J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1462

Published by EMS Press

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Non-classical solutions of the p-Laplace equation

Received February 25, 2022

Abstract. In this paper we settle Iwaniec and Sbordone's 1994 conjecture concerning very weak solutions to the p-Laplace equation. Namely, on the one hand we show that distributional solutions of the p-Laplace equation in $W^{1,r}$ for $p \neq 2$ and $r > \max\{1, p-1\}$ are classical weak solutions if their weak derivatives belong to certain cones. On the other hand, we construct via convex integration non-energetic distributional solutions if this cone condition is not met, thus disproving Iwaniec and Sbordone's conjecture in general.

Keywords: very weak solutions, p-Laplace, convex integration.

1. Introduction

The p-Laplace equation

$$\Delta_p u := \operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{in } \Omega, \tag{1.1}$$

which formally corresponds to the Euler–Lagrange equation of the energy

$$\int_\Omega |Du|^p \, dx,$$

is one of the most well studied problems in the calculus of variations. The classical regularity theory has been developed in a series of papers of N. N. Ural'tseva [43], K. Uhlenbeck [42], and L. C. Evans [14] for $p \geq 2$, and of J. L. Lewis [29] and P. Tolksdorf [41] for $p > 1$ (see also [10, 11]). To mention some of the milestones obtained for p-laplacian-type problems, without pretending to be exhaustive, we may cite the counterexamples to regularity of vectorial problems, the Harnack inequality, the partial regularity theory for vectorial problems, the estimates of the singular set, and Calderón–Zygmund theory (see [8, 12, 13, 28, 31–33, 38, 39] and the references cited therein). The results for the p-Laplace

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Mathematics Subject Classification 2020: 35D30 (primary); 35J15, 35J60, 35J70 (secondary).

equation have become a paradigm to attack several more complex problems, including the associated parabolic equation, fractional versions of the same equation, functionals with variable exponent $p(x)$, double phase functionals, free boundary problems involving p-energies, to mention but a few.

Given a bounded open set $\Omega \subseteq \mathbb{R}^n$, every distributional solution of the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega$$

is known by Weyl's Lemma to be a classical, and therefore also analytic, solution. For the p-Laplace equation (1.1) an analogous result was conjectured by T. Iwaniec and C. Sbordone [26] (see also [31, Section 9]). Indeed, they conjectured that distributional solutions of the p-Laplace equation in $W^{1,r}$ for $p \neq 2$ and $r > \max\{1, p-1\}$ (notice that for such solutions one can give the natural distributional meaning to the equation) are of finite energy and hence belong, in the interior of the domain, to $C^{1,\alpha}$ for some $\alpha > 0$, according to the classical regularity theory . This conjecture has been proven true when r is sufficiently close to p, namely for $p - \delta < r < p$ for some δ which depends only on n and p. This result was first obtained in [26] via a quantitative version of the Hodge decomposition theorem, previously introduced by Iwaniec [23] in the context of quasiregular mappings (see also [24, 25]). A different approach was then followed by Lewis [30], based on a quantitative version of the Lipschitz truncation.

Our first main result, which is the content of Section 2, gives a positive answer to the conjecture of [26], under the additional condition (1.3).

Theorem 1.1. *Let $p > 1$, Ω be an open and bounded subset of $\mathbb{R}^n$ and $f \in L^1(\Omega)$. Suppose $u \in W^{1,\max\{1,p-1\}}(\Omega)$ solves*

$$\operatorname{div}(|Du|^{p-2}Du) = f \tag{1.2}$$

in the weak sense on Ω and, for all $1 \leq i \leq n$, there exist constants $\sigma_i \in \{1, -1\}$, $L_i \in \mathbb{R}$ such that

$$\sigma_i \partial_i u \geq L_i \quad \text{a.e. in } \Omega. \tag{1.3}$$

Then $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ and for all open Ω' compactly contained in Ω there exists a constant $C > 0$ depending on n, R, and $\mathrm{d}(\Omega', \partial\Omega)$, where $R > 0$ is such that $\Omega \subset B_R(0)$, for which

$$\int_{\Omega'} |Du|^p(x)\,dx \leq C(\|f\|_{L^1(\Omega)} + \|Du\|^{p-1}_{L^{p-1}(\Omega)})(\|u\|_{L^1(\Omega)} + |(L_1, \dots, L_n)|). \tag{1.4}$$

Unexpectedly, Theorem 1.1 shows a connection, at least when $n = 2$, with the field of elliptic estimates *by compensation*. Roughly speaking, these are estimates on vector fields $v \in L^1(\mathbb{R}^m, \mathbb{R}^n)$ satisfying some underdetermined system of linear PDEs, $\mathcal{A}(v) = 0$ in the sense of distributions, which is compensated by the fact that v satisfies nonlinear inclusion constraints, $v(x) \in K \subset \mathbb{R}^n$ (see [19]). This classical line of research has led to striking achievements in the last few years; see [2, 9] and references therein.

Iwaniec and Sbordone's conjecture has a similar flavor to the well understood problem for elliptic equations

$$\operatorname{div}(A(x)Du(x)) = 0 \quad \text{in } \Omega \tag{1.5}$$

with $\lambda I \leq A \leq \Lambda I$. For every $q \in (1, 2)$ and $n \geq 2$, J. Serrin [36] provided a striking example of an equation of the form (1.5) which has an *unbounded* solution $u \in W^{1,q}$. Hence, such a solution cannot belong to $W^{1,2}(\Omega)$ in view of the results of E. De Giorgi and J. Nash. However, in this context subsequent results [1, 4, 20, 27] showed that the situation is completely different as long as suitable continuity of A is assumed: in this case, every $W^{1,1}$ solution is necessarily of finite energy, namely $W^{1,2}(\Omega)$, and the regularity theory applies. These results do not provide any clear intuition on the problem we are considering for the p-laplacian, for two main reasons: first, because the freedom in the choice of A is missing in our context, and secondly because it is not clear how to interpret the positive results about the continuous coefficients.

Our second main result, which occupies the rest of the article, shows the sharpness of assumption (1.3) (see (1.6)), and provides a negative answer to the above-stated conjecture in its full generality.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a ball. For every $p \in (1, \infty)$, $p \neq 2$, there exists $\varepsilon = \varepsilon(p) > 0$ and a continuous $u \in W^{1,p-1+\varepsilon}(\Omega)$ such that u is affine on $\partial\Omega$,*

$$\frac{3}{4} \leq \partial_y u \leq \frac{5}{4} \quad \text{a.e. on } \Omega, \tag{1.6}$$

$$\operatorname{div}(|Du|^{p-2} Du) = 0 \tag{1.7}$$

in the sense of distributions, but for all open $B \subset \Omega$,

$$\int_B |Du|^p \, dx = \infty. \tag{1.8}$$

For fixed $\alpha \in (0, 1)$, one can even construct $u \in C^\alpha(\overline{\Omega})$; see [3, Lemma 2.1] for details. In order not to add further technical details to the construction, we will content ourselves with proving Theorem 1.2. The method we employ to show Theorem 1.2 is convex integration. Specifically, we are going to use the *staircase laminate* construction, which, to the best of our knowledge, was introduced by D. Faraco [15]. Since then, this type of construction has been used in several contexts [3, 5, 6, 16–18], and is tailored to tackle problems in which *concentration* phenomena appear. Similar techniques were developed to find counterexamples in which *oscillation* phenomena are the issue. Namely, while the staircase laminate construction deals, roughly speaking, with finding maps which are in some $W^{1,p}$ space but no better, similar methods can be used, for instance, to find maps which are Lipschitz and not C^1 on any open set; see for instance [7, 22, 35, 37, 40]. Instead of outlining our proof here, we defer the discussion to Section 3, where the structure of the part of the paper devoted to the construction of the counterexample will be explained in detail.

2. When are very weak solutions classical solutions?

This section is devoted to the proof of Theorem 1.1, which confirms Iwaniec and Sbordone's conjecture under the additional condition that each component of Du has a one-sided, uniform bound (1.3).

Proof of Theorem 1.1. Fix compactly contained open sets

$$\Omega' \subset \Omega'' \subset \Omega''' \subset \Omega$$

with

$$\mathrm{d}(\Omega', \partial\Omega) \le 2\,\mathrm{d}(\Omega''', \partial\Omega). \tag{2.1}$$

Consider a radial and positive smooth mollification kernel ρ_ε with support in $B_\varepsilon(0)$ for each $\varepsilon > 0$, and define the convolution of u with ρ_ε,

$$u_\varepsilon(x) := (u \star \rho_\varepsilon)(x) \quad \text{for every } x \in \Omega'''.$$

This is well defined as long as $\varepsilon < \mathrm{d}(\Omega''', \partial\Omega)$. Observe that (1.3) still holds for u_ε.

Let $\psi \in C_c^\infty(\Omega'')$ be such that $\psi(x) \in [0, 1]$ for $x \in \Omega$, $\psi \equiv 1$ on Ω' and

$$\|D\psi\|_{L^\infty(\Omega)} \le c\,\mathrm{d}(\Omega', \partial\Omega)^{-1}.$$

We test the weak form of (1.2) with the test function $u_\varepsilon\psi$ to obtain

$$\sum_{i=1}^n \int_\Omega |Du|^{p-2}(x)\partial_i u(x)\partial_i(u_\varepsilon(x)\psi(x))\,dx = \int_\Omega f u_\varepsilon \psi\,dx,$$

which is equivalent to

$$\sum_{i=1}^n \int_{\Omega\cap\{|Du|<1\}} |Du|^{p-2}\partial_i u \partial_i u_\varepsilon \psi\,dx + \sum_{i=1}^n \int_{\Omega\cap\{|Du|\ge 1\}} |Du|^{p-2}\partial_i u \partial_i u_\varepsilon \psi\,dx = \int_\Omega (f\psi + g)u_\varepsilon\,dx, \tag{2.2}$$

where

$$g := -\sum_{i=1}^n |Du|^{p-2}\partial_i u \partial_i \psi.$$

Notice that the artificial splitting of Ω into $\Omega \cap \{|Du| < 1\}$ and $\Omega \cap \{|Du| \ge 1\}$ is necessary for the subsequent argument since, if $p < 2$, one may have $|Du|^{p-2} \notin L^1(\Omega)$. The first term on the left-hand side of (2.2) converges since $\partial_i u_\varepsilon \to \partial_i u$ in $L^1(\Omega''')$ as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} \sum_{i=1}^n \int_{\Omega\cap\{|Du|<1\}} |Du|^{p-2}\partial_i u \partial_i u_\varepsilon \psi\,dx = \int_{\Omega\cap\{|Du|<1\}} |Du|^p \psi\,dx. \tag{2.3}$$

Finally, we exploit (1.3) by writing

$$\sum_{i=1}^{n}\int_{\Omega\cap\{|Du|\ge 1\}}|Du|^{p-2}\partial_i u\partial_i u_\varepsilon\psi\,dx$$
$$=\sum_{i=1}^{n}\int_{\Omega\cap\{|Du|\ge 1\}}|Du|^{p-2}(\partial_i u-\sigma_i L_i)(\partial_i u_\varepsilon-\sigma_i L_i)\psi\,dx$$
$$+\sum_{i=1}^{n}\sigma_i L_i\int_{\Omega\cap\{|Du|\ge 1\}}|Du|^{p-2}(\partial_i u_\varepsilon+\partial_i u-\sigma_i L_i)\psi\,dx.$$

Using our choice of $\Omega\cap\{|Du|\ge 1\}$ and $Du\in L^{\max\{1,p-1\}}(\Omega)$, the last summand converges to

$$\sum_{i=1}^{n}\sigma_i L_i\int_{\Omega\cap\{|Du|\ge 1\}}|Du|^{p-2}(2\partial_i u-\sigma_i L_i)\psi\,dx.$$

Moreover, by (1.3), we can use Fatou's Lemma to bound from below the first summand. Thus, we find

$$\int_{\Omega'\cap\{|Du|\ge 1\}}|Du|^p\,dx\le\int_{\Omega\cap\{|Du|\ge 1\}}|Du|^p\psi\,dx$$
$$\le\liminf_{\varepsilon\to 0+}\sum_{i=1}^{n}\int_{\Omega\cap\{|Du|\ge 1\}}|Du|^{p-2}\partial_i u\partial_i u_\varepsilon\psi\,dx. \tag{2.4}$$

We now estimate the right-hand side of (2.2). Let $R>0$ be such that $\Omega\subset B_R(0)$. Using the definition of g, we find a constant $C'=C'(R,\mathrm{d}(\Omega',\partial\Omega))$ such that for all $\varepsilon>0$,

$$\left|\int_\Omega(f\psi+g)u_\varepsilon\,dx\right|\le\|f\|_{L^1(\Omega)}\|u_\varepsilon\|_{L^\infty(\Omega'')}+C'\|Du\|_{L^{p-1}(\Omega)}^{p-1}\|u_\varepsilon\|_{L^\infty(\Omega'')}. \tag{2.5}$$

We claim the existence of a constant $C=C(n,R,\mathrm{d}(\Omega',\partial\Omega))$ such that for all $0<\varepsilon<\mathrm{d}(\Omega''',\partial\Omega)$,

$$\|u_\varepsilon\|_{L^\infty(\Omega'')}\le C\|u_\varepsilon\|_{L^1(\Omega''')}+C|(L_1,\dots,L_n)|. \tag{2.6}$$

Indeed, we define

$$L:=(\sigma_1 L_1,\dots,\sigma_n L_n),\quad u'_\varepsilon=u_\varepsilon-(L,x),$$

and we show that

$$\|u'_\varepsilon\|_{L^\infty(\Omega'')}\le C\|u'_\varepsilon\|_{L^1(\Omega''')}, \tag{2.7}$$

which yields (2.6) for a possibly larger constant C. The advantage now is that for all $x\in\Omega'''$ and all $\varepsilon\le\mathrm{d}(\Omega''',\partial\Omega)$, by (1.3),

$$\sigma_i\partial_i u'_\varepsilon\ge 0.$$

We consider the set

$$S:=\Big\{h=(h_1,\dots,h_n):\sigma_i h_i\ge 0,\forall i,\max_i|h_i|\le\mathrm{d}(\Omega'',\partial\Omega)/2\Big\}.$$

Notice that its volume $|S| = 2^{-n}\mathrm{d}(\Omega'', \partial\Omega)^n$ is bounded from below by $4^{-n}\mathrm{d}(\Omega', \partial\Omega)^n$. For every $z_0 \in \Omega''$ and $h \in S$, we estimate $u'_\varepsilon(z_0 - h) \le u'_\varepsilon(z_0) \le u'_\varepsilon(z_0 + h)$. Integrating on S we obtain

$$\int_S u'_\varepsilon(z_0 - h)\, dh \le |S| u'_\varepsilon(z_0) \le \int_S u'_\varepsilon(z_0 + h)\, dh,$$

which yields (2.6). We now make use of (2.6) in (2.5) to write, for C possibly larger than the C and C' above,

$$\begin{aligned}\limsup_{\varepsilon\to 0} &\left| \int_\Omega (f\psi + g) u_\varepsilon \, dx \right| \\ &\le C(\|f\|_{L^1(\Omega)} + \|Du\|^{p-1}_{L^{p-1}(\Omega)}) \Big(\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^1(\Omega''')} + |(L_1, \dots, L_n)| \Big) \\ &\le C(\|f\|_{L^1(\Omega)} + \|Du\|^{p-1}_{L^{p-1}(\Omega)})(\|u\|_{L^1(\Omega)} + |(L_1, \dots, L_n)|). \end{aligned} \tag{2.8}$$

Hence by letting $\varepsilon \to 0$ in (2.2) and combining the lower bound for the left-hand side (2.3)–(2.4) and the upper bound for the right-hand side (2.8), we obtain (1.4). ■

Before ending this section, we wish to record a connection between our Theorem 1.1 and the compensation results appearing in the very recent paper [19]. A first observation in this regard, which will also be at the basis of the next sections, is to rewrite (1.7) as a differential inclusion, namely to translate the differential problem (1.7) to that of finding a solution $w \in W^{1,1}(\Omega, \mathbb{R}^2)$ to

$$Dw(x) \in K \quad \text{for a.e. } x \in \Omega, \tag{2.9}$$

for some suitable set $K \subset \mathbb{R}^{2\times 2}$. The set we will consider is

$$K_p := \left\{ \begin{pmatrix} x & y \\ |(x,y)|^{p-2} y & -|(x,y)|^{p-2} x \end{pmatrix} : x, y \in \mathbb{R} \right\}. \tag{2.10}$$

The equivalence between (1.7) and (2.9) is achieved once we interpret (1.7) in the equivalent form:

$$0 = \operatorname{div}(|Du|^{p-2} Du) = \operatorname{curl}(|Du|^{p-2} JDu), \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A simple application of Poincaré's Lemma then yields the following equivalence.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be convex and $u \in W^{1,\max\{1,p-1\}}(\Omega)$. Then*

$$\operatorname{div}(|Du|^{p-2} Du) = 0$$

in the sense of distributions if and only if there exists $v \in W^{1,1}(\Omega)$ such that $w := (u, v) \in W^{1,1}(\Omega, \mathbb{R}^2)$ solves

$$Dw \in K_p \quad \text{a.e. in } \Omega. \tag{2.11}$$

Moreover, for all $q \ge \max\{1, p-1\}$, $u \in W^{1,q}(\Omega)$ if and only if $w \in W^{1,\frac{q}{\max\{1,p-1\}}}$, and there exist positive constants $c_1 < c_2$ such that

$$c_1 \|u\|_{W^{1,q}(\Omega)} \le \|w\|_{W^{1,\frac{q}{\max\{1,p-1\}}}(\Omega,\mathbb{R}^2)} \le c_2 \|u\|_{W^{1,q}(\Omega)}. \tag{2.12}$$

We will sketch how a slightly simplified version of Theorem 1.1 for $n = 2$ can be deduced from [19, Corollary 4.5]. This states that if

$$A = A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^{2\times 2})$$

with $a_{11} \geq 0$, $a_{22} \leq 0$ everywhere on $\mathbb{R}^2$, then

$$-\int_{\mathbb{R}^2} \det(A) \leq \|\mathrm{curl}\, A_1\|_{L^1(\mathbb{R}^2)} \|\mathrm{curl}\, A_2\|_{L^1(\mathbb{R}^2)}, \tag{2.13}$$

where A_i denotes the i-th row of A. Let us define the convex sets

$$\begin{aligned} Y &= \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{11}, a_{12}, a_{21} \geq 0,\ a_{22} \leq 0 \right\} \\ &\subset X = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{11} \geq 0,\ a_{22} \leq 0 \right\}. \end{aligned}$$

Suppose now that we are given $u \in W^{1,\max\{1,p-1\}}(\Omega)$ satisfying (1.2) for $f = 0$ and (1.3) with $\sigma_1 = \sigma_2 = 1$, $L_1 = L_2 = 0$. Assumption (1.3) implies that $Dw \in Y \subset X$, but in order to apply [19, Corollary 4.5] we need to mollify and localize w. Consider $\Omega' \subset \Omega'' \subset \Omega$ and a cut-off function $\psi \in C_c^\infty(\Omega'')$ as in the proof above, and define $w_\varepsilon := w \star \rho_\varepsilon$. Finally, let

$$A_\varepsilon := \psi Dw \star \rho_\varepsilon = D(\psi w_\varepsilon) - w_\varepsilon \otimes D\psi.$$

We still have $A_\varepsilon(x) \in Y$ for all $x \in \mathbb{R}^2$, and we estimate

$$\limsup_{\varepsilon\to 0^+} \|\mathrm{curl}\, A_\varepsilon\|_{L^1(\mathbb{R}^2)} \overset{(2.12)}{\leq} C$$

for some $C = C(\Omega'', \|Du\|^{\max\{1,p-1\}}_{L^{\max\{1,p-1\}}(\Omega)}) > 0$, Therefore, we can employ (2.13) to write

$$-\int_\Omega \det(A_\varepsilon)\, dx \leq C. \tag{2.14}$$

Moreover $A_\varepsilon \in Y$ implies $\det(A_\varepsilon) \leq 0$ in Ω. Thus we can use Fatou's Lemma to conclude from (2.14) that

$$-\int_\Omega \psi^2 \det(Dw)\, dx \leq C.$$

As $Dw \in K_p$, we have $\det(Dw) = -|Du|^p$ a.e. in Ω. This gives an alternative proof of Theorem 1.1 in dimension $n = 2$.

3. Convex integration: outline of the strategy of the proof of Theorem 1.2

The first step in this type of convex integration is to rewrite (1.7) as a differential inclusion as in Proposition 2.1. In order to find nontrivial solutions w to (2.9), we can exploit Faraco's staircase laminates. Given an open domain $\omega \subset \mathbb{R}^2$, two distinct $\mathbb{R}^{2\times 2}$ matrices B, C with $\det(B - C) = 0$ and $\lambda \in (0, 1)$, it is possible for all $\varepsilon > 0$ to construct a highly oscillatory Lipschitz and piecewise affine map f_ε which coincides with any given affine map with gradient $A = \lambda B + (1 - \lambda)C$ on $\partial\omega$, and such that

$$Df_\varepsilon \in B_\varepsilon(B) \cup B_\varepsilon(C) \quad \text{a.e. on } \omega \tag{3.1}$$

with a precise estimate on the set of points where $Df_\varepsilon \in B_\varepsilon(B)$ in terms of λ; see Lemma 4.1. The measure $\mu = \lambda\delta_B + (1 - \lambda)\delta_C$ is called a *laminate*. By *splitting* B in another rank-one direction, one obtains a probability measure supported on three (or four) points, which is called a *laminate of finite order* (of order 2). For every laminate of finite order, one can construct a nontrivial family of maps as f_ε above; see Definition 4.2 and Lemma 4.3.

A staircase laminate is an element μ_n of a sequence of laminates of finite order which is constructed in the following way. Start from a point $A_1 \in \mathbb{R}^{2\times 2}$. First A_1 is split into two points, B_1 and E_1, with $B_1 \in K_p$ and E_1 an auxiliary point, again split into $C_1 := A_2$ and $D_1 \in K_p$. This yields

$$\mu_1 = \lambda_{B_1}\delta_{B_1} + \lambda_{E_1}\lambda_{A_2}\delta_{A_2} + \lambda_{E_1}\lambda_{D_1}\delta_{D_1}.$$

Now the *error term* A_2 is again split with the same rule through points $B_2 \in K_p$, $D_2 \in K_p$ and a new error term $C_2 := A_3$, which allows us to define μ_2. Inductively, one finds μ_n. In our specific problem, we need μ_n to fulfill

$$\int_{\mathbb{R}^{2\times 2}} |X|^{1+\varepsilon}\, d\mu_n(X) < \infty, \qquad \int_{\mathbb{R}^{2\times 2}} |X|^{\frac{p}{p-1}}\, d\mu_n(X) = \infty, \tag{3.2}$$

and

$$\mu_n(\{A_n\}) \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Conditions (3.2) yield the desired integrability of the solution, while (3.3) tells us that the measure $\mu_\infty := \lim_n \mu_n$ is supported precisely on K_p. Unfortunately, as we shall see, this discussion needs some additions to yield an exact solution to the differential inclusion at hand, but it is helpful to understand whether such a laminate can be found in a simpler subset of K_p, for instance $K_p \cap \operatorname{diag}(2)$, where $\operatorname{diag}(2)$ denotes the space of 2×2 diagonal matrices.

We have

$$K_p \cap \operatorname{diag}(2) = \left\{ \begin{pmatrix} x & 0 \\ 0 & -|x|^{p-2}x \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

Identify diagonal matrices with points (x, y) of $\mathbb{R}^2$, so that for instance every point of $K_p \cap \operatorname{diag}(2)$ is described by the graph $(x, -|x|^{p-2}x)$. Notice that in the space of diagonal matrices rank-one lines are precisely the horizontal and vertical lines. Take a sequence of points $A_i := (x_i, y_i)$ with

$$x_{i+1} > x_i > 0, \quad y_{i+1} > y_i > 0, \quad \forall i \in \mathbb{N},$$

to be chosen later. We start by splitting A_1 into a vertical direction until we reach a point $B_1 \in K_p \cap \operatorname{diag}(2)$, and the auxiliary point E_1. Then E_1 is split horizontally so that it lies in the segment with endpoints A_2 and $D_1 \in K_p \cap \operatorname{diag}(2)$. This defines μ_1. Next, given $A_3 = (x_3, y_3)$, we reiterate the reasoning starting from (x_2, y_2). By direct computation, one sees that the first condition in (3.2) and condition (3.3) are then equivalent to finding a suitable sequence $\{(x_i, y_i)\}_{i\in\mathbb{N}}$ as above with $x_i, y_i \to \infty$ as $i \to \infty$ such that the following holds, for some small $\varepsilon > 0$:

$$\lim_{n\to\infty} |(x_n, y_n)|^{1+\varepsilon} \prod_{k=1}^{n} \frac{x_{k-1} + y_k^{1/(p-1)}}{x_k + y_k^{1/(p-1)}} y_{k-1} + x_{k-1}^{p-1}/(y_k + x_{k-1}^{p-1}) < \infty.$$

Notice that for $p = 2$ this is impossible, which reflects the validity of Weyl's Lemma. It turns out that the choice $x_i = ai^2$, $y_i = i^{2(p-1)}$ for a suitable positive a that depends on p yields the previous property, and one can directly check that this choice also yields the second condition of (3.2). The parameter a is chosen so that a certain function $G_p(a)$ enjoys particular properties; see Section 5.1. Other sequences x_i and y_i are probably correct, for instance x_i and y_i could be chosen to be of exponential growth, as long as they contain this multiplicative parameter a. In our work, the careful choice is exploited in Proposition 6.2, which in turn yields the crucial final estimates of Lemma 9.3.

As already mentioned, finding only one staircase laminate is, in general, not enough to find an exact solution to (2.9). This is due to the fact that the maps constructed through laminates introduce errors in their gradient distribution; see (3.1). This is the reason why one considers *in-approximations* of the set K_p, using the terminology of [35], originally due to M. Gromov. Namely, one needs to find a sequence of open sets U_n which *converge* in a suitable sense (see Lemma 7.2) to K_p with the additional property that every point of U_n is the barycenter of a laminate of finite order with suitable integrability properties, supported in U_{n+1}. The fact that these sets are open allows one to *absorb* the errors made by property (3.1). Once one has a sequence of Lipschitz maps f_n with $Df_n \in U_n$ a.e. which in addition converge strongly in $W^{1,1}$ to a map f, it is possible to conclude that $Df \in K_p$ a.e. The (subsequential) pointwise convergence of the gradients is crucial and is necessary to exploit the convergence of $\{U_n\}$ to K_p. It is usually achieved with a mollification trick; see Proposition 9.13 and the proof of Theorem 9.4.

We will not use the term *in-approximation* in this paper, but our whole effort is precisely based on finding these open sets. The idea is to consider a family of staircase laminates with initial point $P \in Q \subset \mathbb{R}^{2\times 2}$, with Q open. In this case, one obtains endpoints of these laminates of finite order $A_i, B_i, C_i, D_i, A_{i+1} = C_i$, which depend on P. The definition of these quantities is in Section 5. As above, $B_i(P) \in K_p$ and $D_i(P) \in K_p$ for all $P \in Q$, but we will actually need to consider staircase laminates with endpoints lying on the segments connecting A_i, B_i and C_i, D_i, which do not reach *exactly* B_i and D_i; see the definition of the maps $\Phi^k_{i,t}$ of (5.16)–(5.17). This is necessary in order to deal with the errors of (3.1). The definition of the laminates and the necessary quantitative estimates are the content of Section 8. In order to find these open sets, we need to prove the openness of the mappings A_i and $\Phi^k_{i,t}$. This is the most technical part of the paper, and occupies Section 6.

Once the openness is shown, we will finally be able to prove the existence of a solution to (2.11) in Section 9. To deduce that the solution w we construct has the required degeneracy property

$$\int_B |Dw|^{\frac{p}{p-1}}\,dx = \infty \tag{3.4}$$

for all nonempty, open $B \subset \Omega$, we will show that our construction yields w non-C^1 on any open set, and we will deduce (3.4) from regularity results concerning the p-Laplace equation. This is based on Lemma 7.1, which, together with the aforementioned Lemma 7.2, constitutes Section 7.

4. Laminates of finite order

In this section we introduce elementary splittings and laminates of finite order. We say that $A, B \in \mathbb{R}^{2\times 2}$ are *rank-one connected* if

$$\operatorname{rank}(A - B) = 1.$$

Furthermore, we will denote by $[A, B]$ the segment connecting $A, B \in \mathbb{R}^{2\times 2}$, and we will call it a *rank-one segment* if $\operatorname{rank}(A - B) = 1$. In that case, $B - A$ is a *rank-one direction* and the line containing $[A, B]$ is a *rank-one line*. We have the following (see for instance [3, Lemma 2.1]).

Lemma 4.1. *Let $A, B, C \in \mathbb{R}^{2\times 2}$ with $\operatorname{rank}(B - C) = 1$ and $A = \lambda B + (1-\lambda)C$ for some $\lambda \in [0, 1]$. Let also $\Omega \subset \mathbb{R}^2$ be a fixed open domain and $b \in \mathbb{R}^2$. Then for every $\varepsilon > 0$, one can find a Lipschitz piecewise affine map $f_\varepsilon : \Omega \to \mathbb{R}^2$ such that*

(1) *$f_\varepsilon(x) = f_0(x) = Ax + b$ on $\partial\Omega$ and $\|f_\varepsilon - f_0\|_\infty \le \varepsilon$;*

(2) *$Df_\varepsilon(x) \in B_\varepsilon(B) \cup B_\varepsilon(C)$;*

(3) *$|\{x \in \Omega : Df_\varepsilon(x) \in B_\varepsilon(B)\}| = \lambda|\Omega|$ and $|\{x \in \Omega : Df_\varepsilon(x) \in B_\varepsilon(C)\}| = (1-\lambda)|\Omega|$.*

Denote by $\mathscr{P}(U)$ the space of probability measures with support in $U \subset \mathbb{R}^{2\times 2}$.

Definition 4.2. Let $\nu, \mu \in \mathscr{P}(U)$, $U \subset \mathbb{R}^{2\times 2}$ open. Let $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i}$. We say that μ can be obtained from ν via *elementary splitting* if for some $i \in \{1, \dots, N\}$, there exist $B, C \in U$ and $\lambda \in [0, 1]$ such that for some $s \in (0, 1)$,

$$\operatorname{rank}(B - C) = 1, \quad [B, C] \subset U, \quad A_i = sB + (1-s)C,$$

and

$$\mu = \nu + \lambda\lambda_i(-\delta_{A_i} + s\delta_B + (1-s)\delta_C).$$

A measure $\nu = \sum_{i=1}^r \lambda_i \delta_{A_i} \in \mathscr{P}(U)$ is called a *laminate of finite order* if there exist a finite number of measures $\nu_1, \dots, \nu_r \in \mathscr{P}(U)$ such that

$$\nu_1 = \delta_X, \quad \nu_r = \nu$$

and ν_{j+1} can be obtained from ν_j via elementary splitting for every $j \in \{1, \dots, N-1\}$.

Using the definition of elementary splitting to iterate Lemma 4.1 as in [35, Lemma 3.2], one can prove the following.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ be an open domain. Let $U \subset \mathbb{R}^{2\times 2}$ be an open set and let $\nu = \sum_{i=1}^r \lambda_i \delta_{A_i} \in \mathcal{P}(U)$ be a laminate of finite order with barycenter $A \in \mathbb{R}^{2\times 2}$, i.e.*

$$A = \int_{\mathbb{R}^{2\times 2}} X \, d\nu(X).$$

Then for any $b \in \mathbb{R}^2$ and $\varepsilon > 0$, the map $f_0(x) := Ax + b$ admits on Ω an approximation by piecewise affine, equi-Lipschitz maps $f_\varepsilon \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ with the following properties:

(1) *$f_\varepsilon(x) = Ax + b$ on $\partial\Omega$ and $\|f_\varepsilon - Ax - b\|_\infty \le \varepsilon$;*

(2) *$Df_\varepsilon(x) \in \bigcup_i B_\varepsilon(A_i)$;*

(3) *$|\{x \in \Omega : Df_\varepsilon(x) \in B_\varepsilon(A_i)\}| = \lambda_i |\Omega|$ for all i.*

5. Definition of the main quantities

5.1. Choice of the sequences and coordinates

For any $a > 0$, consider the following sequence, which is increasing in i:

$$x_i = ai^2 + x_0, \quad y_i = i^{2(p-1)} + y_0, \quad \forall i \ge 0. \tag{5.1}$$

Here a, x_0 and y_0 will be used as coordinates in our construction. Together with these, we will need a fourth parameter $w \in \mathbb{R}$, as will be clear in the next subsection. The space of parameters is, for any $p > 1$,

$$Q(c) := (c, 2c) \times \left(\frac{3}{4}, \frac{5}{4}\right) \times \left(\frac{3}{4}, \frac{5}{4}\right) \times \left(\frac{3}{4}, \frac{5}{4}\right).$$

The parameter $c > 0$ will be chosen small if $p > 2$ and large if $p < 2$. Instead of fixing c here, we will update it in various technical results of the next sections. We introduce a function that will be crucial in the following:

$$G_p(a) = \frac{a}{a+1} + (p-1)\frac{1}{a^{p-1}+1}. \tag{5.2}$$

We start introducing conditions on c: in particular, we claim that the inequality

$$\max\{1, p-1\} < G_p(a) < \max\{1, p-1\} + 1/2 \quad \text{for every } a \in [c, 2c] \tag{5.3}$$

holds for c sufficiently small if $p > 2$ and for c sufficiently large if $1 < p < 2$.

5.1.1. The case $p > 2$. The (right) derivative of $c \mapsto G_p(c)$ at 0 is 1. Hence, $G_p(c)$ is strictly increasing in a neighborhood of 0, and since $G_p(0) = p - 1$, for c sufficiently small we have

$$p - 1 < G_p(c) \le G_p(a) \le G_p(2c) < p - 1/2, \quad \forall a \in [c, 2c]. \tag{5.4}$$

5.1.2. The case $1 < p < 2$. We have $\lim_{c\to\infty} G_p(c) = 1$ and for all c sufficiently large, $G_p(c)$ is decreasing. Indeed,

$$\begin{aligned}\frac{d}{dc}G_p(c) &= \frac{1}{(c+1)^2} - (p-1)^2\frac{c^{p-2}}{(c^{p-1}+1)^2}\\ &= \frac{c^{p-2}}{(c^{p-1}+1)^2}\left(\frac{(c^{p-1}+1)^2}{c^{p-2}(c+1)^2} - (p-1)^2\right).\end{aligned}$$

As $1 < p < 2$, $\lim_{c\to\infty}(\frac{(c^{p-1}+1)^2}{c^{p-2}(c+1)^2} - (p-1)^2) = -(p-1)^2 < 0$. Thus, we can take c sufficiently large that

$$1 < G_p(2c) \le G_p(a) \le G_p(c) < 3/2, \quad \forall a \in [c, 2c]. \tag{5.5}$$

Remark 5.1. The function $G_p(a)$ behaves very differently for $p = 2$ and $p \neq 2$. The strict inequalities in (5.3) are the key to the convergence of some quantities in Section 9. If $p = 2$ instead, then $G_p(a) \equiv 1$ for all a and our strategy fails, as expected in view of Weyl's Lemma recalled in the introduction.

5.2. *Parametrization of the laminates*

Let $p \in (1, \infty)$ be fixed and recall that

$$K_p = \left\{\begin{pmatrix} x & y \\ |(x,y)|^{p-2}y & -|(x,y)|^{p-2}x\end{pmatrix} : x, y \in \mathbb{R}\right\}.$$

We denote points of $Q(c)$ by P, but we will almost always drop the dependence of the functions x_i, y_i, v_i, z_i, etc., on P. For readability, we also introduce the functions

$$h_w(x) = |(x,w)|^{p-2}w, \quad g_w(x) = |(x,w)|^{p-2}x.$$

The counterexample is built in an inductive way. As explained in Section 3, given

$$A_i(P) = \begin{pmatrix} x_{i-1} & w \\ z_{i-1} & y_{i-1}\end{pmatrix}, \quad w \neq 0,$$

we need to split it into points $B_i, D_i, C_i = A_{i+1}$ where $B_i, D_i \in K_p$ and A_{i+1} is the next step of the iteration. Here, $\{x_i\}$ and $\{y_i\}$ are the sequences of functions introduced in Section 5.1, while z_i cannot be chosen, and will instead be built during the construction of the laminates. We start by considering

$$A_i(P) := \begin{pmatrix} x_{i-1} & w \\ z_{i-1} & y_{i-1}\end{pmatrix}. \tag{5.6}$$

Split it into the rank-one direction

$$\begin{pmatrix} 0 & 0 \\ m & n\end{pmatrix}$$

for some suitable $m, n \in \mathbb{R}$, and call the endpoints B_i and E_i. We require that $B_i \in K_p$

and E_i has $(2, 2)$ component equal to y_i. Simple computations show that then B_i and E_i are

$$B_i(P) := \begin{pmatrix} x_{i-1} & w \\ h_w(x_{i-1}) & -g_w(x_{i-1}) \end{pmatrix} \in K_p, \quad E_i(P) := \begin{pmatrix} x_{i-1} & w \\ v_i & y_i \end{pmatrix}, \tag{5.7}$$

where

$$v_i = \frac{y_i + g_w(x_{i-1})}{y_{i-1} + g_w(x_{i-1})} z_{i-1} - \frac{y_i - y_{i-1}}{y_{i-1} + g_w(x_{i-1})} h_w(x_{i-1}). \tag{5.8}$$

In particular, we can write

$$A_i = \lambda_{B_i} B_i + \lambda_{E_i} E_i$$

with

$$\lambda_{B_i} = \frac{y_i - y_{i-1}}{y_i + g_w(x_{i-1})}, \quad \lambda_{E_i} = \frac{y_{i-1} + g_w(x_{i-1})}{y_i + g_w(x_{i-1})}. \tag{5.9}$$

We further split E_i into the rank-one direction

$$\begin{pmatrix} m' & 0 \\ n' & 0 \end{pmatrix}$$

for suitable $m', n' \in \mathbb{R}$, in order to reach the endpoints

$$C_i(P) = A_{i+1}(P) = \begin{pmatrix} x_i & w \\ z_i & y_i \end{pmatrix}, \quad D_i(P) := \begin{pmatrix} -g_w^{-1}(y_i) & w \\ h_w(g_w^{-1}(y_i)) & y_i \end{pmatrix} \in K_p. \tag{5.10}$$

We have $E_i = \lambda_{C_i} C_i + \lambda_{D_i} D_i$ for

$$\lambda_{C_i} = \frac{x_{i-1} + g_w^{-1}(y_i)}{x_i + g_w^{-1}(y_i)}, \quad \lambda_{D_i} = \frac{x_i - x_{i-1}}{x_i + g_w^{-1}(y_i)}. \tag{5.11}$$

As above, z_i is defined by the requirements $C_i(P) = A_{i+1}(P)$ and $D_i(P) \in K_p$:

$$z_i(P) = \frac{x_i + g_w^{-1}(y_i)}{x_{i-1} + g_w^{-1}(y_i)} v_i - \frac{x_i - x_{i-1}}{x_{i-1} + g_w^{-1}(y_i)} h_w(g_w^{-1}(y_i)). \tag{5.12}$$

Notice, though, that we have the freedom of choosing z_0, which we will fix as a function $z_0 = z_0(P)$ in Section 6.1.

Remark 5.2. By construction, the probability measure

$$\nu = \lambda_{B_i(P)} \delta_{B_i(P)} + \lambda_{E_i(P)} \lambda_{D_i(P)} \delta_{D_i(P)} + \lambda_{E_i(P)} \lambda_{C_i(P)} \delta_{C_i(P)}$$

is a laminate of finite order for all $P \in Q(c)$ (see Definition 4.2). This fact will be exploited in Section 8.

To conclude, we introduce maps that will be useful in our constructions. For $t \in [0, 1]$ and $P \in Q(c)$, define

$$\Phi^1_{i,t}(a, x_0, y_0, w) := A_i(P) + t\lambda_{E_i}(P)(B_i(P) - E_i(P)) \in \mathbb{R}^{2\times 2}, \tag{5.13}$$

$$\Phi^2_{i,t}(a, x_0, y_0, w) := E_i(P) + t\lambda_{C_i}(P)(D_i(P) - C_i(P)) \in \mathbb{R}^{2\times 2}. \tag{5.14}$$

Notice that for $t \in (0,1)$ the maps $\Phi^1_{i,t}$ and $\Phi^2_{i,t}$ interpolate between the known maps

$$\Phi^1_{i,0}(P) = A_i(P), \quad \Phi^2_{i,0}(P) = E_i(P), \quad \Phi^1_{i,1}(P) = B_i(P), \quad \Phi^2_{i,1} = D_i(P). \tag{5.15}$$

More explicitly, we rewrite (5.13)–(5.14) as

$$\Phi^1_{i,t}(a, x_0, y_0, w) := \begin{pmatrix} x_{i-1} & w \\ z_{i-1} + t(h_w(x_{i-1}) - z_{i-1}) & y_{i-1} - t(y_{i-1} + g_w(x_{i-1})) \end{pmatrix}, \tag{5.16}$$

$$\Phi^2_{i,t}(a, x_0, y_0, w) := \begin{pmatrix} x_{i-1} - t(g_w^{-1}(y_i) + x_{i-1}) & w \\ v_i + t(h_w(g_w^{-1}(y_i)) - v_i) & y_i \end{pmatrix}. \tag{5.17}$$

6. Openness of the mappings

The goal of this section is to show that the maps A_i and $\Phi^k_{i,t}$ are well-defined, continuous and open. These properties will be shown in Proposition 6.2.

6.1. *Explicit formula for z_i*

For all $i \geq 1$, define

$$S_i(P) := \prod_{\ell=1}^{i} \frac{x_\ell + g_w^{-1}(y_\ell)}{x_{\ell-1} + g_w^{-1}(y_\ell)} \frac{y_\ell + g_w(x_{\ell-1})}{y_{\ell-1} + g_w(x_{\ell-1})}, \tag{6.1}$$

and for all $\ell \geq 1$,

$$\begin{aligned} H_\ell(P) &:= H^1_\ell(P) + H^2_\ell(P) \\ &:= \frac{x_\ell + g_w^{-1}(y_\ell)}{x_{\ell-1} + g_w^{-1}(y_\ell)} \frac{y_\ell - y_{\ell-1}}{y_{\ell-1} + g_w(x_{\ell-1})} h_w(x_{\ell-1}) + \frac{x_\ell - x_{\ell-1}}{x_{\ell-1} + g_w^{-1}(y_\ell)} h_w(g_w^{-1}(y_\ell)). \end{aligned} \tag{6.2}$$

We can use formulas (5.8), (5.12) to find that

$$\begin{aligned} z_i(P) = &\frac{x_i + g_w^{-1}(y_i)}{x_{i-1} + g_w^{-1}(y_i)} \left(\frac{y_i + g_w(x_{i-1})}{y_{i-1} + g_w(x_{i-1})} z_{i-1} - \frac{y_i - y_{i-1}}{y_{i-1} + g_w(x_{i-1})} h_w(x_{i-1}) \right) \\ &- \frac{x_i - x_{i-1}}{x_{i-1} + g_w^{-1}(y_i)} h_w(g_w^{-1}(y_i)). \end{aligned}$$

Using this formula recursively, we find an expression for $z_i(P)$:

$$\begin{aligned} z_i(P) &= S_i(P) z_0 - \sum_{\ell=1}^{i} H_\ell(P) \prod_{k=\ell+1}^{i} \frac{x_k + g_w^{-1}(y_k)}{x_{k-1} + g_w^{-1}(y_k)} \frac{y_k + g_w(x_{k-1})}{y_{k-1} + g_w(x_{k-1})} \\ &= S_i(P) z_0 - S_i(P) \sum_{\ell=1}^{i} \frac{H_\ell(P)}{S_\ell(P)} = S_i(P)\left(z_0 - \sum_{\ell=1}^{i} \frac{H_\ell(P)}{S_\ell(P)} \right). \end{aligned}$$

Notice that we work with the convention

$$\prod_{k=r}^{s} p_k = 1 \tag{6.3}$$

if $r > s$ and $\{p_n\}_n$ is any sequence. We choose

$$z_0 = z_0(P) = \sum_{\ell=1}^{\infty} \frac{H_\ell(P)}{S_\ell(P)}.$$

This yields

$$z_i(P) = S_i(P) \sum_{\ell=i+1}^{\infty} \frac{H_\ell(P)}{S_\ell(P)}. \tag{6.4}$$

This choice requires showing the convergence of $\sum_{\ell=1}^{\infty} \frac{H_\ell(P)}{S_\ell(P)}$, which will be proved in Proposition 6.2 below, after having made some necessary estimates.

6.2. *Sufficient conditions for the openness of the mappings*

We are now interested in providing a sufficient condition to ensure that the mappings $A_{i+1}, \Phi^k_{i,t}$ are open for all $k = 1, 2, t \in [0, 1)$ and $i \geq I$ for I large to be fixed later. To do so, we wish to use the invariance of domain theorem (see [21, Theorem 2B.3]) that tells us that we only need to check that the above maps are continuous injections. Continuity is immediate for most of the components of the above maps, and it is only non-trivial for z_i and v_i. It will be shown for z_i in Proposition 6.2, and from this and (5.8) it readily follows for v_i too. We only need to verify injectivity. The estimates we need to show in order to prove well-posedness and continuity of z_i and the injectivity of the aforementioned maps will be given in this section.

Lemma 6.1. *Let $Q(c) = (c, 2c) \times (\frac{3}{4}, \frac{5}{4}) \times (\frac{3}{4}, \frac{5}{4}) \times (\frac{3}{4}, \frac{5}{4})$ with c such that Proposition 6.2 (a) holds. If $1 < p \neq 2$ and if for some $I \geq 2$ and all $i \geq I$, the limit*

$$(\partial_a z_i - i^2 \partial_{x_0} z_i)(P) := \lim_{s \to 0} \frac{z_i(P + s(1, -i^2, 0, 0)) - z_i(P)}{s}$$

exists, is continuous on $Q(c)$ and

$$\partial_a z_i(P) - i^2 \partial_{x_0} z_i(P) > 0, \quad \forall P \in Q(c), \tag{6.5}$$

then, for all $i \geq I$, $k = 1, 2$ and $t \in [0, 1)$, A_{i+1} and $\Phi^k_{i+1,t}$ are injective mappings and hence $A_{i+1}(Q(c))$ and $\Phi^k_{i+1,t}(Q(c))$ are open sets.

Proof. By the above discussion, we only need to show the injectivity of the three mappings at hand.

A_{i+1} is injective: Let $P_j = (a^j, x_0^j, y_0^j, w^j)$ for $j = 1, 2$, and suppose that for some $i \geq I$,

$$A_{i+1}(P_1) = A_{i+1}(P_2). \tag{6.6}$$

We need to show that $P_1 = P_2$. From (5.6), we immediately see that $y_0^1 = y_0^2$ and $w^1 = w^2$. Furthermore, from the equality

$$x_i(P_1) = x_i(P_2)$$

we infer

$$a^1 i^2 + x_0^1 = a^2 i^2 + x_0^2, \quad \text{i.e.} \quad x_0^1 - x_0^2 = -i^2(a^1 - a^2). \tag{6.7}$$

Now consider the segment

$$\begin{aligned}\sigma(s) &:= (a^1, x_0^1, y_0^1, w) + s(a^2 - a^1, x_0^2 - x_0^1, 0, 0)\\ &= (a^1, x_0^1, y_0^1, w) + (a^2 - a^1)s(1, -i^2, 0, 0).\end{aligned}$$

As $Q(c)$ is convex, $(\sigma(s), y_0, w) \in Q(c)$ for all $s \in [0, 1]$. Moreover, by assumption (6.5), $t \mapsto z_i(\sigma(t))$ is a C^1 function. By (6.6), we can write

$$\begin{aligned}0 = z_i(P_1) - z_i(P_2) &= \int_0^1 \frac{d}{dt} z_i(\sigma(t))\, dt\\ &= (a^2 - a^1) \int_0^1 [\partial_a z_i(\sigma(t)) - i^2 \partial_{x_0} z_i(\sigma(t))]\, dt.\end{aligned}$$

Assumption (6.5) shows $a^1 = a^2$. Through (6.7) we also infer $x_0^1 = x_0^2$, which concludes the proof.

$\Phi^1_{i+1,t}$ *is injective:* Fix $t \in [0, 1)$. Assume with the same notation as above that $\Phi^1_{i+1,t}(P_1) = \Phi^1_{i+1,t}(P_2)$. Then by (5.16) we easily infer $w^1 = w^2$ and

$$a^1 i^2 + x_0^1 = a^2 i^2 + x_0^2.$$

Moreover, since $t \neq 1$, equating the $(2, 2)$ components of $\Phi^1_{i,t}(P_1)$ and $\Phi^1_{i,t}(P_2)$, we also find $y_0^1 = y_0^2$. Since $x_i(P_1) = x_i(P_2)$ and $w^1 = w^2$, the $(2, 1)$ component now gives (again since $t \neq 1$)

$$z_i(P_1) = z_i(P_2),$$

and we can conclude as in the previous case using (6.5).

$\Phi^2_{i+1,t}$ *is injective:* Fix $t \in [0, 1)$. Assume with the same notation as above that $\Phi^2_{i+1,t}(P_1) = \Phi^2_{i+1,t}(P_2)$. Then from (5.17) we easily infer $w^1 = w^2$ and $y_0^1 = y_0^2$. With these observations, and the fact that $t \neq 1$, equality of the $(1, 1)$ components yields once again

$$a^1 i^2 + x_0^1 = a^2 i^2 + x_0^2,$$

and equality of the $(2, 1)$ components yields

$$v_{i+1}(P_1) = v_{i+1}(P_2). \tag{6.8}$$

Combining these facts with (5.8), we easily see that (6.8) is equivalent to $z_i(P_1) = z_i(P_2)$ and once again we conclude analogously to the first case. ∎

6.3. *Computation of the derivatives of z_i*

From Lemma 6.1, we know that we need to compute the first order derivatives of z_i. In particular, let

$$\delta_i = \partial_a - i^2 \partial_{x_0}.$$

Since

$$z_i(P) = S_i(P) \sum_{\ell=i+1}^{\infty} \frac{H_\ell^1(P) + H_\ell^2(P)}{S_\ell(P)},$$

and it is easy to see that

$$P \mapsto \frac{S_i(P)(H_\ell^1(P) + H_\ell^2(P))}{S_\ell(P)}$$

is a smooth mapping, our aim is to estimate from above and below

$$\begin{aligned}\delta_i\left(\frac{S_i(P)H_\ell(P)}{S_\ell(P)}\right) &= \delta_i\left(\frac{S_i(P)}{S_\ell(P)}\right)(H_\ell^1(P) + H_\ell^2(P)) \\ &\quad + \frac{S_i(P)}{S_\ell(P)}\delta_i(H_\ell^1(P) + H_\ell^2(P)). \end{aligned} \tag{6.9}$$

Repeatedly using the fact that

$$\delta_i \prod_{i=1}^{N} f_i g_i^{-1} = \prod_{i=1}^{N} f_i g_i^{-1} \sum_{i=1}^{N} (f_i^{-1}\delta_i f_i - g_i^{-1}\delta_i g_i),$$

we get

$$\begin{aligned}\delta_i H_\ell^1(P) &= \delta_i\left(\frac{x_\ell + g_w^{-1}(y_\ell)}{x_{\ell-1} + g_w^{-1}(y_\ell)}\, \frac{y_\ell - y_{\ell-1}}{y_{\ell-1} + g_w(x_{\ell-1})} h_w(x_{\ell-1})\right) \\ &= H_\ell^1(P)\left(\frac{\delta_i x_\ell}{x_\ell + g_w^{-1}(y_\ell)} - \frac{\delta_i x_{\ell-1}}{x_{\ell-1} + g_w^{-1}(y_\ell)} + \frac{h_w'(x_{\ell-1})\delta_i x_{\ell-1}}{h_w(x_{\ell-1})} - \frac{g_w'(x_{\ell-1})\delta_i x_{\ell-1}}{y_{\ell-1} + g_w(x_{\ell-1})}\right) \\ &= H_\ell^1(P)\left(\frac{\delta_i(x_\ell - x_{\ell-1})}{x_\ell + g_w^{-1}(y_\ell)} - \frac{\delta_i x_{\ell-1}(x_\ell - x_{\ell-1})}{(x_\ell + g_w^{-1}(y_\ell))(x_{\ell-1} + g_w^{-1}(y_\ell))} \right.\\ &\qquad\left. + \frac{h_w'(x_{\ell-1})\delta_i x_{\ell-1}}{h_w(x_{\ell-1})} - \frac{g_w'(x_{\ell-1})\delta_i x_{\ell-1}}{y_{\ell-1} + g_w(x_{\ell-1})}\right) \\ &=: H_\ell^1(P)\rho_{i,\ell}^1, \end{aligned} \tag{6.10}$$

and

$$\begin{aligned}\delta_i H_\ell^2(P) &= \delta_i\left(\frac{x_\ell - x_{\ell-1}}{x_{\ell-1} + g_w^{-1}(y_\ell)} h_w(g_w^{-1}(y_\ell))\right) \\ &= H_\ell^2(P)\left(\frac{\delta_i(x_\ell - x_{\ell-1})}{x_\ell - x_{\ell-1}} - \frac{\delta_i x_{\ell-1}}{x_{\ell-1} + g_w^{-1}(y_\ell)}\right) \\ &=: H_\ell^2(P)\rho_{i,\ell}^2. \end{aligned} \tag{6.11}$$

Similarly, we get

$$\delta_i\left(\frac{S_i(P)}{S_\ell(P)}\right) = \delta_i\left(\prod_{k=i+1}^{\ell}\frac{x_{k-1}+g_w^{-1}(y_k)}{x_k+g_w^{-1}(y_k)}\frac{y_{k-1}+g_w(x_{k-1})}{y_k+g_w(x_{k-1})}\right)$$

$$= \frac{S_i}{S_\ell}\sum_{r=i+1}^{\ell}\Bigg[-\frac{\delta_i x_r}{x_r+g_w^{-1}(y_r)}+\frac{\delta_i x_{r-1}}{x_{r-1}+g_w^{-1}(y_r)} - \frac{g_w'(x_{r-1})\delta_i x_{r-1}}{y_r+g_w(x_{r-1})}+\frac{g_w'(x_{r-1})\delta_i x_{r-1}}{y_{r-1}+g_w(x_{r-1})}\Bigg]$$

$$= \frac{S_i}{S_\ell}\sum_{r=i+1}^{\ell}\Bigg[-\frac{\delta_i(x_r-x_{r-1})}{x_r+g_w^{-1}(y_r)}+\frac{(x_r-x_{r-1})\delta_i x_{r-1}}{(x_{r-1}+g_w^{-1}(y_r))(x_r+g_w^{-1}(y_r))} + \frac{(y_r-y_{r-1})g_w'(x_{r-1})\delta_i x_{r-1}}{(y_r+g_w(x_{r-1}))(y_{r-1}+g_w(x_{r-1}))}\Bigg]$$

$$=: \frac{S_i}{S_\ell}\sigma_{i,\ell}. \tag{6.12}$$

Overall, with the newly defined $\rho^1_{i,\ell}, \rho^2_{i,\ell}, \sigma_{i,\ell}$ introduced in (6.10)–(6.12), we obtain

$$\delta_i z_i = \sum_{\ell=i+1}^{\infty}\frac{S_i}{S_\ell}[\sigma_{i,\ell}H_\ell + H^1_\ell\rho^1_{i,\ell} + H^2_\ell\rho^2_{i,\ell}]. \tag{6.13}$$

6.4. *Asymptotic behavior of the terms composing $\delta_i z_i$*

Let us start by estimating $\sigma_{i,\ell}$ (see (6.12)). Throughout this section, we will make a list of claims concerning the asymptotics of the terms involved. We start by claiming that

$$-\frac{\delta_i(x_r-x_{r-1})}{x_r+g_w^{-1}(y_r)} \sim -\frac{2}{a+1}\frac{1}{r}, \tag{Asymp. 1}$$

$$\frac{x_r-x_{r-1}}{(x_{r-1}+g_w^{-1}(y_r))(x_r+g_w^{-1}(y_r))} \sim \frac{1}{r^3}\frac{2a}{(a+1)^2}, \tag{Asymp. 2}$$

$$\frac{(y_r-y_{r-1})g_w'(x_{r-1})}{(y_r+g_w(x_{r-1}))(y_{r-1}+g_w(x_{r-1}))} \sim \frac{1}{r^3}\frac{2(p-1)^2a^{p-2}}{(1+a^{p-1})^2}. \tag{Asymp. 3}$$

Let us explain the notation we used in the previous lines. Given two sequences $\{a_j\}_{j\geq 1}$ and $\{b_j\}_{j\geq 1}$ of functions defined on $\overline{Q(c)}$, the symbol $a_j \sim b_j$ means

$$\sup_{P\in\overline{Q(c)}}\left|\frac{a_j(P)}{b_j(P)}-1\right| \to 0 \quad \text{as } j\to\infty.$$

The proofs of (Asymp. 1)–(Asymp. 3), as well as of the following asymptotic estimates, are easy and similar to each other. Therefore, let us only show (Asymp. 1). Using the definition

$$|(g_w^{-1}(y_r), w)|^{p-2}g_w^{-1}(y_r) = y_r = r^{2(p-1)} + y_0,$$

we see that

$$1+\frac{y_0}{r^{2(p-1)}}=\left|\left(\frac{g_w^{-1}(y_r)}{r^2},\frac{w}{r^2}\right)\right|^{p-2}\frac{g_w^{-1}(y_r)}{r^2},$$

which shows that

$$g_w^{-1}(y_r)\sim r^2. \tag{6.14}$$

We can now prove (Asymp. 1) with the help of (6.14):

$$\begin{aligned}\frac{\delta_i(x_r-x_{r-1})}{x_r+g_w^{-1}(y_r)}\frac{r(a+1)}{2}&=\frac{r^2-(r-1)^2}{a+x_0r^{-2}+r^{-2}g_w^{-1}(y_r)}\frac{a+1}{2r}\\&=\frac{(2-r^{-1})(a+1)}{2(a+x_0r^{-2}+r^{-2}g_w^{-1}(y_r))}\sim 1.\end{aligned}$$

Estimates (Asymp. 1)–(Asymp. 3) imply that if we define

$$\tilde{\sigma}_{i,\ell}:=\sum_{r=i+1}^{\ell}\left[-\frac{2}{a+1}\frac{1}{r}+\frac{(r-1)^2-i^2}{r^3}\frac{2a}{(a+1)^2}+\frac{(r-1)^2-i^2}{r^3}\frac{2(p-1)^2a^{p-2}}{(1+a^{p-1})^2}\right]$$

and we fix $\varepsilon=\varepsilon(c)>0$, then if $I=I(c,\varepsilon)\in\mathbb{N}$ is sufficiently large and $i\geq I$, then for all $P\in\overline{Q(c)}$,

$$|\sigma_{i,\ell}(P)-\tilde{\sigma}_{i,\ell}(P)|\leq C\varepsilon\sum_{r=i+1}^{\ell}\left[\frac{1}{r}+\frac{(r-1)^2-i^2}{r^3}\right]\leq C\varepsilon\sum_{r=i+1}^{\ell}\frac{1}{r}. \tag{6.15}$$

Here and below, $C=C(c)>0$ is a constant depending solely on $c>0$ (and on p, which is fixed). This constant may change from line to line, but we will always denote it with the same letter. We can further simplify (6.15) and $\tilde{\sigma}_{i,\ell}$. Indeed, by integral comparison, one can see that, for all $\ell\geq i+1$ and $\alpha>0$,

$$\left|\sum_{k=i+1}^{\ell}\frac{1}{k}-\ln\left(\frac{\ell}{i+1}\right)\right|\leq\frac{1}{i+1}, \tag{6.16}$$

$$\left|\sum_{k=i+1}^{\ell}\frac{1}{k^{\alpha+1}}-\frac{1}{\alpha}\left[\frac{1}{(i+1)^{\alpha}}-\frac{1}{\ell^{\alpha}}\right]\right|\leq\frac{1}{(i+1)^{\alpha+1}}. \tag{6.17}$$

Furthermore, letting

$$\begin{aligned}\bar{\sigma}_{i,\ell}:=&-\frac{2}{a+1}\ln\left(\frac{\ell}{i+1}\right)\\&+\left[\ln\left(\frac{\ell}{i+1}\right)+\frac{i^2}{2}\left(\frac{1}{\ell^2}-\frac{1}{(i+1)^2}\right)\right]\left[\frac{2a}{(a+1)^2}+\frac{2(p-1)^2a^{p-2}}{(1+a^{p-1})^2}\right],\end{aligned} \tag{6.18}$$

and using (6.16)–(6.17), we see that if I is sufficiently large, (6.15) becomes, for all $P\in\overline{Q(c)}$ and $i\geq I$,

$$\begin{aligned}|\sigma_{i,\ell}(P)-\bar{\sigma}_{i,\ell}(P)|&\leq|\sigma_{i,\ell}(P)-\tilde{\sigma}_{i,\ell}(P)|+|\tilde{\sigma}_{i,\ell}(P)-\bar{\sigma}_{i,\ell}(P)|\\&\leq C\left[\varepsilon\ln\left(\frac{\ell}{i+1}\right)+\frac{1}{i}\right]\leq C\varepsilon\left[\ln\left(\frac{\ell}{i+1}\right)+1\right].\end{aligned} \tag{6.19}$$

Let us now turn to H^1_ℓ and H^2_ℓ:

$$\begin{aligned} H^1_\ell(P) &= \frac{x_\ell + g_w^{-1}(y_\ell)}{x_{\ell-1} + g_w^{-1}(y_\ell)} \frac{y_\ell - y_{\ell-1}}{y_{\ell-1} + g_w(x_{\ell-1})} h_w(x_{\ell-1}) \\ &\sim 1 \cdot \frac{2(p-1)}{1+a^{p-1}} \frac{1}{\ell} \cdot a^{p-2} \ell^{2(p-2)} w \\ &=: \bar{H}^1_\ell(P), \qquad \text{(Asymp. 4)} \\ H^2_\ell(P) &= \frac{x_\ell - x_{\ell-1}}{x_{\ell-1} + g_w^{-1}(y_\ell)} h_w(g_w^{-1}(y_\ell)) \\ &\sim \frac{2a}{a+1} \frac{1}{\ell} \cdot \ell^{2(p-2)} w \\ &=: \bar{H}^2_\ell(P). \qquad \text{(Asymp. 5)} \end{aligned}$$

Hence

$$|H^1_\ell(P) - \bar{H}^1_\ell(P)| + |H^2_\ell(P) - \bar{H}^2_\ell(P)| \le C\varepsilon \ell^{2(p-2)-1} \tag{6.20}$$

for all $\ell \ge i+1$ provided $i \ge I$. Define also $\bar{H}_\ell := \bar{H}^1_\ell + \bar{H}^2_\ell$. We now turn to $\rho^1_{i,\ell}$ and $\rho^2_{i,\ell}$. Similarly to the previous terms, we can compute the asymptotic behavior of each of the summands in the definition of $\rho^1_{i,\ell}, \rho^2_{i,\ell}$, given in (6.10), (6.11). We introduce the functions

$$\begin{aligned} \bar{\rho}^1_{i,\ell} &:= \frac{2}{(1+a)\ell} - \frac{2a((\ell-1)^2 - i^2)}{(a+1)^2 \ell^3} + \frac{(p-2)((\ell-1)^2 - i^2)}{a\ell^2} \\ &\quad - \frac{(p-1)a^{p-2}}{(1+a^{p-1})} \frac{(\ell-1)^2 - i^2}{\ell^2}, \\ \bar{\rho}^2_{i,\ell} &:= \frac{1}{a} - \frac{1}{1+a} \frac{(\ell-1)^2 - i^2}{\ell^2}, \end{aligned}$$

in which every summand is given by the asymptotic behavior of the corresponding term of $\rho^1_{i,\ell}$ and $\rho^2_{i,\ell}$. Then choosing I possibly larger, if $i \ge I$ then for all $P \in \overline{Q(c)}$,

$$|\rho^1_{i,\ell}(P) - \bar{\rho}^1_{i,\ell}(P)| + |\rho^2_{i,\ell}(P) - \bar{\rho}^2_{i,\ell}(P)| \le C\varepsilon \left(1 + \frac{(\ell-1)^2 - i^2}{\ell^2}\right) \le C\varepsilon. \tag{6.21}$$

6.5. *Asymptotic behavior of S_i/S_ℓ*

We claim that for all $p \in (1,\infty) \setminus \{2\}$, if c satisfies (5.4)–(5.5), then for all $\varepsilon > 0$ there exists $I = I(\varepsilon, c)$ such that

$$\sup_{P \in \overline{Q(c)}} \left| \frac{S_i(P)}{S_\ell(P)} \left(\frac{\ell}{i+1}\right)^{2G_p(a)} - 1 \right| \le \varepsilon \quad \textit{for all } \ell - 1 \ge i \ge I. \tag{6.22}$$

For all $p \in (1,\infty)$, we let

$$\bar{S}_{i,\ell} = \bar{S}_{i,\ell}(a) := \left(\frac{i+1}{\ell}\right)^{2G_p(a)}. \tag{6.23}$$

Fix $\varepsilon > 0$. For $\ell \geq i+1$, we have

$$\begin{aligned}\frac{S_i}{S_\ell}(P) &= \prod_{k=i+1}^{\ell} \frac{x_{k-1}+g_w^{-1}(y_k)}{x_k+g_w^{-1}(y_k)}\,\frac{y_{k-1}+g_w(x_{k-1})}{y_k+g_w(x_{k-1})} \\ &= e^{\sum_{k=i+1}^{\ell}[\ln(x_{k-1}+g_w^{-1}(y_k))-\ln(x_k+g_w^{-1}(y_k))]} \\ &\quad\times e^{\sum_{k=i+1}^{\ell}[\ln(y_{k-1}+g_w(x_{k-1}))-\ln(y_k+g_w(x_{k-1}))]}.\end{aligned} \tag{6.24}$$

Using a first order Taylor expansion and the concavity of the logarithm, we see that

$$\frac{x_{k-1}-x_k}{x_{k-1}+g_w^{-1}(y_k)} \leq \ln(x_{k-1}+g_w^{-1}(y_k)) - \ln(x_k+g_w^{-1}(y_k)) \leq \frac{x_{k-1}-x_k}{x_k+g_w^{-1}(y_k)}. \tag{6.25}$$

We claim that the first and last terms on (6.25) are both $-\frac{2a}{a+1}\frac{1}{k}$ up to lower order corrections, namely

$$\left|\frac{x_{k-1}-x_k}{x_{k-1}+g_w^{-1}(y_k)} + \frac{2a}{a+1}\frac{1}{k}\right| + \left|\frac{x_{k-1}-x_k}{x_{k-1}+g_w^{-1}(y_k)} + \frac{2a}{a+1}\frac{1}{k}\right| \leq \frac{C}{k^{1+\min\{1,2(p-1)\}}}. \tag{6.26}$$

We estimate the first term on the left-hand side of (6.26); the second term can be treated similarly or by comparing it with the first term. First, by the implicit function theorem, the function $w \mapsto g_w^{-1}(y)$ is C^1 and

$$\partial_w(g_w^{-1}(y)) = -(p-2)\frac{w g_w^{-1}(y)}{w^2+(p-1)(g_w^{-1}(y))^2}, \quad \forall w, y \in \mathbb{R}.$$

Therefore, for all $P \in \overline{Q(c)}$, by (6.14),

$$|g_w^{-1}(y_k) - g_0^{-1}(y_k)| \leq C/k^2, \quad \forall k \geq 1. \tag{6.27}$$

Since $g_0^{-1}(y) = y^{1/(p-1)}$ for all $y \in \mathbb{R}^+$, as follows from the definition of $g_w(x)$, we estimate

$$\begin{aligned} g_0^{-1}(y_k) - k^2 &= (k^{2(p-1)}+y_0)^{1/(p-1)} - k^{2(p-1)^{1/(p-1)}} \\ &\leq \begin{cases} y_0^{1/(p-1)} & \text{if } p > 2, \\ Ck^{2(2-p)} & \text{if } p < 2. \end{cases}\end{aligned} \tag{6.28}$$

Hence, for all $i+1 \leq k \leq \ell$ we write

$$\begin{aligned} &\left|\frac{x_{k-1}-x_k}{x_{k-1}+g_w^{-1}(y_k)} + \frac{2a}{a+1}\frac{1}{k}\right| \\ &\quad= a\left|\frac{(-2k+1)(a+1)k + 2a(k-1)^2 + 2x_0 + 2g_w^{-1}(y_k)}{(x_{k-1}+g_w^{-1}(y_k))(a+1)k}\right| \\ &\quad\leq a\frac{|-3ak+k+2a+2x_0| + 2|g_w^{-1}(y_k)-g_0^{-1}(y_k)| + 2|g_0^{-1}(y_k)-k^2|}{(x_{k-1}+g_w^{-1}(y_k))(a+1)k} \\ &\quad\leq C\frac{k+k^{\max\{0,2(2-p)\}}}{k^3} \leq \frac{C}{k^{1+\min\{1,2(p-1)\}}}.\end{aligned}$$

This proves the estimate of the first term in our claim (6.26). With (6.26) at hand we obtain

$$\begin{aligned}
&\left|\sum_{k=i+1}^{\ell}[\ln(x_{k-1}+g_w^{-1}(y_k))-\ln(x_k+g_w^{-1}(y_k))]+\frac{2a}{a+1}\ln\left(\frac{i+1}{\ell}\right)\right| \\
&\qquad \le \left|\sum_{k=i+1}^{\ell}\left(\ln(x_{k-1}+g_w^{-1}(y_k))-\ln(x_k+g_w^{-1}(y_k))+\frac{2a}{(a+1)k}\right)\right| \\
&\qquad\quad + \left|-\sum_{k=i+1}^{\ell}\frac{2a}{(a+1)k}+\frac{2a}{a+1}\ln\left(\frac{i+1}{\ell}\right)\right| \\
&\qquad \overset{(6.26),(6.16)}{\le} \sum_{k=i+1}^{\ell}\frac{C}{k^{1+\min\{1,2(p-1)\}}}+\frac{C}{i+1} \overset{(6.17)}{\le} \frac{C}{(i+1)^{\min\{1,2(p-1)\}}}. \qquad (6.29)
\end{aligned}$$

Analogously, we can prove that

$$\begin{aligned}
&\left|\sum_{k=i+1}^{\ell}[\ln(y_{k-1}+g_w(x_{k-1}))-\ln(y_k+g_w(x_{k-1}))]+(p-1)\frac{2}{a^{p-1}+1}\ln\left(\frac{i+1}{\ell}\right)\right| \\
&\qquad\qquad \le \frac{C}{(i+1)^{\min\{1,2(p-1)\}}}. \qquad (6.30)
\end{aligned}$$

Estimates (6.29)–(6.30) combined with (6.24) and the definition of $G_p(a)$ (see (5.2)), prove (6.22).

6.6. *Estimates of $\delta_i z_i$*

We now use our previous asymptotic estimates to bound $\delta_i z_i$ in (6.13). We will observe that the closeness estimates (6.19)–(6.22) of $\sigma_{i,\ell}$ and $\bar\sigma_{i,\ell}$, S_i/S_ℓ and $\bar S_{i,\ell}$, H_ℓ^i and $\bar H_\ell^i$, $\rho_{i,\ell}^j$ and $\bar\rho_{i,\ell}^j$ enable us to neglect the contributions of $\sigma_{i,\ell}-\bar\sigma_{i,\ell}$, $\frac{S_i}{S_\ell}-\bar S_{i,\ell}$, $H_\ell^i-\bar H_\ell^i$, $\rho_{i,\ell}^j-\bar\rho_{i,\ell}^j$ in the computation of $\delta_i z_i$ in (6.13). Indeed, these estimates imply that the growth of $\sigma_{i,\ell}$ is the same as the growth of $\bar\sigma_{i,\ell}$, and similarly for the other quantities. We have

$$\begin{aligned}
&[\sigma_{i,\ell}H_\ell+H_\ell^1\rho_{i,\ell}^1+H_\ell^2\rho_{i,\ell}^2]-[\bar\sigma_{i,\ell}\bar H_\ell+\bar H_\ell^1\bar\rho_{i,\ell}^1+\bar H_\ell^2\bar\rho_{i,\ell}^2] \\
&\qquad =(\sigma_{i,\ell}-\bar\sigma_{i,\ell})H_\ell+H_\ell^1(\rho_{i,\ell}^1-\bar\rho_{i,\ell}^1)+H_\ell^2(\rho_{i,\ell}^2-\bar\rho_{i,\ell}^2)+\bar\sigma_{i,\ell}(H_\ell-\bar H_\ell) \\
&\qquad\quad +(H_\ell^1-\bar H_\ell^1)\bar\rho_{i,\ell}^1+(H_\ell^2-\bar H_\ell^2)\bar\rho_{i,\ell}^2.
\end{aligned}$$

Thus, by the triangle inequality, (6.19)–(6.21) yield, uniformly in $P\in\overline{Q(c)}$,

$$\begin{aligned}
&|[\sigma_{i,\ell}H_\ell+H_\ell^1\rho_{i,\ell}^1+H_\ell^2\rho_{i,\ell}^2]-[\bar\sigma_{i,\ell}\bar H_\ell+\bar H_\ell^1\bar\rho_{i,\ell}^1+\bar H_\ell^2\bar\rho_{i,\ell}^2]| \\
&\qquad\qquad \le C\varepsilon\left(1+\ln\left(\frac{\ell}{i+1}\right)\right)\ell^{2(p-2)-1}. \qquad (6.31)
\end{aligned}$$

We notice moreover, using the definitions of $\bar{\sigma}_{i,\ell}$, $\bar{H}_\ell$ and $\bar{\rho}_{i,\ell}$, that

$$\begin{aligned} |\bar{\sigma}_{i,\ell}\bar{H}_\ell + \bar{H}^1_\ell\bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell\bar{\rho}^2_{i,\ell}| &\le |\bar{\sigma}_{i,\ell}\bar{H}_\ell| + |\bar{H}^1_\ell\bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell\bar{\rho}^2_{i,\ell}| \\ &\le C\left(1+\ln\left(\frac{\ell}{i+1}\right)\right)\ell^{2(p-2)-1}. \end{aligned} \tag{6.32}$$

Similarly, exploiting (6.22) and (6.31)–(6.32), we can estimate, uniformly in $P \in \overline{Q(c)}$,

$$\begin{aligned} \left|\frac{S_i}{S_\ell}[\sigma_{i,\ell}H_\ell + H^1_\ell\rho^1_{i,\ell} + H^2_\ell\rho^2_{i,\ell}] - \bar{S}_{i,\ell}[\bar{\sigma}_{i,\ell}\bar{H}_\ell + \bar{H}^1_\ell\bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell\bar{\rho}^2_{i,\ell}]\right| \\ \le C\varepsilon\bar{S}_{i,\ell}\left(1+\ln\left(\frac{\ell}{i+1}\right)\right)\ell^{2(p-2)-1}. \end{aligned} \tag{6.33}$$

We now need to divide into the cases $p > 2$ and $1 < p < 2$. We will show that for all c sufficiently small if $p > 2$, and c sufficiently large if $p < 2$, and for a possibly larger I,

$$\left(\sum_{\ell=i+1}^{\infty}\frac{S_i}{S_\ell}(\sigma_{i,\ell}H_\ell + H^1_\ell\rho^1_{i,\ell} + H^2_\ell\rho^2_{i,\ell})\right)(P) \ge C' i^{2(p-2)-1} \tag{6.34}$$

for all $i+1 \ge I$, for a positive constant $C' > 0$. Before dividing into the two cases, we use integral comparison to find that, for all $i \ge 2$ and $a \in [c, 2c]$, with c chosen such that (5.4)–(5.5) hold,

$$\left|\sum_{\ell=i+1}^{\infty}\bar{S}_{i,\ell}\ln\left(\frac{\ell}{i+1}\right)\ell^{2(p-2)-1} - \frac{(i+1)^{2(p-2)}}{4(G_p(a)-p+2)^2}\right| \le 2\ln(i+1)(i+1)^{2(p-2)-1}, \tag{6.35}$$

$$\begin{aligned} \Bigg|\sum_{\ell=i+1}^{\infty}\bar{S}_{i,\ell}\left(\frac{i^2}{(i+1)^2}-\frac{i^2}{\ell^2}\right)\ell^{2(p-2)-1} \\ -\left[\frac{1}{2(2-p)+2G_p(a)} - \frac{1}{2(2-p)+2G_p(a)+2}\right](i+1)^{2(p-2)}\Bigg| \\ \le 10(i+1)^{2(p-2)-1}, \end{aligned} \tag{6.36}$$

$$\begin{aligned} \Bigg|\sum_{\ell=i+1}^{\infty}\bar{S}_{i,\ell}\ell^{2(p-2)-1}\frac{(\ell-1)^2-i^2}{\ell^2} \\ -\left[\frac{1}{2(2-p)+2G_p(a)} - \frac{1}{2(2-p)+2G_p(a)+2}\right](i+1)^{2(p-2)}\Bigg| \\ \le 10(i+1)^{2(p-2)-1}. \end{aligned} \tag{6.37}$$

The case $p > 2$: According to (5.3), (6.17), (6.33) and (6.35), to show (6.34) it is sufficient to prove

$$\sum_{\ell=i+1}^{\infty}\bar{S}_{i,\ell}(\bar{\sigma}_{i,\ell}\bar{H}_\ell + \bar{H}^1_\ell\bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell\bar{\rho}^2_{i,\ell})(P) \ge C' i^{2(p-2)}. \tag{6.38}$$

We see that there exists a constant $L > 0$ (which may change from line to line), independent of c once we choose for instance $c < 1$, such that for all $\ell \geq i + 1 \geq 2$ and $P \in Q(c)$,

$$|\bar{H}_\ell \bar{\sigma}_{i,\ell}|(P) \leq c^{\min\{1,p-2\}} L\left(1 + \ln\left(\frac{\ell}{i+1}\right)\right)\ell^{2(p-2)-1}. \tag{6.39}$$

On the other hand, if $m = \min_{\overline{Q(c)}} \frac{2w}{a+1} > 0$, then

$$\begin{aligned}(\bar{H}^1_\ell \bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell \bar{\rho}^2_{i,\ell})(P) &\geq \frac{2w}{a+1}\ell^{2(p-2)-1} - Lc^{\min\{1,p-1,2(p-2)\}}\ell^{2(p-2)-1} \\ &\geq (m - Lc^{\min\{1,2(p-2)\}})\ell^{2(p-2)-1}. \end{aligned} \tag{6.40}$$

Combining (6.17), (6.35), (6.39), (6.40) with (5.3) we can estimate, for all $P \in \overline{Q(c)}$,

$$\begin{aligned}\sum_{\ell=i+1}^{\infty} &\bar{S}_{i,\ell}(\bar{\sigma}_{i,\ell}\bar{H}_\ell + \bar{H}^1_\ell \bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell \bar{\rho}^2_{i,\ell})(P) \\ &\geq (L_1 m - L_2(c^{\min\{1,2(p-2)\}} + c^{\min\{1,p-2\}}))(i+1)^{2(p-2)}\end{aligned}$$

for some constants $L_1(p), L_2(p) > 0$. If c and ε of (6.33) are chosen sufficiently small, the last inequality implies (6.38) and hence (6.34).

The case $1 < p < 2$*:* We wish to show that for all c sufficiently large there exists a constant $C' = C'(c) > 0$ such that for all $P \in \overline{Q(c)}$ and all $i \geq I = I(c)$,

$$\sum_{\ell=i+1}^{\infty} a\bar{S}_{i,\ell}(\bar{\sigma}_{i,\ell}\bar{H}_\ell + \bar{H}^1_\ell \bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell \bar{\rho}^2_{i,\ell})(P) \geq C' i^{2(p-2)}. \tag{6.41}$$

This would show, since $a \in (c, 2c)$, that

$$\sum_{\ell=i+1}^{\infty} \bar{S}_{i,\ell}(\bar{\sigma}_{i,\ell}\bar{H}_\ell + \bar{H}^1_\ell \bar{\rho}^1_{i,\ell} + \bar{H}^2_\ell \bar{\rho}^2_{i,\ell})(P) \geq \frac{C'}{2c} i^{2(p-2)}.$$

As above, by (5.3), (6.33) and (6.35), the latter would imply (6.34), once ε of (6.33) is chosen small enough in terms of $\frac{C'}{2c}$. To show (6.41), we denote by $g_k(a)$, for $k \in \mathbb{N}$, continuous functions $g_k : \mathbb{R}^+ \to \mathbb{R}$ such that for all a sufficiently large,

$$|g_k(a)| \leq \frac{L_k}{a^{p-1}}$$

for some constant $L_k > 0$. With this notation, we rewrite for all $a \in \mathbb{R}^+$, $\ell \geq i + 1 \geq 1$,

$$a\bar{\sigma}_{i,\ell} = \frac{a^2}{(1+a)^2}\left[\frac{i^2}{\ell^2} - \frac{i^2}{(i+1)^2}\right] + g_1(a)\ln\left(\frac{\ell}{i+1}\right) + g_2(a)\left[\frac{i^2}{\ell^2} - \frac{i^2}{(i+1)^2}\right],$$

and

$$\bar{H}_\ell = \frac{2aw}{a+1}\ell^{2(p-2)-1} + g_3(a)\ell^{2(p-2)-1}.$$

Therefore,

$$a\bar{\sigma}_{i,\ell}\bar{H}_\ell = \frac{2a^3 w}{(1+a)^3}\ell^{2(p-2)-1}\left[\frac{i^2}{\ell^2} - \frac{i^2}{(i+1)^2}\right] + \left(g_5(a)\ln\left(\frac{\ell}{i+1}\right) + g_6(a)\left[\frac{i^2}{\ell^2} - \frac{i^2}{(i+1)^2}\right]\right)\ell^{2(p-2)-1}. \qquad (6.42)$$

The term $a(\bar{H}_\ell^1\bar{\rho}_{i,\ell}^1 + \bar{H}_\ell^2\bar{\rho}_{i,\ell}^2)$ is more complicated, but it admits a similar representation:

$$a(\bar{H}_\ell^1\bar{\rho}_{i,\ell}^1 + \bar{H}_\ell^2\bar{\rho}_{i,\ell}^2) = \frac{2aw}{a+1}\ell^{2(p-2)-1} - \frac{2a^2 w}{(a+1)^2}\ell^{2(p-2)-1}\frac{(\ell-1)^2 - i^2}{\ell^2} + g_6(a)\ell^{2(p-2)-2} + (g_7(a) + \ell g_8(a))\ell^{2(p-2)-2}\frac{(\ell-1)^2 - i^2}{\ell^2}. \qquad (6.43)$$

Now we can exploit (6.17), (6.42), (6.43), (6.36), (6.37), and the definition of $\bar{S}_{i,\ell}$ in (6.23) to write

$$\sum_{\ell=i+1}^{\infty} a\bar{S}_{i,\ell}\bar{\sigma}_{i,\ell}\bar{H}_\ell(P) = f_1(a)w(i+1)^{2(p-2)} + R_{1,i}(a,w)$$
$$= \left[\frac{1}{2(2-p)+2+2G_p(a)} - \frac{1}{2(2-p)+2G_p(a)}\right]\frac{2a^3 w}{(1+a)^3}(i+1)^{2(p-2)} + R_{1,i}(a,w),$$

and

$$\sum_{\ell=i+1}^{\infty} a\bar{S}_{i,\ell}(\bar{H}_\ell^1\bar{\rho}_{i,\ell}^1 + \bar{H}_\ell^2\bar{\rho}_{i,\ell}^2)(P)$$
$$= f_2(a)w(i+1)^{2(p-2)} + f_3(a)w(i+1)^{2(p-2)} + R_{2,i}(a,w)$$
$$= \left[\frac{1}{2(2-p)+2G_p(a)}\right]\frac{2aw(i+1)^{2(p-2)}}{a+1} - \left[\frac{1}{2(2-p)+2G_p(a)} - \frac{1}{2(2-p)+2+2G_p(a)}\right]\frac{2a^2 w(i+1)^{2(p-2)}}{(a+1)^2} + R_{2,i}(a,w).$$

Here, $R_{j,i}(a,w)$ for all $j = 1,2$ and $i \in \mathbb{N}$ are continuous functions with the property that

$$|R_{1,i}(a,w)| + |R_{2,i}(a,w)| \le \frac{k(i+1)^{2(p-2)}}{a^{p-1}},$$

for all $i \in \mathbb{N}$, $w \in (\frac{3}{4}, \frac{5}{4})$ and $a > 0$ sufficiently large, and for a positive constant k depending solely on p once c is chosen large enough. Notice that $R_{3,i}(a,w) := R_{a,i}(a,w) + R_{2,i}(a,w)$ enjoys the same bounds. As $w \in (\frac{3}{4}, \frac{5}{4})$, in order to show (6.41), it is sufficient to prove that

$$\liminf_{a\to\infty}[f_1(a) + f_2(a) + f_3(a)] > 0.$$

A direct computation that exploits $\lim_{a\to\infty} G_p(a) = 1$ (see (5.2)) shows that

$$\lim_{a\to\infty}[f_1(a) + f_2(a) + f_3(a)] = \frac{2(2-p)}{(2(2-p)+4)(2(2-p)+2)},$$

which is positive for all $p < 2$. This concludes the proof of (6.41).

We collect the results we just obtained in the following proposition.

Proposition 6.2. *Let $p \in (1,\infty)\setminus\{2\}$ and S_i, H_ℓ, $\rho^j_{i,\ell}$, $\sigma_{i,\ell}$ be defined in (6.1), (6.2), (6.10), (6.11), (6.12) respectively. Recall that $Q(c) = (c, 2c) \times (\frac{3}{4}, \frac{5}{4}) \times (\frac{3}{4}, \frac{5}{4}) \times (\frac{3}{4}, \frac{5}{4})$. Then for all $c > 0$ sufficiently small if $p > 2$ and $c > 0$ sufficiently large if $p < 2$, there exists $I = I(c) \in \mathbb{N}$ such that for all $i \geq I$:*

(a) *The convergence of the series $\sum_{\ell=1}^{\infty}\frac{H_\ell(P)}{S_\ell(P)}$ is uniform in $\overline{Q(c)}$. In particular,*

$$P \mapsto z_i(P) = S_i(P)\sum_{\ell=i+1}^{\infty}\frac{H_\ell(P)}{S_\ell(P)}$$

is well-defined for all $P \in \overline{Q(c)}$ and continuous on $\overline{Q(c)}$.

(b) $\sup_{P\in\overline{Q(c)}}|z_i(P)| \leq C i^{2(p-2)}$ *for all $i \geq 1$, and for some positive $C > 0$;*

(c) *The convergence of the series $\sum_{\ell=1}^{\infty}\frac{1}{S_\ell}[H_\ell\sigma_{i,\ell} + +H^1_\ell\rho^1_{i,\ell} + H^2_\ell\rho^2_{i,\ell}]$ is uniform in $Q(c)$. In particular,*

$$(\partial_a z_i - i^2\partial_{x_0}z_i)(P)$$

exists at all points $P \in Q(c)$, is continuous, and is represented precisely by

$$(\partial_a z_i - i^2\partial_{x_0}z_i)(P) = \sum_{\ell=1}^{\infty}\frac{S_i}{S_\ell}[H_\ell\sigma_{i,\ell} + H^1_\ell\rho^1_{i,\ell} + H^2_\ell\rho^2_{i,\ell}].$$

(d) *For all $P \in Q(c)$,*

$$(\partial_a z_i - i^2\partial_{x_0}z_i)(P) > 0.$$

In particular, the assumptions of Lemma 6.1 hold true and the mappings $\Phi^1_{i,t}$, $\Phi^2_{i,t}$, A_i are open for all $t \in [0,1)$ and $i \geq I$.

Proof. We see from (6.20) that

$$\max_{P\in\overline{Q(c)}}|H_\ell(P)| \leq C\ell^{2(p-2)-1}. \tag{6.44}$$

Having chosen c such that (5.3) hold, for all $P = (a, x_0, y_0, w) \in \overline{Q(c)}$ we estimate

$$\begin{aligned}\left|\frac{S_i(P)}{S_\ell(P)}H_\ell(P)\right| &\overset{(6.22),(6.44)}{\leq} C\left(\frac{i+1}{\ell}\right)^{2G_p(a)}\ell^{2(p-2)-1}\\ &\overset{(5.3)}{\leq} C\left(\frac{i+1}{\ell}\right)^{2\max\{1,p-1\}}\ell^{2(p-2)-1}\end{aligned}$$

for all $\ell \geq i+1$. Through (6.4) and (6.17), we conclude that z_i is well-defined and continuous for all $i \geq 0$, which is (a), and that it enjoys property (b). Finally, (c) readily follows from (6.32)–(6.33), (6.22) and (6.17), (6.35), and (d) is the content of (6.34). ∎

In what follows, c is chosen so that (5.4)–(5.5) and Proposition 6.2 hold, and will be considered a fixed parameter.

7. Some further properties of $\Phi^k_{i,t}$ and A_i

As in the previous section, the domain of the maps we will consider is $Q(c)$, where $c > 0$ is fixed by (5.4)–(5.5) and Proposition 6.2. We also let $I = I(c)$ be the index for which the conclusion of the aforementioned proposition holds. This index will be made (possibly) larger in the next lemma, but will still be denoted by I. Denote, as before, points of $Q(c)$ by P.

Lemma 7.1. *There exists* $0 < t_0 = t_0(c) < 1$ *such that if* $t \in [t_0, 1]$, *the sets*

$$\Phi^1_{i_1,t}(\overline{Q(c)}), \quad \Phi^1_{i_2,t}(\overline{Q(c)}), \quad A_{i_3}(\overline{Q(c)})$$

are pairwise disjoint for all $i_1, i_2, i_3 \geq I$, *where* $I = I(c)$ *may be larger than the one of Proposition* 6.2.

Proof. We first show the following auxiliary statements. There exist $k > 0$, $I \in \mathbb{N}$ and $0 < t_0 < 1$ such that for all $i \geq I$, $t \in [t_0, 1]$ and $P = (a, x_0, y_0, w) \in \overline{Q(c)}$,

$$\Phi^1_{i,t}(x_0, y_0, z_0, w) \subset \left\{\begin{pmatrix} x & w \\ z & y \end{pmatrix} \in \mathbb{R}^{2\times 2} : x \geq k,\ y \leq -k\right\}, \tag{7.1}$$

$$\Phi^2_{i,t}(x_0, y_0, z_0, w) \subset \left\{\begin{pmatrix} x & w \\ z & y \end{pmatrix} \in \mathbb{R}^{2\times 2} : x \leq -k,\ y \geq k\right\}, \tag{7.2}$$

$$A_i(a, x_0, y_0, w) \subset \left\{\begin{pmatrix} x & w \\ z & y \end{pmatrix} \in \mathbb{R}^{2\times 2} : x \geq k,\ y \geq k\right\}. \tag{7.3}$$

Proof of (7.1). We only need to show that $y_{i-1} - t(y_{i-1} + g_w(x_{i-1}))$ is (uniformly) negative for all $P \in \overline{Q(c)}$ if t_0 is sufficiently close to 1 and $i \geq I$. Indeed, the fact that $P \mapsto x_{i-1}(P)$ is (uniformly) positive is an immediate consequence of the definition. We have

$$\begin{aligned} &y_{i-1} - t(y_{i-1} + g_w(x_{i-1})) \\ &\quad = (1-t)\big((i-1)^{2(p-1)} + y_0\big) - t|(a(i-1)^2 + x_0, w)|^{p-2}(a(i-1)^2 + x_0). \end{aligned}$$

If we divide the above by $i^{2(p-1)}$ and let $i \to \infty$, we obtain, uniformly in $P \in \overline{Q(c)}$,

$$(1-t) - a^{p-1}t,$$

and we can estimate

$$(1-t) - a^{p-1}t \leq (1-t) - c^{p-1}t.$$

Therefore, if t_0 is sufficiently close to 1, we see that (7.1) holds for I large.

Proof of (7.2). Analogously to the proof of (7.1), we find that y_i is always positive, and hence we shall only prove that $x_{i-1} - t(g_w^{-1}(y_i) + x_{i-1})$ is (uniformly) negative for all $P \in \overline{Q(c)}$ if t_0 is sufficiently close to 1 and $i \geq I$. To do so, we write

$$x_{i-1} - t(g_w^{-1}(y_i) - x_{i-1}) = (1-t)x_{i-1} - t g_w^{-1}(y_i).$$

Recalling (6.14), we divide the above by i^2 and we let $i \to \infty$ to obtain, uniformly in $P \in \overline{Q(c)}$,

$$(1-t)a - t \leq (1-t)2c - t.$$

Therefore, if t_0 is sufficiently close to 1, (7.2) holds for I large and uniformly in $P \in \overline{Q(c)}$.

Proof of (7.3). This is immediate, since $\{x_i\}, \{y_i\}$ are uniformly positive by their definition.

Clearly, (7.1)–(7.3) imply that

$$\begin{aligned}
&\Phi^1_{i,t}(\overline{Q(c)}) \cap \Phi^2_{j,t}(\overline{Q(c)}) = \emptyset, \\
&\Phi^1_{i,t}(\overline{Q(c)}) \cap A_j(\overline{Q(c)}) = \emptyset, \quad \Phi^2_{i,t}(\overline{Q(c)}) \cap A_j(\overline{Q(c)}) = \emptyset
\end{aligned}$$

provided $i, j \geq I$, $t \in [t_0, 1]$, I is large enough and t_0 is close enough to 1. To conclude the proof of the lemma, we now claim that if I is sufficiently large, then for all $i > j > I$,

$$x_i(x_0) > x_j(x_0') + 1, \quad y_i(y_0) > y_j(y_0') + 1, \quad \forall i > j \geq I,\ \forall x_0, y_0, x_0', y_0' \in \left(\tfrac{3}{4}, \tfrac{5}{4}\right). \tag{7.4}$$

Using the definitions of $\Phi^k_{i,t}$, it is easy to see that (7.4) implies that for all $k = 1, 2$, $i > j \geq I$ and $t \in [t_0, 1]$,

$$\mathrm{d}(\Phi^k_{i,t}(\overline{Q(c)}), \Phi^k_{j,t}(\overline{Q(c)})) \geq 1, \quad \mathrm{d}(A_i(\overline{Q(c)}), A_j(\overline{Q(c)})) \geq 1, \tag{7.5}$$

which yields the claim.

We only need to show (7.4). We can estimate

$$ai^2 - aj^2 \geq 2aj(i-j) \geq 2aj \geq 2aI, \quad \forall i > j \geq I.$$

Thus, recalling that $x_0, x_0' \in (\frac{3}{4}, \frac{5}{4})$,

$$x_i(x_0) - x_j(x_0') = a(i^2 - j^2) + x_0 - x_0' \geq 2aI - 1/2 \geq 2cI - 1/2.$$

If $I = I(c) > 0$ is sufficiently large, we see that the first part of (7.4) holds. To show the second part, i.e. the analogous estimate for y_i, we repeat the same proof with small modifications. ∎

We end this section with one last technical lemma. First, define, for $k = 1, 2$, $i, q \geq 1$ and $t_q := 1 - \frac{1-t_0}{2^q}$,

$$U^k_{i,q} := \Phi^k_{i,t_q}(Q(c)) \quad \text{and} \quad U^3_i := A_i(Q(c)).$$

The reason to update $t_q \to 1$ is that, as $q \to \infty$, we wish the sets $U^k_{i,q}$ to converge to subsets of K_p in a sense that is made rigorous in the next lemma. Recall that $I \in \mathbb{N}$ is a sufficiently large index so that Proposition 6.2 holds together with Lemma 7.1. Finally, for $n \geq 1$ set

$$V_n := \bigcup_{i=I}^{I+n-1} U^1_{i,I+n-1} \cup \bigcup_{i=I}^{I+n-1} U^2_{i,I+n-1} \cup U^3_{I+n}. \tag{7.6}$$

Notice that by Lemma 6.1 and Proposition 6.2, each of the above sets is open and, by Lemma 7.1, the union defining V_n is disjoint, for all $n \geq I + 1$. We will still need to update the value of I in some of the next results. We modify the definition of V_n accordingly.

Lemma 7.2. *Let $Z_n \in V_n$ for all $n \geq 1$. Suppose that a subsequence $\{Z_{n_j}\}_j$ converges to Z. Then*

$$Z \in \bigcup_{i=I}^{\infty} \Phi^1_{i,1}(\overline{Q(c)}) \cup \bigcup_{i=I}^{\infty} \Phi^2_{i,1}(\overline{Q(c)}) \overset{(5.15)}{=} \bigcup_{i=I}^{\infty} B_i(Q(c)) \cup \bigcup_{i=I}^{\infty} D_i(Q(c)) \overset{(5.7),(5.10)}{\subset} K_p. \tag{7.7}$$

Proof. Denote by $B_R(0) \subset \mathbb{R}^{2\times 2}$ the ball of radius R centered at $0 \in \mathbb{R}^{2\times 2}$. Since

$$\inf_{P \in \overline{Q(c)}} x_i(P) \to \infty \quad \text{and} \quad \inf_{P \in \overline{Q(c)}} y_i(P) \to \infty$$

as $i \to \infty$, we see that given any $R > 0$,

$$A_i(Q(c)) \cap B_R(0) = \emptyset, \quad \Phi^k_{i,t}(Q(c)) \cap B_R(0) = \emptyset \tag{7.8}$$

for all $t \geq t_0$, $k = 1, 2$ and $i \geq \bar{I} = \bar{I}(R) \geq 1$. Since $\{Z_{n_j}\}_j$ is convergent, it is bounded, and from $Z_{n_j} \in V_{n_j}$ for all j, and (7.8), we infer

$$Z_{n_j} \in \bigcup_{i=I}^{N} U^1_{i,n_j} \cup \bigcup_{i=I}^{N} U^2_{i,n_j} \subset V_{n_j}, \quad \forall j \geq J,$$

for some large J, N, the latter independent of j. Now, for either $k = 1$ or $k = 2$, we can select a (nonrelabeled) subsequence of $\{Z_{n_j}\}$ with the property that, for some $i_0 \in \{I, \dots, N\}$,

$$Z_{n_j} \in U^k_{i_0,n_j}.$$

Let $k = 1$ for simplicity; otherwise the proof is analogous. By definition of $U^k_{i_0,n_j}$, we find a sequence $P_j = (a_j, x^j_0, y^j_0, w_j) \in Q(c)$ such that, for all $j \in \mathbb{N}$,

$$Z_{n_j} = \Phi^1_{i_0,t_{n_j}}(P_j).$$

Up to passing to a subsequence, the limit $\lim_j P_j = P \in \overline{Q(c)}$ exists. Now observe that the map $([0,1] \times \overline{Q}(c)) \ni (t, a, x_0, y_0, w) \mapsto \Phi^k_{i_0,t}(x_0, y_0, z_0, w)$ is continuous for $k = 1, 2$ on $\overline{Q(c)}$. This can be inferred from the definitions (5.16)–(5.17), the continuity of z_i, proved in Propositions 6.2, and (5.8). The continuity implies that $Z = \Phi^1_{i_0,1}(P)$ and concludes the proof. ∎

8. Properties of the laminates

The aim of this section is to build the laminates that will be used in the proof of the inductive proposition of Section 9. In what follows, c is a fixed parameter that has been chosen in (5.4)–(5.5) and Proposition 6.2. Moreover,

$$t_q = 1 - \frac{1-t_0}{2^q}$$

for all $q \in \mathbb{N}$, for t_0 of Lemma 7.1. Finally, we denote by $I = I(c) \in \mathbb{N}$ the index for which Proposition 6.2 and Lemma 7.1 hold, for all $i \geq I$. We will update this index a few more times in this section, and this will fix I for the last section.

Lemma 8.1. *Let $i, q \geq 1$. For all $M \in Q(c)$, the matrix $\Phi^1_{i,t_q}(M)$ belongs to the rank-one segment $[\Phi^1_{i,t_{q+1}}(M), A_i(M)]$:*

$$\Phi^1_{i,t_q}(M) = \mu_{1,q}\Phi^1_{i,t_{q+1}}(M) + \mu_{2,q}A_i(M)$$

with

$$\mu_{1,q} = \frac{t_q}{t_{q+1}} \quad \textit{and} \quad \mu_{2,q} = 1 - \frac{t_q}{t_{q+1}}.$$

Proof. By the definitions, we immediately see that $\det(\Phi^1_{i,t_{q+1}}(M) - A_i(M)) = 0$. Therefore $[\Phi^1_{i,t_{q+1}}(M), A_i(M)]$ is a rank-one segment. Using (5.13), we compute

$$\begin{aligned}
&\mu_{1,q}\Phi^1_{i,t_{q+1}}(M) + \mu_{2,q}A_i(M) \\
&\quad = \frac{t_q}{t_{q+1}}(A_i(M) + t_{q+1}\lambda_{E_i(M)}(B_i(M) - E_i(M))) + \frac{t_{q+1}-t_q}{t_{q+1}}A_i(M) \\
&\quad = A_i(M) + t_q\lambda_{E_i(M)}(B_i(M) - E_i(M)) \\
&\quad = \Phi^1_{i,t_q}(M).
\end{aligned}$$

■

Lemma 8.2. *Let $i, q \geq 1$. For all $M \in Q(c)$, the matrix $\Phi^2_{i,t_q}(M)$ belongs to the rank-one segment $[\Phi^2_{i,t_{q+1}}(M), C_i(M)]$:*

$$\Phi^2_{i,t_q}(M) = \mu^1_{i,q}(M)\Phi^2_{i,t_{q+1}}(M) + \mu^2_{i,q}(M)C_i(M) \tag{8.1}$$

with

$$\mu^1_{i,q}(M) = \frac{\lambda_{D_i(M)} + t_q\lambda_{C_i(M)}}{\lambda_{D_i(M)} + t_{q+1}\lambda_{C_i(M)}} \quad \textit{and} \quad \mu^2_{i,q}(M) = \frac{(t_{q+1}-t_q)\lambda_{C_i(M)}}{\lambda_{D_i(M)} + t_{q+1}\lambda_{C_i(M)}}.$$

Furthermore, if I is sufficiently large, there exists a dimensional constant $C > 0$ such that for all $q \geq I$, $i \geq 1$ and $M \in Q(c)$,

$$1 - \frac{C}{2^q} \leq \mu^1_{i,q}(M) < 1 \quad \textit{and} \quad 0 < \mu^2_{i,q}(M) \leq \frac{C}{2^q}. \tag{8.2}$$

Proof. All the assertions can be checked by direct computation. First, using the definitions, we see that $\det(\Phi^2_{i,t_{q+1}}(M) - C_i(M)) = 0$. Therefore, $[\Phi^2_{i,t_{q+1}}(M), C_i(M)]$ is a rank-one segment. Moreover, using (5.14), one directly checks (8.1). We only need to show (8.2). The first estimate follows from the second and the fact that $\mu^1_{i,q} + \mu^2_{i,q} = 1$. Recall that $t_q = 1 - \frac{1-t_0}{2^q} < 1$. To show the second estimate of (8.2), we simply write

$$0 < \mu^2_{i,q}(M) = (t_{q+1} - t_q)\frac{\lambda_{C_i(M)}}{\lambda_{D_i(M)} + t_{q+1}\lambda_{C_i(M)}} \le \frac{t_{q+1} - t_q}{t_{q+1}} = \frac{1 - t_0}{t_{q+1}2^{q+1}}.$$

Since $\lim_{q\to\infty} t_q = 1$, we conclude the validity of (8.2). ∎

Proposition 8.3. *Let $q \ge j_2 - 2$, $j_2 > j_1 \ge 2$. Let moreover $P \in Q(c)$. Then $A_{j_1}(P)$ is the barycenter of a laminate of finite order $\nu_{j_1,j_2,q}(P)$ with*

$$\operatorname{spt}(\nu_{j_1,j_2,q}(P)) \subset \bigcup_{k=j_1}^{j_2-1} U^1_{k,q} \cup \bigcup_{k=j_1}^{j_2-1} U^2_{k,q} \cup U^3_{j_2}. \tag{8.3}$$

Furthermore,

$$0 < \nu_{j_1,j_2,q}(P)(U^3_{j_2}). \tag{8.4}$$

The proof of the previous proposition is inductive. We record the base case separately.

Lemma 8.4. *Let $i \ge I + 1$, $q \ge I$, and $P \in Q(c)$. Then $A_i(P)$ is the barycenter of the laminate of finite order*

$$\nu_{i,q}(P) := \lambda^1_{i,q}(P)\delta_{\Phi^1_{i,t_q}(P)} + \lambda^2_{i,q}(P)\delta_{\Phi^2_{i,t_q}(P)} + \lambda^3_{i,q}(P)\delta_{A_{i+1}(P)},$$

where

$$\lambda^1_{i,q}(P) = \frac{\lambda_{B_i(P)}}{\lambda_{B_i(P)} + t_q\lambda_{E_i(P)}}, \tag{8.5}$$

$$\lambda^2_{i,q}(P) = \frac{t_q\lambda_{E_i(P)}}{\lambda_{B_i(P)} + t_q\lambda_{E_i(P)}}\frac{\lambda_{D_i(P)}}{t_q\lambda_{C_i(P)} + \lambda_{D_i(P)}}, \tag{8.6}$$

$$\lambda^3_{i,q}(P) = \frac{t_q\lambda_{E_i(P)}}{\lambda_{B_i(P)} + t_q\lambda_{E_i(P)}}\frac{t_q\lambda_{C_i(P)}}{t_q\lambda_{C_i(P)} + \lambda_{D_i(P)}}. \tag{8.7}$$

Furthermore, independently of P, if I is large enough, there exist positive constants C and $k_1 < k_2$ such that

$$\frac{k_1}{i} \le \lambda^1_{i,q}(P) \le \frac{k_2}{i}, \tag{8.8}$$

$$\frac{k_1}{i} \le \lambda^2_{i,q}(P) \le \frac{k_2}{i}, \tag{8.9}$$

$$0 < \lambda^3_{i,q}(P) \le e^{C/i^\gamma} e^{-2\min\{G_p(c), G_p(2c)\}/i}, \tag{8.10}$$

where $\gamma = 1 + \min\{1, 2(p-1)\}$.

Proof. From $\delta_{A_i(P)}$ we pass to $\nu_{i,q}(P)$ via two consecutive elementary splittings, in the sense of Definition 4.2. These splittings were already mentioned when introducing the maps under consideration (see Remark 5.2). First, we split $A_i(P)$ into a rank-one segment with direction $B_i(P) - E_i(P)$. The segment has endpoints $E_i(P)$ and $\Phi^1_{i,t_q}(P)$. Then we split again $\delta_{E_i(P)}$ into a rank-one segment with direction $C_i(P) - D_i(P)$, with endpoints $C_i(P) = A_{i+1}(P)$ and $\Phi^2_{i,t_q}(P)$. Weights (8.5)–(8.7) are obtained via direct computation. We now turn to the proof of (8.8)–(8.10). First, we notice that as $t_q = 1 - \frac{1-t_0}{2^q}$, $t_q \to 1$ as $q \to \infty$. Therefore, if I is sufficiently large,

$$1/2 \le t_q \le 1.$$

Furthermore, as $\lambda_{E_i(P)} + \lambda_{B_i(P)} = 1$, it follows that

$$1 \ge \lambda_{B_i(P)} + t_q \lambda_{E_i(P)} \ge t_q \ge 1/2. \tag{8.11}$$

Analogously,

$$1 \ge \lambda_{D_i(P)} + t_q \lambda_{C_i(P)} \ge t_q \ge 1/2. \tag{8.12}$$

Combining these estimates, we find the first bounds

$$\begin{aligned} &k_1 \lambda_{B_i(P)} \le \lambda^1_{i,q}(P) \le k_2 \lambda_{B_i(P)}, \\ &k_1 \lambda_{E_i(P)} \lambda_{D_i(P)} \le \lambda^2_{i,q}(P) \le k_2 \lambda_{E_i(P)} \lambda_{D_i(P)}. \end{aligned}$$

Using the notation of Section 5, it is easy to see that

$$\lambda_{B_i(P)} \sim \frac{2(p-1)}{1+a^{p-1}} \frac{1}{i}, \quad \lambda_{E_i(P)} \lambda_{D_i(P)} \sim \frac{a}{a+1} \frac{1}{i},$$

whence (8.8)–(8.9) follow.

We now turn to the proof of (8.10). The bound from below is immediate. Moreover, by (8.11)–(8.12),

$$0 < \lambda^3_{i,q}(P) \le \lambda_{E_i(P)} \lambda_{C_i(P)} = \frac{y_{i-1} + g_w(x_{i-1})}{y_i + g_w(x_{i-1})} \frac{x_{i-1} + g_w^{-1}(y_i)}{x_i + g_w^{-1}(y_i)}.$$

We need to show that, for $\gamma = 1 + \min\{1, 2(p-1)\}$, some $C > 0$ and large i,

$$\frac{y_{i-1} + g_w(x_{i-1})}{y_i + g_w(x_{i-1})} \frac{x_{i-1} + g_w^{-1}(y_i)}{x_i + g_w^{-1}(y_i)} \le e^{C/i^\gamma} e^{-2\min\{G_p(c), G_p(2c)\}/i}.$$

This was already proved in Section 6.5, and we sketch the estimate for the convenience of the reader. We only treat the estimate of $\lambda_{C_i(P)}$, the proof of the other term being analogous. Write

$$\frac{x_{i-1} + g_w^{-1}(y_i)}{x_i + g_w^{-1}(y_i)} = e^{\ln(x_{i-1} + g_w^{-1}(y_i)) - \ln(x_i + g_w^{-1}(y_i))}.$$

Now use (6.25) to estimate

$$\ln(x_{i-1} + g_w^{-1}(y_i)) - \ln(x_i + g_w^{-1}(y_i)) \le \frac{x_{i-1} - x_i}{x_i + g_w^{-1}(y_i)}.$$

Estimate (6.26) gives

$$e^{\ln(x_{i-1}+g_w^{-1}(y_i))-\ln(x_i+g_w^{-1}(y_i))} \le e^{C/i^\nu} e^{-\frac{2a}{a+1}\frac{1}{i}}.$$

A similar proof yields

$$\frac{y_{i-1}+g_w(x_{i-1})}{y_i+g_w(x_{i-1})} \le e^{C/i^\nu} e^{-\frac{2(p-1)}{1+a^{p-1}}\frac{1}{i}}.$$

Combining the last two inequalities, we find, for all $P \in \overline{Q(c)}$,

$$0 < \lambda^3_{i,q}(P) \le e^{C/i^\nu} e^{-2G_p(a)/i} \overset{(5.4),(5.5)}{\le} e^{C/i^\nu} e^{-2\min\{G_p(c),G_p(2c)\}/i}. \qquad \blacksquare$$

Proof of Proposition 8.3. The construction is inductive. If $j_2 = j_1 + 1$, then we set

$$\nu_{j_1,j_2,q}(P) := \nu_{j_1,q}(P),$$

where $\nu_{j_1,q}$ was introduced in Lemma 8.4. If $j_2 > j_1 + 1$, then $\nu_{j_1,j_2,q}(P)$ is obtained in $j_2 - j_1 \ge 2$ steps. First, $\mu^1 := \nu_{j_1,q}(P)$. By definition, μ^1 contains a Dirac delta at $C_{j_1}(P) = A_{j_1+1}(P)$ with weight $\lambda^3_{j_1,q}(P)$. Now $A_{j_1+1}(P)$ is again the barycenter of $\nu_{j_1+1,q}(P)$ of Lemma 8.4. Hence we set

$$\mu^2 := \mu^1 - \lambda^3_{j_1,q}\delta_{A_{j_1+1}(P)} + \lambda^3_{j_1,q}\nu_{j_1+1,q}(P).$$

If $j_2 - j_1 = 2$, we stop. Otherwise, we continue iteratively, defining for $k \in \{2,\dots, j_2 - j_1\}$,

$$\mu^k := \mu^{k-1} - \Big(\prod_{r=0}^{k-2}\lambda^3_{j_1+r,q}(P)\Big)\delta_{A_{j_1+k-1}(P)} + \Big(\prod_{r=0}^{k-2}\lambda^3_{j_1+r,q}(P)\Big)\nu_{j_1+k-1,q}(P). \tag{8.13}$$

Finally, the required laminate is $\nu_{j_1,j_2,q}(P) := \mu^{j_2-j_1}$. By construction, $\nu_{j_1,j_2,q}(P)$ is a laminate of finite order. The fact that the barycenter of $\nu_{j_1,j_2,q}(P)$ is $A_{j_1}(P)$ and (8.3) also follow by construction. By Lemma 7.1, all the sets $U^1_{k,q}$, $U^2_{k,q}$ and $U^3_{j_2}$ for $k \in \{j_1,\dots,j_2-1\}$ are pairwise disjoint. Therefore,

$$\nu_{j_1,j_2,q}(P)(U^3_{j_2}) = \Big(\prod_{r=0}^{j_2-j_1-2}\lambda^3_{j_1+r,q}(P)\Big)\nu_{j_2-1,q}(P)(U^3_{j_2}) \overset{(8.10)}{>} 0. \qquad \blacksquare$$

We combine Lemmas 8.1–8.2 and Proposition 8.3 to prove the last two propositions of this section.

Proposition 8.5. *Let $q \ge j_2 - 2$ and $j_2 > j_1$. There exists a sufficiently large $I \in \mathbb{N}$ such that if $j_1 \ge I$, then there exists a universal constant $C > 0$ such that the following holds. For all $P \in Q(c)$, $\Phi^1_{j_1,t_q}(P)$ is the barycenter of a laminate of finite order $\nu^1_{j_1,j_2,q}(P)$ with*

$$\operatorname{spt}(\nu^1_{j_1,j_2,q}(P)) \subset \bigcup_{i=j_1}^{j_2-1} U^1_{i,q+1} \cup \bigcup_{i=j_1}^{j_2-1} U^2_{i,q+1} \cup U^3_{j_2}$$

and the following estimates hold:

$$\nu^1_{j_1,j_2,q}(P)(U^1_{j_1,q+1}) \geq 1 - \frac{C}{2^{j_2}}, \tag{8.14}$$

$$\nu^1_{j_1,j_2,q}(P)\Big(\bigcup_{i=j_1+1}^{j_2-1} U^1_{i,q+1} \cup \bigcup_{i=j_1}^{j_2-1} U^2_{i,q+1} \cup U^3_{j_2}\Big) \leq \frac{C}{2^{j_2}}, \tag{8.15}$$

$$\nu^1_{j_1,j_2,q}(P)(U^3_{j_2}) > 0. \tag{8.16}$$

Proof. Combining Lemma 8.1 with Proposition 8.3, we define the laminate of finite order

$$\nu^1_{j_1,j_2,q}(P) = \frac{t_q}{t_{q+1}}\delta_{\Phi^1_{j_1,t_{q+1}}(P)} + \left(1 - \frac{t_q}{t_{q+1}}\right)\nu_{j_1,j_2,q+1}(P).$$

We check as in Lemma 8.1 that $\Phi^1_{j_1,t_q}(P)$ is the barycenter of $\nu^1_{j_1,j_2,q}(P)$. It is immediate to see that

$$\operatorname{spt}(\nu^1_{j_1,j_2,q}(P)) \subset \bigcup_{i=j_1}^{j_2-1} U^1_{i,t_{q+1}} \cup \bigcup_{i=j_1}^{j_2-1} U^2_{i,t_{q+1}} \cup U^3_{j_2}.$$

Recall that the union is disjoint if I is sufficiently large by Lemma 7.1. We now come to the required estimates. First,

$$\nu^1_{j_1,j_2,q}(P)(U^1_{j_1,q+1}) = \frac{t_q}{t_{q+1}} + \left(1 - \frac{t_q}{t_{q+1}}\right)\nu_{j_1,j_2,q+1}(P)(U^1_{j_1,q+1}) \geq \frac{t_q}{t_{q+1}}.$$

Since $q \geq j_2 - 2$ and $t_q = 1 - \frac{1-t_0}{2^q}$, we find a constant $C > 0$ such that

$$\frac{t_q}{t_{q+1}} \geq 1 - \frac{C}{2^{j_2}}.$$

It also follows that

$$\nu^1_{j_1,j_2,q}(P)\Big(\bigcup_{i=j_1+1}^{j_2-1} U^1_{j_1,q+1} \cup \bigcup_{i=j_1}^{j_2-1} U^2_{j_1,q+1} \cup U^3_{j_2}\Big) \leq \frac{C}{2^{j_2}}.$$

Finally, (8.16) follows from (8.4). This finishes the proof. ∎

Proposition 8.6. *Let $q \geq j_2 - 2$ and $j_2 > j_1 + 1$. There exists a sufficiently large $I \in \mathbb{N}$ such that if $j_1 \geq I$, then there exists a universal constant $C > 0$ such that the following holds. For all $P \in Q(c)$, $\Phi^2_{j_1,t_q}(P)$ is the barycenter of a laminate of finite order $\nu^2_{j_1,j_2,q}(P)$ with*

$$\operatorname{spt}(\nu^2_{j_1,j_2,q}(P)) \subset \bigcup_{i=j_1+1}^{j_2-1} U^1_{i,q+1} \cup \bigcup_{i=j_1}^{j_2-1} U^2_{i,q+1} \cup U^3_{j_2}$$

and the following estimates hold:

$$\nu^2_{j_1,j_2,q}(P)(U^2_{j_1,q+1}) \geq 1 - \frac{C}{2^{j_2}}, \tag{8.17}$$

$$\nu^2_{j_1,j_2,q}(P)\Big(\bigcup_{i=j_1+1}^{j_2-1} U^1_{i,q+1} \cup \bigcup_{i=j_1+1}^{j_2-1} U^2_{i,q+1} \cup U^3_{j_2}\Big) \leq \frac{C}{2^{j_2}}, \tag{8.18}$$

$$\nu^2_{j_1,j_2,q}(P)(U^3_{j_2}) > 0. \tag{8.19}$$

Proof. The proof is analogous to the one of Proposition 8.5, and is based on a combination of Lemma 8.2 and Proposition 8.3. We omit the details. ∎

9. The inductive proposition and conclusion

This section is devoted to the proof of Theorem 1.2. As in every convex integration-type argument, the construction of the exact solution is inductive. Let $\Omega \subset \mathbb{R}^2$ be any convex, bounded and open domain. Define countably many families of sets

$$\mathcal{F}_n := \{\Omega_{1,n}, \dots, \Omega_{N_n,n}\}$$

with the following properties:

- for all n, $\{\Omega_{j,n}\}_j$ are pairwise disjoint, open sets, whose union is Ω up to a set of zero measure;
- for every $j \in \{1, \dots, N_{n+1}\}$, if $\Omega_{j,n+1} \cap \Omega_{k(j),n} \neq \emptyset$ for some $k(j) \in \{1, \dots, N_n\}$, then $\Omega_{j,n+1} \subset \Omega_{k(j),n}$;
- $\operatorname{diam}(\Omega_{j,n}) \leq 1/n$ for all $j \in \{1, \dots, N_n\}$ and $n \in \mathbb{N}$.

At the end we fixed of Section 6 the parameter c so that (5.4)–(5.5) and Proposition 6.2 hold. Consequently, we let I be an index for which Propositions 6.2, 8.5, 8.6 and Lemmas 7.1, 8.2, 8.4 hold, and t_0 is fixed by Lemma 7.1. Furthermore, let as usual $Q(c)$ be the open set of parameters of our constructions, whose points are denoted by P. Further, we will always denote

$$\gamma := 1 + \min\{1, 2(p-1)\}. \tag{9.1}$$

The inductive construction will be formalized in Proposition 9.1, and we illustrate here its first step. Start with any affine map $w_0 = Mx$ on Ω, where $M \in A_I(Q(c))$, say $M = A_I(P)$ with $P \in Q(c)$. We write $A_I(P)$ for the barycenter of the laminate of finite order $\nu_{I,I}(P)$ of Lemma 8.4. In every subset $\Omega_{j,I} \in \mathcal{F}_I$, use Proposition 4.3 to find a piecewise affine Lipschitz map w_1 with the following properties:

(1) $w_1 = Mx$ on $\partial\Omega$;

(2) $\|w_1 - w_0\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq 1/2$;

(3) $Dw_1 \in V_1$ a.e.;

(4) the following estimates hold for all $j \in \{1, \dots, N_1\}$:

   (a) $k_1|\Omega_{j,I}|/I \leq |\{x \in \Omega_{j,I} : Dw_1(x) \in U^1_{I,I}\}| \leq k_2|\Omega_{j,I}|/I$;

   (b) $k_1|\Omega_{j,I}|/I \leq |\{x \in \Omega_{j,I} : Dw_1(x) \in U^2_{I,I}\}| \leq k_2|\Omega_{j,I}|/I$;

   (c) $0 < |\{x \in \Omega_{j,I} : Dw_1(x) \in U^3_{I+1}\}| \leq e^{C/I^\gamma} e^{-2\min\{G_p(c), G_p(2c)\}/I} |\Omega_{j,I}|$.

Notice that (3)–(4) are consequences of the openness of V_1, $U^1_{I,I}$, $U^2_{I,I}$ and U^3_{I+1}. We can now move on to the inductive proposition. Recall that

$$V_n = \bigcup_{i=I}^{I+n-1} U^1_{i,I+n-1} \cup \bigcup_{i=I}^{I+n-1} U^2_{i,I+n-1} \cup U^3_{I+n}.$$

In the proof, we will consider the usual positive and radial mollifier $\rho \in C_c^\infty(B_1)$, and its associated mollification kernel ρ_δ.

Proposition 9.1. *Let $n \geq 1$. Suppose we are given piecewise affine, Lipschitz maps $w_q : \Omega \to \mathbb{R}^2$ for $q \in \{1, \dots, n\}$, and positive decreasing numbers $\delta_1, \dots, \delta_n$ with $\delta_n \leq 2^{-n}$ enjoying the following properties:*

(1) $w_q = Mx$ *on* $\partial\Omega$*;*

(2) $\|w_q \star \rho_{\delta_q} - w_q\|_{W^{1,1}(\mathbb{R}^2,\mathbb{R}^2)} \leq 2^{-q}$ *for all* $q \in \{1, \dots, n\}$*;*

(3) $\|w_{q+1} - w_q\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^2)} \leq \delta_q 2^{-q}$ *for all* $q \in \{1, \dots, n-1\}$*;*

(4) $Dw_q \in V_q$ *for all* $q \in \{1, \dots, n\}$.

Then there exist a piecewise affine, Lipschitz map w_{n+1} and a number $0 < \delta_{n+1} < \min\{\delta_n, 2^{-n-1}\}$ with the following properties:

(i) $w_{n+1} = Mx$ *on* $\partial\Omega$*;*

(ii) $\|w_{n+1} \star \rho_{\delta_{n+1}} - w_{n+1}\|_{W^{1,1}(\mathbb{R}^2,\mathbb{R}^2)} \leq 2^{-n-1}$*;*

(iii) $\|w_{n+1} - w_n\|_{L^\infty(\mathbb{R}^2,\mathbb{R}^2)} \leq \delta_n 2^{-n-1}$*;*

(iv) $Dw_{n+1} \in V_{n+1}$*;*

(v) *for* $k = 1, 2$*,* $i \in \{0, \dots, n-1\}$ *and* $j \in \{1, \dots, N_{n+1}\}$*,*

$$\begin{aligned}
&\left(1 - \frac{C}{2^n}\right)|\{x \in \Omega_{j,n+1} : Dw_n \in U^k_{I+i,I+n-1}\}| \\
&\qquad\leq |\{x \in \Omega_{j,n+1} : Dw_{n+1} \in U^k_{I+i,I+n}\}| \\
&\qquad\leq |\{x \in \Omega_{j,n+1} : Dw_n \in U^k_{I+i,I+n-1}\}| + \frac{Cn}{2^n}|\Omega_{j,n+1}|, \qquad (9.2)
\end{aligned}$$

$$\begin{aligned}
&\frac{k_1}{n}|\{x \in \Omega_{j,n+1} : Dw_n \in U^3_{I+n}\}| \leq |\{x \in \Omega_{j,n+1} : Dw_{n+1} \in U^k_{I+n,I+n}\}| \\
&\qquad\leq \frac{k_2}{n}|\{x \in \Omega_{j,n+1} : Dw_n \in U^3_{I+n}\}| + \frac{Cn}{2^n}|\Omega_{j,n+1}|, \qquad (9.3)
\end{aligned}$$

$$\begin{aligned}
&0 < |\{x \in \Omega_{j,n+1} : Dw_{n+1} \in U^3_{I+n+1}\}| \\
&\leq e^{C/n^\gamma} e^{\frac{-2\min\{G_P(c), G_P(2c)\}}{I+n}}|\{x \in \Omega_{j,n+1} : Dw_n \in U^3_{I+n}\}| + \frac{Cn}{2^n}|\Omega_{j,n+1}|. \qquad (9.4)
\end{aligned}$$

Notice that in the previous statement, in order to write quantities like

$$\|w_{n+1} \star \rho_{\delta_{n+1}} - w_{n+1}\|_{W^{1,1}(\mathbb{R}^2,\mathbb{R}^2)},$$

we assume to have extended all maps w_q as $w_q = Mx$ outside Ω. By (1) and (i), this extension preserves the Lipschitzianity of those maps, hence we are allowed to do so.

Proof of Proposition 9.1. Let $\Omega' \subset \Omega_{j,n+1} \subset \Omega_{k(j),n}$ be an open set where w_n is affine, i.e. $w_n(x) = Sx + b$ for some $S \in V_n$ and $b \in \mathbb{R}^2$. By Lemma 7.1, V_n is a disjoint union, hence there are only three cases: either

$$S \in \bigcup_{i=I}^{I+n-1} U^1_{i,I+n-1}, \tag{9.5}$$

$$S \in \bigcup_{i=I}^{I+n-1} U^2_{i,I+n-1}, \quad \text{or} \tag{9.6}$$

$$S \in U^3_{I+n}. \tag{9.7}$$

Assume (9.5) holds, so $S \in U^1_{I+i,I+n-1}$ for some $i \in \{0, \dots, n-1\}$. By definition, there exists $P \in Q(c)$ such that

$$S = \Phi^1_{I+i,t_{I+n-1}}(P).$$

We can then use the laminate $\nu^1_{I+i,I+n+1,I+n-1}(P)$ of Proposition 8.5 combined with Proposition 4.3 to find a Lipschitz and piecewise affine map f with the following properties:

- $f|_{\partial\Omega'} = w_n|_{\partial\Omega'}$;
- almost everywhere on Ω',

$$Df \in \bigcup_{j=I+i}^{I+n} U^1_{j,I+n} \cup \bigcup_{j=I+i}^{I+n} U^2_{j,I+n} \cup U^3_{I+n+1}$$

  with

$$\left(1 - \frac{C}{2^n}\right)|\Omega'| \le |\{x \in \Omega' : Df \in U^1_{I+i,I+n}\}| \le |\Omega'|, \tag{9.8}$$

$$\left|\left\{x \in \Omega' : Df \in \bigcup_{j=I+i+1}^{I+n} U^1_{j,I+n} \cup \bigcup_{j=I+i}^{I+n} U^2_{j,I+n} \cup U^3_{I+n+1}\right\}\right| \le \frac{C}{2^n}|\Omega'|, \tag{9.9}$$

$$0 < |\{x \in \Omega' : Df \in U^3_{I+n+1}\}|; \tag{9.10}$$

- $\|f - w_n\|_{L^\infty(\Omega')} \le \delta_n 2^{-n-1}$.

On such a set Ω', we replace $w_n|_{\Omega'}$ by f. Analogously, if (9.6) holds, i.e. $S \in U^2_{I+i,I+n-1}$ for some $i \in \{0, \dots, n-1\}$, then, by definition, $S = \Phi^2_{I+i,t_{I+n-1}}(P)$ for some $P \in Q(c)$, and we can then use the laminate $\nu^2_{I+i,I+n+1,I+n-1}(P)$ of Proposition 8.6 combined with Proposition 4.3 to find a Lipschitz and piecewise affine map g with the following properties:

- $g|_{\partial\Omega'} = w_n|_{\partial\Omega'}$;
- almost everywhere on Ω',

$$Dg \in \bigcup_{j=I+i+1}^{I+n} U^1_{j,I+n} \cup \bigcup_{j=I+i}^{I+n} U^2_{j,I+n} \cup U^3_{I+n+1}$$

with

$$\left(1-\frac{C}{2^n}\right)|\Omega'| \le |\{x\in\Omega' : Dg\in U^2_{I+i,I+n}\}| \le |\Omega'|, \tag{9.11}$$

$$\left|\left\{x\in\Omega' : Dg\in \bigcup_{j=I+i+1}^{I+n} U^1_{j,I+n} \cup \bigcup_{j=I+i+1}^{I+n} U^2_{j,I+n}\cup U^3_{I+n+1}\right\}\right| \le \frac{C}{2^n}|\Omega'|, \tag{9.12}$$

$$0<|\{x\in\Omega' : Dg\in U^3_{I+n+1}\}|; \tag{9.13}$$

- $\|g-w_n\|_{L^\infty(\Omega')}\le \delta_n 2^{-n-1}$.

On such a set Ω', we replace $w_n|_{\Omega'}$ by g. Finally, in the third case, (9.7), we have $S=A_{I+n}(P)$ for some $P\in Q(c)$. We use Lemma 8.4 combined with Proposition 4.3 to find a Lipschitz and piecewise affine map h with the following properties:

- $h|_{\partial\Omega'}=w_n|_{\partial\Omega'}$;
- almost everywhere on Ω',

$$Dh\in U^1_{I+n,I+n}\cup U^2_{I+n,I+n}\cup U^3_{I+n+1}$$

with

$$\frac{k_1}{n}|\Omega'|\le |\{x\in\Omega' : Dh\in U^1_{I+n,I+n}\}|\le \frac{k_2}{n}|\Omega'|, \tag{9.14}$$

$$\frac{k_1}{n}|\Omega'|\le |\{x\in\Omega' : Dh\in U^2_{I+n,I+n}\}|\le \frac{k_2}{n}|\Omega'|, \tag{9.15}$$

$$0<|\{x\in\Omega' : Dh\in U^3_{I+n+1}\}|\le e^{C/n^\nu}e^{\frac{-2\min\{G_P(c),G_P(2c)\}}{I+n}}|\Omega'|; \tag{9.16}$$

- $\|h-w_n\|_{L^\infty(\Omega')}\le \delta_n 2^{-n-1}$.

Finally, for a set Ω' of this type, we replace $w_n|_{\Omega'}$ by h. These replacements precisely give w_{n+1}. Notice that in all of the above estimates the quantities 2^n, n and e^{C/n^ν} should have been 2^{n+I+1}, $n+I+1$ and $e^{C/(I+n)^\nu}$, but, since they are all comparable, we can simply reabsorb the errors in making these substitutions inside the constants C, k_1, k_2.

By construction, (i), (iii) and (iv) hold. The existence of δ_{n+1} as in (ii) is guaranteed by the Lipschitzianity of w_{n+1}. We can now show the estimates asserted in (v). Let first $i\in\{0,\dots,n-1\}$ and $j\in\{1,\dots,N_{n+1}\}$. Then

$$\begin{aligned}
\left(1-\frac{C}{2^n}\right)|\{x\in\Omega_{j,n+1} : Dw_n\in U^1_{I+i,I+n-1}\}| &\overset{(9.8)}{\le} |\{x\in\Omega_{j,n+1} : Dw_{n+1}\in U^1_{I+i,I+n}\}| \\
&\overset{(9.8),(9.9),(9.12)}{\le} |\{x\in\Omega_{j,n+1} : Dw_n\in U^1_{I+i,I+n-1}\}| \\
&\quad+\sum_{\ell=0}^{i-1}\frac{C}{2^n}|\{x\in\Omega_{j,n+1} : Dw_n\in U^1_{I+\ell,I+n-1}\}| \\
&\quad+\sum_{\ell=0}^{i-1}\frac{C}{2^n}|\{x\in\Omega_{j,n+1} : Dw_n\in U^2_{I+\ell,I+n-1}\}| \\
&\le |\{x\in\Omega_{j,n+1} : Dw_n\in U^1_{I+i,I+n-1}\}| + \frac{Cn}{2^n}|\Omega_{j,n+1}|.
\end{aligned}$$

With analogous computations, for the same ranges of i and j, we find

$$\left(1-\frac{C}{2^n}\right)|\{x\in\Omega_{j,n+1}:Dw_n\in U^2_{I+i,I+n-1}\}| \overset{(9.11)}{\le} |\{x\in\Omega_{j,n+1}:Dw_{n+1}\in U^2_{I+i,I+n}\}|$$
$$\overset{(9.11),(9.9),(9.12)}{\le} |\{x\in\Omega_{j,n+1}:Dw_n\in U^2_{I+i,I+n-1}\}|+\frac{Cn}{2^n}|\Omega_{j,n+1}|.$$

We still need to write the estimates for $i=n$:

$$\frac{k_1}{n}|\{x\in\Omega_{j,n+1}:Dw_n\in U^3_{I+n}\}| \overset{(9.14)}{\le} |\{x\in\Omega_{j,n+1}:Dw_{n+1}\in U^1_{I+n,I+n}\}|$$
$$\overset{(9.14),(9.9),(9.12)}{\le} \frac{k_2}{n}|\{x\in\Omega_{j,n+1}:Dw_n\in U^3_{I+n}\}|+\frac{Cn}{2^n}|\Omega_{j,n+1}|,$$

and

$$\frac{k_1}{n}|\{x\in\Omega_{j,n+1}:Dw_n\in U^3_{I+n}\}| \overset{(9.14)}{\le} |\{x\in\Omega_{j,n+1}:Dw_{n+1}\in U^2_{I+n,I+n}\}|$$
$$\overset{(9.15),(9.9),(9.12)}{\le} \frac{k_2}{n}|\{x\in\Omega_{j,n+1}:Dw_n\in U^3_{I+n}\}|+\frac{Cn}{2^n}|\Omega_{j,n+1}|.$$

Finally,

$$0 \overset{(9.10),(9.13),(9.16)}{<} |\{x\in\Omega_{j,n+1}:Dw_{n+1}\in U^3_{I+n+1}\}|$$
$$\overset{(9.16),(9.9),(9.12)}{\le} e^{C/n^\nu}e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+n}}|\{x\in\Omega_{j,n+1}:Dw_n\in U^3_{I+n}\}|+\frac{Cn}{2^n}|\Omega_{j,n+1}|.$$

This concludes the proof of the inductive proposition. ∎

We now show that the sequence $\{w_n\}$ is equibounded in $W^{1,1+\varepsilon}(\Omega,\mathbb{R}^2)$ for some $\varepsilon=\varepsilon(p)>0$.

Proposition 9.2. *The sequence $\{w_n\}$ is equibounded in $W^{1,1+\varepsilon}(\Omega,\mathbb{R}^2)$.*

We first show the following.

Lemma 9.3. *There exists a constant $C=C(p)>0$ such that the following estimates hold for all $n\ge 1$ and $0\le i\le n-1$:*

$$|\{x\in\Omega:Dw_n\in U^3_{I+n}\}|\le Cn^{-2\min\{G_p(c),G_p(2c)\}}, \tag{9.17}$$
$$|\{x\in\Omega:Dw_n\in U^1_{I+i,I+n-1}\}|\le Ci^{-2\min\{G_p(c),G_p(2c)\}-1}, \tag{9.18}$$
$$|\{x\in\Omega:Dw_n\in U^2_{I+i,I+n-1}\}|\le Ci^{-2\min\{G_p(c),G_p(2c)\}-1}. \tag{9.19}$$

Proof. Sum inequality (9.4) over $j\in\{1,\dots,N_{n+1}\}$ to find

$$|\{x\in\Omega:Dw_{n+1}\in U^3_{I+n+1}\}|\le e^{C/n^\nu}e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+n}}|\{x\in\Omega:Dw_n\in U^3_{I+n}\}| + \frac{Cn}{2^n}.$$

Now we can apply this relation recursively to discover that, for all $n \in \mathbb{N}$,

$$|\{x \in \Omega : Dw_n \in U^3_{I+n}\}| \\ \le C\Big(\prod_{\ell=1}^{n-1} e^{C/\ell^\gamma} e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+\ell}} + \sum_{\ell=1}^{n-1}\Big(\prod_{j=\ell+1}^{n-1} e^{C/j^\gamma} e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+j}}\Big)\frac{\ell}{2^\ell}\Big).$$

In the previous expression, we have used the convention introduced in (6.3). Using (6.16), we can estimate

$$\prod_{\ell=1}^{n-1} e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+\ell}} = e^{-2\min\{G_p(c),G_p(2c)\}\sum_{\ell=1}^{n-1}\frac{1}{I+\ell}} \le Cn^{-2\min\{G_p(c),G_p(2c)\}}.$$

On the other hand, $\gamma > 1$ by (9.1). Therefore, $\prod_{\ell=1}^{n} e^{C/\ell^\gamma}$ is uniformly bounded in n. Concerning the second summand, we start by estimating, for all $\ell + 1 \le n - 1$,

$$\prod_{j=\ell+1}^{n-1} e^{C/j^\gamma} e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+j}} \le e^{\sum_{\ell=1}^{\infty} C/j^\gamma} e^{\sum_{j=\ell+1}^{n-1}\frac{-2\min\{G_p(c),G_p(2c)\}}{I+j}} \\ \overset{(6.16)}{\le} Ce^{-2\min\{G_p(c),G_p(2c)\}\ln(\frac{I+n-1}{I+\ell+1})} \le C\left(\frac{I+\ell+1}{I+n-1}\right)^{2\min\{G_p(c),G_p(2c)\}}.$$

Thus,

$$\sum_{\ell=1}^{n-1}\prod_{j=\ell+1}^{n-1} e^{C/j^\gamma} e^{\frac{-2\min\{G_p(c),G_p(2c)\}}{I+j}}\frac{\ell}{2^\ell} \\ \le C\left(\frac{1}{I+n-1}\right)^{2\min\{G_p(c),G_p(2c)\}}\sum_{\ell=1}^{n-1}(I+\ell+1)^{2\min\{G_p(c),G_p(2c)\}}\frac{\ell}{2^\ell} \\ \le Cn^{-2\min\{G_p(c),G_p(2c)\}},$$

which concludes the proof of (9.17).

Using (9.17) and (9.3), for $k = 1, 2$ and all $n \in \mathbb{N}$ we have

$$|\{x \in \Omega : Dw_n \in U^k_{I+n-1,I+n-1}\}| \le Cn^{-2\min\{G_p(c),G_p(2c)\}-1} + \frac{Cn}{2^n}. \tag{9.20}$$

Through (9.2) and a simple inductive reasoning, we also find, for all $n \ge 1$ and $0 \le i \le n-1$,

$$|\{x \in \Omega : Dw_{n+1} \in U^k_{I+i,I+n}\}| \le |\{x \in \Omega : Dw_n \in U^k_{I+i,I+n-1}\}| + \frac{Cn}{2^n} \\ \overset{(9.20)}{\le} C(i+1)^{-2\min\{G_p(c),G_p(2c)\}-1} + C\sum_{\ell=i+1}^{\infty}\frac{\ell}{2^\ell} \le C(i+1)^{-2\min\{G_p(c),G_p(2c)\}-1}.$$

This concludes the proof. ∎

Proof of Proposition 9.2. Let $\Omega_n := \{x \in \Omega : Dw_n \in U^3_{I+n}\}$, $\Omega^1_{i,n} := \{x \in \Omega : Dw_n \in U^1_{I+i,I+n-1}\}$, $\Omega^2_{i,n} := \{x \in \Omega : Dw_n \in U^2_{I+i,I+n-1}\}$, for $i \in \{0, \dots, n-1\}$. Notice that, by definition,

$$\sup\{|M| : M \in U^1_{I+i,I+n-1}\} = \sup_{P\in Q(c)} |\Phi^1_{I+i,t_{I+n-1}}(P)|,$$
$$\sup\{|M| : M \in U^2_{I+i,I+n-1}\} = \sup_{P\in Q(c)} |\Phi^2_{I+i,t_{I+n-1}}(P)|,$$
$$\sup\{|M| : M \in U^3_{I+n}\} = \sup_{P\in Q(c)} |A_{I+n}(P)|.$$

Thus we can estimate, for any $q \in [1,\infty)$,

$$\|Dw_n\|^q_{L^q(\Omega,\mathbb{R}^2)} \le \sum_{i=1}^n \sup_{P\in Q(c)} |\Phi^1_{I+i,t_{I+n-1}}(P)|^q |\Omega^1_{i,n}| + \sum_{i=1}^n \sup_{P\in Q(c)} |\Phi^2_{I+i,t_{I+n-1}}(P)|^q |\Omega^2_{i,n}| + \sup_{P\in Q(c)} |A_{I+n}(P)|^q |\Omega_n|. \tag{9.21}$$

Recall that by Proposition 6.2 (b),

$$\sup_{P\in \overline{Q(c)}} |z_i(P)| \le C i^{2(p-2)}, \quad \forall i \ge 1.$$

Through (5.8), we see that also

$$\sup_{P\in \overline{Q(c)}} |v_i(P)| \le C i^{2(p-2)}, \quad \forall i \ge 1.$$

Using the definitions of $\Phi^1_{i,t}$ and $\Phi^2_{i,t}$ and the previous estimates, we can bound

$$\sup_{P\in Q(c)} |\Phi^1_{I+i,t_{I+n+1}}(P)| \le C i^{\max\{2,2(p-1)\}},$$
$$\sup_{P\in Q(c)} |\Phi^2_{I+i,t_{I+n+1}}(P)| \le C i^{\max\{2,2(p-1)\}},$$
$$\sup_{P\in Q(c)} |A_{I+n+1}(P)| \le C n^{\max\{2,2(p-1)\}}.$$

Combining these with the estimates of Lemma 9.3, we can continue (9.21) as

$$\|Dw_n\|^q_{L^q(\Omega,\mathbb{R}^2)} \le C \sum_{i=1}^n i^{q\max\{2,2(p-1)\}} i^{-2\min\{G_p(c),G_p(2c)\}-1} + C n^{q\max\{2,2(p-1)\}} n^{-2\min\{G_p(c),G_p(2c)\}}.$$

Our choices (5.4) and (5.5) imply (5.3), which yields precisely

$$\max\{1, p-1\} - \min\{G_p(c), G_p(2c)\} < 0.$$

Therefore, for some $q > 1$, we also have

$$q \max\{1, p-1\} - \min\{G_p(c), G_p(2c)\} < 0,$$

which proves the present proposition. ■

We are finally in a position to prove Theorem 1.2, which we restate here.

Theorem 9.4. *Let $\Omega \subset \mathbb{R}^2$ be a ball. For every $p \in (1,\infty)$, $p \neq 2$, there exists $\varepsilon = \varepsilon(p) > 0$ and a continuous $u \in W^{1,p-1+\varepsilon}(\Omega)$ such that u is affine on $\partial\Omega$,*

$$\frac{3}{4} \le \partial_y u \le \frac{5}{4} \quad \text{a.e. on } \Omega, \tag{9.22}$$

$$\operatorname{div}(|Du|^{p-2}Du) = 0 \tag{9.23}$$

in the sense of distributions, but for all open $B \subset \Omega$,

$$\int_B |Du|^p \, dx = \infty. \tag{9.24}$$

Proof. Let $\{w_n\}$ be the sequence constructed in Proposition 9.1. By (iii), the limit $w_\infty = \lim_{n\to\infty} w_n$ exists in the $L^\infty(\mathbb{R}^2,\mathbb{R}^2)$ topology. Since every w_n is continuous, so is w_∞. By Proposition 9.2, w_∞ is also the weak limit of w_n in $W^{1,1+\varepsilon}(\Omega,\mathbb{R}^2)$. We shall now prove that the convergence is strong. This is a crucial but standard point of this type of constructions (see [34, Theorem 5.3]). Indeed, using the notation of Proposition 9.1, for all n we have

$$\begin{aligned}\|Dw_n - Dw_\infty\|_{L^1} \le \|Dw_n - Dw_n \star \rho_{\delta_n}\|_{L^1} + \|Dw_\infty - Dw_\infty \star \rho_{\delta_n}\|_{L^1} \\ + \|Dw_\infty \star \rho_{\delta_n} - Dw_n \star \rho_{\delta_n}\|_{L^1}.\end{aligned}$$

By our choice of δ_n (see Proposition 4.3 (ii)) and the fact that $Dw_\infty \in W^{1,1+\varepsilon}$, the first two summands on the right-hand side converge to 0. Concerning the third, we can employ standard estimates on mollification to bound

$$\begin{aligned}\|Dw_\infty \star \rho_{\delta_n} - Dw_n \star \rho_{\delta_n}\|_{L^1} &\le \frac{C}{\delta_n}\|w_\infty - w_n\|_{L^\infty} \le \frac{C}{\delta_n}\sum_{j=n}^{\infty}\|w_{j+1} - w_j\|_{L^\infty} \\ &\overset{\text{(iii)}}{\le} \frac{C}{\delta_n}\sum_{j=n}^{\infty}\delta_j 2^{-j-1} \le C\sum_{j=n}^{\infty}2^{-j-1}.\end{aligned}$$

We infer the strong convergence of Dw_n to Dw_∞. Hence a subsequence of $\{Dw_n\}$ converges pointwise a.e. Through Proposition 9.1 (iv) and Lemma 7.2, we deduce that for almost all $x \in \Omega$,

$$Dw_\infty(x) \in K_p.$$

Proposition 2.1 tells us that if $w_\infty = (w_\infty^1, w_\infty^2)$, then $u := w_\infty^1$ has the right integrability and fulfills (9.23). We now turn to (9.22), which is straightforward: by definition of V_n it follows that if

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in V_n$$

for some n, then $\frac{3}{4} < x_{12} < \frac{5}{4}$. Since by Proposition 4.3 (iv) we have, a.e. on Ω,

$$Dw_n \in V_n, \quad \forall n,$$

it follows that if $w_n = (w_n^1, w_n^2)$ then a.e. on Ω,

$$\frac{3}{4} \le \partial_y w_n^1 \le \frac{5}{4}.$$

Therefore, w_∞ enjoys the same property.

We shall now show (9.24). In order to do so, we show that Dw_∞ is (essentially) discontinuous on any open subset $B \subset \Omega$. We claim this is enough to deduce (9.24). Indeed, suppose for contradiction that $Du \in L^p(\Omega')$ for some open $\Omega' \subset \Omega$. Then since u solves (9.23), we see that it is a *weak* solution of the p-Laplace equation, and by the classical regularity theory for that equation, Du is continuous in Ω', and hence so is Dw_∞, which would result in a contradiction.

Thus, we conclude the proof of the present theorem by showing the above discontinuity. Fix $B \subset \Omega$ open. Since

$$\sup_j \{\mathrm{diam}(\Omega_{j,n}) : \Omega_{j,n} \in \mathcal{F}_n\} \le 1/n,$$

we find n_0 and $j_0 \in \{1, \dots, N_{n_0}\}$ such that $\Omega_{j_0,n_0} \subset B$. Recall that, by Lemma 7.2 and the pointwise convergence of a subsequence of $\{Dw_n\}_n$,

$$Dw_\infty \in \bigcup_{i \ge I} B_i(Q(c)) \cup \bigcup_{i \ge I} D_i(Q(c)).$$

Our aim is to show that

$$\left|\left\{x \in \Omega_{j_0,n_0} : Dw_\infty(x) \in \bigcup_{i \ge I} B_i(Q(c))\right\}\right| > 0, \tag{9.25}$$

$$\left|\left\{x \in \Omega_{j_0,n_0} : Dw_\infty(x) \in \bigcup_{i \ge I} D_i(Q(c))\right\}\right| > 0. \tag{9.26}$$

Since, by Lemma 7.1, $\bigcup_{i \ge I} B_i(Q(c)) \cap \bigcup_{i \ge I} D_i(Q(c)) = \emptyset$, this would prove that Dw_∞ is not continuous in B. We claim that, in order to prove (9.25)–(9.26), it is sufficient to show that there exists a constant $c_0 = c_0(j_0, n_0) > 0$ such that for all $k = 1, 2$ and $n \ge n_0$,

$$\left|\left\{x \in \Omega_{j_0,n_0} : Dw_n(x) \in \bigcup_{i=I}^{I+n-1} U^k_{i,I+n-1}\right\}\right| \ge c_0. \tag{9.27}$$

Indeed, let $\Omega_n^k := \{x \in \Omega_{j_0,n_0} : Dw_n(x) \in \bigcup_{i=I}^{I+n-1} U^k_{i,I+n-1}\}$. If (9.27) holds, then

$$\int_{\Omega_{j_0,n_0}} \mathrm{d}\Big(Dw_n(x), \bigcup_{i=I}^{n-1} B_i(Q(c))\Big)\, dx \ge \int_{\Omega_n^2} \mathrm{d}\Big(Dw_n(x), \bigcup_{i=I}^{n-1} B_i(Q(c))\Big)\, dx \ge c' c_0,$$

where in the last inequality we have used the fact that

$$\mathrm{d}\Big(M, \bigcup_{i=I}^{n-1} B_i(Q(c))\Big) \geq c' > 0, \quad \forall M \in \bigcup_{i=I}^{I+n-1} U^2_{i,I+n-1},$$

as can be easily seen by properties (7.1)–(7.2). By the strong convergence $Dw_n \to Dw_\infty$ in $L^1(\Omega)$, it follows that

$$\int_{\Omega_{j_0,n_0}} \mathrm{d}\Big(Dw_\infty(x), \bigcup_{i=I}^{\infty} B_i(Q(c))\Big)\,dx \geq c'c_0 > 0,$$

and we deduce (9.26). Analogously, one infers (9.25) from (9.27) for $k = 1$.

Hence, we only need to prove (9.27). Pick any $n > n_0$. Recall that we chose the partitions $\{\Omega_{j,n}\}$ with the property that for every $j \in \{1, \dots, N_{n+1}\}$, if $\Omega_{j,n+1} \cap \Omega_{k(j),n} \neq \emptyset$ for some $k(j) \in \{1, \dots, N_n\}$, then $\Omega_{j,n+1} \subset \Omega_{k(j),n}$. Therefore, we can sum over a suitable subset of indices $j \in \{1, \dots, N_n\}$ to rewrite, for all $n \geq n_0 + 2$, $i = n_0 + 1$ and $k = 1, 2$, the bounds from below of (9.2)–(9.4) as

$$\Big(1 - \frac{C}{2^n}\Big)|\{x \in \Omega_{j_0,n_0} : Dw_n \in U^k_{I+n_0+1,I+n-1}\}| \\ \leq |\{x \in \Omega_{j_0,n_0} : Dw_{n+1} \in U^k_{I+n_0+1,I+n}\}|, \tag{9.28}$$

$$\frac{k_1}{n}|\{x \in \Omega_{j_0,n_0} : Dw_n \in U^3_{I+n}\}| \leq |\{x \in \Omega_{j_0,n_0} : Dw_{n+1} \in U^k_{I+n,I+n}\}|, \tag{9.29}$$

$$0 < |\{x \in \Omega_{j_0,n_0} : Dw_{n-1} \in U^3_{I+n-1}\}|. \tag{9.30}$$

We use (9.28) inductively to find that if $n \geq n_0 + 3$, then

$$\prod_{i=n_0+2}^{n-1}\Big(1 - \frac{C}{2^i}\Big)|\{x \in \Omega_{j_0,n_0} : Dw_{n_0+2} \in U^k_{I+n_0+1,I+n_0+1}\}| \\ \leq |\{x \in \Omega_{j_0,n_0} : Dw_n \in U^k_{I+n_0+1,I+n-1}\}|. \tag{9.31}$$

We also have

$$0 \overset{(9.30)}{<} \frac{k_1}{n_0+1}|\{x \in \Omega_{j_0,n_0} : Dw_{n_0+1} \in U^3_{I+n_0+1}\}| \\ \overset{(9.29)}{\leq} |\{x \in \Omega_{j_0,n_0} : Dw_{n_0+2} \in U^k_{I+n_0+1,I+n_0+1}\}|. \tag{9.32}$$

Since

$$\prod_{i=n_0+2}^{\infty}\Big(1 - \frac{C}{2^i}\Big) > 0,$$

from (9.31)–(9.32) we infer (9.27), and we conclude the proof of the theorem. ■

Funding. The authors have been supported by the SNF Grant 182565. MC was supported by the Swiss State Secretariat for Education, Research and Innovation (SERI) under contract number M822.00034. This work was accomplished while RT was affiliated with EPFL.

References

[1] Ancona, A.: Elliptic operators, conormal derivatives and positive parts of functions. J. Funct. Anal. **257**, 2124–2158 (2009) Zbl 1173.58007 MR 2548032

[2] Arroyo-Rabasa, A., De Philippis, G., Hirsch, J., Rindler, F.: Dimensional estimates and rectifiability for measures satisfying linear PDE constraints. Geom. Funct. Anal. **29**, 639–658 (2019) Zbl 1420.49046 MR 3962875

[3] Astala, K., Faraco, D., Székelyhidi, L., Jr.: Convex integration and the L^p theory of elliptic equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **7**, 1–50 (2008) Zbl 1164.30014 MR 2413671

[4] Brezis, H.: On a conjecture of J. Serrin. Rend. Lincei Mat. Appl. **19**, 335–338 (2008) Zbl 1197.35079 MR 2465684

[5] Conti, S., Faraco, D., Maggi, F.: A new approach to counterexamples to L^1 estimates: Korn's inequality, geometric rigidity, and regularity for gradients of separately convex functions. Arch. Ration. Mech. Anal. **175**, 287–300 (2005) Zbl 1080.49026 MR 2118479

[6] Conti, S., Faraco, D., Maggi, F., Müller, S.: Rank-one convex functions on 2×2 symmetric matrices and laminates on rank-three lines. Calc. Var. Partial Differential Equations **24**, 479–493 (2005) Zbl 1135.26301 MR 2180863

[7] Cordoba, D., Faraco, D., Gancedo, F.: Lack of uniqueness for weak solutions of the incompressible porous media equation. Arch. Ration. Mech. Anal. **200**, 725–746 (2011) Zbl 1241.35156 MR 2796131

[8] De Filippis, C., Mingione, G.: Nonuniformly elliptic Schauder theory. Invent. Math. **234**. 1109–1196 (2023) Zbl 07770156 MR 4665778

[9] De Philippis, G., Rindler, F.: On the structure of $\mathcal{A}$-free measures and applications. Ann. of Math. (2) **184**, 1017–1039 (2016) Zbl 1352.49050 MR 3549629

[10] DiBenedetto, E.: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal. **7**, 827–850 (1983) Zbl 0539.35027 MR 0709038

[11] DiBenedetto, E., Trudinger, N. S.: Harnack inequalities for quasiminima of variational integrals. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**, 295–308 (1984) Zbl 0565.35012 MR 0778976

[12] Duzaar, F., Kristensen, J., Mingione, G.: The existence of regular boundary points for non-linear elliptic systems. J. Reine Angew. Math. **602**, 17–58 (2007) Zbl 1214.35021 MR 2300451

[13] Duzaar, F., Mingione, G.: Regularity for degenerate elliptic problems via p-harmonic approximation. Ann. Inst. H. Poincaré C Anal. Non Linéaire **21**, 735–766 (2004) Zbl 1112.35078 MR 2086757

[14] Evans, L. C.: A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. J. Differential Equations **45**, 356–373 (1982) Zbl 0508.35036 MR 0672713

[15] Faraco, D.: Milton's conjecture on the regularity of solutions to isotropic equations. Ann. Inst. H. Poincaré C Anal. Non Linéaire **20**, 889–909 (2003) Zbl 1029.30012 MR 1995506

[16] Faraco, D., Lindberg, S., Székelyhidi, L., Jr.: Magnetic helicity, weak solutions and relaxation of ideal MHD. Comm. Pure Appl. Math. **77**, 2387–2412 (2024) Zbl 07807925 MR 4705296

[17] Faraco, D., Mora-Corral, C., Oliva, M.: Sobolev homeomorphisms with gradients of low rank via laminates. Adv. Calc. Var. **11**, 111–138 (2018) Zbl 1396.26023 MR 3778577

[18] Fernández-Real, X., Tione, R.: Improved regularity of second derivatives for subharmonic functions. Proc. Amer. Math. Soc. **151**, 5283–5297 (2023) Zbl 07748564 MR 4648925

[19] Guerra, A., Raiţă, B., Schrecker, M.: Compensation phenomena for concentration effects via nonlinear elliptic estimates. arXiv:2112.10657 (2021)

[20] Hager, R. A., Ross, J.: A regularity theorem for linear second order elliptic divergence equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **26**, 283–290 (1972) Zbl 0256.35027 MR 0374643

[21] Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002) Zbl 1044.55001 MR 1867354

[22] Hirsch, J., Tione, R.: On the constancy theorem for anisotropic energies through differential inclusions. Calc. Var. Partial Differential Equations **60**, article no. 86, 52 pp. (2021) Zbl 1462.49070 MR 4248559

[23] Iwaniec, T.: p-harmonic tensors and quasiregular mappings. Ann. of Math. (2) **136**, 589–624 (1992) Zbl 0785.30009 MR 1189867

[24] Iwaniec, T., Martin, G.: Quasiregular mappings in even dimensions. Acta Math. **170**, 29–81 (1993) Zbl 0785.30008 MR 1208562

[25] Iwaniec, T., Martin, G.: Riesz transforms and related singular integrals. J. Reine Angew. Math. **473**, 25–57 (1996) Zbl 0847.42015 MR 1390681

[26] Iwaniec, T., Sbordone, C.: Weak minima of variational integrals. J. Reine Angew. Math. **454**, 143–161 (1994) Zbl 0802.35016 MR 1288682

[27] Jin, T., Maz'ya, V., Van Schaftingen, J.: Pathological solutions to elliptic problems in divergence form with continuous coefficients. C. R. Math. Acad. Sci. Paris **347**, 773–778 (2009) Zbl 1198.35055 MR 2543981

[28] Kuusi, T., Mingione, G.: Vectorial nonlinear potential theory. J. Eur. Math. Soc. **20**, 929–1004 (2018) Zbl 1394.35206 MR 3779689

[29] Lewis, J. L.: Regularity of the derivatives of solutions to certain degenerate elliptic equations. Indiana Univ. Math. J. **32**, 849–858 (1983) Zbl 0554.35048 MR 0721568

[30] Lewis, J. L.: On very weak solutions of certain elliptic systems. Comm. Partial Differential Equations **18**, 1515–1537 (1993) Zbl 0796.35061 MR 1239922

[31] Lindqvist, P.: Notes on the p-Laplace equation. Report 102, Department of Mathematics and Statistics, University of Jyväskylä (2006) Zbl 1256.35017 MR 2242021

[32] Mingione, G.: Regularity of minima: an invitation to the dark side of the calculus of variations. Appl. Math. (Praha) **51**, 355–426 (2006) Zbl 1164.49324 MR 2291779

[33] Mingione, G.: Gradient potential estimates. J. Eur. Math. Soc. **13**, 459–486 (2011) Zbl 1217.35077 MR 2746772

[34] Müller, S.: Variational models for microstructure and phase transitions. In: Calculus of variations and geometric evolution problems (Cetraro, 1996), Lecture Notes in Math. 1713, Springer, Berlin, 85–210 (1999) Zbl 0968.74050 MR 1731640

[35] Müller, S., Šverák, V.: Convex integration for Lipschitz mappings and counterexamples to regularity. Ann. of Math. (2) **157**, 715–742 (2003) Zbl 1083.35032 MR 1983780

[36] Serrin, J.: Pathological solutions of elliptic differential equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **18**, 385–387 (1964) Zbl 0142.37601 MR 0170094

[37] Sorella, M., Tione, R.: The four-state problem and convex integration for linear differential operators. J. Funct. Anal. **284**, article no. 109785, 44 pp. (2023) Zbl 1518.35240 MR 4515812

[38] Šverák, V., Yan, X.: A singular minimizer of a smooth strongly convex functional in three dimensions. Calc. Var. Partial Differential Equations **10**, 213–221 (2000) Zbl 1013.49027 MR 1756327

[39] Šverák, V., Yan, X.: Non-Lipschitz minimizers of smooth uniformly convex functionals. Proc. Nat. Acad. Sci. USA **99**, 15269–15276 (2002) Zbl 1106.49046 MR 1946762

[40] Székelyhidi, L., Jr.: The regularity of critical points of polyconvex functionals. Arch. Ration. Mech. Anal. **172**, 133–152 (2004) Zbl 1049.49017 MR 2048569

[41] Tolksdorf, P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations **51**, 126–150 (1984) Zbl 0488.35017 MR 0727034

[42] Uhlenbeck, K.: Regularity for a class of non-linear elliptic systems. Acta Math. **138**, 219–240 (1977) Zbl 0372.35030 MR 0474389

[43] Ural'tseva, N. N.: Degenerate quasilinear elliptic systems. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7**, 184–222 (1968) (in Russian) Zbl 0196.12501 MR 0244628