta} \sum_{p=2j+n_1+\dots+n_{l-1}+1}^{\sigma_p} \varphi(x_p)} - \delta_{n_l, 2}) \right\rangle_{\text{GFF}(\sqrt{t}, m)}, \end{aligned} \quad (5.49)$$

with the interpretation that if $n = 2j$, then the k -sum equals 1.

Proof. From Corollary 4.3, we have (for our choice of t)

$$\mathcal{Z}(z|\varepsilon, m) = F_1(z)F_2(z), \quad (5.50)$$

where

$$F_1(z) := e^{-\frac{z^2}{2} \int d\xi_1 d\xi_2 \zeta(x_1)\zeta(x_2)(A(\xi_1, \xi_2|\varepsilon, m) + \tilde{v}_t^2(\xi_1, \xi_2|\varepsilon, m))}, \quad (5.51)$$

$$F_2(z) := \left\langle e^{-\sum_{n=1}^{\infty} \frac{z^n}{n!} \int d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} (e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}) \right\rangle_{\text{GFF}(\sqrt{t}, m)}. \quad (5.52)$$

Note that F_1 is entire and nonvanishing, so by Lemma 5.5, also F_2 is entire, and for any fixed $R > 0$ we have

$$F_2^{(k)}(0) = \frac{k!}{2\pi i} \oint_{|z|=R} dz \frac{F_2(z)}{z^{k+1}}. \quad (5.53)$$

By Fubini and the proof of Proposition 5.3 (more precisely, Fubini is readily justified by controlling the $n \neq 2$ terms with Proposition 4.1 and the $n = 2$ term with Lemma 5.2), we find that if R is chosen to satisfy (5.26), then

$$F_2^{(k)}(0) = \frac{k!}{2\pi i} \left\langle \oint_{|z|=R} \frac{dz}{z^{k+1}} \times e^{-\sum_{n=1}^{\infty} \frac{z^n}{n!} \int d\xi \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} (e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}) \right\rangle_{\text{GFF}(\sqrt{t}, m)}. \quad (5.54)$$

Moreover, again by Proposition 4.1, the z -integrand is an entire (random) function and we find

$$\begin{aligned} F_2^{(k)}(0) &= \left\langle \frac{d^k}{dz^k} \Big|_{z=0} e^{-\sum_{n=1}^{\infty} \frac{z^n}{n!} \int d\xi \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{v}_t^n(\xi_1, \dots, \xi_n|\varepsilon, m)} (e^{i\sqrt{\beta} \sum_{j=1}^n \sigma_j \varphi(x_j) - \delta_{n,2}}) \right\rangle_{\text{GFF}(\sqrt{t}, m)} \\ &= \sum_{l=1}^k \frac{(-1)^l}{l!} \sum_{\substack{1 \leq n_1, \dots, n_l \leq k \\ n_1 + \dots + n_l = k}} \frac{k!}{n_1! \cdots n_l!} \left\langle \prod_{l=1}^k \left(\int_{(\mathbb{R}^2 \times \{-1, 1\})^{n_l}} d\xi_1 \cdots d\xi_{n_l} \zeta(\xi_1) \cdots \zeta(\xi_{n_l}) \right. \right. \\ &\quad \left. \left. \times \tilde{v}_t^{n_l}(\xi_1, \dots, \xi_{n_l}|\varepsilon, m) (e^{i\sqrt{\beta} \sum_{j=1}^{n_l} \sigma_j \varphi(x_j) - \delta_{n_l, 2}}) \right) \right\rangle_{\text{GFF}(\sqrt{t}, m)} \end{aligned} \quad (5.55)$$

with the interpretation that if $k = 0$, then $F_2^{(k)}(0) = 1$. The claim now follows from noting that

$$\mathcal{M}_n(\zeta|\varepsilon, m) = \frac{d^n}{dz^n} \Big|_{z=0} F_1(z)F_2(z) \quad (5.56)$$

and relabeling integration variables suitably. \blacksquare

To conclude this section, we use this representation to prove that $\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m)$ has an integrable upper bound which is independent of ε, m (allowing the use of dominated convergence).

Lemma 5.8. For $\beta \in (0, 6\pi)$, $n \geq 1$, and $t > 0$ there exists $g_t \in L^1_{\text{loc}}((\mathbb{R}^2 \times \{-1, 1\})^n)$, independent of ε, m , such that for $0 < \varepsilon^2 < t < m^{-2}$,

$$|\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n | \varepsilon, m)| \leq g_t(\xi_1, \dots, \xi_n). \quad (5.57)$$

Proof. By Lemma 5.7, we see that $|\tilde{\mathcal{M}}|$ can be bounded by a sum of products of terms of the form

$$G_1(\hat{\xi}_1, \hat{\xi}_2 | \varepsilon, m) := |A(\hat{\xi}_1, \hat{\xi}_2 | \varepsilon, m) + \tilde{v}_t^2(\hat{\xi}_1, \hat{\xi}_2 | \varepsilon, m)|, \quad (5.58)$$

$$G_2(\xi'_1, \dots, \xi'_{n'} | \varepsilon, m) := |\tilde{v}_t^{n'}(\xi'_1, \dots, \xi'_{n'} | \varepsilon, m)| \quad (5.59)$$

with $2 \neq n' \leq n$, and

$$G_3(\tilde{\xi}_1, \dots, \tilde{\xi}_{2k} | \varepsilon, m) := \left\langle \prod_{l=1}^k |\tilde{v}_t^2(\tilde{\xi}_{2l-1}, \tilde{\xi}_{2l} | \varepsilon, m) (e^{i\sqrt{\beta}(\tilde{\sigma}_{2l-1}\varphi(\tilde{x}_{2l-1}) + \tilde{\sigma}_{2l}\varphi(\tilde{x}_{2l}))} - 1)| \right\rangle_{\text{GFF}(\sqrt{t}, m)} \quad (5.60)$$

where we have written $\hat{\xi}, \xi', \tilde{\xi}$ to indicate that these variables are subsets (depending on the term in the sum) of the actual integration variables ξ_i of $\tilde{\mathcal{M}}$. Note that in each product of the G_i , the factors depend on disjoint sets of the ξ_i . Thus it is enough to prove the corresponding integrability bounds for the G_i terms separately.

For G_1 , this is simply Lemma 5.4. For G_2 , this follows from Proposition 4.1. Finally, for G_3 , we note that if $\tilde{\sigma}_{2l-1} = \tilde{\sigma}_{2l}$, we can bound

$$|e^{i\sqrt{\beta}(\tilde{\sigma}_{2l-1}\varphi(\tilde{x}_{2l-1}) + \tilde{\sigma}_{2l}\varphi(\tilde{x}_{2l}))} - 1| \leq 2$$

and from (4.60), one readily gets the uniform bound

$$|\tilde{v}_t^2(\tilde{\xi}_{2l-1}, \tilde{\xi}_{2l} | \varepsilon, m)| \leq e^\beta \int_0^1 ds \frac{1-e^{-m^2s}}{4\pi s} \leq e^\beta \int_0^1 ds \frac{1-e^{-s}}{4\pi s}. \quad (5.61)$$

It remains to control the quantities with $\tilde{\sigma}_{2l-1} \neq \tilde{\sigma}_{2l}$. For these we note (as in the proof of Proposition 5.3), that for x_{2l-1}, x_{2l} in a given compact set $\Lambda \subset \mathbb{R}^2$, we find from (4.60) that

$$\begin{aligned} & |\tilde{v}_t^2(\tilde{\xi}_{2l-1}, \tilde{\xi}_{2l} | \varepsilon, m) (e^{i\sqrt{\beta}(\varphi(\tilde{x}_{2l-1}) - \varphi(\tilde{x}_{2l}))} - 1)| \\ & \leq \sqrt{\beta} \|\nabla\varphi\|_{L^\infty(\Lambda)} |\tilde{x}_{2l-1} - \tilde{x}_{2l}| e^\beta \int_0^1 ds \frac{1-e^{-s}}{4\pi s} |e^{\beta c_l^{m^2}(\tilde{x}_{2l-1} - \tilde{x}_{2l})} - 1| \\ & \leq \sqrt{\beta} \|\nabla\varphi\|_{L^\infty(\Lambda)} |\tilde{x}_{2l-1} - \tilde{x}_{2l}| e^\beta \int_0^1 ds \frac{1-e^{-s}}{4\pi s} \left(1 + \left(\frac{|\tilde{x}_{2l-1} - \tilde{x}_{2l}|}{\sqrt{t}} \wedge 1 \right)^{-\frac{\beta}{2\pi}} \right). \end{aligned} \quad (5.62)$$

By Lemma 5.2, arbitrary moments of $\|\nabla\varphi\|_{L^\infty(\Lambda)}$ under $\nu^{\text{GFF}(\sqrt{t}, m)}$ are bounded uniformly in m , so using the fact that $|x - y|^{1-\beta/2\pi}$ is locally integrable (for $\beta < 6\pi$), one gets a locally integrable upper bound which is independent of ε, m also for G_3 . This concludes the proof. \blacksquare

5.4. Convergence of the renormalized partition function

We now turn to the convergence of $\mathcal{Z}(\zeta|\varepsilon, m)$ as $\varepsilon, m \rightarrow 0$. With the uniform bounds from Section 5.3 in place, the main step is to show that $\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m)$ has a pointwise limit as $\varepsilon, m \rightarrow 0$. This is the content of the following lemma. For $\xi_1, \dots, \xi_n \in \mathbb{R} \times \{\pm 1\}$ distinct, let

$$\begin{aligned} \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n) &= \frac{1}{n!} \sum_{\tau \in S_n} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{n!}{j!(n-2j)!} (-1)^{n-j} 2^{-j} \\ &\quad \times \left(\prod_{l=1}^j \langle :e^{i\sqrt{\beta} \sigma_{\tau_{2l-1}} \varphi(x_{\tau_{2l-1}})} : : e^{i\sqrt{\beta} \sigma_{\tau_{2l}} \varphi(x_{\tau_{2l}})} : \rangle_{\text{GFF}} \right) \\ &\quad \times \left\langle \prod_{l'=2j+1}^n :e^{i\sqrt{\beta} \sigma_{\tau_{l'}} \varphi(x_{\tau_{l'}})} : \right\rangle_{\text{GFF}} \end{aligned} \quad (5.63)$$

and define $\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|m)$ analogously with $\text{GFF}(m)$ instead of GFF .

Lemma 5.9. *If $\xi_i \neq \xi_j$ for $i \neq j$, then*

$$\lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m) = \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|m), \quad (5.64)$$

$$\lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n|\varepsilon, m) = \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n). \quad (5.65)$$

Moreover, $\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n) = 0$ if n is odd.

Proof. The convergence follows immediately from the definition of $\tilde{\mathcal{M}}(\cdot|\varepsilon, m)$ in (5.38) and Lemma 2.5 for the charge correlation functions of the GFF.

That $\tilde{\mathcal{M}}(\xi_1, \dots, \xi_n) = 0$ if n is odd follows from the case $n \neq n'$ in Lemma 2.5, i.e., the fact that the massless GFF charge correlation functions vanish for nonneutral charge. ■

With this result in hand, we can prove Theorem 5.1.

Proof of Theorem 5.1. We only consider the statements for $\varepsilon \rightarrow 0$ and $m \rightarrow 0$; the ones with $m > 0$ fixed are completely analogous. Let us begin by defining our limit candidate. By Lemmas 5.8–5.9, with $\tilde{\mathcal{M}}$ defined in (5.63), we see that

$$\lim_{m \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \mathcal{M}_n(\zeta|\varepsilon, m) = \int d\xi_1 \cdots d\xi_n \zeta(\xi_1) \cdots \zeta(\xi_n) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_n) =: \mathcal{M}_n(\zeta). \quad (5.66)$$

and this quantity is finite for all $\zeta \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$. In fact, this argument also shows that if $\zeta_\alpha(\cdot|\varepsilon, m) \rightarrow \zeta_\alpha$ in $L^\infty(\mathbb{R}^2 \times \{-1, 1\})$ and uniformly in α in some compact set, then also $\mathcal{M}_n(\zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m) \rightarrow \mathcal{M}_n(\zeta_\alpha)$ uniformly in α . Moreover, from (5.39), we see that if α is in some fixed compact set $K \subset \mathbb{C}^N$, then for each $\delta > 0$, $|\mathcal{M}_n(\zeta_\alpha)| \leq C(\delta, K) \delta^n n!$. In particular,

$$\mathcal{Z}(z\zeta_\alpha) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathcal{M}_n(\zeta_\alpha) \quad (5.67)$$

defines an entire function and we have, for any fixed $R > 0$,

$$\sup_{|z| < R} \sup_{\alpha \in K} |\mathcal{Z}(z \zeta_\alpha)| < \infty. \quad (5.68)$$

Moreover, when $m = 0$, $\mathcal{Z}(z \zeta_\alpha)$ is even since $\mathcal{M}_n(\zeta_\alpha) = 0$ if n is odd.

Let us then turn to the convergence claim. Fix any compact set $K \subset \mathbb{C}^N$ as in the statement of item (iv). By Cauchy's integral formula, for any $A \geq 0$ and $\alpha \in K$,

$$\begin{aligned} & |\mathcal{Z}(\zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m) - \mathcal{Z}(\zeta_\alpha)| \\ & \leq \sum_{n=0}^A \frac{1}{n!} |\mathcal{M}_n(\zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m) - \mathcal{M}_n(\zeta_\alpha)| \\ & \quad + \sum_{n=A+1}^{\infty} \left| \oint_{|w|=2} \frac{\mathcal{Z}(w \zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m)}{w^{n+1}} \frac{dw}{2\pi i} \right| + \sum_{n=A+1}^{\infty} \left| \oint_{|w|=2} \frac{\mathcal{Z}(w \zeta_\alpha)}{w^{n+1}} \frac{dw}{2\pi i} \right| \\ & \leq \sum_{n=0}^A \frac{1}{n!} |\mathcal{M}_n(\zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m) - \mathcal{M}_n(\zeta_\alpha)| \\ & \quad + \left(\sup_{|w|=2} |\mathcal{Z}(w \zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m)| + \sup_{|w|=2} |\mathcal{Z}(w \zeta_\alpha)| \right) \left(\frac{1}{2} \right)^A. \end{aligned} \quad (5.69)$$

Uniform convergence now follows readily from our uniform bounds for the partition functions as well as our remark that $\mathcal{M}_n(\zeta_\alpha(\cdot|\varepsilon, m)|\varepsilon, m)$ converges uniformly. This takes care of statements (i), (ii), and (iv) in Theorem 5.1. The final positivity claim, claim (iii), follows from Proposition 5.3 (iii). ■

5.5. Proof of Lemma 2.8

Before we go into the proof of Theorem 3.1, we point out here that Lemma 5.8 can be used to give a proof of Lemma 2.8.

Proof of Lemma 2.8. Let us begin by noting that from multilinearity of the truncated correlation functions as well as the definition (A.1), for any $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$ we have

$$\begin{aligned} & \int_{(\mathbb{R}^2 \times \{-1, 1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} :_\varepsilon \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ & = \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi \cdot_\varepsilon}(f_k) \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ & = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \Bigg|_{t=0} \log \left\langle e^{\sum_{k=1}^n t_k :e^{i\sqrt{\beta} \sigma_k \varphi \cdot_\varepsilon}(f_k)} \right\rangle_{\text{GFF}(\varepsilon, m)} \\ & = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \Bigg|_{t=0} \log Z(-t_1 f_1 \cdots -t_n f_n | \varepsilon, m), \end{aligned} \quad (5.70)$$

where for $f \in L_c^\infty(\mathbb{R}^2 \times \{-1, 1\})$, recall that we understand $:e^{i\sqrt{\beta} \sigma \varphi \cdot_\varepsilon}(f)$ as shorthand notation for $\int_{\mathbb{R}^2 \times \{-1, 1\}} d\xi f(\xi) :e^{i\sqrt{\beta} \sigma \varphi(x)} :_\varepsilon$. For $n \geq 3$, we can write this in terms of the

renormalized partition function as

$$\begin{aligned} \int_{(\mathbb{R}^2 \times \{-1,1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} :_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \Big|_{t=0} \log \mathcal{Z}(-t_1 f_1 \cdots - t_n f_n | \varepsilon, m). \end{aligned} \quad (5.71)$$

On the other hand, by using the expansion (with the notation (5.66)–(5.67))

$$\mathcal{Z}(-t_1 f_1 - \cdots - t_n f_n | \varepsilon, m) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \mathcal{M}_i(t_1 f_1 + \cdots + t_n f_n | \varepsilon, m), \quad (5.72)$$

we have (for small enough $|t_k|$)

$$\begin{aligned} \log \mathcal{Z}(-t_1 f_1 \cdots - t_n f_n | \varepsilon, m) \\ = \log \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_{(\mathbb{R}^2 \times \{-1,1\})^i} d\xi_1 \cdots d\xi_i \prod_{k=1}^i \left(\sum_{l=1}^n t_l f_l(\xi_k) \right) \tilde{\mathcal{M}}(\xi_1, \dots, \xi_i | \varepsilon, m) \right). \end{aligned} \quad (5.73)$$

Carrying out the t -derivatives, we see that

$$\begin{aligned} \int_{(\mathbb{R}^2 \times \{-1,1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} :_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}^T \\ = \int_{(\mathbb{R}^2 \times \{-1,1\})^n} d\xi_1 \cdots d\xi_n f_1(\xi_1) \cdots f_n(\xi_n) \mathcal{P}(\xi_1, \dots, \xi_n | \varepsilon, m), \end{aligned} \quad (5.74)$$

where \mathcal{P} can be expressed in terms of $\tilde{\mathcal{M}}$, i.e., for some constants c_P , we have

$$\mathcal{P}(\xi_1, \dots, \xi_n | \varepsilon, m) = \sum_{P \in \mathfrak{P}_n} c_P \prod_j \tilde{\mathcal{M}}((\xi_l)_{l \in P_j} | \varepsilon, m). \quad (5.75)$$

As the f_k are arbitrary, this allows us to identify (for $n \geq 3$)

$$\left\langle \prod_{k=1}^n :e^{i\sqrt{\beta} \sigma_k \varphi(x_k)} :_{\varepsilon} \right\rangle_{\text{GFF}(\varepsilon, m)}^T = \mathcal{P}(\xi_1, \dots, \xi_n | \varepsilon, m), \quad (5.76)$$

and the convergence and local integrability, for $n \geq 3$, follow immediately from Lemma 5.9 respectively Lemma 5.8.

It remains to consider the $n = 1$ and $n = 2$ cases. The $n = 1$ case is trivial, while for $n = 2$ the statements are straightforward to check due to the assumptions that K is supported away from the diagonal respectively that the test functions have disjoint support; we omit further details. ■

5.6. Analysis of the sine-Gordon correlation functions

We are finally in a position to prove our main result concerning the sine-Gordon correlation functions, i.e., prove Theorem 3.1.

In preparation of the proof, one readily checks that since we are dealing with Gaussian random variables and bounded random variables, for any $\varepsilon, m > 0$ and $\Lambda \subset \mathbb{R}^2$ compact, the function

$$\begin{aligned} & (\mu_1, \dots, \mu_n, \nu_1, \dots, \nu_q, \eta_1, \dots, \eta_{q'}, z) \\ & \mapsto \log \left\langle \exp \left[\sum_{k=1}^n \mu_k : e^{i\sqrt{\beta} \sigma_k \varphi} :_{\varepsilon}(f_k) + \sum_{j=1}^q \nu_j \partial \varphi(g_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial} \varphi(h_{j'}) \right] \right\rangle_{SG(\beta, z | \varepsilon, m, \Lambda)} \end{aligned} \quad (5.77)$$

is analytic in some neighborhood of the origin (which may a priori depend on ε, m, Λ) and the correlation function of interest is obtained from it by differentiating once with respect to each $\mu_k, \nu_j, \eta_{j'}$, and then setting these parameters to zero. Our goal is to prove that (after suitable renormalization) this function is actually analytic in a larger domain that does not depend on ε, m , and that it converges uniformly in the relevant parameters. The limiting function will then automatically be analytic in the given domain, and also the relevant derivatives will converge. In particular, as we will eventually see, this will imply the convergence of the correlation functions, and that they are also analytic in z .

We begin by applying the Girsanov–Cameron–Martin theorem in a similar way as in Lemma 2.6. We now need the following version for Gaussian fields on \mathbb{R}^2 : Let φ be a smooth Gaussian field on \mathbb{R}^2 and let Y be a Gaussian random variable measurable with respect to $(\varphi(x))_{x \in \mathbb{R}^d}$. Then

$$\mathbb{E}(F(\varphi) e^{Y - \mathbb{E}Y^2}) = \mathbb{E}(F(\varphi + \mathbb{E}(\varphi Y))), \quad (5.78)$$

where $\mathbb{E}(\varphi Y)$ stands for the function $x \mapsto \mathbb{E}(\varphi(x) Y)$; see, e.g., [18, Theorem 2.8] for a more general setting. This implies that for real z (the application for complex arguments in the exponential below is justified as in the proof of Lemma 2.6)

$$\begin{aligned} & \left\langle \exp \left[\sum_{k=1}^n \mu_k : e^{i\sqrt{\beta} \sigma_k \varphi} :_{\varepsilon}(f_k) + \sum_{j=1}^q \nu_j \partial \varphi(g_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial} \varphi(h_{j'}) \right] \right\rangle_{SG(\beta, z | \varepsilon, m, \Lambda)} \\ & = \frac{1}{Z(z | \varepsilon, m, \Lambda)} \left\langle \exp \left[- \int d\xi : e^{i\sqrt{\beta} \sigma \varphi(x)} :_{\varepsilon} \zeta_{\mu, \nu, \eta, z, \Lambda}(\xi | \varepsilon, m) \right] \right\rangle_{GFF(\varepsilon, m)} \\ & \quad \times \exp \left[\frac{1}{2} \left(\sum_{j=1}^q \nu_j \partial \varphi(g_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial} \varphi(h_{j'}) \right)^2 \right]_{GFF(\varepsilon, m)} \end{aligned} \quad (5.79)$$

where we have introduced the notation (recall $\xi = (x, \sigma)$)

$$\begin{aligned} & \zeta_{\mu, \nu, \eta, z, \Lambda}(\xi | \varepsilon, m) \\ & = -z \mathbf{1}_{\Lambda}(x) e^{i\sqrt{\beta} \sigma \sum_{j=1}^q \nu_j \langle \varphi(x) \partial \varphi(g_j) \rangle_{GFF(\varepsilon, m)} + i\sqrt{\beta} \sigma \sum_{j'=1}^{q'} \eta_{j'} \langle \varphi(x) \bar{\partial} \varphi(h_{j'}) \rangle_{GFF(\varepsilon, m)}} \\ & \quad - \sum_{k=1}^n \mu_k f_k(x) e^{i\sqrt{\beta} \sigma_k \sum_{j=1}^q \nu_j \langle \varphi(x) \partial \varphi(g_j) \rangle_{GFF(\varepsilon, m)} + i\sqrt{\beta} \sigma_k \sum_{j'=1}^{q'} \eta_{j'} \langle \varphi(x) \bar{\partial} \varphi(h_{j'}) \rangle_{GFF(\varepsilon, m)}} \delta_{\sigma, \sigma_k}. \end{aligned} \quad (5.80)$$

In terms of the renormalized partition function (5.5), we may write (5.79) as

$$\begin{aligned} & \frac{\mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m, \Lambda)}{\mathcal{Z}(z|\varepsilon, m, \Lambda)} \\ & \times \exp \left[\frac{1}{2} \int d\xi_1 d\xi_2 \zeta_{\mu,v,\eta,z,\Lambda}(\xi_1|\varepsilon, m) \zeta_{\mu,v,\eta,z,\Lambda}(\xi_2|\varepsilon, m) A(\xi_1, \xi_2|\varepsilon, m) \right] \\ & \times \exp \left[-\frac{z^2}{2} \int d\xi_1 d\xi_2 \mathbf{1}_\Lambda(x_1) \mathbf{1}_\Lambda(x_2) A(\xi_1, \xi_2|\varepsilon, m) \right] \\ & \times \exp \left[\frac{1}{2} \left\langle \left(\sum_{j=1}^q v_j \partial\varphi(g_j) + \sum_{j'=1}^{q'} \eta_{j'} \bar{\partial}\varphi(h_{j'}) \right)^2 \right\rangle_{\text{GFF}(\varepsilon, m)} \right]. \end{aligned} \quad (5.81)$$

To study the $\varepsilon, m \rightarrow 0$ limit, we define $\zeta_{\mu,v,\eta,z,\Lambda}(\xi)$ exactly as $\zeta_{\mu,v,\eta,z,\Lambda}(\xi|\varepsilon, m)$ in (5.80) with $\text{GFF}(\varepsilon, m)$ replaced by GFF and where $\langle \varphi(x) \partial\varphi(g) \rangle_{\text{GFF}} = \int dy g(y) \langle \varphi(x) \partial\varphi(y) \rangle_{\text{GFF}}$ is given by (2.28). Indeed, Lemma 2.4 implies that, as $\varepsilon, m \rightarrow 0$, one has

$$\zeta_{\mu,v,\eta,z,\Lambda}(\xi|\varepsilon, m) \rightarrow \zeta_{\mu,v,\eta,z,\Lambda}(\xi) \quad (5.82)$$

uniformly on compact sets in ξ , and $\mu_k, v_j, \eta_{j'}, z$ (where Λ is fixed). Moreover, since $\zeta_{\mu,v,\eta,z,\Lambda}(\xi|\varepsilon, m)$ has uniformly compact support in ξ , the convergence is in fact uniform in $\xi \in \mathbb{R}^2 \times \{\pm 1\}$. We conclude from Theorem 5.1 (iv) and (5.82) that

$$\mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m) \rightarrow \mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}). \quad (5.83)$$

Once again, from the fact that we are dealing with bounded random variables, one readily checks that $\mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m)$ extends to an entire function of $\mu_k, v_j, \eta_{j'}, z$. Thus our uniform convergence implies that also $\mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda})$ extends to an entire function of the variables.

We now consider the cases $n > 2$ and $n = 0, 1, 2$ of Theorem 3.1 (i)–(iii) separately. The arguments are all very similar.

Proof of Theorem 3.1 (i)–(iii) for $n > 2$. For $n > 2$, only $\mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m)$ in (5.81) plays a role for the correlation functions – the other terms vanish when we take logarithmic derivatives and set the various parameters to zero. Indeed,

$$\begin{aligned} & \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi} :_{\varepsilon}(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)}^T \\ & = \prod_{k=1}^n \frac{\partial}{\partial\mu_k} \Big|_{\mu_k=0} \prod_{j=1}^q \frac{\partial}{\partial v_j} \Big|_{v_j=0} \prod_{j'=1}^{q'} \frac{\partial}{\partial \eta_{j'}} \Big|_{\eta_{j'}=0} \log \mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m). \end{aligned} \quad (5.84)$$

Now recall from (5.83) that

$$\mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m) \rightarrow \mathcal{Z}(\zeta_{\mu,v,\eta,z,\Lambda}) \quad (5.85)$$

and that the right-hand side is entire in $\mu_k, \nu_j, \eta_{j'}, z$. Moreover, by Theorem 5.1 (iii), we know that $\mathcal{Z}(\zeta_{0,0,0,z,\Lambda}) > 0$ for $z \in \mathbb{R}$, so we see that there exists some complex neighborhood of the origin $N \subset \mathbb{C}$ and some neighborhood of the real axis $\mathbb{R} \subset N' \subset \mathbb{C}$ such that $\log \mathcal{Z}(\zeta_{\mu,\nu,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m) \rightarrow \log \mathcal{Z}(\zeta_{\mu,\nu,\eta,z,\Lambda})$ uniformly in $\mu_k, \nu_j, \eta_{j'} \in N$ and z in a compact subset of N' , and that the limit is analytic in this domain. This implies that also the μ, ν, η derivatives of this logarithm evaluated at zero converge and are analytic in $z \in N'$. We have thus proven items (i) and (ii) of Theorem 3.1 for $n > 2$. Let us turn to item (iii).

Again, since we know that $\log \mathcal{Z}(\zeta_{\mu,\nu,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m)$ converges uniformly (and is analytic in a suitable domain), we know that also its derivatives converge. In particular, going back in our argument, our remaining task is to evaluate the $\varepsilon \rightarrow 0, m \rightarrow 0$ limit of

$$\begin{aligned} & \frac{d^l}{dz^l} \Big|_{z=0} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi} :_{\varepsilon}(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{SG(\beta,z|\varepsilon,m,\Lambda)}^T \\ &= \sum_{\tau_1, \dots, \tau_l \in \{-1, 1\}} \left\langle \prod_{k=1}^n :e^{i\sqrt{\beta}\sigma_k\varphi} :_{\varepsilon}(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \prod_{s=1}^l :e^{i\sqrt{\beta}\tau_s\varphi} :_{\varepsilon}(\mathbf{1}_{\Lambda}) \right\rangle_{GFF(\varepsilon,m)}^T \end{aligned} \quad (5.86)$$

for $l \geq 0$. The claim for item (iii) for $n > 2$ now follows from Lemma 2.10. \blacksquare

Proof of Theorem 3.1 (i)–(iii) for $n = 2$. For $n = 2$, also the term

$$\frac{1}{2} \int d\xi_1 d\xi_2 \zeta_{\mu,\nu,\eta,z,\Lambda}(\xi_1|\varepsilon, m) \zeta_{\mu,\nu,\eta,z,\Lambda}(\xi_2|\varepsilon, m) A(\xi_1, \xi_2|\varepsilon, m) \quad (5.87)$$

in (5.81) and the contribution from the Girsanov transform need to be taken into account. More precisely, one finds (recalling (5.4)) that

$$\left\langle \prod_{k=1}^2 :e^{i\sqrt{\beta}\sigma_k\varphi} :_{\varepsilon}(f_k) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{SG(\beta,z|\varepsilon,m,\Lambda)}^T \quad (5.88)$$

equals

$$\begin{aligned} & \prod_{k=1}^2 \frac{\partial}{\partial\mu_k} \Big|_{\mu_k=0} \prod_{j=1}^q \frac{\partial}{\partial\nu_j} \Big|_{\nu_j=0} \prod_{j'=1}^{q'} \frac{\partial}{\partial\eta_{j'}} \Big|_{\eta_{j'}=0} \log \mathcal{Z}(\zeta_{\mu,\nu,\eta,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m) \\ &+ \int_{(\mathbb{R}^2)^2} dx_1 dx_2 \langle :e^{i\sqrt{\beta}\sigma_1\varphi(x_1)} :_{\varepsilon} :e^{i\sqrt{\beta}\sigma_2\varphi(x_2)} :_{\varepsilon} \rangle_{GFF(\varepsilon,m)}^T f_1(x_1) f_2(x_2) \\ &\times \prod_{j=1}^q (i\sqrt{\beta}\sigma_1\langle\varphi(x_1)\partial\varphi(g_j)\rangle_{GFF(\varepsilon,m)} + i\sqrt{\beta}\sigma_2\langle\varphi(x_2)\partial\varphi(g_j)\rangle_{GFF(\varepsilon,m)}) \\ &\times \prod_{j'=1}^{q'} (i\sqrt{\beta}\sigma_1\langle\varphi(x_1)\bar{\partial}\varphi(h_{j'})\rangle_{GFF(\varepsilon,m)} + i\sqrt{\beta}\sigma_2\langle\varphi(x_2)\bar{\partial}\varphi(h_{j'})\rangle_{GFF(\varepsilon,m)}). \end{aligned} \quad (5.89)$$

The \mathcal{Z} -term again converges uniformly and gives rise to a function analytic in its parameters. The remaining term on the other hand is readily seen (as in the proof of Lemma 2.10) to converge if $q + q' \geq 1$ or if f_1, f_2 have disjoint supports. This reasoning proves items (i) and (ii) of Theorem 3.1 for $n = 2$, and item (iii) for $n = 2$ is verified in the same manner as for $n > 2$. ■

Since $n = 1$ is relevant also to item (iv), let us next consider $n = 0$, and then conclude the proof with the case $n = 1$.

Proof of Theorem 3.1 (i)–(iii) for $n = 0$. The proof for $n = 0$ is again similar, but now with a further contribution from the third exponential in (5.81):

$$\left\langle \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z | \varepsilon, m, \Lambda)}^T \quad (5.90)$$

equals

$$\begin{aligned} & \prod_{j=1}^q \frac{\partial}{\partial v_j} \Big|_{v_j=0} \prod_{j'=1}^{q'} \frac{\partial}{\partial \eta_{j'}} \Big|_{\eta_{j'}=0} \log \mathcal{Z}(\zeta_{0, v, \eta, z, \Lambda}(\cdot | \varepsilon, m) | \varepsilon, m) \\ & + \frac{z^2}{2} \sum_{\tau_1, \tau_2 \in \{-1, 1\}} \int_{\Lambda^2} dx_1 dx_2 \langle :e^{i\sqrt{\beta}\tau_1\varphi(x_1)} :_{\varepsilon} :e^{i\sqrt{\beta}\tau_2\varphi(x_2)} :_{\varepsilon} \rangle_{\text{GFF}(\varepsilon, m)}^T \\ & \quad \times \prod_{j=1}^q (i\sqrt{\beta}\tau_1 \langle \varphi(x_1) \partial\varphi(g_j) \rangle_{\text{GFF}(\varepsilon, m)} + i\sqrt{\beta}\tau_2 \langle \varphi(x_2) \partial\varphi(g_j) \rangle_{\text{GFF}(\varepsilon, m)}) \\ & \quad \times \prod_{j'=1}^{q'} (i\sqrt{\beta}\tau_1 \langle \varphi(x_1) \bar{\partial}\varphi(h_{j'}) \rangle_{\text{GFF}(\varepsilon, m)} + i\sqrt{\beta}\tau_2 \langle \varphi(x_2) \bar{\partial}\varphi(h_{j'}) \rangle_{\text{GFF}(\varepsilon, m)}) \\ & + \delta_{q,1} \delta_{q',1} \langle \partial\varphi(g_1) \bar{\partial}\varphi(h_1) \rangle_{\text{GFF}(\varepsilon, m)} \\ & + \delta_{q,2} \delta_{q',0} \langle \partial\varphi(g_1) \partial\varphi(g_2) \rangle_{\text{GFF}(\varepsilon, m)} + \delta_{q,0} \delta_{q',2} \langle \bar{\partial}\varphi(h_1) \bar{\partial}\varphi(h_2) \rangle_{\text{GFF}(\varepsilon, m)}. \end{aligned} \quad (5.91)$$

The \mathcal{Z} -term can be treated as before and the last three terms converge by Lemma 2.9. For the z^2 -term, we can argue exactly as in the proof of Lemma 2.10 and conclude that also in this case, the correlation functions converge and define analytic functions of z . Thus we have proven items (i) and (ii) of Theorem 3.1 in the case $n = 0$. Item (iii) is verified in the same way as for $n > 2$.

Finally, to see that the limit of right-hand side is symmetric in z , note that $\zeta_{0, v, \eta, z}$ is proportional to z (since $n = 0$) and that \mathcal{Z} is even in ζ in the $\varepsilon, m \rightarrow 0$ limit. ■

Proof of Theorem 3.1 (i)–(iii) for $n = 1$. We finally consider $n = 1$. Let us first look at the situation where $q + q' \geq 1$ where the quantity

$$\left\langle :e^{i\sqrt{\beta}\sigma_1\varphi} :_{\varepsilon}(f_1) \prod_{j=1}^q \partial\varphi(g_j) \prod_{j'=1}^{q'} \bar{\partial}\varphi(h_{j'}) \right\rangle_{\text{SG}(\beta, z | \varepsilon, m, \Lambda)}^T \quad (5.92)$$

equals

$$\begin{aligned}
& \frac{\partial}{\partial \mu_1} \Big|_{\mu_1=0} \prod_{j=1}^q \frac{\partial}{\partial v_j} \Big|_{v_j=0} \prod_{j'=1}^{q'} \frac{\partial}{\partial \eta_{j'}} \Big|_{\eta_{j'}=0} \log \mathcal{Z}(\zeta_{\mu, v, \eta, z, \Lambda}(\cdot | \varepsilon, m) | \varepsilon, m) \\
& + \sum_{\tau \in \{-1, 1\}} \int_{\mathbb{R}^2 \times \Lambda} dx_1 dx_2 \langle :e^{i\sqrt{\beta} \sigma_1 \varphi(x_1)} :_{\varepsilon} ; e^{i\sqrt{\beta} \tau \varphi(x_2)} :_{\varepsilon} \rangle_{\text{GFF}(\varepsilon, m)}^T f_1(x_1) \\
& \quad \times \prod_{j=1}^q (i\sqrt{\beta} \sigma_1 \langle \varphi(x_1) \partial \varphi(g_j) \rangle_{\text{GFF}(\varepsilon, m)} + i\sqrt{\beta} \tau \langle \varphi(x_2) \partial \varphi(g_j) \rangle_{\text{GFF}(\varepsilon, m)}) \\
& \quad \times \prod_{j'=1}^{q'} (i\sqrt{\beta} \sigma_1 \langle \varphi(x_1) \bar{\partial} \varphi(h_{j'}) \rangle_{\text{GFF}(\varepsilon, m)} + i\sqrt{\beta} \tau \langle \varphi(x_2) \bar{\partial} \varphi(h_{j'}) \rangle_{\text{GFF}(\varepsilon, m)}). \quad (5.93)
\end{aligned}$$

Finiteness and convergence of this quantity is again argued analogously as in the proof of Lemma 2.10, so we have the proof of items (i) and (ii) also in the $n = 1$ case. Item (iii) follows by the same argument as before. \blacksquare

Proof of Theorem 3.1 (iv). The only thing that remains is thus item (iv). For this, we find with similar reasoning as before (recall we chose f to be supported in Λ)

$$\begin{aligned}
& \langle :e^{i\sqrt{\beta} \sigma_1 \varphi} :_{\varepsilon}(f) \rangle_{\text{SG}(\beta, z | \varepsilon, m, \Lambda)}^T \\
& = \frac{\partial}{\partial \mu_1} \Big|_{\mu_1=0} \log \mathcal{Z}(\zeta_{\mu_1, 0, 0, z, \Lambda}(\cdot | \varepsilon, m) | \varepsilon, m) \\
& \quad + \sum_{\tau \in \{-1, 1\}} z \int_{\Lambda^2} dx_1 dx_2 \langle :e^{i\sqrt{\beta} \sigma_1 \varphi(x_1)} :_{\varepsilon} ; e^{i\sqrt{\beta} \tau \varphi(x_2)} :_{\varepsilon} \rangle_{\text{GFF}(\varepsilon, m)}^T f(x_1). \quad (5.94)
\end{aligned}$$

The first term once again has a finite limit as $\varepsilon, m \rightarrow 0$, but as we now prove, the second term blows up. For the second term, if $\tau = \sigma_1$, then everything is bounded, but for $\tau \neq \sigma_1$, the leading order behavior (in ε) is given by (making use of asymptotics e.g. from Lemma 2.4)

$$\begin{aligned}
& \int_{\Lambda^2} dx_1 dx_2 \varepsilon^{-\frac{\beta}{2\pi}} e^{-\beta \langle \varphi(0)^2 \rangle_{\text{GFF}(\varepsilon, m)}} e^{\beta \langle \varphi(x_1) \varphi(x_2) \rangle_{\text{GFF}(\varepsilon, m)}} f(x_1) \\
& = (1 + o(1)) \int_{\Lambda^2} dx_1 dx_2 m^{\frac{\beta}{2\pi}} e^{\frac{\beta}{4\pi} \gamma} e^{\beta \int_{\varepsilon^2}^{\infty} ds \frac{e^{-m^2 s}}{4\pi s}} e^{-\frac{|x_1 - x_2|^2}{4s}} f(x_1) \\
& = (1 + o(1) + O(m^2)) \\
& \quad \times \int_{\Lambda^2} dx_1 dx_2 e^{\beta \int_{\varepsilon^2}^1 ds \frac{e^{-m^2 s} - 1}{4\pi s}} e^{-\frac{|x_1 - x_2|^2}{4s}} + \beta \int_{\varepsilon^2}^1 ds \frac{1}{4\pi s} e^{-\frac{|x_1 - x_2|^2}{4s}} \\
& \quad \times e^{\beta \int_1^{\infty} ds \frac{e^{-m^2 s}}{4\pi s}} (e^{-\frac{|x_1 - x_2|^2}{4s}} - 1) f(x_1), \quad (5.95)
\end{aligned}$$

from which we see that the leading order terms (as $\varepsilon, m \rightarrow 0$) are given by

$$\begin{aligned} & \int_{\Lambda^2} dx_1 dx_2 e^{\beta \int_{\varepsilon^2}^1 ds \frac{1}{4\pi s} e^{-\frac{|x_1-x_2|^2}{4s}} + \beta \int_1^\infty ds \frac{1}{4\pi s} (e^{-\frac{|x_1-x_2|^2}{4s}} - 1)} f(x_1) \\ &= e^{-\frac{\gamma\beta}{4\pi}} \int_{\Lambda^2} dx_1 dx_2 |x_1 - x_2|^{-\frac{\beta}{2\pi}} e^{-\frac{\beta}{4\pi} \Gamma(0, \frac{|x_1-x_2|^2}{\varepsilon^2})} f(x_1) \\ &= e^{-\frac{\gamma\beta}{4\pi}} \int_{\Lambda} dx f(x) \int_{x-\varepsilon u \in \Lambda} du \varepsilon^{2-\frac{\beta}{2\pi}} |u|^{-\frac{\beta}{2\pi}} e^{-\frac{\beta}{4\pi} \Gamma(0, |u|^2)}. \end{aligned} \quad (5.96)$$

For $\beta > 4\pi$, $u \mapsto |u|^{-\frac{\beta}{2\pi}} e^{-\frac{\beta}{4\pi} \Gamma(0, |u|^2)} \in L^1(\mathbb{R}^2)$, and we see that

$$\begin{aligned} & e^{-\frac{\gamma\beta}{4\pi}} \int_{\Lambda} dx f(x) \int_{x-\varepsilon u \in \Lambda} du \varepsilon^{2-\frac{\beta}{2\pi}} |u|^{-\frac{\beta}{2\pi}} e^{-\frac{\beta}{4\pi} \Gamma(0, |u|^2)} \\ &= (1 + o(1)) \varepsilon^{2-\frac{\beta}{2\pi}} 2\pi e^{-\frac{\gamma\beta}{4\pi}} \int_{\Lambda} dx f(x) \int_0^\infty dr r^{-\frac{\beta}{2\pi}+1} e^{-\frac{\beta}{4\pi} \Gamma(0, r^2)}, \end{aligned} \quad (5.97)$$

which also concludes the proof of item (iv) for $\beta > 4\pi$.

For $\beta = 4\pi$, on the other hand, we obtain a logarithmic singularity from the long range behavior of the u -integral and one finds for the relevant asymptotics

$$e^{-\gamma} \int_{\Lambda} dx f(x) \int_{x-\varepsilon u \in \Lambda} du |u|^{-2} e^{-\Gamma(0, |u|^2)} = (1 + o(1)) e^{-\gamma} 2\pi \log \varepsilon^{-1} \cdot \int_{\Lambda} dx f(x). \quad (5.98)$$

This concludes the proof of item (iv) for $\beta = 4\pi$ as well, and also the proof of the theorem. \blacksquare

5.7. Existence of φ field

Finally, we prove Theorem 3.2. Since the proof is essentially identical to that of Theorem 3.1, we will be somewhat brief.

Proof of Theorem 3.2. We are interested in the function

$$(w, z) \mapsto \langle e^{w\varphi(f)} \rangle_{\text{SG}(\beta, z|\varepsilon, m, \Lambda)} \quad (5.99)$$

which we may again write using Girsanov's theorem as

$$\begin{aligned} & \frac{\mathcal{Z}(\zeta_{w,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m, \Lambda)}{\mathcal{Z}(z|\varepsilon, m, \Lambda)} \\ & \times \exp \left[\frac{1}{2} \int d\xi_1 d\xi_2 \zeta_{w,z,\Lambda}(\xi_1|\varepsilon, m) \zeta_{w,z,\Lambda}(\xi_2|\varepsilon, m) A(\xi_1, \xi_2|\varepsilon, m) \right] \\ & \times \exp \left[-\frac{z^2}{2} \int d\xi_1 d\xi_2 \mathbf{1}_{\Lambda}(x_1) \mathbf{1}_{\Lambda}(x_2) A(\xi_1, \xi_2|\varepsilon, m) \right] \\ & \times \exp \left[\frac{w^2}{2} \langle \varphi(f)^2 \rangle_{\text{GFF}(\varepsilon, m)} \right] \end{aligned} \quad (5.100)$$

where now

$$\zeta_{w,z,\Lambda}(\xi|\varepsilon, m) = -z \mathbf{1}_\Lambda(x) e^{i\sqrt{\beta} \sigma w (\varphi(x)\varphi(f))_{\text{GFF}(\varepsilon, m)}}. \quad (5.101)$$

We only consider the $\varepsilon, m \rightarrow 0$ limit; the argument for $\varepsilon \rightarrow 0$ with $m > 0$ fixed is analogous. Thus let $f \in L_c^\infty(\mathbb{R}^2)$ with $\int f = 0$. Then $\langle \varphi(f)^2 \rangle_{\text{GFF}(\varepsilon, m)}$ converges as $\varepsilon, m \rightarrow 0$ by Lemma 2.4. Moreover, again using Lemma 2.4, we see that $\zeta_{w,z,\Lambda}(\xi|\varepsilon, m) \rightarrow \zeta_{w,z,\Lambda}(\xi)$ uniformly in $\xi \in \mathbb{R}^2 \times \{\pm 1\}$ and uniformly on compact sets of z, w , and thus $\mathcal{Z}(\zeta_{w,z,\Lambda}(\cdot|\varepsilon, m) | \varepsilon, m) \rightarrow \mathcal{Z}(\zeta_{w,z,\Lambda})$ and the limit is entire in z, w . As in the proof of Theorem 3.1, the same is true for the other terms. ■

6. Estimates for free fermions with finite volume mass

In this section, we prove Theorem 3.3. Most of our work goes into the construction and analysis of the fundamental solution (Green's function) of the Dirac operator with a finite volume mass term. We state these estimates in Section 6.1, then deduce Theorem 3.3 in Section 6.2, and finally prove the estimates stated in Section 6.1 in the remainder of Section 6.

6.1. Statement of estimates on the Green's function

Recall that we are considering $\Lambda_L = \{x = (x_0, x_1) \in \mathbb{R}^2 : |x| \leq L\}$, and that we identify \mathbb{R}^2 with \mathbb{C} . We are interested in the Dirac operator

$$D = D_{\mu, \Lambda_L} = \not{\partial} + \mu \mathbf{1}_{\Lambda_L} := \begin{pmatrix} \mu \mathbf{1}_{\Lambda_L} & 2\bar{\partial} \\ 2\partial & \mu \mathbf{1}_{\Lambda_L} \end{pmatrix} \quad (6.1)$$

where $\partial = \frac{1}{2}(-i\partial_0 + \partial_1)$ and $\bar{\partial} = \frac{1}{2}(i\partial_0 + \partial_1)$. For each $y \in \text{int}(\Lambda_L) = \{z \in \mathbb{C} : |z| < L\}$, we are looking for a continuous function $S_{\mu \mathbf{1}_{\Lambda_L}}(\cdot, y) : \mathbb{C} \setminus \{y\} \rightarrow \mathbb{C}^{2 \times 2}$ such that

$$DS_{\mu \mathbf{1}_{\Lambda_L}}(\cdot, y) = \delta_y \quad \text{and} \quad \lim_{|x| \rightarrow \infty} S_{\mu \mathbf{1}_{\Lambda_L}}(x, y) = 0. \quad (6.2)$$

Our results for this function $S_{\mu \mathbf{1}_{\Lambda_L}}$ are summarized in the following theorem. In the statement, we also use S_0 and S_μ to denote the explicit infinite volume Dirac Green's function (1.8):

$$S_0(x, y) = \frac{1}{2\pi} \begin{pmatrix} 0 & 1/(\bar{x} - \bar{y}) \\ 1/(x - y) & 0 \end{pmatrix}, \quad (6.3)$$

$$S_\mu(x, y) = -\frac{1}{2\pi} \begin{pmatrix} -\mu K_0(|\mu||x - y|) & 2\bar{\partial}_x K_0(|\mu||x - y|) \\ 2\partial_x K_0(|\mu||x - y|) & -\mu K_0(|\mu||x - y|) \end{pmatrix} \quad (\mu \neq 0), \quad (6.4)$$

where K_0 is the 0th modified Bessel function of the second kind. It is well known that S_0 and S_μ really are the fundamental solutions of $i\not{\partial}$ and $i\not{\partial} + \mu$ on \mathbb{R}^2 and it also follows from the well-known asymptotics of K_0 that $S_\mu(x, y) \rightarrow S_0(x, y)$ as $\mu \rightarrow 0$ when $x \neq y$. For a matrix S , we will denote by $|S|$ a submultiplicative matrix norm of S .

Theorem 6.1. *For each $L \geq 1$, $y \in \text{int}(\Lambda_L)$ and $\mu \in \mathbb{R}$, there exists a continuous function $S_{\mu\mathbf{1}_{\Lambda_L}}(\cdot, y) : \mathbb{C} \setminus \{y\} \rightarrow \mathbb{C}^{2 \times 2}$ that satisfies (6.2) and has the following properties for some polynomial $P = P(L, |\mu|)$ in both variables (which does not depend on any of the arguments below).*

(i) *For all $x, y \in \text{int}(\Lambda_L)$, $x \neq y$, and $L \geq 1$, we have (with S_0 as in (6.3))*

$$|S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) - S_0(x, y)| \leq P(L, |\mu|)(1 + |\log|x - y||). \quad (6.5)$$

(ii) *For all $x \in \Lambda_L^c$, $y \in \text{int}(\Lambda_L)$, and $L \geq 1$, we have*

$$|S_{\mu\mathbf{1}_{\Lambda_L}}(x, y)| \leq \frac{P(L, |\mu|)}{L - |y|}. \quad (6.6)$$

(iii) *For each $x, y \in \text{int}(\Lambda_L)$ with $x \neq y$, the function $\mu \mapsto S_{\mu\mathbf{1}_{\Lambda_L}}(x, y)$ has an analytic continuation into some L -dependent neighborhood of the real axis, this analytic continuation also satisfies the estimate (6.5), and*

$$\lim_{\mu \rightarrow 0} S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) = S_0(x, y) = \frac{1}{2\pi} \begin{pmatrix} 0 & 1/(\bar{x} - \bar{y}) \\ 1/(x - y) & 0 \end{pmatrix}. \quad (6.7)$$

(iv) *For any $x, y \in \text{int}(\Lambda_L)$ with $x \neq y$, for $\mu \in \mathbb{R}$ we have*

$$\partial_\mu S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) = - \int_{\Lambda_L} du S_{\mu\mathbf{1}_{\Lambda_L}}(x, u) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y). \quad (6.8)$$

(v) *For each fixed $\mu \in \mathbb{R} \setminus \{0\}$, uniformly on compact subsets of $x \neq y \in \mathbb{R}^2$, as $L \rightarrow \infty$,*

$$S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) \rightarrow S_\mu(x, y) = -\frac{1}{2\pi} \begin{pmatrix} -\mu K_0(|\mu||x - y|) & 2\bar{\partial}_x K_0(|\mu||x - y|) \\ 2\partial_x K_0(|\mu||x - y|) & -\mu K_0(|\mu||x - y|) \end{pmatrix}. \quad (6.9)$$

6.2. Proof of Theorem 3.3

For Theorem 3.3, our function $S_{\mu\mathbf{1}_{\Lambda_L}}$ is of course the Green's function of Theorem 6.1. We will denote the components of the 2×2 matrix $S_{\mu\mathbf{1}_{\Lambda_L}}(x, y)$ by $S_{\mu\mathbf{1}_{\Lambda_L};ij}(x, y)$ where $i, j \in \{1, 2\}$. We also recall the definition of the truncated correlation functions from (3.12), as well as the truncated two-point functions with singularity subtracted from (3.13). Let us begin with the proof of Theorem 3.3 (i). We formulate this as the following lemma.

Lemma 6.2. *For $n \geq 3$ and $f_1, \dots, f_n \in L_c^\infty(\Lambda_L)$,*

$$\mu \mapsto \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T \quad (6.10)$$

has an analytic continuation to an L -dependent neighborhood of \mathbb{R} (with $S_{\mu\mathbf{1}_{\Lambda_L}} = S_0$ for $\mu = 0$). For $n = 2$, the same holds if f_1 and f_2 have disjoint compact supports or if the truncated two-point function is replaced by (3.13).

Proof. From Theorem 6.1 (i), which implies $|S_{\mu\mathbf{1}_\Lambda}(x, y)| \leq P(L, |\mu|)/|x - y|$ for $x, y \in \Lambda_L$, and the representation (3.12), we see that the smeared truncated correlation functions exist for all $\mu \in \mathbb{R}$ and $n \geq 3$ (the claim about what happens at $\mu = 0$ following from Theorem 6.1 (iii)). Here we use the fact that for, any compact $K \subset \mathbb{R}^2$,

$$\int_K du \frac{1}{|x - u| |u - y|} \leq C_K(1 + |\log |x - y||). \tag{6.11}$$

For $n = 2$, in the same way, the truncated two-point function with subtracted singularity (3.13) exists; or, alternatively, if f_1 and f_2 have disjoint compact supports, the truncated two-point function also exists trivially. Moreover, Theorem 6.1 (iii) (in particular, the analogue of (6.5) for complex μ) allows us to construct a candidate for the analytic continuation of the truncated correlation functions (with subtracted singularity for $n = 2$). More precisely, we define the candidate by (3.12) though now using the analytic continuation of the Green’s function provided by Theorem 6.1 (iii). Using Theorem 6.1 (iii) (or more precisely, the bound analogous to (6.5) for complex μ), a routine dominated convergence argument shows that this candidate for the analytic continuation is continuous in μ (in this L -dependent neighborhood of the real axis). By Morera’s theorem, it remains to prove that for any closed loop γ (in our L -dependent neighborhood of the real axis), we have

$$\oint_\gamma d\mu \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T = 0, \tag{6.12}$$

where we have used the $\langle \cdot \rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}$ notation for our candidate for the analytic continuation.

Now using the analogue of (6.5) provided by Theorem 6.1 (iii), one can use Fubini to translate this into a contour integral over suitable products of $S_{\mu\mathbf{1}_{\Lambda_L}}$ at distinct points. By Theorem 6.1 (iii) and Cauchy’s integral theorem, this contour integral vanishes, and we are done. ■

We next turn to item (ii) of Theorem 3.3 which we formulate as the following lemma.

Lemma 6.3. *For $l \geq 1$ and $n \geq 3$ and $f_1, \dots, f_n \in L_c^\infty(\Lambda_L)$,*

$$\begin{aligned} \frac{d^l}{d\mu^l} \Big|_{\mu=0} \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu\mathbf{1}_{\Lambda_L})}^T \\ = \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) (\bar{\psi}_1 \psi_1(\mathbf{1}_{\Lambda_L}) + \bar{\psi}_2 \psi_2(\mathbf{1}_{\Lambda_L})) \right\rangle_{\text{FF}(0)}^T. \end{aligned} \tag{6.13}$$

For $n = 2$, the same holds if f_1 and f_2 have disjoint compact supports or if the truncated two-point function on the left-hand side is replaced by (3.13).

Before the proof, let us just mention that this derivative is finite by the massless correspondence Corollary 2.11 and Lemma 2.10, which implies that the corresponding bosonic correlation functions are integrable, and thus these smeared correlation functions exist.

Proof of Lemma 6.3. First assume that $n \geq 3$. We begin by noting that due to Theorem 6.1 (iv) (interchanging the order of integration and differentiation follows from a routine Cauchy-integral formula/Fubini argument utilizing Lemma 6.2 and Theorem 6.1)

$$\begin{aligned}
& \frac{d}{d\mu} \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu \mathbf{1}_{\Lambda_L})}^T \\
&= (-1)^{n+2} \sum_{\pi \in C_n} \sum_{j=1}^n \sum_{\alpha_{n+1}=1}^2 \sum_{\beta_{n+1}=1}^2 \mathbf{1}_{\alpha_{n+1}=\beta_{n+1}} \int_{\Lambda_L^{n+1}} dx_1 \cdots dx_{n+1} f_1(x_1) \cdots f_n(x_n) \\
&\quad \times S_{\mu \mathbf{1}_{\Lambda_L}; \alpha_{\pi^j(1)} \beta_{n+1}}(x_{\pi^j(1)}, x_{n+1}) S_{\mu \mathbf{1}_{\Lambda_L}; \alpha_{n+1} \beta_{\pi^{j+1}(1)}}(x_{n+1}, x_{\pi^{j+1}(1)}) \\
&\quad \times \prod_{i:i \neq j} S_{\mu \mathbf{1}_{\Lambda_L}; \alpha_{\pi^i(1)} \beta_{\pi^{i+1}(1)}}(x_{\pi^i(1)}, x_{\pi^{i+1}(1)}). \tag{6.14}
\end{aligned}$$

Note that $(\pi, j) \in C_n \times [n]$ defines a cyclic permutation $\sigma \in C_{n+1}$ in terms of which the right-hand side is

$$\begin{aligned}
& (-1)^{n+2} \sum_{\sigma \in C_{n+1}} \sum_{\alpha_{n+1}=1}^2 \sum_{\beta_{n+1}=1}^2 \mathbf{1}_{\alpha_{n+1}=\beta_{n+1}} \int_{\Lambda_L^{n+1}} dx_1 \cdots dx_{n+1} f_1(x_1) \cdots f_n(x_n) \\
&\quad \times \prod_{i=1}^{n+1} S_{\mu \mathbf{1}_{\Lambda_L}; \alpha_{\sigma^i(1)} \beta_{\sigma^{i+1}(1)}}(x_{\sigma^i(1)}, x_{\sigma^{i+1}(1)}). \tag{6.15}
\end{aligned}$$

In particular, this implies (recalling (1.10)) that

$$\begin{aligned}
& \frac{d}{d\mu} \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu \mathbf{1}_{\Lambda_L})}^T \\
&= \sum_{\alpha_{n+1}=1}^2 \sum_{\beta_{n+1}=1}^2 \mathbf{1}_{\alpha_{n+1}=\beta_{n+1}} \\
&\quad \times \int_{\Lambda_L^{n+1}} dx_1 \cdots dx_{n+1} f_1(x_1) \cdots f_n(x_n) \left\langle \prod_{i=1}^{n+1} \bar{\psi}_{\alpha_i} \psi_{\beta_i}(x_i) \right\rangle_{\text{FF}(\mu \mathbf{1}_{\Lambda_L})}^T \\
&= \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) (\bar{\psi}_1 \psi_1(\mathbf{1}_{\Lambda}) + \bar{\psi}_2 \psi_2(\mathbf{1}_{\Lambda})) \right\rangle_{\text{FF}(\mu \mathbf{1}_{\Lambda_L})}^T. \tag{6.16}
\end{aligned}$$

Setting $\mu = 0$ (note that this uses Theorem 6.1 (iii)), we find that

$$\frac{d}{d\mu} \Big|_{\mu=0} \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu \mathbf{1}_{\Lambda_L})}^T = \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) (\bar{\psi}_1 \psi_1(\mathbf{1}_{\Lambda}) + \bar{\psi}_2 \psi_2(\mathbf{1}_{\Lambda})) \right\rangle_{\text{FF}(0)}^T, \tag{6.17}$$

which is the claim when $l = 1$. The case of general l follows by induction. For $n = 2$, assuming that the truncated two-point function is replaced by (3.13) (or alternatively that

f_1 and f_2 have disjoint compact supports), we note that the argument is completely analogous. The subtracted singularity ensures the integrability of the left-hand sides, but does not contribute to the derivatives. ■

The final statement of Theorem 3.3 is the following lemma. Recall that, on the right-hand side, the correlation functions are given by (smeared versions of) (1.10) now with the propagator (1.8) (with infinite volume mass term).

Lemma 6.4. *For any $\mu \in \mathbb{R}$, $n \geq 3$, and $f_1, \dots, f_n \in L_c^\infty(\mathbb{R}^2)$, as $L \rightarrow \infty$,*

$$\left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu 1_{\Lambda_L})}^T \rightarrow \left\langle \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i}(f_i) \right\rangle_{\text{FF}(\mu)}^T. \quad (6.18)$$

For $n = 2$, the same holds if f_1 and f_2 have disjoint compact supports or if the truncated two-point function on the left-hand side is replaced by (3.13) and analogously on the right-hand side.

Proof. This is immediate from the uniform convergence of Theorem 6.1 (v). (The modification for $n = 2$ is again only used to guarantee integrability.) ■

Combining these lemmas yields the proof of Theorem 3.3, so we are done.

6.3. Facts about the Laplacian Green's function and eigenfunctions in a disk

Our proof of Theorem 6.1 relies on relating $S_{\mu 1_{\Lambda_L}}$ to the Green's function of the Laplacian in the disk as well as expansions in terms of the eigenfunctions of this Laplacian. We begin by collecting some well-known facts about these. First, we recall that

$$G_{\Lambda_L}(x, y) = \frac{1}{2\pi} \log \frac{1}{|x - y|} - \frac{1}{2\pi} \log \frac{1}{|L - \frac{\bar{x}y}{L}|} \quad (6.19)$$

is the Green's function for the (positive) Laplacian with zero Dirichlet boundary conditions:

$$-\Delta_x G_{\Lambda_L}(x, y) = \delta_y(x) \quad \text{for } x, y \in \text{int}(\Lambda_L), \quad (6.20)$$

$$G_{\Lambda_L}(x, y) = 0 \quad \text{for } x \in \partial\Lambda_L, y \in \text{int}(\Lambda_L). \quad (6.21)$$

In (6.19) we wrote $\bar{x} = x_1 - ix_0$ for $x = x_1 + ix_0$, while in (6.21) we wrote Δ_x for the Laplacian acting on the x -variable. We also recall that the eigenfunctions of $-\Delta$ on Λ_L (with zero boundary conditions) can be written explicitly in terms of Bessel functions and Fourier modes. More precisely, if for $n \geq 0$, J_n is the n th Bessel function of the first kind and for $k \geq 1$, $j_{n,k}$ is the k th positive zero of J_n (recall that $J_n(0) = 0$ for $n > 0$, so we do not count this zero), then for $n \in \mathbb{Z}$ and $k \geq 1$,

$$e_{n,k}(x) = \frac{1}{\sqrt{\pi} L J_{|n|+1}(j_{|n|,k})} J_{|n|}(j_{|n|,k} r/L) e^{in\theta} \quad (6.22)$$

are the eigenfunctions of $-\Delta$ on Λ_L (with zero boundary conditions), normalized so that they form an orthonormal basis of $L^2(\Lambda_L)$. Here we have written $x = re^{i\theta}$. In particular,

$$\int_{\Lambda_L} dx e_{n,k}(x) \overline{e_{m,l}(x)} = \delta_{n,m} \delta_{k,l}. \quad (6.23)$$

To simplify notation, we set $j_{n,k} = j_{|n|,k}$ for $n < 0$. The eigenvalue associated to $e_{n,k}$ is then $j_{n,k}^2/L^2$:

$$-\Delta e_{n,k} = \frac{j_{n,k}^2}{L^2} e_{n,k}. \quad (6.24)$$

In terms of the eigenfunctions and eigenvalues, the Laplacian Green's function is

$$G_{\Lambda_L}(x, y) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\infty} \frac{L^2}{j_{n,k}^2} e_{n,k}(x) \overline{e_{n,k}(y)}, \quad (6.25)$$

understood in the sense that for $g \in L^2(\Lambda_L)$ written as $g = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\infty} g_{n,k} e_{n,k}$ (with convergence in $L^2(\Lambda_L)$ since the $e_{n,k}$ form an orthonormal basis of $L^2(\Lambda_L)$), we have

$$\int_{\Lambda_L} dy G_{\Lambda_L}(x, y) g(y) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^{\infty} \frac{L^2}{j_{n,k}^2} g_{n,k} e_{n,k}(x), \quad (6.26)$$

again with convergence in $L^2(\Lambda_L)$.

Since the Dirac Green's function is related to derivatives of the Laplacian Green's function, our construction of the Dirac Green's function also involves another family of functions (which are also Laplacian eigenfunctions, but with different boundary conditions): for $n, k \geq 1$, we define

$$f_{n,k}(x) = -2 \frac{L}{j_{n-1,k}} \bar{\partial} e_{n-1,k}(x). \quad (6.27)$$

The following lemma collects the properties of the $e_{n,k}$ and $f_{n,k}$ we need. The stated estimates are not all optimal, but sufficient for our purposes. We write $\nabla^p g$ for the vector of all combinations of p derivatives of g and $\|\nabla^p g\|_{L^\infty(K)}$ for the maximum of the $L^\infty(K)$ -norm of all combinations of p derivatives of g .

Lemma 6.5. *For $n \in \mathbb{Z}$ and $k \geq 1$, the eigenvalues (up to the factor L^2) satisfy*

$$j_{n,k}^2 \geq n^2 + \left(k - \frac{1}{4}\right)^2 \pi^2. \quad (6.28)$$

The eigenfunctions satisfy (for some universal constant C)

$$\|e_{n,k}\|_{L^\infty(\Lambda_L)} \leq C \frac{j_{n,k}}{L}, \quad \|\nabla e_{n,k}\|_{L^\infty(\Lambda_L)} \leq C \frac{j_{n,k}^2}{L^2} \quad (n \in \mathbb{Z}, k \geq 1), \quad (6.29)$$

$$\|f_{n,k}\|_{L^\infty(\Lambda_L)} \leq C \frac{j_{n-1,k}}{L}, \quad \|\nabla f_{n,k}\|_{L^\infty(\Lambda_L)} \leq C \frac{j_{n-1,k}^3}{L^3} L \quad (n, k \geq 1). \quad (6.30)$$

Moreover, for any $p, q \geq 0$, any compact $K \subset \text{int}(\Lambda_L)$, and any $f \in C_c^\infty(\text{int}(\Lambda_L))$, there are constants $C_{p,K,L}$ and $C_{p,q,f,L}$ such that

$$\|\nabla^p e_{n,k}\|_{L^\infty(K)} + \|\nabla^p f_{n,k}\|_{L^\infty(K)} \leq C_{p,K,L} j_{n,k}^{1+p}, \quad (6.31)$$

$$\left| \int dx f(x) \nabla^p e_{n,k}(x) \right| + \left| \int dx f(x) \nabla^p f_{n,k}(x) \right| \leq C_{p,q,f,L} j_{n,k}^{-q}. \quad (6.32)$$

Proof. The bounds on the $j_{n,k}$ follow, for example, from [40, Theorem 3] and the main result of [51]. The bounds (6.29) on the eigenfunctions $e_{n,k}$ and their derivatives $\nabla e_{n,k}$ follow, for example, from [38, Theorem 1] and [56, Corollary 1.1] (as well as scaling by L). The claim for $\|f_{n,k}\|_{L^\infty(\Lambda_L)}$ in (6.30) follows directly from the definition of $f_{n,k}$ in (6.27) combined with the gradient estimate from (6.29).

For the bound on the gradient of $f_{n,k}$ in (6.30), we note that since

$$2\partial f_{n,k} = -\frac{L}{j_{n-1,k}} \Delta e_{n-1,k} = \frac{j_{n-1,k}}{L} e_{n-1,k}, \quad (6.33)$$

by (6.29) we have $\|\partial f_{n,k}\|_{L^\infty(\Lambda_L)} \leq C j_{n-1,k}^2 / L^2$, so it suffices to control $\|\bar{\partial} f_{n,k}\|_{L^\infty(\Lambda_L)}$. For this purpose, using the eigenfunction property (6.24), we see from (6.26) that

$$\begin{aligned} f_{n,k}(x) &= -2 \frac{j_{n-1,k}}{L} \bar{\partial} \int_{\Lambda_L} dy G_{\Lambda_L}(x, y) e_{n-1,k}(y), \\ &= \frac{j_{n-1,k}}{L} \int_{\Lambda_L} dy \left(\frac{1}{2\pi} \frac{1}{\bar{x} - \bar{y}} - \frac{1}{2\pi} \frac{y}{L} \frac{1}{\bar{x}y/L - L} \right) e_{n-1,k}(y) \\ &= -\frac{j_{n-1,k}}{L} \int_{\Lambda_L} dy \left(\frac{1}{\pi} \bar{\partial}_y \log|x - y| + \frac{1}{\pi} \frac{y^2}{L^2} \bar{\partial}_y \log|x - L^2/\bar{y}| \right) e_{n-1,k}(y) \\ &= \frac{j_{n-1,k}}{L} \int_{\Lambda_L} dy \frac{1}{\pi} \log|x - y| \bar{\partial} e_{n-1,k}(y) \\ &\quad + \frac{1}{\pi L^2} \frac{j_{n-1,k}}{L} \int_{\Lambda_L} dy \log|x - L^2/\bar{y}| \partial(y^2 e_{n-1,k}(y)) \end{aligned} \quad (6.34)$$

where in the last step we integrated by parts and made use of the fact that $e_{n-1,k}$ vanishes on the boundary. Thus we find, for some universal constant $C > 0$,

$$\begin{aligned} |\bar{\partial} f_{n,k}(x)| &\leq C \frac{j_{n-1,k}}{L} \|\nabla e_{n-1,k}\|_{L^\infty(\Lambda_L)} \int_{\Lambda_L} dy \frac{1}{|x - y|} \\ &\quad + C \frac{j_{n-1,k}}{L^3} (L \|e_{n-1,k}\|_{L^\infty(\Lambda_L)} + L^2 \|\nabla e_{n-1,k}\|_{L^\infty(\Lambda_L)}) \int_{\Lambda_L} dy \frac{1}{|x - L^2/\bar{y}|}. \end{aligned} \quad (6.35)$$

Since $x \in \Lambda_L$, for the first integral we readily get the bound

$$\int_{\Lambda_L} dy \frac{1}{|x - y|} \leq \int_{|x-y| \leq 2L} dy \frac{1}{|x - y|} \leq CL \quad (6.36)$$

for a universal constant C . For the second integral, one finds on the other hand by rotation invariance that

$$\int_{\Lambda_L} dy \frac{1}{|x - L^2/\bar{y}|} \leq \int_{\Lambda_L} dy \frac{1}{|L - L^2/\bar{y}|} = L \int_{|u| \leq 1} \frac{du}{|1 - 1/\bar{u}|}. \quad (6.37)$$

The last integral here is simply some finite constant. Putting everything together and using (6.29) (and (6.28) to deduce that $j_{n-1,k}^2 \leq j_{n-1,k}^3$), we see that for some universal constant $C > 0$,

$$\|\bar{\partial} f_{n,k}\|_{L^\infty(\Lambda_L)} \leq CL \frac{j_{n-1,k}^3}{L^3}, \quad (6.38)$$

which leads to the claim, as discussed before.

The bounds (6.31) on the higher derivatives in the interior are a standard consequence of elliptic regularity theory for the Laplace operator. For example, one may apply [35, (4.19)] iteratively.

To see the decay of (6.32), by integrating by parts, it suffices to check this for $p = 0$ and for $e_{n,k}$ only. In this case, since $e_{n,k}$ is a Laplace eigenfunction, integration by parts shows that, for any q ,

$$\begin{aligned} \left| \int dx f(x) e_{n,k}(x) \right| &= \left(\frac{L^2}{j_{n,k}^2} \right)^q \left| \int dx (-\Delta)^q f(x) e_{n,k}(x) \right| \\ &\leq C \left(\frac{L^2}{j_{n,k}^2} \right)^{q-1/2} \|\Delta^q f\|_{L^\infty}, \end{aligned} \quad (6.39)$$

which gives the claimed bound. ■

6.4. The building blocks of the Dirac Green's function – I

We next introduce the key building blocks of our construction of the Dirac Green's function with a finite volume mass term. We begin with the following function which is the projection of the Laplacian Green's function to nonpositive Fourier modes related to the x -variable. More precisely, for $x, y \in \Lambda_L$, let

$$E_1(x, y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^2}{j_{n,k}^2} e_{-n,k}(x) \overline{e_{-n,k}(y)}, \quad (6.40)$$

where convergence is understood in $L^2(\Lambda_L \times \Lambda_L)$. We then define inductively, for $j \geq 1$, the functions

$$E_{j+1}(x, y) = \int_{\Lambda_L} du G_{\Lambda_L}(x, u) E_j(u, y), \quad (6.41)$$

$$F_j(x, y) = 4 \bar{\partial}_x \partial_y \overline{E_{j+1}(x, y)}. \quad (6.42)$$

That the derivatives indeed exist is a consequence of the explicit formulas we will derive below. These show that, for $y \in \Lambda_L$, the functions $E_1(x, y)$ and $F_1(x, y)$ are defined

pointwise for $x \neq y$, and that $E_j(x, y)$ and $F_j(x, y)$ with $j > 1$ are defined pointwise for all $x, y \in \Lambda_L$. We also note that, by (6.25),

$$E_j(x, y) = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \left(\frac{L^2}{j_{n,k}^2} \right)^j e_{-n,k}(x) \overline{e_{-n,k}(y)}, \quad (6.43)$$

as an element of $L^2(\Lambda_L \times \Lambda_L)$. Based on the definition of F_j and $f_{n,k}$, one then expects that also

$$F_j(x, y) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{L^2}{j_{n-1,k}^2} \right)^j f_{n,k}(x) \overline{f_{n,k}(y)}. \quad (6.44)$$

This is indeed true, and we prove it in Lemma 6.8 (for $j \geq 3$).

We begin calculating E_1 and F_1 using (6.19) in the next lemma.

Lemma 6.6. *For (almost every) $x, y \in \text{int}(\Lambda_L)$ with $x \neq y$,*

$$E_1(x, y) = \begin{cases} -\frac{1}{2\pi} \log |y| - \frac{1}{4\pi} \log \left(1 - \frac{\bar{x}}{y}\right) + \frac{1}{2\pi} \log L + \frac{1}{4\pi} \log \left(1 - \frac{\bar{x}y}{L^2}\right), & |x| < |y|, \\ -\frac{1}{2\pi} \log |x| - \frac{1}{4\pi} \log \left(1 - \frac{y}{x}\right) + \frac{1}{2\pi} \log L + \frac{1}{4\pi} \log \left(1 - \frac{\bar{x}y}{L^2}\right), & |x| > |y|, \end{cases} \quad (6.45)$$

$$F_1(x, y) = \begin{cases} -\frac{1}{4\pi} \log \left(1 - \frac{x}{y}\right), & |x| < |y|, \\ -\frac{1}{4\pi} \log \left(1 - \frac{y}{x}\right), & |x| > |y|, \end{cases} \quad (6.46)$$

where the branches of the logarithms are understood to be given by the series expansion of $\log(1+z)$ for $|z| < 1$, and, for $|x| = |y|$ with $x \neq y$, E_1 and F_1 are defined by continuity. In particular,

$$E_1(x, y) + F_1(x, y) = -\frac{1}{2\pi} \log |x - y| + \frac{1}{2\pi} \log(L^2 - \bar{x}y). \quad (6.47)$$

Proof. Let us write \tilde{E}_1 for the right-hand side of the claim. Using (6.19), we see that

$$G_{\Lambda_L}(x, y) - \tilde{E}_1(x, y) = \begin{cases} -\frac{1}{4\pi} \log \left(1 - \frac{x}{y}\right) + \frac{1}{4\pi} \log \left(1 - \frac{x\bar{y}}{L^2}\right), & |x| < |y|, \\ -\frac{1}{4\pi} \log \left(1 - \frac{y}{x}\right) + \frac{1}{4\pi} \log \left(1 - \frac{x\bar{y}}{L^2}\right), & |x| > |y|. \end{cases} \quad (6.48)$$

Going into polar coordinates, one can readily check from this (since there are only strictly positive Fourier modes when one expands the logarithms) that for $n \geq 0$ and $k \geq 1$,

$$\int_{\Lambda_L} dx (G_{\Lambda_L}(x, y) - \tilde{E}_1(x, y)) \overline{e_{-n,k}(x)} = 0. \quad (6.49)$$

Similarly one finds in polar coordinates that for $n > 0$ and $k \geq 1$ (again since there are only nonpositive Fourier modes in the expansion of the logarithms)

$$\int_{\Lambda_L} dx \tilde{E}_1(x, y) \overline{e_{n,k}(x)} = 0. \quad (6.50)$$

From these two facts, one finds immediately that $\tilde{E}_1 = E_1$.

For the claim for F_1 , note that from the identity for E_1 , for $y, u \in \text{int}(\Lambda_L)$,

$$\partial_y \overline{E_1(u, y)} = -\mathbf{1}\{|u| < |y|\} \frac{1}{4\pi} \frac{1}{y-u} = -\mathbf{1}\{|u| < |y|\} \frac{1}{4\pi} \frac{1}{y} \sum_{j=0}^{\infty} \left(\frac{u}{y}\right)^j. \quad (6.51)$$

Moreover, for $x, u \in \text{int}(\Lambda_L)$ we have

$$\bar{\partial}_x G_{\Lambda_L}(x, u) = -\frac{1}{4\pi} \frac{1}{\bar{x} - \bar{u}} - \frac{u}{4\pi L^2} \frac{1}{1 - \frac{\bar{x}u}{L^2}}. \quad (6.52)$$

In polar coordinates, one readily checks that the second term on the right-hand side is orthogonal to $\partial_y \overline{E_1(u, y)}$ (when integrated over u). One then finds

$$\begin{aligned} F_1(x, y) &= \int_{|u| < |y|} du \frac{1}{4\pi^2} \frac{1}{\bar{x} - \bar{u}} \frac{1}{y-u} \\ &= \begin{cases} \frac{1}{\bar{x}y} \frac{1}{4\pi^2} \sum_{j,k=0}^{\infty} \frac{1}{\bar{x}^j y^k} \int_{|u| < |y|} du \bar{u}^j u^k, & |x| > |y|, \\ \frac{1}{\bar{x}y} \frac{1}{4\pi^2} \sum_{j,k=0}^{\infty} \frac{1}{\bar{x}^j y^k} \int_{|u| < |x|} du \bar{u}^j u^k \\ \quad - \frac{1}{y} \frac{1}{4\pi^2} \sum_{j,k=0}^{\infty} \frac{\bar{x}^j}{y^k} \int_{|x| < |u| < |y|} du \bar{u}^{-j-1} u^k, & |x| < |y|, \end{cases} \\ &= \begin{cases} \frac{1}{4\pi} \sum_{j=0}^{\infty} \frac{1}{\bar{x}^{j+1} y^{j+1}} \frac{1}{j+1} |y|^{2j+2}, & |x| > |y|, \\ \frac{1}{4\pi} \sum_{j=0}^{\infty} \frac{1}{\bar{x}^{j+1} y^{j+1}} \frac{1}{j+1} |x|^{2j+2}, & |x| < |y|, \end{cases} \\ &= \begin{cases} -\frac{1}{4\pi} \log\left(1 - \frac{\bar{y}}{\bar{x}}\right), & |x| > |y|, \\ -\frac{1}{4\pi} \log\left(1 - \frac{x}{y}\right), & |x| < |y|. \end{cases} \end{aligned} \quad (6.53)$$

The claim that the values of $E_1(x, y)$ and $F_1(x, y)$ for $|x| = |y|$ with $x \neq y$ are given by continuity follows by noting that (6.41) and (6.42) are continuous away from the diagonal. Finally, (6.47) is a direct computation. This concludes the proof. ■

Lemma 6.7. *There exists a polynomial $P = P(L)$ such that for $L \geq 1$ and all $x, y \in \text{int}(\Lambda_L)$ with $x \neq y$,*

$$|E_1(x, y)| + |F_1(x, y)| \leq P(L)(1 + |\log|x - y||), \quad (6.54)$$

$$|\partial_x E_1(x, y)| + |\partial_x F_1(x, y)| \leq \frac{P(L)}{|x - y|}, \quad (6.55)$$

$$|E_2(x, y)| + |F_2(x, y)| + |\partial_x E_2(x, y)| + |\partial_x F_2(x, y)| \leq P(L). \quad (6.56)$$

Proof. We begin by bounding E_1 . By symmetry (up to complex conjugation), we can assume that $|x| < |y|$. We start from the elementary inequality

$$\left| -\frac{1}{2\pi} \log|y| - \frac{1}{4\pi} \log\left(1 - \frac{\bar{x}}{\bar{y}}\right) \right| \leq C + C|\log|y|| + C|\log|x - y||. \quad (6.57)$$

Since $|y| > |x|$, we have $|y| \geq \frac{1}{2}|y - x|$, so we conclude that for some (possibly different) $C > 0$,

$$\left| -\frac{1}{2\pi} \log|y| - \frac{1}{4\pi} \log\left(1 - \frac{\bar{x}}{\bar{y}}\right) \right| \leq C + C \log L + C|\log|x - y||. \quad (6.58)$$

Similarly, $|x| < |y|$ implies $|L^2 - x\bar{y}| = |y| \left| \frac{L^2}{|y|^2} y - x \right| \geq |y| |y - x| \geq \frac{1}{2} |y - x|^2$, which leads to

$$\begin{aligned} \left| \frac{1}{2\pi} \log L + \frac{1}{4\pi} \log \left(1 - \frac{\bar{x}y}{L^2} \right) \right| &\leq C + C |\log |L^2 - \bar{x}y|| \\ &\leq C + C \log L + C |\log |x - y||. \end{aligned} \tag{6.59}$$

Similar reasoning readily proves the analogous bound for F_1 . For $\partial_x E_1$ and $\partial_x F_2$, we obtain the required bound by noting that these derivatives are explicitly given by $1/(4\pi(x - y))$ or 0 by differentiating (6.45) and (6.46).

In order to bound E_2 , we start from the definition

$$E_2(x, y) = \int_{\Lambda_L} du G_{\Lambda_L}(x, u) E_1(u, y). \tag{6.60}$$

Using the bound for E_1 and that $|G_{\Lambda_L}(x, u)| \leq C(1 + |\log |x - u||)$, we readily see that E_2 is uniformly bounded by a polynomial in L . For the derivative, using $|\partial_x G_{\Lambda_L}(x, u)| \leq P(L)/|x - u|$, it similarly follows from the above bound for E_1 that

$$|\partial_x E_2(x, y)| \leq P(L) \int_{\Lambda_L} du \frac{1}{|x - u|} E_1(u, y) \leq P(L), \tag{6.61}$$

where the two polynomials $P(L)$ can be different. Again the bounds for F_2 are similar. ■

We next show that the E_j and F_j are bounded for $j \geq 3$.

Lemma 6.8. *For $j \geq 3$, E_j and F_j are given by the series (6.43) and (6.44), which converge uniformly in $\Lambda_L \times \Lambda_L$.*

Proof. For E_j , we already saw that it agrees with the series in an L^2 -sense. Let us now argue that the series converge uniformly and the E_j series can be differentiated termwise (which implies that F_j will be given by the corresponding series). This follows immediately from applying the bounds (6.29), (6.30), and (6.28) in the series representations (6.43) and (6.44). ■

Next we note that the E_j and F_j are smooth when tested against a smooth test function that is compactly supported in Λ_L .

Lemma 6.9. *For any $j \geq 1$ and $f \in C_c^\infty(\text{int}(\Lambda_L))$,*

$$\begin{aligned} y \mapsto \int_{\Lambda_L} dx f(x) E_j(x, y) &\in C^\infty(\text{int}(\Lambda_L)), \\ y \mapsto \int_{\Lambda_L} dx f(x) F_j(x, y) &\in C^\infty(\text{int}(\Lambda_L)). \end{aligned} \tag{6.62}$$

Proof. By (6.32) and (6.31), it follows that for any $p > 0$, $f \in C_c^\infty(\text{int}(\Lambda_L))$, and $K \subset \text{int}(\Lambda_L)$, there are constants $C_{p,f,K,L}$ such that

$$\sup_{y \in K} \left| \nabla_y^p \int dx f(x) e_{-n,k}(x) \overline{e_{-n,k}(y)} \right| \leq C_{p,f,K,L} j_{n,k}^{-p}, \tag{6.63}$$

and analogously for such expressions with $e_{-n,k}$ replaced by $f_{n,k}$. From this, the claim follows again by differentiating the series term by term. ■

As a final property of the functions E_j and F_j , we record the following recursion properties.

Lemma 6.10. *For $j, k \geq 1$ and $x, y \in \text{int}(\Lambda_L)$, we have*

$$E_{j+k}(x, y) = \int_{\Lambda_L} du E_j(x, u) E_k(u, y), \quad (6.64)$$

$$F_{j+k}(x, y) = \int_{\Lambda_L} du F_j(x, u) F_k(u, y), \quad (6.65)$$

$$\int_{\Lambda_L} du E_j(x, u) F_k(u, y) = \int_{\Lambda_L} du F_k(x, u) E_j(u, y) = 0. \quad (6.66)$$

Proof. The claim for E_{j+k} follows immediately from continuity and the representation (6.43) which implies that the two functions are the same as elements of $L^2(\Lambda_L \times \Lambda_L)$ (and thus in particular for almost all $x, y \in \text{int}(\Lambda_L)$).

For F_{j+k} , we integrate by parts (note that $\bar{\partial}_x \overline{E_{j+1}(x, u)}$ vanishes for $u \in \partial\Lambda_L$ by (6.41) and the fact that $E_j(v, u)$ vanishes for $u \in \partial\Lambda_L$ – which follows e.g. from the explicit representation of E_1 from Lemma 6.6 and (6.41)) and find

$$\begin{aligned} \int_{\Lambda_L} du F_j(x, u) F_k(u, y) &= 4\bar{\partial}_x \partial_y \int_{\Lambda_L} du \overline{E_{j+1}(x, u)} (-\Delta u) \overline{E_{k+1}(u, y)} \\ &= 4\bar{\partial}_x \partial_y \overline{E_{j+k+1}(x, y)} = F_{j+k}(x, y). \end{aligned} \quad (6.67)$$

where we have used the fact that $-\Delta_x E_{j+1}(x, y) = E_j(x, y)$ by (6.41) and the first claim of this lemma.

For the final claim, the fact that the first integral vanishes follows immediately from the remark that considering $E_j(x, u)$ and $F_k(u, y)$ in polar coordinates for u , E_j has only nonpositive Fourier modes while F_k has only strictly positive Fourier modes, so the claim follows from Fourier orthogonality. The vanishing of the second integral follows by a similar argument. ■

6.5. The building blocks of the Dirac Green's function – II

The functions E_j and F_j constructed above will turn out to be responsible for the singular behavior of our Dirac Green's function. To understand the behavior in μ , we introduce the following functions: for $m \geq 1$, let

$$\begin{aligned} R_{m;11}(x, y; \mu, L) &= (-1)^m \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m+1}}{\left(1 + \frac{\mu^2 L^2}{j_{n,k}^2}\right) j_{n,k}^{2(m+1)}} e_{-n,k}(x) \overline{e_{-n,k}(y)} \\ &\quad + (-1)^m \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m+1}}{\left(1 + \frac{\mu^2 L^2}{j_{n-1,k}^2}\right) j_{n-1,k}^{2(m+1)}} f_{n,k}(x) \overline{f_{n,k}(y)} \end{aligned} \quad (6.68)$$

and

$$\begin{aligned}
R_{m;21}(x, y; \mu, L) &= (-1)^{m+1} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n,k}^2}\right) j_{n,k}^{2(m+1)}} (2\partial e_{-n,k})(x) \overline{e_{-n,k}(y)} \\
&\quad + (-1)^{m+1} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n-1,k}^2}\right) j_{n-1,k}^{2(m+1)}} (2\partial f_{n,k})(x) \overline{f_{n,k}(y)}. \tag{6.69}
\end{aligned}$$

A priori, it may not be clear in what sense these series converge, but we now describe the basic facts we will need about these functions – including regularity.

Lemma 6.11. *For any $m \geq 3$ and $y \in \text{int}(\Lambda_L)$, the functions $x \mapsto R_{m;11}(x, y)$ and $x \mapsto R_{m;21}(x, y)$ are continuously differentiable in $\text{int}(\Lambda_L)$ and have the following properties:*

- (i) $\frac{1}{\mu} 2\partial_x R_{m;11}(x, y) = -R_{m;21}(x, y)$ for all $x, y \in \text{int}(\Lambda_L)$.
- (ii) $-2\bar{\partial}_x R_{m;21}(x, y) - \mu R_{m;11}(x, y) = (-1)^{m+1} \mu^{2m} (E_m(x, y) + F_m(x, y))$ for all $x, y \in \text{int}(\Lambda_L)$, with E_m as in (6.41) and F_m as in (6.42).
- (iii) *There exists a polynomial $P_m = P_m(L, |\mu|)$, which does not depend on x, y , such that*

$$\sup_{x, y \in \text{int}(\Lambda_L)} |R_m(x, y)| \leq P_m(L, |\mu|). \tag{6.70}$$

- (iv) *For any $f \in C_c^\infty(\text{int}(\Lambda_L))$, $y \mapsto \int_{\Lambda_L} dx f(x) R_m(x, y) \in C^\infty(\text{int}(\Lambda_L))$.*

Proof. For continuous differentiability, let us first consider $R_{m;11}$. Using (6.29) and (6.30), we find that, for some $C(L, \mu) > 0$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n,k}^2}\right) j_{n,k}^{2(m+1)}} \|\nabla e_{-n,k}\|_{L^\infty(\Lambda_L)} \|e_{-n,k}\|_{L^\infty(\Lambda_L)} \\
&\quad + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n-1,k}^2}\right) j_{n-1,k}^{2(m+1)}} \|\nabla f_{n,k}\|_{L^\infty(\Lambda_L)} \|f_{n,k}\|_{L^\infty(\Lambda_L)} \\
&\leq C(L, \mu) \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} j_{n,k}^{-2(m+1)+4}. \tag{6.71}
\end{aligned}$$

By (6.28), this series is convergent for $m \geq 3$, so standard results concerning uniform convergent series (involving continuity and differentiability) yields continuous differentiability. For R_{21} , we point out that, by (6.27) and (6.24),

$$2\partial e_{-n,k} = -\frac{j_{n,k}}{L} \overline{f_{n+1,k}}, \quad 2\partial f_{n,k} = -\frac{L}{j_{n-1,k}} \Delta e_{n-1,k} = \frac{j_{n-1,k}}{L} e_{n-1,k}. \tag{6.72}$$

Thus the same argument making use of (6.28)–(6.30) implies the continuous differentiability. (Now we end up with the series $\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} j_{n,k}^{-2(m+1)+5}$, which is still convergent for $m \geq 3$.)

We now turn to statement (i). This follows immediately from our preceding argument for continuous differentiability as it allows us to differentiate term by term.

For (ii), we note that again by our continuous differentiability argument, we can differentiate term by term. We find (using (6.27)) that

$$\begin{aligned}
-2\bar{\partial}_x R_{m;21}(x, y) &= (-1)^{m+1} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n,k}^2}\right) j_{n,k}^{2(m+1)}} (-\Delta e_{-n,k})(x) \overline{e_{-n,k}(y)} \\
&\quad + (-1)^{m+1} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n-1,k}^2}\right) j_{n-1,k}^{2(m+1)}} (-\Delta f_{n,k})(x) \overline{f_{n,k}(y)} \\
&= (-1)^{m+1} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2m} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n,k}^2}\right) j_{n,k}^{2m}} e_{-n,k}(x) \overline{e_{-n,k}(y)} \\
&\quad + (-1)^{m+1} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2m} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n-1,k}^2}\right) j_{n-1,k}^{2m}} f_{n,k}(x) \overline{f_{n,k}(y)} \quad (6.73)
\end{aligned}$$

and

$$\begin{aligned}
-2\bar{\partial}_x R_{m;21}(x, y) - \mu R_{m;11}(x, y) &= (-1)^{m+1} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n,k}^2}\right) j_{n,k}^{2(m+1)}} \left(\mu^2 + \frac{j_{n,k}^2}{L^2}\right) e_{-n,k}(x) \overline{e_{-n,k}(y)} \\
&\quad + (-1)^{m+1} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{L^{2(m+1)} \mu^{2m}}{\left(1 + \frac{\mu^2 L^2}{j_{n-1,k}^2}\right) j_{n-1,k}^{2(m+1)}} \left(\mu^2 + \frac{j_{n-1,k}^2}{L^2}\right) f_{n,k}(x) \overline{f_{n,k}(y)} \\
&= (-1)^{m+1} \mu^{2m} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \left(\frac{L^2}{j_{n,k}^2}\right)^m e_{-n,k}(x) \overline{e_{-n,k}(y)} \\
&\quad + (-1)^{m+1} \mu^{2m} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{L^2}{j_{n-1,k}^2}\right)^m f_{n,k}(x) \overline{f_{n,k}(y)}. \quad (6.74)
\end{aligned}$$

Recalling (6.43) and Lemma 6.8, this concludes the proof of claim (ii).

For (iii), we will prove the claim for $R_{m;21}$, the proof for $R_{m;11}$ being similar. Using (6.29) and (6.30), we find that, for some constant $C > 0$ (independent of m, μ, L)

$$\|R_{m;21}\|_{L^\infty(\Lambda_L \times \Lambda_L)} \leq CL^{2m-1} \mu^{2m} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} j_{n,k}^{-2m+2} \quad (6.75)$$

and the claim follows from (6.28).

Finally, to prove (iv), let $f \in C_c^\infty(\Lambda_L)$. Then (6.63) holds and an analogous bound holds with $e_{-n,k}$ replaced by $f_{n,k}$ or derivatives of these. Substituting this into the definition of $R_{m;ij}$, we see that all y -derivatives of the series that defines $\int dx f(x) R_{m;ij}(x, y)$ converge, and thus $\int dx f(x) R_{m;ij}(x, y)$ is smooth in y .

This concludes the proof. ■

6.6. The proof of Theorem 6.1

In this section, we first define our Dirac Green's function, then prove it satisfies the bounds stated in items (i) and (ii) of Theorem 6.1, then prove item (iii) of the theorem, namely analyticity in a neighborhood of the real axis, item (iv) of the theorem – a kind of resolvent identity – and finally prove convergence as $L \rightarrow \infty$. We split this section into parts where these tasks are carried out.

6.6.1. Constructing the Green's function. We begin by defining the function that will be our $S_{\mu 1_{\Lambda_L}}$ in Theorem 6.1, and then prove that it satisfies (6.2). For this definition, recall first the key building blocks E_j , F_j , $R_{m;11}$ and $R_{m;12}$ from (6.41), (6.42), (6.68), and (6.69).

Definition 6.12. For $\mu \in \mathbb{R}$, $L \geq 1$, and $y \in \text{int}(\Lambda_L)$ and $x \in \Lambda_L$, let

$$\begin{aligned} S_{\mu 1_{\Lambda_L};11}(x, y) &= \sum_{l=0}^2 (-1)^l \mu^{2l+1} (E_{l+1}(x, y) + F_{l+1}(x, y)) + R_{3;11}(x, y), \\ S_{\mu 1_{\Lambda_L};21}(x, y) &= 2\partial_x \sum_{l=0}^2 (-1)^{l+1} \mu^{2l} (E_{l+1}(x, y) + F_{l+1}(x, y)) + R_{3;21}(x, y), \quad (6.76) \\ S_{\mu 1_{\Lambda_L};12}(x, y) &= \overline{S_{\mu 1_{\Lambda_L};21}(x, y)}, \\ S_{\mu 1_{\Lambda_L};22}(x, y) &= \overline{S_{\mu 1_{\Lambda_L};11}(x, y)}, \end{aligned}$$

and for $y \in \text{int}(\Lambda_L)$ but $x = re^{i\theta} \in \Lambda_L^c$, define

$$S_{\mu 1_{\Lambda_L}}(x, y) = \frac{r^2 - L^2}{2\pi} \int_0^{2\pi} d\phi \frac{1}{r^2 + L^2 - 2rL \cos(\theta - \phi)} S_{\mu 1_{\Lambda_L}}(Le^{i\phi}, y). \quad (6.77)$$

In particular, for $x \in \Lambda_L^c$, $S_{\mu 1_{\Lambda_L}}(x, y)$ is the harmonic extension of $S_{\mu 1_{\Lambda}}(\cdot, y)|_{\partial\Lambda_L}$. We now show that this is indeed a Green's function for the problem we are considering.

Proposition 6.13. $S_{\mu 1_{\Lambda_L}}$ defined in Definition 6.12 satisfies (6.2).

Proof. Our goal is to show that $S_{\mu 1_{\Lambda_L}}(\cdot, y)$ as defined in Definition 6.12 vanishes at infinity and that for each $f \in C_c^\infty(\mathbb{R}^2)$ and $y \in \text{int}(\Lambda_L)$,

$$\begin{aligned} \mu \int_{\Lambda_L} dx S_{\mu 1_{\Lambda_L}}(x, y) f(x) - 2 \int_{\mathbb{R}^2} dx \begin{pmatrix} \bar{\partial} f(x) S_{\mu 1_{\Lambda_L};21}(x, y) & \bar{\partial} f(x) S_{\mu 1_{\Lambda_L};22}(x, y) \\ \partial f(x) S_{\mu 1_{\Lambda_L};11}(x, y) & \partial f(x) S_{\mu 1_{\Lambda_L};12}(x, y) \end{pmatrix} \\ = \begin{pmatrix} f(y) & 0 \\ 0 & f(y) \end{pmatrix}. \quad (6.78) \end{aligned}$$

Writing out $DS_{\mu 1_L}$ from (6.2) explicitly, for $x, y \in \text{int}(\Lambda_L)$ we have

$$\begin{aligned} DS_{\mu 1_L}(x, y) \\ = \begin{pmatrix} 2\bar{\partial} S_{\mu 1_{\Lambda_L};21}(x, y) + \mu S_{\mu 1_{\Lambda_L};11}(x, y) & 2\bar{\partial} \overline{S_{\mu 1_{\Lambda_L};11}(x, y)} + \mu \overline{S_{\mu 1_{\Lambda_L};21}(x, y)} \\ 2\partial S_{\mu 1_{\Lambda_L};11}(x, y) + \mu S_{\mu 1_{\Lambda_L};21}(x, y) & 2\partial \overline{S_{\mu 1_{\Lambda_L};21}(x, y)} + \mu \overline{S_{\mu 1_{\Lambda_L};11}(x, y)} \end{pmatrix}. \quad (6.79) \end{aligned}$$

We can thus focus on the first column. Let us first consider the 21-entry. Using Definition 6.12 and Lemma 6.11, we find immediately that

$$2\partial_x S_{\mu 1_{\Lambda_L}; 11}(x, y) + \mu S_{\mu 1_{\Lambda_L}; 21}(x, y) = 0 \quad \text{for } x \in \text{int}(\Lambda_L). \quad (6.80)$$

For the 11-entry, we have similarly, using Definition 6.12 and Lemma 6.11,

$$\begin{aligned} & 2\bar{\partial}_x S_{\mu 1_{\Lambda_L}; 21}(x, y) + \mu S_{\mu 1_{\Lambda_L}; 11}(x, y) \\ &= \sum_{l=0}^2 (-1)^l \mu^{2l} (\mu^2 - \Delta_x) (E_{l+1}(x, y) + F_{l+1}(x, y)) \\ &\quad + 2\bar{\partial}_x R_{3; 21}(x, y) - \mu R_{3; 11}(x, y) \\ &= \sum_{l=0}^2 (-1)^l \mu^{2l} (\mu^2 - \Delta_x) (E_{l+1}(x, y) + F_{l+1}(x, y)) \\ &\quad - \mu^6 (E_3(x, y) + F_3(x, y)). \end{aligned} \quad (6.81)$$

Note that for $l \geq 1$, we have, from (6.41) and (6.42),

$$-\Delta_x (E_{l+1}(x, y) + F_{l+1}(x, y)) = E_l(x, y) + F_l(x, y). \quad (6.82)$$

Thus we see that there are cancellations in the sum and we have in fact

$$2\bar{\partial}_x S_{\mu 1_{\Lambda_L}; 21}(x, y) + \mu S_{\mu 1_{\Lambda_L}; 11}(x, y) = -\Delta_x (E_1(x, y) + F_1(x, y)). \quad (6.83)$$

By Lemma 6.6, we see that

$$E_1(x, y) + F_1(x, y) = \frac{1}{2\pi} \log \frac{1}{|x-y|} + \frac{1}{2\pi} \log L + \frac{1}{4\pi} \log \left(1 - \frac{\bar{x}y}{L^2} \right). \quad (6.84)$$

As the latter term here is harmonic (or actually anti-analytic) in $\text{int}(\Lambda_L)$, we have

$$\begin{aligned} & 2\bar{\partial}_x S_{\mu 1_{\Lambda_L}; 21}(x, y) + \mu S_{\mu 1_{\Lambda_L}; 11}(x, y) = -\Delta_x (E_1(x, y) + F_1(x, y)) \\ &= -\Delta_x \frac{1}{2\pi} \log \frac{1}{|x-y|} = \delta_y(x). \end{aligned} \quad (6.85)$$

Let us consider now the case $x \in \Lambda_L^\circ$. Note that for $x \in \Lambda_L^\circ$ and $y \in \text{int}(\Lambda_L)$, to prove (6.2) we need to show that

$$\begin{pmatrix} 2\bar{\partial}_x S_{\mu 1_{\Lambda_L}; 21}(x, y) & 2\bar{\partial}_x \overline{S_{\mu 1_{\Lambda_L}; 11}(x, y)} \\ 2\partial_x S_{\mu 1_{\Lambda_L}; 11}(x, y) & 2\partial_x \overline{S_{\mu 1_{\Lambda_L}; 21}(x, y)} \end{pmatrix} = 0 \quad (6.86)$$

and that $x \mapsto S_{\mu 1_{\Lambda_L}}(x, y)$ vanishes at infinity. For $x \in \Lambda_L^\circ$, we note that $S_{\mu 1_{\Lambda_L}}(x, y)$ is defined as the harmonic extension of $S_{\mu 1_{\Lambda_L}}(\cdot, y)|_{\partial\Lambda_L}$ (where the boundary values are understood as being given by a limit from the interior of Λ_L). Thus our goal is equivalent to $S_{\mu 1_{\Lambda_L}; 21}(\cdot, y)|_{\partial\Lambda_L}$ having only (strictly) negative Fourier modes and $S_{\mu 1_{\Lambda_L}; 11}(\cdot, y)|_{\partial\Lambda_L}$ only (strictly) positive Fourier modes.

Recalling that $e_{-n,k}$ vanishes on $\partial\Lambda_L$ while $f_{n,k}(Le^{i\theta})$ is proportional to $e^{in\theta}$, we see from (6.68) that $R_{3;11}|_{\partial\Lambda_L}$ has only positive Fourier modes. Similarly $E_j(\cdot, y)$ vanishes on $\partial\Lambda_L$ while $F_j(\cdot, y)|_{\partial\Lambda_L}$ has only positive Fourier modes (this follows from (6.42) since E_{j+1} has only negative Fourier modes), so we see indeed that $S_{\mu 1_{\Lambda_L};11}(\cdot, y)|_{\partial\Lambda_L}$ has only positive Fourier modes. Thus $S_{\mu 1_{\Lambda_L};11}(\cdot, y)$ is of the correct form. The argument for $S_{\mu 1_{\Lambda_L};21}(\cdot, y)$ is similar, but makes use of the fact that $\partial e_{-n+1,k} \propto \overline{f_{n,k}}$ and $\partial f_{n,k} \propto \Delta e_{n-1,k} \propto e_{n-1,k}$ – we omit the details. We conclude that, for $x \in \Lambda_L^c$ and $y \in \text{int}(\Lambda_L)$,

$$DS_{\mu 1_{\Lambda_L}}(x, y) = 0. \quad (6.87)$$

The claim (6.78) now follows by splitting the integral over \mathbb{R}^2 into that over Λ_L and Λ_L^c , integrating by parts – the boundary terms cancel due to continuity of $x \mapsto S_{\mu 1_{\Lambda_L}}(x, y)$ across the boundary – and combining (6.80), (6.85), and (6.87). Vanishing at infinity follows from the fact that the entries of $S_{\mu 1_{\Lambda_L}}(\cdot, y)|_{\partial\Lambda_L}$ had only strictly positive or negative Fourier modes (and the corresponding entries were given by either antiholomorphic or holomorphic continuation of these boundary values), so $S_{\mu 1_{\Lambda_L}}(x, y)$ decays at worst like $|x|^{-1}$ as $x \rightarrow \infty$. ■

We now turn to proving that $S_{\mu 1_{\Lambda_L}}$ satisfies the bounds we are after.

6.6.2. Proof of Theorem 6.1 (i)–(ii): Bounds on the Green's function. The goal of this section is to prove the following proposition which is precisely item (i) and item (ii) of Theorem 6.1.

Proposition 6.14. *For $L \geq 1$ and $\mu \in \mathbb{R}$, we have for some polynomial $P = P(L, |\mu|)$, which is independent of $x \in \mathbb{C}$ and $y \in \text{int}(\Lambda_L)$, the estimates*

$$|S_{\mu 1_{\Lambda_L}}(x, y) - S_0(x, y)| \leq P(L, |\mu|)(1 + |\log|x - y||) \quad \text{for } x \in \Lambda_L, \quad (6.88)$$

$$|S_{\mu 1_{\Lambda_L}}(x, y)| \leq \frac{P(L, |\mu|)}{L - |y|} \quad \text{for } x \in \Lambda_L^c. \quad (6.89)$$

Proof. For $x, y \in \text{int}(\Lambda_L)$, it follows from Lemma 6.7 that

$$|E_1(x, y)| + |F_1(x, y)| \leq P(L)(1 + |\log|x - y||), \quad (6.90)$$

$$|\partial_x E_1(x, y)| + |\partial_x F_1(x, y)| \leq \frac{P(L)}{|x - y|}, \quad (6.91)$$

$$|E_2(x, y)| + |F_2(x, y)| + |\partial_x E_2(x, y)| + |\partial_x F_2(x, y)| \leq P(L), \quad (6.92)$$

and note that S_0 is given by

$$S_{0;11}(x, y) = S_{0;22}(x, y) = 0, \quad S_{0;21}(x, y) = -2\partial_x(E_1(x, y) + F_1(x, y)), \quad (6.93)$$

and complex conjugation for the 12-entry. Hence the singular $\partial_x(E_1 + F_1)$ term in the definition of $S_{\mu 1_{\Lambda};21}$ in Definition 6.12 is canceled by $S_{0;21}$ (and analogously for the 21-entry). The above bounds on the E_1, F_1, E_2, F_2 together with the bounds from

Lemma 6.11 for R_3 thus readily imply the required bounds for $S_{\mu 1_{\Lambda_L}}(x, y)$ when $x, y \in \text{int}(\Lambda_L)$.

For $x \in \Lambda_L^c$, the claim follows from the maximum principle. Indeed, $S_{1_{\Lambda_L}}(\cdot, y)$ is harmonic in Λ_L^c , vanishes at infinity, and is continuous up to the boundary. ■

6.6.3. *Proof of Theorem 6.1 (iii): Analyticity in μ .* The goal of this section is to prove item (iii) of Theorem 6.1, which is implied by the following proposition.

Proposition 6.15. *For $x, y \in \text{int}(\Lambda_L)$ with $x \neq y$, the function $\mu \mapsto S_{\mu 1_{\Lambda_L}}(x, y)$ has an analytic continuation into some L -dependent neighborhood of the real axis. In this neighborhood, the estimate (6.5) continuous to hold. Moreover, in an L -dependent neighborhood of the origin, we have*

$$S_{\mu 1_{\Lambda_L};11}(x, y) = \sum_{l=0}^{\infty} (-1)^l \mu^{2l+1} (E_{l+1}(x, y) + F_{l+1}(x, y)), \quad (6.94)$$

$$S_{\mu 1_{\Lambda_L};21}(x, y) = \sum_{l=0}^{\infty} (-1)^{l+1} \mu^{2l} 2\partial_x (E_{l+1}(x, y) + F_{l+1}(x, y)), \quad (6.95)$$

where E_{l+1} and F_{l+1} are as in (6.41) and (6.42). In particular,

$$\lim_{\mu \rightarrow 0} S_{\mu 1_{\Lambda_L}}(x, y) = \frac{1}{2\pi} \begin{pmatrix} 0 & 1/(\bar{x} - \bar{y}) \\ 1/(x - y) & 0 \end{pmatrix}. \quad (6.96)$$

Proof. By Definition 6.12, to prove analyticity, it is enough to prove analyticity of R_{11} and R_{21} . For this purpose, consider $\mu \in \mathbb{C}$ with $|\text{Im}(\mu)| < \frac{1}{2} \frac{3\pi}{4L}$. For such μ we have $\frac{\text{Im}(\mu)^2 L^2}{j_{n,k}^2} < \frac{1}{4}$ by (6.28) and

$$\left| 1 + \frac{\mu^2 L^2}{j_{n,k}^2} \right|^2 = \left(1 + \frac{(\text{Re}(\mu)^2 - \text{Im}(\mu)^2) L^2}{j_{n,k}^2} \right)^2 + \left(\frac{2 \text{Re}(\mu) \text{Im}(\mu) L^2}{j_{n,k}^2} \right)^2 \geq \frac{9}{16}. \quad (6.97)$$

Thus retracing our proof of Lemma 6.11, we can check that the series defining R_{11} and R_{21} converge uniformly in μ in such a complex strip. It then follows, for example, by Morera's theorem that R_{11} and R_{21} are analytic functions in μ on such a strip.

For the analogue of (6.5), we note that the proof of Proposition 6.14 works in this setting as well, and we recover our bounds.

The expansion in terms of E_l and F_l in a neighborhood of the origin follows readily from similar arguments and the definition of R_{11} and R_{21} along with (6.43) and Lemma 6.8.

For the claim about the $\mu \rightarrow 0$ limit, we see from the expansions that $\lim_{\mu \rightarrow 0} S_{\mu 1_{\Lambda_L};11} = 0$, while

$$\lim_{\mu \rightarrow 0} S_{\mu 1_{\Lambda_L};21}(x, y) = -2\partial_x (E_1(x, y) + F_1(x, y)) \quad (6.98)$$

and the claim for the 21-entry follows from (6.47). The claim for the 12- and 22-entries follows simply by complex conjugation (recalling Definition 6.12). ■

Our next goal is to establish a type of resolvent identity for $S_{1_{\Lambda_L}}$.

6.6.4. *Proof of Theorem 6.1 (iv): A resolvent identity.* The goal of this section is to prove item (iv) in Theorem 6.1, that is, the following result.

Proposition 6.16. *For any $L \geq 1$ and $x, y \in \Lambda_L$ with $x \neq y$, we have for $\mu \in \mathbb{R}$,*

$$\partial_\mu S_{\mu 1_{\Lambda_L}; ij}(x, y) = - \sum_{k \in \{1, 2\}} \int_{\Lambda_L} du S_{\mu 1_{\Lambda_L}; ik}(x, u) S_{\mu 1_{\Lambda_L}; kj}(u, y). \quad (6.99)$$

Proof. By the definition of $S_{\mu 1_{\Lambda_L}; 22}(x, y)$ and $S_{\mu 1_{\Lambda_L}; 12}(x, y)$ from Definition 6.12, it is sufficient to prove the claim for $S_{\mu 1_{\Lambda_L}; 11}(x, y)$ and $S_{\mu 1_{\Lambda_L}; 21}(x, y)$. Moreover, as one readily checks from Proposition 6.15 that both sides are analytic functions of μ in a neighborhood of the real axis, it is enough for us to verify the claim for μ in the neighborhood of the origin where we can use the series expansion of Proposition 6.15. Our key tool in the proof will be Lemma 6.10.

Let us begin with $S_{\mu 1_{\Lambda_L}; 11}$. Using the expansion of Proposition 6.15, we have

$$\partial_\mu S_{\mu 1_{\Lambda_L}; 11}(x, y) = \sum_{l=0}^{\infty} (2l+1)(-1)^l \mu^{2l} (E_{l+1}(x, y) + F_{l+1}(x, y)). \quad (6.100)$$

Also, using again Proposition 6.15 for the expansions and Lemma 6.10 to calculate the integrals, we get

$$\begin{aligned} & \sum_{k \in \{1, 2\}} \int_{\Lambda_L} du S_{\mu 1_{\Lambda_L}; 1k}(x, u) S_{\mu 1_{\Lambda_L}; k1}(u, y) \\ &= \sum_{l, m=0}^{\infty} (-1)^{l+m} \mu^{2l+2m+2} \int_{\Lambda_L} du (E_{l+1}(x, u) + F_{l+1}(x, u)) \\ & \quad \times (E_{m+1}(u, y) + F_{m+1}(u, y)) \\ & \quad + \sum_{l, m=0}^{\infty} (-1)^{l+m} \mu^{2l+2m} \int_{\Lambda_L} du (2\bar{\partial}_x \overline{E_{l+1}(x, u)} + 2\bar{\partial}_x \overline{F_{l+1}(x, u)}) \\ & \quad \times (2\partial_u E_{m+1}(u, y) + 2\partial_u F_{m+1}(u, y)). \end{aligned} \quad (6.101)$$

The integrals without the derivatives can be evaluated immediately from Lemma 6.10. For the derivative terms, note that

$$\begin{aligned} \bar{\partial}_x \overline{F_{l+1}(x, u)} &= \bar{\partial}_x 4\partial_x \bar{\partial}_u E_{l+2}(x, u) = -\bar{\partial}_u E_{l+1}(x, u), \\ \partial_u F_{m+1}(u, y) &= -\partial_y \overline{E_{m+1}(u, y)}. \end{aligned} \quad (6.102)$$

Thus, integrating by parts and recalling that E_j vanishes on $\partial\Lambda_L$ (with respect to either variable), and using Lemma 6.10, we find

$$\begin{aligned} & \int_{\Lambda_L} du (2\bar{\partial}_x \overline{E_{l+1}(x, u)} + 2\bar{\partial}_x \overline{F_{l+1}(x, u)}) (2\partial_u E_{m+1}(u, y) + 2\partial_u F_{m+1}(u, y)) \\ &= - \int_{\Lambda_L} du F_l(x, u) E_{m+1}(u, y) - 4\bar{\partial}_x \partial_y \int_{\Lambda_L} du \overline{E_{l+1}(x, u)} E_{m+1}(u, y) \\ & \quad - \int_{\Lambda_L} du E_{l+1}(x, u) E_m(u, y) - \int_{\Lambda_L} du E_{l+1}(x, u) F_m(u, y) \\ &= -F_{l+m+1}(x, y) - \overline{E_{l+m+1}(x, y)}. \end{aligned} \quad (6.103)$$

We conclude that

$$\begin{aligned}
& \sum_{k \in \{1,2\}} \int_{\Lambda_L} du S_{\mu 1_{\Lambda_L}; 1k}(x, u) S_{\mu 1_{\Lambda_L}; k1}(u, y) \\
&= \sum_{l,m=0}^{\infty} (-1)^{l+m} \mu^{2l+2m+2} (E_{l+m+2}(x, y) + F_{l+m+2}(x, y)) \\
&\quad - \sum_{l,m=0}^{\infty} (-1)^{l+m} \mu^{2l+2m} (E_{l+m+1}(x, y) + F_{l+m+1}(x, y)) \\
&= \sum_{n=0}^{\infty} \left[\sum_{l,m=0}^{\infty} \mathbf{1}\{l+m+1=n\} \right] (-1)^{n+1} \mu^{2n} (E_{n+1}(x, y) + F_{n+1}(x, y)) \\
&\quad + \sum_{n=0}^{\infty} \left[\sum_{l,m=0}^{\infty} \mathbf{1}\{l+m=n\} \right] (-1)^{n+1} \mu^{2n} (E_{n+1}(x, y) + F_{n+1}(x, y)). \quad (6.104)
\end{aligned}$$

Noting that

$$\sum_{l,m=0}^{\infty} \mathbf{1}\{l+m+1=n\} + \sum_{l,m=0}^{\infty} \mathbf{1}\{l+m=n\} = n + (n+1) = 2n+1, \quad (6.105)$$

we see that

$$- \sum_{k \in \{1,2\}} \int_{\Lambda_L} du S_{\mu 1_{\Lambda_L}; 1k}(x, u) S_{\mu 1_{\Lambda_L}; k1}(u, y) = \partial_{\mu} S_{\mu 1_{\Lambda_L}; 11}(x, y) \quad (6.106)$$

as was required.

We now turn to the 21-entry. For this, we begin with the remark (from Proposition 6.15) that

$$\partial_{\mu} S_{\mu 1_{\Lambda_L}; 21}(x, y) = \sum_{l=1}^{\infty} 2l (-1)^{l+1} \mu^{2l-1} 2\partial_x (E_{l+1}(x, y) + F_{l+1}(x, y)). \quad (6.107)$$

On the other hand, we have

$$\begin{aligned}
& \sum_{k \in \{1,2\}} \int_{\Lambda_L} du S_{\mu 1_{\Lambda_L}; 2k}(x, u) S_{\mu 1_{\Lambda_L}; k1}(u, y) \\
&= \sum_{l,m=0}^{\infty} (-1)^{l+m+1} \mu^{2l+2m+1} \int_{\Lambda_L} du 2\partial_x (E_{l+1}(x, u) + F_{l+1}(x, u)) \\
&\quad \times (E_{m+1}(u, y) + F_{m+1}(u, y)) \\
&\quad + \sum_{l,m=0}^{\infty} (-1)^{l+m+1} \mu^{2l+2m+1} \int_{\Lambda_L} du (\overline{E_{l+1}(x, u)} + \overline{F_{l+1}(x, u)}) \\
&\quad \times 2\partial_u (E_{m+1}(u, y) + F_{m+1}(u, y)). \quad (6.108)
\end{aligned}$$

The first integrals can again be evaluated directly by taking the x -derivative outside the integral and using Lemma 6.10. For the second integrals, we treat various terms in different ways: the \overline{E} - E term we integrate by parts and note as before that

$$-\partial_u \overline{E_{l+1}}(x, u) = \partial_x F_{l+1}(x, u), \quad (6.109)$$

which by Lemma 6.10 leads to a term which integrates to zero.

In the \overline{F} - E term we write $\overline{F_{l+1}}(x, u) = 4\partial_x \overline{\partial_u E_{l+2}}(x, u)$ and integrate by parts the u -derivative, which (by Lemma 6.10) leads to

$$2\partial_x \int_{\Lambda_L} du E_{l+2}(x, u)(-\Delta_u)E_{m+1}(u, y) = 2\partial_x E_{l+m+2}(x, y). \quad (6.110)$$

For the \overline{E} - F and \overline{F} - F terms we note that $2\partial_u F_{m+1}(u, y) = -2\partial_y \overline{E_{m+1}}(u, y)$. Thus by Lemma 6.10 (and a similar argument as before utilizing the definition of F_j), the \overline{F} - F term integrates to zero while

$$\int_{\Lambda_L} du \overline{E_{l+1}}(x, u)2\partial_u F_{m+1}(u, y) = -2\partial_y \overline{E_{l+m+2}}(x, y) = 2\partial_x F_{l+m+2}(x, y). \quad (6.111)$$

Putting everything together, we conclude that both types of integrals have the same total contribution and

$$\begin{aligned} & \sum_{k \in \{1, 2\}} \int_{\Lambda_L} du S_{\mu 1_{\Lambda_L}; 2k}(x, u) S_{\mu 1_{\Lambda_L}; k1}(u, y) \\ &= 2 \sum_{l, m=0}^{\infty} (-1)^{l+m+1} \mu^{2l+2m+1} 2\partial_x (E_{l+m+2}(x, y) + F_{l+m+2}(x, y)) \\ &= 2 \sum_{n=0}^{\infty} \left[\sum_{l, m=0}^{\infty} \mathbf{1}\{l+m=n\} \right] (-1)^n \mu^{2n+1} 2\partial_x (E_{n+2}(x, y) + F_{n+2}(x, y)) \\ &= \sum_{n=0}^{\infty} 2(n+1)(-1)^{n+1} \mu^{2n+1} 2\partial_x (E_{n+2}(x, y) + F_{n+2}(x, y)) \\ &= \sum_{n=1}^{\infty} 2n(-1)^n \mu^{2n-1} 2\partial_x (E_{n+1}(x, y) + F_{n+1}(x, y)), \end{aligned} \quad (6.112)$$

which is precisely of the desired form and we are thus done. \blacksquare

Finally, we turn to convergence as $L \rightarrow \infty$.

6.6.5. Proof of Theorem 6.1 (v): The $L \rightarrow \infty$ limit. In this section, we prove item (v) of Theorem 6.1. We state this separately as the following proposition.

Proposition 6.17. *For $\mu \neq 0$, as $L \rightarrow \infty$,*

$$S_{\mu 1_{\Lambda_L}}(x, y) \rightarrow -\frac{1}{2\pi} \begin{pmatrix} -\mu K_0(|\mu| |x-y|) & 2\overline{\partial}_x K_0(|\mu| |x-y|) \\ 2\partial_x K_0(|\mu| |x-y|) & -\mu K_0(|\mu| |x-y|) \end{pmatrix} =: S_\mu(x, y) \quad (6.113)$$

uniformly on compact subsets of $\{(x, y) \in \mathbb{C}^2 : x \neq y\}$.

For the proof of Proposition 6.17, we will need the following result which can also be interpreted as a resolvent identity.

Lemma 6.18. For $x, y \in \text{int}(\Lambda_L)$, $x \neq y$,

$$S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) - S_\mu(x, y) = \mu \int_{\Lambda_L^c} du S_\mu(x, u) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y). \quad (6.114)$$

Proof. It is of course sufficient for us to prove that for any $f, g \in C_c^\infty(\text{int}(\Lambda_L))$ with disjoint supports,

$$\begin{aligned} & \int_{\Lambda_L \times \Lambda_L} dx dy f(x) g(y) [S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) - S_\mu(x, y)] \\ &= \mu \int_{\Lambda_L \times \Lambda_L} dx dy f(x) g(y) \int_{\Lambda_L^c} du S_\mu(x, u) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y). \end{aligned} \quad (6.115)$$

Using the disjointness of the supports of f and g , Proposition 6.14 for $S_{\mu\mathbf{1}_{\Lambda_L}}$, and routine asymptotics of Bessel functions for S_μ , we see that the integrands here are L^1 -functions. By Fubini, we can thus perform the integrals in any order we wish. We now claim that, on the left-hand side of (6.115),

$$\begin{aligned} y &\mapsto g(y) \int_{\Lambda_L} dx f(x) S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) \in C_c^\infty(\text{int}(\Lambda_L)), \\ x &\mapsto f(x) \int_{\Lambda_L} dy g(y) S_\mu(x, y) \in C_c^\infty(\text{int}(\Lambda_L)). \end{aligned} \quad (6.116)$$

The fact that these functions have compact supports follows from f and g having compact support. The smoothness of the $S_{\mu\mathbf{1}_{\Lambda_L}}$ term follows from Lemma 6.11 (iv) and (6.62). The smoothness of the S_μ term follows immediately from the explicit expression of S_μ which is smooth off the diagonal.

By definition of the Green's functions, we have $(i\partial_x + \mu)S_\mu(x, u) = \delta(x - u)$ and $(i\partial_x + \mu\mathbf{1}_{\Lambda_L}(x))S_{\mu\mathbf{1}_{\Lambda_L}}(x, u) = \delta(x - u)$. Since $S_\mu(x, u)$ is a function of $x - u$, also $(-i\partial_u + \mu)S_\mu(x, u) = \delta(x - u)$. Thus the above smoothness (and integration by parts) implies that

$$\begin{aligned} & \int_{\Lambda_L \times \Lambda_L} dx dy f(x) g(y) [S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) - S_\mu(x, y)] \\ &= \int_{\mathbb{R}^2} du \left[(-i\partial_u - \mu) \int_{\Lambda_L} dx f(x) S_\mu(x, u) \right] \left[\int_{\Lambda_L} dy g(y) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y) \right] \\ & \quad - \int_{\mathbb{R}^2} du \left[\int_{\Lambda_L} dx f(x) S_\mu(x, u) \right] \left[(i\partial_u + \mu\mathbf{1}_{\Lambda_L}(u)) \int_{\Lambda_L} dy g(y) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y) \right] \\ &= \int_{\mathbb{R}^2} du \left[\int_{\Lambda_L} dx f(x) S_\mu(x, u) \right] \left[(i\partial_u + \mu) \int_{\Lambda_L} dy g(y) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y) \right] \\ & \quad - \int_{\mathbb{R}^2} du \left[\int_{\Lambda_L} dx f(x) S_\mu(x, u) \right] \left[(i\partial_u + \mu\mathbf{1}_{\Lambda_L}(u)) \int_{\Lambda_L} dy g(y) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y) \right] \\ &= \int_{\mathbb{R}^2} du \left[\int_{\Lambda_L} dx f(x) S_\mu(x, u) \right] (\mu - \mu\mathbf{1}_{\Lambda_L}(u)) \left[\int_{\Lambda_L} dy g(y) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y) \right], \end{aligned} \quad (6.117)$$

which is the right-hand side of (6.114). ■

We now turn to the proof of the final claim of Theorem 6.1.

Proof of Proposition 6.17. We can assume that L is so large that $x, y \in \text{int}(\Lambda_L)$. By Lemma 6.18,

$$S_{\mu\mathbf{1}_{\Lambda_L}}(x, y) - S_\mu(x, y) = \mu \int_{\Lambda_L^c} du S_\mu(x, u) S_{\mu\mathbf{1}_{\Lambda_L}}(u, y). \quad (6.118)$$

Using the fact that for any fixed $a > 0$, the Bessel function K_0 satisfies, for $|x| \geq a$, $|K_0(|\mu||x|)| \leq C_a e^{-|\mu||x|}$ for some constant C_a (independent of μ, x) and a similar bound for $\partial K_0(|\mu||x|)$, we find from Proposition 6.14 that for some polynomial $P = P(L, |\mu|)$,

$$|S_\mu(x, y) - S_{\mu\mathbf{1}_{\Lambda_L}}(x, y)| \leq \frac{P(L, |\mu|)}{L - |y|} \int_{\Lambda_L^c} du e^{-|\mu||x-u|}. \quad (6.119)$$

As we take x, y in a fixed compact subset B of \mathbb{C} , $|\int_{\Lambda_L^c} e^{-|\mu||x-u|} du| \leq e^{-\alpha|\mu|L}$ uniformly in $x \in B$ for some $\alpha > 0$ depending only on B . We thus deduce that given a fixed compact subset $K \subset \{(x, y) \in \mathbb{C}^2 : x \neq y\}$ (independent of L) and $\mu \neq 0$,

$$\lim_{L \rightarrow \infty} \sup_{(x, y) \in K} |S_\mu(x, y) - S_{\mu\mathbf{1}_{\Lambda_L}}(x, y)| = 0, \quad (6.120)$$

which was the claim. \blacksquare

Putting together the propositions from this section also concludes our proof of Theorem 6.1, and thus that of Theorem 3.3.

Appendix A. Truncated and free fermion correlations

In this appendix, we collect some well-known properties of truncated correlations (joint cumulants) and free fermion correlations.

A.1. Truncated correlations

For arbitrary random variables A_i , the truncated correlations are defined by

$$\langle A_1 \cdots A_n \rangle_t^T = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \Big|_{t=0} \log \langle e^{\sum_{i=1}^n t_i A_i} \rangle \quad (A.1)$$

when the right-hand side exists. For $N \in \mathbb{N}$ and $t = (t_1, \dots, t_N)$, it is often convenient to define the tilted measure with expectation $\langle \cdot \rangle_t$ by

$$\langle F \rangle_t = \frac{\langle F e^{tA} \rangle}{\langle e^{tA} \rangle}, \quad e^{tA} = e^{\sum_{i=1}^N t_i A_i}, \quad (A.2)$$

when these expressions exist. For $1 \leq n \leq N - 1$, it then follows from (A.1) that

$$\langle A_1 \cdots A_{n+1} \rangle_t^T = \frac{\partial}{\partial t_{n+1}} \langle A_1 \cdots A_n \rangle_t^T. \quad (A.3)$$

The next lemma shows that the definition (A.1) is consistent with (1.6).

Lemma A.1. *Assume that A_1, \dots, A_n are random variables. Then*

$$\langle A_1 \cdots A_n \rangle^T = \langle A_1 \cdots A_n \rangle - \sum_{P \in \mathfrak{P}_n} \prod_j \left\langle \prod_{i \in P_j} A_i \right\rangle^T \quad (\text{A.4})$$

assuming all expectations exist.

Proof. It suffices to show the claim with $\langle \cdot \rangle$ replaced by $\langle \cdot \rangle_t$ where $t = (t_1, \dots, t_N)$ and $n \leq N$. This is clear for $n = 1$. To advance the induction, note that

$$\begin{aligned} \langle A_1 \cdots A_{n+1} \rangle_t^T &= \frac{\partial}{\partial t_{n+1}} \langle A_1 \cdots A_n \rangle_t^T \\ &= \frac{\partial}{\partial t_{n+1}} \left[\langle A_1 \cdots A_n \rangle_t - \sum_{P \in \mathfrak{P}_n} \prod_j \left\langle \prod_{i \in P_j} A_i \right\rangle_t^T \right] \\ &= \langle A_1 \cdots A_{n+1} \rangle_t - \langle A_1 \cdots A_n \rangle_t \langle A_{n+1} \rangle_t \\ &\quad - \sum_{P \in \mathfrak{P}_n} \sum_k \left\langle \prod_{i \in P_k \cup \{n+1\}} A_i \right\rangle_t^T \prod_{j \neq k} \left\langle \prod_{i \in P_j} A_i \right\rangle_t^T \\ &= \langle A_1 \cdots A_{n+1} \rangle_t - \sum_{P \in \mathfrak{P}_{n+1}} \prod_j \left\langle \prod_{i \in P_j} A_i \right\rangle_t^T \end{aligned} \quad (\text{A.5})$$

as needed. ■

A.2. Grassmann integrals

Let \bigwedge^{2N} be the exterior algebra (Grassmann algebra) on $2N$ generators $\bar{\psi}_1, \psi_1, \dots, \bar{\psi}_N, \psi_N$ over \mathbb{C} . The bars only have notational meaning here and for notational simplicity we drop the \wedge from the product notation, e.g., $\bar{\psi}_i \wedge \psi_j \equiv \bar{\psi}_i \psi_j$. Thus elements $F \in \bigwedge^{2N}$ are noncommutative polynomials in the generators of degree at most $2N$. An element $F \in \bigwedge^{2N}$ is called *even* if it is a linear combination of even monomials (i.e., ones with an even number of factors of the generators). Let $\partial_{\bar{\psi}_j}$ and ∂_{ψ_j} be the antiderivations on \bigwedge^{2N} defined by

$$\partial_{\bar{\psi}_j}(\bar{\psi}_j F) = F, \quad \partial_{\bar{\psi}_j} F = 0 \quad (\text{A.6})$$

for any (noncommutative) monomial $F \in \bigwedge^{2N}$ that does not contain a factor $\bar{\psi}_j$, and analogously for the ∂_{ψ_j} . For any $F \in \bigwedge^{2N}$ the *Grassmann integral* of F is then defined by

$$\int d\psi d\bar{\psi} F := \partial_{\psi} \partial_{\bar{\psi}} F := \partial_{\psi_N} \partial_{\bar{\psi}_N} \cdots \partial_{\psi_1} \partial_{\bar{\psi}_1} F. \quad (\text{A.7})$$

Note that the right-hand side is a scalar. For any even elements A_1, \dots, A_n of \bigwedge^{2N} and any smooth function $g \in C^\infty(\mathbb{R}^n)$, we define an element $g(A_1, \dots, A_n) \in \bigwedge^{2N}$ by the

truncation of the formal Taylor expansion of g at order $2N$. For example, using the above definitions, we write, for any $N \times N$ matrix M ,

$$e^{-\psi M \bar{\psi}} = \sum_{n=0}^N \frac{(-1)^n}{n!} \left(\sum_{i,j=1}^N \psi_i M_{ij} \bar{\psi}_j \right)^n, \quad (\text{A.8})$$

and then have

$$\int d\psi d\bar{\psi} e^{-\psi M \bar{\psi}} = \frac{(-1)^N}{N!} \int d\psi d\bar{\psi} (\psi M \bar{\psi})^N = \frac{(-1)^N}{N!} \partial_{\psi} \partial_{\bar{\psi}} (\psi M \bar{\psi})^N = \det M, \quad (\text{A.9})$$

by the anticommutativity of the generators and the definition of the determinant.

The following lemma is a variant of Wick's theorem for Grassmann integrals.

Lemma A.2. *Let K be an invertible $N \times N$ matrix. Then*

$$\det K \cdot \int d\psi d\bar{\psi} \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i} e^{-\psi K^{-1} \bar{\psi}} = \det (K_{\alpha_i \beta_j})_{i,j=1}^n. \quad (\text{A.10})$$

Remark A.3. Note that the Grassmann integral representation of the determinant, (A.10), can be used in the context of (1.9) and (2.2). For finitely many points $x_1, \dots, x_n, y_1, \dots, y_n$ one may indeed apply this lemma to the matrix defined by $K_{ij} = S_{\alpha_i \beta_j}(x_i, y_j)$ for $i \neq j$ and $K_{ii} = C$ for a sufficiently large constant C such that K is invertible.

Proof of Lemma A.2. For an invertible $N \times N$ matrix K , the fermionic Gaussian integration by parts formula holds:

$$\int d\psi d\bar{\psi} \bar{\psi}_i F e^{-\psi K^{-1} \bar{\psi}} = \sum_j K_{ij} \int d\psi d\bar{\psi} (\partial_{\psi_j} F) e^{-\psi K^{-1} \bar{\psi}}. \quad (\text{A.11})$$

Indeed, it follows from the definitions that

$$\partial_{\psi_j} e^{-\psi K^{-1} \bar{\psi}} = - \sum_i (K^{-1})_{ji} \bar{\psi}_i e^{-\psi K^{-1} \bar{\psi}}, \quad (\text{A.12})$$

and hence

$$\bar{\psi}_i e^{-\psi K^{-1} \bar{\psi}} = - \sum_j K_{ij} \partial_{\psi_j} e^{-\psi K^{-1} \bar{\psi}}. \quad (\text{A.13})$$

Note that we may assume that F is odd in (A.11) as otherwise both sides are 0. Therefore

$$\int d\psi d\bar{\psi} \bar{\psi}_i F e^{-\psi K^{-1} \bar{\psi}} = - \int d\psi d\bar{\psi} F \bar{\psi}_i e^{-\psi K^{-1} \bar{\psi}} = \sum_j K_{ij} \int d\psi d\bar{\psi} F \partial_{\psi_j} e^{-\psi K^{-1} \bar{\psi}}. \quad (\text{A.14})$$

The claim now follows from the fact that, since F is odd, for any G one has

$$0 = \int d\psi d\bar{\psi} \partial_{\psi_j} (FG) = \int d\psi d\bar{\psi} [(\partial_{\psi_j} F)G - F(\partial_{\psi_j} G)]. \quad (\text{A.15})$$

Note that for any monomial $F \in \bigwedge^{2N}$ with n factors of $\bar{\psi}_1, \dots, \bar{\psi}_N$ we have $F = \frac{1}{n} \sum_i \bar{\psi}_i \partial_{\bar{\psi}_i} F$. Thus if F has degree $2n$ then

$$\begin{aligned} \int d\psi d\bar{\psi} F e^{-\psi K^{-1} \bar{\psi}} &= \frac{1}{n} \sum_{i,j} K_{ij} \int d\psi d\bar{\psi} (\partial_{\psi_j} \partial_{\bar{\psi}_i} F) e^{-\psi K^{-1} \bar{\psi}} \\ &= \frac{1}{n} \int d\psi d\bar{\psi} (\Delta_K F) e^{-\psi K^{-1} \bar{\psi}} \end{aligned} \quad (\text{A.16})$$

where

$$\Delta_K F = \sum_{i,j} K_{ij} \partial_{\psi_j} \partial_{\bar{\psi}_i} F. \quad (\text{A.17})$$

Iterating this, for F of degree $2n$ we get

$$\int d\psi d\bar{\psi} F e^{-\psi K^{-1} \bar{\psi}} = \frac{1}{n!} \Delta_K^n F \int d\psi d\bar{\psi} e^{-\psi K^{-1} \bar{\psi}}. \quad (\text{A.18})$$

In particular,

$$\det K \cdot \int d\psi d\bar{\psi} \bar{\psi}_i \psi_j e^{-\psi K^{-1} \bar{\psi}} = K_{ij} \quad (\text{A.19})$$

and repeated application gives

$$\det K \cdot \int d\psi d\bar{\psi} \prod_{i=1}^n \bar{\psi}_{\alpha_i} \psi_{\beta_i} e^{-\psi K^{-1} \bar{\psi}} = \det (K_{\alpha_i \beta_j})_{i,j=1}^n. \quad (\text{A.20})$$

as claimed. ■

Given an invertible $N \times N$ matrix K , we now write

$$\langle F \rangle = \det K \cdot \int d\psi d\bar{\psi} e^{-\psi K^{-1} \bar{\psi}} F. \quad (\text{A.21})$$

From this representation, it is also easy to deduce the following properties of the fermionic correlation functions. Using Remark A.3, we make use of the properties in Section 2.1.

Lemma A.4. *For any $\sigma \in S_n$,*

$$\left\langle \prod_{k=1}^n \bar{\psi}_{i_k} \psi_{j_k} \right\rangle = (-1)^\sigma \left\langle \prod_{k=1}^n \bar{\psi}_{i_k} \psi_{j_{\sigma(k)}} \right\rangle \quad (\text{A.22})$$

Moreover, if $F, G \in \bigwedge^{2N}$ are monomials such that for every factor $\bar{\psi}_i$ in F and every factor ψ_j in G one has $K_{ij} = 0$ and for every factor ψ_i in F and every factor $\bar{\psi}_j$ in G one also has $K_{ij} = 0$ then

$$\langle FG \rangle = \langle F \rangle \langle G \rangle. \quad (\text{A.23})$$

As in (A.1), for even elements $A_i \in \bigwedge^{2N}$ the truncated correlations are defined by

$$\langle A_1 \cdots A_n \rangle^T = \frac{\partial^n}{\partial t_1 \cdots \partial t_n} \Big|_{t=0} \log \langle e^{\sum_{i=1}^n t_i A_i} \rangle. \quad (\text{A.24})$$

The next lemma gives equivalent characterizations of the truncated correlation functions.

Lemma A.5. Assume that A_1, \dots, A_n are even elements of \bigwedge^{2N} . Then

$$\langle A_1 \cdots A_n \rangle^T = \langle A_1 \cdots A_n \rangle - \sum_{P \in \mathfrak{P}_n} \prod_j \left\langle \prod_{i \in P_j} A_i \right\rangle^T. \quad (\text{A.25})$$

Moreover, if the A_i are of the form $A_i = \bar{\psi}_{\alpha_i} \psi_{\beta_i}$, then

$$\langle A_1 \cdots A_n \rangle^T = (-1)^{n+1} \sum_{\pi \in C_n} \prod_{i=1}^n K_{\alpha_{\pi^i(1)} \beta_{\pi^{i+1}(1)}}. \quad (\text{A.26})$$

Proof. The proof of (A.25) is identical to that of Lemma A.1. To see (A.26), we assume by induction that the identity holds for every invertible matrix K . If $n = 1$, this claim is

$$\langle A_1 \rangle = \langle \bar{\psi}_{\alpha_1} \psi_{\beta_1} \rangle = K_{\alpha_1 \beta_1}, \quad (\text{A.27})$$

which is true by Lemma A.2. To advance the induction, for t sufficiently small, set $K(t) = (\mathbf{1} + \sum_i t_i K \mathbf{1}_{\beta_i \alpha_i})^{-1} K$ where $\mathbf{1}_{ij}$ is the matrix with value 1 in entry ij and 0 in all other entries, and define $\langle \cdot \rangle_t$ as in (A.21) with $K(t)$ instead of K . Since

$$K(t)^{-1} = K^{-1} \left(\mathbf{1} + \sum_i t_i K \mathbf{1}_{\beta_i \alpha_i} \right) = K^{-1} + \sum_i t_i \mathbf{1}_{\beta_i \alpha_i} \quad (\text{A.28})$$

this definition is consistent with $\langle \cdot \rangle_t$ defined as in (A.2), i.e.,

$$\langle F \rangle_t = \frac{\langle F e^{-\sum t_i \psi_{\beta_i} \bar{\psi}_{\alpha_i}} \rangle}{\langle e^{-\sum t_i \psi_{\beta_i} \bar{\psi}_{\alpha_i}} \rangle} = \frac{\langle F e^{\sum t_i \bar{\psi}_{\alpha_i} \psi_{\beta_i}} \rangle}{\langle e^{\sum t_i \bar{\psi}_{\alpha_i} \psi_{\beta_i}} \rangle}. \quad (\text{A.29})$$

Also note that

$$\frac{\partial}{\partial t_j} K(t) = -K(t) \mathbf{1}_{\beta_j \alpha_j} K(t) \quad (\text{A.30})$$

as follows from

$$\frac{\partial}{\partial t_j} \left(\mathbf{1} + \sum_i t_i K \mathbf{1}_{\beta_i \alpha_i} \right)^{-1} = - \left(\mathbf{1} + \sum_i t_i K \mathbf{1}_{\beta_i \alpha_i} \right)^{-1} K \mathbf{1}_{\beta_j \alpha_j} \left(\mathbf{1} + \sum_i t_i K \mathbf{1}_{\beta_i \alpha_i} \right)^{-1}. \quad (\text{A.31})$$

By the induction hypothesis, now

$$\langle A_1 \cdots A_n \rangle_t^T = (-1)^{n+1} \sum_{\pi \in C_n} \prod_{i=1}^n K_{\alpha_{\pi^i(1)} \beta_{\pi^{i+1}(1)}}(t), \quad (\text{A.32})$$

and the claim follows from (A.3) and (A.30). \blacksquare

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References

- [1] Adler, R. J., Taylor, J. E.: Random Fields and Geometry. Springer Monogr. Math., Springer, New York (2007) Zbl [1149.60003](#) MR [2319516](#)
- [2] Bahns, D., Fredenhagen, K., Rejzner, K.: Local nets of von Neumann algebras in the sine-Gordon model. *Comm. Math. Phys.* **383**, 1–33 (2021) Zbl [1461.81067](#) MR [4236060](#)
- [3] Bauerschmidt, R., Bodineau, T.: Log-Sobolev inequality for the continuum sine-Gordon model. *Comm. Pure Appl. Math.* **74**, 2064–2113 (2021) Zbl [1475.60144](#) MR [4303014](#)
- [4] Bauerschmidt, R., Hofstetter, M.: Maximum and coupling of the sine-Gordon field. *Ann. Probab.* **50**, 455–508 (2022) Zbl [07512869](#) MR [4399156](#)
- [5] Benfatto, G.: An iterated Mayer expansion for the Yukawa gas. *J. Statist. Phys.* **41**, 671–684 (1985) MR [814849](#)
- [6] Benfatto, G., Falco, P., Mastropietro, V.: Functional integral construction of the massive Thirring model: verification of axioms and massless limit. *Comm. Math. Phys.* **273**, 67–118 (2007) Zbl [1124.81031](#) MR [2308750](#)
- [7] Benfatto, G., Falco, P., Mastropietro, V.: Massless sine-Gordon and massive Thirring models: proof of Coleman’s equivalence. *Comm. Math. Phys.* **285**, 713–762 (2009) Zbl [1178.35326](#) MR [2461991](#)
- [8] Benfatto, G., Gallavotti, G., Nicolò, F.: On the massive sine-Gordon equation in the first few regions of collapse. *Comm. Math. Phys.* **83**, 387–410 (1982) Zbl [0492.60100](#) MR [649810](#)
- [9] Benfatto, G., Mastropietro, V.: On the density-density critical indices in interacting Fermi systems. *Comm. Math. Phys.* **231**, 97–134 (2002) Zbl [1016.82023](#) MR [1947693](#)
- [10] Benfatto, G., Mastropietro, V.: Ward identities and vanishing of the beta function for $d = 1$ interacting Fermi systems. *J. Statist. Phys.* **115**, 143–184 (2004) Zbl [1157.82328](#) MR [2070092](#)
- [11] Bernard, D., LeClair, A.: Differential equations for sine-Gordon correlation functions at the free fermion point. *Nuclear Phys. B* **426**, 534–558 (1994) Zbl [1049.81620](#) MR [1297289](#)
- [12] Brydges, D. C., Federbush, P.: Debye screening. *Comm. Math. Phys.* **73**, 197–246 (1980) MR [574172](#)
- [13] Brydges, D. C., Kennedy, T.: Mayer expansions and the Hamilton–Jacobi equation. *J. Statist. Phys.* **48**, 19–49 (1987) MR [914427](#)
- [14] Camia, F., Garban, C., Newman, C. M.: Planar Ising magnetization field I. Uniqueness of the critical scaling limit. *Ann. Probab.* **43**, 528–571 (2015) Zbl [1332.82012](#) MR [3305999](#)
- [15] Chelkak, D., Hongler, C., Izyurov, K.: Conformal invariance of spin correlations in the planar Ising model. *Ann. of Math. (2)* **181**, 1087–1138 (2015) Zbl [1318.82006](#) MR [3296821](#)
- [16] Coleman, S.: Quantum sine-Gordon equation as the massive Thirring model. *Phys. Rev. D* **11**, 2088–2097 (1975)
- [17] Cornu, F., Jancovici, B.: On the two-dimensional Coulomb gas. *J. Statist. Phys.* **49**, 33–56 (1987) Zbl [0960.82502](#) MR [923851](#)
- [18] Da Prato, G.: An Introduction to Infinite-Dimensional Analysis. Universitext, Springer, Berlin (2006) Zbl [1109.46001](#) MR [2244975](#)
- [19] Dashen, R., Hasslacher, B., Neveu, A.: Particle spectrum in model field theories from semi-classical functional integral techniques. *Phys. Rev. D* **11**, 3424–3450 (1975)
- [20] Destri, C., de Vega, H. J.: Non-linear integral equation and excited-states scaling functions in the sine-Gordon model. *Nuclear Phys. B* **504**, 621–664 (1997) Zbl [0925.81029](#) MR [1488586](#)
- [21] Dimock, J.: Bosonization of massive fermions. *Comm. Math. Phys.* **198**, 247–281 (1998) Zbl [0926.46060](#) MR [1672504](#)
- [22] Dimock, J., Hurd, T. R.: Construction of the two-dimensional sine-Gordon model for $\beta < 8\pi$. *Comm. Math. Phys.* **156**, 547–580 (1993) Zbl [0815.35098](#) MR [1240586](#)

- [23] Dimock, J., Hurd, T. R.: Sine-Gordon revisited. *Ann. Henri Poincaré* **1**, 499–541 (2000) Zbl [1002.81035](#) MR [1777310](#)
- [24] Dubédat, J.: Dimers and families of Cauchy–Riemann operators I. *J. Amer. Math. Soc.* **28**, 1063–1167 (2015) MR [3369909](#)
- [25] Fateev, V., Fradkin, D., Lukyanov, S., Zamolodchikov, A., Zamolodchikov, A.: Expectation values of descendent fields in the sine-Gordon model. *Nuclear Phys. B* **540**, 587–609 (1999) Zbl [0942.81039](#) MR [1672130](#)
- [26] Federbush, P., Kennedy, T.: Surface effects in Debye screening. *Comm. Math. Phys.* **102**, 361–423 (1985) Zbl [0608.35065](#) MR [818828](#)
- [27] Fröhlich, J.: Quantized “sine-Gordon” equation with a nonvanishing mass term in two space-time dimensions. *Phys. Rev. Lett.* **34**, 833–836 (1975) MR [421409](#)
- [28] Fröhlich, J.: Classical and quantum statistical mechanics in one and two dimensions: two-component Yukawa- and Coulomb systems. *Comm. Math. Phys.* **47**, 233–268 (1976) Zbl [1092.82505](#) MR [434278](#)
- [29] Fröhlich, J.: Quantum sine-Gordon equation and quantum solitons in two space-time dimensions. In: *Renormalization Theory (Erice, 1975)*, NATO Adv. Study Inst. Ser. C Math. Phys. Sci. 23, Reidel, Dordrecht, 371–414 (1976) MR [0523016](#)
- [30] Fröhlich, J., Marchetti, P.: Bosonization, topological solitons and fractional charges in two-dimensional quantum field theory. *Comm. Math. Phys.* **116**, 127–173 (1988) Zbl [0651.58044](#) MR [937363](#)
- [31] Fröhlich, J., Park, Y. M.: Remarks on exponential interactions and the quantum sine-Gordon equation in two space-time dimensions. *Helv. Phys. Acta* **50**, 315–329 (1977) MR [456220](#)
- [32] Fröhlich, J., Park, Y. M.: Correlation inequalities and the thermodynamic limit for classical and quantum continuous systems. *Comm. Math. Phys.* **59**, 235–266 (1978) MR [496191](#)
- [33] Fröhlich, J., Seiler, E.: The massive Thirring–Schwinger model (QED₂): convergence of perturbation theory and particle structure. *Helv. Phys. Acta* **49**, 889–924 (1976) MR [443693](#)
- [34] Gaudin, M.: L’isotherme critique d’un plasma sur réseau ($\beta = 2$, $d = 2$, $n = 2$). *J. Physique* **46**, 1027–1042 (1985) MR [799499](#)
- [35] Gilbarg, D., Trudinger, N. S.: *Elliptic Partial Differential Equations of Second Order*. *Classics Math.*, Springer, Berlin (2001) Zbl [1042.35002](#) MR [1814364](#)
- [36] Giuliani, A., Mastropietro, V., Toninelli, F. L.: Height fluctuations in interacting dimers. *Ann. Inst. Henri Poincaré Probab. Statist.* **53**, 98–168 (2017) Zbl [1381.82008](#) MR [3606736](#)
- [37] Giuliani, A., Mastropietro, V., Toninelli, F. L.: Non-integrable dimers: universal fluctuations of tilted height profiles. *Comm. Math. Phys.* **377**, 1883–1959 (2020) Zbl [1446.82010](#) MR [4121614](#)
- [38] Grieser, D.: Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary. *Comm. Partial Differential Equations* **27**, 1283–1299 (2002) Zbl [1034.35085](#) MR [1924468](#)
- [39] Guillarmou, C., Kupiainen, A., Rhodes, R., Vargas, V.: Conformal bootstrap in Liouville theory. [arXiv:2005.11530](#) (2020)
- [40] Hethcote, H. W.: Bounds for zeros of some special functions. *Proc. Amer. Math. Soc.* **25**, 72–74 (1970) Zbl [0189.34404](#) MR [255909](#)
- [41] Hongler, C., Viklund, F., Kytölä, K.: Conformal field theory at the lattice level: discrete complex analysis and Virasoro structure. *Comm. Math. Phys.* **395**, 1–58 (2022) Zbl [07587135](#) MR [4483015](#)
- [42] Itzykson, C., Drouffe, J.-M.: *Statistical Field Theory*. Vol. 2. *Cambridge Monogr. Math. Phys.*, Cambridge Univ. Press, Cambridge (1989) Zbl [0825.81001](#) MR [1175176](#)
- [43] Junnila, J., Saksman, E., Webb, C.: Imaginary multiplicative chaos: moments, regularity and connections to the Ising model. *Ann. Appl. Probab.* **30**, 2099–2164 (2020) Zbl [1472.60064](#) MR [4149524](#)

- [44] Kroschinsky, W., Marchetti, D. H. U.: On the Mayer series of two-dimensional Yukawa gas at inverse temperature in the interval of collapse. *J. Statist. Phys.* **177**, 324–364 (2019) Zbl [1483.31010](#) MR [4010781](#)
- [45] Kupiainen, A., Rhodes, R., Vargas, V.: Integrability of Liouville theory: proof of the DOZZ formula. *Ann. of Math. (2)* **191**, 81–166 (2020) Zbl [1432.81055](#) MR [4060417](#)
- [46] Lacoïn, H., Rhodes, R., Vargas, V.: Complex Gaussian multiplicative chaos. *Comm. Math. Phys.* **337**, 569–632 (2015) Zbl [1322.60065](#) MR [3339158](#)
- [47] Lacoïn, H., Rhodes, R., Vargas, V.: A probabilistic approach of ultraviolet renormalization in the boundary Sine-Gordon model. *Probab. Theory Related Fields* **185**, 1–40 (2023) MR [4528966](#)
- [48] Lukyanov, S., Zamolodchikov, A.: Exact expectation values of local fields in the quantum sine-Gordon model. *Nuclear Phys. B* **493**, 571–587 (1997) Zbl [0909.58064](#) MR [1453266](#)
- [49] Lukyanov, S., Zamolodchikov, A.: Form factors of soliton-creating operators in the sine-Gordon model. *Nuclear Phys. B* **607**, 437–455 (2001) Zbl [0969.81566](#) MR [1850795](#)
- [50] Mattis, D. C., Lieb, E. H.: Exact solution of a many-fermion system and its associated Boson field. *J. Math. Phys.* **6**, 304–312 (1965) MR [172638](#)
- [51] McCann, R. C.: Lower bounds for the zeros of Bessel functions. *Proc. Amer. Math. Soc.* **64**, 101–103 (1977) Zbl [0364.33009](#) MR [442316](#)
- [52] Nicolò, F., Renn, J., Steinmann, A.: On the massive sine-Gordon equation in all regions of collapse. *Comm. Math. Phys.* **105**, 291–326 (1986) Zbl [0636.35072](#) MR [849210](#)
- [53] Park, Y. M.: Massless quantum sine-Gordon equation in two space-time dimensions: correlation inequalities and infinite volume limit. *J. Math. Phys.* **18**, 2423–2426 (1977) MR [475380](#)
- [54] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness.* Academic Press, New York (1975) Zbl [0308.47002](#) MR [0493420](#)
- [55] Seiler, R., Uhlenbrock, D. A.: On the massive Thirring model. *Ann. Physics* **105**, 81–110 (1977) MR [446210](#)
- [56] Shi, Y., Xu, B.: Gradient estimate of a Dirichlet eigenfunction on a compact manifold with boundary. *Forum Math.* **25**, 229–240 (2013) Zbl [1327.35275](#) MR [3031783](#)
- [57] Simon, B.: *The $P(\phi)_2$ Euclidean (Quantum) Field Theory.* Princeton Ser. Phys., Princeton Univ. Press, Princeton, NJ (1974) Zbl [1175.81146](#) MR [0489552](#)
- [58] Stone, M.: *Bosonization.* World Sci. (1994)
- [59] Tselik, A. M.: *Quantum Field Theory in Condensed Matter Physics.* 2nd ed., Cambridge Univ. Press, Cambridge (2003) Zbl [1059.81005](#) MR [2021021](#)
- [60] Yang, W.-S.: Debye screening for two-dimensional Coulomb systems at high temperatures. *J. Statist. Phys.* **49**, 1–32 (1987) Zbl [0960.82501](#) MR [923850](#)
- [61] Zamolodchikov, A.: Mass scale in the sine-Gordon model and its reductions. *Int. J. Modern Phys. A* **10**, 1125–1150 (1995)