J. Eur. Math. Soc. (Online first) DOI 10.4171/JEMS/1486

Teng Wang · Yi Wang

Nonlinear stability of planar viscous shock wave to three-dimensional compressible Navier–Stokes equations

Received 12 April 2022

Abstract. We prove the nonlinear stability of the planar viscous shock up to a time-dependent shift for the three-dimensional (3D) compressible Navier–Stokes equations under the generic perturbations, in particular, without zero mass conditions. Moreover, the time-dependent shift function keeps the shock profile shape time-asymptotically. Our stability result is unconditional for the weak planar Navier–Stokes shock. Our proof is motivated by the a-contraction method (a kind of weighted L^2-relative entropy method) with time-dependent shift for the stability of viscous shock in the one-dimensional (1D) case. Instead of the classical anti-derivative techniques, we perform the stability analysis of the planar Navier–Stokes shock in the original H^2-perturbation framework and therefore zero mass conditions are not necessarily needed, which, in turn, brings out the essential difficulties due to the compressibility of viscous shock. Furthermore, compared with 1D case, there are additional difficulties coming from the wave propagations along the multi-dimensional transverse directions and their interactions with the viscous shock. To overcome these difficulties, a multi-dimensional version of the sharp weighted Poincaré inequality, a-contraction techniques with time-dependent shift, and some essential physical structures of the multi-dimensional Navier–Stokes system are fully used.

Keywords: compressible Navier–Stokes equations, planar viscous shock wave, time-asymptotic stability.

Contents

1. Introduction 2
2. Planar viscous shock and main result 4
3. Proof of main result 8
4. Uniform-in-time H^2-estimates 16
References 51

Teng Wang: School of Mathematics, Statistics and Mechanics, Beijing University of Technology, 100 Pingleyuan, 100124 Beijing, P. R. China; tengwang@amss.ac.cn

Yi Wang: Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, No. 55, Zhongguancun East Road, 100190 Beijing; School of Mathematical Sciences, University of Chinese Academy of Sciences, 19A Yuquan Road, 100049 Beijing, P. R. China; wangyi@amss.ac.cn

Mathematics Subject Classification 2020: 76E19 (primary); 76N06, 76L05, 76E30, 76N30 (secondary).

1. Introduction

We are concerned with the time-asymptotic stability of planar viscous shock wave for the 3D compressible Navier–Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \Omega, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p(\rho) = \mu \Delta_x u + (\mu + \lambda)\nabla_x \mathrm{div}_x u. \end{cases} \tag{1.1}$$

Here $\rho = \rho(t,x)\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, $u = u(t,x) = (u_1, u_2, u_3)^t(t,x)\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}^3$ represent the mass density and the velocity of a fluid in $\Omega \subset \mathbb{R}^3$, respectively, and $p(\rho) = b\rho^\gamma$ ($b > 0$, $\gamma > 1$) stands for the classical γ-law pressure, and both constants μ and λ are viscosity coefficients satisfying the physical constraints

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

Without loss of generality, we normalize $b = 1$ from now. We are concerned with the Cauchy problem of the 3D Navier–Stokes system (1.1) in $x = (x_1, x_2, x_3)^t \in \Omega := \mathbb{R} \times \mathbb{T}^2$ with $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$. The initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \to (\rho_\pm, u_\pm) \quad \text{as } x_1 \to \pm\infty \tag{1.2}$$

is prescribed with the far-fields conditions $\rho_\pm > 0$ and $u_\pm = (u_{1\pm}, 0, 0)^t$ as $x_1 \to \pm\infty$, and the periodic boundary conditions are imposed on $(x_2, x_3) \in \mathbb{T}^2$ for the solution (ρ, u).

The large-time asymptotic behavior of the solutions to the 3D compressible Navier–Stokes system (1.1)–(1.2) with different end states $(\rho_\pm, u_\pm)$ without shear is conjectured to be determined by the planar Riemann problem of the corresponding 3D Euler system

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p(\rho) = 0, \\ (\rho, u)(0, x) = \begin{cases} (\rho_-, u_-), & x_1 < 0, \\ (\rho_+, u_+), & x_1 > 0. \end{cases} \end{cases} \tag{1.3}$$

The solution to the Riemann problem (1.3) in general contains two nonlinear waves, i.e., shock and rarefaction waves and the above conjecture towards the time-asymptotic stability of the Riemann solutions is well established in the 1D case. In 1960, Il'in–Oleĭnik [7] first proved the stability of shock and rarefaction wave for the 1D scalar Burgers equation. Then Matsumura–Nishihara [22] proved the stability of viscous shock wave for the 1D compressible Navier–Stokes system with physical viscosity under the zero mass condition. Independently, Goodman [1] proved the same result for a general system with "artificial" diffusions. Then Liu [18], Szepessy–Xin [26] and Liu–Zeng [19] removed the crucial zero mass condition in [1, 22] by introducing the constant shift on the viscous shock, the diffusion waves and the coupled diffusion waves in the transverse characteristic fields. Masica–Zumbrun [20, 21] proved the spectral stability of viscous shock for the 1D compressible Navier–Stokes system under a spectral condition, which is slightly weaker than the zero mass condition. Huang–Matsumura [5] proved the stability of a composite wave consisting of two viscous shocks for the 1D full compressible Navier–Stokes equations with non-zero initial mass and the condition that the strengths of two viscous shocks are suitably small with same order. Note that all the above results for the stability of shocks are based on the classical anti-derivative techniques, which are essentially suitable to the 1D case and seems not applicable to the multi-dimensional system (1.1) directly.

On the other hand, the time-asymptotic stability of rarefaction wave for the 1D compressible Navier–Stokes system was proved by Matsumura–Nishihara [23, 24] by using the direct L^2-energy methods due to the expanding property of rarefaction wave. Very recently, Kang–Vasseur–Wang [11, 12] proved the stability of the combination wave of viscous shock and rarefaction (even with viscous contact) for the 1D compressible Navier–Stokes system by using a-contraction methods with the time-dependent shifts to overcome the difficulties caused by the incompatibility of viscous shock and rarefaction.

In multi-dimensions, Goodman [2] first proved the time-asymptotic stability of weak planar viscous shock for the scalar viscous equation by the anti-derivative techniques with the shift function depending on both time and spatially transverse directions, and then Hoff–Zumbrun [3, 4] extended Goodman's result to the large amplitude shock case. Recently, Kang–Vasseur–Wang [10] proved L^2-contraction of large planar viscous shocks up to a shift function depending on both time and spatial variables.

Comparatively speaking, there are very few results on the nonlinearly time-asymptotic stability of planar viscous shocks for the multi-dimensional Navier–Stokes system (1.1) due to the substantial difficulties in the high-dimensional propagation of shocks and the nonlinearities of the system. In 2017, Humpherys–Lyng–Zumbrun [6] proved the spectral stability of the planar viscous Navier–Stokes shocks under the spectral assumptions by the numerical Evans-function methods, and one can refer to the survey paper by Zumbrun [28] for the related results and the references therein.

The aim of this paper is to prove the nonlinearly time-asymptotic stability of planar viscous shock wave up to a time-dependent shift for the 3D compressible Navier–Stokes system (1.1) by using the weighted energy method under the generic H^2-perturbations without the zero mass conditions.

The compressibility of viscous shock, which substantially causes the "bad" sign terms in L^2 elementary entropy estimates, is the main difficulty in proving its time-asymptotic stability by energy methods. In the 1D case, the classical anti-derivative technique was developed to make full use of the compressibility of viscous shock, and then the zero mass conditions, or generic perturbations with non-zero mass distribution but with the constant shift on the viscous shock and the diffusion waves on transverse characteristic fields, are necessarily needed to clearly define the anti-derivative variables for the perturbation around the viscous shock [1, 18, 19, 22, 26]. However, the above anti-derivative techniques cannot be applied directly to prove the stability of planar viscous shocks for the multi-dimensional Navier–Stokes system (1.1). Alternatively, Kang–Vasseur [8, 9] developed the a-contraction method (a kind of weighted L^2-relative entropy method) with time-dependent shift to obtain L^2-contraction of shock wave to the viscous conservation laws. One of the advantages of a-contraction method is that it is not necessary to introduce the anti-derivative variables for the perturbation and fully use the time-dependent shift in the original perturbation to control the compressibility of shock. The idea can also be applied to prove the stability of planar viscous shock to the multi-dimensional scalar conservation laws [10] and the stability of the combination wave of viscous shock and rarefaction for the 1D compressible Navier–Stokes system [11].

Our proof of the time-asymptotic stability for the multi-dimensional Navier–Stokes shock is motivated by the a-contraction method. However, compared with 1D stability, there are several new difficulties:

(i) We need to establish a 3D version of the sharp weighted Poincaré inequality (see Lemma 3.1) together with the time-dependent shift $\mathbf{X}(t)$ defined in (3.9) to control the compressibility of planar shock.

(ii) For the stability analysis of the 1D Navier–Stokes system in [8, 9], Lagrangian structure of the system is fully utilized. However, this structure cannot be kept in Eulerian coordinates, especially in the multi-dimensional case. Therefore, we need to find a new effective velocity $h := u - (2\mu + \lambda)\nabla_\xi v$ (see also (4.1)) in 3D Eulerian coordinates and the rewritten system (see (4.2)) has the similar stability structure as one in Lagrangian coordinates.

(iii) Some physical underlying structures of the multi-dimensional Navier–Stokes system (1.1) are used. We use the Hodge decomposition to decompose the diffusive term $\Delta_\xi u$ into the irrotational part $\nabla_\xi \text{div}_\xi u$ and the rotational part $\nabla_\xi \times \nabla_\xi \times u$ and borrow some ideas from the stability of planar rarefaction wave in [14–17] to overcome the wave propagations along the transverse directions and their interactions with the planar viscous shock.

The rest part of the paper is organized as follows. In Section 2, we first list the properties of the planar viscous shock and then state our main result. In Section 3, we first present some useful functional inequalities, and then construct the shift function and give the proof of our main theorem based on the local existence in Proposition 3.5 and uniform in time a priori estimates in Proposition 3.6. In Section 4, we reformulate the problem in new variable function (v, h) first, and then prove the uniform in time H^2-estimates in Proposition 3.6.

2. Planar viscous shock and main result

In this section, we first describe the planar viscous shock and then state our main result on the time-asymptotic stability of planar viscous shock for the 3D compressible Navier–Stokes equations (1.1) under generic H^2-perturbations without zero mass conditions.

2.1. Viscous shock wave

First we depict planar viscous shock. For definiteness, we consider 2-shock and 1-shock case that can be treated similarly.

It is well known that the Riemann problem of 1D compressible Euler system

$$\begin{cases} \partial_t \rho + \partial_{x_1}(\rho u_1) = 0, \\ \partial_t(\rho u_1) + \partial_{x_1}(\rho u_1^2 + p(\rho)) = 0, \end{cases} \tag{2.1}$$

with Riemann initial data

$$(\rho, u_1)(0, x_1) = (\rho_0, u_{10})(x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < 0, \\ (\rho_+, u_{1+}), & x_1 > 0, \end{cases}$$

determined by the far-field states (1.2), admits a 2-shock wave solution with the speed σ

$$(\rho, u_1)(t, x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < \sigma t, \\ (\rho_+, u_{1+}), & x_1 > \sigma t. \end{cases}$$

This provided that the following Rankine–Hugoniot condition:

$$\begin{cases} -\sigma(\rho_+ - \rho_-) + (\rho_+ u_{1+} - \rho_- u_{1-}) = 0, \\ -\sigma(\rho_+ u_{1+} - \rho_- u_{1-}) + (\rho_+ u_{1+}^2 - \rho_- u_{1-}^2) + (p(\rho_+) - p(\rho_-)) = 0, \end{cases} \tag{2.2}$$

and the Lax entropy condition

$$\Lambda_2(\rho_+, u_{1+}) < \sigma < \Lambda_2(\rho_-, u_{1-})$$

with $\Lambda_2(\rho, u_1) = u_1 + \sqrt{p'(\rho)}$ being the second eigenvalue of the Jacobi matrix of the Euler system (2.1), hold true. Denote by δ the 2-shock wave strength $\delta := |p(v_-) - p(v_+)| \sim |v_+ - v_-| \sim |u_{1-} - u_{1+}|$ and set

$$\xi = (\xi_1, \xi_2, \xi_3) \quad \text{with } \xi_1 = x_1 - \sigma t \text{ and } \xi_i = x_i,\ i = 2, 3.$$

Correspondingly, planar 2-viscous shock wave $(\rho^s, u^s)(\xi_1)$ with $u^s(\xi_1) := (u_1^s(\xi_1), 0, 0)^t$, connecting (ρ_-, u_-) and (ρ_+, u_+), for the 3D compressible Navier–Stokes equations satisfies the ODE system

$$\begin{cases} -\sigma(\rho^s)' + (\rho^s u_1^s)' = 0, \quad (\cdot)' := \dfrac{d(\cdot)}{d\xi_1}, \\ -\sigma(\rho^s u_1^s)' + (\rho^s (u_1^s)^2)' + p(\rho^s)' = (2\mu + \lambda)(u_1^s)'', \end{cases} \tag{2.3}$$

for the far-field conditions

$$(\rho^s, u_1^s)(-\infty) = (\rho_-, u_{1-}), \quad (\rho^s, u_1^s)(+\infty) = (\rho_+, u_{1+}). \tag{2.4}$$

Using new variables (t, ξ), we can rewrite system (1.1) as

$$\begin{cases} \partial_t \rho - \sigma \partial_{\xi_1} \rho + \mathrm{div}_\xi(\rho u) = 0, \\ \partial_t(\rho u) - \sigma \partial_{\xi_1}(\rho u) + \mathrm{div}_\xi(\rho u \otimes u) + \nabla_\xi p(\rho) = \mu \Delta_\xi u + (\mu + \lambda)\nabla_\xi \mathrm{div}_\xi u. \end{cases} \tag{2.5}$$

If we introduce the volume function $v = \frac{1}{\rho}$, then we can further rewrite system (2.5) as

$$\begin{cases} \rho(\partial_t v - \sigma \partial_{\xi_1} v + u \cdot \nabla_\xi v) = \mathrm{div}_\xi u, \\ \rho(\partial_t u - \sigma \partial_{\xi_1} u + u \cdot \nabla_\xi u) + \nabla_\xi p(v) = (2\mu + \lambda)\nabla_\xi \mathrm{div}_\xi u - \mu \nabla_\xi \times \nabla_\xi \times u, \end{cases} \tag{2.6}$$

where the pressure is defined by $p(v) = bv^{-\gamma}$ and we have used the identity

$$\Delta_\xi u = \nabla_\xi \mathrm{div}_\xi u - \nabla_\xi \times \nabla_\xi \times u$$

for the viscosity term.

Similarly, by using the volume function

$$v^s := \frac{1}{\rho^s},$$

the ODE system (2.3) is transformed into

$$\begin{cases} \rho^s(-\sigma(v^s)' + u_1^s(v^s)') = (u_1^s)', \\ \rho^s(-\sigma(u_1^s)' + u_1^s(u_1^s)') + p(v^s)' = (2\mu + \lambda)(u_1^s)'', \end{cases} \tag{2.7}$$

where $p(v^s) = (v^s)^{-\gamma}$. Integrating $(2.3)_1$ on $(-\infty, \xi_1)$, we get

$$-\sigma\rho^s + \rho^s u_1^s = -\sigma\rho_- + \rho_- u_{1-} =: -\sigma_*. \tag{2.8}$$

Therefore, system (2.7) and far-field conditions (2.4) can be rewritten as

$$\begin{cases} -\sigma_*(v^s)' = (u_1^s)', \\ -\sigma_*(u_1^s)' + p(v^s)' = (2\mu + \lambda)(u_1^s)'', \end{cases} \tag{2.9}$$

and

$$(v^s, u_1^s)(-\infty) = (v_-, u_{1-}), \quad (v^s, u_1^s)(+\infty) = (v_+, u_{1+}), \quad v_\pm = \frac{1}{\rho_\pm}. \tag{2.10}$$

By (2.9) (or (2.2)) and (2.8), it holds that

$$\begin{cases} -\sigma_*(v_+ - v_-) = u_{1+} - u_{1-}, \\ -\sigma_*(u_{1+} - u_{1-}) + p(v_+) - p(v_-) = 0. \end{cases}$$

Therefore, we have

$$\sigma_* = \sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}} > 0$$

for 2-shock.

The existence and properties of the 2-viscous shock wave $(v^s, u_1^s)(\xi_1)$ can be summarized in the following lemma, while its proof can be found in [22].

Lemma 2.1. *Fix the right end state (v_+, u_{1+}) and for any left end state $(v_-, u_{1-}) \in S_2(v_+, u_{1+})$, there exists a unique (up to a constant shift) solution $(v^s, u_1^s)(\xi_1)$ to system (2.9), (2.10), and moreover the following hold:*

$$v^s_{\xi_1} > 0, \quad u^s_{1\xi_1} = -\sigma_* v^s_{\xi_1} < 0,$$

and

$$\begin{aligned}
|v^s(\xi_1) - v_-| &\le C\delta e^{-C\delta|\xi_1|} && \forall \xi_1 < 0,\\
|v^s(\xi_1) - v_+| &\le C\delta e^{-C\delta|\xi_1|} && \forall \xi_1 > 0,\\
|(v^s_{\xi_1}, u^s_{1\xi_1})| &\le C\delta^2 e^{-C\delta|\xi_1|} && \forall \xi_1 \in \mathbb{R},\\
|(v^s_{\xi_1\xi_1}, u^s_{1\xi_1\xi_1})| &\le C\delta |(v^s_{\xi_1}, u^s_{1\xi_1})| && \forall \xi_1 \in \mathbb{R},\\
|(v^s_{\xi_1\xi_1\xi_1}, u^s_{1\xi_1\xi_1\xi_1})| &\le C\delta^2 |(v^s_{\xi_1}, u^s_{1\xi_1})| && \forall \xi_1 \in \mathbb{R}.
\end{aligned}$$

Remark 2.2. In fact, Lemma 2.1 can be proved by using ODE (2.3)–(2.4) for $(\rho^s, u_1^s)(\xi_1)$, and then to $(v^s, u_1^s)(\xi_1)$, not directly from system (2.9)–(2.10).

2.2. *Main result*

Now we can state the main result as follows.

Theorem 2.3. *Let $(\rho^s, u^s)(x_1 - \sigma t)$ be the planar 2-viscous shock wave defined in (2.3)–(2.4) with $u^s := (u_1^s, 0, 0)^t$. Then there exist positive constants δ_0, ε_0 such that if the shock wave strength $\delta \le \delta_0$, and the initial data (ρ_0, u_0) satisfies*

$$\|(\rho_0(x) - \rho^s(x_1), u_0(x) - u^s(x_1))\|_{H^2(\mathbb{R}\times\mathbb{T}^2)} \le \varepsilon_0,$$

then the 3D compressible Navier–Stokes equations (1.1) admit a unique global in time solution $(\rho, u)(t, x)$, and there exists an absolutely continuous shift $\mathbf{X}(t)$ such that

$$\begin{aligned}
\rho(t,x) - \rho^s(x_1 - \sigma t - \mathbf{X}(t)) &\in C([0,+\infty); H^2(\mathbb{R}\times\mathbb{T}^2)),\\
u(t,x) - u^s(x_1 - \sigma t - \mathbf{X}(t)) &\in C([0,+\infty); H^2(\mathbb{R}\times\mathbb{T}^2)),\\
\nabla_x(\rho(t,x) - \rho^s(x_1 - \sigma t - \mathbf{X}(t))) &\in L^2(0,+\infty; H^1(\mathbb{R}\times\mathbb{T}^2)),\\
\nabla_x(u(t,x) - u^s(x_1 - \sigma t - \mathbf{X}(t))) &\in L^2(0,+\infty; H^2(\mathbb{R}\times\mathbb{T}^2)).
\end{aligned} \tag{2.11}$$

Furthermore, the planar 2-viscous shock wave $(\rho^s, u^s)(x_1 - \sigma t)$ is time-asymptotically stable with the time-dependent shift $\mathbf{X}(t)$,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}\times\mathbb{T}^2} |(\rho, u)(t,x) - (\rho^s, u^s)(x_1 - \sigma t - \mathbf{X}(t))| = 0, \tag{2.12}$$

and the shift function $\mathbf{X}(t)$ satisfies the time-asymptotic behavior

$$\lim_{t\to\infty} |\dot{\mathbf{X}}(t)| = 0. \tag{2.13}$$

Remark 2.4. Theorem 2.3 states that if the two far-fields states $(\rho_\pm, u_\pm)$ in (1.2) are connected by the shock wave, then the solution to the 3D compressible Navier–Stokes equations (1.1), or equivalently (2.6), tends to the corresponding planar viscous shock with the time-dependent shift $\mathbf{X}(t)$ under the generic H^2-perturbations, in particular, without zero mass conditions.

Remark 2.5. The shift function $\mathbf{X}(t)$ is proved to satisfy the time-asymptotic behavior (2.13), which implies that

$$\lim_{t\to+\infty} \frac{\mathbf{X}(t)}{t} = 0,$$

that is, the shift function $\mathbf{X}(t)$ grows at most sub-linearly with respect to the time t, and therefore, the shifted planar viscous shock wave $(\rho^s, u^s)(x_1 - \sigma t - \mathbf{X}(t))$ keeps the original traveling wave profile time-asymptotically.

Remark 2.6. Theorem 2.3 is the first analytical result on the time-asymptotic stability of planar viscous shock wave to the multi-dimensional system (1.1) with physical viscosities as far as we know. Moreover, our stability result is unconditional for the weak planar Navier–Stokes shock in the 3D case.

Notation. Throughout this paper, several positive generic constants are denoted by C without confusion. We define

$$x' := (x_2, x_3), \quad dx' := dx_2 dx_3, \quad \text{and} \quad \xi' := (\xi_2, \xi_3), \quad d\xi := d\xi_2 d\xi_3.$$

For $1 \le r \le \infty$, we denote $L^r := L^r(\Omega) = L^r(\mathbb{R} \times \mathbb{T}^2)$ and use the notation $\|\cdot\| := \|\cdot\|_{L^2}$. For a non-negative integer m, the space $H^m(\Omega)$ denotes the m-th order Sobolev space over Ω in the L^2-sense with the norm

$$\|f\|_{H^m} := \left(\sum_{l=0}^{m} \|\nabla^l f\|^2 \right)^{\frac{1}{2}},$$

$$\|f\| := \left(\int_\Omega |f|^2 d\xi \right)^{\frac{1}{2}} = \left(\int_{\mathbb{T}^2} \int_{\mathbb{R}} |f|^2 d\xi_1 d\xi' \right)^{\frac{1}{2}}.$$

Also, we denote

$$\|(f, g)\|_{H^m} = \|f\|_{H^m} + \|g\|_{H^m}.$$

3. Proof of main result

3.1. Some functional inequalities and local existence of solution

We first present a 3D weighted sharp Poincaré type inequality, which is a 3D version of the 1D weighted Poincaré inequality in [9] and plays a very important role in our stability analysis.

Lemma 3.1. *For any* $f\colon[0,1]\times\mathbb{T}^2\to\mathbb{R}$ *satisfying*

$$\int_{\mathbb{T}^2}\int_0^1\Big[y_1(1-y_1)|\partial_{y_1}f|^2+\frac{|\nabla_{y'}f|^2}{y_1(1-y_1)}\Big]dy_1dy'<\infty,$$

the following inequality holds:

$$\begin{aligned}\int_{\mathbb{T}^2}\int_0^1|f-\overline{f}|^2dy_1dy' &\le \frac{1}{2}\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1}f|^2dy_1dy'\\ &\quad+\frac{1}{16\pi^2}\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'}f|^2}{y_1(1-y_1)}dy_1dy',\end{aligned}\tag{3.1}$$

where $\overline{f}=\int_{\mathbb{T}^2}\int_0^1 f\,dy_1dy'$ *and* $dy'=dy_2dy_3$.

Proof. The proof is motivated by that of the 1D weighted Poincaré inequality in [9], and here we need to concern the transverse directions $(y_2,y_3)\in\mathbb{T}^2$ additionally. Let $\{P_n\colon[-1,1]\to\mathbb{R}\}_{n\in\mathbb{Z},n\ge 0}$ be an orthogonal basis of the Legendre polynomials, that are the solutions to the Legendre differential equations

$$\frac{d}{dx_1}\Big((1-x_1^2)\frac{d}{dx_1}P_n(x_1)\Big)=-n(n+1)P_n(x_1),\tag{3.2}$$

and satisfy the orthogonality in $L^2[-1,1]$, i.e., $\int_{-1}^1 P_iP_j\,dx_1=\delta_{ij}$. Then for fixed $x'=(x_2,x_3)\in\mathbb{T}^2$ and any $w(\cdot,x')\in L^2[-1,1]$, we have

$$w(x_1,x')=\sum_{i=0}^\infty c_i(x')P_i(x_1),\quad c_i(x')=\int_{-1}^1 w(x_1,x')P_i(x_1)dx_1.$$

In particular, we see that $P_0(x_1)=\frac{1}{\sqrt{2}}=:P_0$, thus

$$\int_{\mathbb{T}^2}c_0(x')P_0(x_1)dx'=P_0\int_{\mathbb{T}^2}c_0(x')dx'=\frac{1}{2}\int_{\mathbb{T}^2}\int_{-1}^1 w(x_1,x')dx_1dx'=:\overline{w}.$$

If we set $\overline{c}_0:=\int_{\mathbb{T}^2}c_0(x')dx'$, then

$$\begin{aligned}w(x_1,x')-\overline{w}&=\sum_{i=0}^\infty c_i(x')P_i(x_1)-\int_{\mathbb{T}^2}c_0(x')P_0dx'\\ &=\sum_{i=1}^\infty c_i(x')P_i(x_1)+(c_0(x')-\overline{c}_0)P_0.\end{aligned}$$

Then we have

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{-1}^1|w-\overline{w}|^2dx_1dx'&=\sum_{i=1}^\infty\int_{\mathbb{T}^2}\int_{-1}^1 c_i^2(x')P_i^2(x_1)dx_1dx'+\int_{\mathbb{T}^2}|c_0-\overline{c}_0|^2dx'\\ &=:A_1+A_2.\end{aligned}$$

By the Legendre differential equations (3.2), we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\int_{-1}^{1}(1-x_1^2)|\partial_{x_1} w|^2 dx_1 dx' \\
&\quad = -\int_{\mathbb{T}^2}\int_{-1}^{1}\partial_{x_1}((1-x_1^2)\partial_{x_1} w) w dx_1 dx' \\
&\quad = -\int_{\mathbb{T}^2}\int_{-1}^{1}\partial_{x_1}((1-x_1^2)\partial_{x_1} w)(w-\overline{w}) dx_1 dx' \\
&\quad = -\int_{\mathbb{T}^2}\int_{-1}^{1}\partial_{x_1}\left(\sum_{i=1}^{\infty} c_i(x')(1-x_1^2)\frac{d}{dx_1}P_i(x_1)\right)\sum_{j=1}^{\infty} c_j(x')P_j(x_1) dx_1 dx' \\
&\qquad -\int_{\mathbb{T}^2}\int_{-1}^{1}\partial_{x_1}\left(\sum_{i=1}^{\infty} c_i(x')(1-x_1^2)\frac{d}{dx_1}P_i(x_1)\right)(c_0(x')-\overline{c}_0)P_0 dx_1 dx' \\
&\quad = \sum_{i=1}^{\infty}\int_{\mathbb{T}^2}\int_{-1}^{1} i(i+1)c_i^2(x')P_i^2(x_1) dx_1 dx' \\
&\quad \geq 2\sum_{i=1}^{\infty}\int_{\mathbb{T}^2}\int_{-1}^{1} c_i^2(x')P_i^2(x_1) dx_1 dx' = 2A_1,
\end{aligned}
$$

which implies that

$$
A_1 \leq \frac{1}{2}\int_{\mathbb{T}^2}\int_{-1}^{1}(1-x_1^2)|\partial_{x_1} w|^2 dx_1 dx'.
$$

For A_2, by using the Poincaré inequality, we get

$$
\begin{aligned}
A_2 &\leq \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}|\nabla_{x'} c_0|^2 dx' \\
&= \frac{1}{4\pi^2}\int_{\mathbb{T}^2}\left(\left(\int_{-1}^{1}\partial_{x_2} w P_0 dx_1\right)^2 + \left(\int_{-1}^{1}\partial_{x_3} w P_0 dx_1\right)^2\right) dx' \\
&= \frac{1}{8\pi^2}\int_{\mathbb{T}^2}\left(\left(\int_{-1}^{1}\partial_{x_2} w dx_1\right)^2 + \left(\int_{-1}^{1}\partial_{x_3} w dx_1\right)^2\right) dx' \\
&\leq \frac{1}{4\pi^2}\int_{\mathbb{T}^2}\int_{-1}^{1}(|\partial_{x_2} w|^2 + |\partial_{x_3} w|^2) dx_1 dx' \\
&= \frac{1}{4\pi^2}\int_{\mathbb{T}^2}\int_{-1}^{1}|\nabla_{x'} w|^2 dx_1 dx'.
\end{aligned}
$$

Combining the above two estimates on A_1 and A_2, we have

$$
\begin{aligned}
\int_{\mathbb{T}^2}\int_{-1}^{1}|w-\overline{w}|^2 dx_1 dx' \leq{}& \frac{1}{2}\int_{\mathbb{T}^2}\int_{-1}^{1}(1-x_1^2)|\partial_{x_1} w|^2 dx_1 dx' \\
&+ \frac{1}{4\pi^2}\int_{\mathbb{T}^2}\int_{-1}^{1}|\nabla_{x'} w|^2 dx_1 dx'.
\end{aligned}
$$

By a change of variables $y_1 := \frac{x_1+1}{2}$, $y' := x'$, $W(y_1, y') := w(2y_1 - 1, y') = w(x_1, x')$, we get

$$\int_{\mathbb{T}^2}\int_0^1 |W - \bar{W}|^2 dy_1 dy' \le \frac{1}{2}\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1} W|^2 dy_1 dy' + \frac{1}{4\pi^2}\int_{\mathbb{T}^2}\int_0^1 |\nabla_{y'} W|^2 dy_1 dy',$$

where $\bar{W} := \int_{\mathbb{T}^2}\int_0^1 W dy_1 dy'$. Notice that $0 \le y_1(1-y_1) \le \frac{1}{4}$ for $y_1 \in [0,1]$, and so

$$\frac{1}{y_1(1-y_1)} \ge 4.$$

Then we can prove (3.1). ■

Now we present a 3D Gagliardo–Nirenberg inequality in the domain $\Omega = \mathbb{R} \times \mathbb{T}^2$, whose proof can be found in [13, 27].

Lemma 3.2. *It holds for $g(x) \in H^2(\Omega)$ with $x = (x_1, x_2, x_3) \in \Omega := \mathbb{R} \times \mathbb{T}^2$ that*

$$\|g\|_{L^\infty(\Omega)} \le \sqrt{2}\|g\|_{L^2(\Omega)}^{\frac{1}{2}}\|\partial_{x_1} g\|_{L^2(\Omega)}^{\frac{1}{2}} + C\|\nabla_x g\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla_x^2 g\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{3.3}$$

where $C > 0$ is a positive constant.

Then we list several estimates of the relative quantities. For any function F defined on $\mathbb{R}_+$, we define the associated relative quantity for v, $w \in \mathbb{R}_+$ as

$$F(v|w) = F(v) - F(w) - F'(w)(v - w).$$

We gather, in the following lemma, some useful inequalities on the relative quantities associated with the pressure $p(v) = v^{-\gamma}$ and the internal energy $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$. The proofs are based on the Taylor expansions and can be found in [9].

Lemma 3.3. *For given constants $\gamma > 1$ and $v_- > 0$, there exist constants $C, \delta_* > 0$ such that the following hold true:*

(1) *For any v, w such that $0 < w < 2v_-$, $0 < v \le 3v_-$,*

$$|v - w|^2 \le CQ(v|w), \quad |v - w|^2 \le Cp(v|w). \tag{3.4}$$

(2) *For any $v, w > \frac{v_-}{2}$,*

$$|p(v) - p(w)| \le C|v - w|. \tag{3.5}$$

(3) *For any $0 < \delta < \delta_*$, and for any $(v, w) \in \mathbb{R}_+^2$ satisfying $|p(v) - p(w)| < \delta$, and $|p(w) - p(v_-)| < \delta$,*

$$p(v|w) \le \Big(\frac{\gamma+1}{2\gamma}\frac{1}{p(w)} + C\delta\Big)|p(v) - p(w)|^2, \tag{3.6}$$

$$Q(v|w) \ge \frac{|p(v) - p(w)|^2}{2\gamma p^{1+\frac{1}{\gamma}}(w)} - \frac{1+\gamma}{3\gamma^2}\frac{(p(v) - p(w))^3}{p^{2+\frac{1}{\gamma}}(w)}, \tag{3.7}$$

$$Q(v|w) \le \Big(\frac{1}{2\gamma p^{1+\frac{1}{\gamma}}(w)} + C\delta\Big)|p(v) - p(w)|^2. \tag{3.8}$$

Finally, we give an estimate involving the inverse of the pressure function $p(v) = v^{-\gamma}$, while its proof can be found in [9], and the local existence of classical solution to the 3D compressible Navier–Stokes equations (1.1).

Lemma 3.4. *Fix $v_- > 0$. Then there exist $\delta_0 > 0$ and $C > 0$ such that for any $v_+ > 0$, such that $0 < \delta := p(v_-) - p(v_+) \le \delta_0$, $v_- \le v \le v_+$, we have*

$$\Big|\frac{v - v_-}{p(v) - p(v_-)} + \frac{v - v_+}{p(v_+) - p(v)} + \frac{1}{2}\frac{p''(v_-)}{p'(v_-)^2}(v_- - v_+)\Big| \le C\delta^2.$$

For the classical solution, system (1.1) for $(\rho, u)(t, x)$ is equivalent to system (2.6) for $(v, u)(t, \xi)$. We then work with system (2.6) for $(v, u)(t, \xi)$.

Proposition 3.5 (Local existence). *Let $(v^s, u^s)(\xi_1)$ be the planar 2-viscous shock wave with $u^s(\xi_1) := (u_1^s(\xi_1), 0, 0)^t$. For any $\Xi > 0$, suppose the initial data $(v_0, u_0)(x)$ satisfies*

$$\|(v_0(x) - v^s(x_1), u_0(x) - u^s(x_1))\|_{H^2(\mathbb{R}\times\mathbb{T}^2)} \le \Xi.$$

Then there exists a positive constant T_0 depending on Ξ such that the 3D compressible Navier–Stokes system (2.6) has a unique solution $(v, u)(t, \xi)$ on $(0, T_0)$ satisfying

$$\begin{aligned} v - v^s &\in C([0, T_0]; H^2(\mathbb{R}\times\mathbb{T}^2)), \quad \nabla_\xi(v - v^s) \in L^2(0, T_0; H^1(\mathbb{R}\times\mathbb{T}^2)),\\ u - u^s &\in C([0, T_0]; H^2(\mathbb{R}\times\mathbb{T}^2)), \quad \nabla_\xi(u - u^s) \in L^2(0, T_0; H^2(\mathbb{R}\times\mathbb{T}^2)), \end{aligned}$$

and for $t \in [0, T_0]$, it holds that

$$\begin{aligned} &\sup_{\tau\in[0,t]} \|(v - v^s, u - u^s)(\tau)\|_{H^2}^2 + \int_0^t (\|\nabla_\xi(v - v^s)\|_{H^1}^2 + \|\nabla_\xi(u - u^s)\|_{H^2}^2)d\tau\\ &\quad \le 4\|(v_0 - v^s, u_0 - u^s)\|_{H^2}^2. \end{aligned}$$

Proposition 3.5 can be proved by a standard way, see [25].

3.2. *Construction of shift function* $\mathbf{X}(t)$

For notational simplification for any function $f(\xi_1)$, denote

$$f^{-\mathbf{X}}(\xi_1) := f(\xi_1 - \mathbf{X}(t)),$$

where the shift function $\mathbf{X}(t)$ is defined in (3.9).

The definition of the shift function $\mathbf{X}(t)$ depends on the weight function $a\colon \mathbb{R} \to \mathbb{R}$ defined in (4.8). For now, we will only assume that $\|a\|_{C^1(\mathbb{R})} \le 2$. Then we can define the shift $\mathbf{X}(t)$ as a solution to the ODE

$$\begin{cases} \dot{\mathbf{X}}(t) = -\dfrac{M}{\delta}\Big[\displaystyle\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{a^{-\mathbf{X}}(\xi_1)}{\sigma_*}\rho(h_1^s)_{\xi_1}^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))d\xi_1 d\xi'\\ \qquad\qquad - \displaystyle\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(\xi_1)\rho p'((v^s)^{-\mathbf{X}})(v - (v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} d\xi_1 d\xi'\Big],\\ \mathbf{X}(0) = 0, \end{cases} \tag{3.9}$$

where the function $h_1^s := u_1^s - (2\mu + \lambda)\partial_{\xi_1} v^s$ defined in (4.4) and the constant $M := \frac{5}{4}\frac{\gamma+1}{2\gamma}\frac{\sigma_-^3 v_-^2}{p(v_-)}$ with $\sigma_- = \sqrt{-p'(v_-)}$.

Let $Z(t, \mathbf{X}(t))$ be the right-hand side of equation $(3.9)_1$. Thanking to the facts that $\|a\|_{C^1(\mathbb{R})} \le 2$, $\|v^s\|_{C^2(\mathbb{R})} \le v_+$, and $\|v^s_{\xi_1}\|_{L^1(\mathbb{R})} \le C\delta$, we can find some constant $C > 0$, such that

$$\sup_{\mathbf{X}\in\mathbb{R}} |Z(t, \mathbf{X})| \le \frac{C}{\delta}\|a\|_{C^1}(\|v\|_{L^\infty} + \|(v^s)^{-\mathbf{X}}\|_{L^\infty}) \int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}| d\xi_1 d\xi' \le C, \quad (3.10)$$

and

$$\sup_{\mathbf{X}\in\mathbb{R}} |\partial_{\mathbf{X}} Z(t, \mathbf{X})| \le \frac{C}{\delta}\|a\|_{C^1}(\|v\|_{L^\infty} + \|(v^s)^{-\mathbf{X}}\|_{L^\infty}) \int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}| d\xi_1 d\xi' \le C.$$

Then ODE (3.9) has a unique absolutely continuous solution $\mathbf{X}(t)$ defined on any interval in time $[0, T]$ by the well-known Cauchy–Lipschitz theorem. In particular, since $|\dot{\mathbf{X}}(t)| \le C$ by (3.10), we can obtain $|\mathbf{X}(t)| \le Ct$ $\forall t \in [0, T]$.

3.3. *Proof of Theorem 2.3*

In order to prove Theorem 2.3, we shall combine a local existence result from Proposition 3.5 together with a priori estimates from Proposition 3.6 by continuity arguments.

Proposition 3.6 (A priori estimates). *Suppose that $(v, u)(t, \xi)$ is the solution to (2.6) on $[0, T]$ for some $T > 0$, and $((v^s)^{-\mathbf{X}}, (u^s)^{-\mathbf{X}})(\xi_1)$ is the solution to (2.7) with the shift function $\mathbf{X} = \mathbf{X}(t)$, which is an absolutely continuous solution to (3.9). Then there exist positive constants $\delta_0 \le 1$, $\chi_0 \le 1$ and C_0 independent of T, such that if the shock wave strength $\delta < \delta_0$ and*

$$\begin{aligned} &v - (v^s)^{-\mathbf{X}} \in C([0, T]; H^2(\mathbb{R}\times\mathbb{T}^2)), \quad \nabla_\xi(v - (v^s)^{-\mathbf{X}}) \in L^2(0, T; H^1(\mathbb{R}\times\mathbb{T}^2)),\\ &u - (u^s)^{-\mathbf{X}} \in C([0, T]; H^2(\mathbb{R}\times\mathbb{T}^2)), \quad \nabla_\xi(u - (u^s)^{-\mathbf{X}}) \in L^2(0, T; H^2(\mathbb{R}\times\mathbb{T}^2)) \end{aligned}$$

with

$$\chi := \sup_{0\le t\le T} \|(v - (v^s)^{-\mathbf{X}}, u - (u^s)^{-\mathbf{X}})(t, \cdot)\|_{H^2} \le \chi_0, \quad (3.11)$$

then the following estimate holds:

$$\begin{aligned} &\sup_{0\le t\le T} \|(v - (v^s)^{-\mathbf{X}}, u - (u^s)^{-\mathbf{X}})(t, \cdot)\|_{H^2}^2 + \delta\int_0^T |\dot{\mathbf{X}}(t)|^2 dt\\ &+ \int_0^T (\|\sqrt{|(v^s)^{-\mathbf{X}}_{\xi_1}|}(v - (v^s)^{-\mathbf{X}})\|^2 + \|\nabla_\xi(v - (v^s)^{-\mathbf{X}})\|_{H^1}^2 + \|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|_{H^2}^2) dt\\ &\le C_0\|(v_0 - v^s, u_0 - u^s)\|_{H^2}^2. \end{aligned} \quad (3.12)$$

In addition, by (3.9), we have

$$|\dot{\mathbf{X}}(t)| \le C_0\|v - (v^s)^{-\mathbf{X}}(t, \cdot)\|_{L^\infty} \quad \forall t \le T. \quad (3.13)$$

Proposition 3.6 will be proved in Section 4.

Based on Propositions 3.5 and 3.6, we can prove the global existence part of Theorem 2.3 by the continuity arguments, while the time-asymptotic behavior part will be left at the end of the paper.

Proof of Theorem 2.3. We first prove (2.11) in Theorem 2.3 by the continuation method based on Propositions 3.5 and 3.6. Consider the maximal existence time of the solution

$$T_{\max} := \sup\{t > 0 \mid \sup_{\tau\in[0,t]} \|(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})(\tau)\|_{H^2} \le \chi_0\}. \tag{3.14}$$

We shall show the maximal existence time $T_{\max} = +\infty$ by the following steps. We define

$$\varepsilon_0 = \min\Big\{\frac{\chi_0}{4}, \frac{\chi_0}{8\sqrt{C_0}}\Big\}, \quad \Xi = \frac{\chi_0}{4},$$

where χ_0 and C_0 are given in Proposition 3.6.

Step 1: Suppose that $\|(v_0-v^s, u_0-u^s)\|_{H^2} \le \varepsilon_0 \le \frac{\chi_0}{4} (=\Xi)$, by local existence result in Proposition 3.5, there is a positive constant $T_0 = T_0(\Xi)$ such that a unique solution exists on $[0, T_0]$ and satisfies $\|(v-v^s, u-u^s)(t)\|_{H^2} \le 2\|(v_0-v^s, u_0-u^s)\|_{H^2} \le 2\Xi = \frac{\chi_0}{2}$ for $t \in [0, T_0]$. Without loss of generality, we can assume $T_0 \le 1$. Then the Sobolev inequality implies $\|(v-v^s)(t)\|_{L^\infty} \le C\chi_0$ for $t \in [0, T_0]$. Using $v_- < v^s(\xi_1) < v_+$ and the smallness of χ_0 in Proposition 3.6, we get $\frac{v_-}{2} < v(t,\xi) < 2v_+$ for $(t,\xi) \in [0, T_0]\times\Omega$. Therefore, we can see that (3.10) holds for $t \in [0, T_0]$, and we can deduce from (3.9) that $|\mathbf{X}(t)| \le Ct$ for $t \in [0, T_0]$. Then by the mean value theorem, we obtain

$$\|(v^s-(v^s)^{-\mathbf{X}}, u^s-(u^s)^{-\mathbf{X}})(t)\|_{H^2} = |\mathbf{X}(t)|\|(v^s_{\xi_1}, u^s_{\xi_1})\|_{H^2} \le C\delta^{\frac{3}{2}} t \le C\delta_0 \le \frac{\chi_0}{8}$$

for suitably small δ_0. Therefore, it holds for $t \in [0, T_0]$ that

$$\begin{aligned}&\|(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})(t)\|_{H^2}\\ &\quad\le \|(v-v^s, u-u^s)(t)\|_{H^2} + \|(v^s-(v^s)^{-\mathbf{X}}, u^s-(u^s)^{-\mathbf{X}})(t)\|_{H^2}\\ &\quad\le \frac{\chi_0}{2} + \frac{\chi_0}{8} < \chi_0.\end{aligned}$$

Hence, we can apply the a priori estimates from Proposition 3.6 with $T = T_0$ and get the estimate

$$\|(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})(t)\|_{H^2} \le \sqrt{C_0}\|(v_0-v^s, u_0-u^s)\|_{H^2} \le \sqrt{C_0}\varepsilon_0 \le \frac{\chi_0}{8}$$

for $t \in [0, T_0]$.

Step 2: If the maximal existence time $T_{\max} < +\infty$, then there is a positive integer $N \ge 1$, which may depend on χ_0, such that $T_{\max} \in ((N-1)T_0, NT_0]$. We can choose the small constant δ_0 satisfying $\sqrt{\delta_0} \le \frac{1}{N+1}$. We know from step 1 that

$$\begin{aligned}&\|(v-v^s, u-u^s)(T_0)\|_{H^2}\\ &\quad\le \|(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})(T_0)\|_{H^2} + \|(v^s-(v^s)^{-\mathbf{X}}, u^s-(u^s)^{-\mathbf{X}})(T_0)\|_{H^2}\\ &\quad\le \frac{\chi_0}{8} + \frac{\chi_0}{8} = \frac{\chi_0}{4}\ (=\Xi).\end{aligned}$$

Hence, we can apply local existence result in Proposition 3.5 by taking $t = T_0$ as the new initial time. Then we have a unique solution on $[T_0, 2T_0]$ with the estimate

$$\|(v - v^s, u - u^s)(t)\|_{H^2} \le 2\|(v - v^s, u - u^s)(T_0)\|_{H^2} \le 2\Xi = \frac{\chi_0}{2}$$

for $t \in [T_0, 2T_0]$. This together with step 1 implies that $\|(v - v^s, u - u^s)(t)\|_{H^2} \le \frac{\chi_0}{2}$ holds for $t \in [0, 2T_0]$. Similarly to step 1, we can show that $|\mathbf{X}(t)| \le Ct$ holds for $t \in [0, 2T_0]$. Using the smallness of δ_0, we have

$$\begin{aligned}\|(v^s - (v^s)^{-\mathbf{X}}, u^s - (u^s)^{-\mathbf{X}})(t)\|_{H^2} &= |\mathbf{X}(t)|\|(v^s_{\xi_1}, u^s_{\xi_1})\|_{H^2} \\ &\le C\delta^{\frac{3}{2}} t \le 2C\delta_0\sqrt{\delta_0} \le C\delta_0 \le \frac{\chi_0}{8}.\end{aligned}$$

Therefore, it holds for $t \in [0, 2T_0]$ that

$$\begin{aligned}&\|(v - (v^s)^{-\mathbf{X}}, u - (u^s)^{-\mathbf{X}})(t)\|_{H^2} \\ &\quad \le \|(v - v^s, u - u^s)(t)\|_{H^2} + \|(v^s - (v^s)^{-\mathbf{X}}, u^s - (u^s)^{-\mathbf{X}})(t)\|_{H^2} \\ &\quad \le \frac{\chi_0}{2} + \frac{\chi_0}{8} < \chi_0.\end{aligned}$$

Hence, we can apply the a priori estimates from Proposition 3.6 again with $T = 2T_0$ and get the estimate

$$\|(v - (v^s)^{-\mathbf{X}}, u - (u^s)^{-\mathbf{X}})(t)\|_{H^2} \le \sqrt{C_0}\|(v_0 - v^s, u_0 - u^s)\|_{H^2} \le \sqrt{C_0}\varepsilon_0 \le \frac{\chi_0}{8}$$

for $t \in [0, 2T_0]$.

Step 3: Thus, repeating this continuation process, we can extend the solution to the interval $[0, NT_0]$ successively. At the time $t = NT_0$, it holds

$$\begin{aligned}&\|(v - v^s, u - u^s)(NT_0)\|_{H^2} \\ &\quad \le \|(v - (v^s)^{-\mathbf{X}}, u - (u^s)^{-\mathbf{X}})(NT_0)\|_{H^2} + \|(v^s - (v^s)^{-\mathbf{X}}, u^s - (u^s)^{-\mathbf{X}})(NT_0)\|_{H^2} \\ &\quad \le \frac{\chi_0}{8} + \frac{\chi_0}{8} = \frac{\chi_0}{4}\ (= \Xi).\end{aligned}$$

Hence, we can apply Proposition 3.5 by taking $t = NT_0$ as the new initial time. Then we have a unique solution on $[NT_0, (N+1)T_0]$ with the estimate

$$\|(v - v^s, u - u^s)(t)\|_{H^2} \le 2\|(v - v^s, u - u^s)(NT_0)\|_{H^2} \le 2\Xi = \frac{\chi_0}{2}$$

for $t \in [NT_0, (N+1)T_0]$, which implies that $\|(v - v^s, u - u^s)(t)\|_{H^2} \le \frac{\chi_0}{2}$ holds for $t \in [0, (N+1)T_0]$. Meanwhile, we can also show that $|\mathbf{X}(t)| \le Ct$ holds for $t \in [0, (N+1)T_0]$. Using the smallness of δ_0, we have

$$\begin{aligned}\|(v^s - (v^s)^{-\mathbf{X}}, u^s - (u^s)^{-\mathbf{X}})(t)\|_{H^2} &= |\mathbf{X}(t)|\|(v^s_{\xi_1}, u^s_{\xi_1})\|_{H^2} \\ &\le C\delta^{\frac{3}{2}} t \le C\delta_0\sqrt{\delta_0}(N+1) \le C\delta_0 \le \frac{\chi_0}{8}.\end{aligned}$$

Therefore, it holds for $t \in [0, (N+1)T_0]$ that

$$\begin{aligned}
&\|(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})(t)\|_{H^2} \\
&\quad \le \|(v-v^s, u-u^s)(t)\|_{H^2} + \|(v^s-(v^s)^{-\mathbf{X}}, u^s-(u^s)^{-\mathbf{X}})(t)\|_{H^2} \\
&\quad \le \frac{\chi_0}{2} + \frac{\chi_0}{8} < \chi_0.
\end{aligned}$$

Hence, we can apply Proposition 3.6 again with $T = (N+1)T_0$ and get the estimate

$$\|(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})(t)\|_{H^2} \le \sqrt{C_0}\|(v_0-v^s, u_0-u^s)\|_{H^2} \le \sqrt{C_0}\varepsilon_0 \le \frac{\chi_0}{8}$$

for $t \in [0, (N+1)T_0]$. This indicates that the solution has been extended to the interval $[0, (N+1)T_0]$, which contradicts that $T_{\max}(\le NT_0)$ is the maximum existence time. Therefore, the maximum existence time defined in formula (3.14) is infinity, that is, $T_{\max} = +\infty$. ■

4. Uniform-in-time H^2-estimates

Throughout this section, C denotes a positive constant which may change from line to line, but which stays independent on δ (the shock strength) and κ (the total variation of the function $a(\xi_1)$). We will consider two smallness conditions, one on δ, and the other on $\frac{\delta}{\kappa}$. In the argument, δ will be far smaller than $\frac{\delta}{\kappa}$.

4.1. Reformulation of the problem

We introduce a new multi-dimensional effective velocity

$$h := u - (2\mu+\lambda)\nabla_\xi v. \tag{4.1}$$

Then the system (2.6) is transformed into

$$\begin{cases}
\rho(\partial_t v - \sigma\partial_{\xi_1} v + u\cdot\nabla_\xi v) - \mathrm{div}_\xi h = (2\mu+\lambda)\Delta_\xi v, \\
\rho(\partial_t h - \sigma\partial_{\xi_1} h + u\cdot\nabla_\xi h) + \nabla_\xi p(v) = R,
\end{cases} \tag{4.2}$$

where

$$R = \frac{2\mu+\lambda}{v}(\nabla_\xi u\cdot\nabla_\xi v - \mathrm{div}_\xi u\nabla_\xi v) - \mu\nabla_\xi\times\nabla_\xi\times u. \tag{4.3}$$

We also set

$$h_1^s := u_1^s - (2\mu+\lambda)\partial_{\xi_1} v^s, \quad h^s := (h_1^s, 0, 0)^t. \tag{4.4}$$

We use here a change of variable $\xi_1 \to \xi_1 - \mathbf{X}(t)$, then system (2.7) can be rewritten as

$$\begin{cases}
(\rho^s)^{-\mathbf{X}}(-\sigma\partial_{\xi_1}(v^s)^{-\mathbf{X}} + (u_1^s)^{-\mathbf{X}}\partial_{\xi_1}(v^s)^{-\mathbf{X}}) - \partial_{\xi_1}(h_1^s)^{-\mathbf{X}} \\
\qquad = (2\mu+\lambda)\partial_{\xi_1}^2(v^s)^{-\mathbf{X}}, \\
(\rho^s)^{-\mathbf{X}}(-\sigma\partial_{\xi_1}(h_1^s)^{-\mathbf{X}} + (u_1^s)^{-\mathbf{X}}\partial_{\xi_1}(h_1^s)^{-\mathbf{X}}) + \partial_{\xi_1}p((v^s)^{-\mathbf{X}}) = 0.
\end{cases} \tag{4.5}$$

It follows from (4.2) and (4.5) that

$$
\begin{cases}
\rho\partial_t(v-(v^s)^{-\mathbf{X}})-\sigma\rho\partial_{\xi_1}(v-(v^s)^{-\mathbf{X}})+\rho u\cdot\nabla_\xi(v-(v^s)^{-\mathbf{X}}) \\
\qquad -\operatorname{div}_\xi(h-(h^s)^{-\mathbf{X}})-\dot{\mathbf{X}}(t)\rho\partial_{\xi_1}(v^s)^{-\mathbf{X}}+F\partial_{\xi_1}(v^s)^{-\mathbf{X}} \\
\qquad =(2\mu+\lambda)\Delta_\xi(v-(v^s)^{-\mathbf{X}}), \\
\rho\partial_t(h-(h^s)^{-\mathbf{X}})-\sigma\rho\partial_{\xi_1}(h-(h^s)^{-\mathbf{X}})+\rho u\cdot\nabla_\xi(h-(h^s)^{-\mathbf{X}}) \\
\qquad +\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}}))-\dot{\mathbf{X}}(t)\rho\partial_{\xi_1}(h^s)^{-\mathbf{X}}+F\partial_{\xi_1}(h^s)^{-\mathbf{X}}=R,
\end{cases}
\tag{4.6}
$$

where

$$
\begin{aligned}
F &= -\sigma(\rho-(\rho^s)^{-\mathbf{X}})+\rho u_1-(\rho^s u_1^s)^{-\mathbf{X}} \\
&= -\frac{\sigma_*}{(\rho^s)^{-\mathbf{X}}}(\rho-(\rho^s)^{-\mathbf{X}})+\rho(u_1-(u_1^s)^{-\mathbf{X}}) \\
&= \sigma_*\frac{v-(v^s)^{-\mathbf{X}}}{v}+\frac{h_1-(h_1^s)^{-\mathbf{X}}}{v}+(2\mu+\lambda)\frac{\partial_{\xi_1}(v-(v^s)^{-\mathbf{X}})}{v}.
\end{aligned}
\tag{4.7}
$$

We define the weight function $a(\xi_1)$ by

$$
a(\xi_1)=1+\frac{\kappa}{\delta}(p(v_-)-p(v^s(\xi_1))),
\tag{4.8}
$$

where the constant κ is chosen to be small but far bigger than δ such that

$$
\delta \ll \kappa \le C\sqrt{\delta}.
\tag{4.9}
$$

For definiteness and simplicity, we can choose $\kappa=\sqrt{\delta}$.

Notice that

$$
1<a(\xi_1)<1+\kappa,
\tag{4.10}
$$

and

$$
a'(\xi_1)=-\frac{\kappa}{\delta}p'(v^s)v^s_{\xi_1}>0,\quad |a'|\sim\frac{\kappa}{\delta}|v^s_{\xi_1}|.
\tag{4.11}
$$

Lemma 4.1. *Let $a(\xi_1)$ be the weighted function defined by (4.8), then*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(Q(v|(v^s)^{-\mathbf{X}})+\frac{1}{2}|h-(h^s)^{-\mathbf{X}}|^2\Big)d\xi_1 d\xi' \\
&\qquad =\dot{\mathbf{X}}(t)\mathbf{Y}(t)+\mathbf{B}(t)-\mathbf{G}(t)-\mathbf{D}(t),
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
\mathbf{Y}(t) := &-\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\rho\Big(Q(v|(v^s)^{-\mathbf{X}})+\frac{1}{2}|h-(h^s)^{-\mathbf{X}}|^2\Big)d\xi_1 d\xi' \\
&-\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho p'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' \\
&+\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho(h_1^s)_{\xi_1}^{-\mathbf{X}}(h_1-(h_1^s)^{-\mathbf{X}})d\xi_1 d\xi', \\
\mathbf{B}(t) := &\sum_{i=1}^{9}\mathbf{B}_i(t)
\end{aligned}
$$

with

$$\mathbf{B}_1(t) := \frac{1}{2\sigma_*}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}|p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi',$$

$$\mathbf{B}_2(t) := \sigma_*\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}} p(v|(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} d\xi_1 d\xi',$$

$$\mathbf{B}_3(t) := \frac{\delta}{\kappa}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{\sigma_* v} a_{\xi_1}^{-\mathbf{X}}(h_1-(h_1^s)^{-\mathbf{X}})^2 d\xi_1 d\xi',$$

$$\mathbf{B}_4(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} F a_{\xi_1}^{-\mathbf{X}}\Big(Q(v|(v^s)^{-\mathbf{X}}) + \frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big) d\xi_1 d\xi',$$

$$\mathbf{B}_5(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\frac{2\mu+\lambda}{v} p'((v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} \times \partial_{\xi_1}(v-(v^s)^{-\mathbf{X}})\Big(v-(v^s)^{-\mathbf{X}} - \frac{h_1-(h_1^s)^{-\mathbf{X}}}{\sigma_*}\Big) d\xi_1 d\xi',$$

$$\mathbf{B}_6(t) := -(2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\partial_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}})) \times \partial_{\xi_1} p((v^s)^{-\mathbf{X}})\gamma^{-1}(p^{-1-\frac{1}{\gamma}}(v) - p^{-1-\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})) d\xi_1 d\xi',$$

$$\mathbf{B}_7(t) := -(2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\gamma^{-1}p^{-1-\frac{1}{\gamma}}(v) \times (p(v)-p((v^s)^{-\mathbf{X}}))\partial_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}})) d\xi_1 d\xi',$$

$$\mathbf{B}_8(t) := -(2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}(p(v)-p((v^s)^{-\mathbf{X}}))\partial_{\xi_1} p((v^s)^{-\mathbf{X}}) \times \gamma^{-1}(p^{-1-\frac{1}{\gamma}}(v) - p^{-1-\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})) d\xi_1 d\xi',$$

$$\mathbf{B}_9(t) := \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(h-(h^s)^{-\mathbf{X}})\cdot R\, d\xi_1 d\xi',$$

and

$$\begin{aligned}\mathbf{G}(t) &= \sigma_*\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}} Q(v|(v^s)^{-\mathbf{X}}) d\xi_1 d\xi' + \sigma_*\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\frac{h_2^3+h_3^2}{2} d\xi_1 d\xi' \\ &\quad + \frac{\sigma_*}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\Big|h_1-(h_1^s)^{-\mathbf{X}} - \frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|^2 d\xi_1 d\xi' \\ &\quad + \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\frac{\sigma_*}{v}|p'((v^s)^{-\mathbf{X}})|(v^s)_{\xi_1}^{-\mathbf{X}}(v-(v^s)^{-\mathbf{X}})^2 d\xi_1 d\xi' \\ &:= \sum_{i=1}^{4}\mathbf{G}_i(t),\end{aligned}$$

$$\mathbf{D}(t) = (2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\gamma^{-1}p^{-1-\frac{1}{\gamma}}(v)|\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}}))|^2 d\xi_1 d\xi'.$$

Remark 4.2. Since $\sigma_* a_{\xi_1}^{-\mathbf{X}} > 0$ and $a^{-\mathbf{X}} > 1$, $\mathbf{G}(t)$ consists of four terms with good sign, while $\mathbf{B}(t)$ consists of bad terms.

Proof of Lemma 4.1. We set $a^{-\mathbf{X}} := a(\xi_1 - \mathbf{X}(t))$. Multiplying $(4.6)_1$ by $-a^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))$, we obtain

$$\begin{aligned}
&\partial_t(a^{-\mathbf{X}}\rho Q(v|(v^s)^{-\mathbf{X}})) - \sigma\partial_{\xi_1}(a^{-\mathbf{X}}\rho Q(v|(v^s)^{-\mathbf{X}})) \\
&\quad + \mathrm{div}_\xi(a^{-\mathbf{X}}\rho u Q(v|(v^s)^{-\mathbf{X}})) + a^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))\mathrm{div}_\xi(h - (h^s)^{-\mathbf{X}}) \\
&\quad = -\dot{\mathbf{X}}(t)a_{\xi_1}^{-\mathbf{X}}\rho Q(v|(v^s)^{-\mathbf{X}}) - \dot{\mathbf{X}}(t)a^{-\mathbf{X}}\rho p'((v^s)^{-\mathbf{X}})(v - (v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} \\
&\qquad + (-\sigma\rho + \rho u_1)a_{\xi_1}^{-\mathbf{X}}Q(v|(v^s)^{-\mathbf{X}}) - (-\sigma\rho + \rho u_1)a^{-\mathbf{X}}p(v|(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} \\
&\qquad + a^{-\mathbf{X}}F(p(v) - p((v^s)^{-\mathbf{X}}))(v^s)_{\xi_1}^{-\mathbf{X}} \\
&\qquad - (2\mu + \lambda)\mathrm{div}_\xi(a^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))\nabla_\xi(v - (v^s)^{-\mathbf{X}})) \\
&\qquad + (2\mu + \lambda)\nabla_\xi(a^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))) \cdot \nabla_\xi(v - (v^s)^{-\mathbf{X}}).
\end{aligned} \tag{4.13}$$

Using (4.7) the definition of F, and (2.8) the definition of σ_*, we get

$$\begin{aligned}
-\sigma\rho + \rho u_1 &= -\sigma(\rho^s)^{-\mathbf{X}} + (\rho^s u_1^s)^{-\mathbf{X}} - \sigma(\rho - (\rho^s)^{-\mathbf{X}}) + (\rho u_1 - (\rho^s u_1^s)^{-\mathbf{X}}) \\
&= -\sigma_* + F.
\end{aligned} \tag{4.14}$$

Thus, we have

$$\begin{aligned}
&(-\sigma\rho + \rho u_1)a_{\xi_1}^{-\mathbf{X}}Q(v|(v^s)^{-\mathbf{X}}) = (-\sigma_* + F)a_{\xi_1}^{-\mathbf{X}}Q(v|(v^s)^{-\mathbf{X}}), \\
&- (-\sigma\rho + \rho u_1)a^{-\mathbf{X}}p(v|(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} + a^{-\mathbf{X}}F(p(v) - p((v^s)^{-\mathbf{X}}))(v^s)_{\xi_1}^{-\mathbf{X}} \\
&\quad = \sigma_* a^{-\mathbf{X}}p(v|(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} + Fa^{-\mathbf{X}}p'((v^s)^{-\mathbf{X}})(v - (v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}}.
\end{aligned}$$

Notice that

$$\nabla_\xi v = \frac{\nabla_\xi p(v)}{p'(v)} = \frac{\nabla_\xi p(v)}{-\gamma p^{1+\frac{1}{\gamma}}(v)}.$$

Hence, the last term on the right-hand side of (4.13) can be rewritten as

$$\begin{aligned}
&(2\mu + \lambda)\nabla_\xi(a^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))) \cdot \nabla_\xi(v - (v^s)^{-\mathbf{X}}) \\
&\quad = (2\mu + \lambda)a^{-\mathbf{X}}\nabla_\xi(p(v) - p((v^s)^{-\mathbf{X}})) \cdot \Big(\frac{\nabla_\xi p(v)}{-\gamma p^{1+\frac{1}{\gamma}}(v)} - \frac{\nabla_\xi p((v^s)^{-\mathbf{X}})}{-\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\Big) \\
&\qquad + (2\mu + \lambda)a_{\xi_1}^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))\Big(\frac{\partial_{\xi_1} p(v)}{-\gamma p^{1+\frac{1}{\gamma}}(v)} - \frac{\partial_{\xi_1} p((v^s)^{-\mathbf{X}})}{-\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\Big) \\
&\quad = -(2\mu + \lambda)a^{-\mathbf{X}}\frac{|\nabla_\xi(p(v) - p((v^s)^{-\mathbf{X}}))|^2}{\gamma p^{1+\frac{1}{\gamma}}(v)} - (2\mu + \lambda)a^{-\mathbf{X}}\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}})) \\
&\qquad \times \partial_{\xi_1} p((v^s)^{-\mathbf{X}})\Big(\frac{1}{\gamma p^{1+\frac{1}{\gamma}}(v)} - \frac{1}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\Big) \\
&\qquad - (2\mu + \lambda)a_{\xi_1}^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))\frac{\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))}{\gamma p^{1+\frac{1}{\gamma}}(v)} \\
&\qquad - (2\mu + \lambda)a_{\xi_1}^{-\mathbf{X}}(p(v) - p((v^s)^{-\mathbf{X}}))\partial_{\xi_1} p((v^s)^{-\mathbf{X}}) \\
&\qquad \times \Big(\frac{1}{\gamma p^{1+\frac{1}{\gamma}}(v)} - \frac{1}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\Big).
\end{aligned} \tag{4.15}$$

Multiplying (4.6)$_2$ by $a^{-\mathbf{X}}(h-(h^s)^{-\mathbf{X}})$, we have

$$\begin{aligned}
&\partial_t\Big(a^{-\mathbf{X}}\rho\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big)-\sigma\partial_{\xi_1}\Big(a^{-\mathbf{X}}\rho\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big)\\
&\quad+\mathrm{div}_\xi\Big(a^{-\mathbf{X}}\rho u\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big)+\mathrm{div}_\xi(a^{-\mathbf{X}}(p(v)-p((v^s)^{-\mathbf{X}}))(h-(h^s)^{-\mathbf{X}}))\\
&\quad-a^{-\mathbf{X}}(p(v)-p((v^s)^{-\mathbf{X}}))\mathrm{div}_\xi(h-(h^s)^{-\mathbf{X}})\\
&\quad=-\dot{\mathbf{X}}(t)a^{-\mathbf{X}}_{\xi_1}\rho\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}+\dot{\mathbf{X}}(t)a^{-\mathbf{X}}\rho(h_1^s)^{-\mathbf{X}}_{\xi_1}(h_1-(h_1^s)^{-\mathbf{X}})\\
&\qquad+a^{-\mathbf{X}}_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))(h_1-(h_1^s)^{-\mathbf{X}})+(-\sigma\rho+\rho u_1)a^{-\mathbf{X}}_{\xi_1}\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\\
&\qquad-Fa^{-\mathbf{X}}(h_1^s)^{-\mathbf{X}}_{\xi_1}(h_1-(h_1^s)^{-\mathbf{X}})+a^{-\mathbf{X}}(h-(h^s)^{-\mathbf{X}})\cdot R.
\end{aligned}\tag{4.16}$$

Before we add (4.13) and (4.16) together, direct calculations yield

$$\begin{aligned}
&-\sigma_*a^{-\mathbf{X}}_{\xi_1}\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}+a^{-\mathbf{X}}_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))(h_1-(h_1^s)^{-\mathbf{X}})\\
&\quad=-\frac{\sigma_*}{2}a^{-\mathbf{X}}_{\xi_1}\Big|h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|^2\\
&\qquad+a^{-\mathbf{X}}_{\xi_1}\frac{|p(v)-p((v^s)^{-\mathbf{X}})|^2}{2\sigma_*}-\sigma_*a^{-\mathbf{X}}_{\xi_1}\frac{h_2^2+h_3^2}{2}.
\end{aligned}\tag{4.17}$$

We treat the perturbed flux term in the Eulerian coordinates along the shock wave propagation direction, which is different from that in the Lagrangian coordinates. It follows from (4.5)$_2$ that $\sigma_*\partial_{\xi_1}(h_1^s)^{-\mathbf{X}}=\partial_{\xi_1}p((v^s)^{-\mathbf{X}})$. Hence, using (4.7), we have

$$\begin{aligned}
&Fa^{-\mathbf{X}}p'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}-Fa^{-\mathbf{X}}(h_1^s)^{-\mathbf{X}}_{\xi_1}(h_1-(h_1^s)^{-\mathbf{X}})\\
&\quad=a^{-\mathbf{X}}\frac{\sigma_*}{v}p'((v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}(v-(v^s)^{-\mathbf{X}})^2+\frac{\delta}{\kappa}\frac{a^{-\mathbf{X}}}{\sigma_*v}a^{-\mathbf{X}}_{\xi_1}(h_1-(h_1^s)^{-\mathbf{X}})^2\\
&\qquad+a^{-\mathbf{X}}\frac{2\mu+\lambda}{v}p'((v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}\partial_{\xi_1}(v-(v^s)^{-\mathbf{X}})\\
&\qquad\times\Big(v-(v^s)^{-\mathbf{X}}-\frac{h_1-(h_1^s)^{-\mathbf{X}}}{\sigma_*}\Big).
\end{aligned}\tag{4.18}$$

Adding (4.13) and (4.16) together, integrating the resultant equation by parts over $\Omega:=\mathbb{R}\times\mathbb{T}^2$, and using (4.15), (4.17) and (4.18), we can obtain (4.12). The proof of Lemma 4.1 is completed. ■

In order to derive the a-contraction property of the viscous shock wave, we decompose the function $\mathbf{Y}(t)$ in Lemma 4.1 as

$$\mathbf{Y}(t):=\sum_{i=1}^{5}\mathbf{Y}_i(t),$$

where

$$
\begin{aligned}
\mathbf{Y}_1(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{\sigma_*}\rho(h_1^s)^{-\mathbf{X}}_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))d\xi_1 d\xi',\\
\mathbf{Y}_2(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho p'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}d\xi_1 d\xi',\\
\mathbf{Y}_3(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho(h_1^s)^{-\mathbf{X}}_{\xi_1}\Big(h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big)d\xi_1 d\xi',\\
\mathbf{Y}_4(t) &:= -\frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}\rho\Big(h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big)\\
&\qquad\qquad \times\Big(h_1-(h_1^s)^{-\mathbf{X}}+\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big)d\xi_1 d\xi',\\
\mathbf{Y}_5(t) &:= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}\rho\Big(Q(v|(v^s)^{-\mathbf{X}})+\frac{h_2^2+h_3^2}{2}\Big)d\xi_1 d\xi'\\
&\quad -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}\rho\frac{|p(v)-p((v^s)^{-\mathbf{X}})|^2}{2\sigma_*^2}d\xi_1 d\xi'.
\end{aligned}
$$

Notice that

$$
\dot{\mathbf{X}}(t) = -\frac{M}{\delta}(\mathbf{Y}_1(t)+\mathbf{Y}_2(t)), \tag{4.19}
$$

and so

$$
\dot{\mathbf{X}}(t)\mathbf{Y}(t) = -\frac{\delta}{M}|\dot{\mathbf{X}}(t)|^2+\dot{\mathbf{X}}(t)\sum_{i=3}^{5}\mathbf{Y}_i(t).
$$

Then we have the following lemma.

Lemma 4.3. *There exists uniform in time $C>0$ such that for $\forall t\in[0,T]$,*

$$
\begin{aligned}
&-\frac{\delta}{2M}|\dot{\mathbf{X}}(t)|^2+\mathbf{B}_1(t)+\mathbf{B}_2(t)+\mathbf{B}_3(t)-\mathbf{G}_1(t)-\mathbf{G}_4(t)-\frac{3}{4}\mathbf{D}(t)\\
&\quad\le -C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}||p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi'\\
&\qquad + C\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}|p(v)-p((v^s)^{-\mathbf{X}})|^3 d\xi_1 d\xi' + \frac{1}{40}\mathbf{G}_3(t).
\end{aligned} \tag{4.20}
$$

Proof. We now rewrite the above functions with respect to the following variables:

$$
\begin{aligned}
w &:= p(v)-p((v^s)^{-\mathbf{X}}),\\
y_1 &:= \frac{p(v_-)-p((v^s(\xi_1))^{-\mathbf{X}})}{\delta},\\
y' &:= (y_2,y_3) = (\xi_2,\xi_3) =: \xi'.
\end{aligned} \tag{4.21}
$$

We use a change of variable $\xi_1\in\mathbb{R}\mapsto y_1\in[0,1]$. Then it follows from (4.8) that $a^{-\mathbf{X}}(\xi_1) = 1+\kappa y_1$ and

$$
\frac{dy_1}{d\xi_1} = -\frac{1}{\delta}p((v^s)^{-\mathbf{X}})_{\xi_1},\quad a^{-\mathbf{X}}_{\xi_1} = \kappa\frac{dy_1}{d\xi_1},\quad |a^{-\mathbf{X}}-1|\le\kappa=\sqrt{\delta}. \tag{4.22}
$$

To perform the sharp estimates, we will consider the $O(1)$-constants

$$\sigma_- := \sqrt{-p'(v_-)}, \quad \alpha_- := \frac{\gamma+1}{2\gamma\sigma_- p(v_-)},$$

which are indeed independent of the small constant δ, since $\frac{v_+}{2} \le v_- \le v_+$. Note that

$$|\sigma_* - \sigma_-| \le C\delta, \tag{4.23}$$

which together with $\sigma_-^2 = -p'(v_-) = \gamma p^{1+\frac{1}{\gamma}}(v_-)$ implies

$$|\sigma_-^2 + p'((v^s)^{-\mathbf{X}})| \le C\delta, \quad \left|\frac{1}{\sigma_-^2} - \frac{1}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\right| \le C\delta. \tag{4.24}$$

• *Estimate on* $-\frac{\delta}{2M}|\dot{\mathbf{X}}(t)|^2$. To do this, we will control $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ due to (4.19). Using (2.8), system (4.5) is transformed into

$$\begin{cases} -\sigma_*(v^s)^{-\mathbf{X}}_{\xi_1} - (h_1^s)^{-\mathbf{X}}_{\xi_1} = (2\mu+\lambda)(v^s)^{-\mathbf{X}}_{\xi_1\xi_1}, \\ -\sigma_*(h_1^s)^{-\mathbf{X}}_{\xi_1} + p((v^s)^{-\mathbf{X}})_{\xi_1} = 0. \end{cases} \tag{4.25}$$

Using $(4.25)_2$ and the new variable (4.22), we obtain

$$\begin{aligned} \mathbf{Y}_1(t) &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{\sigma_*^2 v} p((v^s)^{-\mathbf{X}})_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))d\xi_1 d\xi' \\ &= -\frac{\delta}{\sigma_*^2}\int_{\mathbb{T}^2}\int_0^1 a^{-\mathbf{X}}\frac{w}{v} dy_1 dy'. \end{aligned}$$

Using (4.23) and $|a^{-\mathbf{X}} - 1| \le \kappa$, we have

$$\left|\mathbf{Y}_1(t) + \frac{\delta}{\sigma_-^2 v_-}\int_{\mathbb{T}^2}\int_0^1 w\, dy_1 dy'\right| \le C\delta(\kappa+\delta+\chi)\int_{\mathbb{T}^2}\int_0^1 |w| dy_1 dy'. \tag{4.26}$$

For

$$\begin{aligned} \mathbf{Y}_2(t) &= -\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho p((v^s)^{-\mathbf{X}})_{\xi_1}(v - (v^s)^{-\mathbf{X}})d\xi_1 d\xi' \\ &= \delta\int_{\mathbb{T}^2}\int_0^1 a^{-\mathbf{X}}\frac{v - (v^s)^{-\mathbf{X}}}{v} dy_1 dy', \end{aligned}$$

we observe that since (by considering $v = p(v)^{-\frac{1}{\gamma}}$)

$$\left|v - (v^s)^{-\mathbf{X}} + \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\right| \le C|p(v) - p((v^s)^{-\mathbf{X}})|^2,$$

then it holds that

$$\left|v - (v^s)^{-\mathbf{X}} + \frac{1}{\sigma_-^2}(p(v) - p((v^s)^{-\mathbf{X}}))\right| \le C(\delta+\chi)|p(v) - p((v^s)^{-\mathbf{X}})|.$$

This implies

$$\left|\mathbf{Y}_2(t) + \frac{\delta}{\sigma_-^2 v_-}\int_{\mathbb{T}^2}\int_0^1 w\,dy_1dy'\right| \le C\delta(\kappa+\delta+\chi)\int_{\mathbb{T}^2}\int_0^1 |w|dy_1dy'. \tag{4.27}$$

By (4.19), (4.26) and (4.27), we have

$$\begin{aligned}\left|\dot{\mathbf{X}}(t) - \frac{2M}{\sigma_-^2 v_-}\int_{\mathbb{T}^2}\int_0^1 w\,dy_1dy'\right| &= \left|\sum_{i=1}^{2}\frac{M}{\delta}\Big(\mathbf{Y}_i(t) + \frac{\delta}{\sigma_-^2 v_-}\int_{\mathbb{T}^2}\int_0^1 w\,dy_1dy'\Big)\right|\\ &\le C(\kappa+\delta+\chi)\int_{\mathbb{T}^2}\int_0^1 |w|dy_1dy',\end{aligned}$$

which yields

$$\begin{aligned}\left(\left|\frac{2M}{\sigma_-^2 v_-}\int_{\mathbb{T}^2}\int_0^1 w\,dy_1dy'\right| - |\dot{\mathbf{X}}(t)|\right)^2 &\le C(\kappa+\delta+\chi)^2\Big(\int_{\mathbb{T}^2}\int_0^1 |w|dy_1dy'\Big)^2\\ &\le C(\kappa+\delta+\chi)^2\int_{\mathbb{T}^2}\int_0^1 |w|^2dy_1dy',\end{aligned}$$

which together with the algebraic inequality

$$\frac{p^2}{2} - q^2 \le (p-q)^2$$

for all $p, q \ge 0$ indicate

$$\frac{2M^2}{\sigma_-^4 v_-^2}\Big(\int_{\mathbb{T}^2}\int_0^1 w\,dy_1dy'\Big)^2 - |\dot{\mathbf{X}}(t)|^2 \le C(\kappa+\delta+\chi)^2\int_{\mathbb{T}^2}\int_0^1 |w|^2dy_1dy'.$$

Thus, we can get

$$\begin{aligned}-\frac{\delta}{2M}|\dot{\mathbf{X}}(t)|^2 \le\; &-\frac{M\delta}{\sigma_-^4 v_-^2}\Big(\int_{\mathbb{T}^2}\int_0^1 w\,dy_1dy'\Big)^2\\ &+ C\delta(\kappa+\delta+\chi)^2\int_{\mathbb{T}^2}\int_0^1 |w|^2dy_1dy'.\end{aligned} \tag{4.28}$$

• *Change of variables for* $\mathbf{B}_i(t)$ $(i = 1, 2, 3)$. By the change of variables, using (4.23), we have

$$\begin{aligned}\mathbf{B}_1(t) &= \frac{\kappa}{2\sigma_*}\int_{\mathbb{T}^2}\int_0^1 w^2dy_1dy'\\ &= \frac{\kappa}{2\sigma_-}\int_{\mathbb{T}^2}\int_0^1 w^2dy_1dy' + \frac{\kappa}{2}\Big(\frac{1}{\sigma_*} - \frac{1}{\sigma_-}\Big)\int_{\mathbb{T}^2}\int_0^1 w^2dy_1dy'\\ &\le \frac{\kappa}{2\sigma_-}\int_{\mathbb{T}^2}\int_0^1 w^2dy_1dy' + C\kappa\delta\int_{\mathbb{T}^2}\int_0^1 w^2dy_1dy'.\end{aligned} \tag{4.29}$$

For $\mathbf{B}_2(t)$, by the change of variables and using (3.6), we obtain

$$\begin{aligned}
\mathbf{B}_2(t) &= \sigma_*\delta \int_{\mathbb{T}^2}\int_0^1 (1+\kappa y_1) p(v|(v^s)^{-\mathbf{X}}) \frac{1}{|p'((v^s)^{-\mathbf{X}})|} dy_1 dy' \\
&\le \sigma_*\delta(1+\kappa) \int_{\mathbb{T}^2}\int_0^1 \Big(\frac{\gamma+1}{2\gamma}\frac{1}{p((v^s)^{-\mathbf{X}})} + C\chi\Big) \frac{|p(v)-p((v^s)^{-\mathbf{X}})|^2}{|p'((v^s)^{-\mathbf{X}})|} dy_1 dy' \\
&= \sigma_*\delta(1+\kappa) \int_{\mathbb{T}^2}\int_0^1 \Big(\alpha_- \frac{\sigma_- p(v_-)}{p((v^s)^{-\mathbf{X}})} + C\chi\Big) \frac{w^2}{|p'((v^s)^{-\mathbf{X}})|} dy_1 dy' \\
&\le \delta\alpha_-(1+C(\kappa+\delta+\chi)) \int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy', \qquad (4.30)
\end{aligned}$$

where in the last inequality we have used (4.23) and (4.24).

For $\mathbf{B}_3(t)$, using the algebraic inequality $p^2 = (q+p-q)^2 \le (1+\vartheta)q^2 + (1+\frac{1}{\vartheta})\times(p-q)^2$ for $\vartheta > 0$, it holds that

$$\mathbf{B}_3(t) \le (1+\vartheta)\frac{\delta}{\kappa}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{\sigma_*^3 v} a_{\xi_1}^{-\mathbf{X}} |p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' + C\Big(1+\frac{1}{\vartheta}\Big)\frac{\delta}{\kappa}\mathbf{G}_3(t).$$

Since

$$\frac{1}{\sigma_-^3 v_- \alpha_-} = \frac{1}{\sigma_-^3 v_-}\frac{2\gamma\sigma_- p(v_-)}{\gamma+1} = \frac{2}{\gamma+1},$$

by the change of variables and using (4.23), we have

$$\begin{aligned}
&\frac{\delta}{\kappa}(1+\vartheta)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{\sigma_*^3 v} a_{\xi_1}^{-\mathbf{X}} |p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\
&\quad = \delta(1+\vartheta)\int_{\mathbb{T}^2}\int_0^1 \frac{(1+\kappa y_1)}{\sigma_-^3 v_-}\frac{\sigma_-^3 v_-}{\sigma_*^3 v} w^2 dy_1 dy' \\
&\quad \le \delta(1+\vartheta)(1+C(\kappa+\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 \frac{1}{\sigma_-^3 v_-} w^2 dy_1 dy' \\
&\quad = \frac{2}{\gamma+1}(1+\vartheta)\delta\alpha_-(1+C(\kappa+\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy'.
\end{aligned}$$

Thus, the following estimate holds:

$$\begin{aligned}
\mathbf{B}_3(t) \le{}& \frac{2}{\gamma+1}(1+\vartheta)\delta\alpha_-(1+C(\kappa+\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' \\
&+ C(1+\frac{1}{\vartheta})\frac{\delta}{\kappa}\mathbf{G}_3(t). \qquad (4.31)
\end{aligned}$$

• *Change of variables for* $\mathbf{G}_1(t)$, $\mathbf{G}_4(t)$. For $\mathbf{G}_1(t)$, we first use (3.7) to split it into two parts,

$$\begin{aligned}
\mathbf{G}_1(t) \ge{}& \underbrace{\sigma_* \int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}} \frac{|p(v)-p((v^s)^{-\mathbf{X}})|^2}{2\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})} d\xi_1 d\xi'}_{\mathbf{G}_{1,1}(t)} \\
&- \sigma_* \int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}} \frac{1+\gamma}{3\gamma^2}\frac{(p(v)-p((v^s)^{-\mathbf{X}}))^3}{p^{2+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})} d\xi_1 d\xi'. \qquad (4.32)
\end{aligned}$$

We only need to do the change of variables for the good term $\mathbf{G}_{1,1}(t)$ as follows. By (4.23), (4.24) and the change of variables,

$$\begin{aligned}\mathbf{G}_{1,1}(t) &\geq \frac{\sigma_*}{2\sigma_-^2}(1-C\delta)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}|p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\ &\geq \frac{\kappa}{2\sigma_-}(1-C\delta)\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy',\end{aligned}$$

which together with (4.29) yields

$$\mathbf{B}_1(t)-\mathbf{G}_{1,1}(t)\leq C\kappa\delta\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy'. \tag{4.33}$$

For $\mathbf{G}_4(t)$, using the mean value theorem,

$$v-(v^s)^{-\mathbf{X}} = \frac{p(v)-p((v^s)^{-\mathbf{X}})}{p'(\zeta)} \quad \text{for } \zeta \text{ between } v \text{ and } (v^s)^{-\mathbf{X}}.$$

Using (4.22) and the change of variables, we get

$$\begin{aligned}\mathbf{G}_4(t) &= \frac{\delta}{\kappa}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}} a_{\xi_1}^{-\mathbf{X}}\frac{\sigma_*}{v}|v-(v^s)^{-\mathbf{X}}|^2 d\xi_1 d\xi' \\ &\geq \delta\int_{\mathbb{T}^2}\int_0^1 \frac{\sigma_*}{v}\frac{w^2}{|p'(v_-)|^2}\left|\frac{p'(v_-)}{p'(\zeta)}\right|^2 dy_1 dy' \\ &\geq \delta(1-C(\delta+\chi))\frac{1}{\sigma_-^3 v_-}\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' \\ &= \frac{2}{\gamma+1}\delta\alpha_-(1-C(\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy'.\end{aligned} \tag{4.34}$$

• *Change of variables for* $\mathbf{D}(t)$. First, by (4.10) $a^{-\mathbf{X}} > 1$, and then using the change of variables, we obtain

$$\begin{aligned}\mathbf{D}(t) &\geq (2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{|\partial_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))|^2}{\gamma p^{1+\frac{1}{\gamma}}(v)} d\xi_1 d\xi' \\ &\quad + (2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{|\nabla_{\xi'}(p(v)-p((v^s)^{-\mathbf{X}}))|^2}{\gamma p^{1+\frac{1}{\gamma}}(v)} d\xi_1 d\xi' \\ &= \underbrace{(2\mu+\lambda)\int_{\mathbb{T}^2}\int_0^1 \frac{|\partial_{y_1} w|^2}{\gamma p^{1+\frac{1}{\gamma}}(v)}\left(\frac{dy_1}{d\xi_1}\right) dy_1 dy'}_{\mathbf{D}_{\mathrm{I}}(t)} \\ &\quad + \underbrace{(2\mu+\lambda)\int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'} w|^2}{\gamma p^{1+\frac{1}{\gamma}}(v)}\left(\frac{d\xi_1}{dy_1}\right) dy_1 dy'}_{\mathbf{D}_{\mathrm{II}}(t)}.\end{aligned}$$

On the one hand, integrating (4.25) over $(-\infty,\xi]$ yields

$$(2\mu+\lambda)(v^s)_{\xi_1}^{-\mathbf{X}} = -\sigma_*((v^s)^{-\mathbf{X}}-v_-) - \frac{1}{\sigma_*}(p((v^s)^{-\mathbf{X}})-p(v_-)).$$

On the other hand,

$$(v^s)^{-\mathbf{X}}_{\xi_1} = \frac{p((v^s)^{-\mathbf{X}})_{\xi_1}}{p'((v^s)^{-\mathbf{X}})} = \frac{\delta}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})} \frac{dy_1}{d\xi_1}.$$

Hence, we have

$$\begin{aligned}(2\mu+\lambda)\frac{\delta}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{dy_1}{d\xi_1} &= -\sigma_*((v^s)^{-\mathbf{X}} - v_-) - \frac{1}{\sigma_*}(p((v^s)^{-\mathbf{X}}) - p(v_-)) \\ &= \frac{-1}{\sigma_*}(\sigma_*^2((v^s)^{-\mathbf{X}} - v_-) + (p((v^s)^{-\mathbf{X}}) - p(v_-))),\end{aligned}$$

which together with $\sigma_*^2 = -\frac{p(v_-)-p(v_+)}{v_- - v_+}$ leads to

$$\begin{aligned}(2\mu+\lambda)\frac{\delta}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{dy_1}{d\xi_1} &= \frac{-1}{\sigma_*(v_+ - v_-)}\Big((p(v_-) - p(v_+))((v^s)^{-\mathbf{X}} - v_-) \\ &\qquad + (p((v^s)^{-\mathbf{X}}) - p(v_-))(v_+ - v_-)\Big) \\ &= \frac{-1}{\sigma_*(v_+ - v_-)}\Big((p((v^s)^{-\mathbf{X}}) - p(v_+))((v^s)^{-\mathbf{X}} - v_-) \\ &\qquad - ((v^s)^{-\mathbf{X}} - v_+)(p((v^s)^{-\mathbf{X}}) - p(v_-))\Big).\end{aligned}$$

Recall that

$$y_1 = \frac{p(v_-) - p((v^s)^{-\mathbf{X}})}{\delta} \quad \text{and} \quad 1 - y_1 = \frac{p((v^s)^{-\mathbf{X}}) - p(v_+)}{\delta},$$

and it follows that

$$\begin{aligned}&\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{dy_1}{d\xi_1} \\ &\quad = \frac{\delta}{\sigma_*(v_+ - v_-)}\Big(\frac{(v^s)^{-\mathbf{X}} - v_-}{p((v^s)^{-\mathbf{X}}) - p(v_-)} - \frac{(v^s)^{-\mathbf{X}} - v_+}{p((v^s)^{-\mathbf{X}}) - p(v_+)}\Big).\end{aligned}$$

Then

$$\begin{aligned}&\Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{dy_1}{d\xi_1} - \frac{\delta p''(v_-)}{2\sigma_-(p'(v_-))^2}\Big| \\ &\quad \le \Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{dy_1}{d\xi_1} - \frac{\delta p''(v_-)}{2\sigma_*(p'(v_-))^2}\Big| + \frac{\delta p''(v_-)}{2(p'(v_-))^2}\Big|\frac{1}{\sigma_*} - \frac{1}{\sigma_-}\Big| \\ &\quad =: I_1 + I_2.\end{aligned}$$

Using Lemma 3.4, we have

$$\begin{aligned}I_1 = \Big|\frac{\delta}{\sigma_*(v_+ - v_-)}\Big(&\frac{(v^s)^{-\mathbf{X}} - v_-}{p((v^s)^{-\mathbf{X}}) - p(v_-)} + \frac{(v^s)^{-\mathbf{X}} - v_+}{p(v_+) - p((v^s)^{-\mathbf{X}})} \\ &+ \frac{1}{2}\frac{p''(v_-)}{(p'(v_-))^2}(v_- - v_+)\Big)\Big| \le C\delta^2.\end{aligned}$$

Since it follows from (4.23) that $I_2 \le C\delta^2$, we get

$$\Big|\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{dy_1}{d\xi_1}-\frac{\delta p''(v_-)}{2\sigma_-(p'(v_-))^2}\Big|\le C\delta^2. \tag{4.35}$$

In addition,

$$\Big|\Big(\frac{p((v^s)^{-\mathbf{X}})}{p(v)}\Big)^{1+\frac{1}{\gamma}}-1\Big|\le C|v-(v^s)^{-\mathbf{X}}|\le C\chi.$$

Thus, it holds

$$\begin{aligned}\mathbf{D}_{\mathrm{I}}(t)&=\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1}w|^2\Big(\frac{p((v^s)^{-\mathbf{X}})}{p(v)}\Big)^{1+\frac{1}{\gamma}}\\&\qquad\times\frac{1}{y_1(1-y_1)}\frac{2\mu+\lambda}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\Big(\frac{dy_1}{d\xi_1}\Big)dy_1dy'\\&\ge(1-C\chi)\Big(\frac{\delta p''(v_-)}{2\sigma_-(p'(v_-))^2}-C\delta^2\Big)\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1}w|^2dy_1dy'.\end{aligned}$$

Since

$$\frac{p''(v_-)}{2\sigma_-(p'(v_-))^2}=\frac{\gamma+1}{2\gamma\sigma_-p(v_-)}=\alpha_-,$$

we have

$$\mathbf{D}_{\mathrm{I}}(t)\ge\alpha_-\delta(1-C(\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1}w|^2dy_1dy'.$$

We can deduce from (4.35) that

$$y_1(1-y_1)\frac{d\xi_1}{dy_1}\ge\frac{2\mu+\lambda}{\gamma p^{1+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})}\frac{1}{\alpha_-\delta+C\delta^2}\ge\frac{2\mu+\lambda}{2\alpha_-\delta|p'(v_-)|}.$$

Hence, we obtain

$$\begin{aligned}\mathbf{D}_{\mathrm{II}}(t)&=(2\mu+\lambda)\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'}w|^2}{y_1(1-y_1)}\Big(\frac{p(v_-)}{p(v)}\Big)^{1+\frac{1}{\gamma}}\frac{y_1(1-y_1)}{\gamma p^{1+\frac{1}{\gamma}}(v_-)}\Big(\frac{d\xi_1}{dy_1}\Big)dy_1dy'\\&\ge(1-C(\delta+\chi))(2\mu+\lambda)\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'}w|^2}{y_1(1-y_1)}\frac{2\mu+\lambda}{|p'(v_-)|^2}\frac{1}{2\alpha_-\delta}dy_1dy'\\&=(1-C(\delta+\chi))\frac{\sigma_-}{\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)}\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'}w|^2}{y_1(1-y_1)}dy_1dy'.\end{aligned}$$

Combining the estimates on $\mathbf{D}_{\mathrm{I}}(t)$ and $\mathbf{D}_{\mathrm{II}}(t)$, we have

$$\begin{aligned}\mathbf{D}(t)&\ge\alpha_-\delta(1-C(\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1}w|^2dy_1dy'\\&\quad+(1-C(\delta+\chi))\frac{\sigma_-}{\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)}\int_{\mathbb{T}^2}\int_0^1\frac{|\nabla_{y'}w|^2}{y_1(1-y_1)}dy_1dy'.\end{aligned}\tag{4.36}$$

• *Proof of Lemma* 4.3. First, by (4.30), (4.31), (4.33), (4.34) and (4.36), we obtain

$$
\begin{aligned}
&\mathbf{B}_1(t) + \mathbf{B}_2(t) + \mathbf{B}_3(t) - \mathbf{G}_{1,1}(t) - \mathbf{G}_4(t) - \frac{3}{4}\mathbf{D}(t) \\
&\quad \leq \delta\alpha_-\Big[1 + C(\kappa+\delta+\chi) + \frac{2(1+\vartheta)}{\gamma+1}(1+C(\kappa+\delta+\chi)) - \frac{2}{\gamma+1}(1-C(\delta+\chi))\Big] \\
&\qquad \times \int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' + C(1+\frac{1}{\vartheta})\frac{\delta}{\kappa}\mathbf{G}_3(t) \\
&\qquad - \frac{3}{4}\delta\alpha_-(1-C(\delta+\chi))\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1} w|^2 dy_1 dy' \\
&\qquad - \frac{3}{4}(1-C(\delta+\chi))\frac{\sigma_-}{\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)}\int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'} w|^2}{y_1(1-y_1)} dy_1 dy'.
\end{aligned}
$$

Choosing κ, δ, χ and ϑ suitably small, we have

$$
\begin{aligned}
&\mathbf{B}_1(t) + \mathbf{B}_2(t) + \mathbf{B}_3(t) - \mathbf{G}_{1,1}(t) - \mathbf{G}_4(t) - \frac{3}{4}\mathbf{D}(t) \\
&\quad \leq \frac{6}{5}\delta\alpha_-\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' - \frac{5}{8}\delta\alpha_-\int_{\mathbb{T}^2}\int_0^1 y_1(1-y_1)|\partial_{y_1} w|^2 dy_1 dy' \\
&\qquad - \frac{5}{8}\frac{\sigma_-}{\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)}\int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'} w|^2}{y_1(1-y_1)} dy_1 dy' + C\frac{\delta}{\kappa}\mathbf{G}_3(t).
\end{aligned}
$$

Using (3.1) and the fact that

$$
\overline{w} := \int_{\mathbb{T}^2}\int_0^1 w dy_1 dy',
$$

it holds

$$
\int_{\mathbb{T}^2}\int_0^1 |w-\overline{w}|^2 dy_1 dy' = \int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' - \overline{w}^2.
$$

By (3.1), we have

$$
\begin{aligned}
&\mathbf{B}_1(t) + \mathbf{B}_2(t) + \mathbf{B}_3(t) - \mathbf{G}_{1,1}(t) - \mathbf{G}_4(t) - \frac{3}{4}\mathbf{D}(t) \\
&\quad \leq \frac{6}{5}\delta\alpha_-\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' - \frac{5}{4}\delta\alpha_-\int_{\mathbb{T}^2}\int_0^1 |w-\overline{w}|^2 dy_1 dy' \\
&\qquad - \Big(\frac{5\sigma_-}{8\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)} - \frac{5\delta\alpha_-}{64\pi^2}\Big)\int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'} w|^2}{y_1(1-y_1)} dy_1 dy' + C\frac{\delta}{\kappa}\mathbf{G}_3(t) \\
&\quad = -\frac{\delta\alpha_-}{20}\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' + \frac{5}{4}\delta\alpha_-\Big(\int_{\mathbb{T}^2}\int_0^1 w dy_1 dy'\Big)^2 \\
&\qquad - \frac{5}{8}\Big(\frac{\sigma_-}{\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)} - \frac{\delta\alpha_-}{8\pi^2}\Big)\int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'} w|^2}{y_1(1-y_1)} dy_1 dy' + C\frac{\delta}{\kappa}\mathbf{G}_3(t). \quad (4.37)
\end{aligned}
$$

Choosing $M = \frac{5}{4}\alpha_-\sigma_-^4 v_-^2$, and combining (4.28), (4.32) and (4.37), we obtain

$$
\begin{aligned}
&-\frac{\delta}{2M}|\dot{\mathbf{X}}(t)|^2 + \mathbf{B}_1(t) + \mathbf{B}_2(t) + \mathbf{B}_3(t) - \mathbf{G}_1(t) - \mathbf{G}_4(t) - \frac{3}{4}\mathbf{D}(t) \\
&\quad \le \Big(-\frac{\delta\alpha_-}{20} + C\delta(\kappa+\delta+\chi)^2\Big)\int_{\mathbb{T}^2}\int_0^1 w^2 dy_1 dy' \\
&\qquad - \frac{5}{8}\Big(\frac{\sigma_-}{\delta}\frac{(2\mu+\lambda)^2}{p''(v_-)} - \frac{\delta\alpha_-}{8\pi^2}\Big)\int_{\mathbb{T}^2}\int_0^1 \frac{|\nabla_{y'}w|^2}{y_1(1-y_1)} dy_1 dy' \\
&\qquad + \sigma_*\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\frac{1+\gamma}{3\gamma^2}\frac{|p(v)-p((v^s)^{-\mathbf{X}})|^3}{p^{2+\frac{1}{\gamma}}((v^s)^{-\mathbf{X}})} d\xi_1 d\xi' + C\frac{\delta}{\kappa}\mathbf{G}_3(t),
\end{aligned}
$$

which indicates the desired inequality (4.20) by using (4.9). The proof of Lemma 4.3 is completed. ∎

Lemma 4.4. *Under the hypotheses of Proposition 3.6, there exists a constant $C > 0$ independent of κ, δ, χ and T, such that for all $t \in [0, T]$, it holds*

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\Big(Q(v|(v^s)^{-\mathbf{X}}) + \frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big) d\xi_1 d\xi' \\
&\qquad + \delta\int_0^t |\dot{\mathbf{X}}(\tau)|^2 d\tau + \int_0^t (G_2(\tau) + G_3(\tau) + G^s(\tau) + D(\tau)) d\tau \\
&\qquad \le C\int_{\mathbb{T}^2}\int_{\mathbb{R}} \Big(Q(v_0|v^s) + \frac{|h_0 - h^s|^2}{2}\Big) d\xi_1 d\xi' \\
&\qquad\quad + C(\delta+\chi)\int_0^t \|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{H^1}^2 d\tau,
\end{aligned} \tag{4.38}
$$

where

$$h_0(\xi) = u_0(\xi) - (2\mu+\lambda)\nabla_\xi v_0(\xi)$$

and

$$
\begin{aligned}
G_2(t) &:= \frac{\kappa}{\delta}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)_{\xi_1}^{-\mathbf{X}}|(h_2^2 + h_3^2) d\xi_1 d\xi', \\
G_3(t) &:= \frac{\kappa}{\delta}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)_{\xi_1}^{-\mathbf{X}}|\Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|^2 d\xi_1 d\xi', \\
G^s(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)_{\xi_1}^{-\mathbf{X}}||p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi', \\
D(t) &:= \int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}}))|^2 d\xi_1 d\xi'.
\end{aligned} \tag{4.39}
$$

Note that by (4.10), (4.11) and the uniform lower and upper boundedness of the volume function v, we have

$$\mathbf{G}_2(t) \sim G_2(t), \quad \mathbf{G}_3(t) \sim G_3(t), \quad \mathbf{D}(t) \sim D(t),$$

uniform in time $t \in [0, T]$.

Proof of Lemma 4.4. First of all, we use (4.12) to have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(Q(v|(v^s)^{-\mathbf{X}})+\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big)d\xi_1 d\xi' \\
&\quad = -\frac{\delta}{2M}|\dot{\mathbf{X}}(t)|^2+\mathbf{B}_1(t)+\mathbf{B}_2(t)+\mathbf{B}_3(t)-\mathbf{G}_1(t)-\mathbf{G}_4(t)-\frac{3}{4}\mathbf{D}(t) \\
&\quad\quad -\frac{\delta}{2M}|\dot{\mathbf{X}}(t)|^2+\dot{\mathbf{X}}(t)\sum_{i=3}^{5}\mathbf{Y}_i(t)+\sum_{i=4}^{9}\mathbf{B}_i(t)-\sum_{i=2}^{3}\mathbf{G}_i(t)-\frac{1}{4}\mathbf{D}(t).
\end{aligned}$$

Using Lemma 4.3 and the Cauchy inequality, we find that there exist positive constants C_1 and C such that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(Q(v|(v^s)^{-\mathbf{X}})+\frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big)d\xi_1 d\xi' \\
&\quad\le -C_1\underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}|p(v)-p((v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi'}_{\mathbf{G}^s(t)} \\
&\quad\quad + C\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}|p(v)-p((v^s)^{-\mathbf{X}})|^3 d\xi_1 d\xi' \\
&\quad\quad -\frac{\delta}{4M}|\dot{\mathbf{X}}(t)|^2+\frac{C}{\delta}\sum_{i=3}^{5}|\mathbf{Y}_i(t)|^2+\sum_{i=4}^{9}\mathbf{B}_i(t)-\sum_{i=2}^{3}\mathbf{G}_i(t)-\frac{1}{4}\mathbf{D}(t)+\frac{1}{40}\mathbf{G}_3(t).
\end{aligned}$$

In what follows, to control the above bad terms, we will use the above good terms $\mathbf{G}_i(t)$ $(i=2,3)$, $\mathbf{D}(t)$ and $\mathbf{G}^s(t)$. In the following, we control the terms on the right-hand side of the above inequality one by one. First, for simplicity, we use the notation $w=p(v)-p((v^s)^{-\mathbf{X}})$ as in (4.21). Using (4.22), the 3D Gagliardo–Nirenberg inequality (3.3) in strip domain and assumption (3.11), we get

$$\begin{aligned}
&C\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}|p(v)-p((v^s)^{-\mathbf{X}})|^3 d\xi_1 d\xi' \le C\frac{\kappa}{\delta}\|w\|^2_{L^\infty}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}|w|d\xi_1 d\xi' \\
&\quad\le C\frac{\kappa}{\delta}(\|w\|\|\partial_{\xi_1}w\|+\|\nabla_\xi w\|\|\nabla^2_\xi w\|)\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}w^2 d\xi_1 d\xi'\Big)^{\frac12} \\
&\quad\quad\times\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}d\xi_1 d\xi'\Big)^{\frac12} \\
&\quad\le C\frac{\kappa}{\sqrt{\delta}}(\|w\|+\|\nabla^2_\xi w\|)\|\nabla_\xi w\|\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}w^2 d\xi_1 d\xi'\Big)^{\frac12} \\
&\quad\le C\frac{\kappa}{\sqrt{\delta}}\chi\|\nabla_\xi w\|\sqrt{\mathbf{G}^s(t)}\le\frac{1}{80}(\mathbf{D}(t)+C_1\mathbf{G}^s(t)).
\end{aligned}\tag{4.40}$$

Here and further in the paper, the norm $\|\cdot\|$ always denotes $\|\cdot\|_{L^2(\mathbb{R}\times\mathbb{T}^2)}$.

- *Estimates on the terms* $\mathbf{Y}_i(t)$ $(i=3,4,5)$. Since

$$|\mathbf{Y}_3(t)|\le C\frac{\delta}{\kappa}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}_{\xi_1}\Big|h_1-(h^s_1)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|d\xi_1 d\xi'\le C\frac{\delta}{\sqrt{\kappa}}\sqrt{\mathbf{G}_3(t)},$$

then

$$\frac{C}{\delta}|\mathbf{Y}_3(t)|^2 \le C\frac{\delta}{\kappa}\mathbf{G}_3(t) \le \frac{1}{40}\mathbf{G}_3(t).$$

For $\mathbf{Y}_4(t)$, we first use the definition of h and $h^s = (h_1^s, 0, 0)^t$ to estimate $h - (h^s)^{-\mathbf{X}}$ in terms of $u - (u^s)^{-\mathbf{X}}$ and $v - (v^s)^{-\mathbf{X}}$ as follows:

$$h - (h^s)^{-\mathbf{X}} = u - (u^s)^{-\mathbf{X}} - (2\mu + \lambda)\nabla_\xi(v - (v^s)^{-\mathbf{X}}), \tag{4.41}$$

which together with assumption (3.11) implies

$$\|h - (h^s)^{-\mathbf{X}}\| \le C(\|u - (u^s)^{-\mathbf{X}}\| + \|\nabla_\xi(v - (v^s)^{-\mathbf{X}})\|) \le C\chi.$$

This together with $\|a_{\xi_1}^{-\mathbf{X}}\|_{L^\infty} \le C\kappa\delta$ and assumption (3.11) leads to

$$|\mathbf{Y}_4(t)| \le C\sqrt{\mathbf{G}_3(t)}\|a_{\xi_1}^{-\mathbf{X}}\|_{L^\infty}^{\frac{1}{2}}(\|u_1 - (u_1^s)^{-\mathbf{X}}\| + \|v - (v^s)^{-\mathbf{X}}\|_{H^1}) \le C\chi(\kappa\delta)^{\frac{1}{2}}\sqrt{\mathbf{G}_3(t)},$$

which implies

$$\frac{C}{\delta}|\mathbf{Y}_4(t)|^2 \le C\chi^2\kappa\mathbf{G}_3(t) \le \frac{1}{40}\mathbf{G}_3(t).$$

Using (3.8) and assumption (3.11), we have

$$\begin{aligned}
\frac{C}{\delta}|\mathbf{Y}_5(t)|^2 &\le \frac{C}{\delta}\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}} w^2 d\xi_1 d\xi'\Big)^2 + \frac{C}{\delta}\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\frac{h_2^2 + h_3^2}{2} d\xi_1 d\xi'\Big)^2 \\
&\le C\frac{\kappa^2}{\delta}\|w\|^2\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)_{\xi_1}^{-\mathbf{X}} w^2 d\xi_1 d\xi' \\
&\quad + C\kappa\|h - (h^s)^{-\mathbf{X}}\|^2\int_{\mathbb{T}^2}\int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}}\frac{h_2^2 + h_3^2}{2} d\xi_1 d\xi' \\
&\le C\chi^2\mathbf{G}^s(t) + C\kappa\chi^2\mathbf{G}_2(t) \le \frac{1}{40}(C_1\mathbf{G}^s(t) + \mathbf{G}_2(t)).
\end{aligned}$$

• *Estimates on the terms* $\mathbf{B}_i(t)$ $(i = 4, \ldots, 9)$. Recalling the definition of F in (4.7) and assumption (3.11), we get

$$\|F\|_{L^\infty} \le C(\|v - (v^s)^{-\mathbf{X}}\|_{L^\infty} + \|u_1 - (u_1^s)^{-\mathbf{X}}\|_{L^\infty}) \le C\chi. \tag{4.42}$$

For $\mathbf{B}_4(t)$, using (3.8) and (4.42), we have

$$\begin{aligned}
\mathbf{B}_4(t) &= \int_{\mathbb{T}^2}\int_{\mathbb{R}} F a_{\xi_1}^{-\mathbf{X}}\Big(Q(v|(v^s)^{-\mathbf{X}}) + \frac{(h_1 - (h_1^s)^{-\mathbf{X}})^2}{2} + \frac{h_2^2 + h_3^2}{2}\Big) d\xi_1 d\xi' \\
&\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |F| a_{\xi_1}^{-\mathbf{X}}(Q(v|(v^s)^{-\mathbf{X}}) + |p(v) - p((v^s)^{-\mathbf{X}})|^2) d\xi_1 d\xi' \\
&\quad + C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |F| a_{\xi_1}^{-\mathbf{X}}\Big(\frac{h_2^2 + h_3^2}{2} + \Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|^2\Big) d\xi_1 d\xi' \\
&\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |F| a_{\xi_1}^{-\mathbf{X}} w^2 d\xi_1 d\xi' + C\chi(\mathbf{G}_2(t) + \mathbf{G}_3(t)).
\end{aligned}$$

By the definition of F (4.7), we obtain

$$C\int_{\mathbb{T}^2}\int_{\mathbb{R}}|F|a_{\xi_1}^{-\mathbf{X}}w^2d\xi_1d\xi' \le \underbrace{C\int_{\mathbb{T}^2}\int_{\mathbb{R}}a_{\xi_1}^{-\mathbf{X}}|w|^3d\xi_1d\xi'}_{\mathbf{B}_{4,1}(t)} + \underbrace{C\int_{\mathbb{T}^2}\int_{\mathbb{R}}a_{\xi_1}^{-\mathbf{X}}|h_1-(h_1^s)^{-\mathbf{X}}|w^2d\xi_1d\xi'}_{\mathbf{B}_{4,2}(t)} + \underbrace{C\int_{\mathbb{T}^2}\int_{\mathbb{R}}a_{\xi_1}^{-\mathbf{X}}|\partial_{\xi_1}(v-(v^s)^{-\mathbf{X}})|w^2d\xi_1d\xi'}_{\mathbf{B}_{4,3}(t)}.$$

The same as in (4.40), it holds

$$\mathbf{B}_{4,1}(t)\le\frac{1}{80}(\mathbf{D}(t)+C_1\mathbf{G}^s(t)).$$

Using the Gagliardo–Nirenberg inequality (3.2) in strip domain and Lemma 2.1, we have

$$\begin{aligned}
\mathbf{B}_{4,2}(t) &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}a_{\xi_1}^{-\mathbf{X}}\Big|h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|w^2d\xi_1d\xi'\\
&\quad + C\int_{\mathbb{T}^2}\int_{\mathbb{R}}a_{\xi_1}^{-\mathbf{X}}|w|^3d\xi_1d\xi'\\
&\le C\|w\|_{L^\infty}^2\sqrt{\mathbf{G}_3(t)}\sqrt{\frac{\kappa}{\delta}}\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)_{\xi_1}^{-\mathbf{X}}d\xi_1d\xi'\Big)^{\frac12}+\frac{1}{80}(\mathbf{D}(t)+C_1\mathbf{G}^s(t))\\
&\le C(\|w\|+\|\nabla_\xi^2w\|)\|\nabla_\xi w\|\sqrt{\mathbf{G}_3(t)}\sqrt{\kappa}+\frac{1}{80}(\mathbf{D}(t)+C_1\mathbf{G}^s(t))\\
&\le C\chi\sqrt{\kappa}\sqrt{\mathbf{D}(t)}\sqrt{\mathbf{G}_3(t)}+\frac{1}{80}(\mathbf{D}(t)+C_1\mathbf{G}^s(t))\\
&\le \frac{1}{40}\mathbf{D}(t)+\frac{1}{80}(\mathbf{G}_3(t)+C_1\mathbf{G}^s(t)).
\end{aligned}$$

Using the fact that

$$\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}})) = p'(v)\nabla_\xi(v-(v^s)^{-\mathbf{X}})+\nabla_\xi(v^s)^{-\mathbf{X}}(p'(v)-p'((v^s)^{-\mathbf{X}})),$$

we obtain

$$\begin{aligned}
|\nabla_\xi(v-(v^s)^{-\mathbf{X}})| &\le C|\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}}))|+C|(v^s)_{\xi_1}^{-\mathbf{X}}||v-(v^s)^{-\mathbf{X}}|\\
&\le C|\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}}))|+C|(v^s)_{\xi_1}^{-\mathbf{X}}||p(v)-p((v^s)^{-\mathbf{X}})|. \quad (4.43)
\end{aligned}$$

Using (4.43), we have

$$\begin{aligned}
\mathbf{B}_{4,3}(t) &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}a_{\xi_1}^{-\mathbf{X}}(|\partial_{\xi_1}w|+(v^s)_{\xi_1}^{-\mathbf{X}}|w|)w^2d\xi_1d\xi'\\
&\le \|w\|_{L^\infty}^2\|\partial_{\xi_1}w\|\|a_{\xi_1}^{-\mathbf{X}}\|+C\|w\|_{L^\infty}\mathbf{G}^s(t)\\
&\le C(\|w\|+\|\nabla_\xi^2w\|)\|\nabla_\xi w\|^2\sqrt{\delta\kappa}+C\chi\mathbf{G}^s(t)\\
&\le C\chi\sqrt{\delta\kappa}\mathbf{D}(t)+C\chi\mathbf{G}^s(t)\le\frac{1}{80}(\mathbf{D}(t)+C_1\mathbf{G}^s(t)).
\end{aligned}$$

Combining the above estimates, we get

$$\mathbf{B}_4(t) \leq \frac{1}{16}\mathbf{D}(t) + \frac{1}{20}(\mathbf{G}_2(t) + \mathbf{G}_3(t) + C_1\mathbf{G}^s(t)).$$

Notice that

$$\partial_{\xi_1}(v - (v^s)^{-\mathbf{X}}) = \frac{\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))}{p'(v)} + \partial_{\xi_1} p((v^s)^{-\mathbf{X}})\Big(\frac{1}{p'(v)} - \frac{1}{p'((v^s)^{-\mathbf{X}})}\Big).$$

Using (4.11), we obtain

$$\begin{aligned}
\mathbf{B}_5(t) &\leq C\int_{\mathbb{T}^2}\int_{\mathbb{R}} (v^s)^{-\mathbf{X}}_{\xi_1}|\partial_{\xi_1} w|\Big(|w| + \Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|\Big) d\xi_1 d\xi' \\
&\quad + C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}|^2 |w|\Big(|w| + \Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|\Big) d\xi_1 d\xi' \\
&\leq C\delta\|\partial_{\xi_1} w\|\sqrt{\mathbf{G}^s(t)} + C\delta\|\partial_{\xi_1} w\|\sqrt{\frac{\delta}{\kappa}}\sqrt{\mathbf{G}_3(t)} \\
&\quad + C\delta^2\mathbf{G}^s(t) + C\delta^2\sqrt{\mathbf{G}^s(t)}\sqrt{\frac{\delta}{\kappa}}\sqrt{\mathbf{G}_3(t)} \\
&\leq \frac{1}{80}(\mathbf{D}(t) + C_1\mathbf{G}^s(t) + \mathbf{G}_3(t)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathbf{B}_6(t) + \mathbf{B}_7(t) + \mathbf{B}_8(t) \\
&\quad \leq C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}||\partial_{\xi_1} w||w| d\xi_1 d\xi' + C\frac{\kappa}{\delta}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}||\partial_{\xi_1} w||w| d\xi_1 d\xi' \\
&\qquad + C\frac{\kappa}{\delta}\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)^{-\mathbf{X}}_{\xi_1}|^2 w^2 d\xi_1 d\xi' \\
&\quad \leq C\delta\|\partial_{\xi_1} w\|\sqrt{\mathbf{G}^s(t)} + C\kappa\|\partial_{\xi_1} w\|\sqrt{\mathbf{G}^s(t)} + C\kappa\delta\mathbf{G}^s(t) \\
&\quad \leq \frac{1}{80}(\mathbf{D}(t) + C_1\mathbf{G}^s(t)).
\end{aligned}$$

Finally, we control the last term $\mathbf{B}_9(t)$. Using the definition of R in (4.3), we obtain

$$\begin{aligned}
\mathbf{B}_9(t) &= (2\mu + \lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(h - (h^s)^{-\mathbf{X}}) \cdot \frac{\nabla_\xi u \cdot \nabla_\xi v - \mathrm{div}_\xi u \nabla_\xi v}{v} d\xi_1 d\xi' \\
&\quad - \mu\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}(h - (h^s)^{-\mathbf{X}}) \cdot \nabla_\xi \times \nabla_\xi \times u d\xi_1 d\xi' \\
&=: \mathbf{B}_{9,1}(t) + \mathbf{B}_{9,2}(t).
\end{aligned}$$

To control $\mathbf{B}_{9,1}(t)$, we set

$$u' = (u_2, u_3), \quad \nabla_{\xi'} = (\partial_{\xi_2}, \partial_{\xi_3}) \quad \text{and} \quad \nabla_{\xi'} \cdot u' = \partial_{\xi_2} u_2 + \partial_{\xi_3} u_3.$$

Notice that the first component of $\nabla_\xi u \cdot \nabla_\xi v - \mathrm{div}_\xi u \nabla_\xi v$ is

$$\begin{aligned}\partial_{\xi_1} u \cdot \nabla_\xi v - \mathrm{div}_\xi u \partial_{\xi_1} v &= \partial_{\xi_1} u' \cdot \nabla_{\xi'} v - \nabla_{\xi'} \cdot u' \partial_{\xi_1} v \\ &= \partial_{\xi_1} u' \cdot \frac{\nabla_{\xi'} p(v)}{-\gamma p^{1+\frac{1}{\gamma}}(v)} - \nabla_{\xi'} \cdot u' \frac{\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))}{-\gamma p^{1+\frac{1}{\gamma}}(v)} \\ &\quad - \nabla_{\xi'} \cdot u' \frac{\partial_{\xi_1} p((v^s)^{-\mathbf{X}})}{-\gamma p^{1+\frac{1}{\gamma}}(v)}.\end{aligned}$$

By assumption (3.11) and the Sobolev inequality, we have

$$\begin{aligned}\|h_1 - (h_1^s)^{-\mathbf{X}}\|_{L^3} &\le C\|h_1 - (h_1^s)^{-\mathbf{X}}\|_{H^1} \\ &\le C(\|u - (u^s)^{-\mathbf{X}}\|_{H^1} + \|v - (v^s)^{-\mathbf{X}}\|_{H^2}) \le C\chi. \end{aligned} \tag{4.44}$$

Thus, using (4.44), the first part of the integrand in $\mathbf{B}_{9,1}(t)$ can be controlled as

$$\begin{aligned}&(2\mu + \lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{v} (h_1 - (h_1^s)^{-\mathbf{X}})(\partial_{\xi_1} u \cdot \nabla_\xi v - \mathrm{div}_\xi u \partial_{\xi_1} v) d\xi_1 d\xi' \\ &\le C\|h_1 - (h_1^s)^{-\mathbf{X}}\|_{L^3} \|\nabla_\xi (u - (u^s)^{-\mathbf{X}})\|_{L^6} \sqrt{\mathbf{D}(t)} \\ &\quad + C\delta\Big(\sqrt{\frac{\delta}{\kappa}}\sqrt{\mathbf{G}_3(t)} + \sqrt{\mathbf{G}^s(t)}\Big)\|\nabla_{\xi'}(u - (u^s)^{-\mathbf{X}})\| \\ &\le C(\chi + \delta)(\mathbf{G}_3(t) + C_1\mathbf{G}^s(t) + \mathbf{D}(t) + \|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|_{H^1}^2) \\ &\le \frac{1}{80}(\mathbf{G}_3(t) + C_1\mathbf{G}^s(t) + \mathbf{D}(t)) + C(\chi + \delta)\|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|_{H^1}^2.\end{aligned}$$

Similarly, the second and third parts of integrand in $\mathbf{B}_{9,1}(t)$ can be treated as

$$\begin{aligned}&(2\mu + \lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \frac{a^{-\mathbf{X}}}{v} h' \cdot (\nabla_{\xi'} u \cdot \nabla_\xi v - \mathrm{div}_\xi u \nabla_{\xi'} v) d\xi_1 d\xi' \\ &\le C\|h'\|_{L^3} \|\nabla_\xi (u - (u^s)^{-\mathbf{X}})\|_{L^6} \sqrt{\mathbf{D}(t)} \\ &\quad + C\delta\sqrt{\frac{\delta}{\kappa}}\sqrt{\mathbf{G}_2(t)}(\|\nabla_{\xi'}(u - (u^s)^{-\mathbf{X}})\| + \sqrt{\mathbf{D}(t)}) \\ &\le C(\chi + \delta)(\mathbf{G}_2(t) + \mathbf{D}(t) + \|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|_{H^1}^2) \\ &\le \frac{1}{80}(\mathbf{G}_2(t) + \mathbf{D}(t)) + C(\chi + \delta)\|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|_{H^1}^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathbf{B}_{9,1}(t) &\le \frac{1}{40}(\mathbf{G}_2(t) + \mathbf{G}_3(t) + C_1\mathbf{G}^s(t) + \mathbf{D}(t)) \\ &\quad + C(\chi + \delta)\|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|_{H^1}^2.\end{aligned} \tag{4.45}$$

Using (4.41) and direct calculations yield

$$\begin{aligned}\mathbf{B}_{9,2}(t) &= -\mu \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}}(u - (u^s)^{-\mathbf{X}}) \cdot \nabla_\xi \times \nabla_\xi \times u \, d\xi_1 d\xi' \\ &\quad + \mu(2\mu + \lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} \nabla_\xi (v - (v^s)^{-\mathbf{X}}) \cdot \nabla_\xi \times \nabla_\xi \times u \, d\xi_1 d\xi' \\ &=: \mathbf{B}_{9,2}^1(t) + \mathbf{B}_{9,2}^2(t).\end{aligned}$$

By integration by parts over $\mathbb{R} \times \mathbb{T}^2$, we obtain

$$
\begin{aligned}
\mathbf{B}_{9,2}^1(t) &= -\mu \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\
&\quad - \mu \int_{\mathbb{T}^2} \int_{\mathbb{R}} a_{\xi_1}^{-\mathbf{X}} (u_2(\partial_{\xi_1} u_2 - \partial_{\xi_2} u_1) - u_3(\partial_{\xi_3} u_1 - \partial_{\xi_1} u_3)) d\xi_1 d\xi' \\
&\le -\frac{3\mu}{4} \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\
&\quad + C \int_{\mathbb{T}^2} \int_{\mathbb{R}} |a_{\xi_1}^{-\mathbf{X}}|^2 (h_2^2 + h_3^2) d\xi_1 d\xi' + C \int_{\mathbb{T}^2} \int_{\mathbb{R}} |a_{\xi_1}^{-\mathbf{X}}|^2 |\nabla_{\xi'} v|^2 d\xi_1 d\xi' \\
&\le -\frac{3\mu}{4} \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' + C\kappa\delta \mathbf{G}_2(t) + C(\kappa\delta)^2 \mathbf{D}(t) \\
&\le -\frac{3\mu}{4} \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' + \frac{1}{80} (\mathbf{G}_2(t) + \mathbf{D}(t)).
\end{aligned}
$$

By integration by parts over $\mathbb{R} \times \mathbb{T}^2$, we have

$$
\begin{aligned}
\mathbf{B}_{9,2}^2(t) &= -\mu(2\mu + \lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} (v - (v^s)^{-\mathbf{X}}) \nabla_\xi \times \nabla_\xi \times u \cdot \nabla_\xi a^{-\mathbf{X}} d\xi_1 d\xi' \\
&= -\mu(2\mu + \lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \big[(v - (v^s)^{-\mathbf{X}}) \big(\partial_{\xi_2} (\partial_{\xi_1} u_2 - \partial_{\xi_2} u_1) \\
&\qquad\qquad - \partial_{\xi_3} (\partial_{\xi_3} u_1 - \partial_{\xi_1} u_3) \big) a_{\xi_1}^{-\mathbf{X}} \big] d\xi_1 d\xi' \\
&= \mu(2\mu + \lambda) \int_{\mathbb{T}^2} \int_{\mathbb{R}} \big[\big(\partial_{\xi_2} (v - (v^s)^{-\mathbf{X}}) (\partial_{\xi_1} u_2 - \partial_{\xi_2} u_1) \\
&\qquad\qquad - \partial_{\xi_3} (v - (v^s)^{-\mathbf{X}}) (\partial_{\xi_3} u_1 - \partial_{\xi_1} u_3) \big) a_{\xi_1}^{-\mathbf{X}} \big] d\xi_1 d\xi' \\
&\le C\kappa\delta\mu \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\
&\quad + C\kappa\delta \int_{\mathbb{T}^2} \int_{\mathbb{R}} |\nabla_{\xi'} (v - (v^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\
&\le \frac{\mu}{4} \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' + \frac{1}{80} \mathbf{D}(t).
\end{aligned}
$$

Combining the above two estimates, we get

$$
\begin{aligned}
\mathbf{B}_{9,2}(t) \le &- \frac{\mu}{2} \int_{\mathbb{T}^2} \int_{\mathbb{R}} a^{-\mathbf{X}} |\nabla_\xi \times (u - (u^s)^{-\mathbf{X}})|^2 d\xi_1 d\xi' \\
&+ \frac{1}{80} \mathbf{G}_2(t) + \frac{1}{40} \mathbf{D}(t).
\end{aligned}
\tag{4.46}
$$

Combination (4.45) and (4.46) yields

$$
\begin{aligned}
\mathbf{B}_9(t) \le &\frac{3}{80} (\mathbf{G}_2(t) + \mathbf{G}_3(t) + C_1 \mathbf{G}^s(t)) + \frac{1}{20} \mathbf{D}(t) \\
&+ C(\chi + \delta) \|\nabla_\xi (u - (u^s)^{-\mathbf{X}})\|_{H^1}^2.
\end{aligned}
$$

• *Conclusion.* Combining the above estimates, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}} a^{-\mathbf{X}}\rho\Big(Q(v|(v^s)^{-\mathbf{X}}) + \frac{|h-(h^s)^{-\mathbf{X}}|^2}{2}\Big)d\xi_1 d\xi' \\
&\quad\le -\frac{\delta}{4M}|\dot{\mathbf{X}}(t)|^2 - \frac{1}{2}\mathbf{G}_2(t) - \frac{1}{2}\mathbf{G}_3(t) - \frac{1}{16}\mathbf{D}(t) - \frac{1}{2}C_1\mathbf{G}^s(t) \\
&\quad\quad + C(\chi+\delta)\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{H^1}^2.
\end{aligned}
$$

Integrating the above inequality over $[0,t]$ for any $t\in[0,T]$, using (4.43) and noting that

$$
1 < a^{-\mathbf{X}} < 1+\kappa, \quad \rho_0 \le C,
$$

we can obtain the desired inequality (4.38) with the new notations (4.39).

This completes the proof of Lemma 4.4. ■

4.2. *Estimates for* $\|u-(u^s)^{-\mathbf{X}}\|$ *and* $\|v-(v^s)^{-\mathbf{X}}\|_{H^1}$

In this subsection, we shall obtain the zero-th order energy estimates for function (v,u).

Lemma 4.5. *Under the hypotheses of Proposition 3.6, there exists a constant $C>0$ independent of κ, δ, χ and T, such that for all $t\in[0,T]$, it holds*

$$
\begin{aligned}
&\|(v-(v^s)^{-\mathbf{X}})(t)\|_{H^1}^2 + \|(u-(u^s)^{-\mathbf{X}})(t)\|^2 + \delta\int_0^t |\dot{\mathbf{X}}(\tau)|^2 d\tau \\
&+\int_0^t (G^s(\tau)+D(\tau)+\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|^2)d\tau \\
&\le C(\|v_0-v^s\|_{H^1}^2 + \|u_0-u^s\|^2) + C(\delta+\chi)\int_0^t \|\nabla_\xi^2(u-(u^s)^{-\mathbf{X}})\|^2 d\tau, \quad (4.47)
\end{aligned}
$$

where G^s and D are as in (4.39).

Proof. From systems (2.6) and (2.7), we can get the perturbed system for $(v-(v^s)^{-\mathbf{X}}, u-(u^s)^{-\mathbf{X}})$ as

$$
\begin{cases}
\rho\partial_t(v-(v^s)^{-\mathbf{X}}) - \sigma\rho\partial_{\xi_1}(v-(v^s)^{-\mathbf{X}}) + \rho u\cdot\nabla_\xi(v-(v^s)^{-\mathbf{X}}) \\
\qquad -\dot{\mathbf{X}}(t)\rho\partial_{\xi_1}(v^s)^{-\mathbf{X}} + F\partial_{\xi_1}(v^s)^{-\mathbf{X}} = \mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}}), \\
\rho\partial_t(u-(u^s)^{-\mathbf{X}}) - \sigma\rho\partial_{\xi_1}(u-(u^s)^{-\mathbf{X}}) + \rho u\cdot\nabla_\xi(u-(u^s)^{-\mathbf{X}}) \\
\qquad + \nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}})) - \dot{\mathbf{X}}(t)\rho\partial_{\xi_1}(u^s)^{-\mathbf{X}} + F\partial_{\xi_1}(u^s)^{-\mathbf{X}} \\
\qquad = \mu\Delta_\xi(u-(u^s)^{-\mathbf{X}}) + (\mu+\lambda)\nabla_\xi\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}}),
\end{cases}
\quad (4.48)
$$

where F is defined in (4.7). Multiplying $(4.48)_1$ by $-(p(v)-p((v^s)^{-\mathbf{X}}))$, using (4.14), we obtain

$$
\begin{aligned}
&\partial_t(\rho Q(v|(v^s)^{-\mathbf{X}})) - \sigma\partial_{\xi_1}(\rho Q(v|(v^s)^{-\mathbf{X}})) + \mathrm{div}_\xi(\rho u Q(v|(v^s)^{-\mathbf{X}})) \\
&\quad = -(p(v)-p((v^s)^{-\mathbf{X}}))\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}}) - \dot{\mathbf{X}}(t)\rho p'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} \\
&\quad\quad + \sigma_* p(v|(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}} + Fp'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)_{\xi_1}^{-\mathbf{X}}. \quad (4.49)
\end{aligned}
$$

Multiplying (4.48)$_2$ by $u-(u^s)^{-\mathbf{X}}$, using (4.14), we get

$$
\begin{aligned}
&\partial_t\Big(\rho\frac{|u-(u^s)^{-\mathbf{X}}|^2}{2}\Big)-\sigma\partial_{\xi_1}\Big(\rho\frac{|u-(u^s)^{-\mathbf{X}}|^2}{2}\Big)+\mathrm{div}_\xi\Big(\rho u\frac{|u-(u^s)^{-\mathbf{X}}|^2}{2}\Big)\\
&\quad+\mathrm{div}_\xi((p(v)-p((v^s)^{-\mathbf{X}}))(u-(u^s)^{-\mathbf{X}}))-\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})(p(v)-p((v^s)^{-\mathbf{X}}))\\
&\quad=\dot{\mathbf{X}}(t)\rho(u_1^s)^{-\mathbf{X}}_{\xi_1}(u_1-(u_1^s)^{-\mathbf{X}})-F(u_1^s)^{-\mathbf{X}}_{\xi_1}(u_1-(u_1^s)^{-\mathbf{X}})-\mu|\nabla_\xi(u-(u^s)^{-\mathbf{X}})|^2\\
&\qquad-(\mu+\lambda)(\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}}))^2+\mu\mathrm{div}_\xi(\nabla_\xi(u-(u^s)^{-\mathbf{X}})\cdot(u-(u^s)^{-\mathbf{X}}))\\
&\qquad+(\mu+\lambda)\mathrm{div}_\xi(\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})(u-(u^s)^{-\mathbf{X}})).
\end{aligned}
\tag{4.50}
$$

Adding (4.49) and (4.50) together, and integrating the resultant equation over $\mathbb{R}\times\mathbb{T}^2$, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\Big(Q(v|(v^s)^{-\mathbf{X}})+\frac{|u-(u^s)^{-\mathbf{X}}|^2}{2}\Big)d\xi_1d\xi'\\
&\qquad+\underbrace{\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\mu|\nabla_\xi(u-(u^s)^{-\mathbf{X}})|^2+(\mu+\lambda)(\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}}))^2)d\xi_1d\xi'}_{\mathbf{D}_1(t)}\\
&\qquad=\dot{\mathbf{X}}(t)\mathcal{Y}(t)+\sum_{i=1}^{3}I_i(t),
\end{aligned}
\tag{4.51}
$$

where

$$
\begin{aligned}
\mathcal{Y}(t)=&-\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho p'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}d\xi_1d\xi'\\
&+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho(u_1^s)^{-\mathbf{X}}_{\xi_1}(u_1-(u_1^s)^{-\mathbf{X}})d\xi_1d\xi'=:\mathcal{Y}_1(t)+\mathcal{Y}_2(t),
\end{aligned}
$$

and

$$
\begin{aligned}
I_1(t)&=\sigma_*\int_{\mathbb{T}^2}\int_{\mathbb{R}}p(v|(v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}d\xi_1d\xi',\\
I_2(t)&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}Fp'((v^s)^{-\mathbf{X}})(v-(v^s)^{-\mathbf{X}})(v^s)^{-\mathbf{X}}_{\xi_1}d\xi_1d\xi',\\
I_3(t)&=-\int_{\mathbb{T}^2}\int_{\mathbb{R}}F(u_1^s)^{-\mathbf{X}}_{\xi_1}(u_1-(u_1^s)^{-\mathbf{X}})d\xi_1d\xi'.
\end{aligned}
$$

From (3.4) and (3.6), it follows that

$$
\begin{aligned}
|\mathcal{Y}_1(t)|&\le\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}d\xi_1d\xi'\Big)^{\frac12}\Big(\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)^{-\mathbf{X}}_{\xi_1}|v-(v^s)^{-\mathbf{X}}|^2d\xi_1d\xi'\Big)^{\frac12}\\
&\le C\sqrt{\delta}\sqrt{G^s(t)}.
\end{aligned}
$$

Hence, we have

$$
|\dot{\mathbf{X}}(t)||\mathcal{Y}_1(t)|\le\frac{\delta}{8}|\dot{\mathbf{X}}(t)|^2+CG^s(t).
$$

By (4.41), it holds

$$u_1 - (u_1^s)^{-\mathbf{X}} = h_1 - (h_1^s)^{-\mathbf{X}} + (2\mu + \lambda)\partial_{\xi_1}(v - (v^s)^{-\mathbf{X}}). \tag{4.52}$$

To control $\mathcal{Y}_2(t)$, we use $(2.9)_1$ and (4.43) and get

$$\begin{aligned}
|\mathcal{Y}_2(t)| &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}} |(v^s)_{\xi_1}^{-\mathbf{X}}|\Big(\Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big| + |p(v) - p((v^s)^{-\mathbf{X}})| \\
&\qquad + |\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))| + |(v^s)_{\xi_1}^{-\mathbf{X}}||v - (v^s)^{-\mathbf{X}}|\Big)d\xi_1 d\xi' \\
&\le C\Big(\frac{\delta}{\sqrt{\kappa}}\sqrt{G_3(t)} + \sqrt{\delta}\sqrt{G^s(t)} + \delta^{\frac{3}{2}}\sqrt{D(t)}\Big).
\end{aligned}$$

Thus, it holds

$$|\dot{\mathbf{X}}(t)||\mathcal{Y}_2(t)| \le \frac{\delta}{8}|\dot{\mathbf{X}}(t)|^2 + C\frac{\delta}{\kappa}G_3(t) + CG^s(t) + C\delta^2 D(t).$$

For $I_1(t)$, using (3.6), we obtain

$$|I_1(t)| \le CG^s(t).$$

For $I_2(t)$, using Lemma 2.1, and the definition of F (4.7), it holds

$$\begin{aligned}
I_2(t) &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}} (v^s)_{\xi_1}^{-\mathbf{X}}\Big(\Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big| + |p(v) - p((v^s)^{-\mathbf{X}})| \\
&\qquad + |\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))| + (v^s)_{\xi_1}^{-\mathbf{X}}|v - (v^s)^{-\mathbf{X}}|\Big) \\
&\qquad \times |p(v) - p((v^s)^{-\mathbf{X}})|d\xi_1 d\xi' \\
&\le C\Big(\sqrt{\frac{\delta}{\kappa}}\sqrt{G_3(t)}\sqrt{G^s(t)} + G^s(t) + \delta\sqrt{D(t)}\sqrt{G^s(t)}\Big) \\
&\le C\frac{\delta}{\kappa}G_3(t) + CG^s(t) + C\delta^2 D(t).
\end{aligned}$$

Using (4.52) and $(2.9)_1$, similarly to $I_2(t)$, we get

$$I_3(t) \le C\frac{\delta}{\kappa}G_3(t) + CG^s(t) + C\delta^2 D(t).$$

Integrating (4.51) over $[0, t]$ for any $t \le T$ and combining the above estimates, we can find that for some constant $C_2 > 0$,

$$\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\Big(Q(v|(v^s)^{-\mathbf{X}}) + \frac{|u - (u^s)^{-\mathbf{X}}|^2}{2}\Big)d\xi_1 d\xi' + \int_0^t \mathbf{D}_1(\tau)d\tau \\
&\quad \le C\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho_0\Big(Q(v_0|v^s) + \frac{|u_0 - u^s|^2}{2}\Big)d\xi_1 d\xi' \\
&\qquad + \int_0^t \Big(\frac{\delta}{4}|\dot{\mathbf{X}}(\tau)|^2 + C\frac{\delta}{\kappa}G_3(\tau) + C_2 G^s(\tau) + C\delta^2 D(\tau)\Big)d\tau,
\end{aligned}$$

which together with (4.38) yields

$$\begin{aligned}&\|(v-(v^s)^{-\mathbf{X}})(\tau)\|^2+\|(h-(h^s)^{-\mathbf{X}})(\tau)\|^2+\|(u-(u^s)^{-\mathbf{X}})(\tau)\|^2\\&\quad+\delta\int_0^t|\dot{\mathbf{X}}(\tau)|^2d\tau+\int_0^t(G_2(\tau)+G_3(\tau)+G^s(\tau)+D(\tau)+\mathbf{D}_1(\tau))d\tau\\&\quad\le C(\|v_0-v^s\|^2+\|h_0-h^s\|^2+\|u_0-u^s\|^2)\\&\qquad+C(\delta+\chi)\int_0^t\|\nabla_\xi^2(u-(u^s)^{-\mathbf{X}})\|^2d\tau,\end{aligned}\tag{4.53}$$

where by Lemma 3.3, we have used the fact that

$$C^{-1}|v-(v^s)^{-\mathbf{X}}|^2\le Q(v|(v^s)^{-\mathbf{X}})\le C|v-(v^s)^{-\mathbf{X}}|^2.$$

Finally, from (4.41), it holds

$$\|\nabla_\xi(v-(v^s)^{-\mathbf{X}})\|\le C(\|h-(h^s)^{-\mathbf{X}}\|+\|u-(u^s)^{-\mathbf{X}}\|)\tag{4.54}$$

and

$$\|h_0-h^s\|\le C(\|u_0-u^s\|+\|\nabla_\xi(v_0-v^s)\|).\tag{4.55}$$

Thus, combining (4.53), (4.54) and (4.55), and using the fact that

$$\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|^2\sim\mathbf{D}_1,$$

we can obtain the desired inequality (4.47). The proof of Lemma 4.5 is completed. ■

4.3. Estimates for $\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|$

Lemma 4.6. *Under the hypotheses of Proposition 3.6, there exists a constant $C>0$ independent of κ, δ, χ and T, such that for all $t\in[0,T]$, it holds*

$$\begin{aligned}&\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})(t)\|^2+\int_0^t\|\nabla_\xi^2(u-(u^s)^{-\mathbf{X}})\|^2d\tau\\&\quad\le C(\|v_0-v^s\|_{H^1}^2+\|u_0-u^s\|_{H^1}^2).\end{aligned}\tag{4.56}$$

Proof. Multiplying $(4.48)_2$ by $-v\Delta_\xi(u-(u^s)^{-\mathbf{X}})$ and integrating the resultant equations over $\mathbb{R}\times\mathbb{T}^2$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\nabla_\xi(u-(u^s)^{-\mathbf{X}})|^2}{2}d\xi_1d\xi'\\&\quad+\underbrace{\mu\int_{\mathbb{T}^2}\int_{\mathbb{R}}v|\Delta_\xi(u-(u^s)^{-\mathbf{X}})|^2d\xi_1d\xi'+(\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}v|\nabla_\xi\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})|^2d\xi_1d\xi'}_{\mathbf{D}_2(t)}\\&\quad=-\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Big(\nabla_\xi u\cdot\nabla_\xi(u-(u^s)^{-\mathbf{X}})\cdot\nabla_\xi(u-(u^s)^{-\mathbf{X}})-\mathrm{div}_\xi u\frac{|\nabla_\xi(u-(u^s)^{-\mathbf{X}})|^2}{2}\Big)d\xi_1d\xi'\end{aligned}$$

$$
\begin{aligned}
&-\dot{\mathbf{X}}(t)\int_{\mathbb{T}^2}\int_{\mathbb{R}}(u_1^s)_{\xi_1}^{-\mathbf{X}}\Delta_\xi(u_1-(u_1^s)^{-\mathbf{X}})d\xi_1 d\xi' \\
&+\int_{\mathbb{T}^2}\int_{\mathbb{R}} vF(u_1^s)_{\xi_1}^{-\mathbf{X}}\Delta_\xi(u_1-(u_1^s)^{-\mathbf{X}})d\xi_1 d\xi' \\
&+(\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\Delta_\xi(u-(u^s)^{-\mathbf{X}})-\nabla_\xi \mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})) \\
&\qquad\qquad \times\nabla_\xi v\,\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})d\xi_1 d\xi' \\
&+\int_{\mathbb{T}^2}\int_{\mathbb{R}} v\Delta_\xi(u-(u^s)^{-\mathbf{X}})\cdot\nabla_\xi(p(v)-p((v^s)^{-\mathbf{X}}))d\xi_1 d\xi' =: \sum_{i=1}^{5} J_i(t).
\end{aligned}
$$

Using assumption (3.11) and the Sobolev inequality, we have $\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{L^3}\le C\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{H^1}\le C\chi$. Thus, it holds

$$
\begin{aligned}
J_1(t) &\le C\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{L^3}\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{L^6}\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\| \\
&\quad + C\delta^2\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|^2 \\
&\le C\chi\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|_{H^1}\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\| + C\delta^2\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|^2 \\
&\le C(\delta+\chi)(\mathbf{D}_2(t)+\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|^2).
\end{aligned}
$$

We notice $-\sigma_*(v^s)_{\xi_1}^{-\mathbf{X}}=(u_1^s)_{\xi_1}^{-\mathbf{X}}$ from $(2.9)_1$ and use Lemma 2.1, then we have

$$
J_2(t)\le|\dot{\mathbf{X}}(t)|\|(u_1^s)_{\xi_1}^{-\mathbf{X}}\|_{L^2(\mathbb{R})}\sqrt{\mathbf{D}_2(t)}\le|\dot{\mathbf{X}}(t)|\delta^{\frac{3}{2}}\sqrt{\mathbf{D}_2(t)}\le C\delta^2|\dot{\mathbf{X}}(t)|^2+C\delta\mathbf{D}_2(t).
$$

For $J_3(t)$, using $-\sigma_*(v^s)_{\xi_1}^{-\mathbf{X}}=(u_1^s)_{\xi_1}^{-\mathbf{X}}$ and Lemma 2.1, and the definition of F (4.7), it holds

$$
\begin{aligned}
J_3(t) &\le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(v^s)_{\xi_1}^{-\mathbf{X}}\Big(\Big|h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|+|p(v)-p((v^s)^{-\mathbf{X}})| \\
&\quad +|\partial_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))|+(v^s)_{\xi_1}^{-\mathbf{X}}|v-(v^s)^{-\mathbf{X}}|\Big)|\Delta_\xi(u_1-(u_1^s)^{-\mathbf{X}})|d\xi_1 d\xi' \\
&\le C\Big(\delta\sqrt{\frac{\delta}{\kappa}}\sqrt{G_3(t)}+\delta\sqrt{G^s(t)}+\delta^2\sqrt{D(t)}\Big)\sqrt{\mathbf{D}_2(t)} \\
&\le C\frac{\delta}{\kappa}G_3(t)+CG^s(t)+C\delta^2 D(t)+C\delta^2\mathbf{D}_2(t).
\end{aligned}
$$

For $J_4(t)$, using assumption we have $\|\nabla_\xi(v-(v^s)^{-\mathbf{X}})\|_{L^3}\le C\|\nabla_\xi(v-(v^s)^{-\mathbf{X}})\|_{H^1}\le C\chi$. Then we get

$$
\begin{aligned}
J_4(t) &\le C(\|\Delta_\xi(u-(u^s)^{-\mathbf{X}})\|+\|\nabla_\xi\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})\|)\|(v^s)_{\xi_1}^{-\mathbf{X}}\|_{L^\infty} \\
&\qquad \times\|\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})\| \\
&\qquad + C(\|\Delta_\xi(u-(u^s)^{-\mathbf{X}})\|+\|\nabla_\xi\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})\|) \\
&\qquad\quad \times\|\nabla_\xi(v-(v^s)^{-\mathbf{X}})\|_{L^3}\|\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})\|_{L^6} \\
&\le C\delta^2\sqrt{\mathbf{D}_2(t)}\|\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})\|+C\chi\sqrt{\mathbf{D}_2(t)}\|\mathrm{div}_\xi(u-(u^s)^{-\mathbf{X}})\|_{H^1} \\
&\le C(\delta+\chi)(\mathbf{D}_2(t)+\|\nabla_\xi(u-(u^s)^{-\mathbf{X}})\|^2).
\end{aligned}
$$

By the Cauchy inequality, we have

$$J_5(t) \leq \frac{1}{4}\mathbf{D}_2(t) + C\|\nabla_\xi(p(v) - p((v^s)^{-\mathbf{X}}))\|^2 = \frac{1}{4}\mathbf{D}_2(t) + CD(t).$$

Therefore, the combination of the above estimates yields

$$\begin{aligned} \frac{d}{dt}\|\nabla_\xi(u - (u^s)^{-\mathbf{X}})(t)\|^2 + \mathbf{D}_2(t) \leq C\delta^2|\dot{\mathbf{X}}(t)|^2 + C\frac{\delta}{\kappa}G_3(t) + CG^s(t) + CD(t) \\ + C(\delta + \chi)\|\nabla_\xi(u - (u^s)^{-\mathbf{X}})\|^2. \end{aligned}$$

Integrating the above inequality over $[0, t]$ for any $t \leq T$, and using Lemmas 4.4 and 4.5 and the fact that

$$\|\nabla_\xi^2(u - (u^s)^{-\mathbf{X}})(t)\|^2 \sim \mathbf{D}_2(t),$$

we can obtain the desired inequality (4.56). The proof of Lemma 4.6 is completed. ■

4.4. Estimates for $\|\nabla_\xi^2(v - (v^s)^{-\mathbf{X}})\|$

Lemma 4.7. *Under the hypotheses of Proposition 3.6, there exists a constant $C > 0$ independent of κ, δ, χ and T, such that for all $t \in [0, T]$, it holds*

$$\begin{aligned} &\|\nabla_\xi^2(v - (v^s)^{-\mathbf{X}})(t)\|^2 + \int_0^t \|\nabla_\xi^2(v - (v^s)^{-\mathbf{X}})\|^2 d\tau \\ &\quad \leq C(\|v_0 - v^s\|_{H^2}^2 + \|u_0 - u^s\|_{H^1}^2) + C(\delta + \chi)\int_0^t \|\nabla_\xi^3(u - (u^s)^{-\mathbf{X}})\|^2 d\tau. \end{aligned} \tag{4.57}$$

Proof. We set $\varphi := v - (v^s)^{-\mathbf{X}}$, $\psi := u - (u^s)^{-\mathbf{X}}$ for notational simplicity, and rewrite (4.48) as

$$\begin{cases} \partial_t\varphi - \sigma\partial_{\xi_1}\varphi + u\cdot\nabla_\xi\varphi - \dot{\mathbf{X}}(t)(v^s)_{\xi_1}^{-\mathbf{X}} + vF(v^s)_{\xi_1}^{-\mathbf{X}} = v\mathrm{div}_\xi\psi, \\ \partial_t\psi - \sigma\partial_{\xi_1}\psi + u\cdot\nabla_\xi\psi + vp'(v)\nabla_\xi\varphi + v(p'(v) - p'((v^s)^{-\mathbf{X}}))\nabla_\xi(v^s)^{-\mathbf{X}} \\ \qquad - \dot{\mathbf{X}}(t)(u^s)_{\xi_1}^{-\mathbf{X}} + vF(u^s)_{\xi_1}^{-\mathbf{X}} = \mu v\Delta_\xi\psi + (\mu+\lambda)v\nabla_\xi\mathrm{div}_\xi\psi. \end{cases} \tag{4.58}$$

Applying $\nabla_\xi\partial_{\xi_i}$ $(i = 1, 2, 3)$ to $(4.58)_1$, and ∂_{ξ_i} $(i = 1, 2, 3)$ to $(4.58)_2$, we have

$$\begin{cases} \partial_t\nabla_\xi\partial_{\xi_i}\varphi - \sigma\partial_{\xi_1}\nabla_\xi\partial_{\xi_i}\varphi + u\cdot\nabla_\xi(\nabla_\xi\partial_{\xi_i}\varphi) - \dot{\mathbf{X}}(t)\nabla_\xi\partial_{\xi_i}(v^s)_{\xi_1}^{-\mathbf{X}} \\ \quad + vF\nabla_\xi\partial_{\xi_i}(v^s)_{\xi_1}^{-\mathbf{X}} + \nabla_\xi\partial_{\xi_i}u\cdot\nabla_\xi\varphi + \nabla_\xi u\cdot\nabla_\xi\partial_{\xi_i}\varphi + \partial_{\xi_i}u\cdot\nabla_\xi(\nabla_\xi\varphi) \\ \quad + \nabla_\xi\partial_{\xi_i}(vF)(v^s)_{\xi_1}^{-\mathbf{X}} + \nabla_\xi(vF)\partial_{\xi_i}(v^s)_{\xi_1}^{-\mathbf{X}} + \partial_{\xi_i}(vF)\nabla_\xi(v^s)_{\xi_1}^{-\mathbf{X}} \\ \quad = v\nabla_\xi\partial_{\xi_i}\mathrm{div}_\xi\psi + \nabla_\xi\partial_{\xi_i}v\mathrm{div}_\xi\psi + \partial_{\xi_i}v\nabla_\xi\mathrm{div}_\xi\psi + \nabla_\xi v\partial_{\xi_i}\mathrm{div}_\xi\psi, \\ \partial_t\partial_{\xi_i}\psi - \sigma\partial_{\xi_1}\partial_{\xi_i}\psi + u\cdot\nabla_\xi\partial_{\xi_i}\psi + vp'(v)\nabla_\xi\partial_{\xi_i}\varphi + \partial_{\xi_i}u\cdot\nabla_\xi\psi \\ \quad + \partial_{\xi_i}(vp'(v))\nabla_\xi\varphi + \partial_{\xi_i}(v(p'(v) - p'((v^s)^{-\mathbf{X}})))\nabla_\xi(v^s)^{-\mathbf{X}} \\ \quad + v(p'(v) - p'((v^s)^{-\mathbf{X}}))\nabla_\xi\partial_{\xi_i}(v^s)^{-\mathbf{X}} - \dot{\mathbf{X}}(t)\partial_{\xi_i}(u^s)_{\xi_1}^{-\mathbf{X}} \\ \quad + vF\partial_{\xi_i}(u^s)_{\xi_1}^{-\mathbf{X}} + \partial_{\xi_i}(vF)(u^s)_{\xi_1}^{-\mathbf{X}} \\ \quad = \mu v\Delta_\xi\partial_{\xi_i}\psi + (\mu+\lambda)v\nabla_\xi\partial_{\xi_i}\mathrm{div}_\xi\psi + \partial_{\xi_i}v(\mu\Delta_\xi\psi + (\mu+\lambda)\nabla_\xi\mathrm{div}_\xi\psi). \end{cases} \tag{4.59}$$

Multiplying (4.59)$_1$ by $\rho(2\mu+\lambda)\nabla_\xi\partial_{\xi_i}\varphi$, and summating by i from 1 to 3, then integrating the resultant equations over $\mathbb{R}\times\mathbb{T}^2$, we get

$$
\begin{aligned}
&(2\mu+\lambda)\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\frac{|\nabla_\xi^2\varphi|^2}{2}d\xi_1 d\xi' - (2\mu+\lambda)\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla_\xi\partial_{\xi_i}\varphi\cdot\nabla_\xi\partial_{\xi_i}\mathrm{div}_\xi\psi d\xi_1 d\xi' \\
&\quad = (2\mu+\lambda)\dot{\mathbf{X}}(t)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_1}^2\varphi(v^s)_{\xi_1\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' \\
&\qquad - (2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}F\partial_{\xi_1}^2\varphi(v^s)_{\xi_1\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' \\
&\qquad - (2\mu+\lambda)\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_i}\varphi\cdot\big[\nabla_\xi\partial_{\xi_i}u\cdot\nabla_\xi\varphi+\nabla_\xi u\cdot\nabla_\xi\partial_{\xi_i}\varphi \\
&\qquad\qquad + \partial_{\xi_i}u\cdot\nabla_\xi(\nabla_\xi\varphi)\big]d\xi_1 d\xi' \\
&\qquad - (2\mu+\lambda)\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\big(\nabla_\xi\partial_{\xi_i}\varphi\cdot\nabla_\xi\partial_{\xi_i}(vF)(v^s)_{\xi_1}^{-\mathbf{X}} \\
&\qquad\qquad + \partial_{\xi_1}\partial_{\xi_i}\varphi\partial_{\xi_i}(vF)(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}\big)d\xi_1 d\xi' \\
&\qquad - (2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_1}\varphi\cdot\nabla_\xi(vF)(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' \\
&\qquad + (2\mu+\lambda)\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_i}\varphi\cdot\big[\nabla_\xi\partial_{\xi_i}v\mathrm{div}_\xi\psi+\partial_{\xi_i}v\nabla_\xi\mathrm{div}_\xi\psi \\
&\qquad\qquad + \nabla_\xi v\partial_{\xi_i}\mathrm{div}_\xi\psi\big]d\xi_1 d\xi'.
\end{aligned}\tag{4.60}
$$

Multiplying (4.59)$_2$ by $-\rho\nabla_\xi\partial_{\xi_i}\varphi$, and summating by i from 1 to 3, then integrating the resultant equations over $\mathbb{R}\times\mathbb{T}^2$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}^2}\int_{\mathbb{R}}-p'(v)|\nabla_\xi^2\varphi|^2 d\xi_1 d\xi' + (2\mu+\lambda)\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\nabla_\xi\partial_{\xi_i}\varphi\cdot\nabla_\xi\partial_{\xi_i}\mathrm{div}_\xi\psi d\xi_1 d\xi' \\
&\quad = \frac{d}{dt}\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}\psi\cdot\nabla_\xi\partial_{\xi_i}\varphi d\xi_1 d\xi' + \sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_i}\varphi\cdot\partial_{\xi_i}u\cdot\nabla_\xi\psi d\xi_1 d\xi' \\
&\qquad - \sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}\psi\cdot[\nabla_\xi\partial_{\xi_i}\partial_t\varphi-\sigma\partial_{\xi_1}\nabla_\xi\partial_{\xi_i}\varphi+u\cdot\nabla_\xi(\nabla_\xi\partial_{\xi_i}\varphi)]d\xi_1 d\xi' \\
&\qquad + \sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}(vp'(v))\nabla_\xi\partial_{\xi_i}\varphi\cdot\nabla_\xi\varphi d\xi_1 d\xi' \\
&\qquad - \dot{\mathbf{X}}(t)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_1}^2\varphi(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' \\
&\qquad + \sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}(v(p'(v)-p'((v^s)^{-\mathbf{X}})))\partial_{\xi_1}\partial_{\xi_i}\varphi(v^s)_{\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' \\
&\qquad + \int_{\mathbb{T}^2}\int_{\mathbb{R}}(p'(v)-p'((v^s)^{-\mathbf{X}}))\partial_{\xi_1}^2\varphi(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi'
\end{aligned}
$$

$$+\int_{\mathbb{T}^2}\int_{\mathbb{R}} F\partial_{\xi_1}^2\varphi(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' + \sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}(vF)\partial_{\xi_1}\partial_{\xi_i}\varphi(u_1^s)_{\xi_1}^{-\mathbf{X}}d\xi_1 d\xi'$$
$$-\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}v(\mu\Delta_\xi\psi+(\mu+\lambda)\nabla_\xi \mathrm{div}_\xi\psi)\cdot\nabla_\xi\partial_{\xi_i}\varphi d\xi_1 d\xi'. \tag{4.61}$$

Adding (4.60) and (4.61) together, and integrating the resultant equations over $[0,t]$ for any $t\in[0,T]$, we have

$$(2\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\frac{|\nabla_\xi^2\varphi|^2}{2}d\xi_1 d\xi'\Big|_{\tau=0}^{\tau=t}+\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|p'(v)||\nabla_\xi^2\varphi|^2 d\xi_1 d\xi' d\tau$$
$$=\sum_{j=1}^{8}K_j(t), \tag{4.62}$$

where

$$K_1(t)=\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}\psi\cdot\nabla_\xi\partial_{\xi_i}\varphi d\xi_1 d\xi'\Big|_{\tau=0}^{\tau=t},$$
$$K_2(t)=\int_0^t\dot{\mathbf{X}}(\tau)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_1}^2\varphi[(2\mu+\lambda)(v^s)_{\xi_1\xi_1\xi_1}^{-\mathbf{X}}-(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}]d\xi_1 d\xi' d\tau,$$
$$K_3(t)=-\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}F\partial_{\xi_1}^2\varphi[(2\mu+\lambda)(v^s)_{\xi_1\xi_1\xi_1}^{-\mathbf{X}}-(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}]d\xi_1 d\xi' d\tau,$$
$$K_4(t)=-(2\mu+\lambda)\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\big[\rho(\nabla_\xi\partial_{\xi_i}\varphi\cdot\nabla_\xi\partial_{\xi_i}(vF)(v^s)_{\xi_1}^{-\mathbf{X}}$$
$$+\partial_{\xi_1}\partial_{\xi_i}\varphi\partial_{\xi_i}(vF)(v^s)_{\xi_1\xi_1}^{-\mathbf{X}})\big]d\xi_1 d\xi' d\tau$$
$$-(2\mu+\lambda)\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_1}\varphi\cdot\nabla_\xi(vF)(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' d\tau$$
$$+\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_1}\partial_{\xi_i}\varphi\partial_{\xi_i}(vF)(u_1^s)_{\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' d\tau,$$
$$K_5(t)=-\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}\psi\cdot\big[\nabla_\xi\partial_{\xi_i}\partial_t\varphi-\sigma\partial_{\xi_1}\nabla_\xi\partial_{\xi_i}\varphi$$
$$+u\cdot\nabla_\xi(\nabla_\xi\partial_{\xi_i}\varphi)\big]d\xi_1 d\xi' d\tau,$$
$$K_6(t)=-(2\mu+\lambda)\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_i}\varphi\cdot\big[\nabla_\xi\partial_{\xi_i}u\cdot\nabla_\xi\varphi+\nabla_\xi u\cdot\nabla_\xi\partial_{\xi_i}\varphi$$
$$+\partial_{\xi_i}u\cdot\nabla_\xi(\nabla_\xi\varphi)\big]d\xi_1 d\xi' d\tau$$
$$+\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_i}\varphi\cdot[\partial_{\xi_i}u\cdot\nabla_\xi\psi+\partial_{\xi_i}(vp'(v))\nabla_\xi\varphi]d\xi_1 d\xi' d\tau,$$

$$
\begin{aligned}
K_7(t) = &\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\partial_{\xi_i}(v(p'(v)-p'((v^s)^{-\mathbf{X}})))\partial_{\xi_1}\partial_{\xi_i}\varphi(v^s)_{\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' d\tau\\
&+\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}(p'(v)-p'((v^s)^{-\mathbf{X}}))\partial_{\xi_1}^2\varphi(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1 d\xi' d\tau,\\
K_8(t) = &(2\mu+\lambda)\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\nabla_\xi\partial_{\xi_i}\varphi\cdot\big[\nabla_\xi\partial_{\xi_i}v\mathrm{div}_\xi\psi+\partial_{\xi_i}v\nabla_\xi(\mathrm{div}_\xi\psi)\\
&\qquad\qquad\qquad\qquad+\nabla_\xi v\partial_{\xi_i}(\mathrm{div}_\xi\psi)\big]d\xi_1 d\xi' d\tau\\
&-\sum_{i=1}^{3}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\rho\partial_{\xi_i}v\nabla_\xi\partial_{\xi_i}\varphi\cdot[\mu\Delta_\xi\psi+(\mu+\lambda)\nabla_\xi\mathrm{div}_\xi\psi]d\xi_1 d\xi' d\tau.
\end{aligned}
$$

Using the Cauchy inequality, it holds

$$
K_1(t)\le\frac{2\mu+\lambda}{8}\|\sqrt{\rho}\nabla_\xi^2\varphi(t)\|^2+C\|\nabla_\xi\psi(t)\|^2+C(\|\nabla_\xi^2\varphi_0\|^2+\|\nabla_\xi\psi_0\|^2).
$$

Using Lemma 2.1, we have

$$
\begin{aligned}
K_2(t)&\le C\delta\int_0^t|\dot{\mathbf{X}}(\tau)|\|\partial_{\xi_1}^2\varphi\|\,\|(v^s)_{\xi_1}^{-\mathbf{X}}\|_{L^2(\mathbb{R})}d\tau\\
&\le C\delta^2\int_0^t(|\dot{\mathbf{X}}(\tau)|^2+\|\sqrt{|p'(v)|}\partial_{\xi_1}^2\varphi\|^2)d\tau.
\end{aligned}
$$

Using (4.7), we get

$$
\begin{aligned}
K_3(t)\le\ &C\delta\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Big(\Big|h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big|+|p(v)-p((v^s)^{-\mathbf{X}})|\\
&\qquad+|\partial_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))|+|(v^s)_{\xi_1}^{-\mathbf{X}}||p(v)-p((v^s)^{-\mathbf{X}})|\Big)\\
&\qquad\times|\partial_{\xi_1}^2\varphi||(v^s)_\xi^{-\mathbf{X}}|d\xi_1 d\xi' d\tau\\
\le\ &C\delta^2\int_0^t(\sqrt{G_3(\tau)}+\sqrt{G^s(\tau)}+\sqrt{D(\tau)})\|\partial_{\xi_1}^2\varphi\|d\tau\\
\le\ &C\delta^2\int_0^t(G_3(\tau)+G^s(\tau)+D(\tau)+\|\sqrt{|p'(v)|}\partial_{\xi_1}^2\varphi\|^2)d\tau.
\end{aligned}
$$

By using (4.7) again, it holds

$$
vF=\sigma_*(v-(v^s)^{-\mathbf{X}})+u_1-(u_1^s)^{-\mathbf{X}}=\sigma_*\varphi+\psi_1. \tag{4.63}
$$

Thus, we can get

$$
\begin{aligned}
K_4(t)\le\ &C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla_\xi^2\varphi|(|\nabla_\xi^2\varphi|+|\nabla_\xi^2\psi|)|(v^s)_{\xi_1}^{-\mathbf{X}}|d\xi_1 d\xi' d\tau\\
&+C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla_\xi^2\varphi|(|\nabla_\xi\varphi|+|\nabla_\xi\psi|)|(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}|d\xi_1 d\xi' d\tau\\
\le\ &C\delta^2\int_0^t\|\nabla_\xi^2\varphi\|(\|\nabla_\xi\varphi\|_{H^1}+\|\nabla_\xi\psi\|_{H^1})d\tau\\
\le\ &C\delta^2\int_0^t(\|\sqrt{|p'(v)|}\nabla_\xi^2\varphi\|^2+D(\tau)+G^s(\tau)+\|\nabla_\xi\psi\|_{H^1}^2)d\tau.
\end{aligned}
$$

By using equation (4.59)$_1$, we can compute the term $K_5(t)$ as

$$\begin{aligned}
K_5(t) = &-\int_0^t \dot{\mathbf{X}}(\tau)\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\partial_{\xi_1}\psi_1(v^s)^{-\mathbf{X}}_{\xi_1\xi_1\xi_1}\,d\xi_1 d\xi' d\tau \\
&+\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} F\partial_{\xi_1}\psi_1(v^s)^{-\mathbf{X}}_{\xi_1\xi_1\xi_1}\,d\xi_1 d\xi' d\tau \\
&+\sum_{i=1}^3\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\partial_{\xi_i}\psi\cdot\big[\nabla_\xi\partial_{\xi_i}u\cdot\nabla_\xi\varphi+\nabla_\xi u\cdot\nabla_\xi\partial_{\xi_i}\varphi \\
&\qquad\qquad + \partial_{\xi_i}u\cdot\nabla_\xi(\nabla_\xi\varphi)\big]d\xi_1 d\xi' d\tau \\
&+\sum_{i=1}^3\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\partial_{\xi_i}\psi\cdot\big[\nabla_\xi\partial_{\xi_i}(vF)(v^s)^{-\mathbf{X}}_{\xi_1}+\nabla_\xi(vF)\partial_{\xi_i}(v^s)^{-\mathbf{X}}_{\xi_1} \\
&\qquad\qquad + \partial_{\xi_i}(vF)\nabla_\xi(v^s)^{-\mathbf{X}}_{\xi_1}\big]d\xi_1 d\xi' d\tau \\
&-\sum_{i=1}^3\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} \rho\partial_{\xi_i}\psi\cdot\big[\nabla_\xi\partial_{\xi_i}v\,\mathrm{div}_\xi\psi+\partial_{\xi_i}v\nabla_\xi\mathrm{div}_\xi\psi \\
&\qquad\qquad + \nabla_\xi v\partial_{\xi_i}\mathrm{div}_\xi\psi\big]d\xi_1 d\xi' d\tau \\
&-\sum_{i=1}^3\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} \partial_{\xi_i}\psi\cdot\nabla_\xi\partial_{\xi_i}\mathrm{div}_\xi\psi\,d\xi_1 d\xi' d\tau =: \sum_{i=1}^6 K_{5,i}(t).
\end{aligned}$$

Using Lemma 2.1, we obtain

$$K_{5,1}(t) \le C\delta^2\int_0^t |\dot{\mathbf{X}}(\tau)|^2 d\tau + C\delta^2\int_0^t \|\nabla_\xi\psi\|^2 d\tau.$$

Using (4.7) and (4.43), we have

$$\begin{aligned}
K_{5,2}(t) &\le C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla_\xi\psi||(v^s)^{-\mathbf{X}}_{\xi_1\xi_1\xi_1}| \\
&\quad\times\Big[\Big|h_1-(h_1^s)^{-\mathbf{X}}-\frac{p(v)-p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big| + |\partial_{\xi_1}(p(v)-p((v^s)^{-\mathbf{X}}))| \\
&\quad + |p(v)-p((v^s)^{-\mathbf{X}})| + |p(v)-p((v^s)^{-\mathbf{X}})||(v^s)^{-\mathbf{X}}_{\xi_1}|\Big]d\xi_1 d\xi' d\tau \\
&\le C\delta^2\int_0^t (\|\nabla_\xi\psi\|^2 + G_3(\tau) + G^s(\tau) + D(\tau))d\tau.
\end{aligned}$$

By assumption (3.11), the Cauchy inequality, and the Sobolev inequality, we obtain the following estimations:

$$\begin{aligned}
\|\varphi\|_{L^3}+\|\psi\|_{L^3} &\le C\|\varphi\|^{\frac12}\|\varphi\|^{\frac12}_{L^6} + C\|\psi\|^{\frac12}\|\psi\|^{\frac12}_{L^6} \\
&\le C\|\varphi\|_{H^1} + C\|\psi\|_{H^1} \le C\chi, \\
\|\nabla_\xi\varphi\|_{L^3}+\|\nabla_\xi\psi\|_{L^3} &\le C\|\nabla_\xi\varphi\|^{\frac12}\|\nabla_\xi\varphi\|^{\frac12}_{L^6} + C\|\nabla_\xi\psi\|^{\frac12}\|\nabla_\xi\psi\|^{\frac12}_{L^6} \\
&\le C\|\nabla_\xi\varphi\|_{H^1} + C\|\nabla_\xi\psi\|_{H^1} \le C\chi.
\end{aligned} \tag{4.64}$$

Then we have

$$
\begin{aligned}
K_{5,3}(t) &\le C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla_\xi\psi|(|\nabla_\xi^2\psi||\nabla_\xi\varphi|+|\nabla_\xi\psi||\nabla_\xi^2\varphi|)d\xi_1 d\xi' d\tau\\
&\quad + C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla_\xi\psi|(|(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}||\partial_{\xi_1}\varphi|+|(u_1^s)_{\xi_1}^{-\mathbf{X}}||\nabla_\xi\partial_{\xi_1}\varphi|)d\xi_1 d\xi' d\tau\\
&\le C\int_0^t\|\nabla_\xi\psi\|_{L^6}\|\nabla_\xi(\varphi,\psi)\|_{L^3}\|\nabla_\xi^2(\varphi,\psi)\|d\tau\\
&\quad + C\delta^2\int_0^t\|\nabla_\xi\psi\|(\|\partial_{\xi_1}\varphi\|+\|\nabla_\xi\partial_{\xi_1}\varphi\|)d\tau\\
&\le C\chi\int_0^t(\|\nabla_\xi^2\varphi\|^2+\|\nabla_\xi\psi\|_{H^1}^2)d\tau\\
&\quad + C\delta^2\int_0^t(\|\nabla_\xi^2\varphi\|^2+\|\nabla_\xi\psi\|^2+D(\tau)+G^s(\tau))d\tau.
\end{aligned}
$$

We use (4.63) again and Lemma 2.1, then we get

$$
\begin{aligned}
K_{5,4}(t) &\le C\delta^2\int_0^t(\|\nabla_\xi^2(\varphi,\psi)\|^2+\|\nabla_\xi(\varphi,\psi)\|^2)d\tau\\
&\le C\delta^2\int_0^t(\|\nabla_\xi^2\varphi\|^2+\|\nabla_\xi\psi\|_{H^1}^2+D(\tau)+G^s(\tau))d\tau.
\end{aligned}
$$

Similarly to $K_{5,3}(t)$, we have

$$
\begin{aligned}
K_{5,5}(t) &\le C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla_\xi\psi|(|\nabla_\xi\psi||\nabla_\xi^2\varphi|+|\nabla_\xi\varphi||\nabla_\xi^2\psi|)d\xi_1 d\xi' d\tau\\
&\quad + C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\nabla_\xi\psi|(|(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}||\nabla_\xi\psi|+|(v^s)_{\xi_1}^{-\mathbf{X}}||\nabla_\xi^2\psi|)d\xi_1 d\xi' d\tau\\
&\le C\int_0^t\|\nabla_\xi\psi\|_{L^6}\|\nabla_\xi(\varphi,\psi)\|_{L^3}\|\nabla_\xi^2(\varphi,\psi)\|d\tau\\
&\quad + C\delta^2\int_0^t\|\nabla_\xi\psi\|(\|\nabla_\xi\psi\|+\|\nabla_\xi^2\psi\|)d\tau\\
&\le C\chi\int_0^t(\|\nabla_\xi^2\varphi\|^2+\|\nabla_\xi\psi\|_{H^1}^2)d\tau + C\delta^2\int_0^t\|\nabla_\xi\psi\|_{H^1}^2 d\tau.
\end{aligned}
$$

Integration by parts over $\mathbb{R}\times\mathbb{T}^2$ yields

$$
K_{5,6}(t) = \sum_{i=1}^3\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\partial_{\xi_i}\mathrm{div}_\xi\psi)^2 d\xi_1 d\xi' d\tau = \int_0^t\|\nabla_\xi\mathrm{div}_\xi\psi\|^2 d\tau.
$$

Thus, the combination of the above estimates implies

$$
\begin{aligned}
K_5(t) &\le \frac{1}{8}\int_0^t\|\sqrt{|p'(v)|}\nabla_\xi^2\varphi\|^2 d\tau + C\int_0^t\|\nabla_\xi^2\psi\|^2 d\tau + C\delta^2\int_0^t|\dot{\mathbf{X}}(\tau)|^2 d\tau\\
&\quad + C\delta^2\int_0^t(G_3(\tau)+G^s(\tau)+D(\tau))d\tau + C(\delta+\chi)\int_0^t\|\nabla_\xi\psi\|^2 d\tau.
\end{aligned}
$$

Using the Cauchy inequality and (4.64), we get

$$
\begin{aligned}
K_6(t) \le\, & C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla_\xi^2\varphi|[|\nabla_\xi^2\psi||\nabla_\xi\varphi| + |\nabla_\xi\psi|^2 + |\nabla_\xi^2\varphi||\nabla_\xi\psi| + |\nabla_\xi\varphi|^2]d\xi_1 d\xi' d\tau \\
& + C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla_\xi^2\varphi|\big[|\partial_{\xi_1}\varphi||(u_1^s)^{-\mathbf{X}}_{\xi_1\xi_1}| + |\partial_{\xi_1}\psi||(u_1^s)^{-\mathbf{X}}_{\xi_1}| \\
& \qquad + |\nabla_\xi^2\varphi||(u_1^s)^{-\mathbf{X}}_{\xi_1}| + |\nabla_\xi\varphi||(v^s)^{-\mathbf{X}}_{\xi_1}|\big]d\xi_1 d\xi' d\tau \\
\le\, & C\int_0^t \|\nabla_\xi^2\varphi\|\big[\|\nabla_\xi^2\psi\|_{L^6}\|\nabla_\xi\varphi\|_{L^3} + \|\nabla_\xi\psi\|_{L^6}\|\nabla_\xi\psi\|_{L^3} \\
& \qquad + \|\nabla_\xi^2\varphi\|\|\nabla_\xi\psi\|_{L^\infty}\big]d\tau \\
& + C\int_0^t \|\nabla_\xi^2\varphi\|\|\nabla_\xi\varphi\|_{L^3}\|\nabla_\xi\varphi\|_{L^6}d\tau \\
& + C\delta^2\int_0^t \|\nabla_\xi^2\varphi\|(\|\nabla_\xi(\varphi,\psi)\| + \|\nabla_\xi^2\varphi\|)d\tau \\
\le\, & C(\delta+\chi)\int_0^t (\|\sqrt{|p'(v)|}\nabla_\xi^2\varphi\|^2 + D(\tau) + G^s(\tau) + \|\nabla_\xi\psi\|_{H^2}^2)d\tau,
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
\|\nabla_\xi^2\varphi\|^2\|\nabla_\xi\psi\|_{L^\infty} &\le C\|\nabla_\xi^2\varphi\|^2\|\nabla_\xi\psi\|_{H^2} \le C\chi\|\nabla_\xi^2\varphi\|\|\nabla_\xi\psi\|_{H^2} \\
&\le C\chi(\|\sqrt{|p'(v)|}\nabla_\xi^2\varphi\|^2 + \|\nabla_\xi\psi\|_{H^2}^2).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
K_7(t) \le\, & C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} (|\nabla_\xi\varphi||\varphi| + |\varphi||(v^s)^{-\mathbf{X}}_{\xi_1}| + |\nabla_\xi\varphi|)|\nabla_\xi^2\varphi||(v^s)^{-\mathbf{X}}_{\xi_1}|d\xi_1 d\xi' d\tau \\
& + C\delta\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\varphi||\partial_{\xi_1}^2\varphi||(v^s)^{-\mathbf{X}}_{\xi_1}|d\xi_1 d\xi' d\tau \\
\le\, & C\delta\int_0^t (\|\sqrt{|p'(v)|}\nabla_\xi^2\varphi\|^2 + D(\tau) + G^s(\tau))d\tau.
\end{aligned}
$$

Using the Cauchy inequality and (4.64) again, we get

$$
\begin{aligned}
K_8(t) \le\, & C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla_\xi^2\psi||\nabla_\xi^2\varphi|(|\nabla_\xi\varphi| + |(v^s)^{-\mathbf{X}}_{\xi_1}|)d\xi_1 d\xi' d\tau \\
& + C\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}} |\nabla_\xi^2\varphi|(|\nabla_\xi\psi||\nabla_\xi^2\varphi| + |\nabla_\xi\psi||(v^s)^{-\mathbf{X}}_{\xi_1\xi_1}|)d\xi_1 d\xi' d\tau \\
\le\, & C\int_0^t [\|\nabla_\xi\varphi\|_{L^3}\|\nabla_\xi^2\psi\|_{L^6}\|\nabla_\xi^2\varphi\| + \|\nabla_\xi\psi\|_{L^\infty}\|\nabla_\xi^2\varphi\|^2]d\tau \\
& + C\delta^2\int_0^t \|\nabla_\xi^2\varphi\|[\|\nabla_\xi^2\psi\| + \|\nabla_\xi\psi\|]d\tau \\
\le\, & C(\delta+\chi)\int_0^t [\|\sqrt{|p'(v)|}\nabla_\xi^2\varphi\|^2 + \|\nabla_\xi\psi\|_{H^2}^2]d\tau.
\end{aligned}
$$

Substituting the above estimates into (4.62) and using Lemmas 4.4, 4.5 and 4.6, we can obtain the desired inequality (4.57). The proof of Lemma 4.7 is completed. ∎

4.5. Estimates for $\|\nabla_\xi^2(u-(u^s)^{-\mathbf{X}})\|$

Lemma 4.8. *Under the hypotheses of Proposition 3.6, there exists a constant $C>0$ independent of κ, δ, χ and T, such that for all $t\in[0,T]$, it holds*

$$\begin{aligned}&\|\nabla_\xi^2(u-(u^s)^{-\mathbf{X}})(t)\|^2+\int_0^t\|\nabla_\xi^3(u-(u^s)^{-\mathbf{X}})\|^2d\tau\\&\quad\le C(\|v_0-v^s\|_{H^2}^2+\|u_0-u^s\|_{H^2}^2).\end{aligned}\tag{4.65}$$

Proof. Multiplying $(4.59)_2$ by $-\Delta_\xi\partial_{\xi_i}\psi$, and summating by i from 1 to 3, then integrating the resultant equations over $\mathbb{R}\times\mathbb{T}^2$, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\frac{|\nabla_\xi^2\psi|^2}{2}d\xi_1d\xi'\\&\quad+\underbrace{\mu\int_{\mathbb{T}^2}\int_{\mathbb{R}}v|\nabla_\xi\Delta_\xi\psi|^2d\xi_1d\xi'+(\mu+\lambda)\int_{\mathbb{T}^2}\int_{\mathbb{R}}v|\nabla_\xi^2\mathrm{div}_\xi\psi|^2d\xi_1d\xi'}_{\mathbf{D}_3(t)}\\&\quad=:\sum_{i=1}^3L_i(t),\end{aligned}\tag{4.66}$$

where

$$\begin{aligned}L_1(t)&=-\sum_{i,j=1}^3\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{\xi_j}u\cdot\nabla_\xi\partial_{\xi_i}\psi\cdot\partial_{\xi_j}\partial_{\xi_i}\psi d\xi_1d\xi'+\int_{\mathbb{T}^2}\int_{\mathbb{R}}\mathrm{div}_\xi u\frac{|\nabla_\xi^2\psi|^2}{2}d\xi_1d\xi',\\L_2(t)&=\sum_{i=1}^3\int_{\mathbb{T}^2}\int_{\mathbb{R}}vp'(v)\Delta_\xi\partial_{\xi_i}\psi\cdot\nabla_\xi\partial_{\xi_i}\varphi d\xi_1d\xi',\\L_3(t)&=\sum_{i=1}^3\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{\xi_i}u\cdot\nabla_\xi\psi\cdot\Delta\partial_{\xi_i}\psi d\xi_1d\xi'\\&\quad+\sum_{i=1}^3\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{\xi_i}(vp'(v))\Delta_\xi\partial_{\xi_i}\psi\cdot\nabla_\xi\varphi d\xi_1d\xi',\\L_4(t)&=\sum_{i=1}^3\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{\xi_i}(v(p'(v)-p'((v^s)^{-\mathbf{X}})))(v^s)_{\xi_1}^{-\mathbf{X}}\Delta_\xi\partial_{\xi_i}\psi_1d\xi_1d\xi'\\&\quad+\int_{\mathbb{T}^2}\int_{\mathbb{R}}v(p'(v)-p'((v^s)^{-\mathbf{X}}))(v^s)_{\xi_1\xi_1}^{-\mathbf{X}}\Delta_\xi\partial_{\xi_1}\psi_1d\xi_1d\xi',\\L_5(t)&=-\dot{\mathbf{X}}(t)\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Delta_\xi\partial_{\xi_1}\psi_1(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}d\xi_1d\xi',\\L_6(t)&=\int_{\mathbb{T}^2}\int_{\mathbb{R}}vF(u_1^s)_{\xi_1\xi_1}^{-\mathbf{X}}\Delta_\xi\partial_{\xi_1}\psi_1d\xi_1d\xi',\\L_7(t)&=\sum_{i=1}^3\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{\xi_i}(vF)(u_1^s)_{\xi_1}^{-\mathbf{X}}\Delta_\xi\partial_{\xi_i}\psi_1d\xi_1d\xi',\end{aligned}$$

$$L_8(t) = (\mu+\lambda)\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}(\Delta_\xi\partial_{\xi_i}\psi - \nabla_\xi\partial_{\xi_i}\mathrm{div}_\xi\psi)\cdot\nabla_\xi v\,\partial_{\xi_i}\mathrm{div}_\xi\psi d\xi_1 d\xi'$$
$$-\sum_{i=1}^{3}\int_{\mathbb{T}^2}\int_{\mathbb{R}}\partial_{\xi_i}v(\mu\Delta_\xi\psi + (\mu+\lambda)\nabla_\xi\mathrm{div}_\xi\psi)\cdot\Delta\partial_{\xi_i}\psi d\xi_1 d\xi'.$$

Using the Cauchy inequality and (4.64), we get

$$L_1(t) \le C\|\nabla_\xi\psi\|_{L^3}\|\nabla_\xi^2\psi\|_{L^6}\|\nabla_\xi^2\psi\| + C\delta^2\|\nabla_\xi^2\psi\|^2 \le C(\delta+\chi)\|\nabla_\xi^2\psi\|_{H^1}^2$$
$$\le C(\delta+\chi)(\mathbf{D}_3(t) + \|\nabla_\xi^2\psi\|^2).$$

Using the Cauchy inequality, we have

$$L_2(t) \le \frac{1}{8}\mathbf{D}_3(t) + C\|\nabla_\xi^2\varphi\|^2.$$

It follows from the Cauchy inequality and (4.64) that

$$L_3(t) \le C[\|\nabla_\xi\psi\|_{L^3}\|\nabla_\xi\psi\|_{L^6} + \|\nabla_\xi\varphi\|_{L^3}\|\nabla_\xi\varphi\|_{L^6} + \delta^2\|\nabla_\xi(\varphi,\psi)\|]\sqrt{\mathbf{D}_3(t)}$$
$$\le C(\delta+\chi)(\mathbf{D}_3(t) + \|\nabla_\xi\psi\|_{H^1}^2 + \|\nabla_\xi\varphi\|_{H^1}^2)$$
$$\le C(\delta+\chi)(\mathbf{D}_3(t) + G^s(t) + D(t) + \|\nabla_\xi\psi\|_{H^1}^2 + \|\nabla_\xi^2\varphi\|^2).$$

Using Lemma 2.1, we obtain

$$L_4(t) \le C\int_{\mathbb{T}^2}\int_{\mathbb{R}}(|\nabla_\xi\varphi| + |(v^s)_{\xi_1}^{-\mathbf{X}}||\varphi|)|(v^s)_{\xi_1}^{-\mathbf{X}}||\nabla_\xi^3\psi_1|d\xi_1 d\xi'$$
$$+ C\delta\int_{\mathbb{T}^2}\int_{\mathbb{R}}|\varphi||(v^s)_{\xi_1}^{-\mathbf{X}}||\nabla_\xi^3\psi_1|d\xi_1 d\xi'$$
$$\le C\delta(\mathbf{D}_3(t) + G^s(t) + \|\nabla_\xi\varphi\|^2) \le C\delta(\mathbf{D}_3(t) + G^s(t) + D(t)),$$

and

$$L_5(t) \le C\delta|\dot{\mathbf{X}}(t)|\sqrt{\mathbf{D}_3(t)}\|(v^s)_{\xi_1}^{-\mathbf{X}}\|_{L^2(\mathbb{R})}$$
$$\le C\delta^{\frac{5}{2}}|\dot{\mathbf{X}}(t)|\sqrt{\mathbf{D}_3(t)} \le C\delta^{\frac{5}{2}}\mathbf{D}_3(t) + C\delta^{\frac{5}{2}}|\dot{\mathbf{X}}(t)|^2.$$

Using the definition of F (4.7), we have

$$L_6(t) \le C\delta\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}}\Big(\Big|h_1 - (h_1^s)^{-\mathbf{X}} - \frac{p(v) - p((v^s)^{-\mathbf{X}})}{\sigma_*}\Big| + |p(v) - p((v^s)^{-\mathbf{X}})|$$
$$+ |\partial_{\xi_1}(p(v) - p((v^s)^{-\mathbf{X}}))| + |(v^s)_{\xi_1}^{-\mathbf{X}}||p(v) - p((v^s)^{-\mathbf{X}})|\Big)$$
$$\times |(v^s)_{\xi_1}^{-\mathbf{X}}||\Delta_\xi\partial_{\xi_1}\psi_1|d\xi_1 d\xi' d\tau$$
$$\le C\delta^2(\sqrt{G_3(t)} + \sqrt{G^s(t)} + \sqrt{D(t)})\|\Delta\partial_{\xi_1}\psi_1\|$$
$$\le C\delta^2(G_3(t) + G^s(t) + D(t) + \mathbf{D}_3(t)).$$

Using (4.63) and the Cauchy inequality, we get

$$L_7(t) \le C\delta^2 \|\nabla_\xi(\varphi, \psi_1)\| \|\Delta_\xi \partial_{\xi_1} \psi_1\| \le C\delta^2 (\mathbf{D}_3(t) + G^s(t) + D(t) + \|\nabla_\xi \psi_1\|^2).$$

Using the Cauchy inequality and (4.64), we have

$$\begin{aligned} L_8(t) &\le C\sqrt{\mathbf{D}_3(t)} \|\nabla_\xi \varphi\|_{L^3} (\|\Delta_\xi \psi\|_{L^6} + \|\nabla_\xi \mathrm{div}_\xi \psi\|_{L^6}) \\ &\quad + C\delta^2 \sqrt{\mathbf{D}_3(t)} (\|\Delta_\xi \psi\| + \|\nabla_\xi \mathrm{div}_\xi \psi\|) \\ &\le C(\delta + \chi)\sqrt{\mathbf{D}_3(t)} (\|\Delta_\xi \psi\|_{H^1} + \|\nabla_\xi \mathrm{div}_\xi \psi\|_{H^1}) \\ &\le C(\delta + \chi)(\mathbf{D}_3(t) + \|\Delta_\xi \psi\|_{H^1}^2 + \|\nabla_\xi \mathrm{div}_\xi \psi\|_{H^1}^2). \end{aligned}$$

Substituting the above estimates into (4.66), and integrating the resultant equations over $[0, t]$ for any $t \le T$, using Lemmas 4.5, 4.6 and 4.7, we can get the desired inequality (4.65). The proof of Lemma 4.8 is completed. ■

4.6. *Proofs of Proposition 3.6 and Theorem 2.3*

We use (4.43) to have

$$\|\nabla_\xi (v - (v^s)^{-\mathbf{X}})\|^2 \le C(D(t) + G^s(t)),$$

which together with Lemmas 4.4–4.8 yields (3.12). In addition, using (3.9) and $(4.25)_2$ together with Lemma 2.1 and assumption (3.11), we have

$$\begin{aligned} |\dot{\mathbf{X}}(t)| &\le \frac{C}{\delta} (\|(p(v) - p((v^s)^{-\mathbf{X}}))\|_{L^\infty} + \|v - (v^s)^{-\mathbf{X}}\|_{L^\infty}) \int_{\mathbb{T}^2} \int_{\mathbb{R}} (v^s)_{\xi_1}^{-\mathbf{X}} d\xi_1 d\xi' \\ &\le C\|v - (v^s)^{-\mathbf{X}}\|_{L^\infty}, \end{aligned}$$

which implies (3.13). The proof of Proposition 3.6 is completed.

To finish the proof of Theorem 2.3, we remain to justify the time-asymptotic behaviors (2.12) and (2.13). Set

$$g(t) := \|\nabla_\xi \varphi(t)\|^2 + \|\nabla_\xi \psi(t)\|^2,$$

where φ, ψ are defined in Lemma 4.7. The aim is to show that

$$\int_0^{+\infty} (|g(t)| + |g'(t)|) dt < \infty,$$

which implies

$$\lim_{t\to+\infty} g(t) = \lim_{t\to+\infty} (\|\nabla_\xi \varphi(t)\|^2 + \|\nabla_\xi \psi(t)\|^2) = 0. \tag{4.67}$$

First, we can deduce from (3.12) that $\int_0^{+\infty} |g(t)| dt < \infty$. Then we apply ∇_ξ to equation $(4.58)_1$ to get

$$\begin{aligned} \int_0^t \|\nabla_\xi \partial_t \varphi\|^2 d\tau &\le C \int_0^t \|\nabla_\xi^2 (\varphi, \psi)\|^2 d\tau + C(\delta + \chi) \int_0^t \|\nabla_\xi (\varphi, \psi)\|^2 d\tau \\ &\quad + C\delta^2 \int_0^t (|\dot{\mathbf{X}}(\tau)|^2 + G_3(\tau) + G^s(\tau) + D(\tau)) d\tau \le C. \end{aligned}$$

Meanwhile, it follows from $(4.59)_2$ that

$$\int_0^t \|\nabla_\xi \partial_t \psi\|^2 d\tau \le C \int_0^t (\|\nabla_\xi^2(\varphi,\psi)\|^2 + \|\nabla_\xi^3 \psi\|^2) d\tau + C(\delta+\chi) \int_0^t \|\nabla_\xi(\varphi,\psi)\|^2 d\tau + C\delta^2 \int_0^t \Big(|\dot{\mathbf{X}}(\tau)|^2 + G_3(\tau) + G^s(\tau) + D(\tau)\Big) d\tau \le C.$$

Using the above two facts and the Cauchy inequality, we have

$$\int_0^{+\infty} |g'(t)| dt = \int_0^{+\infty} \int_{\mathbb{T}^2} \int_{\mathbb{R}} (2|\nabla_\xi \varphi||\nabla_\xi \partial_t \varphi| + 2|\nabla_\xi \psi||\nabla_\xi \partial_t \psi|) d\xi_1 d\xi' dt \le 2 \int_0^{+\infty} (\|\nabla_\xi \varphi\|\|\nabla_\xi \partial_t \varphi\| + \|\nabla_\xi \psi\|\|\nabla_\xi \partial_t \psi\|) dt < \infty.$$

By the Gagliardo–Nirenberg inequality in Lemma 3.2 and (4.67), we obtain

$$\lim_{t\to+\infty} \|(\varphi,\psi)\|_{L^\infty} \le \lim_{t\to+\infty} \Big(\sqrt{2}\|(\varphi,\psi)\|^{\frac12} \|\partial_{\xi_1}(\varphi,\psi)\|^{\frac12} + C\|\nabla_\xi(\varphi,\psi)\|^{\frac12} \|\nabla_\xi^2(\varphi,\psi)\|^{\frac12}\Big) = 0,$$

which proves (2.12). In addition, by (3.13) and the above large-time behavior, it holds

$$|\dot{\mathbf{X}}(t)| \le C\|v - (v^s)^{-\mathbf{X}}(t,\cdot)\|_{L^\infty} \to 0 \quad \text{as } t \to +\infty,$$

which proves (2.13). Thus, the proof of Theorem 2.3 is completed.

Acknowledgments. The authors would like to thank the anonymous referees for their valuable suggestions that helped improve the manuscript.

Funding. The work of Teng Wang is supported by National Key R&D Program of China (No. 2022YFA1007700) and the NSFC grant (No. 12371215). The work of Yi Wang is supported by National Key R&D Program of China (No. 2021YFA1000800), NSFC grants (No. 12171459, No. 12288201, No. 12090014) and CAS Project for Young Scientists in Basic Research, Grant No. YSBR-031.

References

[1] Goodman, J.: Nonlinear asymptotic stability of viscous shock profiles for conservation laws. Arch. Ration. Mech. Anal. **95**, 325–344 (1986) Zbl 0631.35058 MR 853782

[2] Goodman, J.: Stability of viscous scalar shock fronts in several dimensions. Trans. Amer. Math. Soc. **311**, 683–695 (1989) Zbl 0693.35112 MR 978372

[3] Hoff, D., Zumbrun, K.: Asymptotic behavior of multidimensional scalar viscous shock fronts. Indiana Univ. Math. J. **49**, 427–474 (2000) Zbl 0967.76049 MR 1793680

[4] Hoff, D., Zumbrun, K.: Pointwise Green's function bounds for multidimensional scalar viscous shock fronts. J. Differential Equations **183**, 368–408 (2002) Zbl 1028.35108 MR 1919784

[5] Huang, F., Matsumura, A.: Stability of a composite wave of two viscous shock waves for the full compressible Navier–Stokes equation. Comm. Math. Phys. **289**, 841–861 (2009) Zbl 1172.35054 MR 2511653

[6] Humpherys, J., Lyng, G., Zumbrun, K.: Multidimensional stability of large-amplitude Navier–Stokes shocks. Arch. Ration. Mech. Anal. **226**, 923–973 (2017) Zbl 1386.35326 MR 3712275

[7] Il'in, A. M., Oleĭnik, O. A.: Asymptotic behavior of solutions of the Cauchy problem for some quasi-linear equations for large values of the time. Mat. Sb. (N.S.) **51(93)**, 191–216 (1960) Zbl 0096.06601 MR 120469

[8] Kang, M.-J., Vasseur, A. F.: L^2-contraction for shock waves of scalar viscous conservation laws. Ann. Inst. H. Poincaré C Anal. Non Linéaire **34**, 139–156 (2017) Zbl 1368.35183 MR 3592682

[9] Kang, M.-J., Vasseur, A. F.: Contraction property for large perturbations of shocks of the barotropic Navier–Stokes system. J. Eur. Math. Soc. (JEMS) **23**, 585–638 (2021) Zbl 1460.76614 MR 4195742

[10] Kang, M.-J., Vasseur, A. F., Wang, Y.: L^2-contraction of large planar shock waves for multi-dimensional scalar viscous conservation laws. J. Differential Equations **267**, 2737–2791 (2019) Zbl 1417.35129 MR 3953019

[11] Kang, M.-J., Vasseur, A. F., Wang, Y.: Time-asymptotic stability of composite waves of viscous shock and rarefaction for barotropic Navier–Stokes equations. Adv. Math. **419**, article no. 108963 (2023) Zbl 1512.35442 MR 4560999

[12] Kang, M.-J., Vasseur, A. F., Wang, Y.: Time-asymptotic stability of generic Riemann solutions for compressible Navier–Stokes–Fourier equations. arXiv:2306.05604 (2023)

[13] Li, H.-L., Wang, T., Wang, Y.: Wave phenomena to the three-dimensional fluid-particle model. Arch. Ration. Mech. Anal. **243**, 1019–1089 (2022) Zbl 1507.35155 MR 4367916

[14] Li, L.-A., Wang, D., Wang, Y.: Vanishing viscosity limit to the planar rarefaction wave for the two-dimensional compressible Navier–Stokes equations. Comm. Math. Phys. **376**, 353–384 (2020) Zbl 1439.35372 MR 4093861

[15] Li, L.-A., Wang, D., Wang, Y.: Vanishing dissipation limit to the planar rarefaction wave for the three-dimensional compressible Navier–Stokes–Fourier equations. J. Funct. Anal. **283**, article no. 109499 (2022) Zbl 1489.76041 MR 4406409

[16] Li, L.-A., Wang, T., Wang, Y.: Stability of planar rarefaction wave to 3D full compressible Navier–Stokes equations. Arch. Ration. Mech. Anal. **230**, 911–937 (2018) Zbl 1414.35150 MR 3851052

[17] Li, L.-A., Wang, Y.: Stability of planar rarefaction wave to two-dimensional compressible Navier–Stokes equations. SIAM J. Math. Anal. **50**, 4937–4963 (2018) Zbl 1403.35205 MR 3853607

[18] Liu, T.-P.: Nonlinear stability of shock waves for viscous conservation laws. Mem. Amer. Math. Soc. **56**, v+108 (1985) Zbl 0617.35058 MR 791863

[19] Liu, T.-P., Zeng, Y.: Shock waves in conservation laws with physical viscosity. Mem. Amer. Math. Soc. **234**, vi+168 pp. (2015) Zbl 1338.35369 MR 3244333

[20] Mascia, C., Zumbrun, K.: Stability of large-amplitude viscous shock profiles of hyperbolic-parabolic systems. Arch. Ration. Mech. Anal. **172**, 93–131 (2004) Zbl 1058.35160 MR 2048568

[21] Mascia, C., Zumbrun, K.: Stability of small-amplitude shock profiles of symmetric hyperbolic-parabolic systems. Comm. Pure Appl. Math. **57**, 841–876 (2004) Zbl 1060.35111 MR 2044067

[22] Matsumura, A., Nishihara, K.: On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math. **2**, 17–25 (1985) Zbl 0602.76080 MR 839317

[23] Matsumura, A., Nishihara, K.: Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas. Japan J. Appl. Math. **3**, 1–13 (1986) Zbl 0612.76086 MR 899210

[24] Matsumura, A., Nishihara, K.: Global stability of the rarefaction wave of a one-dimensional model system for compressible viscous gas. Comm. Math. Phys. **144**, 325–335 (1992) Zbl 0745.76069 MR 1152375

[25] Solonnikov, V. A.: The solvability of the initial-boundary value problem for the equations of motion of a viscous compressible fluid. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **56**, 128–142 (1976) Zbl 0338.35078 MR 481666

[26] Szepessy, A., Xin, Z. P.: Nonlinear stability of viscous shock waves. Arch. Ration. Mech. Anal. **122**, 53–103 (1993) Zbl 0803.35097 MR 1207241

[27] Wang, T., Wang, Y.: Nonlinear stability of planar rarefaction wave to the three-dimensional Boltzmann equation. Kinet. Relat. Models **12**, 637–679 (2019) Zbl 1420.35177 MR 3928124

[28] Zumbrun, K.: Planar stability criteria for viscous shock waves of systems with real viscosity. In: Hyperbolic systems of balance laws, Lecture Notes in Math. 1911, Springer, Berlin, 229–326 (2007) Zbl 1138.35061 MR 2348937