

Involutive knot Floer homology and bordered modules

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Abstract. We prove that, up to local equivalences, a suitable truncation of the involutive knot Floer homology of a knot in S^3 and the involutive bordered Heegaard Floer theory of its complement determine each other. In particular, given two knots K_1 and K_2 , we prove that the $\mathbb{F}_2[U, V]/(UV)$ -coefficient involutive knot Floer homology of $K_1 \# -K_2$ is ι_K -locally trivial if $\widehat{CFD}(S^3 \setminus K_1)$ and $\widehat{CFD}(S^2 \setminus K_2)$ satisfy a certain condition which can be seen as the bordered counterpart of ι_K -local equivalence. We further establish an explicit algebraic formula that computes the hat-flavored truncation of the involutive knot Floer homology of a knot from the involutive bordered Floer homology of its complement. It follows that there exists an algebraic satellite operator defined on the local equivalence group of knot Floer chain complexes, which can be computed explicitly up to a suitable truncation.

1. Introduction

Given a closed, connected, and oriented 3-manifold Y , the minus-flavored Heegaard Floer theory, defined by Ozsváth and Szabó [21], associates to Y a chain complex $CF^-(Y)$ over the ring $\mathbb{F}_2[U]$, whose homotopy type is an invariant of the oriented diffeomorphism class of Y . Furthermore, if we are given a knot K inside Y , then the knot Floer theory [24, 29] associates to K a homotopy class of a chain complex $CFK_{UV}(Y, K)$ over the ring $\mathbb{F}_2[U, V]$, from which $CF^-(Y)$ can be recovered by taking the specialization $(U, V) = (1, 0)$, or equivalently, $(U, V) = (0, 1)$.

Like Seiberg–Witten Floer homology, whose intrinsic $\text{Pin}(2)$ -symmetry was used by Manolescu [18] to disprove the triangulation conjecture in high dimensions, Heegaard Floer theory has an intrinsic \mathbb{Z}_2 -symmetry, which is induced by the involution

$$(\Sigma, \alpha, \beta, z) \mapsto (-\Sigma, \beta, \alpha, z)$$

on the space of pointed Heegaard diagrams representing the given 3-manifold Y . This action, which preserves all relevant counts of holomorphic disks, induces a homotopy-involution ι_Y on $CF^-(Y)$, which is well defined up to homotopy, as observed first in [9]. Involutive Heegaard Floer theory exploits this involution to give new 3-manifold

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invariants to define new homology cobordism invariants. Those invariants were then used extensively to solve various problems regarding the structures of homology cobordism groups and knot concordance groups [1, 2, 5, 7, 10, 11].

Moreover, as observed by Hendricks and Manolescu [9], a similar construction can also be applied to knot Floer theory. Recall that knot Floer homology starts by representing a pair (Y, K) of a 3-manifold Y and an oriented knot $K \subset Y$ as a doubly-pointed Heegaard diagram, i.e., Heegaard diagram with two basepoints. Then, we have symmetry

$$(\Sigma, \alpha, \beta, z, w) \mapsto (-\Sigma, \beta, \alpha, w, z)$$

on the space of doubly-pointed Heegaard diagrams representing (Y, K) . However, since the basepoints are swapped to compensate the change of orientation on K that occurred by reversing the given orientation on the Heegaard surface Σ , a half-twist along K is needed to define a well-defined homotopy skew-autoequivalence ι_K of $CFK_{UV}(Y, K)$. Due to the presence of a half-twist in the definition of ι_K , it is no longer a homotopy involution, but squares to the Sarkar map ξ_K up to homotopy. The theory of $CFK_{UV}(Y, K)$ together with ι_K is called involutive knot Floer homology, which was used to prove the existence of a linearly independent infinite family of rationally slice knots in [11].

On the other hand, given a compact oriented 3-manifold M with a suitably parametrized torus boundary, bordered Heegaard Floer theory [16] associates to M a differential module $\widehat{CFD}(M)$ and an A_∞ -module $\widehat{CFA}(M)$ over the torus algebra $\mathcal{A}(T^2)$. When M is the 0-framed exterior $S^3 \setminus K$ of a knot $K \subset S^3$, we know from [13] that the homotopy type of those modules is determined by the homotopy type of the truncation $CFK_{\mathcal{R}}(S^3, K)$ of $CFK_{UV}(S^3, K)$ by taking $UV = 0$, and vice versa. Furthermore, we know from [8] that mimicking the construction of involutive Heegaard Floer theory defines homotopy equivalences

$$\begin{aligned} \iota_M : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M) &\rightarrow \widehat{CFD}(M), \\ \iota_M : \widehat{CFA}(M) \boxtimes \widehat{CFDA}(\mathbf{AZ}) &\rightarrow \widehat{CFA}(M). \end{aligned}$$

Hence, it is natural to ask how the knot involution ι_K on $CFK_{\mathcal{R}}(S^3, K)$, the knot Floer complex of K with coefficients in

$$\mathcal{R} = \mathbb{F}_2[U, V]/(UV),$$

is related to the bordered involution $\iota_{S^3 \setminus K}$ of its 0-framed knot complement. The following theorem answers this question in the coarse affirmative, by showing that ι_K and $\iota_{S^3 \setminus K}$ determine each other up to a certain equivalence relation; this equivalence relation is called the ι_K -local equivalence, which can be seen as the involutive algebraic counterpart of knot concordance.

Theorem 1.1. *Given the two knots K_1 and K_2 , consider the involutions $\iota_{K_1\#-K_2}$ of $CFK_{\mathcal{R}}(S^3, K_1\#-K_2)$, as well as any choice of bordered involutions $\iota_{S^3\setminus K_1} \in \text{Inv}_D(S^3\setminus K_1)$ and $\iota_{S^3\setminus K_2} \in \text{Inv}_D(S^3\setminus K_2)$. Then*

$$(CFK_{\mathcal{R}}(S^3, K_1\#-K_2), \iota_{K_1\#-K_2})$$

is ι_K -locally equivalent to the trivial complex if and only if there exists a type-D morphism

$$g : \widehat{CFD}(S^3\setminus K_1) \rightarrow \widehat{CFD}(S^3\setminus K_2)$$

between type-D modules of 0-framed knot complements such that the diagram

$$\begin{array}{ccc} \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3\setminus K_1) & \xrightarrow{\iota_{S^3\setminus K_1}} & \widehat{CFD}(S^3\setminus K_1) \\ \downarrow \text{id} \boxtimes g & & \downarrow g \\ \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3\setminus K_2) & \xrightarrow{\iota_{S^3\setminus K_2}} & \widehat{CFD}(S^3\setminus K_2) \end{array}$$

is homotopy-commutative and the induced chain map

$$\begin{aligned} \widehat{CF}(S^3) &\simeq \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3\setminus K_1) \\ &\xrightarrow{\text{id} \boxtimes g} \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3\setminus K_2) \simeq \widehat{CF}(S^3) \end{aligned}$$

is a homotopy equivalence, and a similar type-D morphism also exists in the opposite direction. Here, T_∞ denotes the ∞ -framed solid torus, and $S^3\setminus K_1$ and $S^3\setminus K_2$ are endowed with the 0-framing on their boundaries. Furthermore, the statement also holds if “any choice of bordered involutions” is replaced with “some choice of bordered involutions”.

We now consider involutive knot Floer homology for satellite knots. Given two knots K_1 and K_2 whose knot Floer chain complexes are locally equivalent, it is very unclear whether the satellite knots $P(K_1)$ and $P(K_2)$ should also have locally equivalent knot Floer chain complexes, where P is any pattern in $S^1 \times D^2$. Using Theorem 1.1, we prove the existence of a satellite operator in the local equivalence group of knot Floer chain complexes.

Theorem 1.2. *Let K_1, K_2 be knots such that*

$$(CFK_{\mathcal{R}}(S^3, K_1\#-K_2), \iota_{K_1\#-K_2})$$

is ι_K -locally equivalent to the trivial complex. Then, for any pattern $P \subset S^1 \times D^2$,

$$(CFK_{\mathcal{R}}(S^3, P(K_1)\#-P(K_2)), \iota_{P(K_1)\#-P(K_2)})$$

is also ι_K -locally equivalent to the trivial complex.

A very natural question is then how can one explicitly compute ι_K from $\iota_{S^3 \setminus K}$. Using the bordered quasi-stabilization constructions, we prove the following theorem which provides a formula to compute the hat-flavored truncation of ι_K from $\iota_{S^3 \setminus K}$ up to orientation reversal.

Theorem 1.3. *Let ν be the longitudinal knot in the ∞ -framed solid torus T_∞ . Then, there exists a type-D morphism*

$$f : \widehat{CFD}(T_\infty, \nu) \rightarrow \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu)$$

such that, for any knot K and for any choice of $\iota_{S^3 \setminus K} \in \text{Inv}_D(S^3 \setminus K)$, the induced map

$$\begin{aligned} \widehat{CFK}(S^3, K) &\simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\xrightarrow{\iota_{S^3 \setminus K}^{-1} \boxtimes f} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\xrightarrow{\simeq} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(T_\infty, \nu) \simeq \widehat{CFK}(S^3, K) \end{aligned}$$

is homotopic to the truncation of either ι_K or its homotopy inverse ι_K^{-1} to the hat-flavored complex $\widehat{CFK}(S^3, K)$ under the natural identification

$$\widehat{CFA}(T_\infty, \nu) \boxtimes \widehat{CFD}(S^3 \setminus K) \simeq \widehat{CFK}(S^3, K)$$

induced by the pairing theorem [17, Theorem 11.19], where $S^3 \setminus K$ is endowed with the 0-framing on its boundary.

Theorem 1.3 can also be used to explicitly compute $\iota_{S^3 \setminus K}$ for some nontrivial knots K . The case when K is the figure-eight knot is computed in Example 5.8. Note that $\widehat{CFD}(S^3 \setminus K)$ is not rigid, i.e., it has more than one homotopy class of homotopy autoequivalences; Example 5.8 gives the first example of explicitly computing bordered involutive Floer homology for homotopically non-rigid bordered 3-manifolds.

Furthermore, together with the proof of Theorem 1.2, Theorem 1.3 can also be considered as an involutive satellite formula. In particular, given a pattern $P \subset S^1 \times D^2$, if $\widehat{CFDA}((S^1 \times D^2) \setminus P)$ is homotopy-rigid and one already knows the action of $\iota_{S^3 \setminus K}$, then one can explicitly compute the hat-flavored involutive knot Floer homology of the satellite knot $P(K)$.

Remark 1.4. When P is the $(p, 1)$ -cabling pattern for some $p > 0$, the bimodule $\widehat{CFDA}((S^1 \times D^2) \setminus P)$, with respect to some boundary framings, can be computed from the type DAA trimodule of $S^3 \times (\text{pair-of-pants})$, which was explicitly computed in [4, Table 1], by taking a box tensor product on its ρ -boundary with the type D module of the $\frac{1}{p}$ -framed solid torus. It is easy to observe, via manual computation, that the resulting bimodule is homotopy-rigid. Hence, Theorem 1.3 gives a hat-flavored

involutive $(p, 1)$ -cabling formula, which computes the involutive action of the cable knot $K_{p,1}$ from $\iota_{S^3} K$.

Notation

Throughout this article, the symbol \sim denotes homotopy; i.e., $f \sim g$ means that the morphisms f and g are homotopic in a morphism space of a suitable category. Also, the symbol \simeq denotes homotopy equivalence; i.e., $A \simeq B$ means that the objects A and B are homotopy equivalent in a suitable category. Furthermore, we denote by \mathcal{R} the ring $\mathbb{F}_2[U, V]/(UV)$.

Organization

This article is organized as follows. In Section 2, we recall some results regarding involutive Heegaard Floer homology and bordered Floer homology. In Section 3, we develop a theory of involutive knot Floer homology with a free basepoint and discuss its relationship with involutive bordered Floer homology of 0-framed knot complements. In Section 4, we prove Theorem 1.1 and use it to prove Theorem 1.2. Finally, in Section 5, we prove Theorem 1.3 and discuss its explicit applications.

2. Involutive Heegaard Floer homology for knots and 3-manifolds

We assume that the reader is familiar with Heegaard Floer theory [19–22] of knots and 3-manifolds, as well as bordered Heegaard Floer theory [17]. Throughout the paper, we will only work with \mathbb{F}_2 coefficients. Furthermore, we will often consider 3-manifolds M endowed with torsion Spin^c structures. In such cases, the Heegaard Floer chain complexes $CF^-(M, \mathfrak{s})$ and $\widehat{CF}(M, \mathfrak{s})$ are chain complexes of free modules over $\mathbb{F}_2[U]$ and \mathbb{F}_2 , respectively, and absolutely \mathbb{Q} -graded.

2.1. Involutive Heegaard Floer homology and ι -complexes

Recall that the definition of Heegaard Floer homology of any flavor starts with choosing an admissible pointed Heegaard diagram

$$H = (\Sigma, \alpha, \beta, z)$$

representing M . The theory of involutive Heegaard Floer homology, as defined first in [9], starts by considering the conjugate diagram

$$\bar{H} = (-\Sigma, \beta, \alpha, z).$$

Then, we have a canonical identification map

$$\text{conj} : CF^-(H) \xrightarrow{\cong} CF^-(\bar{H}).$$

Since \bar{H} also represents M , it is related to H by a sequence of Heegaard moves. Such a sequence induces a homotopy equivalence

$$\Phi_{\bar{H}, H} : CF^-(\bar{H}) \rightarrow CF^-(H).$$

By the naturality of Heegaard Floer theory [12], the homotopy class of $\Phi_{\bar{H}, H}$ does not depend on our choice of a sequence of Heegaard moves from \bar{H} to H . Thus, the homotopy autoequivalence

$$\iota_M = \Phi_{\bar{H}, H} \circ \text{conj} : CF^-(M) \rightarrow CF^-(M)$$

is well defined up to homotopy, and the image of its restriction $\iota_{\mathfrak{s}}$ to $CF^-(M, \mathfrak{s})$ is $CF^-(M, \bar{\mathfrak{s}})$. In particular, when \mathfrak{s} is self-conjugate, i.e., spin, then $\iota_{M, \mathfrak{s}}$ is a homotopy autoequivalence of $CF^-(M, \mathfrak{s})$.

The involution ι_M satisfies the following properties:

- $\iota_M^2 \sim \text{id}$.
- The localized map $U^{-1}\iota_M$ is homotopic to identity.

Inspired by the above properties, the notion of ι -complex was defined in [10] as follows. An ι -complex is a pair (C, ι) which satisfies the following properties:

- C is a chain complex of finitely generated free modules over $\mathbb{F}_2[U]$ such that the localized complex $U^{-1}C$ has homology $\mathbb{F}_2[U^{\pm 1}]$.
- ι is a homotopy autoequivalence of C such that $\iota^2 \sim \text{id}$.

Furthermore, given two chain complexes C_1 and C_2 of modules over $\mathbb{F}_2[U]$, a chain map $f : C_1 \rightarrow C_2$ is said to be a *local map* if the localized map $U^{-1}f : U^{-1}C_1 \rightarrow U^{-1}C_2$ induces an injective map in homology. Given two ι -complexes (C_1, ι_1) and (C_2, ι_2) , a local map $f : C_1 \rightarrow C_2$ is said to be a *ι -local map* if $\iota_2 \circ f \sim f \circ \iota_1$. If ι -local maps between (C_1, ι_1) and (C_2, ι_2) exist in both directions, we say that the given two ι -complexes are *ι -locally equivalent*. The set of ι -local equivalence classes of ι -complexes forms a group \mathfrak{F} under the tensor product operation, which is called the *local equivalence group*.

The notion of ι -complexes and local equivalences between them can be weakened, as shown in [2], in the following way. An almost ι -complex is a pair (C, ι) which satisfies the following properties:

- C is a chain complex of finitely generated free modules over $\mathbb{F}_2[U]$ such that the localized complex $U^{-1}C$ has homology $\mathbb{F}_2[U^{\pm 1}]$.

- $\iota : C \rightarrow C$ is a chain map of chain complexes of \mathbb{F}_2 -vector spaces such that

$$\text{Im}(\partial\iota + \iota\partial) \subset \text{Im}(U) \quad \text{and} \quad \iota^2 \sim \text{id} \quad \text{mod } U.$$

Given almost local ι -complexes (C_1, ι_1) and (C_2, ι_2) , a local map $f : C_1 \rightarrow C_2$ is an *almost local map* if

$$\iota_2 \circ f + f \circ \iota_1 \sim 0 \quad \text{mod } U.$$

If almost local maps exist in both directions, we say that the given two almost ι -complexes are *almost locally equivalent*. Again, the set of almost local equivalences of almost ι -complexes form a group $\hat{\mathfrak{S}}$, which is called the *almost local equivalence group*. The construction of involutive Heegaard Floer homology gives a canonical map

$$\Phi_{\mathbb{Z}_2}^3 \xrightarrow{f} \mathfrak{S} \xrightarrow{g} \hat{\mathfrak{S}},$$

where $\Phi_{\mathbb{Z}_2}^3$ denotes the homology cobordism group of \mathbb{Z}_2 -homology spheres, f maps a homology cobordism class $[Y]$ of a \mathbb{Z}_2 -homology sphere Y to its involutive Heegaard Floer homology $[CF^-(Y, [0]), \iota_{Y,[0]}]$ for the unique spin structure $[0]$ on Y , and g is the canonical forgetful map.

Remark 2.1. The definition of ι -local maps, local equivalences, and their “almost” versions also work when we drop the condition that $U^{-1}C$ is homotopy equivalent to $\mathbb{F}_2[U^{\pm 1}]$. We will sometimes use this generalized notion throughout this paper.

2.2. Involutive knot Floer homology and ι_K -complexes

The involutive theory for knot Floer homology is a bit more complicated than the 3-manifold case. For simplicity, we only consider knots K in S^3 . Consider a doubly-pointed Heegaard diagram $H = (\Sigma, \alpha, \beta, z, w)$ representing (S^3, K) . By counting holomorphic disks while recording their algebraic intersection numbers with z and w by formal variables U and V , respectively, one gets an absolutely \mathbb{Z} -bigraded (called *Alexander* and *Maslov* gradings, respectively) chain complex $CFK_{UV}(S^3, K)$ of finitely generated free modules over the ring $\mathbb{F}_2[U, V]$.

Consider the conjugate diagram $\bar{H} = (-\Sigma, \beta, \alpha, w, z)$ of H ; note that, in addition to flipping the orientation of Σ and exchanging α and β curves, we are also exchanging the basepoints z and w . Then, as in the 3-manifold case, we have a canonical conjugation map

$$\text{conj} : CFK_{UV}(H) \rightarrow CFK_{UV}(\bar{H}),$$

which is a chain skew-isomorphism, i.e., intertwines the actions of U and V on its domain with the actions of V and U on its codomain. Then, we consider a self-diffeomorphism of S^3 that acts on a tubular neighborhood of K by a “half-twist”,

so that it fixes K setwise and maps z and w to w and z , respectively. It induces a chain isomorphism

$$\phi_* : CFK_{UV}(\bar{H}) \rightarrow CFK_{UV}(\phi(\bar{H})).$$

Now, the diagrams $\phi(\bar{H})$ and H both represent the knot K together with two prescribed basepoints z and w on K , so they are related by a sequence of Heegaard moves. Such a sequence induces a homotopy equivalence

$$\Phi_{\phi(\bar{H}), H} : CFK_{UV}(\phi(\bar{H})) \rightarrow CFK_{UV}(H),$$

whose homotopy class is independent of our choice of a sequence of Heegaard moves from $\phi(\bar{H})$ to H due to naturality. Thus, we have a homotopy skew-equivalence

$$\iota_K = \Phi_{\phi(\bar{H}), H} \circ \phi_* \circ \text{conj} : CFK_{UV}(S^3, K) \rightarrow CFK_{UV}(S^3, K),$$

which is well defined up to homotopy. Note that such a construction can also be applied for links as well; given a link L , where each component $K \subset L$ has one z -basepoint and one w -basepoint (which correspond to formal variables U_K and V_K), following the above construction gives a homotopy skew-equivalence ι_L which intertwines the actions of U_K and V_K for each component K .

The homotopy skew-equivalence ι_K satisfies the following properties, as shown in [27]:

- $\iota_K^2 \sim 1 + \Phi\Psi \sim 1 + \Psi\Phi$, where Φ and Ψ are the formal derivatives of the differential ∂ of $CFK_{UV}(S^3, K)$ with respect to the formal variables U and V , respectively.
- The localized map $(U, V)^{-1}$ is homotopic to identity.

Using the above properties, the notion of ι_K -complexes was defined in [27] as follows. An ι_K -complex is a pair (C, ι_K) which satisfies the following properties:

- C is a chain complex of finitely generated free modules over $\mathbb{F}_2[U, V]$ such that $(U, V)^{-1}C$ has homology $(U, V)^{-1}\mathbb{F}_2[U, V]$.
- ι_K is a homotopy skew-autoequivalence of C such that $\iota_K^2 \sim 1 + \Phi\Psi$, where Φ and Ψ are the formal derivatives of the differential ∂ of C with respect to the formal variables U and V , respectively.

Given two chain complexes C_1 and C_2 of free modules over $\mathbb{F}_2[U, V]$, a chain map $f : C_1 \rightarrow C_2$ is said to be a *local map* if the maps

$$\begin{aligned} U^{-1}f|_{V=0} : C \otimes \mathbb{F}_2[U^{\pm 1}] &\rightarrow D \otimes \mathbb{F}[U^{\pm 1}], \\ V^{-1}f|_{U=0} : C \otimes \mathbb{F}_2[V^{\pm 1}] &\rightarrow D \otimes \mathbb{F}[V^{\pm 1}] \end{aligned}$$

induce injective maps in homology. Given two ι_K -complexes (C_1, ι_1) and (C_2, ι_2) , a local map $f : C_1 \rightarrow C_2$ is said to be a ι_K -local map if $\iota_2 \circ f \sim f \circ \iota_1$. If ι_K -local

maps between two ι_K -complexes exist in both directions, then we say that they are ι_K -locally equivalent. The set of ι_K -local equivalence classes of ι_K -complexes form a group \mathfrak{S}_K when endowed with the addition operation

$$\iota_{C_1 \otimes C_2} = (\text{id} + \Phi \otimes \Psi) \circ (\iota_1 \otimes \iota_2).$$

As in the 3-manifold case, the construction of involutive knot Floer homology gives a canonical map $\mathcal{C} \rightarrow \mathfrak{S}_K$.

We will sometimes work with knot Floer homology with coefficient ring $\mathbb{F}_2[U, V]/(UV)$, which is denoted as \mathcal{R} , rather than the full two-variable ring $\mathbb{F}_2[U, V]$. Note that although ι_K -local maps and ι_K -local equivalences are well defined, it is unclear whether ι_K -local equivalence classes of involutive \mathcal{R} -coefficient knot Floer chain complexes form a well-defined group, since the basepoint actions might not be uniquely determined from the \mathcal{R} -coefficient differential.

2.3. Involutive bordered Floer homology

Let M be a bordered 3-manifold with one boundary; for simplicity, we will assume that ∂M is a torus. Choose a bordered Heegaard diagram $H = (\Sigma, \alpha, \beta, z)$ representing M and consider its conjugate diagram $\bar{H} = (-\Sigma, \beta, \alpha, z)$. Then, we have canonical identification maps

$$\begin{aligned} \text{conj} : \widehat{CFD}(H) &\rightarrow \widehat{CFD}(\bar{H}), \\ \text{conj} : \widehat{CFA}(H) &\rightarrow \widehat{CFA}(\bar{H}) \end{aligned}$$

between the type-D and type-A modules associated to H and \bar{H} , respectively. Note that we are using the same name for the type-D and type-A identification maps for convenience.

In contrast to the case of closed 3-manifolds, there does not exist a sequence of Heegaard moves from \bar{H} to H . The reason is that H is α -bordered, whereas \bar{H} is β -bordered. To remedy this problem, Hendricks and Lipshitz [8] use the Auroux–Zarev piece \mathbf{AZ} and its conjugate $\overline{\mathbf{AZ}}$, which satisfies the property that $\overline{\mathbf{AZ}} \cup \mathbf{AZ}$ represents a trivial cylinder $T^2 \times I$. A Heegaard diagram representing \mathbf{AZ} is shown in Figure 2.1.

One starts with the [15, Theorem 4.6], which implies that $\mathbf{AZ} \cup \bar{H}$ and $\bar{H} \cup \overline{\mathbf{AZ}}$ are related to H by a sequence of Heegaard moves. Choosing such sequences give homotopy equivalences

$$\begin{aligned} \Phi_{\mathbf{AZ} \cup \bar{H}, H} : \widehat{CFD}(\mathbf{AZ} \cup \bar{H}) &\rightarrow \widehat{CFD}(H), \\ \Phi_{\bar{H} \cup \overline{\mathbf{AZ}}, H} : \widehat{CFA}(\bar{H} \cup \overline{\mathbf{AZ}}) &\rightarrow \widehat{CFA}(H). \end{aligned}$$

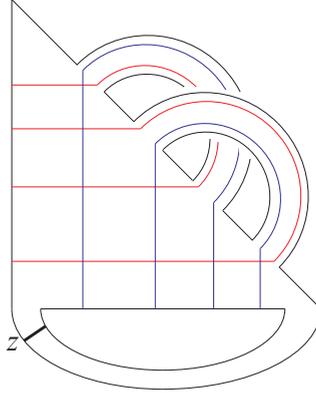


Figure 2.1. A α - β -bordered Heegaard diagram representing the Auroux–Zarev piece \mathbf{AZ} . Note that this diagram is nice in the sense of [17, Definition 8.1].

Recall that we have pairing maps induced by time dilation, as discussed in [17, Chapter 9], which are defined uniquely up to homotopy

$$\begin{aligned} \widehat{CFD}(\mathbf{AZ} \cup \bar{H}) &\xrightarrow{\simeq} \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\bar{H}), \\ \widehat{CFA}(\bar{H} \cup \overline{\mathbf{AZ}}) &\xrightarrow{\simeq} \widehat{CFA}(\bar{H}) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}). \end{aligned}$$

Then, we can define the bordered involution ι_M , in both type-D and type-A modules, as follows:

$$\begin{aligned} \iota_M : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(H) &\xrightarrow{\text{id} \boxtimes \text{conj}} \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\bar{H}) \\ &\xrightarrow{\simeq} \widehat{CFD}(\mathbf{AZ} \cup \bar{H}) \xrightarrow{\Phi_{\mathbf{AZ} \cup \bar{H}, H}} \widehat{CFD}(H), \\ \iota_M : \widehat{CFA}(H) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) &\xrightarrow{\text{conj} \boxtimes \text{id}} \widehat{CFD}(\bar{H}) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \\ &\xrightarrow{\simeq} \widehat{CFA}(\bar{H} \cup \overline{\mathbf{AZ}}) \xrightarrow{\Phi_{\bar{H} \cup \overline{\mathbf{AZ}}, H}} \widehat{CFA}(H). \end{aligned}$$

Now, suppose that we are given a bordered 3-manifold N whose boundary consists of two torus components. Choose an α - α -bordered Heegaard diagram H representing N . Then, it follows again from [15, Theorem 4.6] that

$$\mathbf{AZ} \cup \bar{H} \cup \overline{\mathbf{AZ}}$$

is related to H by a sequence of Heegaard moves. Choosing such a sequence gives a homotopy equivalence

$$\Phi_{\mathbf{AZ} \cup \bar{H} \cup \overline{\mathbf{AZ}}, H} : \widehat{CFDA}(\mathbf{AZ} \cup \bar{H} \cup \overline{\mathbf{AZ}}) \rightarrow \widehat{CFDA}(H).$$

Thus, we can define a bordered involution ι_N as follows:

$$\begin{aligned}
 \iota_N : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(H) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \\
 \xrightarrow{\text{id} \boxtimes \text{conj} \boxtimes \text{id}} \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(\bar{H}) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \\
 \xrightarrow{\simeq} \widehat{CFDA}(\mathbf{AZ} \cup \bar{H} \cup \overline{\mathbf{AZ}}) \\
 \xrightarrow{\Phi_{\mathbf{AZ} \cup \bar{H} \cup \overline{\mathbf{AZ}}, H}} \widehat{CFDA}(H).
 \end{aligned}$$

Unlike the cases of knots and closed 3-manifolds, we do not know whether the homotopy classes of ι_M and ι_N are independent of our choices of sequences of Heegaard moves. This is because a naturality result for bordered Heegaard Floer homology is currently unknown. However, we can instead consider the sets of all possible involutions coming from any possible choices of sequences of Heegaard moves, as shown in the definition below.

Definition 2.2. Given a bordered 3-manifold M with one torus boundary, we denote the set of all possible involutions

$$\begin{aligned}
 \iota_M : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M) &\rightarrow \widehat{CFD}(M), \\
 \iota_M : \widehat{CFA}(M) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) &\rightarrow \widehat{CFD}(M),
 \end{aligned}$$

induced by choosing a sequence of Heegaard moves from $\mathbf{AZ} \cup \bar{H}$ and $\bar{H} \cup \overline{\mathbf{AZ}}$ to H as $\text{Inv}_D(M)$ and $\text{Inv}_A(M)$, respectively. Furthermore, given a bordered 3-manifold N with two torus boundaries, we similarly denote the set of all possible involutions

$$\iota_M : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \rightarrow \widehat{CFDA}(N)$$

induced by choosing a sequence of Heegaard moves from $\mathbf{AZ} \cup \bar{H} \cup \overline{\mathbf{AZ}}$ to H as $\text{Inv}(N)$.

Recall that, given two bordered 3-manifolds M_1 and M_2 , we have a pairing theorem

$$\widehat{CF}(M_1 \cup M_2) \simeq \widehat{CFA}(M_1) \boxtimes \widehat{CFD}(M_2). \quad (2.1)$$

Due to the pairing theorem for triangles [16, Proposition 5.35], it is clear that the homotopy equivalence used in equation (2.1) is well defined up to homotopy [8, Theorem 5.1] tells us that, for any $\iota_1 \in \text{Inv}_D(M_1)$ and $\iota_2 \in \text{Inv}_D(M_2)$, the map

$$\begin{aligned}
 \widehat{CF}(M_1 \cup M_2) &\xrightarrow{\text{pairing}} \widehat{CFA}(M_1) \boxtimes \widehat{CFD}(M_2) \\
 &\xrightarrow{\simeq} \widehat{CFA}(M_1) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M_2) \\
 &\xrightarrow{\iota_1 \boxtimes \iota_2} \widehat{CFA}(M_1) \boxtimes \widehat{CFD}(M_2) \xrightarrow{\text{pairing}} \widehat{CF}(M_1 \cup M_2)
 \end{aligned}$$

is homotopic to the involution $\iota_{M_1 \cup M_2}$ on $\widehat{CF}(M_1 \cup M_2)$.

One also has another pairing formula involving morphism spaces between type-D modules. Given two bordered 3-manifolds M_1 and M_2 with one torus boundary, one can also obtain the hat-flavored Heegaard Floer homology of $-M_1 \cup M_2$ as follows [15, Theorem 1]:

$$\widehat{CF}(-M_1 \cup M_2) \simeq \text{Mor}(\widehat{CFD}(M_1), \widehat{CFD}(M_2)). \quad (2.2)$$

Unlike the box tensor product version of pairing formula, the well-definedness of homotopy equivalence up to homotopy in the above formula is not entirely obvious. This is because its proof relies on the following isomorphism:

$$\begin{aligned} \widehat{CF}(-M_1 \cup M_2) &\xrightarrow{\text{pairing}} \widehat{CFA}(-M_1) \boxtimes \widehat{CFD}(M_2) \\ &\xrightarrow{M_1 \simeq M_1 \cup \mathbf{AZ}} \widehat{CFD}(-M_1) \boxtimes \widehat{CFAA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M_2) \\ &\xrightarrow{\text{canonical isomorphism}} \text{Mor}(\widehat{CFD}(M_1), \widehat{CFD}(M_2)). \end{aligned}$$

In particular, the homotopy equivalence

$$\widehat{CFA}(M_1) \simeq \widehat{CFD}(M_1) \boxtimes \widehat{CFAA}(\mathbf{AZ}),$$

which is induced by a sequence of Heegaard moves from M_1 to $M_1 \cup \mathbf{AZ}$, may not be well defined due to the lack of naturality. However, if we have two such sequences which induce two identification maps

$$F, G : \widehat{CF}(-M_1 \cup M_2) \simeq \text{Mor}(\widehat{CFD}(M_1), \widehat{CFD}(M_2)),$$

then, by the pairing theorem for triangles, the map $G^{-1} \circ F : \widehat{CF}(-M_1 \cup M_2) \rightarrow \widehat{CF}(-M_1 \cup M_2)$ is the homotopy autoequivalence induced by a loop of Heegaard moves, which should be homotopic to identity due to naturality. Therefore, the homotopy equivalence used in equation (2.2) is well defined up to homotopy.

Now, it follows from the proof of [8, Theorem 8.5] that the map

$$\begin{aligned} &\widehat{CF}(-M_1 \cup M_2) \\ &\xrightarrow{\simeq} \text{Mor}(\widehat{CFD}(M_1), \widehat{CFD}(M_2)) \\ &\xrightarrow{f \mapsto \text{id} \boxtimes f} \text{Mor}(\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M_1), \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M_2)) \\ &\xrightarrow{g \mapsto \iota_2 \circ g \circ \iota_1^{-1}} \text{Mor}(\widehat{CFD}(M_1), \widehat{CFD}(M_2)) \\ &\xrightarrow{\simeq} \widehat{CF}(-M_1 \cup M_2) \end{aligned}$$

is homotopic to the involution $\iota_{-M_1 \cup M_2}$ on $\widehat{CF}(-M_1 \cup M_2)$ for any choice of $\iota_1 \in \text{Inv}_D(M_1)$ and $\iota_2 \in \text{Inv}_D(M_2)$.

3. Involutive knot Floer homology with a free basepoint

In this section, we will define the notion of knot Floer homology with a free basepoint, as well as its involutive refinement. This generalization works as a bridge which relates to both involutive knot Floer homology (with no free basepoints) and involutive bordered Heegaard Floer homology of 0-framed knot complements. Roughly speaking, this will be proven by decomposing a doubly-pointed Heegaard diagram (representing a knot) with a free basepoint as the union of two bordered diagrams, where one of them represents the complement of the given knot and the other one is the diagram \mathbb{X} defined in this section. The technical reason for which it is necessary to have a free basepoint in our arguments can be found in Remark 3.2.

Given a knot K , instead of choosing a doubly-pointed Heegaard diagram representing K , we consider a multipointed Heegaard diagram

$$H = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \{z_{\text{free}}\}, z, w),$$

where z and w are points on K and z_{free} is a *free basepoint*, which lies outside K . To make our construction more clear, we will use the notation throughout this paper that the points inside the brackets in Heegaard diagrams are free basepoints.

Given such a diagram, we define its 2-variable knot Floer homology

$$CFK_{UV}(S^3, K, z_{\text{free}}) = CF_{UV}(H) := H_* \left(\bigoplus_{\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \mathbb{F}_2[U, V]_{\mathbf{x}}, \partial \right),$$

where the differential ∂ is defined using the formula

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ n_{z_{\text{free}}}(\phi) = 0 \\ \mu(\phi) = 1}} \# \widetilde{\mathcal{M}}(\phi) \cdot U^{n_z(\phi)} V^{n_w(\phi)} \mathbf{y}.$$

Here, $\widetilde{\mathcal{M}}(\phi)$ denotes the moduli space (modulo the \mathbb{R} -action given by translation) of holomorphic curves representing the given homotopy class ϕ of Whitney disks from \mathbf{x} to \mathbf{y} , and $n_{z_{\text{free}}}(\phi)$, $n_z(\phi)$, and $n_w(\phi)$ denote the algebraic intersection number of ϕ with the codimension 2 submanifolds given by z_{free} , z , and w , respectively. In particular, we are counting disks which do not algebraically intersect with free basepoints. Note that the naturality result for Heegaard Floer homology [12] also applies to this case, so that chain homotopy autoequivalences of $CFK_{UV}(S^3, K, z_{\text{free}})$ induced by any loop of Heegaard moves connecting Heegaard diagrams representing $(S^3, K, z_{\text{free}})$ are homotopic to the identity map.

As in involutive knot Floer homology, we can define the conjugate diagram \bar{H} of H as follows:

$$\bar{H} = (-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \{z_{\text{free}}\}, w, z).$$

We have a canonically defined chain skew-isomorphism

$$\text{conj} : CF_{UV}(H) \rightarrow CF_{UV}(\bar{H}).$$

We then consider the half-twist self-diffeomorphism ϕ of $(S^3, K, z_{\text{free}})$ which maps z and w to w and z , respectively. It induces a diffeomorphism map

$$\phi_* : CF_{UV}(\bar{H}) \rightarrow CF_{UV}(\phi(\bar{H})).$$

Then, since $\phi(\bar{H})$ and H both represent $(S^3, K, z_{\text{free}})$, there exists a sequence of Heegaard moves between them, which induces a homotopy equivalence

$$\Phi_{\phi(\bar{H}), H} : CF_{UV}(\phi(\bar{H})) \rightarrow CF_{UV}(H),$$

which is well defined up to chain homotopy due to naturality. Composing the above three maps thus gives

$$\iota_{K, z_{\text{free}}} = \Phi_{\phi(\bar{H}), H} \circ \phi_* \circ \text{conj},$$

which is again well defined up to chain homotopy.

Given a doubly-pointed Heegaard diagram H_K representing K , we can perform a free-stabilization on H_K near the basepoint z , as shown in the left of Figure 3.1, to get a new diagram H_K^{st} representing $(S^3, K, z_{\text{free}})$. Then, by [29, Lemma 7.1], the differential of $CF_{UV}(H_K^{\text{st}})$ is given by the matrix

$$\partial_{CF_{UV}(H_K^{\text{st}})} = \begin{pmatrix} \partial_{CF_{UV}(H_K)} & UV \\ 0 & \partial_{CF_{UV}(H_K)} \end{pmatrix},$$

where we are using an identification

$$CF_{UV}(H_K^{\text{st}}) \simeq CF_{UV}(H_K) \otimes (\mathbb{F}_2 \cdot \theta^+ \oplus \mathbb{F}_2 \cdot \theta^-)$$

of the chain group. Furthermore, the free-stabilization map $S_{z_{\text{free}}} : CF_{UV}(H_K) \rightarrow CF_{UV}(H_K^{\text{st}})$, defined as

$$S_{z_{\text{free}}}^+(\mathbf{x}) = \mathbf{x} \otimes \theta^+,$$

depends only on the isotopy class of K .

We now assume that K is boundary-parallel to the Heegaard surface Σ and the self-diffeomorphism ϕ acts as identity near the free-stabilization locus. Then, $\phi(\bar{H}_K^{\text{st}})$ is also a free-stabilization on $\phi(\bar{H}_K)$ near the basepoint z , and for any sequence $H_0 = \phi(\bar{H}_K) \rightarrow H_1 \rightarrow \cdots \rightarrow H_n = H_K$ of Heegaard diagrams, we have a corresponding sequence

$$H_0^{\text{st}} = \phi(\bar{H}_K^{\text{st}}) \rightarrow \cdots \rightarrow H_n^{\text{st}} = H_K^{\text{st}}$$

such that, for each i , H_i^{st} is a free-stabilization of H_i near z .

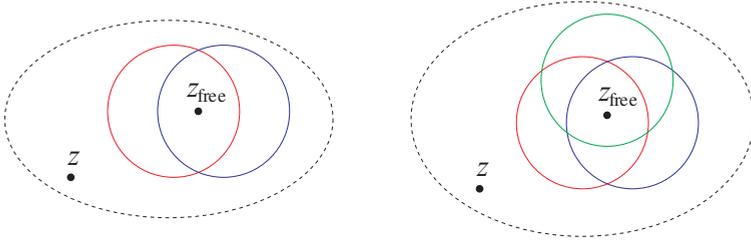


Figure 3.1. Left: A free-stabilization of a Heegaard diagram near a basepoint z . Right: A free-stabilization of a Heegaard triple-diagram near the same basepoint z .

For each i , the Heegaard move $H_i \rightarrow H_{i+1}$ is either an isotopy, a handleslide, or a stabilization. Since we can always start with sufficiently stabilized diagrams and replace an isotopy by a sequence of handleslides, we may further assume that all Heegaard moves that we use are handleslides. Recall that the chain homotopy equivalences associated to handleslides are defined by counting holomorphic triangles in a Heegaard triple diagram. If the homotopy equivalence $f_i : CF_{UV}(H_i) \rightarrow CF_{UV}(H_{i+1})$ is defined by counting triangles in a triple diagram $H_T = (\Sigma, \alpha, \beta, \gamma, z, w)$, then the homotopy equivalence $f_i^{\text{st}} : CF_{UV}(H_i^{\text{st}}) \rightarrow CF_{UV}(H_{i+1}^{\text{st}})$ is defined by counting triangles in a triple diagram H_T^{st} which is obtained by free-stabilizing H_T near z , as shown in the right of Figure 3.1. Thus, by [25, Theorem 6.7], we know that

$$S_{z_{\text{free}}}^+ \circ f_i \sim f_i^{\text{st}} \circ S_{z_{\text{free}}}^+,$$

so we deduce that $S_{z_{\text{free}}}^+$ is well defined up to homotopy and

$$S_{z_{\text{free}}}^+ \circ \iota_k \sim \iota_{K, z_{\text{free}}} \circ S_{z_{\text{free}}}^+.$$

Furthermore, since the truncated map $S_{z_{\text{free}}}^+|_{U=1, V=0}$ is the hat-flavored free-stabilization map

$$\hat{S}_{z_{\text{free}}}^+ : \widehat{CFK}(S^3, z) \rightarrow \widehat{CFK}(S^3, \{z_{\text{free}}\}, z),$$

which is injective in homology, we see that $U^{-1}S_{z_{\text{free}}}^+|_{V=0}$ (and also $V^{-1}S_{z_{\text{free}}}^+|_{U=0}$) also induces an injective map in homology. Therefore, $S_{z_{\text{free}}}^+$ is local.

We now interpret involutive knot Floer theory with a free basepoint in terms of bordered Floer homology. Consider the triply-pointed bordered Heegaard diagram

$$\mathbb{X} = (\Sigma, \alpha, \beta, \{z_{\text{free}}\}, z, w),$$

defined as in Figure 3.2. This diagram represents the longitudinal knot lying inside the ∞ -framed solid torus, together with a prescribed free basepoint z_{free} on the boundary torus.

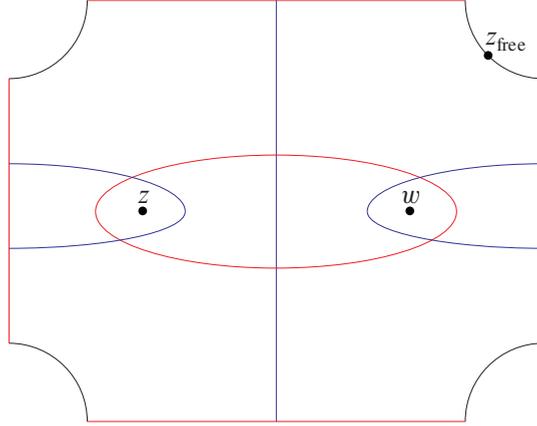


Figure 3.2. The triply-pointed Heegaard diagram \mathbb{X} .

Note that, for any bordered Heegaard diagram H of $M \setminus K$, where K is a framed knot inside a closed 3-manifold M and the framing is denoted as ν , the glued diagram $H \cup \mathbb{X}$ is a Heegaard diagram representing the core curve inside the Dehn surgery $M_\nu(K)$, together with a free basepoint.

We now consider the new diagram $\phi(\bar{\mathbb{X}})$, where $\bar{\mathbb{X}}$ denotes the conjugate diagram of \mathbb{X} , defined as

$$\bar{\mathbb{X}} = (-\Sigma, \beta, \alpha, \{z_{\text{free}}\}, w, z),$$

and ϕ denotes the “half-twist” self-diffeomorphism of Σ along the longitudinal knot, so that it maps z to w and w to z , respectively.

Lemma 3.1. *Consider the α - β -bordered Auroux–Zarev piece \mathbf{AZ} . Then, $\overline{\mathbf{AZ}} \cup \phi(\bar{\mathbb{X}})$ and \mathbb{X} are related by a sequence of Heegaard moves.*

Proof. Denote the bordered Heegaard diagram representing the 0-framed solid torus as H and its conjugate as \bar{H} . It is proven in [15, Figures 8 and 9] that $\overline{\mathbf{AZ}} \cup \bar{H}$ and H are related by a sequence of handleslides and a destabilization. Since H is simply \mathbb{X} without the basepoints z, w and the α - and β -curves surrounding them, it is clear that the sequence of handleslides (and a single destabilization) from $\overline{\mathbf{AZ}} \cup \bar{H}$ to H induces a sequence of handleslides and a single destabilization from $\overline{\mathbf{AZ}} \cup \phi(\bar{\mathbb{X}})$ to \mathbb{X} . A detailed process is drawn in Figure 3.3. ■

Remark 3.2. Since the Heegaard diagram \mathbb{X} represents a longitudinal knot ν inside the ∞ -framed solid torus T_∞ , it is natural to ask whether Lemma 3.1 also holds for doubly-pointed bordered Heegaard diagrams (with no free basepoint) representing the pair (T_∞, ν) . Unfortunately, it does not, as first observed in [15, Remark 5.1].

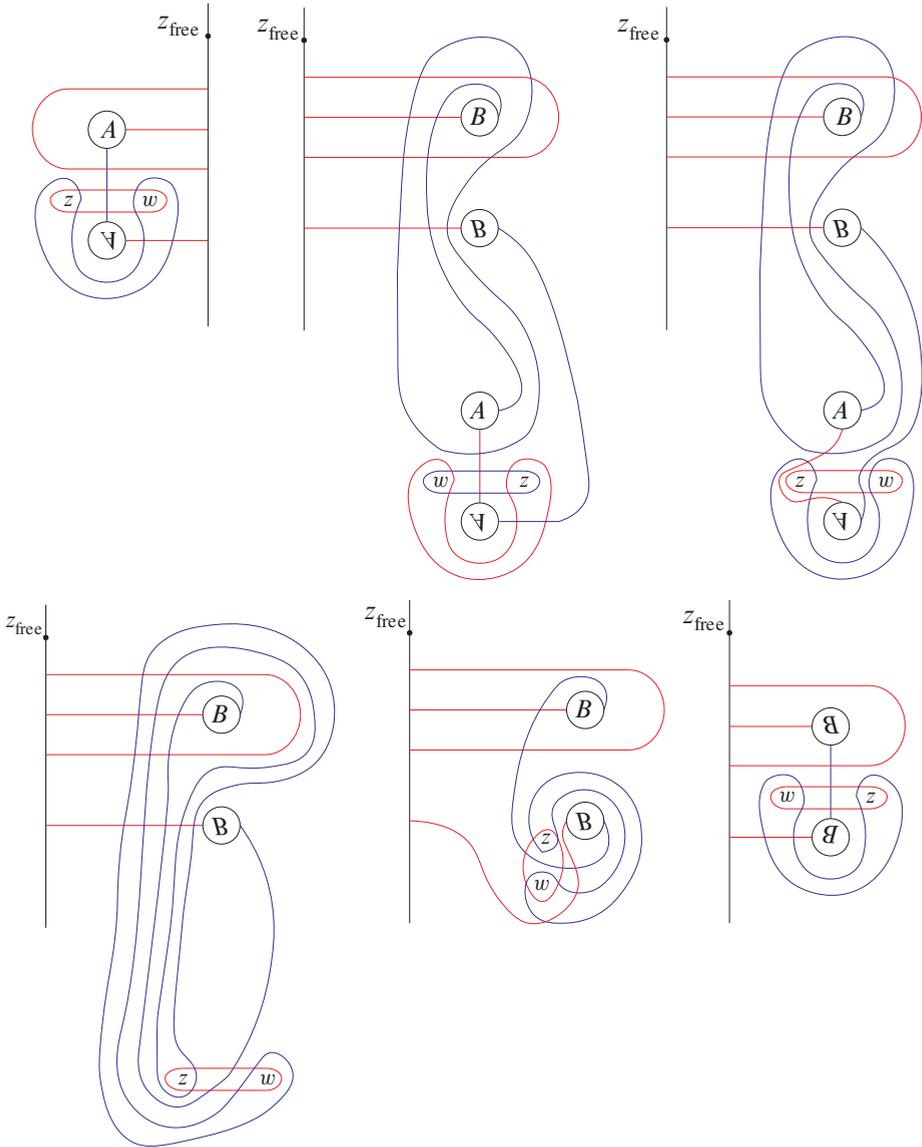


Figure 3.3. Top-left: The diagram \mathbb{X} . Top-middle: The diagram $\overline{\mathbf{AZ}} \cup \overline{\mathbb{X}}$. Top-right: The diagram $\overline{\mathbf{AZ}} \cup \phi(\overline{\mathbb{X}})$. Bottom-left: A diagram obtained from the one on the top-right by a sequence of handleslides, followed by a destabilization. Bottom-middle: A diagram obtained from the one on the bottom-left by another sequence of handleslides. Bottom-right: The diagram obtained by isotopy from the one on the bottom-middle. Note that this is the same as the original diagram \mathbb{X} .

Choose a nice diagram \mathbb{X}_0 which is related to \mathbb{X} by a sequence of Heegaard moves; such a diagram always exists by Sarkar–Wang algorithm [17, Proposition 8.2], and it is always provincially admissible. Then, \mathbb{X}_0 has a well-defined bordered Floer homology. In particular, if we write

$$\mathbb{X}_0 = (\Sigma, \alpha, \beta, \{z_{\text{free}}\}, z, w),$$

then we have a well-defined type-D structure $CFD_{UV}(\mathbb{X}_0)$ and a type-A structure $CFA_{UV}(\mathbb{X}_0)$ over the module $\mathbb{F}_2[U, V]$, defined by counting holomorphic disks which do not intersect algebraically with z_{free} , while recording their algebraic intersection numbers with z and w by formal variables U and V , respectively.

Recall from [17, Chapter 10] that, given a bordered 3-manifold Y with boundary \mathcal{Z} , the associated modules $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$ are graded by a transitive $G(\mathcal{Z})$ -set, and for a doubly-pointed bordered Heegaard diagram $H = (\Sigma, \alpha, \beta, z, w)$ with the same boundary, the associated modules $CFA^-(H)$ and $CFD^-(H)$ admit an enhanced grading by a transitive $(G(\mathcal{Z}) \times \mathbb{Z})$ -set, where the grading on the \mathbb{Z} component is given by $n_w - n_z$. We can define a grading on $CFA_{UV}(\mathbb{X}_0)$ by the group $G(T^2) \times \mathbb{Z}$ in a similar manner as follows.

Write $\mathbb{X}_0 = (\Sigma, \alpha, \beta, \{z_{\text{free}}\}, z, w)$. Then, for any choice of Floer generators \mathbf{x} and \mathbf{y} and a homology class $B \in \pi_2(\mathbf{x}, \mathbf{y})$ of curves connecting \mathbf{x} to \mathbf{y} , we define the relative grading $\tilde{g}(\mathbf{x}, \mathbf{y}) \in G(T^2) \times \mathbb{Z}$ as

$$\tilde{g}(\mathbf{x}, \mathbf{y}) = (\lambda^{-2n_{z_{\text{free}}}(B)} g(\mathbf{x}, \mathbf{y}), n_z(B) - n_w(B)), \quad (3.1)$$

where λ is the central element $(1; 0, 0)$ of $G(T^2)$ and g denotes the quantity determined by [17, equation (10.31)]. This endows $CFA_{UV}(\mathbb{X}_0)$ with a grading by a transitive $(G(T^2) \times \mathbb{Z})$ -set. After taking a box tensor product with $\widehat{CFD}(S^3 \setminus K)$, where K is a knot, the gradings on $\widehat{CFD}(S^3 \setminus K)$ and $CFA_{UV}(\mathbb{X}_0)$ induce a grading on the tensor product.

Lemma 3.3. *Given a knot $K \subset S^3$, denote the bordered 3-manifold representing its 0-framed complement as $S^3 \setminus K$. Then, we have a pairing formula*

$$\widehat{CFD}(S^3 \setminus K) \boxtimes CFA_{UV}(\mathbb{X}_0) \simeq CFK_{UV}(S^3, K, z_{\text{free}}).$$

Furthermore, the induced grading on the left-hand side matches the bigrading (i.e., Maslov and Alexander) on the right-hand side.

Proof. Choose a nice bordered Heegaard diagram \mathcal{H} representing $S^3 \setminus K$. Since the proof of pairing theorem [17, Theorem 1.3] works trivially for admissible diagrams, we have

$$\widehat{CFD}(S^3 \setminus K) \boxtimes CFA_{UV}(\mathbb{X}_0) \simeq CF_{UV}(\mathcal{H} \cup \mathbb{X}_0).$$

The Heegaard diagram $\mathcal{H} \cup \mathbb{X}_0$ represents K ; together with a free basepoint z_{free} lying outside K , we get the desired homotopy equivalence. The statement about gradings follows directly from the arguments used in the proof of [17, Theorem 1.3]. ■

Remark 3.4. In the proof of Lemma 3.3, the term $CF_{UV}(\mathcal{H} \cup \mathbb{X}_0)$ is the Floer chain complex coming from cylindrical reformulation of Heegaard Floer homology due to Lipshitz [14]. The original setting of cylindrical reformation is only for Heegaard diagrams with one basepoint, so it is natural to ask whether it also works for general diagrams $(\Sigma, \alpha, \beta, \mathbf{z})$, where the number of α -curves may exceed the genus of Σ (in which case we have more than one basepoint). Fortunately, the cylindrical reformation also works in those generalized settings; see [23, Section 5.2] for details.

Lemma 3.5. *There exist type-A homotopy equivalences*

$$\begin{aligned} CFA_{UV}(\mathbb{X}_0) \otimes \mathbb{F}[U, V]/(U-1, V) &\simeq CFA_{UV}(\mathbb{X}_0) \otimes \mathbb{F}[U, V]/(U, V-1) \\ &\simeq \widehat{CFA}(0\text{-framed solid torus}) \otimes \mathbb{F}_2^2. \end{aligned}$$

Proof. Write $\mathbb{X}_0 = (\Sigma, \alpha, \beta, \{z_{\text{free}}, z, w\})$. Since truncating by $V = 1$ is equivalent to forgetting the basepoint w , we have the following homotopy equivalence of type-A modules:

$$CFA_{UV}(\mathbb{X}_0) \otimes \mathbb{F}[U, V]/(U, V-1) \simeq \widehat{CFA}(\Sigma, \alpha, \beta, \{z_{\text{free}}, z\}).$$

Since we no longer have w as a basepoint, the bordered Heegaard diagram $(\Sigma, \alpha, \beta, \{z_{\text{free}}, z\})$ is isotopic to the diagram we obtain by stabilizing a bordered Heegaard diagram representing the 0-framed solid torus near its basepoint. Since we are not counting holomorphic disks intersecting the stabilization region, it is clear, even without a neck-stretching argument, that we have a canonical isomorphism

$$\widehat{CFA}(\Sigma, \alpha, \beta, \{z_{\text{free}}, z\}) \simeq \widehat{CFA}(0\text{-framed solid torus}) \otimes \mathbb{F}_2^2,$$

which proves the lemma. ■

Let $\bar{\mathbb{X}}_0$ be the conjugate diagram of \mathbb{X}_0 , defined in the same way as $\bar{\mathbb{X}}$. Then, by Lemma 3.1, we know that $\bar{\mathbf{AZ}} \cup \phi(\bar{\mathbb{X}}_0)$ is related by a sequence of Heegaard moves to \mathbb{X}_0 . As in the proof of Lemma 3.3, it is clear that we have a pairing formula

$$CFA_{UV}(\bar{\mathbb{X}}_0) \boxtimes \widehat{CFDA}(\bar{\mathbf{AZ}}) \simeq CFA_{UV}(\bar{\mathbb{X}}_0 \cup \bar{\mathbf{AZ}}),$$

so any choice of a sequence of Heegaard moves from $\bar{\mathbf{AZ}} \cup \phi(\bar{\mathbb{X}}_0)$ to \mathbb{X}_0 induces a type-A morphism

$$\iota_{\mathbb{X}} : CFA_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFDA}(\bar{\mathbf{AZ}}) \rightarrow CFA_{UV}(\mathbb{X}_0).$$

Note that $\iota_{\mathbb{X}}$ is a homotopy equivalence of type-A modules over \mathbb{F}_2 , but not over $\mathbb{F}_2[U, V]$; this is because it intertwines the actions of U and V . Thus, $\iota_{\mathbb{X}}$ is a type-A homotopy *skew-equivalence*.

The definition of $\iota_{\mathbb{X}}$ depends on the choices that we have made in its construction. Choosing a different sequence of Heegaard moves may result in another homotopy equivalence which is not homotopic to $\iota_{\mathbb{X}}$ due to the lack of naturality for bordered Floer homology. However, it will not affect the results of this paper; we only have to choose one sequence of Heegaard moves once and for all.

Given a knot $K \subset S^3$ and a bordered Heegaard diagram \mathcal{H} for the 0-framed complement of K , recall that we can choose a homotopy equivalence

$$\iota_{S^3 \setminus K} : \widehat{CFD}(\bar{\mathcal{H}} \cup \overline{\mathbf{AZ}}) \rightarrow \widehat{CFD}(\mathcal{H}),$$

which is an element of $\text{Inv}_D(S^3 \setminus K)$. Furthermore, we have the following conjugation maps:

$$\begin{aligned} \text{conj}_{\mathbb{X}} &: CFA_{UV}(\mathbb{X}_0) \rightarrow CFA_{UV}(\bar{\mathbb{X}}_0), \\ \text{conj}_{S^3 \setminus K} &: CFD(\mathcal{H}) \rightarrow CFD(\bar{\mathcal{H}}). \end{aligned}$$

We consider the following composition of homotopy equivalences:

$$\begin{aligned} &CFK_{UV}(S^3, K, z_{\text{free}}) \\ &\simeq CFA_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(\mathcal{H}) \\ &\xrightarrow{\text{conj}_{\mathbb{X}} \boxtimes \text{conj}_{S^3 \setminus K}} CFA_{UV}(\bar{\mathbb{X}}_0) \boxtimes \widehat{CFD}(\bar{\mathcal{H}}) \\ &\xrightarrow{\phi_* \boxtimes \text{id}} CFA_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFD}(\bar{\mathcal{H}}) \\ &\simeq CFA_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\bar{\mathcal{H}}) \\ &\simeq CFA_{UV}(\phi(\bar{\mathbb{X}}_0) \cup \overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\bar{\mathcal{H}}) \\ &\xrightarrow{\iota_{\mathbb{X}} \boxtimes \iota_{S^3 \setminus K}} CFA_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(\mathcal{H}) \simeq CFK_{UV}(S^3, K, z_{\text{free}}). \end{aligned}$$

Lemma 3.6. *For any choice of $\iota_{S^3 \setminus K} \in \text{Inv}_D(S^3 \setminus K)$, the above homotopy equivalence is homotopic to $\iota_{K, z_{\text{free}}}$.*

Proof. One can use the argument used in the proof of [8, Theorem 5.1] verbatim. ■

For later use, we prove the following lemma. We recall, for clarity, that \mathcal{R} denotes the ring $\mathbb{F}_2[U, V]/(UV)$.

Lemma 3.7. *Given a knot K , suppose that there exists a local chain map*

$$f : CFK_{UV}(S^3, K, z_{\text{free}}) \rightarrow CFK_{UV}(S^3, \text{unknot}, z_{\text{free}})$$

which preserves the Alexander and Maslov gradings such that

$$f \circ \iota_{K, z_{\text{free}}} \sim f.$$

Then, there also exists a local (bidegree-preserving) chain map $g : CFK_{\mathcal{R}}(S^3, K) \rightarrow \mathcal{R}$ satisfying

$$g \circ \iota_K \sim g.$$

Proof. Consider the free-stabilization map

$$S_{z_{\text{free}}}^+ : CFK_{UV}(S^3, K) \rightarrow CFK_{UV}(S^3, K, z_{\text{free}}).$$

Then, we have

$$f \circ S_{z_{\text{free}}}^+ \circ \iota_K \sim f \circ \iota_{K, z_{\text{free}}} \circ S_{z_{\text{free}}}^+ \sim f \circ S_{z_{\text{free}}}^+.$$

Since the codomain of $f \circ S_{z_{\text{free}}}^+$ is \mathcal{R} , it induces a chain map

$$\begin{aligned} f_{\mathcal{R}} &: CFK_{\mathcal{R}}(S^3, K) \\ &= CFK_{UV}(S^3, K) \otimes \mathcal{R} \xrightarrow{(f \circ S_{z_{\text{free}}}^+) \otimes \text{id}_{\mathcal{R}}} CFK_{UV}(S^3, \text{unknot}, z_{\text{free}}) \otimes \mathcal{R}. \end{aligned}$$

Since

$$\begin{aligned} U^{-1} f_{\mathcal{R}}|_{V=0} &= U^{-1} (f \circ S_{z_{\text{free}}}^+)|_{V=0}, \\ V^{-1} f_{\mathcal{R}}|_{U=0} &= V^{-1} (f \circ S_{z_{\text{free}}}^+)|_{U=0} \end{aligned}$$

and the maps f and $S_{z_{\text{free}}}^+$ are local, we deduce that $f_{\mathcal{R}}$ is also local.

Recall that the differential on

$$CFK_{UV}(S^3, \text{unknot}, z_{\text{free}}) = (\mathbb{F}_2[U, V]_{\mathbf{x}_+} \oplus \mathbb{F}_2[U, V]_{\mathbf{x}_-}, \partial)$$

is given by

$$\partial_{\mathbf{x}_+} = 0, \quad \partial_{\mathbf{x}_-} = UV_{\mathbf{x}_+}.$$

Since $UV = 0$ in \mathcal{R} , we can define a projection map

$$p : CFK_{UV}(S^3, \text{unknot}, z_{\text{free}}) \otimes \mathcal{R} \rightarrow \mathcal{R}$$

by $p(\mathbf{x}_+) = 1$ and $p(\mathbf{x}_-) = 0$. Then, the composed map $g = p \circ f_{\mathcal{R}}$ satisfies

$$g \circ \iota_K = p \circ f_{\mathcal{R}} \circ \iota_K \sim p \circ f_{\mathcal{R}} = g.$$

Furthermore, g is a local map due to grading reasons. Therefore, g is the desired map. \blacksquare

4. Involutive knot Floer homology and involutive bordered Floer homology

Recall that, for any two bordered 3-manifold M, N with the same boundary, we have a pairing formula

$$\widehat{CF}(M \cup N) \simeq \text{Mor}(\widehat{CFD}(-M), \widehat{CFD}(N)).$$

Note that the cycles in the morphism space correspond to type-D morphisms, and boundaries correspond to nullhomotopic morphisms. Consider the case when M is the 0-framed complement of a knot K and N is the 0-framed solid torus. Then, we have $S_0^3(-K) \simeq -M \cup N$, so the pairing formula induces a homotopy equivalence

$$\widehat{CF}(S_0^3(-K)) \simeq \text{Mor}(\widehat{CFD}(S^3 \setminus K), \widehat{CFD}(T_0)),$$

where T_0 denotes the 0-framed solid torus. Now, by Lemma 3.3, we get a chain map

$$\begin{aligned} \text{ch}_{\mathbb{X}} : \widehat{CF}(S_0^3(-K)) & \\ & \xrightarrow{\simeq} \text{Mor}(\widehat{CFD}(S^3 \setminus K), \widehat{CFD}(T_0)) \\ & \xrightarrow{f \mapsto \text{id} \boxtimes f} \text{Hom}(CFA_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(S^3 \setminus K), CFA_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(T_0)) \\ & \xrightarrow{\text{pairing}} \text{Hom}(CFK_{UV}(S^3, K, z_{\text{free}}), CFK_{UV}(S^3, \text{unknot}, z_{\text{free}})). \end{aligned}$$

On the other hand, by pairing with $\widehat{CFA}(\infty\text{-framed solid torus})$ instead of $CFA_{UV}(\mathbb{X}_0)$, we also get a chain map

$$\text{ch}_{\infty} : \widehat{CF}(S_0^3(-K)) \rightarrow \text{Hom}(\widehat{CF}(S^3), \widehat{CF}(S^3)) = \mathbb{F}_2.$$

Lemma 4.1. *Let $X_0(-K)$ be the punctured 0-trace of the knot $-K$ i.e., the 4-manifold obtained by attaching a 0-framed 2-handle to $S^3 \times I$ along $-K \times \{1\}$. Then, the map $x \mapsto \text{ch}_{\infty}(x)(1) : \widehat{CF}(S_0^3(-K)) \rightarrow \widehat{CF}(S^3)$ is the hat-flavored cobordism map induced by the cobordism $X_0(-K)$ flipped upside-down.*

Proof. Discussions in [15, Section 1.5] tell us that the map $\text{ch}_{\infty} : \widehat{CF}(S_0^3(-K)) \otimes \widehat{CF}(S^3) \rightarrow \widehat{CF}(S^3)$ is the cobordism map induced by the 4-manifold W_0 given by

$$W_0 = (\Delta \times T) \bigcup_{e_1 \times T} (e_1 \times (S^3 \setminus K)) \bigcup_{e_2 \times T} (e_2 \times T_{\infty}) \bigcup_{e_3 \times T} (e_3 \times T_0),$$

where Δ denotes a triangle with edges e_1, e_2, e_3 , and T denotes a torus. Note that W_0 has three boundary components given by

$$S_0^3(-K) = -(S^3 \setminus K) \cup T_0, \quad S^3 = T_0 \cup T_{\infty}, \quad \text{and} \quad S^3 = (S^3 \setminus K) \cup T_{\infty}.$$

Hence, the cobordism map induced by 4-manifold W obtained by gluing a 4-ball to the second boundary, i.e.,

$$W = W_0 \bigcup_{T_0 \cup T_\infty} B^4,$$

is the given map $x \mapsto \text{ch}_\infty(x)(1)$. Since W is diffeomorphic to $X_0(K)$, flipped upside-down, the lemma follows. \blacksquare

The following example explains Lemma 4.1 in the case when K is the unknot.

Example 4.2. Let K be the unknot. Then, $S^3 \setminus K \simeq T_0$, $S_0^3(-K) \simeq S^1 \times S^2$, and $X_0(-K) \simeq D^2 \times S^2$. The type-D module of the 0-framed solid torus T_0 is freely generated over the torus algebra $\mathcal{A}(T^2)$, which is generated (over \mathbb{F}_2) by the set

$$\{\iota_0, \iota_1, \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}\},$$

by a single element x , and the differential is given by $\partial x = \rho_{12}x$. The identity morphism

$$\text{id} : \widehat{CFD}(S^3 \setminus K) = \widehat{CFD}(T_0) \rightarrow \widehat{CFD}(T_0)$$

corresponds to the $\frac{1}{2}$ -graded generator in

$$\widehat{CFD}(S^1 \times S^2) \simeq \mathbb{F}_2 \left[\begin{array}{c} 1 \\ 2 \end{array} \right] \oplus \mathbb{F}_2 \left[-\frac{1}{2} \right].$$

Note that the $-\frac{1}{2}$ -graded generator corresponds to the map $x \mapsto \rho_{12}x$. The cobordism map

$$x \mapsto \text{ch}_\infty(x)(1) : \widehat{CF}(S^1 \times S^2) \rightarrow \widehat{CF}(S^3)$$

induced by $D^2 \times S^2$ which bounds $S^1 \times S^2$ is a map of degree $-\frac{1}{2}$, which maps the $\frac{1}{2}$ -graded generator (which corresponds to the identity morphism) to 1 and the $-\frac{1}{2}$ -graded generator to 0.

Lemma 4.3. *Let K be a knot such that $(CFK_{\mathcal{R}}(S^3, K), \iota_K)$ is locally equivalent to the trivial complex. Then, there exists a cycle $x \in \widehat{HF}(S_0^3(-K))$ of absolute \mathbb{Q} -grading $\frac{1}{2}$, which is mapped to the unique homotopy autoequivalence*

$$[\text{id}] \in H_*(\text{Hom}(\widehat{CF}(S^3), \widehat{CF}(S^3)))$$

under the map F .

Proof. By Lemma 4.1, we know that the map

$$x \mapsto \text{ch}_\infty(x)(1) : \widehat{CF}(S_0^3(-K)) \rightarrow \widehat{CF}(S^3)$$

is the hat-flavored cobordism map induced by the cobordism W by flipping the 0-framed 2-handle attaching map along $-K$ upside-down. Recall from the involutive

mapping cone formula [5, Section 22.9] that the Heegaard Floer homology of $S_0^3(-K)$ is homotopy equivalent to a complex of the form

$$\widehat{CF}(S_0^3(-K), [0]) \simeq \text{Cone}(D_0 : \hat{A}_0 \rightarrow \hat{B}_0),$$

and the involution $\iota_{S_0^3(-K)}$ takes the form $\iota_A + \iota_B + H$, where ι_A and ι_B are the involutions on \hat{A}_0 and \hat{B}_0 , respectively, induced by ι_{-K} , and H is a certain homotopy between $\iota_B \circ D_0$ and $D_0 \circ \iota_A$. Also, it is shown in [6, Theorem 15.1] that the cobordism map $\widehat{CF}(S_0^3(-K), [0]) \rightarrow \widehat{CF}(S^3)$ is given by the projection onto \hat{A}_0 , composed with the inclusion map of \hat{A}_0 into \hat{B}_0 .

Let $g : CFK_{\mathcal{R}}(S^3, \text{unknot}) \rightarrow CFK_{\mathcal{R}}(S^3, K)$ be a local map such that $\iota_K \circ g \sim g \circ \iota_{\text{unknot}}$. Following the proof of [5, Proposition 3.15 (3)] shows that choosing a homotopy between $\iota_K \circ g$ and $g \circ \iota_{\text{unknot}}$ induces a local map $F_g : \widehat{CF}(S^1 \times S^2) \rightarrow \widehat{CF}(S_0^3(-K), [0])$ satisfying $\iota_{S_0^3(-K)} \circ F_g \sim F_g \circ \iota_{S^1 \times S^2}$. Denote by x_0 the unique generator of the $\frac{1}{2}$ -graded piece of $\widehat{HF}(S^1 \times S^2)$. Since projection to A_0 clearly homotopy-commutes with F_g , we see from Example 4.2 that $F_g(x_0)$ is a $\iota_{S_0^3(-K)}$ -invariant element of $\widehat{HF}(S_0^3(-K))$ which is mapped to the generator of $\widehat{HF}(S^3)$ under the cobordism map induced by W , proving the lemma. \blacksquare

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. Given two knots K_1 and K_2 , suppose that $(CFK_{\mathcal{R}}(S^3, K_1\# - K_2), \iota_{K_2\# - K_1})$ is ι_K -locally equivalent to the trivial complex. By Lemma 4.3, there exists a cycle $x \in \widehat{HF}(S_0^3(K_2\# - K_1))$ of absolute \mathbb{Q} -grading $\frac{1}{2}$, which is invariant under the action of $\iota_{S_0^3(K_2\# - K_1)}$ and mapped to the unique homotopy autoequivalence $[\text{id}] \in H_*(\text{Hom}(\widehat{CF}(S^3), \widehat{CF}(S^3)))$ under the map F .

Since we have

$$-(S^3 \setminus K_1) \cup (S^3 \setminus K_2) \simeq S_0^3(K_2\# - K_1),$$

where both $S^3 \setminus K_1$ and $S^3 \setminus K_2$ are 0-framed, we have a pairing theorem

$$\widehat{CF}(S_0^3(K_2\# - K_1)) \simeq \text{Mor}(\widehat{CFD}(S^3 \setminus K_1), \widehat{CFD}(S^3 \setminus K_2)).$$

Denote by $F_x : \widehat{CFD}(S^3 \setminus K_1) \rightarrow \widehat{CFD}(S^3 \setminus K_2)$ the type-D morphism which corresponds to x . Then, we have the following homotopy-commutative diagram for any choice of $\iota_{S^3 \setminus K_1} \in \text{Inv}_D(S^3 \setminus K_1)$ and $\iota_{S^3 \setminus K_2} \in \text{Inv}_D(S^3 \setminus K_2)$:

$$\begin{array}{ccc} \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K_1) & \xrightarrow{\iota_{S^3 \setminus K_1}} & \widehat{CFD}(S^3 \setminus K_1) \\ \downarrow \text{id} \boxtimes g & & \downarrow g \\ \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K_2) & \xrightarrow{\iota_{S^3 \setminus K_2}} & \widehat{CFD}(S^3 \setminus K_2) \end{array}$$

Furthermore, since $\text{ch}_\infty(x)$ corresponds to the identity morphism of $\widehat{CF}(S^3)$, we see that the induced map

$$\begin{aligned} \widehat{CF}(S^3) &\simeq \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3 \setminus K_1) \\ &\xrightarrow{\text{id} \boxtimes g} \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3 \setminus K_2) \simeq \widehat{CF}(S^3) \end{aligned}$$

is homotopic to the identity morphism.

Now, suppose that we have a type-D morphism $g: \widehat{CFD}(S^3 \setminus K_1) \rightarrow \widehat{CFD}(S^3 \setminus K_2)$ which satisfies the conditions of Theorem 1.1 for some choices of

$$\iota_{S^3 \setminus K_1} \in \text{Inv}_D(S^3 \setminus K_1) \quad \text{and} \quad \iota_{S^3 \setminus K_2} \in \text{Inv}_D(S^3 \setminus K_2).$$

By taking a box tensor product with an involution $\iota_{T_\infty \setminus P} \in \text{Inv}(T_\infty \setminus P)$ of the type-DA bimodule $\widehat{CFDA}(T_\infty \setminus P)$ of the exterior of the connected-sum pattern P induced by $-K_1$, we may replace K_1 with $K_1 \sharp - K_1$ and K_2 with $K_2 \sharp - K_1$ without any loss of generality (see the discussion below the proof for details). Then, after pairing with $\widehat{CFDA}(\mathbb{X}_0)$, we get the following homotopy-commutative diagram:

$$\begin{array}{ccc} \text{CFK}_{UV}(S^3, K_1 \sharp - K_1, z_{\text{free}}) & \xrightarrow{\text{ch}_{\mathbb{X}}(x)} & \text{CFK}_{UV}(S^3, K_2 \sharp - K_1, z_{\text{free}}) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{CFA}_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(S^3 \setminus (K_1 \sharp - K_1)) & \xrightarrow{\text{id} \boxtimes F_x} & \text{CFA}_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(S^3 \setminus (K_2 \sharp - K_1)) \\ \downarrow \text{conj}_{\mathbb{X}} \boxtimes \text{id} & & \downarrow \text{conj}_{\mathbb{X}} \boxtimes \text{id} \\ \text{CFA}_{UV}(\bar{\mathbb{X}}_0) \boxtimes \widehat{CFD}(S^3 \setminus (K_1 \sharp - K_1)) & \xrightarrow{\text{id} \boxtimes F_x} & \text{CFA}_{UV}(\bar{\mathbb{X}}_0) \boxtimes \widehat{CFD}(S^3 \setminus (K_2 \sharp - K_1)) \\ \downarrow \phi_* \boxtimes \text{id} & & \downarrow \phi_* \boxtimes \text{id} \\ \text{CFA}_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFD}(S^3 \setminus (K_1 \sharp - K_1)) & \xrightarrow{\text{id} \boxtimes F_x} & \text{CFA}_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFD}(S^3 \setminus (K_2 \sharp - K_1)) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{CFA}_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFDA}(\mathbf{AZ}) & \xrightarrow{\text{id} \boxtimes \text{id} \boxtimes \text{id} \boxtimes F_x} & \text{CFA}_{UV}(\phi(\bar{\mathbb{X}}_0)) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \\ \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus (K_1 \sharp - K_1)) & & \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus (K_2 \sharp - K_1)) \\ \downarrow \iota_{\mathbb{X}} \boxtimes \iota_{S^3 \setminus (K_1 \sharp - K_1)} & & \downarrow \iota_{\mathbb{X}} \boxtimes \iota_{S^3 \setminus (K_2 \sharp - K_1)} \\ \text{CFA}_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(S^3 \setminus (K_1 \sharp - K_1)) & \xrightarrow{\text{id} \boxtimes F_x} & \text{CFA}_{UV}(\mathbb{X}_0) \boxtimes \widehat{CFD}(S^3 \setminus (K_2 \sharp - K_1)) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{CFK}_{UV}(S^3, K_1 \sharp - K_1, z_{\text{free}}) & \xrightarrow{\text{ch}_{\mathbb{X}}(x)} & \text{CFK}_{UV}(S^3, K_2 \sharp - K_1, z_{\text{free}}) \end{array}$$

By Lemma 3.6, the compositions of vertical maps on the two columns of the above diagram are $\iota_{K_1 \sharp - K_1, z_{\text{free}}}$ and $\iota_{K_2 \sharp - K_1, z_{\text{free}}}$, respectively, which implies that

$$\text{ch}_{\mathbb{X}}(x) \circ \iota_{K_1 \sharp - K_1, z_{\text{free}}} \sim \iota_{K_2 \sharp - K_1, z_{\text{free}}} \circ \text{ch}_{\mathbb{X}}(x).$$

Since $K_1\sharp - K_1$ is slice, we should have a local map

$$G : CFK_{UV}(S^3, \text{unknot}, z_{\text{free}}) \rightarrow CFK_{UV}(S^3, K_1\sharp - K_1, z_{\text{free}})$$

satisfying

$$G \sim \iota_{K_1\sharp - K_1, z_{\text{free}}} \circ G.$$

Hence, by Lemma 3.7 (applied dually), we have an ι_K -local chain map

$$f : \mathcal{R} \rightarrow CFK_{\mathcal{R}}(S^3, K_2\sharp - K_1).$$

Now, since our argument can also be applied with the knots K_2 and K_1 swapped, we should also have an ι_K -local chain map

$$f' : CFK_{\mathcal{R}}(S^3, K_2\sharp - K_1) \rightarrow \mathcal{R}.$$

Therefore,

$$[CFK_{\mathcal{R}}(S^3, K_1\sharp - K_2), \iota_{K_1\sharp - K_2}]$$

is locally equivalent to the trivial complex \mathcal{R} , as desired. \blacksquare

Now, suppose that we have two bordered 3-manifolds M and N , where M has one torus boundary ∂M and N has two torus boundaries, $\partial_1 N$ and $\partial_2 N$. Choose any $\iota_M \in \text{Inv}_D(M)$ and $\iota_N \in \text{Inv}(N)$ so that we have type-D and type-DA homotopy equivalences

$$\begin{aligned} \iota_M &: \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M) \rightarrow \widehat{CFD}(M), \\ \iota_N &: \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \rightarrow \widehat{CFDA}(N), \end{aligned}$$

where the boundary components $\partial_1 N$ and $\partial_2 N$ are considered as type-A and type-D boundaries, respectively. Recall that we have a pairing theorem for computing $\widehat{CFD}(M \cup N)$, where we identify ∂M with $\partial_1 N$

$$\widehat{CFD}(M \cup N) \simeq \widehat{CFDA}(N) \boxtimes \widehat{CFD}(M).$$

Then, our choice of ι_M and ι_N induces a homotopy equivalence ι_{M, ι_N} for $\widehat{CFD}(M \cup N)$ as follows:

$$\begin{aligned} \iota_{M, \iota_N} &: \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M \cup N) \\ &\simeq \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFD}(M) \\ &\simeq \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \\ &\quad \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(M) \\ &\xrightarrow{\iota_M \boxtimes \iota_N} \widehat{CFDA}(N) \boxtimes \widehat{CFD}(M) \simeq \widehat{CFD}(M \cup N). \end{aligned}$$

Following the proof of [8, Theorem 5.1], we immediately see that

$$\iota_M, \iota_N \in \text{Inv}_D(M \cup N).$$

Using this fact, we can now prove Theorem 1.2.

Proof of Theorem 1.2. Let K_1 and K_2 be two knots satisfying the given assumptions. Then, by Theorem 1.1, there exists a type-D morphism

$$g : \widehat{CFD}(S^3 \setminus K_1) \rightarrow \widehat{CFD}(S^3 \setminus K_2),$$

which fits into the homotopy-commutative diagram for any choice of $\iota_{S^3 \setminus (K_1 \# - K_2)} \in \text{Inv}_D(S^3 \setminus (K_1 \# - K_2))$

$$\begin{array}{ccc} \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K_1) & \xrightarrow{\iota_{S^3 \setminus K_1}} & \widehat{CFD}(S^3 \setminus K_1) \\ \downarrow \text{id} \boxtimes g & & \downarrow g \\ \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K_2) & \xrightarrow{\iota_{S^3 \setminus K_2}} & \widehat{CFD}(S^3 \setminus K_2) \end{array}$$

Furthermore, the induced chain map

$$\begin{aligned} \widehat{CF}(S^3) &\simeq \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3 \setminus K_1) \\ &\xrightarrow{\text{id} \boxtimes g} \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3 \setminus K_2) \simeq \widehat{CF}(S^3) \end{aligned}$$

is a homotopy equivalence.

Now, let

$$N_1 = T_\infty \setminus P$$

be the 0-framed exterior of the given pattern P inside the ∞ -framed solid torus. Then, the union of N (glued along its 0-framed boundary) with T_∞ is again T_∞ . Hence, if we denote the type-D morphism

$$\begin{aligned} \widehat{CFD}(S^3 \setminus P(K_1)) &\simeq \widehat{CFDA}(N) \boxtimes \widehat{CFD}(S^3 \setminus K_1) \\ &\xrightarrow{\text{id} \boxtimes g} \widehat{CFDA}(N) \boxtimes \widehat{CFD}(S^3 \setminus K_2) \\ &\simeq \widehat{CFD}(S^3 \setminus P(K_2)) \end{aligned}$$

by g_0 , then the induced map

$$\begin{aligned} \widehat{CF}(S^3) &\simeq \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3 \setminus P(K_1 \# K_2 \# - K_2)) \\ &\xrightarrow{\text{id} \boxtimes g_0} \widehat{CFA}(T_\infty) \boxtimes \widehat{CFD}(S^3 \setminus P(K_2)) \\ &\simeq \widehat{CF}(S^3) \end{aligned}$$

is homotopic to identity. Furthermore, we have the following homotopy-commutative diagram:

$$\begin{array}{ccc}
\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus P(K_1)) & \xrightarrow{\text{id} \boxtimes g_0} & \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus P(K_2)) \\
\downarrow \simeq & & \downarrow \simeq \\
\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFD}(S^3 \setminus K_1) & \xrightarrow{\text{id} \boxtimes \text{id} \boxtimes g} & \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFD}(S^3 \setminus K_2) \\
\downarrow \simeq & & \downarrow \simeq \\
\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) & \xrightarrow{\text{id} \boxtimes \text{id} \boxtimes \text{id} \boxtimes \text{id} \boxtimes g} & \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(N) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \\
\boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K_1) & & \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K_2) \\
\downarrow \iota_N \boxtimes \iota_{S^3 \setminus K_1} & & \downarrow \iota_N \boxtimes \iota_{S^3 \setminus K_2} \\
\widehat{CFDA}(N) \boxtimes \widehat{CFD}(S^3 \setminus K_1) & \xrightarrow{\text{id} \boxtimes g} & \widehat{CFDA}(N) \boxtimes \widehat{CFD}(S^3 \setminus K_2) \\
\downarrow \simeq & & \downarrow \simeq \\
\widehat{CFD}(S^3 \setminus P(K_1)) & \xrightarrow{g_0} & \widehat{CFD}(S^3 \setminus P(K_2))
\end{array}$$

The compositions of vertical maps on both sides of the above diagram are $\iota_N, \iota_{S^3 \setminus K_1}$ and $\iota_N, \iota_{S^3 \setminus K_2}$, which are contained in

$$\text{Inv}_D(S^3 \setminus P(K_1)) \quad \text{and} \quad \text{Inv}_D(S^3 \setminus P(K_2)),$$

respectively. Also, since our assumption is symmetric on the choices of K_1 and K_2 , we can repeat our argument with K_1 and K_2 swapped. Hence, by Theorem 1.1, we deduce that

$$(CFK_{\mathcal{R}}(S^3, P(K_1)_{\#} - P(K_2)), \iota_{P(K_1)_{\#} - P(K_2)})$$

is ι_K -locally equivalent to the trivial complex. \blacksquare

5. An explicit formula for the hat-flavored truncation of ι_K

In this section, we give a proof of Theorem 1.3; we provide here its very rough sketch for better readability. We define the ∞ -framed version of \mathbb{X} , which we denote by \mathbb{X}_{∞} , and classify all possible type-D morphisms from $\widehat{CFD}(\mathbb{X}_{\infty})$ to $\widehat{CFD}(T_{\infty}, \nu)$ up to homotopy via gluing formula. It turns out that the space of homotopy classes of morphisms in the “correct” bigrading is small. Actually, it is so small so that the “canonical” map, which roughly corresponds to the trivial saddle cobordism of a knot with its split union with an unknot via gluing with a 0-framed knot complement, can be easily identified up to “orientation reversal”, from which Theorem 1.3 will follow.

Recall that we had the bordered Heegaard diagram \mathbb{X} ; write $\mathbb{X} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \{z_{\text{free}}\}, z, w)$. We can add one more free basepoint w_{free} very close to z_{free} to get a new diagram $\mathbb{Y} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \{z_{\text{free}}, w_{\text{free}}\}, z, w)$. As we modified \mathbb{X} by Heegaard moves to get a nice diagram \mathbb{X}_0 , we can do the same process to \mathbb{Y} to get a nice diagram \mathbb{Y} . By counting holomorphic disks on \mathbb{Y}_0 which does not algebraically intersect z_{free} and w_{free} and recording their algebraic intersection numbers with z and w by formal variables U and V , respectively, we can get a well-defined type-A module $CFA_{UV}(\mathbb{Y}_0)$. Note that, by construction, we have

$$CFA_{UV}(\mathbb{X}_0) \simeq CFA_{UV}(\mathbb{Y}_0).$$

Recall that the proof of the pairing theorem

$$\widehat{CF}(-M_1 \cup M_2) \simeq \text{Mor}(\widehat{CFD}(M_1), \widehat{CFD}(M_2))$$

relies on the observation that

$$-M_1 \cup \overline{\mathbf{AZ}} \simeq -M_1.$$

Denote by \mathbb{Y}_∞ the 4-pointed nice bordered diagram obtained by gluing \mathbb{Y}_0 with a cylinder whose boundaries have framings 0 and ∞ and by \mathbb{X}_∞ the diagram obtained by gluing \mathbb{X}_0 with the same cylinder. Note that the Heegaard diagram obtained by gluing $-\mathbb{X}_\infty$ with a bordered diagram representing (T_∞, ν) , where ν denotes the longitudinal knot inside the ∞ -framed solid torus T_∞ , represents the 2-component link

$$L_2 = \{p, q\} \times S^1 \quad \text{in } S^2 \times S^1$$

for two distinct points $p, q \in S^2$. Since \mathbb{Y}_∞ should also satisfy

$$\widehat{CFD}(-\mathbb{Y}_\infty \cup \overline{\mathbf{AZ}}) \simeq \widehat{CFD}(-\mathbb{Y}_\infty)$$

and the type-D and type-A modules associated to \mathbb{Y}_∞ and \mathbb{X}_∞ are homotopy equivalent, we see that

$$\begin{aligned} \widehat{CFL}(S^2 \times S^1, L_2) &\simeq \widehat{CFA}(-\mathbb{X}_\infty) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\simeq \widehat{CFA}(-\mathbb{Y}_\infty) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\simeq \text{Mor}(\widehat{CFD}(\mathbb{Y}_\infty), \widehat{CFD}(T_\infty, \nu)). \end{aligned}$$

Here, L_2 is endowed with an orientation so that its total homology class $[L_2] \in H_1(S^2 \times S^1; \mathbb{Z})$ vanishes. Hence, L_2 is nullhomologous, which tells us that its link Floer homology (at the unique spin structure of $S^2 \times S^1$) has well-defined \mathbb{Z} -valued Maslov and (collapsed) Alexander gradings; see also [28, Figure 5.1]. These gradings should be compatible with the natural gradings of $\widehat{CFD}(\mathbb{Y}_\infty)$ and $\widehat{CFD}(T_\infty, \nu)$; note that the grading on $\widehat{CFD}(\mathbb{Y}_\infty)$ can be defined as in equation (3.1).

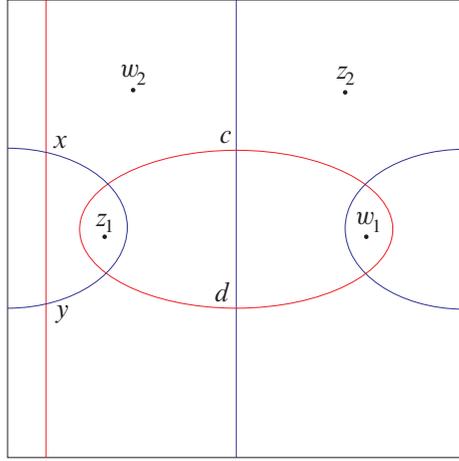


Figure 5.1. A 4-pointed Heegaard diagram representing the 2-component link L_2 .

Lemma 5.1. $\widehat{HFL}(S^2 \times S^1, L_2)$ is generated by four elements, which are supported on the unique spin structure of $S^2 \times S^1$ and lie on bidegrees $(0, 0)$, $(0, 0)$, $(1, 1)$, $(-1, -1)$, respectively, after a suitable grading shift. Here, bidegree is defined as the pair (Maslov grading, collapsed Alexander grading).

Proof. Write $L_2 = A \cup B$ and choose \mathbf{z} -basepoint z_1, z_2 and \mathbf{w} -basepoints w_1, w_2 on L_2 so that $z_1, w_1 \in A$ and $z_2, w_2 \in B$. We will compute the link Floer homology $CFL_{UV}(S^2 \times S^1, L_2)$ of the basepointed link $(L_2, \{z_1, z_2\}, \{w_1, w_2\})$, where the differential records the algebraic intersections of holomorphic disks with the basepoints z_1, w_1, z_2, w_2 by $U, V, 0, 0$, respectively. Note that truncating it by $U = V = 0$ and taking homology gives $\widehat{HFL}(S^2 \times S^1, L_2)$.

Consider the Heegaard diagram in Figure 5.1. Since we are counting disks which do not intersect z_2 and w_2 algebraically, the given diagram is nice, so all relevant holomorphic disks are represented by either bigons or squares which do not contain z_2 and w_2 . Thus, we see that $CFL_{UV}(S^2 \times S^1, L_2)$ is generated by the intersection points xc, xd, yc, yd , and the differential is given by

$$\begin{aligned}\partial(xd) &= Uxc + Vyd, \\ \partial(xc) &= V(xd + yc), \\ \partial(yd) &= U(xd + yc), \\ \partial(xd + yc) &= 0.\end{aligned}$$

Since U and V act on the bigrading by $(-2, -1)$ and $(0, 1)$ and the differential ∂ lowers the Maslov grading by 1 and leaves the collapsed Alexander grading invariant,

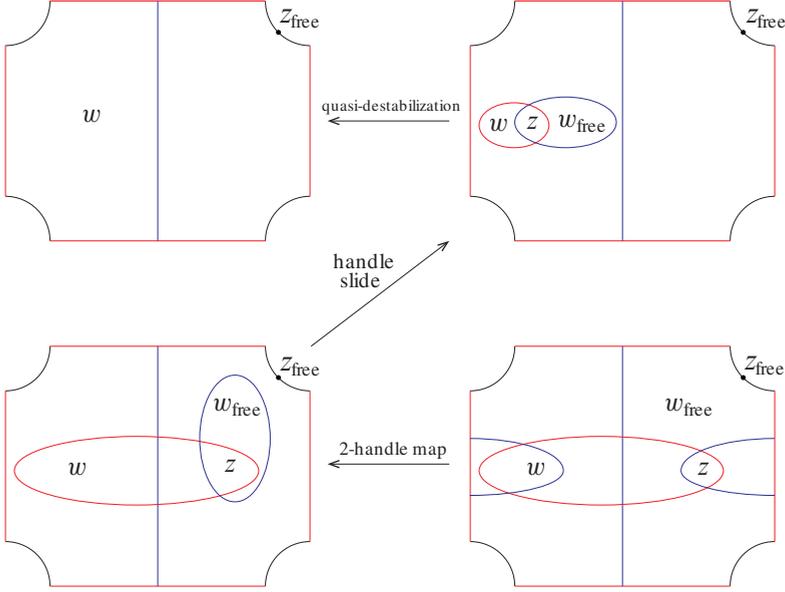


Figure 5.2. Upper left: The diagram H_0 . Upper right: The diagram H_0^{qst} . Lower left: A result of performing a handleslide to H_0^{qst} . Lower right: The diagram \mathbb{Y} .

we see that xd and $xd + yc$ have the same bidegree, say (m, n) , xc has bidegree $(m + 1, n + 1)$, and yd has bidegree $(m - 1, n - 1)$. Therefore, after truncating by $U = V = 0$ and shifting the bigrading by $(-m, -n)$, we get four generators $xd, xc, yd, xd + yc$ of $\widehat{HFL}(S^2 \times S^1, L_2)$, which lie on bidegrees $(0, 0), (0, 0), (1, 1), (-1, -1)$, respectively, as desired. ■

We define a type-D morphism

$$G_\infty : \widehat{CFD}(\mathbb{Y}_\infty) \rightarrow \widehat{CFD}(T_\infty, \nu)$$

as follows. We start with a Heegaard diagram \mathbb{Y}_0 . If we denote by $H_0 = (\Sigma, \alpha, \beta, z_{\text{free}}, w)$ the doubly-pointed Heegaard diagram for the pair (T_∞, ν) and the diagram we get by quasi-stabilizing it as H_0^{qst} , then we have a 2-handle map

$$\widehat{CFD}(\mathbb{Y}_0) \rightarrow \widehat{CFD}(H_0^{\text{qst}}).$$

Furthermore, the proof of [26, Proposition 5.3] tells us that we can define the “quasi-stabilization map”

$$\widehat{CFD}(H_0^{\text{qst}}) \rightarrow \widehat{CFD}(T_\infty, \nu).$$

We define G_∞ as the composition of the above two maps. For a graphical description of this definition, see Figure 5.2.

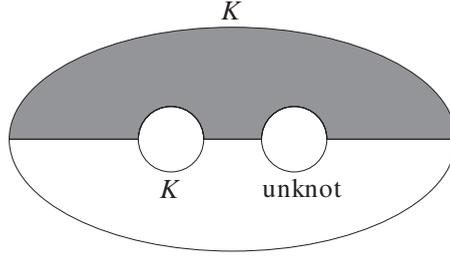


Figure 5.3. A decoration on the trivial saddle cobordism $S(K)$ from $K \cup \text{unknot}$ to K . Note that this cobordism can be seen as the composition of a quasi-stabilization followed by a saddle move.

Then, for any knot K , the induced map

$$\begin{aligned} \widehat{CFL}(S^3, K \cup \text{unknot}) &\simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ &\xrightarrow{\text{id} \boxtimes G_\infty} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\simeq \widehat{CFK}(S^3, K) \end{aligned}$$

is homotopic to the cobordism map $F_{S(K)}$ induced by the trivial saddle cobordism $S(K)$ from $K \cup \text{unknot}$ to K , as drawn in Figure 5.3. Furthermore, we can also define type-D endomorphisms

$$\Phi_{\mathbb{Y}}^D, \Psi_{\mathbb{Y}}^D : \widehat{CFD}(\mathbb{Y}_\infty) \rightarrow \widehat{CFD}(\mathbb{Y}_\infty)$$

using quasi-destabilization maps and (similarly defined) quasi-stabilization maps, as follows. Given a bordered diagram

$$H_Y = (\Sigma, \alpha, \beta, \{z_{\text{free}}, w_{\text{free}}\}, z, w)$$

representing \mathbb{Y}_∞ , we (α -)quasi-stabilize it near the point z to get a new diagram H_Y^{qst} , which introduces a new pair (z', w') of basepoints, and then we quasi-destabilize it to eliminate the basepoints z, w and rename z', w' as z, w , respectively, to obtain H_Y again. We define the resulting map as Ψ_Y , i.e.,

$$\begin{aligned} \Psi_Y : \widehat{CFD}(\mathbb{Y}_\infty) &= \widehat{CFD}(H_Y) \\ &\xrightarrow{\text{quasi-stabilization}} \widehat{CFD}(H_Y^{\text{qst}}) \\ &\xrightarrow{\text{quasi-destabilization}} \widehat{CFD}(H_Y) = \widehat{CFA}(\mathbb{Y}_0). \end{aligned}$$

We omit the construction of $\Phi_{\mathbb{Y}}^D$ since it is similar to the construction of $\Psi_{\mathbb{Y}}^D$. The definitions of $\Phi_{\mathbb{Y}}^D$ and $\Psi_{\mathbb{Y}}^D$ are not natural, i.e., depend on the choices of auxiliary

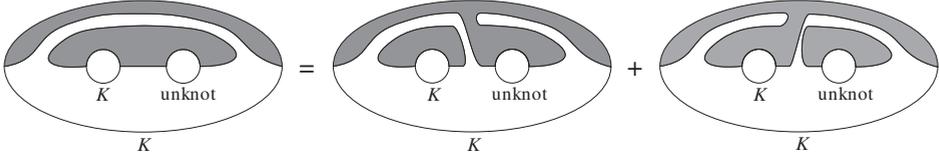


Figure 5.4. A bypass relation applied to the saddle cobordism $S(K)$ from $K \cup \text{unknot}$ to K , with a decoration as shown in Figure 5.3.

data. However, by the pairing theorem for triangles, we know that the map

$$\begin{aligned} \widehat{CFL}(S^3, K \cup \text{unknot}) &\simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ &\xrightarrow{\text{id} \boxtimes \Phi_Y} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ &\simeq \widehat{CFL}(S^3, K \cup \text{unknot}) \end{aligned}$$

is homotopic to the basepoint action $\Phi_{K \cup \text{unknot}, K}$ corresponding to the basepoint z on the link $K \cup \text{unknot}$, for any knot K . A similar statement also holds for $\Psi_{\mathbb{Y}}^D$ as well.

Lemma 5.2. *For any knot K , we have*

$$F_{S(K)} \circ \Phi_{K \cup \text{unknot}, K} \sim \Phi_K \circ F_{S(K)} \quad \text{and} \quad F_{S(K)} \circ \Psi_{K \cup \text{unknot}, K} \sim \Psi_K \circ F_{S(K)}.$$

Proof. Bypass relation [29, Figure 1.3], applied as shown in Figure 5.4, gives the equality

$$\Psi_K \circ F_{S(K)} \sim F_{S(K)} \circ \Psi_{K \cup \text{unknot}, K} + F_{S(K)} \circ \Psi_{K \cup \text{unknot}, \text{unknot}},$$

where $\Psi_{K \cup \text{unknot}, \text{unknot}}$ denotes the basepoint action associated to $w_{\text{free}} \in \text{unknot}$. Since the basepoint actions for the unknot are trivial, the lemma follows. The same argument also proves the commutation result for Φ actions. \blacksquare

Lemma 5.3. *The type-D morphisms G_∞ , $G_\infty \circ \Phi_{\mathbb{Y}}^D$, $G_\infty \circ \Psi_{\mathbb{Y}}^D$, and $G_\infty \circ (1 + \Phi_{\mathbb{Y}}^D \Psi_{\mathbb{Y}}^D)$ form a basis of*

$$H_*(\text{Mor}(\widehat{CFD}(\mathbb{Y}_\infty), \widehat{CFD}(T_\infty, \nu))).$$

Furthermore, they lie on bidegrees $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(0, 0)$, respectively, under the shifting bigrading used in Lemma 5.1.

Proof. By Lemma 5.1, in order to show that the given elements form a basis, we only have to show that the homotopy classes of the given type-D morphisms are linearly independent, so assume that they are linearly dependent. Then, for any knot K , the endomorphisms $F_{S(K)}$, $F_{S(K)} \circ \Phi_{K \cup \text{unknot}, K}$, $F_{S(K)} \circ \Psi_{K \cup \text{unknot}, K}$, and $F_{S(K)} \circ (1 + \Phi_{K \cup \text{unknot}, K} \Psi_{K \cup \text{unknot}, K})$ should be linearly dependent up to homotopy. By Lemma 5.2

and the fact that $F_{S(K)}$ has a homotopy right inverse (which follows from the fact that the trivial saddle cobordism from $K \cup \text{unknot}$ to K has a right inverse) would imply that the endomorphisms

$$\text{id}, \quad \Phi_K, \quad \Psi_K, \quad 1 + \Phi_K \Psi_K$$

of $\widehat{CFK}(S^3, K)$ should also be linearly dependent up to homotopy.

Now, consider the case when K is the figure-eight knot. Then, $\widehat{CFK}(S^3, K)$ is generated by five elements, say, a, b, c, d, x . The basepoint actions are given by

$$\Phi_K(a) = b, \quad \Phi_K(c) = d, \quad \Psi_K(a) = c, \quad \Psi_K(b) = d,$$

and all other generators are mapped to zero. Thus, we see that the endomorphisms $\text{id}, \Phi_K, \Psi_K, 1 + \Phi_K \Psi_K$ are linearly independent up to homotopy, a contradiction. Furthermore, since the basepoint actions Φ_K and Ψ_K are maps of bidegree $(1, 1)$ and $(-1, -1)$, respectively, it is clear that, with respect to the element id , the elements Φ_Y and Ψ_Y have relative bidegrees $(1, 1)$ and $(-1, -1)$, respectively. Hence, the statement about bigrading also follows from Lemma 5.1. \blacksquare

Recall that mimicking the construction of ι_X gives a bordered involution

$$\iota_Y : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \rightarrow \widehat{CFD}(\mathbb{Y}_\infty),$$

which is a homotopy equivalence which satisfies the property that the induced map

$$\begin{aligned} \widehat{CFL}(S^3, K \cup \text{unknot}) &\simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ &\simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ &\xrightarrow{\iota_{S^3 \setminus K} \boxtimes \iota_Y} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ &\simeq \widehat{CFL}(S^3, K \cup \text{unknot}) \end{aligned}$$

is homotopic to the involution $\iota_{K \cup \text{unknot}}$ of the link Floer homology of $K \cup \text{unknot}$.

On the other hand, the type-D module $\widehat{CFD}(T_\infty, \nu)$ is generated by a single element, say, x , and the differential is trivial. This implies that

$$\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu)$$

is not homotopy equivalent to $\widehat{CFD}(T_\infty, \nu)$. In fact, $\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu)$ is homotopy equivalent to a type-D module generated by five elements, say a, b, c, d, e , where the differential is given by

$$\partial a = 0, \quad \partial b = \rho_1 a + \rho_3 c, \quad \partial c = \rho_2 d, \quad \partial d = \rho_1 e, \quad \partial e = 0. \quad (5.1)$$

Since a is a cycle, the map

$$f : \widehat{CFD}(T_\infty, \nu) \rightarrow \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(T_\infty, \nu)$$

defined by $f(x) = a$ commutes with the differential on both sides and thus is a well-defined type-D morphism.

Lemma 5.4. *The type-D morphism $f \circ G_\infty \circ \iota_{\mathbb{Y}}$ is homotopic to either $\text{id} \boxtimes G_\infty$ or $\text{id} \boxtimes (G_\infty \circ (1 + \Phi_{\mathbb{Y}}^D \Psi_{\mathbb{Y}}^D))$.*

Proof. For simplicity, write

$$g = f \circ G_\infty \circ \iota_{\mathbb{Y}}.$$

Then, for any knot K , we have an induced map

$$\begin{aligned} & \widehat{CFL}(S^3, K \cup \text{unknot}) \\ & \simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(\mathbb{Y}_\infty) \\ & \xrightarrow{\text{id} \boxtimes \text{id} \boxtimes g} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty) \\ & \simeq \widehat{CFL}(S^3, K), \end{aligned}$$

which we will denote as g_K . Then, by construction, we have

$$g_K \sim \tilde{f} \circ F_{S(K)} \circ \iota_{K \cup \text{unknot}},$$

where \tilde{f} is the map defined as

$$\begin{aligned} & \widehat{CFK}(S^3, K) \\ & \simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ & \xrightarrow{\iota_{S^3 \setminus K}^{-1} \boxtimes f} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ & \xrightarrow{\text{id} \boxtimes \Omega \boxtimes \text{id}} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(T_\infty, \nu) \simeq \widehat{CFK}(S^3, K), \end{aligned}$$

where Ω is the homotopy equivalence

$$\widehat{CFDA}(\overline{\mathbf{AZ}}) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \simeq \widehat{CFDA}(\mathbb{I}),$$

which is unique up to homotopy due to homotopy rigidity [8, Lemma 4.4]. It is easy to check, using a bypass relation, that

$$F_{S(K)} \circ \iota_{K \cup \text{unknot}} \sim \iota_K \circ F_{S(K)}.$$

Hence, we get

$$g_K \sim \tilde{f} \circ \iota_K \circ F_{S(K)}.$$

We now consider the case when K is the unknot. Then, the 0-framed knot complement $S^3 \setminus K$ is the 0-framed solid torus T_0 . Recall that $\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu)$ is homotopic to the type-D module M_D generated by a, b, c, d, e , where the differential is given as in equation (5.1), and the image of the generator x of $\widehat{CFD}(T_\infty, \nu)$ is a . This means that there exists a type-D homotopy equivalence

$$h_D : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu) \rightarrow M_D$$

such that

$$(h_D \circ f)(x) = a.$$

On the other hand, the type-A module $\widehat{CFA}(S^3 \setminus K)$, which is homotopy equivalent to $\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFDA}(\mathbf{AZ})$ via $\iota_{S^3 \setminus K}$, is generated by one element, say, y , and the A_∞ operations are given by

$$m_{3+i}(y, \rho_2, \overbrace{\rho_{12}, \dots, \rho_{12}}^i, \rho_1) = y \quad \text{for each } i \geq 0.$$

Hence, the chain map

$$\begin{aligned} m : \widehat{CFK}(S^3, K) &\simeq \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\xrightarrow{\text{id} \boxtimes f} \widehat{CFA}(S^3 \setminus K) \boxtimes \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu) \\ &\xrightarrow{\text{id} \boxtimes h_D} \widehat{CFA}(S^3 \setminus K) \boxtimes M_D \end{aligned}$$

maps the generator 1 of $\widehat{CFK}(S^3, K) \simeq \mathbb{F}_2$ to $y \boxtimes a$. Furthermore, the chain complex

$$\widehat{CFA}(S^3 \setminus K) \boxtimes M_D$$

is generated by three elements, namely, $y \boxtimes a$, $y \boxtimes c$, and $y \boxtimes d$ and the differential is given by

$$\partial(y \boxtimes c) = m_3(y, \rho_2, \rho_1) \boxtimes e = y \boxtimes e.$$

Hence, there exists a homotopy equivalence

$$h_C : \widehat{CFA}(S^3 \setminus K) \boxtimes M_D \rightarrow \widehat{CFK}(S^3, K)$$

such that $(h_C \circ m)(1) = 1$. However, since $\iota_{S^3 \setminus K}$ is a homotopy equivalence and any homotopy autoequivalence of $\widehat{CFK}(S^3, K) \simeq \mathbb{F}_2$ is homotopic to the identity, we should have

$$\begin{aligned} \tilde{f}(1) &= ((\text{id} \boxtimes \Omega \boxtimes \text{id}) \circ (\iota_{S^3 \setminus K}^{-1} \boxtimes \text{id}) \circ (\text{id} \boxtimes h_D^{-1}) \circ h_C^{-1})(h_C \circ m)(1) \\ &= (h_C \circ m)(1) = 1. \end{aligned}$$

Therefore, \tilde{f} is homotopic to the identity map. Since it is obvious that ι_{unknot} is also homotopic to the identity map, we get

$$g_{\text{unknot}} \sim F_{\text{unknot}},$$

which implies that g itself should not be nullhomotopic. Since box-tensoring with $\widehat{CFDA}(\mathbf{AZ})$ is an equivalence of categories and g clearly has bidegree $(0, 0)$ in the sense of Lemma 5.1 (otherwise, it follows from the arguments in the proof of Lemma 5.3 that g_{unknot} is nullhomotopic, a contradiction), we can apply Lemma 5.3 to see that g should be chain homotopic to one of the following three morphisms:

$$\text{id} \boxtimes G_\infty, \quad \text{id} \boxtimes (G_\infty \circ \Phi_Y^D \Psi_Y^D), \quad \text{id} \boxtimes (G_\infty \circ (1 + \Phi_Y^D \Psi_Y^D)).$$

Suppose that g is homotopic to $\text{id} \boxtimes (G_\infty \circ \Phi_Y^D \Psi_Y^D)$. Then, we should have

$$g_K \sim F_{S(K)} \circ \Phi_{K \cup \text{unknot}, K} \Psi_{K \cup \text{unknot}, K} \sim \Phi_K \Psi_K \circ F_{S(K)}.$$

We have already seen that g_{unknot} is not nullhomotopic, which is a contradiction since Φ_{unknot} and Ψ_{unknot} are both nullhomotopic. Therefore, g_K is homotopic to either $\text{id} \boxtimes G_\infty$ or $\text{id} \boxtimes (G_\infty \circ (1 + \Phi_Y^D \Psi_Y^D))$, as desired. ■

Now, we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Denote the homotopy autoequivalence of $\widehat{CFK}(S^3, K)$ defined in the theorem as $\tilde{\iota}_K$. By Lemma 5.4, we know that $f \circ G_\infty$ is homotopic to either $(\text{id} \boxtimes G_\infty) \circ \iota_Y^{-1}$ or $(\text{id} \boxtimes G_\infty) \circ \iota_Y^{-1} \circ (1 + \Phi_Y^D \Psi_Y^D)$, so we should have either

$$\tilde{\iota}_K \circ F_{S(K)} \sim F_{S(K)} \circ \iota_{K \cup \text{unknot}}^{-1} \sim \iota_K^{-1} \circ F_{S(K)}$$

or

$$\begin{aligned} \tilde{\iota}_K \circ F_{S(K)} &\sim F_{S(K)} \circ \iota_{K \cup \text{unknot}}^{-1} \circ (1 + \Phi_{K \cup \text{unknot}, K} \Psi_{K \cup \text{unknot}, K}) \\ &\sim \iota_K^{-1} \circ (1 + \Phi_K \Psi_K) \circ F_{S(K)} \\ &\sim \iota_K^{-1} \circ \iota_K^2 \circ F_{S(K)} \\ &\sim \iota_K \circ F_{S(K)}. \end{aligned}$$

Since the trivial saddle cobordism $S(K)$ from $K \cup \text{unknot}$ to K clearly has a right inverse, its associated cobordism map $F_{S(K)}$ admits a homotopy right inverse. Hence, by precomposing with the homotopy right inverse of $F_{S(K)}$, we see that $\tilde{\iota}_K$ should be homotopic to either ι_K or ι_K^{-1} , as desired. ■

Remark 5.5. The proof of the pairing theorem (equation (2.2)) also works in the following way:

$$\text{Mor}(\widehat{CFD}(T_\infty, \nu), \widehat{CFD}(S^3 \setminus K)) \simeq \widehat{CFK}(S^3, K).$$

The reason is that, although $\widehat{CFD}(T_\infty, \nu)$ is not homotopy equivalent to

$$\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu),$$

$\widehat{CFK}(S^3, K)$ is homotopy equivalent to $\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFK}(S^3, K)$. Hence, given an involution $\iota_M \in \text{Inv}_D(S^3 \setminus K)$, one can consider the following map:

$$\begin{aligned} & \widehat{CFK}(S^3, K) \\ & \simeq \text{Mor}(\widehat{CFD}(T_\infty, \nu), \widehat{CFD}(S^3 \setminus K)) \\ & \xrightarrow{\text{id} \boxtimes} \text{Mor}(\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu), \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K)) \\ & \xrightarrow{h \mapsto \iota_{S^3 \setminus K} \circ h \circ f} \text{Mor}(\widehat{CFD}(T_\infty, \nu), \widehat{CFD}(S^3 \setminus K)) \simeq \widehat{CFK}(S^3, K). \end{aligned}$$

Here, f is the type-D morphism given in Theorem 1.3. Following the proof of Theorem 1.3, it is straightforward to see that the above map is homotopic to either ι_K or ι_K^{-1} . This gives a more applicable interpretation of Theorem 1.3 since type-D modules are easier to work with than type-A modules.

Example 5.6. Let K be the left-handed trefoil. The knot Floer chain complex $CFK_{UV}(S^3, K)$ is generated by three elements a, b, c , which lie on bidegrees $(0, 1)$, $(1, 2)$, $(-1, 0)$, respectively, and the differential is given as follows:

$$\begin{array}{ccc} & & a \\ & \xleftarrow{U} & \\ b & & \\ & & \downarrow V \\ & & c \end{array}$$

It is known [9, Section 8] that the action of ι_K is given by the reflection along the diagonal, i.e., fixes a and exchanges b and c .

On the bordered side, we know from [17, Theorem 11.26] that the Floer chain complex of K determines $\widehat{CFD}(S^3 \setminus K)$. Thus, we see that $\widehat{CFD}(S^3 \setminus K)$ is generated by 7 elements $e_0, f_0, f_1, g_0, g_1, h_1, k_1$, where the differential is given as follows:

$$\begin{array}{ccccc} f_0 & \xleftarrow{\rho_2} & f_1 & \xleftarrow{\rho_3} & e_0 \\ \rho_1 \downarrow & & & & \downarrow \rho_1 \\ k_1 & & & & g_1 \\ & & \swarrow \rho_{23} & & \uparrow \rho_{123} \\ & & h_1 & \xleftarrow{\rho_3} & g_0 \end{array}$$

It can be seen via straightforward computation that there are only two homotopy classes of degree-preserving type-D endomorphisms of $\widehat{CFD}(S^3 \setminus K)$, represented by 0 and id. Hence, $\widehat{CFD}(S^3 \setminus K)$ is *homotopy-rigid*; i.e., it admits a unique homotopy class of homotopy autoequivalences. This means that there exists only one homotopy class of homotopy equivalences

$$\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K) \rightarrow \widehat{CFD}(S^3 \setminus K).$$

Since one of such homotopy equivalences can be computed explicitly using the proof of [3, Theorem 37], we deduce that it also gives an explicit description of $\iota_{S^3 \setminus K}$. Applying Theorem 1.3 then recovers the hat-flavored action

$$\iota_K(a) = a, \quad \iota_K(b) = c, \quad \iota_K(c) = b$$

in $\widehat{CFK}(S^3, K)$, which is consistent with the action of ι_K on $CFK_{UV}(S^3, K)$.

Remark 5.7. In general, one can prove that $\widehat{CFD}(S^3 \setminus K)$ is homotopy-rigid whenever K is an L-space knot, which means that one can explicitly compute $\iota_{S^3 \setminus K}$ for such knots by computing the box tensor product $\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K)$ and finding a sequence of homotopy equivalences which connects it to $\widehat{CFD}(S^3 \setminus K)$. One can check using Theorem 1.3 that the hat-flavored action of ι_K is given by “reflection with respect to the diagonal”. This is consistent with the action of ι_K on $CFK_{UV}(S^3, K)$, which was first determined in [9, Section 7].

Theorem 1.3 can also be used in the reverse way to compute $\iota_{S^3 \setminus K}$ from ι_K , as shown in Example 5.8.

Example 5.8. Let K be the figure-eight knot. The knot Floer chain complex $CFK_{UV}(S^3, K)$ is generated by five elements a, b, c, d, x , which lie on bidegrees $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(0, 0)$, $(0, 0)$, respectively, and the differential is given as follows:

$$\begin{array}{ccc} b & \xleftarrow{U} & a \\ \downarrow v & & \downarrow v \\ d & \xleftarrow{U} & c \end{array} \oplus x$$

Furthermore, the involution ι_K is given by

$$\begin{aligned} \iota_K(a) &= a + x, & \iota_K(b) &= c, & \iota_K(c) &= b, \\ \iota_K(d) &= d, & \iota_K(x) &= x + d. \end{aligned}$$

On the other hand, $\widehat{CFD}(S^3 \setminus K)$ is generated by 9 elements $e_0, f_0, g_0, h_0, e_1, f_1, g_1, h_1, z$, where the differential is given as follows:

$$\begin{array}{ccc}
 f_0 & \xleftarrow{\rho_2} & e_1 \xleftarrow{\rho_3} e_0 \\
 \rho_1 \downarrow & & \downarrow \rho_1 \\
 f_1 & & h_1 \\
 \rho_{123} \uparrow & & \uparrow \rho_{123} \\
 g_0 & \xleftarrow{\rho_2} & g_1 \xleftarrow{\rho_3} h_0
 \end{array}
 \oplus
 \begin{array}{c}
 z \xrightarrow{\rho_{12}} z
 \end{array}$$

Unlike the trefoil case (covered in Example 5.6), the type-D module $\widehat{CFD}(S^3 \setminus K)$ is not homotopy-rigid, so we cannot find a random homotopy equivalence between $\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K)$ and $\widehat{CFD}(S^3 \setminus K)$ and claim that it is homotopic to $\iota_{S^3 \setminus K}$. Denote by M and N the type-D submodule of $\widehat{CFD}(S^3 \setminus K)$ generated by z and everything else (i.e., e_0, \dots, h_1), respectively, so that we have a splitting

$$\widehat{CFD}(S^3 \setminus K) \simeq M \oplus N.$$

Using the proof of [3, Theorem 37], one can explicitly construct homotopy equivalences

$$\begin{aligned}
 F_M &: \widehat{CFDA}(\mathbf{AZ}) \boxtimes M \rightarrow M, \\
 F_N &: \widehat{CFDA}(\mathbf{AZ}) \boxtimes N \rightarrow N.
 \end{aligned}$$

Consider $F = F_M \oplus F_N : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K) \rightarrow \widehat{CFD}(S^3 \setminus K)$. Then, $F \circ \iota_{S^3 \setminus K}^{-1}$ is a homotopy autoequivalence of $\widehat{CFD}(S^3 \setminus K)$. Recall that we have a pairing theorem

$$\text{Mor}(\widehat{CFD}(S^3 \setminus K), \widehat{CFD}(S^3 \setminus K)) \simeq \widehat{CF}(-(S^3 \setminus K) \cup (S^3 \setminus K)) \simeq \widehat{CF}(S_0^3(K\sharp - K)).$$

Since F is a homotopy equivalence, it should correspond to a nontrivial element with absolute \mathbb{Q} -grading $\frac{1}{2}$ in $\widehat{HF}(S_0^3(K\sharp - K), \mathfrak{s}_0)$, where \mathfrak{s}_0 denotes the unique spin structure on $S_0^3(K\sharp - K)$. The integral surgery formula for knots [24, Theorem 1.1] tells us that the $\frac{1}{2}$ -graded piece $V_{\frac{1}{2}}$ of $\widehat{HF}(S_0^3(K\sharp - K), \mathfrak{s}_0)$ is 5-dimensional.

Now, we construct an explicit basis of $V_{\frac{1}{2}}$ in terms of type-D endomorphisms of $\widehat{CFD}(S^3 \setminus K)$. Consider the type-D endomorphisms K_1, K_2, K_3 of $\widehat{CFD}(S^3 \setminus K)$, defined as

$$\begin{aligned}
 K_1(e_0) &= z, K_1(h_1) = \rho_2 z, K_1(\text{everything else}) = 0, \\
 K_2(z) &= g_0 + \rho_3 f_1, K_2(\text{everything else}) = 0, \\
 K_3(z) &= z, K_3(\text{everything else}) = 0.
 \end{aligned}$$

We claim that the type-D morphisms id , K_1 , K_2 , $K_2 \circ K_1$, and K_3 are linearly independent up to homotopy and thus form a basis of $V_{\frac{1}{2}}$. To prove the claim, we take a tensor product with $\widehat{CFA}(T_\infty, \nu)$, and consider the maps $\text{id} \boxtimes K_1$ and $\text{id} \boxtimes K_2$, which are now considered as chain endomorphisms of $\widehat{CFK}(S^3, K)$. One can easily see that

$$\begin{aligned} (\text{id} \boxtimes K_1)(a) &= x, & (\text{id} \boxtimes K_1)(\text{everything else}) &= 0, \\ (\text{id} \boxtimes K_2)(x) &= d, & (\text{id} \boxtimes K_2)(\text{everything else}) &= 0, \\ (\text{id} \boxtimes K_3)(x) &= x, & (\text{id} \boxtimes K_3)(\text{everything else}) &= 0. \end{aligned}$$

Hence, we see that $\text{id} \boxtimes g$ for $g = \text{id}, K_1, K_2, K_2 \circ K_1, K_3$ induce linearly independent endomorphisms of $\widehat{HFK}(S^3, K)$, and so, the claim is proven.

Given a type-D morphism

$$m : \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K) \rightarrow \widehat{CFD}(S^3 \setminus K),$$

we define an endomorphism E_m of $\widehat{CFK}(S^3, K)$ as follows:

$$\begin{aligned} E_m &: \widehat{CFK}(S^3, K) \\ &\simeq \text{Mor}(\widehat{CFD}(T_\infty, \nu), \widehat{CFD}(S^3 \setminus K)) \\ &\xrightarrow{\text{id} \boxtimes} \text{Mor}(\widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(T_\infty, \nu), \widehat{CFDA}(\mathbf{AZ}) \boxtimes \widehat{CFD}(S^3 \setminus K)) \\ &\xrightarrow{h \mapsto m \circ h \circ f} \text{Mor}(\widehat{CFD}(T_\infty, \nu), \widehat{CFD}(S^3 \setminus K)) \simeq \widehat{CFK}(S^3, K). \end{aligned}$$

Here, f denotes the type-D morphism appearing in Theorem 1.3. Then, a manual computation tells us that, for the homotopy equivalence F described above, the endomorphism E_F acts on $\widehat{CFK}(S^3, K)$ by

$$a \mapsto a, \quad b \mapsto c, \quad c \mapsto b, \quad d \mapsto d, \quad x \mapsto x.$$

Comparing this with ι_K , we see that $\text{id} \boxtimes (F \circ \iota_{S^3 \setminus K}^{-1})$ acts on $\widehat{CFK}(S^3, K)$ by

$$a \mapsto a + x, \quad x \mapsto x + d, \quad b \mapsto b, \quad c \mapsto c, \quad d \mapsto d.$$

Since $F \circ \iota_{S^3 \setminus K}^{-1}$ is an element of $V_{\frac{1}{2}}$, which is generated by id , K_1 , K_2 , $K_2 \circ K_1$, and K_3 , we deduce that

$$\iota_{S^3 \setminus K} \sim (\text{id} + K_1 + K_2) \circ F.$$

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